EC1410 Topic #3

The Monocentric City Model with Housing

Matthew A. Turner
Brown University
Spring 2022

(Updated December 22, 2021)

Outline

Real Life Land Density Gradients I

One of the most obvious trends in cities since the industrial revolution has been that they are becoming less dense. In a classic study, Clark (1951) looks at census data for many cities from early in the industrial revolution until the mid-20th century. He shows that cities are expanding, becoming less dense, and their density gradients are getting flatter.

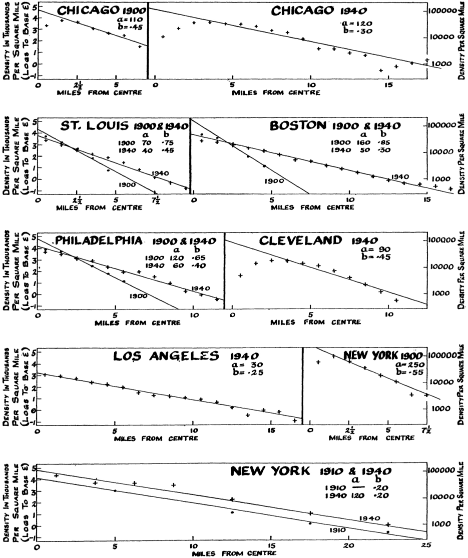
One more time, we're going to be estimating density gradients that look like

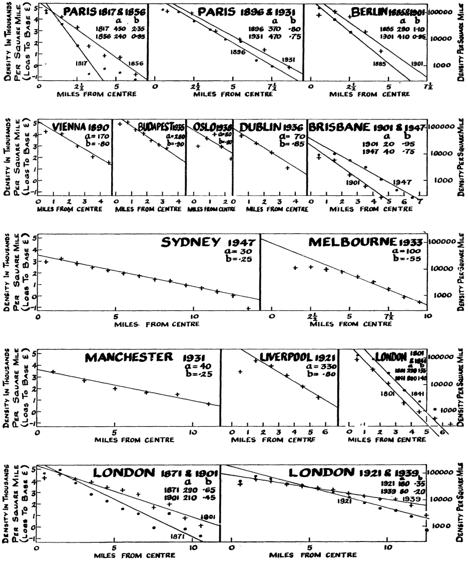
$$\ln y = A + B \ln x + \varepsilon,$$

where y is population density in a census tract(usually a few blocks and 4-5,000 people), and x is distance from the center.

Real Life Land Density Gradients II

This is exactly what we've done before, but we are looking at how population density varies with distance, not rent/prices.


CHICAGO 1900
a=110
b=·45
CHICAGO 1940
a=120
b=·30
ST. LOUIS 1900 & 1940
a b
1900 70 ·75
1940 40 ·45
BOSTON 1900 & 1940
a b
1900 160 ·85
1940 50 ·30
PHILADELPHIA 1900 & 1940
a b
1900 120 ·65
1940 60 ·40
CLEVELAND 1940
a= 90
b= ·45
LOS ANGELES 1940
a= 30
b= ·25
NEW YORK 1900
a=250
b=·55
NEW YORK 1910 & 1940
a b
1910 — ·20
1940 120 ·20
MILES FROM CENTRE
DENSITY IN THOUSANDS PER SQUARE MILE (LOGS TO BASE ε)
DENSITY PER SQUARE MILE
100000
10000
1000


PARIS 1817 & 1856
PARIS 1896 & 1931
VIENNA 1890
DUBLIN 1936
BRISBANE 1901 & 1947
SYDNEY 1947
MELBOURNE 1933
MANCHESTER 1931
LIVERPOOL 1921
LONDON 1871 & 1901
LONDON 1921 & 1939
MILES FROM CENTRE
DENSITY IN THOUSANDS PER SQUARE MILE (LOGS TO BASE ε)
DENSITY PER SQUARE MILE

Parameters in the Expression $y = Ae^{-bx}$ *Relating Density of Resident Population in Thousands per Square Mile to Distance in Miles from the Centre of the City*

Region, City and Date	*A*	*b*	*Region, City and Date*	*A*	*b*
Australia—			Continental Europe (*continued*)—		
Brisbane			Oslo		
1901	20	·95	1938	80	·80
1947	40	·75	Paris		
Melbourne			1817	450	2·35
1933	100	·55	1856	240	·95
Sydney			1896	370	·80
1947	30	·25	1931	470	·75
British Isles—			Vienna		
Dublin			1890	170	·80
1936	70	·85	United States of America—		
Liverpool			Boston		
1921	330	·80	1900	160	·85
London			1940	50	·30
1801	290	1·35	Chicago		
1841	800	1·40	1900	110	·45
1871	290	·65	1940	120	·30
1901	210	·45	Cleveland		
1921	180	·35	1940	90	·45
1939	80	·20	Los Angeles		
Manchester			1940	30	·25
1931	40	·25	New York		
Ceylon—Colombo			1900	250	·55
1946	60	·40	1910	?	·20
Continental Europe—			1940	120	·20
Berlin			Philadelphia		
1885	290	1·10	1900	120	·65
1901	410	·95	1940	60	·40
Budapest			St. Louis		
1935	280	·90	1900	70	·75
			1940	40	·45

Notice:

- All have broadly downward sloping density gradients (except near zero where there is industry).
- All get flatter and lower over time – cities are spreading out.
- Population densities were MUCH higher than they are now. Several large cities recorded densities of 100,000/sq mile. Very few modern cities are anywhere near that dense.

The monocentric city model, so far, assumes land consumption is fixed at $\bar{\ell}$. This means density is constant by assumption.

We now want to generalize the model to allow people to adjust their housing consumption, so we can try to explain these big facts about cities; negative density gradients and lower, flatter density gradients over time.

Before, consumption was fixed at c^*, Now, utility is going to be fixed, but you can achieve this utility with any mix of housing and consumption. As you move closer to the center, land, and hence housing is more expensive, and people substitute away from housing. As you move further from the center, this will give us the opposite effect.

This gives us cities that are less dense as we move out from their center.

How can this explain the reduction in density and the flattening of density gradients? This will have to be a comparative static result. It will turn out that Clark's results look like what happens in the monocentric city model with housing as the cost of transportation decreases.

In the monocentric city model with housing, households solve

$$\max_{c,h,x} c^{\alpha} h^{1-\alpha}$$
$$\text{s.t. } w = c + ph + 2tx$$

at all occupied locations, where c is consumption and h is housing. p is the price of housing. Implicitly, h, c, p all vary with x.

To make this look a little more familiar, let $\widetilde{w} = w - 2tx$. That is, income net of commuting. Then the household's problem becomes,

$$\max_{c,h,x} c^{\alpha} h^{1-\alpha}$$
$$\text{s.t. } \widetilde{w} = c + ph$$

This is the standard statement of the household problem from intermediate micro.

To solve it, reduce it to one variable by solving the constraint for c and substituting this into the objective. This gives us the equivalent, univariate, unconstrained maximization problem.

$$\max_{h,x} (\widetilde{w} - ph)^{\alpha} h^{1-\alpha}$$

Now, set the first order condition equal to zero and solve for h^*.

$$
\begin{aligned}
& -\alpha p(\widetilde{w} - ph)^{\alpha-1} h^{1-\alpha} + (\widetilde{w} - ph)^{\alpha}(1-\alpha) h^{-\alpha} = 0 \qquad (1) \\
\implies & -\alpha p(\widetilde{w} - ph)^{\alpha-1} h + (1-\alpha)(\widetilde{w} - ph)^{\alpha} = 0 \\
\implies & -\alpha p(\widetilde{w} - ph)^{-1} h + (1-\alpha) = 0 \\
\implies & -\alpha ph + (1-\alpha)(\widetilde{w} - ph) = 0 \\
\implies & -ph + (1-\alpha)\widetilde{w} = 0 \\
\implies & h^* = \frac{(1-\alpha)\widetilde{w}}{p}
\end{aligned}
$$

and substituting back into the budget constraint, we get

$$\begin{aligned} c^* &= \widetilde{w} - ph^* \\ &= \widetilde{w} - p[\frac{(1-\alpha)\widetilde{w}}{p}] \\ &= \alpha\widetilde{w}. \end{aligned}$$

Hopefully, this looks familiar from intermediate micro. This problem has exactly the same form as a common teaching example of the 'household problem'.

For an equilibrium, we need households to optimize, and we need 'no one wants to move'. To impose this second condition, we want to (1) calculate the indirect utility function by substituting h^* and c^*

back into the utility function, and then require (2) that indirect utility be constant everywhere.

$$
\begin{aligned}
V(p, \widetilde{w}; x) &= (c^*)^\alpha (h^*)^{1-\alpha} \\
&= \alpha \widetilde{w}^\alpha \left(\frac{(1-\alpha)\widetilde{w}}{p}\right)^{1-\alpha} \\
&= \alpha^\alpha (1-\alpha)^{1-\alpha} \left(\frac{\widetilde{w}}{p^{1-\alpha}}\right) \\
&= \overline{u}
\end{aligned}
$$

If we rearrange, we get

$$
p^* = \left[\frac{\alpha^\alpha (1-\alpha)^{1-\alpha} \widetilde{w}}{\overline{u}}\right]^{1/1-\alpha}
$$

Recalling the definition of $\widetilde{w}$, this is

$$p^* = \left[\frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\overline{u}}\right]^{1/1-\alpha}(w-2tx)^{1/1-\alpha} \tag{2}$$

This is the housing price gradient. As we expect, it decreases with x. Hopefully, this is not surprising. As households spend more on commuting, their total expenditure on housing and consumption must fall. Since the price of consumption is fixed at one by assumption, the only way this can happen is if the price of housing falls as commute costs increase.

If we do a little more work, there is a more useful way to write this.

If we substitute the expression for p^* (2) back into the expression for h^* (1), we get,

$$
\begin{aligned}
h^* &= \frac{(1-\alpha)\widetilde{w}}{p^*} \\
&= (1-\alpha)\widetilde{w}(p^*)^{-1} \\
&= (1-\alpha)\widetilde{w}\left[\frac{\alpha^\alpha(1-\alpha)^{1-\alpha}\widetilde{w}}{\overline{u}}\right]^{-1} \\
&= \left[\frac{\widetilde{w}^\alpha \alpha^\alpha}{\overline{u}}\right]^{-1/1-\alpha}
\end{aligned}
$$

Aside: reorganizing this a little, we have

$$
\begin{aligned}
h^* &= \left[\frac{\alpha^\alpha}{\overline{u}}\right]^{-1/1-\alpha} \widetilde{w}^{-\alpha/1-\alpha} \\
&= \left[\frac{\alpha^\alpha}{\overline{u}}\right]^{-1/1-\alpha} (w - 2tx)^{-\alpha/1-\alpha} \\
&= \left[\frac{\alpha^\alpha}{\overline{u}}\right]^{-1/1-\alpha} \left(\frac{1}{(w - 2tx)}\right)^{-\alpha/1-\alpha}
\end{aligned}
$$

so equilibrium consumption of housing increases in x, too.

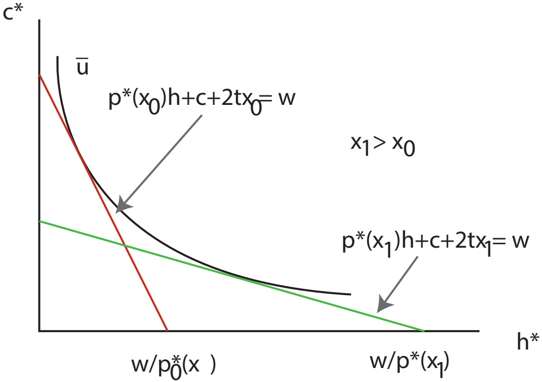
Now for the trick we have been working towards. We're going to differentiate the expression for p^* (2) with respect to x and use the expression for h^* that we just derived to simplify it.

$$\begin{aligned}
\frac{dp^*}{dx} &= \frac{d}{dx}\left(\left[\frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\overline{u}}\right]^{1/1-\alpha}(w-2tx)^{1/1-\alpha}\right) \\
&= \left[\frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\overline{u}}\right]^{1/1-\alpha}\frac{1}{1-\alpha}(w-2tx)^{-1+1/1-\alpha}(-2t) \\
&= \left[\frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\overline{u}}\right]^{1/1-\alpha}\frac{1}{1-\alpha}(\widetilde{w})^{-\alpha/1-\alpha}(-2t)
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}\widetilde{w}^{-\alpha}}{\overline{u}} \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \right]^{1/1-\alpha} (-2t) \\
&= \left[\frac{\alpha^{\alpha}\widetilde{w}^{-\alpha}}{\overline{u}} \right]^{1/1-\alpha} (-2t) \\
&= [h^*]^{-1} (-2t) = \frac{-2t}{h^*}
\end{aligned}
$$

- Since everyone actually buys h^* of housing, this says that total expenditure on housing $p^* h^*$ decreases by $2t$ for each extra unit of commuting,exactly the result we got without housing.
- We've just shown this for a particular example. In fact, *it is true for any utility functions such that indifference curves are concave*.

- Empirically, housing price gradients are log more-or-less log linear. In the monocentric city model without housing, we got a linear housing/land rent gradient, which i s not a good description of what we observe. Does the model with housing do better?



Here is an indifference curve. Suppose $x_1 > x_0$. For any choice of x, we have to be on this indifference curve AND the households must be optimizing, so the budget line must be tangent. Since $x_1 > x_0$ if the household consumes zero housing, they consume less c at x_1 than x_0, so the intercept of this budget line must be smaller. In order for a budget line with a lower intercept to be tangent to the indifference curve, it must be flatter. This means the price of housing is lower. This picture is based on Brueckner (1987).

We set out to incorporate housing choice into the monocentric city model in order allow this model to explain density gradients.

If we think ‘housing’ and ‘land’ are the same thing, then ‘housing’ consumption increases with x, while the amount of land at each locatin stays constant, so population density must go down with distance, and we’ve pretty well succeeded.

We see more land/housing consumption with distance to the CBD, and we still have a downward sloping rent gradient and we get a downward sloping rent gradient.

This is OK, but ...

Note: (top) View of the city of Providence as seen from the dome of the new State House. Drawn by M. D. Mason, published in the Providence Sunday journal, Nov. 15, 1896. (Bottom) New York skyline,

Obviously, we're still not explaining something pretty important.

Monocentric city with actual housing I

- We looked at what happens when people have preferences over housing, but we haven't thought about how housing is made. This is pretty important.
- The thing that is scarce is proximity to the center. The cost of construction varies much less with distance to the center than does the cost of commuting.
- As land gets scarcer near the city, people should (1) substitute away from housing toward consumption, (2) housing should substitute away from land, toward capital. Together this should give us downward sloping density and building height gradients.

Monocentric city with actual housing II

- In order to think about this, we're going to imagine that housing, still h, is built from land, l and physical capital k. The price of housing is p, land is R (like before), and capital is i. Both p and R vary with x, but i does not.

- A housing sector transforms land and capital into housing according to

$$h = k^{\beta} l^{1-\beta}, \quad 0 < \beta < 1$$

  In fact, our best evidence on this point is that this is actually a pretty good description of the housing production function and that $\beta \approx 2/3$ (Combes et al., 2020).

- Assume that housing production is perfectly competitive, so firms maximize and profits are zero. This seems like a pretty defensible assumption for how the real estate market works. Developers are nothing if not profit maximizers.

- Restating the developer's problem in math, we have,

$$\max_{k,l} pk^{\beta}l^{1-\beta} - ik - Rl$$

  where p and R vary with x, but i does not.
- This is a little easier to tackle if we divide through by l and write the problem in terms of profit per unit land, rather than profit. If we let $S = k/l$ be the capital-land ratio, building height, this means we an write the developer's problem as,

$$\max_{s} pS^{\beta} - iS - R$$

- Note: We can use this trick for any housing production function that is Constant Returns to Scale. We're just working an example, but all of these results extend to any constant returns to scale housing production function.

- The first order condition for this problem, in terms of the capital to land ratio, is

$$p\beta S^{\beta-1} - i = 0 \tag{3}$$
$$\Longrightarrow S = \left(\frac{i}{\beta}\right)^{1/1-\beta} p^{1/\beta-1}$$

- Developers enter the market until profits are driven to zero, so we also have

$$pS^{\beta} - iS - R = 0 \tag{4}$$
$$\Longrightarrow R = pS^{\beta} - iS.$$

- Remember, both FOC and zero profit condition have to hold at all locations x, even though we haven't written the x's explicitly.

- If we substitute S from (3) into (4), we can write land rent at a function of housing price,

$$R = p\left[\left(\frac{i}{\beta}\right)^{1/1-\beta} p^{1/\beta-1}\right]^{\beta} - i\left[\left(\frac{i}{\beta}\right)^{1/1-\beta} p^{1/\beta-1}\right]$$

and after a lot of algebra, this gives,

$$R = p^{\beta/1-\beta}\left[\frac{(1+\beta)i}{\beta}\right]^{1/\beta-1} \qquad (5)$$

- We already figured out the price of housing, $p(x)$ was decreasing in x (see (2)).
  - Using this and (5) we have that land rent, $R(x)$, is also decreasing in x.

- Using this and (3) we have that the capital-land ration, $S(x)$, is also decreasing in x.

- Since the gradients for S and R depend on p, all of the comparative statics for p apply to S and R. For example, gradients get flatter at transportation costs decreases.
- We've already shown that housing consumptin increases with x, and that housing price falls. Since $h^* \uparrow$ and $S^* \downarrow$ as $x \uparrow$, it follows that population density, S^* / h^* also falls with x.

Why the monocentric city model is awesome I

I started this series of lectures telling you that the monocentric city model was one of the most successful economic models I have ever seen. We can now see why.

The model is based on the following assumptions about households,

- Commuting is costly.
- People optimize.
- In equilibrium no one wants to move.
- People work in the center.
- Indifference curves for housing and consumption are convex.

and the following assumptions about the production of housing,

- Developers maximize profits

Why the monocentric city model is awesome II

- There is free entry of developers
- Housing production function can be written in terms of the capital-land ratio.

Except for ‘everyone works in the center, these are pretty weak assumptions. Here are some big stylized facts about cities,

- Housing price gradients are downward sloping.
- Land rent gradients are downward sloping.
- Building heights decrease with distance to the center.
- Population density decreases with distance to the center.

Why the monocentric city model is awesome III

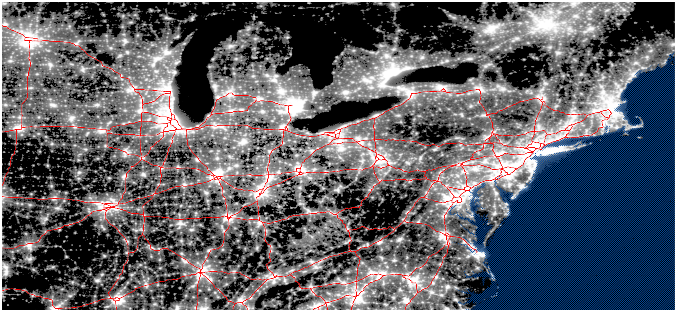
The monocentric city model predicts every one of them.

What about the flattening of the density gradient that Clark (1951) documents? This is just what would happen if transportation costs fell, which pretty clearly happened.

Also, recall the empirical evidence for comparative statics predicted by the model that we have already seen.

Big Failures I

- Cities are not monocentric



Note: 2007 Lights at night image of the Northeastern US. Cities are pretty clearly not all monocentric, though some of them are.

Big Failures II

- Much of the decrease in density with distance to the center occurs because there is more open space, not because people have bigger yards. ‘Leapfrog’ development is important.

Open questions I

- What if people are not all identical?
- Why do people go to the center?
- How do we choose the center?
- How many cities are there?
- Why are cities so different from one another?

These are important questions that the model doesn't really speak to, and that we'll consider during future lectures.

Brueckner, J. K. (1987). The structure of urban equilibria: A unified treatment of the muth-mills model. *Handbook of regional and urban economics*, 2(20):821–845.

Clark, C. (1951). Urban population densities. *Journal of the Royal Statistical Society. Series A (General)*, 114(4):490–496.

Combes, P.-P., Duranton, G., and Gobillon, L. (2020). The production function for housing: Evidence from france. *Journal of Political Economy*.