Assignment 05
Solving Boundary Value Problem
Using Finite Difference Method

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1 Formulation

Boundary value problem is of the form

$$x^{''} = f(t, x, x^{'})$$

$$in\ a \leq t \leq b\ with\ x(a) = \alpha;\ x(b) = \beta$$

Central difference formula:

$$x^{'}(t) = \frac{1}{2h}[x(t+h) - x(t-h)]$$

$$x^{''}(t) = \frac{1}{h^2}[x(t+h) - 2x(t) + x(t-h)]$$

where h is spacing between consecutive data points. For equal spacing

$$h = (\frac{b-a}{n})$$

where n is number of intervals.
Given Problem:

$$x^{''} = xsin(t) + x^{'}cos(t) - e^t$$

$$x(0) = 0;\ x(1) = 1;$$

The problem can be written using central difference formulas as

$$\frac{1}{h^2}[x(t+h) - 2x(t) + x(t-h)] = x_i sin(t_i) + \frac{1}{2h}[x(t+h) - x(t-h)]cos(t_i) - e^{t_i}$$

$$\frac{1}{h^2}(x_{i+1} - 2x_i + x_{i-1}) = x_i sin(t_i) + \frac{1}{2h}(x_{i+1} - x_{i-1})cos(t_i) - e^{t_i}$$

where

$$t_i = a + ih \quad i = 1,\ 2, ..., n-1$$

$$x_i = x(t_i)$$

now collecting $x_{i+1},\ x_i,\ x_{i-1}$ terms on LHS and terms without x on RHS we get

$$(\frac{1}{h^2} - \frac{cos(t_i)}{2h})x_{i+1} - (\frac{2}{h^2} + sin(t_i))x_i + (\frac{1}{h^2} + \frac{cos(t_i)}{2h})x_{i-1} = -e^{t_i}$$

multiplying both side by h^2 we get

$$(1 - \frac{hcos(t_i)}{2})x_{i+1} - (2 + h^2 sin(t_i))x_i + (1 + \frac{hcos(t_i)}{2})x_{i-1} = -h^2 e^{t_i}$$

Then we can write this above equation in the form

$$a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i$$

where

$$a_i = (1 + \frac{hcos(t_i)}{2})$$

$$b_i = -h^2 e^{t_i}$$

$$d_i = -(2 + h^2 sin(t_i))$$

$$c_i = (1 - \frac{hcos(t_i)}{2})$$

as $x_0 = \alpha$ and $x_n = \beta$ are known we can write the above equation as

$$d_1 x_1 + c_1 x_2 = b_1 - a_1 \alpha$$

$$a_2 x_1 + d_2 x_2 + c_2 x_3 = b_2$$

$$a_3 x_2 + d_3 x_3 + c_3 x_4 = b_3$$

$$\vdots \; \vdots$$

$$a_{n-1} x_{n-2} + d_{n-1} x_{n-1} = b_{n-1} - c_{n-1}\beta$$

Now these equations can be written in matrix form as

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 & ... & 0 & 0 \\ a_2 & d_2 & c_2 & 0 & 0 & ... & 0 & 0 \\ 0 & a_3 & d_3 & c_3 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & a_{n-1} & d_{n-1} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{Bmatrix} = \begin{Bmatrix} b_1 - a_1\alpha \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} - c_{n-1}\beta \end{Bmatrix}$$

The matrix is tridiagonal matrix. And the problem is of the form $\mathbf{Ax} = \mathbf{B}$ and it can be solved using gauss elimination method.

2 Algorithms

2.1 Initialization

```
procedure FILL_AB(m, a, b, α, β)
                                        ▷ m = number of intervals - 1
                                        ▷ x(a) = α; x(b) = β
    A ← initialize matriz of size m×3   ▷ Only tridiagonal elements are stored
    B ← initialize vector of size m
    for i←1:1:m do
                                        ▷ a_i, b_i, c_i, d_i are from formulation
        if i > 1 then
            A[i][1] ← a_i
        end if
        A[i][2] ← d_i
        if i < m then
            A[i][3] ← c_i
        end if
        B[i] ← b_i
    end for
    B[0] ← B[0] - a_1 ∗ α
    B[m] ← B[m] - c_m ∗ β
    return A,B
end procedure
```

2.2 Solving Ax = B

```
procedure SOLVE(A[][], B[])
                                        ▷ A[][] is matrix of size m × 3
                                        ▷ B[] is vector of size m × 1
                                        ▷ Elimination Steps
    for i←1:1:m-1 do
        ratio ← A[i+1][1] / A[i][2]
        A[i + 1][2] ← A[i + 1][2] - (ratio × A[i][3])
        B[i + 1] ← B[i + 1] - (ratio × B[i])
    end for
                                        ▷ Backsubstitution
    x ← initialize vector of size m × 1
    x[m] ← B[m] / A[m][2]
    for i←m - 1:-1:1 do
        x[i] ← (B[i] − (A[i][3] × x[i+1])) / A[i][2]
    end for
    return x
end procedure
```

3 Results

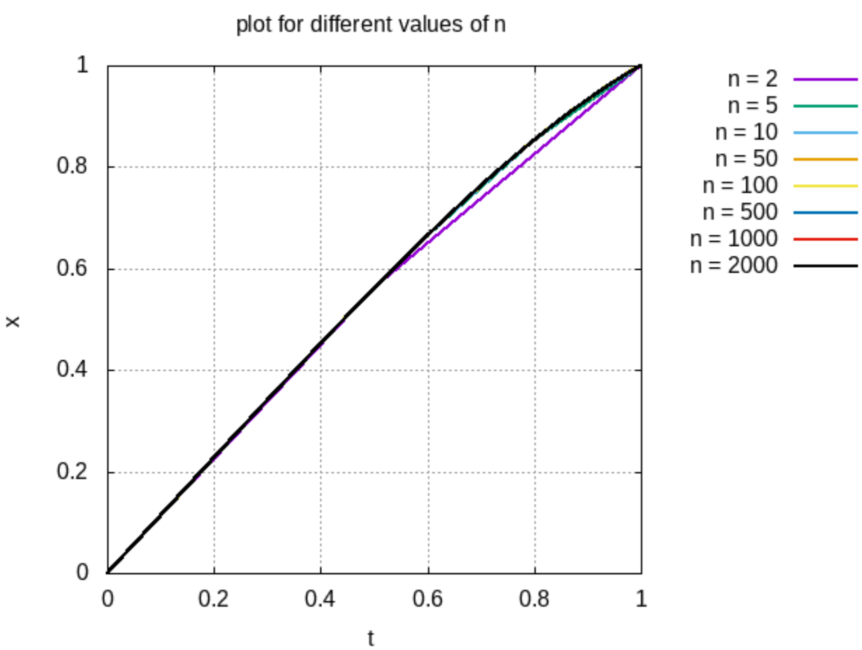
3.1 Solution

t	n = 2	n = 10	n = 100	n == 1000
0	0	0	0	0
0.1	-	0.113591	0.113569	0.113568
0.2	-	0.227566	0.227526	0.227526
0.3	-	0.34091	0.340868	0.340867
0.4	-	0.45252	0.45247	0.452469
0.5	0.562672	0.56111	0.561062	0.561061
0.6	-	0.66524	0.665196	0.665195
0.7	-	0.763251	0.76321	0.76321
0.8	-	0.853224	0.853196	0.853196
0.9	-	0.932975	0.932961	0.932961
1	1	1	1	1

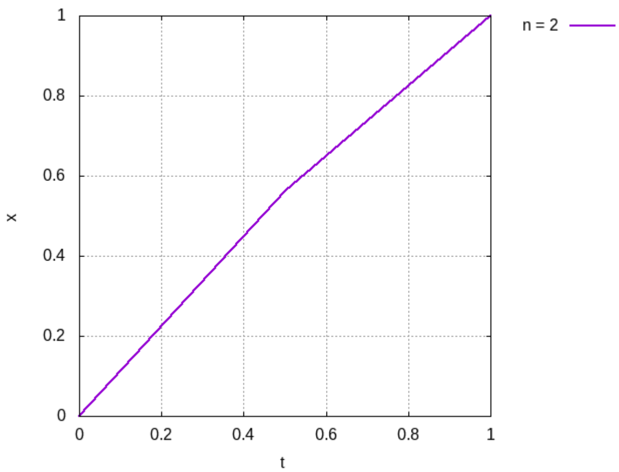
Table 1: Table showing value of x(t) for various n i.e. number of intervals

3.2 Plots

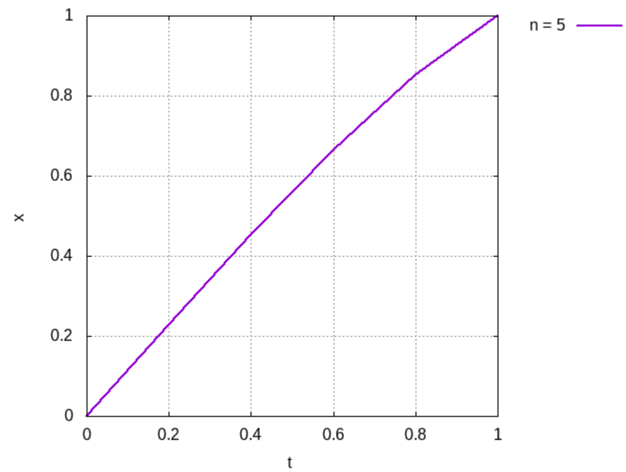
Following plot shows results obtained for different values of n.



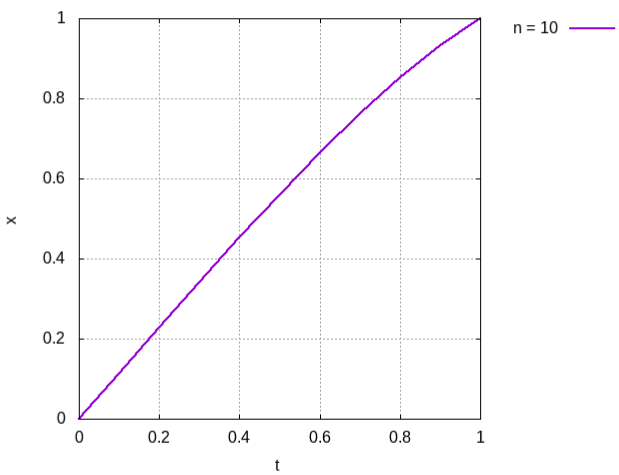
The curves are overlapping because the points obtained for different values of n had very same value up to 4 to 5 decimal places. For n = 2 the curve can be distinguished and for n = 5 the curve is almost overlapping and curves of n = 10, 50, 100, 500, 1000, 2000 are completely overlapping.



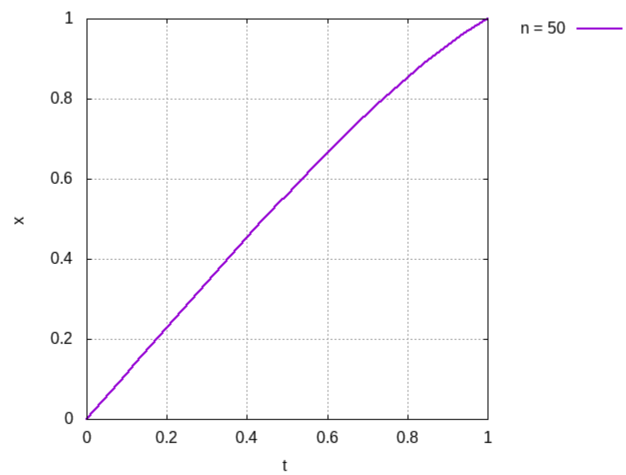
(a)



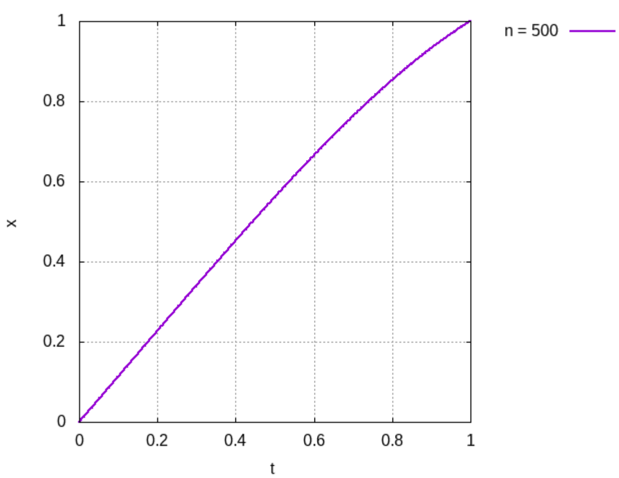
(b)



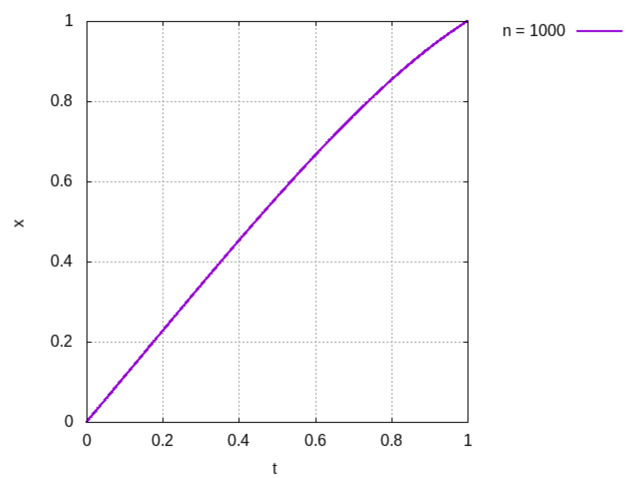
(c)



(d)



(e)



(f)

Table 2: Graphs for different values of n a) 2; b) 5; c) 10; d) 50; e) 500; f) 1000

4 Conclusion

- The given problem can be solved with 10 intervals.
- Time and Space complexity of solving tridiagonal matrix for given problem with gauss elimination is $\mathbf{O(n)}$ where n is number of intervals.