EC1340 Topic #3

How to think about uncertainty and the mystery of the warming hiatus

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Outline

Probability theory I

Uncertainty about climate change is important and pervasive. We need to be able to think about it precisely. So, we'll review probability theory before using it to illustrate a couple of our problems.

Let $M = \{m_1, ...m_K\}$ be a set of events. $m \in M$ is a 'realization' or 'outcome'. Events are exclusive, exactly one can happen. One event must happen. No event can happen that is not in set M.

Let P be a function such that

- $P(m_i) \in [0, 1]$ all $i = 1, ..., K$
- $P(m_i \bigcup m_j) \geq P(m_j)$ for all i, j
- $P(\bigcup_{i=1}^{K} m_i) = 1$.

Say that P is a probability distribution over M, and that $p_i = P(m_i)$ is the probability of event i.

Example

For a fair coin toss, $M = \{H, T\}$ $P(m = H) = P(m = T) = 1/2$, $P(m \in \{H, T\}) = 1$, $P(m \notin \{H, T\}) = 0$.

- A random variable is a function that assigns numbers to events. That is, $X : M \to R$ or $X(m_i) \in (-\infty, +\infty)$ for all $m_i \in M$.
- Let $x_i = X(m_i)$ to make things easier and call x_i a 'realization of X'. Thus, $p_i = P(x_i) = P(m_i)$ and $\sum_{i=1}^{K} p_i = 1$.
- A bet on a coin toss is a random variable. For example, win 1$ for H and 0$ for T. Here, we have $x_H = X(H) = 1$, $x_T = X(T) = 0$ and $p_T = p_H = 1/2$.
- Write $X = (1, 0; 1/2, 1/2)$. Random variables written this way are often referred to as 'lotteries'.

Describing random variables I

Random variables are complicated. Each is described by a vector of probabilities and outcomes.
We'd like to calculate some statistics to describe them more succinctly.

- the expected value of X is $E(X) = \sum_{i=1}^{K} p_i x_i$
- the variance of X is $VAR(X) = E(X - E(X))^2 = \sum_{i=1}^{K} p_i (x_i - E(X))^2$
- the standard deviation of X is $\sqrt{VAR(X)}$

Describing random variables II

In our coin toss example, $X = (1, 0; 1/2, 1/2)$ we have:

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{2} p_i x_i \\
&= \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2} \\
VAR(X) &= \sum_{i=1}^{2} p_i (x_i - E(X))^2 \\
&= \frac{1}{2}(1 - \frac{1}{2})^2 + \frac{1}{2}(0 - \frac{1}{2})^2 = \frac{1}{4} \\
SD(X) &= \sqrt{VAR(X)} = \frac{1}{2}
\end{aligned}
$$

Philosophical aside

- This is a 'formal' story for probability. Competing stories are 'frequentist' and 'subjective'. In frequentist probability, p is the share of realizations in many trials that an event occurs. In subjective probability, probability p is a measure of how likely you think something is.
- Both subjective and frequentist stories are consistent with formal theory. Frequentist is better for measurement error and cards, subjective is better for global warming. Why?

Measurement error I

- In ‘Measuring temperature, the 170 year record’ (optional reading, by climate sceptics) the authors suggest that measurement error should cast doubt on the observed trend in measured temperature. Let’s think about this carefully.

Measurement error II

- Suppose the true temperature is T and two thermometers measure temperature with error,

$$X_1 = (T+2, T-2; 1/2, 1/2)$$
$$X_2 = (T+2, T-2; 1/2, 1/2),$$

  so $E(X_i) = T$, $Var(X_i) = 4$ and $SD(X_i) = 2$ for each of them. Let $x_1 \in \{T+2, T-2\}$ be a 'realization of random variable X_1'. Similarly, x_2.
- Are two noisy thermometers better, worse or the same as one?

Measurement error III

Suppose we pick one of the two thermometers at random? This is a lottery, too

$$
\begin{aligned}
X^* &= (X_1, X_2; 1/2, 1/2) \\
&= ((T+2, T-2; 1/2, 1/2), (T+2, T-2; 1/2, 1/2); 1/2, 1/2) \\
&= (T+2, T-2; 1/2, 1/2)
\end{aligned}
$$

This is exactly what we started with, so this is no better and no worse than having one thermometer.

Measurement error IV

What if we average of the two draws? This is also a lottery.

$$\overline{X} = \frac{1}{2}(X_1 + X_2).$$

What do we know about this average?

$\overline{X}$	X_1	X_2	p^0	p^1	p^2
$T+2$	$T+2$	$T+2$	1/4	1/2	3/8
T	$T-2$	$T+2$	1/4	0	1/8
T	$T+2$	$T-2$	1/4	0	1/8
$T+2$	$T-2$	$T-2$	1/4	1/2	3/8

Each of the three probability distributions p^0, p^1, p^2 is consistent with what we know. Both X_1 and X_2 have equal probabilities of too high or too low a measurement under each.

Measurement error V

- p^0 has a special property, for all realizations, we have $P(x_1 \text{ and } x_2) = P(x_1)P(x_2)$. When a pair of random variables has this property, say they are 'independent'.
- Intuitively, if X_1 and X_2 are independent, my ability to predict X_2 is not improved when I learn the realization of X_1. This means that there is a lot of new information in X_2, even if I know X_1.
- Compare this to p^1 where the two thermometers are clones, and p^1 where they are related.

The Law of Large Numbers I

Now for the big result. Calculate mean, variance and standard deviation for the average when X_1 and X_2 are independent:

$$E(\overline{X}) = \frac{1}{4}(T+2) + \frac{1}{2}(T) + \frac{1}{4}(T-2) = T$$

$$VAR(\overline{X}) = \frac{1}{4}(T+2-E(\overline{X}))^2 + \frac{1}{4}(T-2-E(\overline{X}))^2 = 2$$

and

$$SD(\overline{X}) = \sqrt{2}$$

- The variance of our temperature measure goes down with two thermometers. *This is what we get if we average the readings of both thermometers and the thermometers are independent*.

The Law of Large Numbers II

- This illustrates a general principle. The average of mean zero, independent measurement errors go to zero as you average more and more observations. This is called the ‘law of large numbers’.
- Therefore, contrary to the claim in ‘Measuring temperature, the 170 year record’, the fact that there is measurement error in individual thermometer readings is probably not a problem for measurements of mean surface temperature. The law of large numbers tells us that having lots of noisy independent measurements (what we have) is as good as having a perfect measurement.

Covariance I

- Independence of random variables is a particular case in which the realization of one has ability to predict the other.
- If independence fails, then the realization of one helps predict the other.
- ‘Covariance’ is a way of thinking about ‘how much’ information the realization of one random variable gives about the distribution of another.
- Covariance is defined as:

$$COV(X,Y) = E[(X - E(x))(Y - E(Y))]$$

- Note that

$$\begin{aligned} COV(X,X) &= E[(X - E(x))(X - E(X))] \\ &= E(X - E(x))^2 = Var(X) \end{aligned}$$

Covariance II

Recall our example with thermometers,

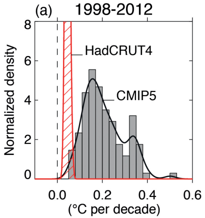
$\overline{X}$	X_1	X_2	p^0	p^1	p^2
$T+2$	$T+2$	$T+2$	1/4	1/2	3/8
T	$T-2$	$T+2$	1/4	0	1/8
T	$T+2$	$T-2$	1/4	0	1/8
$T+2$	$T-2$	$T-2$	1/4	1/2	3/8

Recalling that $E(X_1) = E(X_2) = T$ in all cases,

$$\begin{aligned}
COV^0(X_1, X_2) &= E((X_1 - T)(X_2 - T)) \\
&= \frac{1}{4}((T+2) - T)^2 + \frac{1}{2}((T+2) - T)((T-2) - T)) + \frac{1}{4}((T-2) - T)^2 = 0 \\
COV^1(X_1, X_2) &= \frac{1}{2}((T+2) - T)^2 + \frac{1}{2}((T-2) - T)^2 = 4 \\
COV^2(X_1, X_2) &= \frac{3}{8}((T+2) - T)^2 + \frac{1}{4}((T+2) - T)((T-2) - T)) + \frac{3}{8}((T-2) - T)^2 = 2
\end{aligned}$$

If p^0, then X_1 and X_2 are independent $\Longrightarrow COV^1(X_1, X_2) = 0$. If p^1 or p^2, X_1 and X_2 are not independent $\Longrightarrow COV^1(X_1, X_2) \neq 0$. $COV^1(X_1, X_2) > COV^2(X_1, X_2)$ reflects the fact that the variables are 'more dependent' under p^1 than p^2. (informal)

Another technical aside: Confidence intervals



- Red is mean change in measured temperature from 1988-2012. Grey is the distribution of 114 runs of an IPPC climate model predicting temperature change during this period on the basis of 1988 data (from AR4).
- Less than 5% (probably) of model realizations are as small as observed change.
- We can say with 95% confidence that realized temperature would not have occurred if the model were true.

Model uncertainty I

Now suppose our thermometers are

$$T_1 = \epsilon_1$$
$$T_2 = k + \epsilon_2$$

with $\epsilon_1 = \epsilon_2 = (\frac{1}{2}, \frac{1}{2}, -1, 1)$, but we don't know which thermometer is biased.

If we rely on one thermometer chosen at random then we face the following compound lottery:

$$\begin{aligned} T^* &= ((-1, 1, 1/2, 1/2), (k-1, k+1, 1/2, 1/2); 1/2/12) \\ &= (-1, 1, k-1, k+1; 1/4, 1/4, 1/4, 1/4) \\ E(T^*) &= k/2 \\ V(T^*) &= 1 + k^2 \end{aligned}$$

Model uncertainty II

If we average over thermometers, then we get $\frac{T_1+T_2}{2}$ no matter what. If ϵ's are independent, then each of the four possible outcomes is equally likely. That is,

$$T^{**} = ((\frac{k-2}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k+2}{2}; 1/4, 1/4, 1/4, 1/4)$$
$$E(T^{**}) = k/2$$
$$V(T^{**}) = 1/2 < 1 + k^2 = V(T^*)$$

Therefore, with two thermometers, one biased, averaging is always better than picking one at random. It gives the same reading on average, but less uncertainty.

Model uncertainty III

Suppose instead of two thermometers we have two models predicting future temperature, i.e., two instruments for measuring future temperature conditional of future $\mathrm{CO_2}$.
Suppose our two models for predicting temp from $\mathrm{CO_2}$ are

$$T_1 = 1\mathrm{CO_2} + \epsilon_1$$
$$T_2 = 2 + 3\mathrm{CO_2} + \epsilon_2$$

If $\mathrm{CO_2} = 0$ then this reduces to exactly the same problem as we just solved with biased thermometers. If $\mathrm{CO_2}$ is positive, then we have a slight variation.
Averaging across models is ‘like’ averaging across thermometers.
If we have no information about which model is right, then we think both models are equally likely. If we think one is better, then assign it a higher weight.

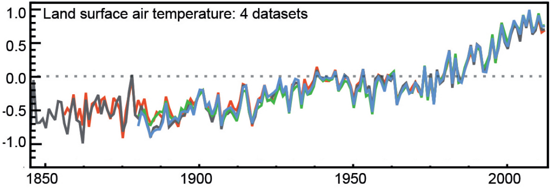
Confidence intervals I

Confidence intervals are one of the main ways that we quantify uncertainty. How do they work?

- Suppose a model is true and generate a distribution of of realizations that follow from that model or hypothesis.
- Find the, say 95% of outcomes that are most likely to occur. This is the confidence interval.
- If we observe a realization not in the confidence interval, it is unlikely to have occured if the model or hypothesis is true.

The warming hiatus

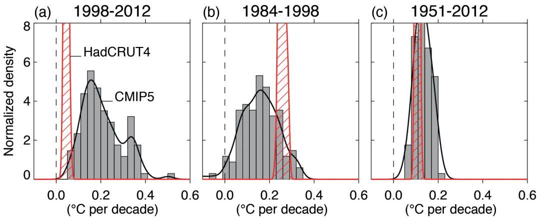
Over the last 10-15 years, the rate of warming has slowed to almost zero.



IPCC 2013 WG1, TS Figure 1

NB: This is GMST, Global Mean Surface Temperature.

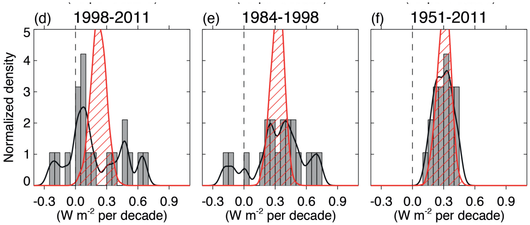
... was not predicted by AR4



IPCC 2013 WG1, TS.3 Figure 1

- Red shows estimated observed temperature. Noise is ‘like’ our example of measuring temperature with two imperfect thermometers. Warming was about zero during 1998-2012.
- Grey shows distribution of predicted temperatures from 114 runs of IPCC ’s favorite climate model. It over predicts warming for 1998-2012.

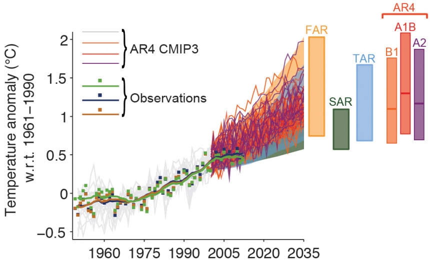
... even though they got forcing about right



IPCC 2013 WG1, TS.3 Figure 1

- Red shows estimated observed forcing. Forcing stayed about constant over the whole 1951-2012 period.
- Grey shows distribution of forcing from 114 runs of IPCC 's favorite climate model. It matches pretty closely, at least at the mean.

This is embarassing.



IPCC 2013 WG1, TS. TFE 3.1

- There was less warming than predicted by any of the previous IPCC reports.
- This is what happens if you don't worry about model uncertainty.
- Compare to 'credibility bounds'.

Conclusion

- Uncertainty is a central part of the global warming problem.
  - We have uncertainty around measurements.
  - We have uncertainty around models.
- Statistics lets us think about both problems carefully, but it's not easy.
- Carelessness about this can lead to fundamental misunderstandings of the data and to really bad predictions.