EC1410-Spring 2022
Problem Set 3 solutions

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1. This problem will examine the change in population density and its gradient over time based on Clark's 1951 study of population density in major metropolitan areas in Australia, the US and Europe.

   (a) Using Table 1 from Clark (1951), calculate the population density of London at the CBD (x=0), and three miles from the CBD, in 1801, 1841, and 1939.

   Population density is given by $y = Ae^{-bx}$, where A and b are given in the table.
   In 1801, $y = 290e^{-1.35x}$

   $$\text{At } x = 0,\ y = 290e^{0} = 290.$$
   $$\text{At } x = 3,\ y = 290e^{-3*1.35} = 290e^{-4.05} \approx 5.05.$$

   In 1841, $y = 800e^{-1.4x}$

   $$\text{At } x = 0,\ y = 800e^{0} = 800.$$
   $$\text{At } x = 3,\ y = 800e^{-3*1.4} = 800e^{-4.2} \approx 12.$$

   (Note this implies 800,000 people, or three-quarters of the population of Rhode Island, lived in the square mile at the London city center).
   In 1939, $y = 80e^{-0.2x}$

   $$\text{At } x = 0,\ y = 80e^{0} = 80.$$
   $$\text{At } x = 3,\ y = 80e^{-3*0.2} = 80e * -0.6 \approx 43.9.$$

   (Note how much the population density at the city center dropped in one century, and how much the population density at $x = 3$ grew.)

   (b) How does the ratio of population density at the CBD to the population density three miles from the CBD change between 1801 and 1939?

   In 1801, the ratio of population densities at the CBD to the population density three miles from the CBD is

   $$\frac{290e^{0}}{290e^{-4.05}} = e^{4.05} \approx 57.4$$

   In 1939, the relevant ratio of population densities is

   $$\frac{80e^{0}}{80e^{-0.6}} = e^{0.6} \approx 1.82$$

   Quite a decline!

2. This problem will examine the relationship between population density and transportation costs.

   In 1, you saw that the population density gradient flattened out (there is relatively more population at $x = 3$ compared to $x = 0$ over time). This problem examines if decreased transportation costs could explain this flattening of the density gradient.

   Assume we have the setup of the monocentric city model with housing, as in the lecture. Assume as well that housing production is perfectly competitive. Let $\overline{u} = 3$.

   (a) Let the household's problem be given by

   $$\max_{c,h,x} c^{1/2}h^{1/2} \text{ subject to } w = c + ph + 2tx$$

   Let $\widetilde{w} = w - 2tx$. Use the first-order condition of the household's problem with respect to h to find h^* in terms of p and $\widetilde{w}$.
   The household's problem is:

   $$\max_{c,h,x} c^{1/2}h^{1/2} \text{ subject to } \widetilde{w} = c + ph$$

   Solving the constraint for c, we have $c = \widetilde{w} - ph$. If we plug this into the expression that we want to maximize, we no longer need to worry about the constraint, but instead can just solve:

   $$\max_{h}(\widetilde{w} - ph)^{1/2}h^{1/2}$$

   We do this, finding the value of h^* that maximizes the above expression, by setting the derivative of this expression with respect to h equal to zero.

   $$\begin{aligned}
   \frac{\partial[(\widetilde{w} - ph)^{1/2}h^{1/2}]}{\partial h} &= 0 \\
   -(1/2)p(\widetilde{w} - ph^*)^{-1/2}h^{*1/2} + (1/2)(\widetilde{w} - ph^*)^{1/2}h^{-*1/2} &= 0 \\
   (1/2)p(\widetilde{w} - ph^*)^{-1/2}h^{*1/2} &= (1/2)(\widetilde{w} - ph^*)^{1/2}h^{*-1/2} \\
   ph^* &= \widetilde{w} - ph^* \\
   2ph^* &= \widetilde{w} \\
   h^* &= \frac{\widetilde{w}}{2p}
   \end{aligned}$$

   (b) Use the fact that utility is $\overline{u} = 3$ everywhere to solve for p^* in terms of $\widetilde{w}$.

$$
\begin{aligned}
\overline{u} &= 3 \\
\overline{u} &= (\widetilde{w} - p^*h^*)^{1/2} h^{*1/2} \\
&= \left(\widetilde{w} - p^* \frac{\widetilde{w}}{2p^*}\right)^{1/2} \left(\frac{\widetilde{w}}{2p^*}\right)^{1/2} \\
&= \left(\frac{\widetilde{w}}{2}\right)^{1/2} \left(\frac{\widetilde{w}}{2p^*}\right)^{1/2} \\
&= \sqrt{\frac{\widetilde{w}^2}{2^2 p^*}} \\
3 &= \frac{\widetilde{w}}{2\sqrt{p^*}} \\
\sqrt{p^*} &= \frac{\widetilde{w}}{2 * 3} \\
p^* &= \frac{\widetilde{w}^2}{36}
\end{aligned}
$$

(c) Substitute your expressions for p^* and $\widetilde{w}$ into your expression for h^* to write h^* in terms of w, t, and x.

$$
\begin{aligned}
h^* &= \frac{\widetilde{w}}{2p} \\
&= \frac{\widetilde{w}}{\widetilde{w}^2/18} \\
&= \frac{18}{\widetilde{w}} \\
&= \frac{18}{w - 2tx}
\end{aligned}
$$

(d) Let the developer's problem be given by

$$\max_S pS^{2/3} - iS - R$$

where S is the capital to land ratio, and p, i and R are the costs of housing, capital, and land, respectively. For the remainder of the problem, let $i = \frac{1}{33}$.
Comment: The technology for producing housing is constant returns to scale and can be written as $h_s(S) = S^{2/3}$. Here, h_s is housing supplied, and is (with constant returns to scale) units of housing per constant area. This is NOT the same as h in the household problem, which is housing units per person.
Use the first-order condition of this problem with respect to S to solve for ${h_s}^*$ in terms of p.

$$
\begin{aligned}
\frac{\partial(pS^{2/3} - iS - R)}{\partial S} &= (2/3)pS^{-1/3} - i \\
(2/3)p^* S^{*-1/3} - i &= 0 \\
(2/3)p^* S^{*-1/3} &= i \\
S^{*-1/3} &= \frac{1.5i}{p^*} \\
S^* &= \left(\frac{p^*}{1.5/33}\right)^3 = (22p^*)^3 \\
{h_s}^* &= (S^*)^{(2/3)} = ((22p^*)^3)^{(2/3)} = (22p^*)^2
\end{aligned}
$$

(e) Substitute in your expression for p^* from earlier to obtain an expression for population density, $\frac{{h_s}^*}{h^*}$, in terms of w, t and x.

$$
\begin{aligned}
\frac{{h_s}^*}{h^*} &= \frac{(22p^*)^2}{\frac{18}{w-2tx}} \\
&= \frac{(22(\widetilde{w}^2/36))^2}{\frac{18}{w-2tx}} \\
&= \frac{\widetilde{w}^4(w-2tx)(11/18)^2}{18} \\
&= \frac{(w-2tx)^5 * 11^2}{18^3}
\end{aligned}
$$

(f) Solve for the population density at $x = 1$ and $x = 2$ for $t = 1$ and $t = 0.5$. How does population density outside of the city center change when transportation costs fall?

$$
\begin{aligned}
\frac{{h_s}^*}{h^*} &= \frac{(w-2tx)^5 * 11^2}{18^3} \\
&= \frac{(w-2)^5 * 11^2}{18^3} \text{ for } x = 1, t = 1 \\
&= \frac{(w-1)^5 * 11^2}{18^3} \text{ for } x = 1, t = 0.5 \\
&= \frac{(w-4)^5 * 11^2}{18^3} \text{ for } x = 2, t = 1 \\
&= \frac{(w-2)^5 * 11^2}{18^3} \text{ for } x = 2, t = 0.5
\end{aligned}
$$

Population density outside the city center increases (less is subtracted from w) when transportation costs fall.

(g) In order to obtain a more general result about the population density gradient and transportation costs, take the derivative of your expression for population density with respect to t.

$$\begin{aligned}
\frac{{h_s}^*}{h^*} &= \frac{(w-2tx)^5 * 11^2}{18^3} \\
\frac{\partial(\frac{{h_s}^*}{h^*})}{\partial t} &= 5(-2x)(w-2tx)^4 * \frac{11^2}{18^3} \\
&= -10x(w-2tx)^4 * \frac{11^2}{18^3} < 0
\end{aligned}$$

(h) Evaluate the derivative from the previous part at $x = 0$. Does your expression for population density at $x = 0$ depend on t?

$$\begin{aligned}
\frac{\partial(\frac{{h_s}^*}{h^*})}{\partial t} &= -10x(w-2tx)^5 * \frac{11^2}{18^3} \\
\frac{\partial(\frac{{h_s}^*}{h^*})}{\partial t}\Big|_{x=0} &= 0 \\
\frac{{h_s}^*}{h^*} &= \frac{(w-2tx)^5 * 11^2}{18^3} \\
\frac{{h_s}^*}{h^*}\Big|_{x=0} &= \frac{w^5 * 11^2}{18^3}
\end{aligned}$$

The population density at $x = 0$ does not change when transportation costs change (intuitively, because the people at $x = 0$ do not pay the transportation costs).

(i) Based on your results to the previous two parts, could this model of falling transportation costs explain the decreasing population density at the center and flattening of the population density gradient that you examined in the previous problem?

This model of falling transportation costs could explain the flattening of the population density gradient, as we saw in the example where we evaluated the population density at $x = 1$ and $x = 2$ as t decreased from 1 to 0.5. However, this model does not explain the decreasing population density at the center.