

The linear city model

Lecture notes #2: EC2410

Matt Turner
Brown University
18 January 2024

The previous lecture provides a framework to think about how people organize themselves in space. We now begin to generalize this framework to allow it to explain some of the major features of how cities are organized.

Figure 1 provides a nice way to pose our problem. This figure is based on ‘lights at night’ data for 2007 and shows the pattern of nighttime lights in the Northeastern US in 2007, together with the interstate highway network. This image suggests a many of the issues we would like to investigate. First, cities often have an approximately circular structure and are denser at their centers. Second, the transportation network seems to be important in determining where people locate, lights generally track the interstate system and also seem to track more minor roads. Finally, the cities in this image appear to be part of a system of cities. While we will begin by studying cities in isolation, this figure suggests the importance of thinking about how they interact.

One of the central features of cities is a density gradient. Cities are densest at their centers and get less dense as we travel radially. One of the earliest demonstrations of such gradients was ? in a remarkable, pre-computer, empirical investigation based on census data from 20 cities, all over the world, from about 1850 until about 1950.

Let y denote population density in a sub-city census enumeration district and x the distance from this location to the city’s center. Using these data, Clark estimates ‘density gradients’,

$$y = Ae^{-bx}$$

or equivalently,

$$\ln y = \ln A - bx.$$

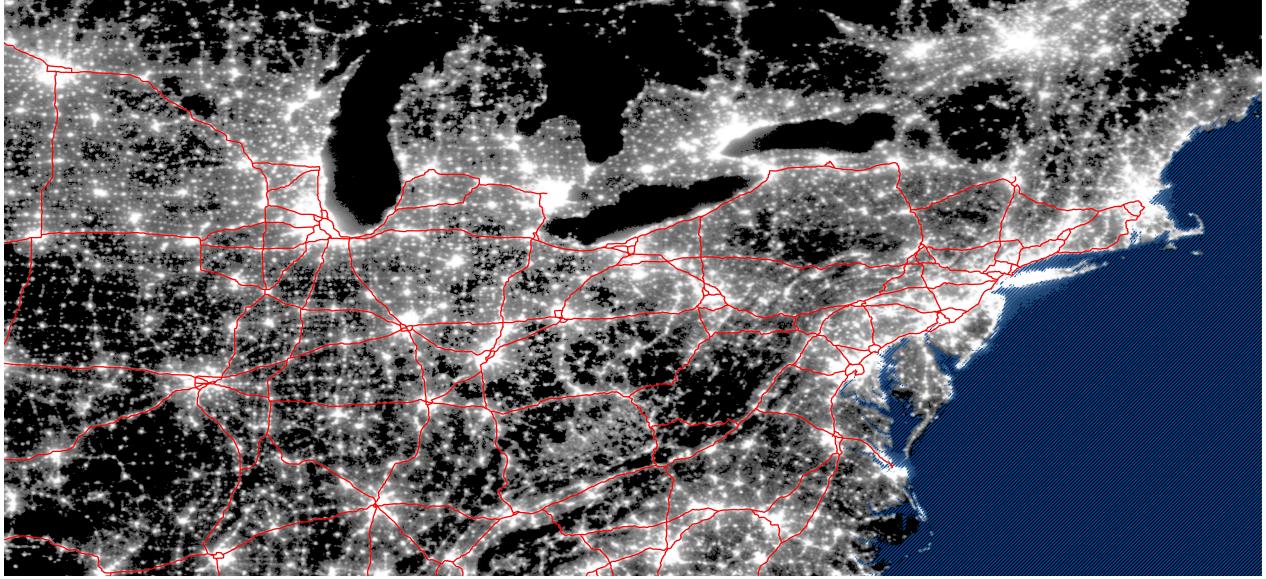
Clark’s results are reproduced in figures 2 and 3.

These results reveal some of the main features about how cities are organized.

- Density gradients are downward sloping, except sometimes near the center where industry is located.
- Density gradients get flatter over time. Cities are spreading out.
- Population densities were much higher in the late 1800’s and early 1900’s than they are now. Several large cities that Clark studies record densities of 100,000/sq mile. Of the about 8m sq kilometers in the US, only 38 have population densities above 30,000 ($\sim 76,000$ / sq. mile). The downtown neighborhood of Toronto that contains the University of Toronto (Ward 20) was about 26,000/ sq. mile in 2010.

Unsurprisingly, there is a corresponding decrease in land rents that approximately parallels that of population density. This is illustrated very nicely in ?. Land rent or sales

Figure 1: Lights at night and Interstate Highways in the Northeast US, 2007.



data has, at least historically, been more difficult to observe than population density, and so it has been less systematically studied.

? gave rise to a large literature examining gradients. Most simply confirm the conclusions suggested by ? or ?. However, ? provide an important refinement, and one that our theory will struggle to explain.

Specifically, ? examine population density as a function of distance from Houston in the late 20th century. They find the same basic patterns documented in ?. Population density is decreasing with distance from the center, and the density gradient is flattening over time. They also look at how lot size changes with distance from the center, and find much less change. For Houston in 1980, residential density decreases by about 6% with a doubling of distance from the center. The fraction of land occupied decreases by nearly 10%. Thus, much of the decrease in population density reflects the fact that more land is vacant further away from the center. The remote occupied locations are clustered together.

Linear city

The linear city model is the simplest version of the monocentric city model and is loosely based on ?. The object of the model is to relate two stylized facts. First, that transportation is costly. Second, that land rent declines with distance to the center.

To begin, imagine a city located on a flat featureless plain. In the example we will develop here, we will actually consider a linear city. This simplifies the mathematics slightly because the area available for residential use is constant as distance to the center varies. Conceptually, the important assumption is radial symmetry. Given this assumption, up to the extra math to keep track of variation in area with radial distance, there are no important changes required to extend this model to the description of a city that occupies any arc.

In addition to assuming a featureless, linear landscape, we assume a center. In particular, a household that resides in the city is assumed to commute from their residential

Figure 2: Density gradients from ?

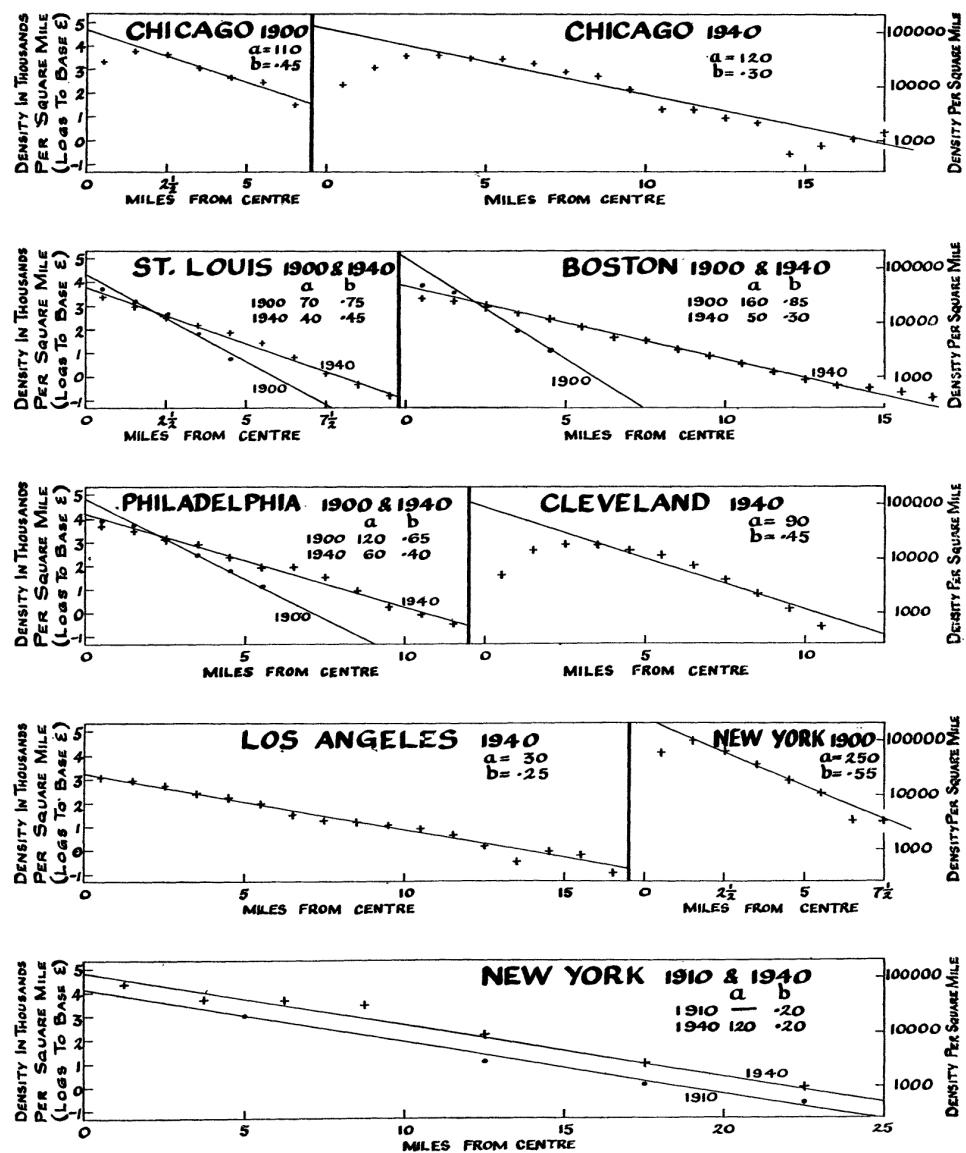
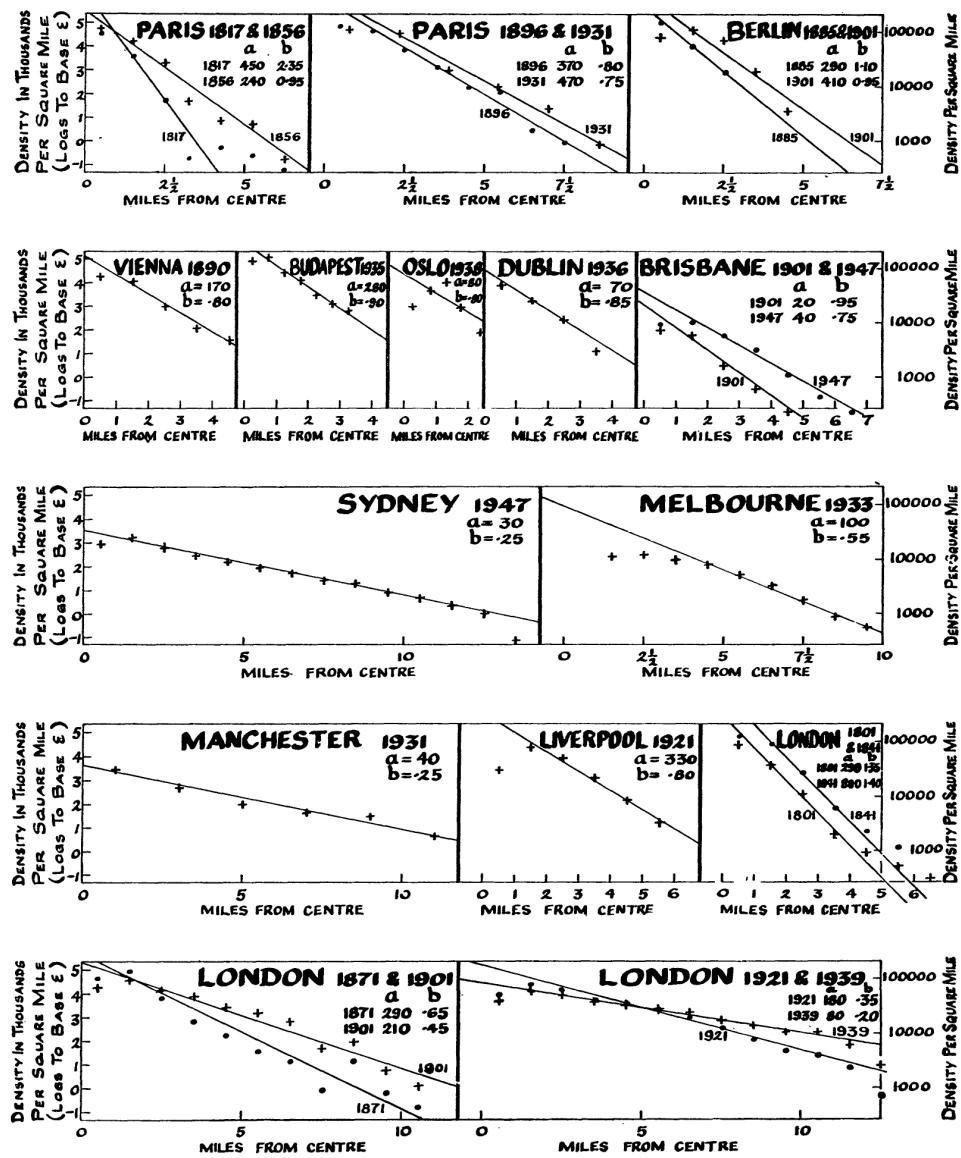


Figure 3: Density gradients from ?



location to the center, usually called the ‘central business district’ or ‘CBD’, where they receive their wage.

The notation required to specify the model is as follows. Each identical household consumes a fixed amount of land, \bar{l} and a composite numeraire consumption good c . Each household chooses the location x where they will live. With radial symmetry, x is a displacement from the CBD at zero. Unit land rent at x is $R(x)$. Each agent commutes to the CBD where they receive wage w . The unit cost of commuting is t . In addition, let N denote the population of the city, \bar{u} the outside option available to all households, and \bar{R} the rent that a landowner receives in an alternative use when his parcel is not occupied by a household, e.g., agriculture. Landowners are, usually, absentee landlords who collect rent but spend it somewhere outside the model.

Each household chooses their location, commutes to work and divides w between commuting and c . More formally, the consumer’s problem is

$$\begin{aligned} & \max_{c,x} u(c) \\ & \text{s.t. } w = c + R(x)\bar{l} + 2t|x| \end{aligned} \tag{1}$$

To complete the model, we must also specify an equilibrium concept. Two are common. The first is an ‘free mobility open city equilibrium’. In an open city equilibrium, we fix the value of the outside option and allow the population of the city to adjust. In this case, an equilibrium is land rent gradient $R(x)$ such that all occupied locations in the city provide utility level \bar{u} and no occupied location rents for less than \bar{R} . The second equilibrium is a ‘free mobility closed city equilibrium’. In this case, an equilibrium is a rent gradient such that all households choose a location, no household wants to move and no parcel rents for less than \bar{R} . The analysis presented here will focus on the open city equilibrium, the closed city equilibrium behaves in much the same way.

In an open city free mobility equilibrium, all households must receive exactly the reservation level of utility. Since households derive utility only from consumption of the numeraire good, this requires that all households consume exactly the amount of c that yields this level of utility. That is, $c^* = u^{-1}(\bar{u})$. Note that u^{-1} is really an expenditure function. We will see later that it can be convenient to treat an expenditure function, rather than a utility function, as the primitive description of preferences.

Putting this together, the household’s budget constraint becomes

$$w - c^* = R(x)\bar{l} + 2tx. \tag{2}$$

That is, with wages and consumption fixed for all households, commuting costs and land rent must vary in such a way that they always sum to a constant.

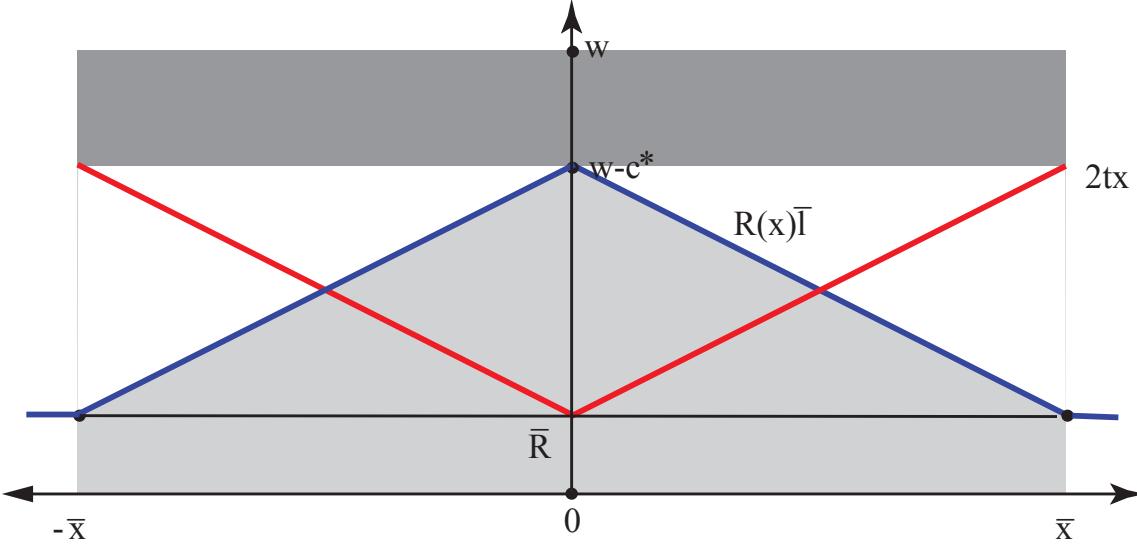
Let \bar{x} denote the most remote occupied location. At this location, we must have

$$w - c^* = \bar{R}\bar{l} + 2t\bar{x}. \tag{3}$$

That is, at the edge of the city, the cost to commute is such that a household can just pay the reservation cost for land and commuting costs, while still preserving a reservation consumption level. Reorganizing, we have

$$\bar{x} = \frac{w - c^* - \bar{R}\bar{l}}{2t}. \tag{4}$$

Figure 4: The linear city model



Since the city extends from $-\bar{x}$ to \bar{x} and each household consumes an exogenously fixed amount of land, it follows that

$$N^* = \frac{2\bar{x}}{\bar{l}}. \quad (5)$$

Using the equilibrium budget constraint (2) and the equilibrium extent of the city, we can solve for the equilibrium rent gradient,

$$R^*(x) = \begin{cases} \frac{w-c^*-2t|x|}{\bar{l}} & \text{if } |x| \leq \bar{x} \\ \bar{R} & \text{if } |x| > \bar{x} \end{cases} \quad (6)$$

Figure 4 illustrates this model. The occupied portion of the city extends from $-\bar{x}$ to \bar{x} . All residents of the city receive the same level of consumption and divide the remainder of their income between land rent and commuting. The outcome is efficient in the sense that each parcel of land is employed in its highest value use, whether residential or agricultural.

Land rent capitalizes access to the CBD in a very precise sense. For each unit decrease in commute costs, land rent increases by exactly the same amount. This is exactly analogous to what we saw in the earlier, much simpler example with two households and two landlords. Land rent conveys information about welfare in exactly the same sense as well. Land rent measures the difference in utility between the occupied location and a household's best alternative. Similarly, aggregate land rent also measures welfare in the same sense as the earlier, simpler example. Aggregate land rent give the total amount of surplus created because households have the opportunity to locate in the city rather than the outside option. Also as in the earlier example, landlords capture the entire surplus.

Discussion

The linear city model provides a parsimonious model of land rent and city size in which the model is driven by rational households trying to limit their costly commuting. The model matches the observed decline in land rent with radial distance to the center and provides a foundation for thinking about the welfare implications of changes in land rent.

With this said, the linear city model is deficient in a number of obvious ways. First, while it predicts the widely observed downward sloping land rent gradient, by construction, it cannot predict the corresponding density gradient. It is also not obviously able to predict the changes in the slope of the density gradient that shows up so strongly in ?. In our next topic, we will consider a monocentric city with housing and find that it is able to predict all of these phenomena.

In closing, it is useful to note a logical incompleteness in the linear city model. Because the city is populated by a continuum of people (really a ‘distance’ or measure of people), each of whom consumes a discrete amount of land, our population cannot fit on a line of finite length. Thus, the nominally linear geography of the linear city model is inconsistent with the assumptions of the model. The common way around this problem is to think of the city as being a ‘ribbon city’. That is, it is located along a line, and has width \bar{l} . This change in geography resolves the problem, but at the cost of another. In particular, in a ribbon city, travel is costly in the x direction, but is free in the other implicit horizontal direction. It is hard to understand how this could be true in a real geography. This problem is pervasive in the monocentric city literature.