EC2410-Spring 2016
Problem Set 2

(Updated 13 February 2018)

Matt Turner

1. Consider a city with measure one of land. All city residents receive a wage w and consume land inelastically so that measure on of residents occupies measure 1 on land, i.e., fills the city up. Residents pay land rent $R \geq 0$ and derive utility from consumption. The set of potential city residents is the set $[0,\Theta]$, with measure Θ and is indexed by θ. Agent θ's utility is,

$$u(\theta) = \begin{cases} w - R & \text{if } \theta \quad \text{in city} \\ \theta & \text{else} \end{cases}$$

   That is, agents get utility from consuming $w - R$ in the city, and an idiosyncratic reservation value outside the city. Consider two cases, $\Theta \geq 1 > w$ and $\Theta \geq w > 1$.

   (a) Characterize a free mobility equilibrium for this economy, and in particular, find land rent for all locations in the city.

   (b) Calculate aggregate land rent and consumers' surplus in equilibrium.

   (c) Is land rent as interesting a measure of welfare in this model as in the linear city model? Explain briefly.

2. Consider a 'partially mixed' land use distribution in a linear city. Specifically, suppose that the central region of the city is mixed, i.e., occupied by firms an households. This central mixed region is surrounded by two symmetric business districts, occupied solely by firms. Finally, these business districts are surrounded by purely residential regions. Suppose that the details of this economy are as described in the Fujita and Ogawa we discussed in class. Can you construct land rent, agglomeration and wage gradients such that this spatial configuration is an equilibrium? Draw a graph to illustrate and explain briefly.

3. This question asks you to find the rent gradient three different ways. Suppose that $u(h,z) = h^{\alpha}z^{1-\alpha}$ where h is housing, z is consumption. Suppose agents choose location x, have income w and pay unit transportation cost t.

   (a) Find $p(x)$ using the Marshallian method.

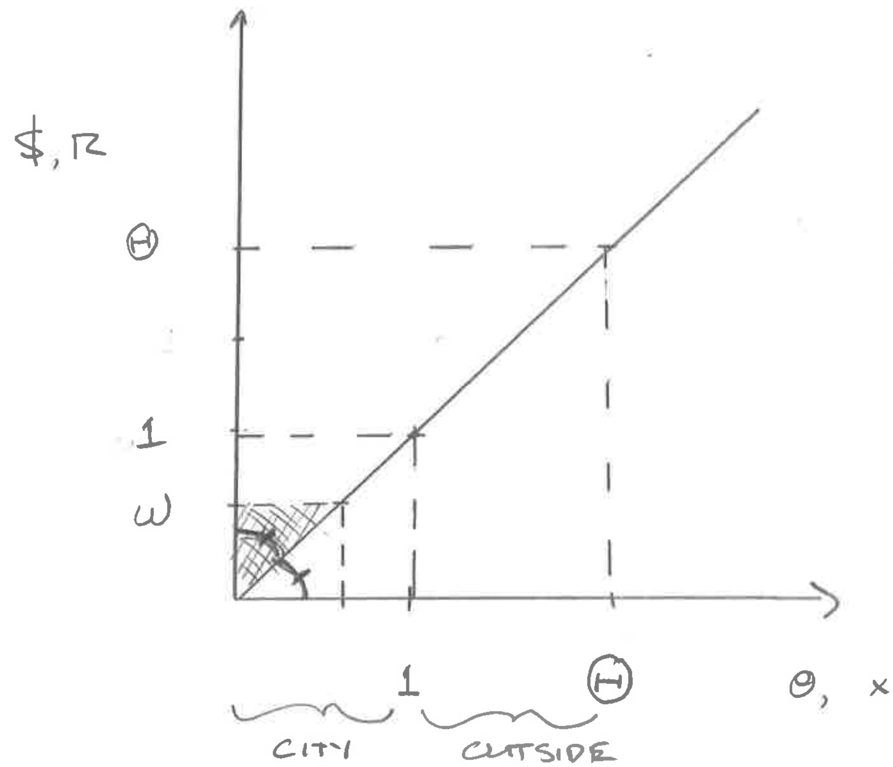
   (b) Find $p(x)$ using the Bid-rent approach.

   (c) Find $\frac{dp}{dx}$ using the expenditure function approach. For extra credit, verify that $p(x)$ you found in the first two parts of this question satisfies this definition of $\frac{dp}{dx}$.

4. (Extra credit) Derive the equation $p_s^* = l^c \frac{dr}{ds} + \frac{dw}{ds}$ from equation (5) of Roback (1982).

① FIRST NOTE THAT AN AGENT WITH A LOW θ WILL ALWAYS OUT-BID AN AGENT WITH A HIGH θ FOR A SPOT IN THE CITY.

THEN, EQUILIBRIUM LOOKS LIKE THIS:



SUPPOSE $\Theta \geq 1 > \omega$ AS DRAWN. THEN IF $R=0$ AND

$\theta \leq \omega$ LIVE IN THE CITY AND $\theta > \omega$ LIVE OUTSIDE, THEN NO ONE WANTS TO MOVE, AND NO LANDLORD CAN INCREASE THEIR RENT. THIS IS A FREE-MOBILITY EQUILIBRIUM WITH THE CITY PARTLY OCCUPIED. AGGREGATE LAND RENT IS ZERO AND CONSUMERS' SURPLUS IS THE HATCHED REGION,

$$\int_0^{\omega} (\omega - \theta)\, d\theta$$

IF $\Theta \geq \omega > 1$ THEN THE CITY IS FULLY OCCUPIED BY $\{\theta \mid 0 \leq \theta \leq 1\}$. THE MARGINAL RESIDENT IS $\theta = 1$. RENT ADJUSTS

SO THAT SHE IS INDIFFERENT BETWEEN THE CITY AND THE OUTSIDE OPTION, SO $R = W - 1$. IN THIS CASE, NO ONE WANTS TO MOVE AND LANDLORDS CAN'T RAISE THEIR RENT.

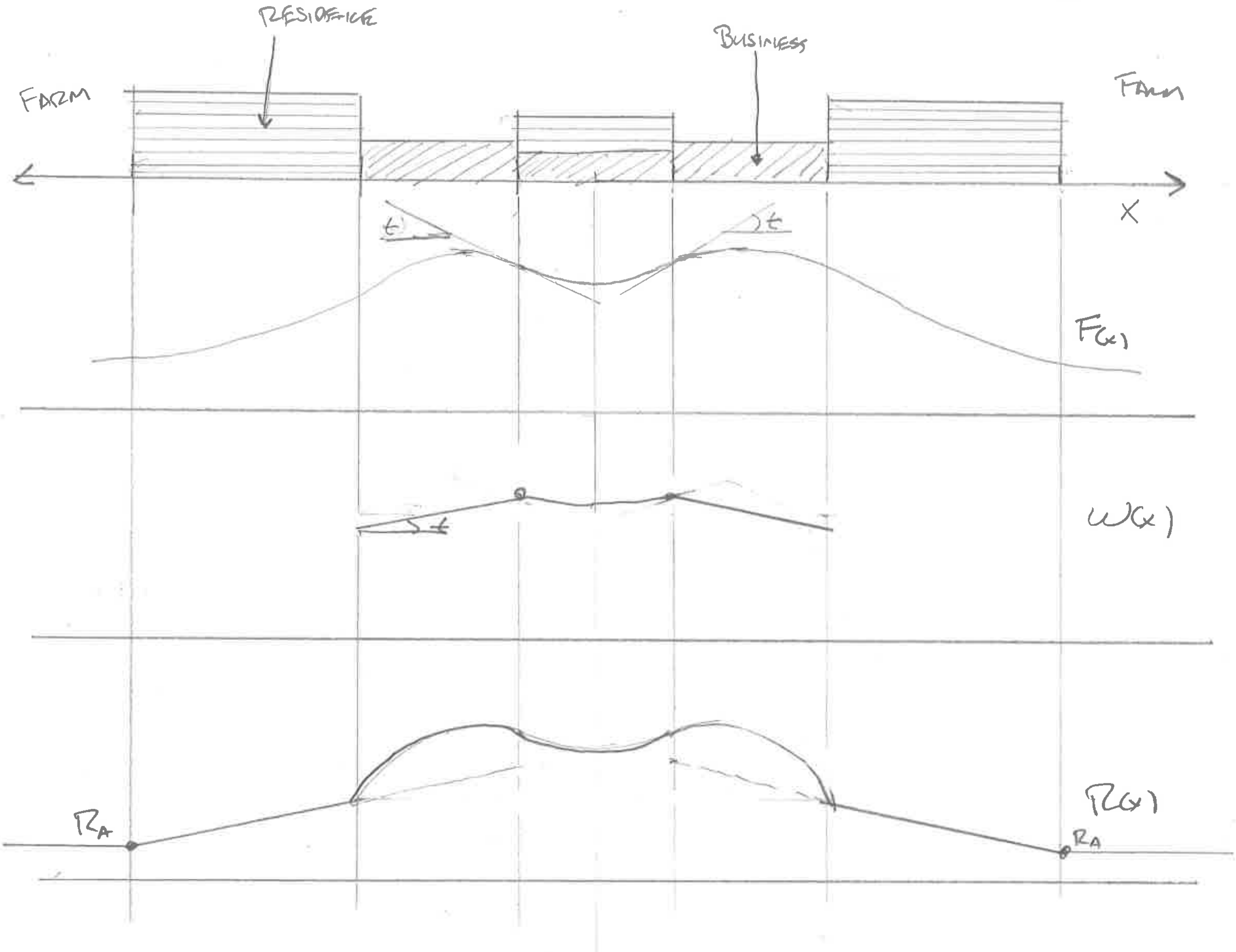
AGGREGATE LAND RENT IS $\int_0^1 (W-1)\,dx = W-1$ AND

AGGREGATE CONSUMERS' SURPLUS IS $\int_0^1 ((W-1) - \theta)\,d\theta$.

IN AN ENVIRONMENT WHERE AGENTS HAVE HETEROGENOUS OUTSIDE OPTIONS <u>OR</u> WHERE THEIR TASTE FOR A LOCATION, WE NEED TO WORRY ABOUT SURPLUS FOR INFRA-MARGINAL AGENTS WHEN WE CALCULATE WELFARE.

IN THE LINEAR CITY MODEL, AGENTS ARE HOMOGENOUS AND THIS ISSUE DOES NOT ARISE.

FUJITA + OGAWA PARTIALLY MIXED EQUIL



1. CENTER IS MIXED ⇒ WORK WHERE YOU LIVE
⇒ SLOPE OF WAGE GRADIENT LESS THAN t.
2. PEOPLE IN ADJACENT BUSINESS DISTRICTS COMMUTE FROM OUTSIDE ⇒ WAGE GRADIENT HAS SLOPE t AND RENT GRADIENT ALWAYS ABOVE tx.

[a] MAX $h^\alpha z^{1-\alpha}$

S.T. $ph + z = \omega - \tau x$

$$\Rightarrow \quad z(p) = (1-\alpha)(\omega - \tau x)$$
$$h(p) = \frac{\alpha}{p}(\omega - \tau x)$$

WITH FREE MOBILITY

$$[h(p)]^\alpha [z(p)]^{1-\alpha} = \underline{u}$$

$$\Rightarrow \quad p(x) = \left[\frac{(\omega - \tau x)\alpha^\alpha (1-\alpha)^{1-\alpha}}{\underline{u}}\right]^{1/\alpha}$$

[b] SEE DURANTON + PUGA HANDBOOK p8

[c] $$\frac{d}{dx} e(p, \underline{u}) = \frac{\partial e}{\partial p} \cdot \frac{\partial p}{\partial x} = -\tau$$

SINCE $\frac{d}{dx}(\omega - \tau x) = -\tau$

BUT $\frac{\partial e}{\partial p} = h$ SO $\frac{\partial p}{\partial x} = \frac{-\tau}{h}$

- $s \in [s_1, s_2]$ ~ AMENITY
  X ~ NUMERAIRE CONSUMPTION GOOD
  l^c ~ RESIDENTIAL LAND
  k ~ UTILITY LEVEL
  w, I ~ WAGE, NON-WAGE INCOME
  r ~ LAND RENT

N ~ LABOR = PEOPLE
l^p ~ PRODUCTION LAND
$X = f(N, l^p; s)$
~ CRS.
(1) $L = l^c + l^p$

FREE-MOBILITY

- CONSUMERS SOLVE: $\max u(x, l^c; s)$ s.t. $x + r l^c = w + I$ $\Rightarrow$ (2) $V(w, r; s) = k$

- FIRMS ~~HAVE~~ UNIT COST FUNCTION
  (3) $c(w, r; s) = 1$
  FREE ENTRY + $\pi = 0$ + $P_X \equiv 1$.

WITH CRS $\Rightarrow$ (4) $C_r = \frac{l^p}{x}$, (5) $C_w = N/X$

AN EQUILIBRIUM MUST SATISFY (1)-(5).

TOTAL DIF (2) + (3) $\Rightarrow$

$$V_w \frac{dw}{ds} + V_r \frac{dr}{ds} = +V_s \quad (6)$$

$$C_w \frac{dw}{ds} + C_r \frac{dr}{ds} = -C_s \quad (7)$$

SOLVE (6)+(7) FOR $\frac{dr}{ds} = \frac{-V_w C_s + V_s C_w}{V_w C_r - V_r C_w} \equiv \frac{V_w C_s - V_s C_w}{\Delta}$

SOLVE (6)+(7) FOR $\frac{dw}{ds} = \frac{-V_s C_r + V_r C_s}{V_w C_r - V_r C_w} \equiv \frac{V_s C_r - V_r C_s}{\Delta}$

USING (4) + (5) WE HAVE

$$\Delta = -V_r C_w + V_w C_r = -V_r \frac{N}{X} + V_w \frac{l^P}{X}$$

$$= V_w \left[-\frac{V_r}{V_w} \frac{N}{X} + \frac{l^P}{X} \right]$$

USING RON'S IDENTITY,
$$= V_w \left[+l^c \frac{N}{X} + \frac{l^P}{X} \right]$$

$$= +V_w \left[\frac{l^c N + l^P}{X} \right]$$

$$= \frac{+V_w L}{X}$$

FROM USING CRS OF $C(\cdot)$ IN (7)

$$\Rightarrow \quad -C_s = \frac{N}{X} \frac{dw}{ds} + \frac{l^P}{X} \frac{dr}{ds}$$

$$\Rightarrow \quad C_s = -\left[\frac{N}{X} \frac{dw}{ds} + \frac{l^P}{X} \frac{dr}{ds} \right] \qquad (8)$$

USING CRS OF $C(\cdot)$ IN (6)

$$\Rightarrow \quad \frac{dw}{ds} + \frac{V_r}{V_w} \frac{dr}{ds} = \frac{V_s}{V_w}$$

(USE RON'S IDENTITY)
$$\left[\frac{V_r}{V_w} = -l^c \right]$$

$$\Rightarrow \quad \frac{dw}{ds} = l^c \frac{dr}{ds} + \frac{V_s}{V_w}$$

$$\Rightarrow \quad \frac{dw}{ds} = l^c \left[\frac{-V_w C_s + V_s C_w}{\Delta} \right] + \frac{V_s}{V_w}$$

$$\Rightarrow \quad \frac{dw}{ds} = \frac{+X l^c}{V_w L} \left[-V_w C_s + V_s \frac{N}{X} \right] + \frac{V_s}{V_w}$$

→

$$\Rightarrow \frac{dw}{ds} = \frac{-X l^c}{L} \cdot c_s - \frac{N l^c}{L} \cdot \frac{V_s}{V_w} + \frac{V_s}{V_w}$$

$$P_s^* \equiv V_s / V_w \Rightarrow \frac{dw}{ds} = -\frac{X l^c}{L} c_s - \frac{N l^c}{L} P_s^* + P_s^*$$

$$\frac{dw}{ds} = -\frac{X l^c}{L} c_s + \frac{L - N l^c}{L} P_s^*$$

$$\frac{dw}{ds} = \frac{-X l^c}{L} c_s + \frac{l^P}{L} P_s^* \qquad (9)$$

USING (8) TO SUBSTITUTE FOR c_s IN (9)

$$\frac{dw}{ds} = +\frac{X l^c}{L}\left[c_w \frac{dw}{ds} + c_r \frac{dr}{ds}\right] + \frac{l^P}{L} P_s^*$$

USING CRS

$$= +\frac{X l^c}{L}\left[\frac{N}{X}\frac{dw}{ds} + \frac{l^P}{X}\frac{dr}{ds}\right] + \frac{l^P}{L} P_s^*$$

$$\frac{l^P}{L} P_s^* = \frac{dw}{ds} - \frac{X l^c}{L}\left[\frac{N}{X}\frac{dw}{ds} + \frac{l^P}{X}\frac{dr}{ds}\right]$$

$$\frac{l^P}{L} P_s^* = \frac{dw}{ds} - \frac{l^c N}{L}\frac{dw}{ds} - \frac{l^c l^P}{L}\frac{dr}{ds}$$

$$P_s^* = \left(\frac{L}{l^P} + \frac{l^c N}{l^P}\right)\frac{dw}{ds} - l^c \frac{dr}{ds}$$

$$\Rightarrow \boxed{P_s^* = \frac{dw}{ds} - l^c \frac{dr}{ds}}$$