EC1410-Spring 2022
Problem Set 6 solutions

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1. To answer these questions, holding rent constant, check whether (1) a household requires more or less income to achieve the reservation utility level when the amenity increases, and (2) whether the firm must pay a higher or lower wage to achieve the zero profit condition when the amenity increases. Using this logic, in panel (a) increases in the amenity are harmful to households – they achieve the same utility with a lower wage under the lower level of A – and harmful to firms – they require a lower wage to produce one unit of output when A increase, holding r constant. In panel (b), the amenity is still harmful to firms, but is beneficial to households.

   See illustrations.

2. This problem asks you to let $U(c,l_c,A) = \overline{u}$. Assume the household problem is given by
$$\max_{c,l_c} U(c, l_c, A) = Ac^{2/3} l_c^{1/3} \text{ such that } w = c + rl_c$$

   (a) Solve the constraint for c. Plug your expression for c into the utility function.

$$\begin{aligned}
w &= c + rl_c \\
c &= w - rl_c \\
U(c,l_c,A) &= Ac^{2/3} l_c^{1/3} \\
&= A(w - rl_c)^{2/3} l_c^{1/3}
\end{aligned}$$

   (b) Solve the maximum problem for l_c.
   Since we have substituted the constraint into the function we wish to maximize, we now have a single-variable unconstrained maximization problem, which we can solve by setting the derivative of the utility function with respect to l_c equal to zero.

$$\begin{aligned}
U(l_c,A) &= A(w - rl_c)^{2/3} l_c^{1/3} \\
\frac{\partial U(l_c,A)}{\partial l_c} &= -(2/3)rA(w - rl_c)^{-1/3} l_c^{1/3} + (1/3)A(w - rl_c)^{2/3} l_c^{-2/3} \\
(2/3)rA(w - rl_c)^{-1/3} l_c^{1/3} &= (1/3)A(w - rl_c)^{2/3} l_c^{-2/3} \\
\frac{2}{3} rl_c &= \frac{1}{3}(w - rl_c) \\
2rl_c &= w - rl_c \\
w &= 3rl_c \\
l_c &= \frac{w}{3r}
\end{aligned}$$

(c) Find the indirect utility function $V(w,r,A)$ by substituting demand for housing and consumption into $U(c,l_c,A)$.

From above,

$$\begin{aligned}
w &= 3rl_c \\
c &= w - rl_c \\
&= 3rl_c - rl_c \\
&= 2rl_c \\
&= \frac{2}{3}w
\end{aligned}$$

Then using our expressions for c and l_c in the utility function:

$$\begin{aligned}
U(c,l_c,A) &= Ac^{2/3}l_c^{1/3} \\
&= A\Big(\frac{2w}{3}\Big)^{2/3}\Big(\frac{w}{3r}\Big)^{1/3} \\
&= A\Big(\frac{2}{3}\Big)^{2/3}\Big(\frac{1}{3}\Big)^{1/3}w^{2/3}w^{1/3}r^{-1/3} \\
&= A\Big(\frac{2}{3}\Big)^{2/3}\Big(\frac{1}{3}\Big)^{1/3}wr^{-1/3} \\
&= V(w,r,A)
\end{aligned}$$

(d) Define an indifference curve by $V(w,r,A) = \overline{u}$. Solve for r in terms of $A, \overline{u},$ and w.

$$\begin{aligned}
V(w,r,A) &= \overline{u} \\
&= A\Big(\frac{2}{3}\Big)^{2/3}\Big(\frac{1}{3}\Big)^{1/3}wr^{-1/3} \\
r &= \frac{1}{3}A^3\Big(\frac{2}{3}\Big)^2\Big(\frac{w}{\overline{u}}\Big)^3
\end{aligned}$$

(e) Evaluate $\frac{\partial r}{\partial w}$. What is the sign of this derivative?

Is A an amenity or dis-amenity from the perspective of the consumer? Explain briefly.

$$\begin{aligned}
r &= \frac{1}{3}A^3\Big(\frac{2}{3}\Big)^2\Big(\frac{w}{\overline{u}}\Big)^3 \\
\frac{\partial r}{\partial w} &= \frac{4}{9}\Big(\frac{A}{\overline{u}}\Big)^3 w^2 > 0
\end{aligned}$$

A is an amenity from the perspective of the consumer. Looking at our expression for r, we can see that increasing A but holding w constant will increase r. That means that more A must be offset by higher rent/lower wages to keep utility constant - so people are essentially willing to "pay" for more A, making it an amenity.

Additionally, notice that as A increases, $\frac{\partial r}{\partial w}$ increases - so the tradeoff between rent and wages becomes steeper. That is, for a given increase in wages, rents have to increase even more than they did before to keep constant utility, due to the utility value of the higher A.

3. This problem asks you to calculate the importance of amenity A in real terms. Assume you have data on rents, wages, and amenity A for a cross-section of cities. That is, your data is $\{r_i, w_i, A_i\}$ for a set of cities $i = 1, ..., J$. You may also assume that housing expenditure is one-third of the city wage. Describe the regressions you would run, and any subsequent analysis you would do, to determine the importance of amenity A in real terms (that is, as a share of the city wage).

   Our statement of the Roback theorem is

$$\frac{p_A}{w} = \frac{l_c r}{w}\frac{\partial \ln r}{\partial A} - \frac{\partial \ln w}{\partial A}$$

   To determine $\frac{\partial \ln r}{\partial A}$ and $\frac{\partial \ln w}{\partial A}$, we can perform the following regressions:

$$\begin{aligned} \ln r_i &= \alpha_1 + \beta_1 A_i + \epsilon_i \\ \ln w_i &= \alpha_2 + \beta_2 A_i + \mu_i \end{aligned}$$

   The regression results will return:

$$\begin{aligned} \beta_1 &= \frac{\partial \ln r}{\partial A} \\ \beta_2 &= \frac{\partial \ln w}{\partial A} \end{aligned}$$

   Since we were given that housing expenditure is one-third of the city wage, we have:

$$\begin{aligned} \frac{p_A}{w} &= \frac{l_c r}{w}\frac{\partial \ln r}{\partial A} - \frac{\partial \ln w}{\partial A} \\ &= \frac{l_c r}{w}\beta_1 - \beta_2 \\ &= \frac{1}{3}\beta_1 - \beta_2 \end{aligned}$$

   And this is how we would compute the importance of amenity A in real terms.