

# EC1410-Spring 2024

## Problem Set 7 solutions

(Updated 12 December 2023)

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1. To answer these questions, holding rent constant, check whether (1) a household requires more or less income to achieve the reservation utility level when the amenity increases, and (2) whether the firm must pay a higher or lower wage to achieve the zero profit condition when the amenity increases. Using this logic, in panel (a) increases in the amenity are harmful to households – they achieve the same utility with a lower wage under the lower level of  $A$  – and harmful to firms – they require a lower wage to produce one unit of output when  $A$  increase, holding  $r$  constant. In panel (b), the amenity is still harmful to firms, but is beneficial to households.

See illustrations.

2. This problem asks you to let  $U(c, l_c, A) = \bar{u}$ . Assume the household problem is given by

$$\max_{c, l_c} U(c, l_c, A) = Ac^{2/3}l_c^{1/3} \text{ such that } w = c + rl_c$$

- (a) Solve the constraint for  $c$ . Plug your expression for  $c$  into the utility function.

$$w = c + rl_c$$

$$c = w - rl_c$$

$$\begin{aligned} U(c, l_c, A) &= Ac^{2/3}l_c^{1/3} \\ &= A(w - rl_c)^{2/3}l_c^{1/3} \end{aligned}$$

- (b) Solve the maximum problem for  $l_c$ .

Since we have substituted the constraint into the function we wish to maximize, we now have a single-variable unconstrained maximization problem, which we can solve by setting the derivative of the utility function with respect to  $l_c$  equal to zero.

$$U(l_c, A) = A(w - rl_c)^{2/3}l_c^{1/3}$$

$$\frac{\partial U(l_c, A)}{\partial l_c} = -(2/3)rA(w - rl_c)^{-1/3}l_c^{1/3} + (1/3)A(w - rl_c)^{2/3}l_c^{-2/3}$$

$$(2/3)rA(w - rl_c)^{-1/3}l_c^{1/3} = (1/3)A(w - rl_c)^{2/3}l_c^{-2/3}$$

$$\frac{2}{3}rl_c = \frac{1}{3}(w - rl_c)$$

$$2rl_c = w - rl_c$$

$$w = 3rl_c$$

$$l_c = \frac{w}{3r}$$

- (c) Find the indirect utility function  $V(w, r, A)$  by substituting demand for housing and consumption into  $U(c, l_c, A)$ .

From above,

$$\begin{aligned} w &= 3rl_c \\ c &= w - rl_c \\ &= 3rl_c - rl_c \\ &= 2rl_c \\ &= \frac{2}{3}w \end{aligned}$$

Then using our expressions for  $c$  and  $l_c$  in the utility function:

$$\begin{aligned} U(c, l_c, A) &= Ac^{2/3}l_c^{1/3} \\ &= A\left(\frac{2w}{3}\right)^{2/3}\left(\frac{w}{3r}\right)^{1/3} \\ &= A\left(\frac{2}{3}\right)^{2/3}\left(\frac{1}{3}\right)^{1/3}w^{2/3}w^{1/3}r^{-1/3} \\ &= A\left(\frac{2}{3}\right)^{2/3}\left(\frac{1}{3}\right)^{1/3}wr^{-1/3} \\ &= V(w, r, A) \end{aligned}$$

- (d) Define an indifference curve by  $V(w, r, A) = \bar{u}$ . Solve for  $r$  in terms of  $A, \bar{u}$ , and  $w$ .

$$\begin{aligned} V(w, r, A) &= \bar{u} \\ &= A\left(\frac{2}{3}\right)^{2/3}\left(\frac{1}{3}\right)^{1/3}wr^{-1/3} \\ r &= \frac{1}{3}A^3\left(\frac{2}{3}\right)^2\left(\frac{w}{\bar{u}}\right)^3 \end{aligned}$$

- (e) Evaluate  $\frac{\partial r}{\partial w}$ . What is the sign of this derivative?

Is  $A$  an amenity or dis-amenity from the perspective of the consumer? Explain briefly.

$$\begin{aligned} r &= \frac{1}{3}A^3\left(\frac{2}{3}\right)^2\left(\frac{w}{\bar{u}}\right)^3 \\ \frac{\partial r}{\partial w} &= \frac{4}{9}\left(\frac{A}{\bar{u}}\right)^3w^2 > 0 \end{aligned}$$

$A$  is an amenity from the perspective of the consumer. Looking at our expression for  $r$ , we can see that increasing  $A$  but holding  $w$  constant will increase  $r$ . That means that more  $A$  must be offset by higher rent/lower wages to keep

utility constant - so people are essentially willing to “pay” for more  $A$ , making it an amenity.

Additionally, notice that as  $A$  increases,  $\frac{\partial r}{\partial w}$  increases - so the tradeoff between rent and wages becomes steeper. That is, for a given increase in wages, rents have to increase even more than they did before to keep constant utility, due to the utility value of the higher  $A$ .

3. This problem asks you to calculate the importance of amenity  $A$  in real terms. Assume you have data on rents, wages, and amenity  $A$  for a cross-section of cities. That is, your data is  $\{r_i, w_i, A_i\}$  for a set of cities  $i = 1, \dots, J$ . You may also assume that housing expenditure is one-third of the city wage. Describe the regressions you would run, and any subsequent analysis you would do, to determine the importance of amenity  $A$  in real terms (that is, as a share of the city wage).

Our statement of the Roback theorem is

$$\frac{p_A}{w} = \frac{l_c r}{w} \frac{\partial \ln r}{\partial A} - \frac{\partial \ln w}{\partial A}$$

To determine  $\frac{\partial \ln r}{\partial A}$  and  $\frac{\partial \ln w}{\partial A}$ , we can perform the following regressions:

$$\ln r_i = \alpha_1 + \beta_1 A_i + \epsilon_i$$

$$\ln w_i = \alpha_2 + \beta_2 A_i + \mu_i$$

The regression results will return:

$$\beta_1 = \frac{\partial \ln r}{\partial A}$$

$$\beta_2 = \frac{\partial \ln w}{\partial A}$$

Since we were given that housing expenditure is one-third of the city wage, we have:

$$\begin{aligned} \frac{p_A}{w} &= \frac{l_c r}{w} \frac{\partial \ln r}{\partial A} - \frac{\partial \ln w}{\partial A} \\ &= \frac{l_c r}{w} \beta_1 - \beta_2 \\ &= \frac{1}{3} \beta_1 - \beta_2 \end{aligned}$$

And this is how we would compute the importance of amenity  $A$  in real terms.