A Unified Theory of Cities*

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Abstract: *How do people arrange themselves when they are free to choose work and residence locations, when commuting is costly, and when increasing returns may affect production? We consider this problem when the location set is discrete and households have heterogenous preferences over workplace-residence pairs. We provide a general characterization of equilibrium throughout the parameter space. The introduction of preference heterogeneity into an otherwise conventional urban model fundamentally changes equilibrium behavior. Multiple equilibria are pervasive although stable equilibria need not exist. Stronger increasing returns to scale need not concentrate economic activity and lower commuting costs need not disperse it. The qualitative behavior of the model as returns to scale increase accords with changes in the patterns of urbanization observed in the Western world between the pre-industrial period and the present.*

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1. Introduction

Two centuries ago, one human out of ten lived in a city, incomes were a fraction of their present level, and commuting occurred on foot. Today, more than half of the world's population is urbanized, and commuting by foot is a curiosity in much of the world. Conventional wisdom holds that this process of urbanization is a contest between increasing returns to scale in production and the cost of commuting. Returns to scale in production leads to highly concentrated employment while commuting allows people to live at lower densities than those at which they work: as returns to scale increase and commuting costs fall, cities grow.

We investigate how the organization of cities is related to the strength of returns to scale and to the cost of commuting. Our problem is to understand how people arrange themselves when they are free to choose work and residence locations, when commuting is costly, and when some form of increasing returns may affect production. We consider this problem in a framework with stylized discrete geography, local increasing returns to scale, and households with heterogenous preferences over workplace-residence pairs. We provide a general characterization of equilibrium throughout the parameter space. In particular, we describe the implications of arbitrarily strong increasing returns to scale for the organization of cities. We investigate the stability of equilibrium and provide a preliminary analysis of more general economies where locations have first nature advantages or where productivity spillovers from one location to another may occur.

The contest between increasing returns to scale and commuting costs does not play out as the conventional wisdom suggests. Qualitative features of an equilibrium city depend sensitively on the strength of returns to scale in production. When production is not constant returns to scale, corner equilibria, where production concentrates in a few locations, must always occur. With constant returns to scale or weak increasing returns, there is a unique interior equilibrium, economic activity is centralized, and stronger returns to scale increase central employment, wages and land rents. When increasing returns to scale are moderate, multiple stable interior equilibria may occur and stronger returns to scale may decrease central employment, wages and land rents. When returns to scale are strong, stronger increasing returns to scale disperse employment and equalize wages and land rent across locations. That is, the conventional intuition that returns to scale are an agglomeration force does not hold for sufficiently strong increasing returns to scale.

Our findings about commuting costs are equally surprising. We find that employment is dispersed both when commuting costs are high and when they are low. It is only at intermediate levels of commuting costs that highly concentrated employment can arise. That is, when the location of production and residence is endogenous, the standard intuition that decreases in commuting costs must disperse economic activity need not apply.

Our investigation is important for a number of reasons. We address a foundational problem of urban economics. While much progress has been made on this problem, existing work relies on strong simplifying assumptions and arbitrary restrictions on the strength of agglomeration forces. For example, the workhorse monocentric city model sets the location of work exogenously, while recent developments based on quantitative spatial models require assumptions prohibiting multiple equilibria. Apart from its stylized geography, our model is general and allows us to provide the complete analytical characterization of equilibrium that has long eluded the literature.

Common theoretical approaches to the economics of cities can be usefully divided into two frameworks, classical urban economics and quantitative spatial modeling (QSM). The classical urban economics literature assumes homogenous households and stylized continuous geographies. In contrast, QSM assumes agents with heterogeneous preferences over workplace-residence pairs; complex, empirically founded discrete geographies; and a flexible description of the first nature advantages of particular locations for work or residence. We consider a hybrid case. We apply a quantitative spatial model to a stylized geography, a discrete linear city on a featureless landscape, much like what is often studied in the classical approach. A single parameter describes household heterogeneity in our model, as in the QSM literature, and so we are able to investigate what happens as the heterogenous households approach the homogeneity of the older urban economics literature. In this way, we unify the two literatures and allow for a more complete understanding of both classes of models and their different implications for how cities behave.

The flexibility of quantitative spatial models permits an analysis of real world comparative statics in estimated or calibrated models that the classical literature does not. However, this flexibility comes at a price. It is not always clear whether such empirically founded comparative statics are primarily a reflection of the data on which the exercise is based, or if, like a 'theorem', they are a direct consequence of model assumptions. We proceed in the tradition of the Krugman model of economic geography and the Hecksher-Ohlin-Samuelsen model of trade and examine a highly stylized geography. By doing this, we hope to develop insight into the way that the basic forces of urban economics – commuting costs, returns to scale and preference heterogeneity – interact to affect equilibrium, and hence to refine our understanding about which comparative statics are empirically ambiguous and which are not.

While a number of our findings are surprising, two are particularly noteworthy. The heterogenous preferences of the QSM framework imply that an average household must prefer central to peripheral work or residence. Such preferences have no analog in the classical literature and they encourage a concentration of work and residence in the central location even in the absence of any other agglomeration force. Second, although the set of interior equilibria is often unique, we find that corner equilibria exist whenever there are increasing returns to scale in production. Both interior and corner equilibria always occur as long as there are increasing returns. Our framework is similar enough to those typical of the QSM literature to suggest that these corner equilibria are a general feature of these models. Given that quantitative implementations of these models often depend on the uniqueness of equilibrium, we suspect this result may have important practical implications for the quantitative literature.

Determining the stability of equilibria is surprisingly subtle. The notion of stability most relevant to the quantitative literature, 'an iterative process will find the equilibrium', turns out not to be well defined. On the other hand, an analysis of a system of differential equations whose steady states coincide with the static equilibria of our model requires that we assume an ad hoc adjustment process and seems intractable. Given this, we rely on a more game theoretic notion of stability, in the spirit of trembling hand perfection. Using this definition, we show that: corner and 'near corner' equilibria are always unstable; for weak increasing returns to scale there is a unique, stable, interior equilibrium; and, in portions of the parameter space no stable equilibria exist.

We extend our analysis to a model where, as is typical in the quantitative literature, locations differ in first nature amenities and productivity. Our results suggest that the qualitative features of equilibrium in our benchmark case generalize to this economy. That is, we find no evidence that the presence of first nature differences in locations gives rise to equilibrium behavior that does not also arise in our benchmark model where locations are identical. Our benchmark model is based on a technology for which increasing returns to scale depends only on employment in a single location. We extend this model to a case where returns to scale can 'spillover' to nearby locations. Our results suggest that corner equilibrium do not persist with small productivity spillovers. They become 'near corner' equilibria that approach a corner as spillovers become weaker. More generally, the equilibria we find in our benchmark model appear to change continuously with the introduction of small spatial productivity spillovers.

Finally, for the cities of the Western world, we present a simple and brief 500 year history of the relationships between urban form and increasing returns to scale. We then compare the qualitative features of this history with the behavior of our model as increasing returns to scale varies over a range consistent with the approximately 13-fold increase income observed over this period. We find a qualitative relationship between the behavior we see in our model and history. Thus, our model appears to offer a simple theory that predicts the qualitative behavior of urban geography in the Western world from the pre-industrial period to the present.

2. Literature

We can usefully partition the literature into an older urban economics literature and a more recent literature on quantitative spatial models. Most papers in classical urban economics rest on the following assumptions. Households are homogenous or there are at most a small number of types. Space is continuous and uniform, whether on a line or in a plane, and equilibrium cities are generally symmetric around a single exogenously selected point. The simplest, and most influential model in this literature is the monocentric city model. It was developed by Alonso (1964), Mills (1967) and Muth (1969), and is articulated with particular clarity in Fujita (1989). This workhorse model rests on the assumption that the location of work is fixed exogenously at the center and households choose only their location of residence, although the model is otherwise quite general. In particular, firms may substitute between land and labor, and households between land and consumption.

Beckmann (1976) presents one of the earliest models to explain the endogenous formation of a city center. He assumes that household utility depends on land consumption and on the average distance to all households with which the household interacts. This leads to a city with a bell-shaped population density distribution and a similarly shaped land rent curve. Thus, the city emerges here as a social magnet. Alternatively, in spirit of Armington (1969) where each variety of a product is differentiated by the place where it is produced, one might think of the density of social interaction between any two locations as the volume of trade between these two locations whose size decreases with distance. In this case, Beckmann's model may be viewed as a one-dimensional reduced form of Allen and Arkolakis (2014).

The first general statement of our problem is due to Ogawa and Fujita (1980). This landmark paper considers a simple setting where firms choose only their location and households choose only their places of work and residence. They introduce the idea that firm productivity benefits from spillovers from every location, with distant spillovers less beneficial than those nearby. This assumption, now conventional, requires that the productivity of any given location responds to a distance weighted mean of employment at all locations. This creates an agglomeration externality, while land scarcity acts as a dispersion force. As the benefits of spillovers increases relative to the cost of commuting, they observe first a uniform, then a duocentric, and finally a monocentric equilibrium.

Fujita and Ogawa (1982) builds on their first paper by considering the case where spillovers decay exponentially rather than linearly with distance. When the spatial decay parameter is small, numerical results indicate that the three possible configurations described above still occur. However, when the spatial decay parameter is larger, the following may occur: (i) there exist equilibria with several centers; (ii) there is multiplicity of equilibria; and (iii) the transition from one equilibrium to another may be catastrophic. Lucas (2001) establishes general existence results in a model where agglomeration economies are not too strong and where firms and households are allowed to choose locations, subject to firms being restricted to a central business district and households to the surrounding region. Lucas and Rossi-Hansberg (2002) revisits the problem posed by Fujita and Ogawa (1982), but allow firms and households to make the same substitutions between labor and land and consumption and land as in a typical monocentric city model. They consider both constant and increasing returns to scale. They establish general existence and uniqueness results, but otherwise rely on numerical methods and restrict attention to 'weak enough' increasing returns that the multiple equilibria observed in Fujita and Ogawa (1982) do not arise.

Recently, a second class of models, 'quantitative spatial models' (QSM), has been brought to bear on problems of urban economics (Redding and Rossi-Hansberg, 2017). The fundamentals of these quantitative models are different from the older urban economics literature. In the QSM literature, model cities consist of discrete sets of locations rather than continuous spaces, and they describe realistic, rather than highly stylized geographies. More importantly, this literature considers heterogenous rather than homogenous agents. Where the older literature tends to focus on analytic solutions and qualitative results, the QSM literature focuses on the numerical evaluation of particular comparative statics in models that describe particular real world locations.

The QSM literature draws on a long history of scholarship on discrete choice models, and on the well-established literature that applies discrete choice models to transportation, location, geography, and trade problems (Anas, 1983; de Palma *et al.*, 1985; Tabuchi and Thisse, 2002; Eaton and Kortum, 2002). With that said, much of the recent work closely follows Ahlfeldt *et al.* (2015).[1] In this model, households have preferences over housing and consumption, as in the older urban economics literature, and commute from home to work. Space is discrete and is described a matrix of pairwise commuting costs. These matrices are typically constructed to describe commuting costs between pairs of neighborhoods in the empirical application of interest. Households have heterogenous preferences over work-residence pairs and each household selects a unique pair. Locations are heterogenous in their amenities and productivity, and the possibility of endogenous agglomeration economies is sometimes considered.

Most of what is known about quantitative spatial models is based on empirically founded and numerically evaluated comparative statics. However, in an important series of papers, Allen and Arkolakis (with coauthors) study the existence and uniqueness of equilibrium in spatial models similar to ours. Briefly, Allen and Arkolakis (2014) considers a model with a reduced form description of the land market and is entirely missing in Allen, Arkolakis and Takahashi (2020), while Allen, Arkolakis and Li (2020) set the shares of residential and commercial land in each neighborhood exogenously. Allen, Arkolakis and Li (2015) is difficult to compare to our model because they "consider a general firm technology (e.g. it could be constant or decreasing returns to scale)" (p.4), which seems to rule out increasing returns to scale (see their equation (10)).

Summing up, while important existence and uniqueness results are available, they are based on models with a land market that is rudimentary compared to what is typical in classical urban economics, or they restrict the level of returns to scale that is allowed. In contrast, our model considers a completely articulated land market and we are particularly interested in what happens as increasing returns to scale increases past the point where the equilibrium must be unique. While some of the Arkolakis and Allen results can probably be adapted to our setting, we do not investigate this possibility because our existence proof is straightforward and because our focus is on a characterization of multiple equilibrium, an issue that the Arkolakis and Allen theorems do not address.

3. A discrete city with heterogenous households

A city consists of a finite set of locations $\mathcal{I}$ with $|\mathcal{I}| = I$ and each location is endowed with one unit of land. The city is populated by a continuum [0,1] of households indexed by ν and by a competitive production sector whose size is endogenous. All households choose a residence $i \in \mathcal{I}$, a workplace $j \in \mathcal{I}$, and their consumption of housing and a tradable produced good.

In the QSM based literature, locations are typically endowed with both employment and residential 'amenities' that scale the payoffs from work and residence in each

[1]For example: Severen (2019); Monte *et al.* (2018); Dingle and Tintlenot (2020); Tsivanidis (2019); Couture *et al.* (2020); Balboni *et al.* (2020), Heblich *et al.* (2020), Herzog (2020) and Allen *et al.* (2015).

location. In the baseline model, we omit them for two reasons. First, it restricts our QSM based model to a stylized, featureless landscape of the sort considered in classical urban economics, and thereby facilitates the comparison of the two literatures. Second, it allows us to concentrate our attention on implications of the fundamental economic forces of the model, commuting costs, returns to scale, and preference dispersion, without the distraction of idiosyncratic exogenous first nature properties of the landscape. This simplification comes at a price. In particular, we cannot investigate whether economic fundamentals and first nature features of the landscape interact to lead to qualitatively different behavior. We investigate the implications of allowing for first nature amenities and productivity in Section 9.

Each household $\nu \in [0,1]$ has a type $\mathbf{z}(\nu) \equiv (z_{ij}(\nu)) \in \mathbb{R}_{+}^{I\times I}$. Thus, a household's type is a vector of non-negative real numbers, one for each possible workplace-residence pair ij. The mapping $\mathbf{z}(\nu) : [0,1] \to \mathbb{R}_{+}^{I\times I}$ is such that the distribution of types is the product measure of I^2 identical Fréchet distributions:

$$F(\mathbf{z}) \equiv \exp\left(-\sum_{i=1}^{I}\sum_{j=1}^{I} z_{ij}^{-\varepsilon}\right). \tag{1}$$

Households have heterogenous preferences over workplace-residence pairs, and household types will parameterize these workplace-residence preferences. Thus, types describe preferences while $\varepsilon \in (0,\infty)$ describes the heterogeneity of preferences. An increase in ε reduces preference heterogeneity and conversely.

Households must commute between workplace and residence. Commuting from i to j involves an iceberg cost $\tau_{ij} \geq 1$. This cost is the same for all households and $\tau_{ij} = 1$ if and only if $i = j$. Commuting costs affect household utility directly.[2]

A household that lives at i and works at j has utility

$$U_{ij}(\nu) = \frac{z_{ij}(\nu)}{\beta^{\beta}(1-\beta)^{1-\beta}} \frac{H_{ij}^{\beta} C_{ij}^{1-\beta}}{\tau_{ij}}.$$

where H_{ij} is household housing consumption and C_{ij} is household consumption of a homogeneous and costlessly tradable numéraire good. To simplify the model, 'housing' consists entirely of land.

Given the choice of workplace and residence, ij, the household budget constraint is,

$$W_j = C_{ij} + R_i H_{ij},$$

[2]These preferences are widely used in quantitative models, e.g., Ahlfeldt *et al.* (2015), Monte *et al.* (2018) and Heblich *et al.* (2020) to mention a few. The focus on psychic rather than real commute costs can be motivated by appeal to the finding in Kahneman *et al.* (2004) that travelling between home and the workplace is reported to be one of the most unpleasant activities in individuals' daily life. Furthermore, as Redding (2020) points out, with a linear homogenous utility function this formulation is equivalent to one in which the budget is $W_j/z_{ij} = C_{ij} + R_i H_{ij}$ and z_{ij} does not appear directly in the utility function. This equivalence also requires that the taste parameters do not enter the market clearing condition for land.

where W_j is the wage paid at location j and R_i the land rent at i. The resulting indirect utility function is

$$V_{ij}(\nu) = z_{ij}(\nu)\frac{W_j}{\tau_{ij}R_i^{\beta}}. \tag{2}$$

Summing up, we have a deterministic model in which each household makes mutually exclusive choices from a finite number of workplace-residence pairs ij in order to maximize their indirect utility (2).

Let

$$S_{ij} = \left\{ \mathbf{z} \in \mathbb{R}_{+}^{I\times I}\,;\, V_{ij}(\mathbf{z}) = \max_{r,s\in\mathcal{I}} V_{rs}(\mathbf{z}) \right\}$$

be the set of types $\mathbf{z}$ such that ij is (weakly) preferred to all other location pairs rs. Then, using (1) and (2), the share s_{ij} of households who choose the location pair ij equals

$$s_{ij} = \mu\left(\mathbf{z}^{-1}(S_{ij})\right) = \frac{\left[W_j/(\tau_{ij}R_i^{\beta})\right]^{\varepsilon}}{\sum_{r\in\mathcal{I}}\sum_{s\in\mathcal{I}}\left[W_s/(\tau_{rs}R_r^{\beta})\right]^{\varepsilon}}, \tag{3}$$

where the last equality stems from the Fréchet distribution assumption and μ is the Lebesgue measure over [0,1].[3] Our model is static and so all choices occur simultaneously.

Because the $V_{ij}(\nu)$ are Fréchet distributed, the average utility $\overline{V}$ across all households equals:

$$\overline{V} \equiv \int_0^1 \max_{i,j=1,\dots I} V_{ij}(\mathbf{z}(\nu))\mathrm{d}\nu = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left\{\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}}\left[W_j/\left(\tau_{ij}R_i^{\beta}\right)\right]^{\varepsilon}\right\}^{1/\varepsilon},$$

where $\Gamma(\cdot)$ is the gamma function.[4] Households that share the same type choose the same location pair[5] ij and reach the same equilibrium utility level, while households who make the same choice may have different types and do not have the same equilibrium utility level. Likewise, households that choose different location pairs do not generally enjoy the same equilibrium utility level. In general, equilibrium utility varies with type.

We have described an explicitly static and deterministic model of heterogeneous households. Existing formulations of this model are less clear about this issue. This can lead to questions about households' expectations and the extent to which those expectations coincide with realized outcomes.[6] Such questions are usually resolved by informal appeals to the law of large numbers, despite the difficulty of formulating the law of large numbers for a continuum (Judd, 1985; Feldman and Gilles, 1985; Uhlig,

[3] Note that there might be types which belong to more than one S_{ij} as they are indifferent between their multiple favorite choices. However, the set of such types always has a zero measure, so that they do not affect the type distribution (1).

[4] For the average utility to be finite, we require $\varepsilon > 1$.

[5] Except for the measure zero set of types that is indifferent between alternatives.

[6] These issues are stated clearly in Dingle and Tintlenot (2020).

1996). By insisting on a deterministic model of heterogenous households, we avoid these (admittedly subtle) issues. There is no need for probabilities and expectations; our outcomes are shares and averages.

We can rewrite (3) as,

$$s_{ij} = \kappa R_i^{-\beta\varepsilon} W_j^{\varepsilon} \tau_{ij}^{-\varepsilon}, \tag{4}$$

for

$$\kappa \equiv \left[\Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\right]^{\varepsilon} \overline{V}^{-\varepsilon}. \tag{5}$$

Let M_i and L_i be the mass of residents and households at i. Labor and land market clearing requires

$$\sum_{i\in\mathcal{I}} L_i = \sum_{j\in\mathcal{I}} M_j = 1, \tag{6}$$

where the residential population at i is

$$M_i \equiv \sum_{j\in\mathcal{I}} s_{ij} = \kappa R_i^{-\beta\varepsilon} \sum_{j\in\mathcal{I}} W_j^{\varepsilon} \tau_{ij}^{-\varepsilon}, \tag{7}$$

while the labor force at j is

$$L_j \equiv \sum_{i\in\mathcal{I}} s_{ij} = \kappa W_j^{\varepsilon} \sum_{i\in\mathcal{I}} R_i^{-\beta\varepsilon} \tau_{ij}^{-\varepsilon}. \tag{8}$$

Because each location i is endowed with one unit of land, land market clearing requires

$$H_i + N_i \leq 1, \tag{9}$$

where H_i is the amount of residential land and N_i is the amount of commercial land at location i. Condition (9) implies that land rent is zero when the whole amount of land at i is not used.[7]

Applying Roy's identity to (2), we have

$$H_i \equiv \sum_{j\in\mathcal{I}} s_{ij} H_{ij} = \sum_{j\in\mathcal{I}} s_{ij} \frac{\beta W_j}{R_i}. \tag{10}$$

Substituting (4) in (10) gives

$$R_i = \left(\frac{\beta\kappa}{H_i} \sum_{j\in\mathcal{I}} W_j^{1+\varepsilon} \tau_{ij}^{-\varepsilon} \right)^{\frac{1}{1+\beta\varepsilon}}. \tag{11}$$

This is the bid rent that households are willing to pay to reside at i, given $(W_1,...,W_I,H_i)$. Land rent accrues to absentee landlords who play no further role in the model.

Assume that the numéraire is produced under perfect competition and the production function at location j is

$$Y_j = A_j L_j^{\alpha} N_j^{1-\alpha},$$

[7] To allow for corner equilibria, this condition should be written as $(H_i + N_i - 1)R_i = 0$.

where A_j is location-specific TFP and $0 < \alpha < 1$. We assume that A_j depends only on the level of employment at j,

$$A_j = L_j^{\gamma}, \tag{12}$$

for $\gamma \geq 0$, which is the same for all j.

To keep our model as simple as possible, our description of returns to scale restricts attention to local scale economies. In Section 9, we consider the possibility of spillovers of the sort that form the basis for important previous work such as Ogawa and Fujita (1980) or Lucas (2001).

If location j hosts a positive share of the production sector, the first-order conditions for the production sector yields the equilibrium wage and land rent as functions of land-labor ratio:

$$W_j = \alpha A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}, \tag{13}$$

$$R_j = (1-\alpha) A_j \left(\frac{L_j}{N_j}\right)^{\alpha}. \tag{14}$$

Dividing (13) by (14) and simplifying,

$$\frac{W_j}{R_j} = \frac{\alpha}{1-\alpha}\frac{N_j}{L_j}. \tag{15}$$

When $\gamma = 0$, note that

$$R_j = (1-\alpha)\left(\frac{N_j}{L_j}\right)^{-\alpha} = (1-\alpha)\left(\frac{W_j}{\alpha}\right)^{-\alpha/(1-\alpha)}. \tag{16}$$

This expression implies that R_j and W_j move in opposite directions under constant returns: a higher land rent at j is equivalent to a lower wage at this location. Moreover, the land rent at j is positive and finite if and only if W_j is positive and finite.

Our city is described by I-vectors of real numbers. To describe them, define a *spatial pattern* to be an element of $\mathbb{R}_+^I$, a vector enumerating a non-negative real number for each location in $\mathcal{I}$. The spatial patterns that describe our model city are: the *residential pattern* $\mathbf{M} \equiv (M_1, M_2, \ldots, M_I)$; the *employment pattern* $\mathbf{L} \equiv (L_1, L_2, \ldots, L_I)$; the *housing pattern* $\mathbf{H} \equiv (H_1, H_2, \ldots, H_I)$; the *commercial pattern* $\mathbf{N} \equiv (N_1, N_2, \ldots, N_I)$; the *wage pattern* $\mathbf{W} \equiv (W_1, W_2, \ldots, W_I)$; and the *land rent pattern* $\mathbf{R} \equiv (R_1, R_2, \ldots, R_I)$. A spatial pattern is *interior* if none of its elements is zero. Otherwise, it is a *corner* pattern.

We can now define an equilibrium for our discrete city with heterogenous agents.

Definition 1 *A (spatial) equilibrium is a vector* $\{\mathbf{M}^*, \mathbf{L}^*, \mathbf{H}^*, \mathbf{N}^*, \mathbf{W}^*, \mathbf{R}^*\}$ *such that:*

i. all households make utility-maximizing choices of workplace, residence, land, and consumption to satisfy (3),

ii. the first-order conditions (13) and (14) are satisfied at every location where the tradable good is produced, *iii. the land market clearing condition (9) holds at each location, and*

iv. the population constraint (6) holds.

We say that an equilibrium is *interior* when all component patterns of $\{\mathbf{M}^*,\mathbf{L}^*,\mathbf{H}^*,\mathbf{N}^*,\mathbf{W}^*,\mathbf{R}^*\}$ are interior. Otherwise, it is a corner equilibrium.

4. A linear city with three locations

Our discrete city with heterogenous households is based on assumptions and functional forms common in the QSM literature. Because the object of the QSM literature is (usually) the numerical evaluation of particular empirically founded comparative statics, this means that our model is more complex than we require to investigate analytic comparative statics or to compare with the more stylized urban economics literature. Therefore, we let $\mathcal{I} \equiv \{-1,0,1\}$ and restrict our discrete city with heterogenous agents to a geography consisting of three identical locations evenly spaced along a line.

This geography is the simplest in which to examine when activities concentrate in the center or disperse to the periphery. Thus, it is the simplest geography in which we can consider when a city forms a central district where activity is concentrated, and when economic activity remains dispersed. It is also qualitatively similar to the geography of the linear monocentric city that forms the basis for much of the urban economics literature.

For tractability and to ease comparison with the urban economics literature, we focus on *symmetric spatial patterns* $x = (x_{-1},x_0,x_1)$ where $x_1 = x_{-1}$, and usually write such patterns as (x_1,x_0,x_1) rather than (x_{-1},x_0,x_1). We say that a symmetric spatial pattern is *bell-shaped*, *flat*, or *U-shaped* as x_0 is greater than, equal to, or less than x_1. Say that a symmetric spatial pattern (x_1,x_0,x_1) is *more centralized* than a symmetric spatial pattern (y_1,y_0,y_1) if and only if $\frac{x_0}{x_1} > \frac{y_0}{y_1}$.

The restriction to three locations and symmetric patterns, allows us to focus our attention on the symmetric, three element versions of the spatial patterns listed above: (M_1,M_0,M_1), (L_1,L_0,L_1), (H_1,H_0,H_1), (N_1,N_0,N_1), (W_1,W_0,W_1), and (R_1,R_0,R_1). That is, the patterns for residence, employment, housing, industry, wages and land rents.

Our analysis below is organized around studying the centrality of these six patterns. To facilitate this, define the corresponding centrality ratios,

$$m \equiv \frac{M_0}{M_1}; \quad \ell \equiv \frac{L_0}{L_1}; \quad h \equiv \frac{H_0}{H_1}; \quad n \equiv \frac{N_0}{N_1}; \quad w \equiv \frac{W_0}{W_1}; \quad r \equiv \frac{R_0}{R_1}.$$

Under symmetry, given an aggregate constraint, e.g. $2x_1 + x_0 = 1$, a spatial pattern is uniquely determined by one more piece of information, such as the ratio x_0/x_1.

For this linear city, the iceberg commuting cost matrix is

$$\begin{pmatrix} \tau_{-1,-1} & \tau_{-1,0} & \tau_{-1,1} \\ \tau_{0,-1} & \tau_{0,0} & \tau_{0,1} \\ \tau_{1,-1} & \tau_{1,0} & \tau_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & \tau & \tau^2 \\ \tau & 1 & \tau \\ \tau^2 & \tau & 1 \end{pmatrix}, \tag{17}$$

where $\tau > 1$. To ease notation, we define

$$\begin{aligned} a &\equiv \frac{\alpha\beta}{1-\alpha} > 0, \\ b &\equiv \frac{(1-\alpha)(1+\varepsilon)}{\alpha\beta\varepsilon} = \frac{1+\varepsilon}{a\varepsilon}, \\ \phi &\equiv \tau^{-\varepsilon}. \end{aligned} \tag{18}$$

The parameter a is important for our analysis. Recall that α and β are the labor share of output and the housing share of consumption. Thus, the denominator of a is the commercial land share and the numerator is the induced residential land share in production. It follows that a measures the relative intensity of the production sector's demand for commercial versus residential land. As a increases, firms are relatively less reliant on commercial land, and conversely. Recalling that the periphery is land rich and the center is land poor, it is not surprising that a plays an important role in determining whether employment or residence is more centralized.

We will sometimes refer to $\phi \in (0,1)$ as the *spatial discount factor*. By inspection of equation (18), ϕ decreases with the level of commuting costs ($\tau \uparrow$) and increases with the heterogeneity of the population ($\varepsilon \downarrow$). Hence, ϕ may be high (resp., low) because either commuting costs are low (resp., high), or the population is very (resp., not very) heterogeneous, or both.

Applying the symmetry assumption to the wage and land rent patterns **W** and **R** and using (4) and (17), we get the following equilibrium conditions:

$$\begin{aligned} s_{0,0} &= \kappa R_0^{-\beta\varepsilon} W_0^{\varepsilon}, \\ s_{0,1} = s_{0,-1} &= \kappa R_0^{-\beta\varepsilon} W_1^{\varepsilon}\tau^{-\varepsilon}, \\ s_{1,1} = s_{-1,-1} &= \kappa R_1^{-\beta\varepsilon} W_1^{\varepsilon}, \\ s_{1,-1} = s_{-1,1} &= \kappa R_1^{-\beta\varepsilon} W_1^{\varepsilon}\tau^{-2\varepsilon}, \\ s_{1,0} = s_{-1,0} &= \kappa R_1^{-\beta\varepsilon} W_0^{\varepsilon}\tau^{-\varepsilon}. \end{aligned} \tag{19}$$

Assume that $\mathbf{s} \equiv (s_{ij})$ is a symmetric *commuting pattern*. Since

$$\sum_{i=-1}^{1} s_{ij} = L_j, \qquad \sum_{j=-1}^{1} s_{ij} = M_i,$$

symmetry implies that we must determine the values of three variables to obtain the equilibrium outcome. For example, it is sufficient to know L_0, M_0, and s_{00} to determine all the s_{ij}, and hence the vector $\{$**M**,**L**,**H**,**N**,**W**,**R**$\}$.

Setting

$$\omega \equiv w^{\varepsilon} \quad \text{and} \quad \rho \equiv r^{-\beta\varepsilon}, \tag{20}$$

the first-order condition (16) becomes $\rho = \omega^a$. Clearly, $\omega > 1$ means that jobs at the central area pay a higher wage than those at the peripheries, while $\rho > 1$ means that land at the central area is cheaper than that at the peripheries.

We can now discuss the economic forces at work in this model. To illustrate ideas, consider the case when wages and land rents are the same in all locations and let $V = W/R^{\beta}$. In this case, using (2), a household's discrete choice problem is

$$\max_{ij} \left\{ \begin{array}{ccc} z_{-1,-1}V, & \frac{z_{-1,0}}{\tau}V, & \frac{z_{-1,1}}{\tau^2}V \\ \frac{z_{0,-1}}{\tau}V, & z_{0,0}V, & \frac{z_{0,1}}{\tau}V \\ \frac{z_{1,-1}}{\tau^2}V, & \frac{z_{1,0}}{\tau}V, & z_{1,1}V \end{array} \right\}.$$

This is the standard way of stating a discrete choice problem, except that we have arranged the nine choices in a matrix so that the row choice corresponds to a choice of residence and choice of column to a choice workplace.

In this case, because the distribution of idiosyncratic tastes is identical for all nine location pairs, the average payoff for a household choosing a central residence is

$$E\left(\max\left\{\frac{z_{0,-1}}{\tau}V, z_{0,0}V, \frac{z_{0,1}}{\tau}V\right\}\right) = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(1+\frac{2}{\tau^{\varepsilon}}\right)^{1/\varepsilon}V. \tag{21}$$

Similarly, the average payoff for a household choosing one of the peripheral locations as a residence is

$$E\left(\max\left\{z_{-1,-1}V, \frac{z_{-1,0}}{\tau}V, \frac{z_{-1,1}}{\tau^2}V\right\}\right) = \Gamma\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(1+\frac{1}{\tau^{\varepsilon}}+\frac{1}{\tau^{2\varepsilon}}\right)^{1/\varepsilon}V. \tag{22}$$

Because $\tau > 1$, it follows that the average payoff for a household choosing the peripheral location is lower than that of an average household choosing the central location. By symmetry, the corresponding statement is also true for the choice of work location. As a result, the average payoff for a household choosing a peripheral work location is lower than that of an average household choosing the central location for work. In this sense, the structure of the above discrete choice problem creates what we will call an *average preference* for residence in the central location, and a similar average preference for work in the central location.

Because the average preference for central work and residence depends on both commuting costs, τ, and population preference dispersion ε, it is not simply the translation into a discrete model of the commuting costs of classical urban economics. When $\varepsilon \to \infty$, (21) and (22) are identical, so that commuting costs alone are not sufficient to create an average preference for central work or residence. Nor are such average preferences an artifact of our stylized geography. If we exclude empirically uninteresting geographies like circles, most remaining location sets will have a center in the sense of this example, and corresponding average preferences for this location.

Average preferences for central work and location are noteworthy for the following reasons. First, even in the absence of more familiar agglomeration effects operating through production, this model has two 'agglomeration' forces, the average preference for central work and the average preference for central residence. Second, these average preferences are not agglomeration forces in the conventional sense. They do not incentivize geographic concentration, rather they incentivize concentration in the central location. Third, the urban economics literature based on homogenous agents takes seriously the possibility of differential labor productivity across locations. However, the possibility of preferences over work locations is never considered. This is a feature of quantitative spatial models without analog in the urban economics literature. Fourth, on the basis of the existing literature, we expect that the preference for central residence will be capitalized into higher central land rents. We will also see that the preference for central work location reduces central wages.

Against the two centralizing forces of average preferences are set two centrifugal forces. There is twice as much land in the periphery as the center. Because land contributes to utility and productivity, the scarcity of central land incentivizes the movement of employment and residence to the periphery.

We can now guess at the form of a symmetric equilibrium under constant returns to scale. In equilibrium, the centrifugal force of land scarcity and the centralizing force of average preferences balance. Access to the center will be scarce and so land rent will be higher and wages lower in the center. Whether the center ends up relatively specialized in residence or work will depend on which of the two activities has the highest demand for land, and this activity will locate disproportionately in the land abundant periphery.

We postpone a discussion of equilibrium under increasing returns to scale until after we provide a more formal characterization of such equilibria.

5. Existence and multiplicity of equilibria

We now turn to the characterization of symmetric equilibria in our three location city. We proceed in three main steps. In the first, we derive the demand for housing, the demand for commercial space, the supply of workers, and the supply of residents as functions of ω and ρ. This done, we derive a system of two equations in wages and land rent that characterize equilibrium. In the third step, we solve this system of equations.

The next proposition shows that in equilibrium all variables can be expressed in terms of just ρ and ω.

Proposition 1 *The equilibrium demand for housing and commercial space and the equilibrium supply of workers and residents are:*

$$M_0 = \frac{\rho(\omega+2\phi)}{\rho(\omega+2\phi)+2(\phi\omega+1+\phi^2)}, \qquad M_1 = \frac{1-M_0}{2}, \tag{23}$$

$$L_0 = \frac{\omega(\rho+2\phi)}{\omega(\rho+2\phi)+2(\phi\rho+1+\phi^2)}, \qquad L_1 = \frac{1-L_0}{2}, \tag{24}$$

$$H_0 = \frac{a\rho(1+2\phi\omega^{-\frac{1+\varepsilon}{\varepsilon}})}{a\rho(1+2\phi\omega^{-\frac{1+\varepsilon}{\varepsilon}})+\rho+2\phi}, \qquad N_0 = 1-H_0, \tag{25}$$

$$H_1 = \frac{a(\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}+1+\phi^2)}{a(\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}+1+\phi^2)+\phi\rho+1+\phi^2}, \qquad N_1 = 1-H_1. \tag{26}$$

Proof: This proposition is a special case of Proposition 11, proved in Appendix K.

The derivation of these functions involves algebraic manipulation of the equilibrium conditions. For example, each of M_i and L_j is derived from the expressions for the share of households choosing each workplace-residence pair ij. In one case, we sum over workplace-residence pairs with a common residence, and in the other over pairs with a common workplace. The identical denominator for each of the four functions described by (23) and (24) is simply the three-location version of the denominator on the right-hand side of equation (3). The expressions for H_i and N_j are more complicated because they must also satisfy the land market clearing condition (9). Notice that we write M_i, L_i, N_j, and H_i exclusively in terms of price ratios, r and w, rather than the prices themselves, R_i and W_j. This simplifies our effort to characterize equilibrium because we need to solve for only two equilibrium quantities instead of four.

We can refine our understanding of how the average preference for central work and residence affect equilibrium by examining the expressions for residential population and employment (23)-(24). Assume that wages and land rents are equal across locations so that $\omega = \rho = 1$. Substituting in the expressions for M_i and L_i we find that $M_0(1) = L_0(1) > M_1(1) = L_1(1)$. Therefore, central employment and residence is greater than in the periphery even though the different locations have the same relative pecuniary appeal. This reflects the average preference for central residence and workplace. Note also that $H_0 = H_1 = a/(a+1)$ for $\omega = \rho = 1$.

We now turn to finding the equilibrium values of ω and ρ. We would like to derive a system of equations involving only ω and ρ that incorporates all of the equilibrium conditions given in Definition 1. The following proposition describes such a system.

Proposition 2 *Assume $\gamma \neq \alpha/\varepsilon$. Then, a pair (ρ^*,ω^*) is an interior equilibrium if and only if it solves the following two equations:*

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho) \equiv \frac{\phi\rho - 2a\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+a)}{(1+a)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - a\phi}, \tag{27}$$

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} = g(\rho;\gamma) \equiv \rho^{\frac{b}{1-\gamma\varepsilon/\alpha}} \left(\frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2}\right)^{\frac{\gamma\varepsilon/\alpha}{1-\gamma\varepsilon/\alpha}\frac{1+\varepsilon}{\varepsilon}}. \tag{28}$$

Proof: This proposition is a special case of proposition 11, proved in Appendix K.

Combining (27) and (28), we arrive at a *single* equation in terms ρ, which determines the interior equilibria. Thus, studying the equilibrium behavior of our discrete linear city reduces to studying the solution(s) of one equation in one variable:

$$f(\rho) = g(\rho;\gamma). \tag{29}$$

We show the existence of an interior equilibrium by showing that (29) has a positive solution ρ^*. We determine the number of possible interior equilibria by determining the number of positive solutions of equation (29).

This argument requires two comments. First, equation (29) is not defined when $\gamma = \alpha/\varepsilon$, so that we will need a specific argument for this case. Second, we will see that $\gamma = \alpha/\varepsilon$ is a threshold value of γ, below which there is a unique interior equilibrium, and above which multiple interior equilibria may occur. To ease exposition of equilibrium behavior around this threshold, define $\gamma_m \equiv \alpha/\varepsilon$.

We now turn to a characterization of equilibrium. To begin, we establish the properties of the functions f and g. Observe that the function f does not involve the parameter γ, and thus remains the same for all values of γ. The following lemma states the main properties of function the f that are important for the characterization of spatial equilibria.

Lemma 1 *The function $f(\rho)$ has a vertical asymptote at $\rho_0 \in (0,1)$ and is equal to 0 at $\rho_1 > 1$. Furthermore, $f(\rho) > 0$ if and only if $\rho_0 < \rho < \rho_1$ and decreases over (ρ_0,ρ_1).*

Proof: (i) The denominator of $f(\rho)$ is an increasing function of ρ, which is unbounded above and negative at $\rho = 0$. Therefore, the equation

$$D(\rho) \equiv (1+a)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - a\phi = 0$$

has a unique positive root ρ_0. Clearly, $D(\rho) > 0$ if and only if $\rho > \rho_0$. Since $D(1) > 0$, it must be that $\rho_0 < 1$.

(ii) It is readily verified using (27) that the numerator of $f(\rho)$ is a concave function which is positive at zero. It increases in the vicinity of 0 and then starts decreasing and goes to $-\infty$ as $\rho \to \infty$. Hence, the equation

$$N(\rho) \equiv \phi\rho - 2a\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+a) = 0$$

has a unique positive root ρ_1. Clearly, $N(\rho) < 0$ if and only if $\rho > \rho_1$. Since $N(1) > 0$, it must be that $\rho_1 > 1$.

(iii) We have $f(\rho) > 0$ if and only if only if the numerator $N(\rho)$ and the denominator $D(\rho)$ have the same sign. This holds if and only if $\rho_0 < \rho < \rho_1$. The interval (ρ_0,ρ_1) is not empty because $\rho_0 < 1 < \rho_1$. As a result, $f(\rho) > 0$ if and only if $\rho \in (\rho_0,\rho_1)$.

(iv) We show in Appendix A that $f(\rho)$ decreases from ∞ to 0 over (ρ_0,ρ_1). Q.E.D.

Unlike f, the function g varies with γ. We show in Lemma 2 that when $\gamma < \gamma_m$, g is an increasing function that converges to an increasing step function as $\gamma \nearrow \gamma_m$. When $\gamma > \gamma_m$, g is a decreasing function that converges to a decreasing step function as $\gamma \searrow \gamma_m$. In both limiting cases, the value ρ_L at which the step occurs solves the equation

$$\rho^{\frac{1}{a}} \frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} = 1. \tag{30}$$

Because the left-hand side of (30) increases with ρ and is equal to 0 (resp., ∞) when $\rho = 0$ (resp., $\rho \to \infty$), ρ_L is unique and $\rho_L < 1$.

The following lemma provides a more formal statement of the relevant properties of g.

Lemma 2 *(i) If $\gamma \neq \gamma_m$, then $g(\rho;\gamma)$ is strictly positive and finite over $[\rho_0,\rho_1]$. (ii) If $\gamma < \gamma_m$, then g is increasing over $[\rho_0,\rho_1]$. (iii) If $\gamma > \gamma_m$, then g is decreasing over $[\rho_0,\rho_1]$. (iv) As γ converges to γ_m, we have:*

$$\lim_{\gamma\nearrow\gamma_m} g(\rho;\gamma) = \begin{cases} 0, & \rho < \rho_L; \\ \left(\frac{\rho_L+2\phi}{1+\phi\rho_L+\phi^2}\right)^{-\frac{1+\varepsilon}{\varepsilon}} & \rho = \rho_L; \\ \infty, & \rho > \rho_L; \end{cases} \qquad \lim_{\gamma\searrow\gamma_m} g(\rho;\gamma) = \begin{cases} \infty, & \rho < \rho_L; \\ \left(\frac{\rho_L+2\phi}{1+\phi\rho_L+\phi^2}\right)^{-\frac{1+\varepsilon}{\varepsilon}} & \rho = \rho_L; \\ 0, & \rho > \rho_L. \end{cases}.$$

Proof: Part (i) follows from combining (28) with $0 < \rho_0 < \rho_1 < \infty$. Parts (ii) and (iii) are obtained by differentiating g with respect to ρ. Part (iv) holds because $g(\rho;\gamma)$ may be rewritten as follows:

$$g(\rho;\gamma) = \left[\left(\rho^{\frac{1}{a}} \frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} \right)^{\frac{\gamma_m}{\gamma_m - \gamma}} \cdot \left(\frac{\rho + 2\phi}{1 + \phi\rho + \phi^2} \right)^{-1} \right]^{\frac{1+\varepsilon}{\varepsilon}}. \tag{31}$$

Q.E.D.

The four panels of figure 1 illustrate the functions f and g for various parameter values. In each panel, the horizontal axis is ρ and the vertical axis is an increasing transformation of ω. The behavior of f, the red line in each panel, is relatively simple. It is a decreasing, continuous function that has a positive asymptote at $\rho_0 < 1$ and declines monotonically to 0 at $\rho_1 > 1$. The behavior of g is more complicated. The two panels on the left each describe g for three different values of γ, with dark blue the smallest, light blue the largest, medium blue in between, and all three less than γ_m. In every case, g is a continuous, increasing function. The right column is the same as the left, but considers larger values of γ. Here, the light blue line traces g for the smallest value of γ, dark blue uses the largest value, medium blue is intermediate value, and all three are greater than γ_m.

This figure makes clear that, in general, f and g cross for a positive value ρ, and when $\gamma > \gamma_m$, they may cross more than once. On this basis, we can surmise that an interior equilibrium exists in our model city throughout much or all of the parameter space.

Lemma 1 shows that f has an asymptote at $\rho_0 < 1$ and is zero at $\rho_1 > 1$. Lemma 2 shows that as $\gamma \to \gamma_m$, g also approaches its singularity at $\rho_L < 1$. While Lemmas 1 and 2 guarantee that $\rho_1 > \rho_L$, they do not allow us to order ρ_L and ρ_0. Unsurprisingly, the equilibrium configuration of our city depends sensitively on whether or not $\rho_L > \rho_0$. Lemma 3 provides necessary and sufficient conditions on commuting costs and the demands for commercial and residential land for $\rho_L > \rho_0$.

Lemma 3 *There exists a function $\overline{\phi}(\beta\varepsilon) \in (0,1)$ and scalar $\overline{a} > 0$ such that if $\phi < \overline{\phi}$ or $a < \overline{a}$ then $\rho_0 < \rho_L$. Conversely, if $\phi > \overline{\phi}$ and $a > \overline{a}$ then $\rho_0 > \rho_L$.*

Proof: See Appendix B.

Restating this lemma informally, we have $\rho_0 < \rho_L$ if either the spatial discount factor is low *or* the demand for commercial land is sufficiently large relative to the demand for residential land. Conversely, if the spatial discount factor is high *and* the demand for commercial land is low, then $\rho_0 > \rho_L$.

The above equilibrium conditions imply the following relationship between land rent and employment, which is surprisingly simple and will be useful in a number of our proofs. Using (24) leads to

$$\ell = \omega \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2}.$$

Combining this with (28), this equation becomes

$$\ell = \left(\rho^{\frac{1}{a}} \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2} \right)^{\frac{\gamma_m}{\gamma_m - \gamma}}. \tag{32}$$

If $\gamma < \gamma_m$, then ℓ increases over (ρ_0,ρ_1) and $\ell(\rho) > 1$ if and only if $\rho > \rho_L$. On the other hand, when $\gamma > \gamma_m$, the opposite holds: ℓ decreases over (ρ_0,ρ_1) and $\ell(\rho) > 1$ if and only if $\rho < \rho_L$.

So far, we have focused on interior equilibria. The following proposition formalizes this discussion and characterizes the corner equilibria.

Proposition 3 *If $\gamma \geq 0$, an interior equilibrium always exists. Furthermore, there exist two corner equilibria if and only if $\gamma > 0$. These corner equilibria are such that the employment patterns are given, respectively, by $(0,1,0)$ and $(1/2,0,1/2)$. In both corner and interior equilibria, each location hosts a positive mass of residents.*

Proof: Using the properties of f and g given in Lemmas 1 and 2, the intermediate value theorem implies the existence of an equilibrium when $\gamma \neq \gamma_m$. The case where $\gamma = \gamma_m$ is discussed in Section 6.1.2. Furthermore, we show in Appendix C that there exist two corner equilibria for all $\gamma > 0$: (i) $\omega^* \to \infty$ and $\rho^* = \rho_1$, and thus employment is evenly distributed between peripheries, and (ii) $\omega^* = 0$ and $\rho^* = \rho_0$, so that employment is fully concentrated at the center $i = 0$.

Finally, we show that $M_i^* > 0$ for all i. Assume that $R_i^* = 0$ at i. Since there is a location $j \in \mathcal{I}$ such that $W_j^* > 0$, workers who choose the pair ij enjoys an infinite utility level, which implies $s_{ij} > 0$. These workers' land demand is thus infinite while the land supply is finite, a contradiction. Q.E.D.

Thus, multiple equilibria always arise when there are increasing returns.

6. Comparative statics

We now turn our attention to an investigation of how the equilibrium behavior of our city changes as fundamental characteristics of the economy change. More specifically, we focus on the implications of changes in returns to scale, γ, commuting costs, τ, and preference dispersion, ε.

A *Returns to scale*

We investigate changes in the equilibrium behavior of our city as returns to scale changes. To ease exposition, we introduce terminology to describe the three important domains of returns to scale. These ranges will correspond to qualitatively different equilibrium behavior.

Definition 2 *Increasing returns to scale (IRS) are:*

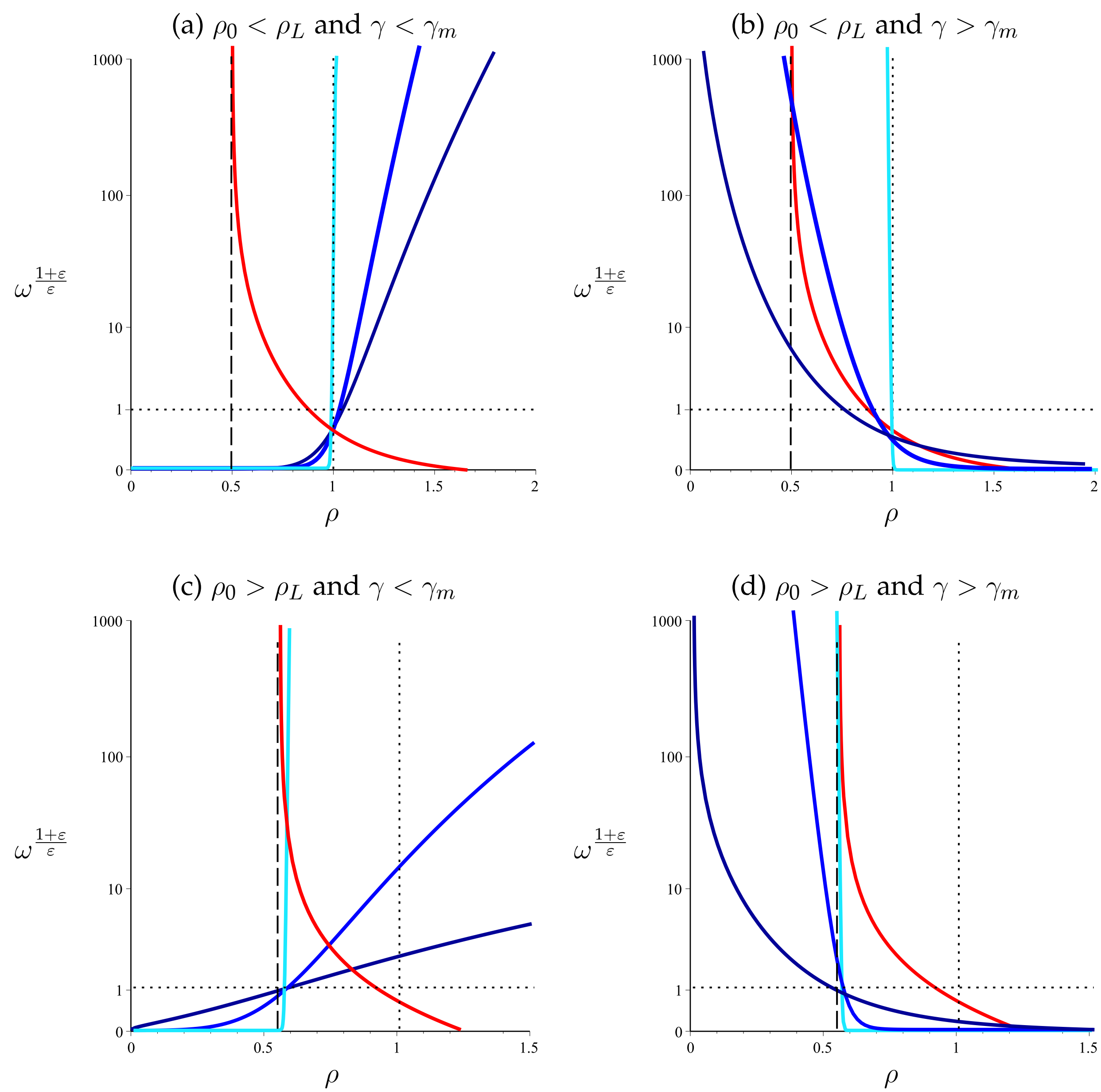
i. weak $\Leftrightarrow 0 < \gamma < \gamma_m \equiv \alpha/\varepsilon$,

ii. moderate $\Leftrightarrow \gamma_m \leq \gamma \leq \gamma_s$,

iii. strong $\Leftrightarrow \gamma > \gamma_s$.

Production is constant returns to scale when $\gamma = 0$.

Figure 1: Graphical demonstration of equilibrium for a range of parameter values.



Notes: These figures illustrate equilibrium in several different cases. In all panels, f is given by the red line. The blue lines describe g. In the left two panels, darker colors of blue indicate smaller values of γ and in the right two panels darker colors of blue indicate larger values of γ.

This definition requires three comments. First, population must be heterogeneous (ε finite) for weak returns to scale to occur. Therefore, we cannot observe the weak increasing returns domain cities with the homogenous households that are standard in the urban economics literature. This establishes that qualitative features of the equilibrium depend on the degree of preference heterogeneity.

Second, we will see that the threshold level of returns to scale at which the possibility of discontinuous changes in equilibrium outcome most obviously arise is γ_m. Recall that $\gamma_m = \alpha/\varepsilon$. In a modern economy, the labor share of production is about 0.6, while the range of commonly used estimates for epsilon is about $[5, 7]$. Taking the ratio of these values, we have γ_m in $[0.085, 0.12]$. Estimates of the wage elasticity of population for modern, developed country cities that control for sorting and first-nature productivity are typically around 0.05. However, the raw correlation between wages and density is larger, as are estimates for developing countries. This back of the envelope calculation, together with results presented below, will suggest that multiple interior equilibria occur in an empirically relevant part of the parameter space.

Third, ρ_0 must be smaller than ρ_L, i.e., commuting costs or the productivity of land sufficiently large, for moderate returns to scale to arise. To see this, solve the equilibrium condition $f(\rho) = g(\rho; \gamma)$ for to γ to get,

$$\gamma(\rho) = \frac{\log\left(\rho^{-b} f(\rho)\right)}{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\left[f(\rho)\right]^{\frac{\varepsilon}{1+\varepsilon}}\right)}.$$

The inverse of this function is the correspondence that specifies the set of interior equilibria for each value of γ.[8] Denote by γ_s the maximum of the function $\gamma(\rho)$ over $[\rho_0, \rho_1]$ subject to the condition $\rho^{-b} f(\rho) \geq 1$. It is readily verified that $\gamma_m = \gamma_s$ when $\rho_0 \geq \rho_L$, while $\gamma_m < \gamma_s$ when $\rho_0 < \rho_L$. In words, the region of moderate increasing returns to scale does not exist unless $\rho_0 < \rho_L$.

Figure 1 summarizes Lemmas 1-3 graphically. The top two panels describe cities where $\rho_0 < \rho_L$, a city where commuting costs are high or land is valuable in production. The bottom two panels describe an economy where $\rho_0 > \rho_L$, a city where commuting costs are low and land has a low value in production. The two panels on the left describe cities where production involves constant or weak increasing returns to scale. The right two panels describe cities where increasing returns are moderate or strong. Each panel evaluates g for three different values of γ. In every panel, the light blue line describes g for a value of γ close to the weak/moderate threshold γ_m, and darker blue lines describe g functions for values of γ that are progressively further from this threshold. Taken together, these figures permit a fairly complete description of the interior equilibria of our linear city with three locations.

Begin with $\rho_0 < \rho_L$. The case of constant or weak increasing returns to scale is illustrated in panel (a) where high commuting costs encourage households to work where they live or land hungry production faces pressure to disperse to the periphery (or both). We expect an equilibrium in such an economy to exhibit low levels of commuting and dispersed production. In fact, regardless of γ, f and g cross slightly below $\rho = 1$ and at a moderate value of ω. Since $\rho_0 < \rho_L$, as $\gamma \nearrow \gamma_m$ and g approaches its asymptote at ρ_L, it does so when f is well away from its asymptote at ρ_0. Hence, the equilibrium value of ω grows with γ but remains bounded, meaning that the ratio of central to peripheral wages can never grow too large. Thus, panel (a) describes a city

[8] Indeed, we use this function to construct the plots of ρ versus γ in figure 2.

where neither employment nor residence are highly concentrated in either the center or periphery.

In panel (b), we first consider the case of moderate increasing returns described by the medium blue line. At this value of γ, g crosses f three times. At the first intersection point, we have $\rho_1^* < 1$ and $\omega_1^* > 1$; at the second one, we see that ρ_2^* approaches ρ_L as γ decreases toward γ_m; at the third intersection point, we have $\rho_3^* > 1$ and $\omega_3^* < 1$. The value ω_1^* (resp., ω_3^*) in turn requires that employment occurs primarily in the center (resp., splits between the two peripheral locations).

The light blue line in panel (b) describes g when γ is just above γ_m. As gamma approaches this threshold, for one of the two new equilibria ω grows without bound (and occurs outside the frame of the figure) while ω approaches zero in the other equilibrium. That is, just above the threshold, these two equilibria approach corner patterns where all employment is either central or peripheral. The dark blue line in panel (b) describes g when returns to scale are strong. As γ increases further, the remaining interior equilibrium involves a moderate value of ω, that is, an equilibrium where employment is more or less evenly distributed across the three locations. Indeed, as households get wealthier, they consume more of the numéraire good and become more sensitive to the cost of commuting. As this occurs, they bid away differences in land rent and distribute themselves more uniformly across the three locations. Thus we arrive at the surprising conclusion that a low degree of increasing returns leads to agglomeration while a high degree fosters dispersion.

Summing up, in a city with $\rho_0 < \rho_L$, as γ increases, the city converges to an interior pattern in which employment does not become highly concentrated in the center or periphery. When returns to scale increases beyond the weak/moderate threshold, two new branches of equilibria arise. These equilibria involve extreme concentration of employment in the center or in the periphery. As γ increases further, these two extreme equilibria become flatter, and ultimately, when γ crosses the moderate/strong threshold, we are left with a single equilibrium where employment is not highly concentrated in the center or periphery.

In contrast, panels (c) and (d) show an economy where $\rho_L \leq \rho_0$. In panel (c), low commuting costs allow households to separate work and residence locations in response to a small wage premium, and productivity is not sensitive to the relatively abundant land of the periphery. In this case, increasing returns to scale compounds the average preference for central employment to concentrate employment in the center, and that households are able to cheaply disperse their residences to the land abundant periphery. In this city, as γ approaches $\gamma_m = \gamma_s$, it does so near the asymptote of f. As a result, the value of ω at which the two curves intersects becomes large. This reflects the fact that peripheral productivity is falling almost to zero as production concentrates in the center and consumes progressively less peripheral land, even as central productivity rises with agglomeration economies.

Panel (d) illustrates what occurs when returns to scale are strong in an economy where commuting costs are low and the share of land in production is relatively low. For such an economy, there is a unique equilibrium, and in this equilibrium most employment occurs in the periphery. As γ decreases toward the threshold, a second equilibrium arises. This equilibrium involves a large value of ω, and hence, employment highly concentrated in the center. As γ increases so that returns to scale are strong, ω gradually increases, just as in panel (b) where $\rho_0 < \rho_L$.

Summarizing, for $\rho_0 < \rho_L$ as returns to scale increase from low levels, employment concentrates in the center. When returns to scale surpass γ_m this bell-shaped equilibrium persists, but two new equilibrium branches involving a higher degree of concentration in the center or periphery arise. As returns to scale increases further the equilibrium branch with a U-shaped employment distribution persists, but the one with centralized employment does not. As γ increases still further, the distribution of employment becomes more uniform as wealthier households arbitrage away small differences in land price. Alternatively, when $\rho_0 \geq \rho_L$, there is always a unique interior equilibrium. Along this equilibrium branch, as γ increases employment first becomes more and more agglomerated, but as γ continues to increase employment becomes more and more dispersed.

We now turn to a formal statement of our results. Propositions 4 to 7 confirm and extend the intuition about equilibrium that we take from figure 1.

a Constant returns to scale

Under constant returns to scale equation (28) becomes $\omega^{\frac{1+\varepsilon}{\varepsilon}} = g(\rho; 0) = \rho^b$ so that (29) may be rewritten as

$$\frac{\phi\rho - 2a\phi\rho^{1+\frac{1}{\beta\varepsilon}} + (1+\phi^2)(1+a)}{(1+a)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - a\phi} = \rho^b. \tag{33}$$

Proposition 4 *Under constant returns to scale, there exists a unique equilibrium. This equilibrium is interior and such that $0 < \rho^* < 1$ and $0 < \omega^* < 1$. Furthermore, if*

$$\frac{\varepsilon}{1+\varepsilon} < \frac{1-\alpha}{\alpha\beta} \tag{34}$$

holds, then the equilibrium employment pattern is bell-shaped.

Proof: Proposition 3 implies that there is no corner equilibria when $\gamma = 0$. Since $g(\rho; 0)$ increases from 0 to 1 as ρ increases over [0,1], Lemma 1 implies that the two curves must cross exactly once. Furthermore, the intersection must occur strictly between ρ_0 and 1, which implies $0 < \rho^* < 1$. Since $g < 1$ over this interval, it must be that $\omega^* < 1$, while (33) implies $\omega^* > 0$. That (34) is sufficient for the equilibrium employment pattern to be bell-shaped is proven Appendix D. Q.E.D.

The inequalities $0 < \rho^* < 1$ and $0 < \omega^* < 1$ imply that all rents and wages are positive and finite. Furthermore, the equilibrium land rent is higher in the center while the equilibrium wage is lower, which reflects the average preference for the center. Hence, even in the absence of increasing returns, a (partial) agglomeration of production occurs at the center when (34) holds. It may seem surprising that this condition does not involve τ. We believe this reflects the fact that this condition is sufficient for $\ell^* > 1$, but not necessary.

When the population is homogeneous ($\varepsilon \to \infty$), the equilibrium condition (33) has a unique solution $\rho^* = 1$, which in turn implies $\omega^* = 1$. Using (23) and (24) shows that

$L_i^* = M_i^* = 1/3$ for $i = -1,0,1$, while it is easy to show that $s_{ij}^* = 0$ for $i \neq j$ and $s_{ii}^* = 1/3$. In other words, the spatial equilibrium is described by a flat world in which each location is an autarky. This should not come as a surprise as there is no agglomeration force.

b Weak increasing returns

The next two propositions describe the equilibrium when increasing returns to scale are weak.

Proposition 5 *Assume that $0 < \gamma < \gamma_m$. Then, there is a unique interior equilibrium and $\rho^* < 1$. Furthermore, if (34) holds, then the equilibrium employment pattern is bell-shaped and such that*

$$\frac{d\ell^*}{d\gamma} > 0 \qquad \frac{d\rho^*}{d\gamma} < 0 < \frac{d\omega^*}{d\gamma}.$$

Proof: Under weak IRS, given Lemmas 1 and 2, f and g must intersect exactly once. Furthermore, because $f(1) < 1 < g(1;\gamma)$, the intersection must occur at $\rho^* < 1$. We refer to Appendix E for a proof of the second part of the proposition. Q.E.D.

The substance of Proposition 5 seems intuitive. As scale economies increase, the central location attracts a larger share of workers, while both the relative land price and relative wage increase as the land rent and wage at the center capitalize the agglomeration force resulting from increasing scale economies.

The comparative statics in Proposition 5 hold whenever $0 < \gamma < \gamma_m$. This is somewhat surprising because, consistent with our earlier discussion of panels (a) and (c) of Figure 1, there are two distinct types of equilibrium behavior with weak increasing returns to scale, depending on whether ρ_0 is larger or smaller than ρ_L. More specifically, when γ is slightly below γ_m, two cases may arise:

(i) if $\rho_0 < \rho_L$, then setting $\gamma = \gamma_m$ in $[f(\rho)]^{1-\gamma\varepsilon/\alpha} = [g(\rho;\gamma)]^{1-\gamma\varepsilon/\alpha}$ yields an expression equivalent to (30) whose unique solution given by $\rho^* = \rho_L$. The equilibrium wage ratio remains bounded and the limiting employment pattern remains interior as $\gamma \nearrow \gamma_m$;

(ii) if $\rho_0 > \rho_L$, then $\omega^* \to \infty$ and the equilibrium employment pattern converges to (0,1,0) as $\gamma \nearrow \gamma_m$, hence $\rho^* = \rho_0$.

The existence of two distinct equilibrium regimes confirms the intuition suggested by figure 1. When commuting costs or land productivity is high and $\rho_0 < \rho_L$, the full concentration of production in the central location does not occur. The productivity advantage of the land abundant periphery is too great, or commuting is too costly to allow such central concentration of employment. On the other hand, when commuting costs are low and land is less productive so that $\rho_0 > \rho_L$, then almost full concentration of employment at the CBD occurs. In this case, there is no interior equilibrium at $\gamma = \gamma_m$ because increasing returns are strong enough to complement the preference for central employment and generate almost full concentration of employment at the center.

c Moderate increasing returns

When $\gamma > \gamma_m$, the function f remains unchanged, but the function g changes from an increasing to a decreasing function. When both f and g are decreasing, we are no longer assured of the existence of a unique equilibrium.

The following proposition formalizes the intuition suggested by figure 1 and confirms its generality.

Proposition 6 *Assume γ is slightly above γ_m.*

i. If $\rho_0 < \rho_L$, then there exist two interior equilibria, (ρ_1^*,ω_1^*) and (ρ_3^*,ω_3^*), such that

$$\omega_1^* > 1 > \omega_3^* \qquad \textit{and} \qquad \rho_1^* < 1 < \rho_3^*,$$

as well as a third interior equilibrium. As $\gamma \searrow \gamma_m$, the first two equilibrium employment patterns converge to $(1/2,0,1/2)$ and $(0,1,0)$, while the third equilibrium is interior.

ii. If $\rho_0 > \rho_L$, then there exists a unique interior equilibrium (ρ^*,ω^*) such that:

$$\omega^* < 1 \qquad \textit{and} \qquad \rho^* > 1.$$

Furthermore, as $\gamma \searrow \gamma_m$, the equilibrium employment pattern converges to $(1/2,0,1/2)$.

Proof: See Appendix F.

Part (i) of Proposition 6 establishes the existence of multiple interior equilibria when either commuting costs are high or the productivity of land in production is relatively low. To be precise, for any γ slightly larger than γ_m, there exists an arbitrary small $\Delta > 0$ such that the equilibrium employment patterns are given respectively by $((1-\Delta)/2,\Delta,(1-\Delta)/2)$ and $(\Delta/2,1-\Delta,\Delta/2)$. This result seems surprising as the assumption of heterogeneous agents is generally sufficient to smooth out discontinuities in equilibrium behavior.

Proposition 6 characterizes equilibrium just above the threshold separating weak and moderate increasing returns, γ_m, while Proposition 7 characterizes equilibrium above $\gamma_s > \gamma_m$, the threshold separating moderate and strong increasing returns. It is natural to expect that the behavior we observe near γ_m persists throughout the full range of moderate increasing returns, (γ_m,γ_s), as we observe in figure 2. We demonstrate that this is sometimes the case in the examples illustrated in figure 2. In fact, we cannot rule out the possibility of more complicated equilibrium behavior for values of γ just below γ_s, although we cannot find a counter example to contradict the conjecture that the results of Proposition 6 hold throughout the range of moderate increasing returns.

d Strong increasing returns

Figure 1 shows that only a single interior equilibrium persists when γ exceeds γ_s. More generally, we have:

Proposition 7 *If $\gamma > \frac{1+\varepsilon}{(1-\beta)\varepsilon} - \alpha \geq \gamma_s$, then there exists a unique interior equilibrium. Furthermore, the equilibrium employment pattern gets flatter as γ rises and converges to the uniform pattern when $\gamma \to \infty$.*

Proof: See Appendix G.

Surprisingly, increasing returns need not foster the concentration of activity. When increasing returns are strong, households are rich. Since land is in fixed supply, this means that the willingness to trade consumption for land must increase. Among other things, this will lead households to live in places where land is less in demand. Since commuting is costly, this peripheral migration of households will be followed by firms who can capitalize lower commute costs into wages, and partly offset the loss in productivity from returns to scale with increased use of commercial land. Both behaviors lead to flatter equilibrium patterns of employment and residence as returns to scale increase.

To our knowledge, Proposition 7 is new to the literature. While it has long been understood that increasing returns to scale could lead to multiple equilibria, the idea that sufficiently high increasing returns leads, once again, to a unique equilibrium is novel. Even more surprising, this equilibrium involves a growing dispersion of production as the degree of increasing returns rises. The intuition behind this result seems straightforward. As γ increases sufficiently, all else equal, wages must rise. This makes households better-off, which in turn increases the utility loss from commuting. As the marginal utility of income falls, households solve this trade-off by reducing the total amount of commuting, which leads to more dispersion.

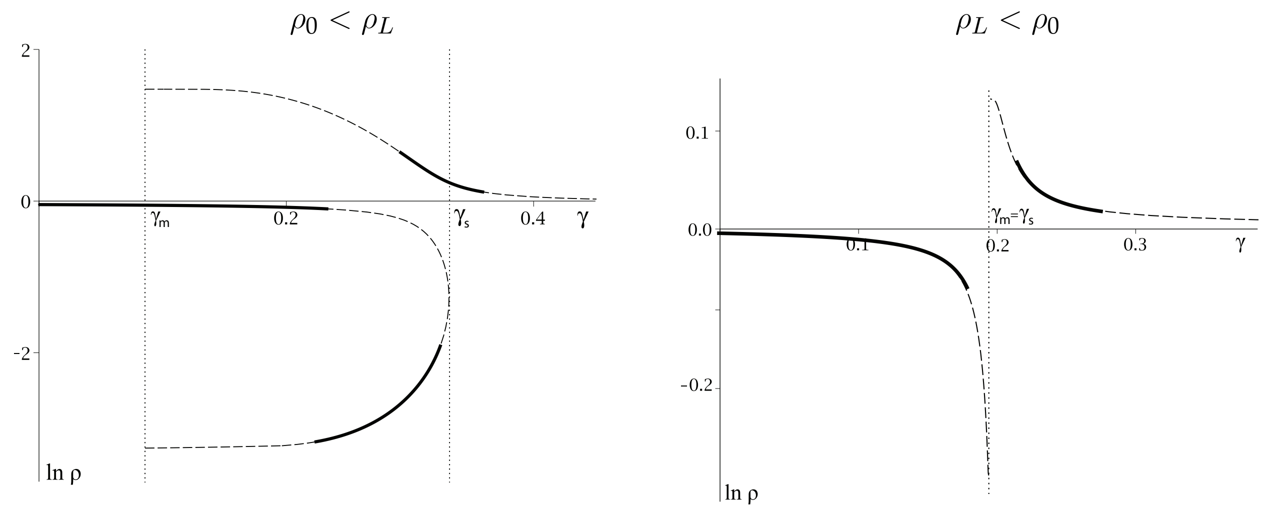
Figure 2 describes the equilibrium correspondence between γ and ρ for numerical examples satisfying $\rho_0 < \rho_L$ in panel (a) and $\rho_L < \rho_0$ in panel (b). In both panels, the x-axis describes γ and the y-axis describes $\ln \rho$. Both figures show all interior equilibria, but not the corner equilibria required by Proposition 3. Both figures anticipate our analysis of stability in Section 7 and indicate stable equilibria with a solid line and unstable equilibria with a dashed line.

Figure 2 illustrates equilibria for the entire range of γ. This is slightly more complete than our propositions. Our propositions establish behavior in a neighborhood above γ_m and for $\gamma > \gamma_s$. They do not eliminate the possibility of a region in (γ_m, γ_s) where qualitatively different equilibrium behavior occurs. As we noted above, while we have not been able to eliminate the possibility of such behavior, we have never seen an example of it in our simulations.

B *Commuting cost and preference dispersion*

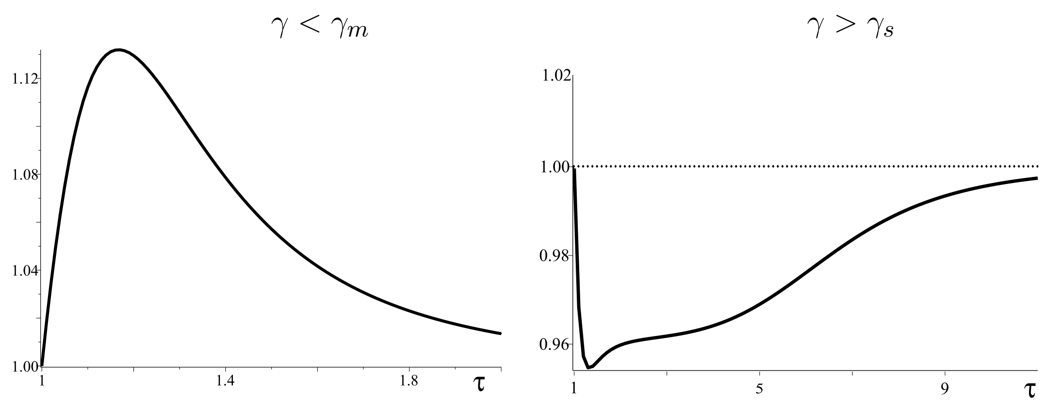
We consider how interior equilibria vary as commuting cost or preference dispersion change, with a special emphasis on the cases of a homogeneous or infinitely heterogeneous population. We begin with figure 3. Both panels of this figure illustrate the equilibrium employment ratio $\ell = L_0/L_1$ as τ varies. The left panel illustrates an example where returns to scale are weak, $\gamma < \gamma_m$, and the right panel an example

Figure 2: Equilibrium correspondence between ρ and γ.



Notes: In both panels the x-axis is γ and the y-axis is $\ln \rho$. The left panel illustrates all interior equilibria as γ varies when $\rho_0 < \rho_L$. The right panel shows the case where $\rho_L < \rho_0$. Solid lines indicate stable equilibria and dashed lines indicate unstable equilibria, where stability is defined as in Section 7.

Figure 3: Employment and residence ratios as commuting cost varies.



Notes: In both panels, the x-axis is τ and the heavy black line shows the equilibrium value of $\ell = L_0/L_1$. In panel (a) $\gamma = 0.07$ and in panel (b) $\gamma = 0.7$. Other parameters used for these calculations are $\alpha = 0.9, \beta = 0.25, \varepsilon = 5$.

where they are strong, $\gamma > \gamma_s$. We omit the intermediate case to avoid the complication of multiple equilibria.

In the case of weak increasing returns, the concentration of employment in the center is increasing in commute costs for low levels of τ and decreasing for high levels of τ. Equilibrium employment is uniform as τ approaches either one or infinity. The peak of the employment ratio locus occurs around $\tau = 1.2$, where commuting results in a 20% utility penalty. For reference, Redding and Turner (2015) report an average round trip commute of about 50 minutes for an average American, this is about 12% of an eight hour work day. While it would be a mistake to take parameter magnitudes too seriously in such a stylized model, this calculation suggests that the complicated comparative statics illustrated in panel (a) may be empirically relevant.

The case of strong returns to scale illustrated in panel (b) is about opposite. For τ to the left of the minimum of the employment ratio curve, the concentration of employment in the periphery increases as commute costs increase. To the right of the peak, this behavior reverses. Like the case of weak increasing returns illustrated in panel (a), as τ approaches one or infinity, the employment pattern becomes flat. Note that the scale of the x-axis in panel (b) is different from that in panel (a) and the peak of the employment ratio curve occurs around $\tau = 2$. The empirically relevant region of panel (b) likely lies to the left of the minimum of the employment ratio curve. In this region, increases in τ decentralize employment.

Because τ and ε so often appear together in the analysis (as ϕ), it is natural to conjecture that the comparative statics for the two parameters are similar. As we show below, this conjecture is correct. With this said, it is more difficult to illustrate the comparative statics of ε because, as inspection of definition 2 shows, changes in ε can cause an economy to change between weak, moderate, and strong returns to scale, and therefore leads to more complicated behavior.

With these examples in place, we turn to a characterization of equilibria as τ approaches one and infinity, and as ε approaches zero or infinity.

Recalling that $\phi = \tau^{-\varepsilon}$, it is easy to see that $\phi \to 1$ when $\tau \to 1$ or $\varepsilon \to 0$, and that $\phi \to 0$ when $\tau \to \infty$ or $\varepsilon \to \infty$. Combining this observation with inspection of equations (21) and (22), we see that the average payoff from the choice of central versus peripheral workplace (or the corresponding payoffs from choice of residential locations) converge toward each other when ϕ goes to either zero or one. This is consistent with the convergence to a flat equilibrium for extreme values of τ that we see in figure 3, and suggests that such convergence may be general.

To show that this is the case, set $\phi = 0$ and $\phi = 1$ in (27) and (28), to obtain:

$$\begin{aligned} f(\rho)|_{\phi=0} &= \rho^{-1-\frac{1}{\beta\varepsilon}}, \qquad f(\rho)|_{\phi=1} = \frac{\rho - 2a\rho^{1+\frac{1}{\beta\varepsilon}} + 2(1+a)}{(1+a)\rho^{1+\frac{1}{\beta\varepsilon}} + 2\rho^{\frac{1}{\beta\varepsilon}} - a}, \\ g(\rho;\gamma)|_{\phi=0} &= \rho^{\frac{1}{a}+\frac{\gamma}{\alpha\varepsilon}}\rho^{\frac{\gamma\varepsilon/\alpha}{1-\gamma\varepsilon/\alpha}\frac{1+\varepsilon}{\varepsilon}}, \qquad g(\rho;\gamma)|_{\phi=1} = \rho^{\frac{b}{1-\gamma\varepsilon/\alpha}}. \end{aligned}$$

Evaluating each of these functions at $\rho = 1$ shows that $f(1)|_{\phi=0} = g(1;\gamma)|_{\phi=0} = 1$ and $f(1)|_{\phi=1} = g(1;\gamma)|_{\phi=1} = 1$. Hence, in both cases $\rho^* = 1$ is an interior equilibrium. As $\varepsilon \to 0$, we have that $\gamma_m \to \infty$. It follows that as $\varepsilon \to 0$ returns to scale must be weak. Therefore, Proposition 5 applies and the interior equilibrium is unique.

When $\varepsilon \to \infty$ in $f(\rho)|_{\phi=0} = g(\rho;\gamma)|_{\phi=0}$, we have $\rho^{-1} = \rho^{\frac{1}{a}-1}$ whose unique interior solution is $\rho^* = 1$. It then follows from (27) that $\omega^* = 1$, which implies $\ell^*(1) = m^*(1) = h^*(1) = n^*(1) = 1$ when $\varepsilon \to \infty$. Using (23) and (24) we have

$L_i^* = M_i^* = 1/3$ for $i = -1,0,1$ and it is easy to show that $s_{ij}^* = 0$ for $i \neq j$ and $s_{ii}^* = 1/3$. Thus, as the population becomes homogeneous, the equilibrium city consists of backyard capitalists who work where they live. In this equilibrium, land is shared between consumption and production according to the same proportion $H_i^*/N_i^* = a = \alpha\beta/(1-\alpha)$ across locations, so that residential land consumption increases with the land share in consumption and decreases with the land share in production. Since this result holds regardless of the value of γ, contrary to general beliefs, increasing returns need not lead to the (partial) agglomeration of production.

Because the domain of weak returns to scale vanishes when $\varepsilon \to \infty$, only the cases of moderate or strong increasing returns are relevant. In the former case, we cannot rule out multiplicity of equilibria. In the latter, we know there is a unique interior equilibrium. Therefore, we may appeal to proposition 7 to say that, when $\varepsilon \to \infty$, the flat pattern is the only interior equilibrium when $\gamma > \frac{1}{1-\beta} - \alpha$. We may thus safely conclude that the interaction between heterogeneity in preferences and increasing returns plays a key role in quantitative models.

The same patterns emerge as the population becomes infinitely heterogeneous, that is, $\varepsilon \to 0$. However, commuting patterns vary between the two types of equilibria. When $\varepsilon \to 0$, the payoffs for each of the nine location pairs become identical. As $\varepsilon \to \infty$, the payoff (22) attached to off-diagonal pairs goes to zero. In the former case, we have symmetric cross-commuting between any location pair, while we have no commuting at all in the latter case.

The following proposition summarizes this discussion.

Proposition 8 *Interior equilibrium employment, residential, housing, and commercial patterns converge to a flat pattern when one of the following conditions holds:*

i. $\tau \to 1$ *or* $\tau \to \infty$ *and* $\gamma < \gamma_m$ *or* $\gamma > \frac{1+\varepsilon}{(1-\beta)\varepsilon} - \alpha$*;*

ii. $\varepsilon \to 0$*;*

iii. $\varepsilon \to \infty$ *and* $\gamma > \frac{1}{1-\beta} - \alpha$*.*

As $\varepsilon \to 0$, preference dispersion increases. As this happens, taste heterogeneity over pairwise choices becomes increasingly important relative to commuting costs and, in the limit, households ignore land price and wage differences, and the distribution of households across pairs is uniform. Likewise, as $\tau \to \infty$ the cost of commuting grows so high that it never makes sense to commute. Since the distribution of types is the same across locations, households must be uniformly distributed across locations. In this case, only autarchy is possible and all households work where they live. In both cases, the distribution of residence and employment is uniform, although the city functions differently in the two cases. When $\varepsilon \to 0$ we have 'urban sprawl', where many people commute and there is no city center. When $\tau \to \infty$ we have a city of backyard capitalists.

As $\varepsilon \to \infty$, preference dispersion disappears and households become homogenous. Simultaneously, the range of weak returns to scale collapses toward $\{0\}$. When returns to scale are constant, there is no longer a centralizing force in the model and the distribution of activity is uniform. The distribution of activity is also uniform when returns to scale are strong. This is less intuitive. When households no longer care about where they live and work, they focus on their own consumption only. In this case, high increasing returns make households rich enough that they prefer to work where they live at a lower wage in order to avoid larger utility losses from commuting. In spite of strong increasing returns to scale, when the households are homogeneous the only interior equilibrium involves a uniform distribution of production activities across locations.

As $\tau \to 1$ something similar occurs. In the absence of increasing returns to scale, there is no reason for firms to concentrate, and so we arrive at a uniform distribution of activity because location does not matter anymore to agents.

Proposition 8 describes what happens as $\varepsilon \to \infty$ and agents become homogeneous. However, it is possible that this limit does not describe what happens when agents are perfectly homogenous because the c.d.f. of the Fréchet distribution is a step function at $z = 0$. Assuming from the outset that households are homogeneous, that is, the utility of a ij-household is given by

$$U_{ij} = \frac{1}{\beta^{\beta}(1-\beta)^{1-\beta}} \frac{H_{ij}^{\beta} C_{ij}^{1-\beta}}{\tau_{ij}},$$

we can show that the equilibrium involves three autarkic locations.[9] This result resembles Starrett's (1978) Spatial Impossibility Theorem. This theorem states that, regardless of the technology (decreasing, constant or increasing returns), any equilibrium (if it exists) in the Arrow-Debreu model of general equilibrium must be such that each location is an autarky. By contrast, if we allow for spatial spillovers across locations like in Osaka and Fujita (1980) and Lucas and Rossi-Hansberg (2002), agglomeration arises, but commuting must be 'inexpensive enough' if it is to occur in equilibrium.

We regard these comparative statics as surprising and important for two reasons. First, in the monocentric city model, decreases in commuting costs lead households to spread out. Our results contradict this intuition. In our heterogenous household model, comparative statics on commuting costs are not monotone. Second, as we describe in our review of the literature, discrete heterogenous agent models, similar to ours, are the basis for a rapidly growing quantitative literature. Often, such quantitative exercises evaluate the effects of counterfactual changes in commuting costs. To the extent that these counterfactual exercises are comparative statics of commuting costs, our results suggest the qualitative features of such counterfactual exercises may change sign in response to changes in incidental parameters.

7. Stability

It is common to appeal to stability as a selection device in the presence of multiple equilibria. This leads to the question of how to define stability. One candidate,

[9] The proof is available from the authors upon request.

particularly relevant for the quantitative literature, is to say that an equilibrium is stable if an iterative process will converge to it. Formally, if equilibria are defined by $f(\rho) = g(\rho)$ then equilibria are fixed points of $h(\rho) = \rho$, for $h(\rho) \equiv f^{-1}(g(\rho))$. It is well known that an iterative process will find a fixed point ρ^* if and only if $|h'(\rho^*)| < 1$. Surprisingly, this notion of stability is not well defined. To understand the problem with this definition, observe that $h(\rho) = \theta\rho + (1-\theta)\rho$ also defines solutions of $f(\rho) = g(\rho)$, and therefore, that fixed points of $\widetilde{h}(\rho) = [(h(\rho) - (1-\theta)\rho)/\theta] = \rho$ are also solutions of $f(\rho) = g(\rho)$. However, the stability properties of this second equation may be different from the original. By choosing θ sufficiently small, we guarantee that $|\widetilde{h}'(\rho^*)| > 1$. Thus, an iterative notion of stability is not invariant to different, equivalent ways of formulating the equilibrium conditions.

A second approach to stability requires that we specify an adjustment process describing how the three state variables for our economy, L_0, M_0 and s_{00}, respond to a perturbation. Stability is then well defined in the resulting dynamic system. This approach is subject to two problems. First, it is likely to be intractable. Second, it must rest on ad hoc descriptions of the adjustment process, and we suspect that the stability of any particular equilibrium is likely to be sensitive to these assumptions.

These difficulties lead us to a more game-theoretic notion of stability. In the spirit of trembling hand perfection, we say that an equilibrium is stable if households want to return to the equilibrium when an arbitrarily small measure of them are displaced. This definition of stability has three advantages. First, like our model, it is static and does not require an explicit description of time. Second, and unlike the other candidate definitions of stability, it has explicit behavioral foundations. Third, as we will see, it is tractable.

Let ij and kl be two arbitrary location pairs; $ij = kl$ (location pairs are equal) when $i = k$ and $j = l$ hold simultaneously and distinct otherwise. We say that an equilibrium is *unstable* if, for some $ij \neq jk$, for any arbitrarily small $\Delta > 0$, there is a subset of individuals of mass Δ who strictly prefer the location pair kl, which differs from their utility-maximizing pair ij, when a perturbation moves them all to kl. In other words, the subset of individuals who have been moved away from ij do not want to move back. Otherwise, the equilibrium is *stable*.

The key issue is to determine the subset of individuals to use to check whether the equilibrium is unstable. In what follows, we assume that this subset is formed by individuals whose types are close to those of an individual indifferent between her equilibrium pair ij and another location pair kl. Lemma 4 establishes that such an individual always exists.

Consider an equilibrium commuting pattern $\mathbf{s}^* \equiv (s^*_{ij})$, which could be interior or corner, and two location pairs, ij and kl, such that $ij \neq kl$ and $s^*_{ij} > 0$. We say that an individual $\nu \in$ [0,1] is *indifferent between ij and kl* if and only if

$$V^*_{ij}(\nu) = V^*_{kl}(\nu) \geq V^*_{od}(\nu), \tag{35}$$

for every location pair od such that $od \neq ij$ and $od \neq kl$. Given this definition, we have:

Lemma 4 *For any two distinct location pairs ij and kl such that $s^*_{ij} > 0$, there exists an individual $\nu \in$ [0,1] with $z_{ij}(\nu) \in S_{ij}$ and $z_{kl}(\nu) > 0$ who is indifferent between ij and kl.*

Proof: See Appendix H.

Definition 3 *Consider an arbitrarily small subset of individuals of measure $\Delta > 0$ who choose ij and have types close to $\mathbf{z}(\nu) \in S_{ij}$ where ν is indifferent between ij and $kl \neq ij$. If this individual is strictly better off when she and her neighboring individuals are relocated from ij to kl, the spatial equilibrium is unstable. Otherwise, the spatial equilibrium is stable.*

The motivation for this definition is as follows. If the relocation of a small group of almost indifferent individuals from ij to kl makes the indifferent agent strictly better off, then, by continuity there is a non-negligible subset of individuals who strictly prefer kl to ij. Hence, these individuals will never switch back to ij. On the contrary, if the indifferent individual never becomes strictly better off for any small subset, no other individual strictly prefers a different location pair. Hence, all the individuals will be willing to switch back to ij.

By relocating a small subset of individuals from ij to kl, the commuting pattern $\mathbf{s}$ becomes different from the equilibrium pattern $\mathbf{s}^*$. Hence, for our definition of stability to make sense, we must be able to compare the equilibrium and off-equilibrium utility levels. For this to be possible, we must determine the conditional equilibrium vectors of wages and land rents $\overline{\mathbf{W}}(\mathbf{s})$ and $\overline{\mathbf{R}}(\mathbf{s})$. We show in Appendices J and K that, for $\alpha > 1/2$, these vectors exist, are unique and continuous in $\mathbf{s}$.

Using the above definition of stability, we are now equipped to study the stability of the equilibria identified in Proposition 9. We start with corner equilibria.

Proposition 9 *The corner equilibria are always unstable.*

Proof: See Appendix I.

This result is easy to understand. Consider the agglomerated corner equilibrium $\mathbf{L}^* = (0,1,0)$. No single individual wants to move to, say, location 1 because her marginal productivity would be zero. This is why $(0,1,0)$ is an equilibrium employment pattern. By contrast, when a small subset of workers happens to be at 1, equation (13) implies the income of those whose tastes are very close to those of the indifferent individual is extremely high. As a consequence, they do not want to move back to location 0.

If we rely on stability to select among multiple equilibria, it means that we can ignore the corner equilibria. This result has the potential to greatly simplify quantitative exercises based on this family of models.

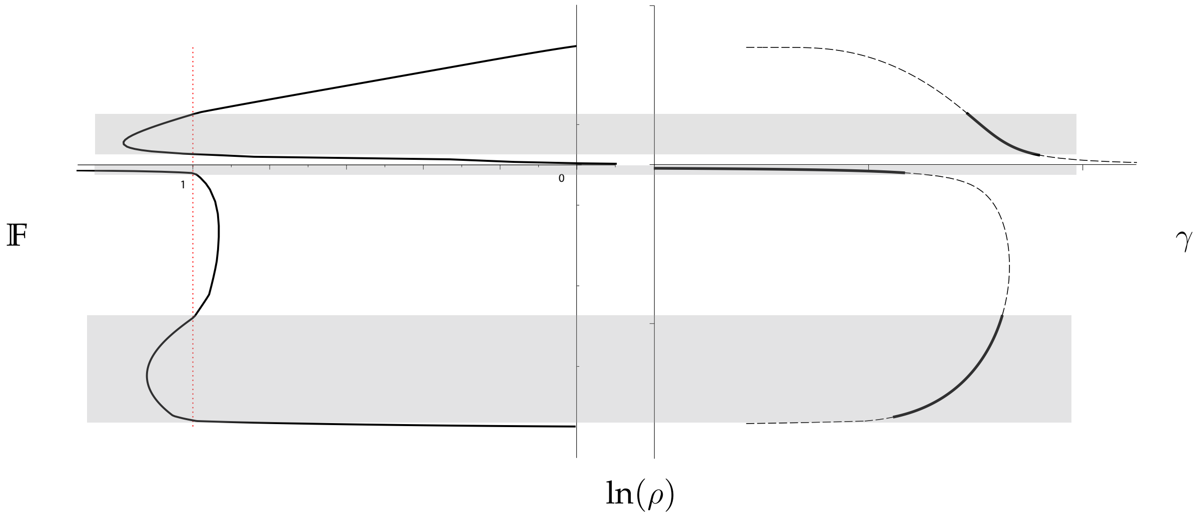
By Proposition 3, interior equilibria always exist. The next proposition provides necessary and sufficient conditions for an interior equilibrium to be stable. Denote by ρ_{CR} be the equilibrium that prevails under constant returns.

Proposition 10 *There exists a function $\mathbb{F}(\rho)$ independent of γ such that an interior equilibrium ρ^* is stable if and only if $\mathbb{F}(\rho^*) > 1$. This function is continuous over (ρ_0, ρ_{CR}) and over (ρ_{CR}, ρ_1), satisfies $\mathbb{F}(\rho_0) = \mathbb{F}(\rho_1) = 0$, and has a vertical asymptote at $\rho = \rho_{CR}$.*

Proof: See Appendix J where we also give the explicit form of function $\mathbb{F}$.

Proposition 10 provides a simple test for checking the stability of any interior equilibria. In fact, it also allows us to draw general conclusions about stability. The left

Figure 4: Graphical representation of Proposition 10.



Notes: The right half of this figure plots the equilibrium correspondence between $\ln(\rho)$ *and* γ *reproduced from figure 2. The x-axis in this half of the figure is γ. As in figure 2, heavy lines indicate stable equilibria and dashed lines indicate unstable equilibria. The left half of this figure uses the same parameter values to plot the function $\mathbb{F}$ (times minus one) from Proposition 10 on the x-axis and* $\ln(\rho)$ *on the y-axis. From Proposition 10, ρ is a stable equilibrium whenever* $\mathbb{F} > 1$. *The* $\mathbb{F} = 1$ *threshold is illustrated by the vertical dashed line in the left panel. Intervals of* $\ln(\rho)$ *where* $\mathbb{F} > 1$ *and equilibrium is stable are indicated by the shaded horizontal bands.*

panel of figure 4 illustrates the function $\mathbb{F}$. In this panel, the value of $\mathbb{F}$ is on the x-axis and $\ln \rho$ on the y-axis (strictly, this is a plot of the correspondence $\mathbb{F}^{-1}$). The critical value $\mathbb{F} = 1$ is indicated by the dotted vertical line. As in figure 2, we transform the y-axis for legibility. From this figure, we can read off the intervals of ρ where the equilibrium is stable by checking whether $\mathbb{F} > 1$. These regions are indicated by the horizontal shaded bands.

In the right panel of figure 4 we repeat the left panel of figure 2. This figure describes the equilibrium correspondence between γ and $\ln \rho$. Since the y-axes on the two panels agree, we can put them next to each other and project the stable intervals onto the equilibrium correspondence between γ and $\ln \rho$ given in the right panel. This done, we can also read off the intervals of γ for which some or all of the possible interior equilibria are stable. As in figure 2, the heavy lines indicate stable equilibria, and dashed lines indicate unstable equilibria.

Although figure 4 is based on a particular numerical example, it illustrates the general stability results implied by Proposition 10. In particular, the fact that $\mathbb{F}$ always has a vertical asymptote at $\gamma = 0$ guarantees that for γ sufficiently close to zero, equilibrium will always be stable. Furthermore, from our discussion of equilibrium in the weak IRS case, we know that ρ^* is close to ρ_0 or ρ_1 if and only if the equilibrium is close enough to a corner equilibrium. Therefore, because $\mathbb{F}(\rho_0) = \mathbb{F}(\rho_1) = 0$, it must be that equilibria close enough to corner equilibria are unstable.

We replicate the evaluation of stability reported in figure 2 for different parameter values. The following additional features of stability, visible in figure 2 recur. First, for γ sufficiently large, the unique strong increasing returns to scale equilibrium always becomes unstable. Second, for $\rho_0 < \rho_L$, the interior 'weak IRS' branch of equilibrium continues to be stable for values of $\gamma > \gamma_m$, but eventually becomes unstable at about the same value of γ at which the highly centralized equilibrium becomes stable. Third, for $\rho_0 < \rho_L$, the centralized moderate γ equilibrium remains stable until γ increases almost to γ_s before becoming unstable. Fourth, for $\rho_0 < \rho_L$, the decentralized branch of moderate and strong returns to scale equilibrium becomes stable for values of γ about the same as those for which the centralized branch becomes unstable, and remains stable until γ is above γ_s.

8. A very short introduction to European urban history

Explaining the evolution of cities and their role in the process of economic growth over the course of history has long been central to urban economics, and the literature has long looked to history as a way to assess the relevance of our theories of cities. In what follows, we examine how an equilibrium city changes as the degree of increasing returns, γ, increases and compare these changes to the history of European cities from the pre-industrial period to the present.

The historical record, surveyed below, indicates that returns to scale have increased over the past several centuries. The evolution of urban density during this time has been non-monotonic. Cities formed and became denser and employment more concentrated, before gradually becoming less dense with employment less concentrated. As figure 2 demonstrates, this basic pattern of increasing, and then decreasing density and concentration of employment can be observed along every one of the possible equilibrium paths predicted by our model as returns to scale increase. Thus, our model rationalizes the joint evolution of returns to scale and these basic features of cities.

To focus on returns to scale, we ignore other important trends, e.g., decreases in transportation cost and in the land intensity of production. We have three reasons for this. First, up until now, the theoretical understanding of the implications of changes in returns to scale has been restricted by the intractability of this problem. This is where our results are the most novel. Second, as we will see, changes in returns to scale alone can explain qualitative features of the history of cities. Finally, as important as other changes have been, increases in productivity are surely more important. By focusing on changes returns to scale we keep attention on the central economic force behind the industrial revolution.

The existence of various types of agglomeration effects in modern cities is well established, even if debate continues about exactly how large they are. We do not have systematic evidence about time series change in the importance of agglomeration effects, but three well-established facts from economic history suggests that returns to scale are increasing. First, during the period around 1500, de Vries (1984) reports 154 European cities with population above 10,000, while Bairoch (1988) reports 89 cities of at least 20,000. At this time, Europe (without Russia) had only between 10 and 12 cities of more than 100,000 inhabitants. By 1800, the count of cities with a population of at least 10,000 and 20,000 increased to 364 and 194, respectively. Similarly, the share of the urban population was low and rose slowly from 10.7 in 1500 to 12.2 percent in 1750. The urbanization rate was still around 12 percent in 1800, but grew rapidly to 19 percent in 1850, 38 percent in 1900, 51 percent in 1950, and 75 percent in 2000 (Bairoch, 1988). Before the industrial revolution, Europe (without Russia) has no city of more than 2 million, but by 1910, four European cities cross this threshold (Berlin, London, Paris, and Vienna). Without a doubt, European cities have become progressively more attractive. While these increases are surely not entirely due to increases in returns to scale in production,[10] it is equally sure that stronger returns to scale are partly responsible.

Second, economic historians have documented many small changes in pre-industrial Europe that contributed to the productivity of cities. For example, Cantoni and Yuchtman (2014) observe the spread of Universities in 14th century Germany and argue that, by spreading knowledge of Roman law, these Universities contributed to larger and more productive market cities. Dittmar (2011) documents the spread of the printing press and finds that it contributed causally to the sizes of cities where it was introduced. Finally, de la Croix *et al.* (2018) argue that apprenticeship may have played a similar role by facilitating the transmission of tacit knowledge. In all, the history of the pre-industrial period points to slow increases in the productivity of cities.

Finally, the nature of production has changed in a way that indicates an increase in returns to scale. For the period prior to the industrial revolution, de Vries (1984) reports on the pervasiveness of proto-industrialization in manufacturing. Under this system, rural households performed manufacturing at home, often of textiles, using materials provided by manufacturers. It is hard to imagine a system of industrial production that more strongly suggests constant or decreasing returns to scale in manufacturing.

Summing up, we can be confident that agglomeration and scale economies are positive in the modern economy and the available evidence strongly suggests that they were small and increasing in the pre-industrial period.[11] Taken together these two observations indicate a trend upwards.

We now turn to a stylized description of the evolution of cities from the pre-industrial period to the present. de Vries (1984) describes cities in pre-industrial Europe as being organized as much for protection as production. Much of the population was employed in agriculture, either within city walls or without, manufacturing was at least as rural as urban, and the preponderance of urban residents were abjectly poor. While the absolute poverty of an average urban resident is clear, there is suggestive evidence from the US for the relative poverty of urban residents as well. Costa (1984) reports in a survey of US soldiers in the US civil war, early in the US industrial revolution, and finds that soldiers with urban backgrounds were shorter than

[10]See Nunn and Qian (2011), for example.

[11]We note the Cesaretti *et al.* (2020) study finding that tax collection per capita increases with city size in England between 1450 and 1670. While this result seems relevant, it is unclear whether it reflects increases in the efficiency or effort of tax collectors, or increases in productivity of city residents, or simply the first nature advantages of larger pre-industrial cities.

those from rural backgrounds.

Clark (1951) is the first effort to provide systematic evidence about patterns of urban density. In this landmark study, Clark collects historical census data for twenty US and European cities. While the precise time period he considers varies from city to city, his data often begins early in the industrial revolution and continues until the early 20^{th} century. His findings are unequivocal: center city density falls and density gradients flatten over time.

Turning to the late 20^{th} and early 21^{st} centuries, Baum-Snow (2007) documents the decline in US central city population from 1950 to 1990, a phenomena earlier described by Meyer *et al.* (1965).[12] Glaeser and Kahn (2004) find that in late 20^{th} century US, employment follows decentralizing population to the suburbs and that peripheral urban residents of US cities tended to have much longer commutes than did central residents. Related to this, Garreau (1992) documents the rise of 'edge cities' in late 20^{th} century US, while McMillen and MacDonald (1998) provide econometric support for this idea using detailed data for Chicago. More recently, Couture and Handbury (2017) and Couture *et al.* (2019) document a resurgence of central cities in the US.

Inspection of figure 2 shows that this basic pattern occurs along any possible equilibrium path as returns to scale increases. In particular, along all equilibrium paths: (1) land rents are nearly the same in the center and the periphery for γ small; (2) as γ increases central land rents enter a region of rapid or discontinuous increase corresponding the industrial revolution; (3) as γ increases past the region of rapid or discontinuous increase, as in the late 20^{th} century, land rents in the center and periphery gradually equalize as employment becomes more suburban. Note that in this region of γ multiple equilibria are possible, and so this is a region where we would expect to see more complex cities, again, like what emerged in the late 20^{th} century.

Summing up, while both our model and our description of history are stylized, important features of the history of cities can be rationalized by our model and a trend upward in the strength of returns to scale.

9. Extensions and discussion

A *Exogenous first-nature amenities and productivity*

The QSM literature typically allows locations to differ in their exogenous productivity and amenities. We have so far considered a geography of identical locations in order to focus attention on the way that commuting costs, returns to scale and preference heterogeneity interact to determine the equilibrium. This leaves open the possibility that economic fundamentals interact with first nature advantages to give rise to qualitatively different behavior than is possible in a homogenous geography. We here introduce such first nature advantages into our model in order to investigate this possibility.

We change the model developed in Section 3 in exactly two ways. First, we introduce a multiplicative location specific residential amenity O_i and work amenity D_j

[12] Angel-Garcia et al. (2015) document the corresponding phenomena in European cities.

for each location. This leads to the indirect utility function

$$V_{ij}(\nu) = z_{ij}(\nu) O_i D_j \frac{W_j}{\tau_{ij} R_i^{\beta}}. \tag{36}$$

This coincides with equation (2) in the special case when $O_i = D_j = 1$. Second, we generalize the expression for location specific productivity to allow for productivity to vary with exogenous first nature differences, T_j, so that

$$A_j = T_j L_j^{\gamma}. \tag{37}$$

This coincides with equation (12) when T_j is one.

Consistent with the symmetry assumption imposed throughout, we require that O, D and T take the same value in each of the peripheral locations, and define

$$o \equiv \left(\frac{O_0}{O_1}\right)^{\varepsilon}, \qquad d \equiv \left(\frac{D_0}{D_1}\right)^{\varepsilon} \qquad \text{and} \quad t \equiv \frac{T_0}{T_1}.$$

In Proposition 11, proved in Appendix K, we derive the functions describing how residence, employment, housing and commercial land use vary with wages and land rent. We find that these functions are qualitatively similar to those described in Proposition 1. We also show that equilibrium in the economy with first nature amenities and productivity is determined by similar functions to those that determine equilibrium in an economy with a homogenous landscape. In particular, proposition 11, presented below, generalizes Proposition 2 to an economy with first nature advantages.

Proposition 11 *Assume $\gamma \neq \alpha/\varepsilon$. Then, a pair (ρ^*,ω^*) is an interior equilibrium if and only if it solves the following two equations:*

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} d \;=\; \widetilde{f}(\rho o) \equiv \frac{\phi \rho o - 2a\phi o^{-\frac{1}{\beta\varepsilon}} (\rho o)^{1+\frac{1}{\beta\varepsilon}} + (1+a)(1+\phi^2)}{(1+a) o^{-\frac{1}{\beta\varepsilon}} (\rho o)^{1+\frac{1}{\beta\varepsilon}} + 2\phi o^{-\frac{1}{\beta\varepsilon}} (\rho o)^{\frac{1}{\beta\varepsilon}} - a\phi}, \tag{38}$$

$$\omega^{\frac{1+\varepsilon}{\varepsilon}} d \;=\; \lambda g(\rho o;\gamma) = \lambda (\rho o)^{\frac{b}{1-\gamma\varepsilon/\alpha}} \left(\frac{\rho o + 2\phi}{\phi \rho o + 1 + \phi^2}\right)^{\frac{\gamma\varepsilon/\alpha}{1-\gamma\varepsilon/\alpha}\frac{1+\varepsilon}{\varepsilon}}, \tag{39}$$

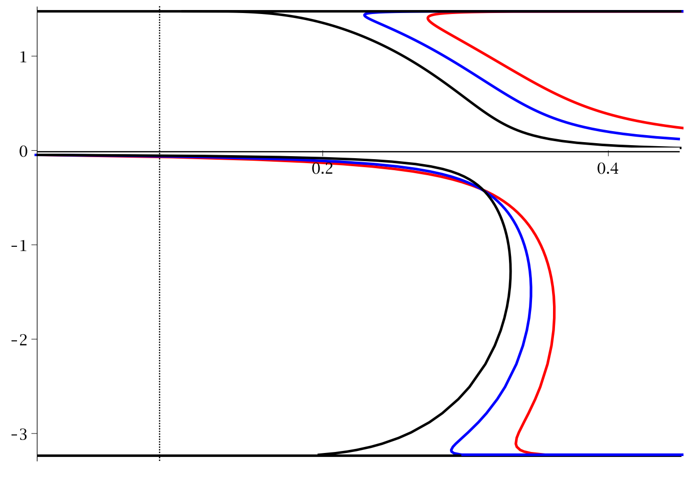
where $\lambda > 0$ is a scalar defined by

$$\lambda \equiv d^{\frac{\gamma/\alpha}{1-\gamma\varepsilon/\alpha}} o^{-\frac{b}{1-\gamma\varepsilon/\alpha}} t^{\frac{1+\varepsilon}{\varepsilon}\frac{\varepsilon/\alpha}{1-\gamma\varepsilon/\alpha}}.$$

Proof: See Appendix K.

In general the equilibrium condition is given by $\widetilde{f}(\rho o) = \lambda g(\rho o,\gamma) = \omega^{\frac{1+\varepsilon}{\varepsilon}} d$, and one can show $\widetilde{f}(\cdot)$, although not identical to $f(\cdot)$, has all the qualitative properties of $f(\cdot)$: it is positive over an open interval bounded away from both 0 and ∞, and decreases from ∞ to 0 over that interval. It follows from this that the threshold at which the slope of g changes sign, $\gamma_m \equiv \alpha/\varepsilon$, is not affected by amenities. This, together with qualitatively similar behavior for the functions f and g means that the basic intuition behind

Figure 5: Equilibrium correspondence between ρ and γ with productivity spillovers.



Notes: Equilibrium correspondences between $\ln(\rho)$ *and* γ*. Black line: reproduced from figure 2a, but also showing corner equilibria. Blue line: like figure 2a, but with* $\delta = 0.02$*. Red line: like figure 2a, but with* $\delta = 0.04$*.*

Proposition 4 and 5 is unchanged by the addition of first nature advantages. In particular, we expect a unique interior equilibrium in this region of the parameter space.

Second, in the absence of residential amenities ($o = 1$), we have $\widetilde{f}(\cdot) = f(\cdot)$ so that the equilibrium condition has the form $f(\rho) = \lambda g(\rho,\gamma)$. Thus, the effect of workplace amenities and first nature productivity is to rescale g. This suggests again that the intuition behind Propositions 6 and 7 should be unaffected.

Finally, we note that the impacts of both workplace amenity d and the production amenity t on ρ^* are isomorphic, since they are both fully captured by the coefficient λ in the equation $\widetilde{f}(\rho o) = \lambda g(\rho o,\gamma)$. However, the impact of the two amenities on ω^* is different, since d enters separately the equilibrium equation $\widetilde{f}(\rho o) = \lambda g(\rho o,\gamma) = \omega^{\frac{1+\varepsilon}{\varepsilon}} d$.

B *Spillovers*

It is common in the urban economics and QSM literatures to model local productivity as a function of the amount of economic activity in a particular location and a distance discounted sum of nearby economic activity. The mathematical representation of this idea is usually called a potential function and was introduced by Fujita and Ogawa (1982), although the idea of productivity enhancing local spillovers is much older. In the model presented in Section 3 we restrict increasing returns to scale to operate only within a location. We here investigate the implications of the more general formulation.

To accomplish this, we make a single change to the model described in Section 3. In place of the definition of local productivity given by equation (12) we assume that productivity at $j = 0,1$ is given by:

$$A_0 = (L_0 + 2\delta L_1)^\gamma, \qquad A_1 = (\delta L_0 + (1+\delta^2)L_1)^\gamma. \tag{40}$$

Here, $\delta \in [0,1)$ describes the rate at which spillovers decay with distance, and $\gamma \geq 0$ continues to describe the rate at which productivity increases with employment in a location. Note that equation (40) converges to equation (12) as $\delta \to 0$.

Recalling that the function f given in Proposition 2 does not depend on γ, it should be unsurprising that this function remains unchanged from equation (27) under the more general technology. The function g, however, does not remain the same. In particular, following logic similar to that used to derive equation (28), one can derive the analogous equation when spillovers operate:

$$\omega = \rho^{\frac{1}{\alpha}} \left(\frac{\omega(\rho + 2\phi) + 2\delta(\phi\rho + 1 + \phi^2)}{(1+\delta^2)(\phi\rho + 1 + \phi^2) + \delta\omega(\rho + 2\phi)} \right)^{\frac{\gamma\varepsilon}{\alpha}}. \tag{41}$$

Note that equation (41) is now transcendental. However, when $\delta = 0$, this expression becomes (K.27) given in Appendix K for our baseline case without spillovers.

We have focussed our efforts on numerical solutions to equations (41) and (27) for small values of δ. This is the empirically relevant range for δ and provides a basis for thinking about whether equilibrium responds continuously to the introduction of small cross location productivity spillovers.

Figure 5 illustrates typical results. This figure is based on the same parameters as figure 2a ($\alpha = 0.6$; $\beta = 0.3$; $\varepsilon = 7$; $\tau = 1.1$). The heavy black line reproduces equilibrium correspondence between ρ and γ from figure 2a, where spillovers are zero. The blue and red lines are identical, but assumes $\delta = 0.02$ and 0.04, respectively. Figure 5 suggests two conclusions. First, that the corner equilibria are not robust to the inclusion of a small spillover. Rather, with small spillovers, these equilibria become 'near corner' interior equilibria. Second, ρ^* at a corner equilibrium responds continuously to the introduction of a small spatial productivity spillover.

Summing up, numerical analysis of the economy with a small amount of spatial productivity spillovers suggests that equilibrium responds continuously to the introduction of such spillovers. In other words, we should not expect qualitatively different equilibrium behavior to arise as a consequence of low levels of productivity spillovers, particularly at the interior equilibria.

C *Geography*

It seems natural to wonder about the extent to which our results are a reflection of restricting households to choose from a discrete set, and from an especially simple discrete geography in particular.

Any quantitative exercise must begin by delimiting the area to be studied. This has two important implications. First, our analysis has been organized around understanding the interplay of three main economic forces, returns to scale, commuting costs, and the average preference for central work and residence. Stipulating to the generality of the preferences for central work and residence, and noting that commuting costs and returns to scale also (usually) operate in the more complicated models that form the basis of quantitative exercises, we should expect that intuition derived from our simple example will often extend to these more complicated settings. Second, the fact that delimiting boundaries and assuming a QSM type discrete choice framework creates preferences for central work and residence means that a researcher's decisions about where to put study area boundaries partly determines the first nature advantages of interior locations by affecting their relative centrality advantage.

In Proposition 3 we see that every location must always be occupied in equilibrium. This result appears to extend to arbitrary geographies, and is one of the more noteworthy differences between the quantitative spatial models and the classical urban economics literature. In the classical literature, cities almost always have an edge beyond which agents will not commute in equilibrium. As important as is the discrete geography to the behavior of our model and its more complex quantitative cousins, we here note that the discrete geography does not appear to responsible for this difference between the two frameworks. There are three reasons for this claim. First, commute costs are typically additive in the classical framework. This means that, if there is a cost of commuting, there is some distance beyond which commuting is impossible because it lies outside the budget set. In our framework, as is typical in quantitative models, commuting costs are multiplicative and so this hard constraint on commuting costs does not occur. Second, in typical quantitative models, the support of individual tastes for workplace-residence pairs is unbounded. That all locations are occupied may hold true in the case of a continuous geography as discrete choice models, such as the logit, may be extended to a continuous choice set (Ben-Akiva *et al.*, 1985). Hence, a discrete geography alone is not necessary to assure an equilibrium where all locations are occupied. Finally, inspection of the indirect utility function on which our analysis is based, equation (2), shows that this function is linear in income and hyperbolic in land prices. Thus, agents can accept an infinitesimal wage in return for zero land rent and still achieve infinite utility. This feature of preferences, also common to quantitative models, must also play a role in assuring that all locations are occupied.

There is no simple way to define what a "realistic" geography should be. Furthermore, the most common alternative is to consider a space without edges, either a circle or an unbounded line or plane. There has been a large literature devoted to the formation of cities on such spaces. One of the hallmarks of this literature is its mathematical sophistication. As a consequence, these models are often too abstract to suggest themselves as a foundation for empirical or quantitative research. How to model space remains an issue in search of a good theory.

10. Conclusion

Understanding how people arrange themselves when they are free to choose work and residence locations, when commuting is costly, and when increasing returns to scale affect production, is one of the defining problems of urban economics. We address this problem by combining the discrete choice models employed by the recent quantitative spatial models literature, with the stylized geographies of classical urban economics. This permits a complete description of equilibria, throughout the parameter space.

Equilibrium behavior is surprising and interesting for a number of reasons. First, we find that corner equilibria exist in much of the parameter space. In these equilibria, employment does not occur in all locations. Given that the support of individual tastes for workplace-residence pairs is unbounded, this result is unexpected. Second, comparative statics as returns to scale increase contradict the conventional wisdom: increasing returns to scale in production can cause dispersion as well as agglomeration. Third, comparative statics on commuting costs also contradict conventional wisdom. As in the monocentric model, reductions in commuting costs can lead to dispersed economic activity, but they can also lead to greater concentration.

By combining classical urban economics and the QSM toolbox, we have also learned about how the two frameworks differ. Introducing heterogeneous agents with commuting costs is not merely a mathematical trick for solving a difficult choice problem. It is a fundamental change from classical urban economics. Introducing agents with heterogenous preferences over work-residence location pairs results in an average preference for central work and residence. These average preferences lead to centralized employment and residence even in the absence of increasing returns to scale. Neither preference is present in the older literature.

Our results also seem to have implications for quantitative exercises. We provide a basis for thinking about the extent to which quantitative comparative statics primarily reflect features of a particular data set, or if they are a direct consequence of the interaction of commute costs, returns to scale, and preference heterogeneity in the model. For example, in the absence of returns to scale our model requires lower central than peripheral wages because wages capitalize the average preference for central employment. To match the modern day empirical regularity of higher central than peripheral wages, we require that returns to scale be high enough to more than offset this capitalization. It follows that the returns to scale parameter in this model does not correspond to the elasticity we obtain from regressing wages on density, a common empirical measure of scale economies.

The existence of corner equilibria whenever there are increasing returns to scale seems likely to generalize to the richer models used for quantitative exercises. To the extent that calibration and estimation techniques rely on the uniqueness of equilibria, this seems to be important. While our notion of stability can exclude these equilibria from consideration, it is less clear whether these equilibria will be attractive to the particular algorithm used to find equilibrium in a quantitative exercise. This concern seems especially relevant given that the results of an iterative solution algorithm need not be robust to different ways of stating equilibrium conditions. Furthermore, it might be worth recalling that the uniqueness of a competitive equilibrium in general equilibrium models is more often the rule than the exception (Arrow and Hahn, 1971). This may explain why very specific assumptions are made in the QSM literature. Yet, we have seen that such assumptions may not be sufficient to get rid of the riddle of multiple stable equilibria.

Third, a back of the envelope calculation suggests than quantitative exercises may well be within estimation error of the weak-moderate returns to scale threshold. This is the region of the parameter space where the model's behavior is most complicated. Inspection of figure 2 indicates that (depending on which regime the economy is in) that catastrophic discontinuities, instability, and stable continuity are all consistent with equilibrium around this threshold. This suggests that at least a rudimentary exploration of what happens when the economy crosses from the weak to moderate increasing returns to scale region is of interest.

Fourth, one of the more common uses of quantitative models is to evaluate comparative statics for commuting costs. Our results make clear that these comparative statics are complicated and may be sensitive to changes in the values of structural parameters. Our simple model does not provide a basis for thinking about how sensitive richer empirical models may be to this problem, but it does suggest that the possibility should at least be considered. This suggests the importance of robustness checks in which the values of these parameters are permitted to vary.

Finally, we show that the qualitative behavior of our model as returns to scale increase (along one of the possible equilibrium paths), can reproduce many of the qualitative features observed over the last 500 years of urban history in the Western world. Urbanization and increasing productivity are surely two of the most important economic phenomena in history and they appear to have been closely linked. That a simple economic geography model relating the two can reproduce basic features that we observe in the history of urbanization is striking, but it should probably not be surprising given that our model describes central issues in urban history.

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Appendix

A. Proof of Lemma 1(iv)

Step 1. We first show that

$$\psi(x,y) \equiv \frac{(1+a)xy + 2a\phi x + 2\phi y}{\phi x + a\phi y + (1+a)(1+\phi^2)} \tag{A.1}$$

is increasing in both x and y for all $(x,y) \in R^2_{++}$. We have

$$\frac{\partial \psi(x,y)}{\partial x} = \frac{(1+a)a\phi y^2 + \left[(1+a)^2(1+\phi^2) + 2a^2\phi^2 - 2\phi^2\right] y + 2a\phi(1+a)(1+\phi^2)}{\left[\phi x + a\phi y + (1+a)(1+\phi^2)\right]^2} > 0,$$

$$\frac{\partial \psi(x,y)}{\partial y} = \frac{(1+a)a\phi x^2 + \left[(1+a)^2(1+\phi^2) - 2a^2\phi^2 + 2\phi^2\right] x + 2a\phi(1+a)(1+\phi^2)}{\left[\phi x + a\phi y + (1+a)(1+\phi^2)\right]^2} > 0,$$

because, as $0 < \phi < 1$,

$$(1+a)^2(1+\phi^2) + 2a^2\phi^2 - 2\phi^2 > 1 + \phi^2 - 2\phi^2 = 1 - \phi^2 > 0,$$

$$(1+a)^2(1+\phi^2) - 2a^2\phi^2 + 2\phi^2 > a^2(1+\phi^2) - 2a^2\phi^2 = a^2(1-\phi^2) > 0.$$

Step 2. It is readily verified that $f(\rho)$ satisfies the following identity:

$$\rho = \left[\frac{(1+a)\rho f(\rho) + 2a\phi\rho + 2\phi f(\rho)}{\phi\rho + a\phi f(\rho) + (1+\phi^2)(1+a)}\right]^{-\beta\varepsilon} = \left[\psi(\rho, f(\rho))\right]^{-\beta\varepsilon},$$

where ψ is defined by (A.1). When ρ increases, the RHS of this expression also increases, which means that $\psi(\rho, f(\rho))$ decreases with ρ. Since ψ increases with ρ and $f(\rho)$, this is possible only if $f(\rho)$ is decreasing. Q.E.D.

B. Proof of Lemma 3

It follows from the proof of Lemma 1 that ρ_0 is the unique solution of

$$D(\rho) \equiv (1+a)\rho^{\frac{1+\beta\varepsilon}{\beta\varepsilon}} + 2\phi\rho^{\frac{1}{\beta\varepsilon}} - a\phi = 0. \tag{B.1}$$

The expressions (30) and (B.1) imply that ρ_L and ρ_0 are functions of a. We next show that ρ_0 and ρ_L vary with a as follows,

$$\lim_{a\to 0} \rho_L = 1, \quad \frac{d\rho_L}{da} < 0, \quad \lim_{a\to\infty} \rho_L = 1 - \phi,$$

$$\lim_{a\to 0} \rho_0 = 0, \quad \frac{d\rho_0}{da} > 0, \quad \lim_{a\to\infty} \rho_0 = \phi^{\frac{\beta\varepsilon}{1+\beta\varepsilon}}.$$

We can show that ρ_0 (resp., ρ_L) increases (resp., decreases) in a by applying the implicit function theorem to $D(\rho) = 0$ (resp., (30)). Observe further that, when $a \to \infty$

(resp., $a \to 0$), dividing $D(\rho) = 0$ by a and taking the limit yields $\rho_0 = \phi^{\beta\varepsilon/(1+\beta\varepsilon)}$ (resp., $\rho_0 = 1$). Last, when $a \to \infty$ (resp., $a \to 0$), taking (B.1) at the power a and the limit yields $\rho_L = 1 - \phi$ (resp., $\rho_L = 1$).

To determine where ρ_0 and ρ_L intersect, we compare $\lim_{a\to\infty} \rho_L$ and $\lim_{a\to\infty} \rho_0$ by considering the equation

$$\phi^{\beta\varepsilon/(1+\beta\varepsilon)} + \phi = 1. \tag{B.2}$$

Differentiating the LHS of (B.2) with respect to ϕ shows that it increases from 0 to 2 when ϕ increases from 0 to 1. The intermediate value theorem then implies that, for any given $\beta\varepsilon$, the equation (B.2) has a unique solution $\overline{\phi}(\beta\varepsilon) \in (0,1)$, which increases with $\beta\varepsilon$.

The inequality $\rho_0 \leq \rho_L$ holds if $\phi^{\beta\varepsilon/(1+\beta\varepsilon)} \leq 1 - \phi$, which amounts to $\phi \leq \overline{\phi}(\beta\varepsilon)$. If $\overline{\phi} < \phi \leq 1$, then there exists a unique value $\overline{a} > 0$ that solves the condition $\rho_L(a) = \rho_0(a)$. Consequently, if $a < \overline{a}$, then $\rho_0 \leq \rho_L$. If $a \geq \overline{a}$, then $\rho_0 > \rho_L$. Summing up, $\rho_0 \leq \rho_L$ if $\phi \leq \overline{\phi}$ or $a \leq \overline{a}$, and $\rho_0 > \rho_L$ when both conditions fail. Q.E.D.

C. Proof of Proposition 3

Only two employment patterns may be symmetric corner equilibria: $(0,1,0)$ and $(1/2,0,1/2)$.

Case 1: $\mathbf{L}^* = (1/2,0,1/2)$. The demand for commercial land at $j = 0$ is zero ($N_0^* = 0$). Hence, the land rent equilibrium condition (11) at $i = 0$ and $i = 1$ amount to

$$R_0 = \left[\frac{\beta\kappa}{H_0} 2\phi W_1^{1+\varepsilon}\right]^{\frac{1}{1+\beta\varepsilon}} > 0, \tag{C.1}$$

$$R_1 = \left[\frac{\beta\kappa}{H_1} (1+\phi^2) W_1^{1+\varepsilon}\right]^{\frac{1}{1+\beta\varepsilon}} > 0. \tag{C.2}$$

Multiplying both sides of the land balance conditions (C.1) and (C.2) by, respectively, R_0 and R_1, we get:

$$R_0 H_0 = R_0, \tag{C.3}$$

$$R_1 H_1 + R_1 N_1 = R_1 H_1 + \frac{1-\alpha}{\alpha} W_1 L_1 = R_1. \tag{C.4}$$

Dividing (C.3) over (C.4), we obtain:

$$r = \frac{R_0}{R_1} = \frac{R_0}{R_1 H_1 + \frac{1-\alpha}{\alpha} W_1 L_1}. \tag{C.5}$$

It then follows from (C.1)-(C.2) that

$$R_0 = \beta\kappa 2\phi W_1^{1+\varepsilon} R_0^{-\beta\varepsilon}, \tag{C.6}$$

$$R_1 H_1 = \beta\kappa (1+\phi^2) W_1^{1+\varepsilon} R_1^{-\beta\varepsilon}. \tag{C.7}$$

The labor market balance condition at $j = 1$ is given by

$$L_1 = s_{11} + s_{01} + s_{-11} = \kappa W_1^{\varepsilon} \left[(1 + \phi^2) R_1^{-\beta\varepsilon} + \phi R_0^{-\beta\varepsilon} \right],$$

so that

$$W_1 L_1 = \kappa \left[\phi W_1^{1+\varepsilon} R_0^{-\beta\varepsilon} + (1 + \phi^2) W_1^{1+\varepsilon} R_1^{-\beta\varepsilon} \right]. \tag{C.8}$$

Plugging (C.6)-(C.7) and (C.8) into (C.5), we get after simplifications:

$$r = \frac{2a\phi r^{-\beta\varepsilon}}{\phi r^{-\beta\varepsilon} + (1 + a)(1 + \phi^2)},$$

or, equivalently,

$$\rho = \left[\frac{2a\phi\rho}{\phi\rho + (1 + a)(1 + \phi^2)} \right]^{-\beta\varepsilon}$$

whose unique solution is $\rho^* = \rho_1$. In other words, employment is evenly concentrated in the two peripheries and the central location is specialized in residential activities.

Case 2: $\mathbf{L}^* = (0,1,0)$. Following the same argument as in Case 1, we end up with the equation:

$$\rho = \left[\frac{(1 + a)\rho + 2\phi}{a\phi} \right]^{-\beta\varepsilon}$$

whose unique solution is given by $\rho^* = \rho_0$. Therefore, $\rho^* = \rho_0$ is an equilibrium if and only if $L_0^* = 1$. Q.E.D.

D. Proof of Proposition 4

Assume $\gamma = 0$. Since $0 < \rho^* < 1$, we may assume throughout that $\rho \in (0,1)$. Note that the equilibrium employment pattern is bell-shaped if and only if $\ell^* > 1$, while (34) is equivalent to $b > 1$.

Some tedious calculations show that the equilibrium condition $f(\rho) = g(\rho; 0)$ may be rewritten as follows:

$$\frac{\left(\frac{a}{1+a}\rho^{-b} + \frac{1}{1+a}\rho^{-1} \right)^{-1} + 2\phi}{\phi \left(\frac{a}{1+a}\rho^{b} + \frac{1}{1+a}\rho \right) + 1 + \phi^2} \left(\frac{a}{1 + a}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1 + a}\rho^{b+\frac{1}{\beta\varepsilon}} \right) = 1. \tag{D.1}$$

Since $1/x$ is convex, for every $\rho < 1$ Jensen's inequality implies

$$\left(\frac{a}{1 + a}\rho^{-b} + \frac{1}{1 + a}\rho^{-1} \right)^{-1} < \frac{a}{1 + a}\rho^{b} + \frac{1}{1 + a}\rho < \rho. \tag{D.2}$$

Plugging (D.2) into (D.1) leads to

$$1 < \frac{\frac{a}{1+a}\rho^{b} + \frac{1}{1+a}\rho + 2\phi}{\phi \left(\frac{a}{1+a}\rho^{b} + \frac{1}{1+a}\rho \right) + 1 + \phi^2} \left(\frac{a}{1 + a}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1 + a}\rho^{b+\frac{1}{\beta\varepsilon}} \right).$$

Using $b > 1$ yields

$$\frac{a}{1+a}\rho^b + \frac{1}{1+a}\rho < \frac{a}{1+a}\rho + \frac{1}{1+a}\rho = \rho.$$

Since the function $\frac{x+2\phi}{\phi x+1+\phi^2}$ is increasing for all $x \geq 0$, we obtain

$$1 < \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2}\left(\frac{a}{1+a}\rho^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+a}\rho^{b+\frac{1}{\beta\varepsilon}}\right). \tag{D.3}$$

As (34) implies

$$\frac{1}{a} < 1 + \frac{1}{\beta\varepsilon} < b + \frac{1}{\beta\varepsilon},$$

while $\rho^* < 1$, we have

$$\frac{a}{1+a}\left(\rho^*\right)^{1+\frac{1}{\beta\varepsilon}} + \frac{1}{1+a}\left(\rho^*\right)^{b+\frac{1}{\beta\varepsilon}} < \left(\rho^*\right)^{\frac{1}{a}}.$$

Replacing the bracketed term in (D.3), we obtain the inequality:

$$1 < \left(\rho^*\right)^{\frac{1}{a}}\frac{\rho^* + 2\phi}{\phi\rho^* + 1 + \phi^2},$$

which is equivalent to $\rho^* > \rho_L$, hence $\ell^* > 1$ (see (32)). Q.E.D.

E. Proof of Proposition 5

Assume $0 < \gamma < \alpha/\varepsilon$. Since $\rho^* > \rho_L$, we have

$$f(\rho_L) > f(\rho^*) = g(\rho^*;\gamma) > g(\rho_L) \tag{E.1}$$

because f is decreasing by Lemma 1 and g is increasing in ρ by Lemma 2. As shown by (31), $g(\rho_L)$ is independent of γ. Combining this with (E.1), we obtain $f(\rho_L) - g(\rho_L;\gamma) > 0$. Since $f(\rho^*) - g(\rho^*;\gamma) = 0$ while $f - g$ is decreasing by Lemmas 1 and 2, we have $\rho_L < \rho^*$ for all $\gamma < \alpha/\varepsilon$, which amounts to $\ell^* > 1$.

We now study the impact of γ on (i) ρ^*, (ii) $\omega^*$and (iii) ℓ^*.

(i) Since $\partial g(\rho;\gamma)/\partial\gamma > 0$, applying the implicit function theorem to (29) leads to

$$\frac{d\rho^*}{d\gamma} = \left.\frac{\partial g(\rho;\gamma)/\partial\gamma}{\partial f'(\rho)/\partial\rho - \partial g(\rho;\gamma)/\partial\rho}\right|_{\rho=\rho^*} < 0,$$

where the numerator is positive because $\rho^* > \rho_L$ while the denominator is negative because $f(\rho)$ is decreasing and $g(\rho;\gamma)$ is increasing in ρ.

(ii) Differentiating (27) with respect to γ, we obtain:

$$\frac{1+\varepsilon}{\varepsilon}\omega^{\frac{1}{\varepsilon}}\frac{d\omega^*}{d\gamma} = \frac{df}{d\rho}\frac{d\rho^*}{d\gamma} > 0.$$

(iii) Observe that, combining (32) with (31), the equilibrium condition (29) can be restated as

$$\ell^{\frac{1+\varepsilon}{\varepsilon}} = \left(\frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2}\right)^{\frac{1+\varepsilon}{\varepsilon}} f(\rho). \tag{E.2}$$

Since $f(\rho)$ can be decomposed as

$$f(\rho) = \rho^{-\frac{1}{\beta\varepsilon}} \cdot \frac{\phi\rho + 1 + \phi^2}{\rho + 2\phi} \cdot \frac{1 + a\frac{1+\phi^2-2\phi\rho^{1+\frac{1}{\beta\varepsilon}}}{\phi\rho+1+\phi^2}}{1 + a\frac{\rho-\phi\rho^{-\frac{1}{\beta\varepsilon}}}{\rho+2\phi}}, \tag{E.3}$$

Plugging (E.3) into (E.2) leads to

$$\ell^{\frac{1+\varepsilon}{\varepsilon}} = \left(\frac{\rho^{-\frac{1-\beta}{\beta}} + 2\phi\rho^{-\frac{1}{\beta}}}{\phi\rho + 1 + \phi^2}\right)^{\frac{1}{\varepsilon}} \cdot \frac{1 + a\frac{1+\phi^2-2\phi\rho^{1+\frac{1}{\beta\varepsilon}}}{\phi\rho+1+\phi^2}}{1 + a\frac{\rho-\phi\rho^{-\frac{1}{\beta\varepsilon}}}{\rho+2\phi}}.$$

The first term in the RHS clearly decreases in ρ. Since the numerator (resp., denominator) of the second term is decreasing (resp., increasing), the second term also decreases in ρ. Hence, the RHS is decreasing in γ. Combining this with $d\rho^*/d\gamma < 0$, we obtain $d\ell^*/d\gamma > 0$. Q.E.D.

F. Proof of Proposition 6

(i) Consider first the case when commuting costs are high. It then follows from (30) and Lemma 3 that $\rho_0 < \rho_L < 1 < \rho_1$. Therefore, for $\Delta > 0$ sufficiently small, we have:

$$\rho_0 + \Delta < \rho_L - \Delta < \rho_L + \Delta < 1 < \rho_1.$$

If γ is sufficiently close to α/ε (but still such that $\gamma > \alpha/\varepsilon$ holds), Lemma 2 implies the following inequalities:

$$\begin{aligned} g(\rho_0 + \Delta; \gamma) &< f(\rho_0 + \Delta), \\ g(\rho_L - \Delta; \gamma) &> f(\rho_L - \Delta), \\ g(\rho_L + \Delta; \gamma) &< f(\rho_L + \Delta), \\ g(\rho_1; \gamma) &> f(\rho_1) = 0, \end{aligned}$$

where the last inequality holds because (28) implies that, for $\gamma > \alpha/\varepsilon$, $g(\rho;\gamma) > 0$ for all $\rho > 0$ while $f(\rho_1) = 0$ for any γ by definition of ρ_1. Therefore, by continuity of f and g, the equation (29) has at least *three* distinct solutions, which we denote as follows:

$$\rho_1^\star > \rho_2^\star > \rho_3^\star.$$

Furthermore, the properties of function g imply the following:

$$\lim_{\gamma\varepsilon \searrow \alpha} \rho_1^\star = \rho_1,$$

$$\lim_{\gamma\varepsilon \searrow \alpha} \rho_2^\star = \rho_L,$$

$$\lim_{\gamma\varepsilon \searrow \alpha} \rho_3^\star = \rho_0.$$

The solution $\rho_2^\star$ matches the equilibrium of Proposition 5. As for the other two solutions, $\rho^\star$ and $\rho_3^\star$, when γ is close enough to α/ε, we have $\rho^\star > 1 > \rho_3^\star$.

As $\gamma \searrow \alpha/\varepsilon$, it follows from Lemma 1 that $f(\rho^\star)$ and $f(\rho_3^\star)$ converge, respectively, to 0 and ∞, which implies:

$$\lim_{\gamma\varepsilon\searrow\alpha} \omega_1^\star = 0 \qquad \text{and} \qquad \lim_{\gamma\varepsilon\searrow\alpha} \omega_3^\star = \infty.$$

Hence, $\omega_1^\star < 1 < \omega_3^\star$ when $\gamma\varepsilon$ is close enough to α. It then follows from (28) that

$$\lim_{\gamma\varepsilon\searrow\alpha} \ell_1^\star = 0 \qquad \text{and} \qquad \lim_{\gamma\varepsilon\searrow\alpha} \ell_3^\star = \infty.$$

(ii) Consider now the case of high commuting costs. Then, we know from the proof of Proposition 7 that there exists a value $\overline{a} \in (0,1)$ such that

$$\rho_L \leq \rho_0 < 1 < \rho_1 \tag{F.1}$$

is satisfied for $a \geq \overline{a}$, and $\rho_0 < \rho_L < 1 < \rho_1$ holds otherwise. Under (F.1), there is a small $\Delta > 0$ such that the following inequalities hold:

$$\begin{gathered} g(\rho_1 - \Delta; \gamma) < f(\rho_1 - \Delta), \\ g(\rho_1; \gamma) > f(\rho_1) = 0. \end{gathered}$$

while $\rho^\star > 1$ when γ slightly exceeds α/ε.

Furthermore,

$$\lim_{\gamma\varepsilon\searrow\alpha} (\omega_1^*)^{\frac{\varepsilon}{1+\varepsilon}} = f(\rho_1) = 0.$$

Since $\lim_{\gamma\varepsilon\searrow\alpha} \omega_1^\star = 0$, $\omega_1^\star < 1$ when $\gamma\varepsilon$ is sufficiently close to α. Last, using (32), we have:

$$\lim_{\gamma\varepsilon\searrow\alpha} \ell_1^\star = 0.$$

Q.E.D.

G. Proof of Proposition 7

First, we show the existence and uniqueness of an equilibrium. Using (E.3) and (31), the equilibrium condition (29) becomes after simplifications:

$$\frac{1}{\phi\rho + 1 + \phi^2} \left(\frac{1 + a\frac{1+\phi^2 - 2\phi\rho^{1+\frac{1}{\beta\varepsilon}}}{\phi\rho + 1 + \phi^2}}{1 + a\frac{\rho - \phi\rho^{-\frac{1}{\beta\varepsilon}}}{\rho + 2\phi}} \right)^{\lambda} = \rho^{\mu} \frac{\rho}{\rho + 2\phi}, \tag{G.1}$$

where λ and μ are defined by

$$\lambda \equiv \frac{\gamma\varepsilon - \alpha}{\gamma + \alpha} > 0 \qquad \text{and} \qquad \mu \equiv \frac{\gamma\varepsilon - \alpha - (1-\alpha)(1+\varepsilon)}{\beta\varepsilon(\gamma + \alpha)}.$$

The first term of the LHS of (G.1) decreases in ρ; the second term also decreases because the numerator decreases while the denominator increases in ρ. Therefore, the LHS of (G.1) is a decreasing function of ρ. Furthermore, the RHS of (G.1) increases from 0 to ∞ in ρ when $\mu > 0$. It is readily verified that $\mu > 0$ if and only if

$$\gamma > \frac{1+\varepsilon}{(1-\beta)\varepsilon} - \alpha.$$

Hence, (G.1) has a unique solution ρ^*.

We now show that ℓ^* converges monotonically toward 1 when $\gamma > \gamma_s$ increases. Using (32), we obtain

$$\log \ell^* = -\frac{1}{\gamma\varepsilon/\alpha - 1} \log\left((\rho^*)^{\frac{1}{a}} \frac{\rho^* + 2\phi}{\phi\rho^* + 1 + \phi^2} \right). \tag{G.2}$$

Since $\rho^* > \rho_L$ under strong increasing returns, the expression under the log is greater than 1 and thus the RHS of (G.2) is negative. Furthermore, as ρ^* decreases with γ, the RHS of (G.2) increases with γ. In addition, the first of the RHS goes to 0 when γ goes to infinity. Consequently, ℓ^* increases and converges to 1. Q.E.D.

H. Proof of Lemma 4

The assumption $s^*_{ij} > 0$ implies $L^*_j > 0$, hence $W^*_j > 0$. Combining this with ??(**??**)?? and (35) implies that any individual $\nu \in [0,1]$ whose type $\mathbf{z}(\nu)$ satisfies

$$z_{ij}(\nu) = z_{kl}(\nu) \frac{W^*_l}{W^*_j} \frac{\tau_{kl}}{\tau_{ij}} \left(\frac{R^*_i}{R^*_k}\right)^\beta \geq z_{od}(\nu) \frac{W^*_d}{W^*_j} \frac{\tau_{od}}{\tau_{ij}} \left(\frac{R^*_o}{R^*_i}\right)^\beta \tag{H.1}$$

is indifferent between ij and kl.

Two cases may arise. First, if $s^*_{kl} > 0$, then $L^*_l > 0$ and $W^*_l > 0$. (H.1) thus implies that any individual ν satisfying

$$z_{kl}(\nu) > 0, \quad z_{ij}(\nu) = z_{kl}(\nu) \frac{W^*_l}{W^*_j} \frac{\tau_{kl}}{\tau_{ij}} \left(\frac{R^*_i}{R^*_k}\right)^\beta, \quad z_{od}(\nu) = 0$$

is indifferent between ij and kl.

Second, if $s^*_{kl} = 0$, then $L^*_l = 0$ and $W^*_l = 0$. Therefore, (H.1) implies that any individual such that $z_{kl}(\nu) > 0$ and $z_{ij}(\nu) = 0$ for any $ij \neq kl$ is indifferent between ij and kl. Q.E.D.

I. Proof of Proposition 9

Step 1. We first show the existence of a unique conditional equilibrium price for a symmetric commuting pattern $\mathbf{s}$ such that either $\mathbf{L}(\mathbf{s}) = (0,1,0)$ or $\mathbf{L}(\mathbf{s}) = (1/2,0,1/2)$, and $M_i(\mathbf{s}) > 0$ for $i = 0, \pm 1$.

We focus on the case of a fully agglomerated labor supply pattern, i.e., such that $L_0(\mathbf{s}) = 1$and $L_{-1}(\mathbf{s}) = L_1(\mathbf{s}) = 0$ (the proof for the fully dispersed labor supply pattern given by $L_0 = 0$ and $L_{-1}(\mathbf{s}) = L_1(\mathbf{s}) = 1/2$ goes along the same lines). Plugging $L_0 = 1$ and $L_{-1} = L_1 = 0$ into the FOCs (13) and (14) at $i = 0$, we obtain

$$W_0 = \alpha N_0^{1-\alpha} \qquad \text{and} \qquad R_0 = (1-\alpha) N_0^{-\alpha}, \tag{I.1}$$

hence,

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha} N_0. \tag{I.2}$$

Observe that $L_1(\mathbf{s}) = L_{-1}(\mathbf{s}) = 0$ implies $s_{i1} = s_{i,-1} = 0$ for all $i \in \{-1,0,1\}$. Combining this with the land market clearing condition and the market residential demand at $i = 0$, we get:

$$H_0 + N_0 = 1 \qquad \text{and} \qquad H_0 = s_{00} \frac{W_0}{R_0},$$

so that

$$N_0 = 1 - H_0 = 1 - s_{00} \frac{W_0}{R_0}. \tag{I.3}$$

Plugging (I.3) into (I.2), we get a linear equation in W_0/R_0:

$$\frac{W_0}{R_0} = \frac{\alpha}{1-\alpha} \left(1 - s_{00} \frac{W_0}{R_0} \right) \quad \Longrightarrow \quad \frac{\overline{W}_0(\mathbf{s})}{\overline{R}_0(\mathbf{s})} = \frac{\alpha}{1 - \alpha + \alpha s_{00}}. \tag{I.4}$$

From (I.3)-(I.4), we get:

$$\overline{N}_0(\mathbf{s}) = \frac{1-\alpha}{1-\alpha+\alpha s_{00}}.$$

Plugging $N_0 = \overline{N}_0(\mathbf{s})$ into the equilibrium condition (I.1) pins down uniquely the conditional equilibrium wage $\overline{W}_0(\mathbf{s})$ and the conditional equilibrium land rent $\overline{R}_0(\mathbf{s})$. As for $\overline{W}_j(\mathbf{s})$ and $\overline{R}_i(\mathbf{s})$ for $i,j = \pm 1$, zero labor supply implies $\overline{W}_j(\mathbf{s}) = 0$ and $\overline{N}_j(\mathbf{s}) = 0$ for $j = \pm 1$. Hence, the land market clearing at the periphery becomes

$$H_i = 1 = s_{i0} \frac{W_0}{R_i} \qquad \text{for} \qquad i = \pm 1,$$

which implies $\overline{R}_i(\mathbf{s}) = s_{i0} \overline{W}_0(\mathbf{s})$ for $i = \pm 1$.

Step 2. We now show that corner equilibria are unstable. Assume that $L_0^* = 1$ (the proof for $L_{-1}^* = L_1^* = 1/2$ goes along the same lines). Consider an individual ν such that, for all $i \in \{-1,0,1\}$, ν's match values satisfy $z_{ij}(\nu) = 0$ for $j = 0, \pm 1$. Clearly, ν is indifferent between working at the center and working at the periphery (in both cases, she enjoys zero utility). Consider a positive-measure set of individuals whose tastes are close to those of ν and whose utility-maximizing choice is $ij = 00$. Relocating them (together with ν) from $ij = 00$ to $kl = 01$, we have $V_{01}(\nu,\mathbf{s}) > 0$ because $\overline{W}_1(\mathbf{s}) > 0$. Using (13), there exists a positive-measure subset of individuals who are strictly better-off working at location 1. As a result, the corner equilibrium $L_0^* = 1$ is an unstable equilibrium. Q.E.D.

J. Proof of Proposition 10

Step 1. We first show the existence of a unique conditional equilibrium price for a symmetric commuting $\mathbf{s}$ such that $s_{ij} > 0$ for all i,j when $\alpha > 1/2$.

Since $L_i > 0$ for $i = 0, \pm 1$, the first-order conditions for the production sector yields the equilibrium conditions:

$$W_j = \alpha A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}, \tag{J.1}$$

$$R_j = (1-\alpha) A_j \left(\frac{L_j}{N_j}\right)^{\alpha}. \tag{J.2}$$

Furthermore, we also know that housing market clearing at location i yields:

$$H_i = \frac{\beta}{R_i} \sum_{j=1}^{n} s_{ij} W_j. \tag{J.3}$$

Plugging (J.1) and (J.2) into (J.3), and using the land market balance condition $N_i + H_i = 1$, we get:

$$H_i = 1 - N_i = \frac{\alpha\beta}{(1-\alpha)A_i} \left(\frac{N_i}{L_i}\right)^{\alpha} \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha},$$

$$(1-\alpha) A_i (1 - N_i) \left(\frac{N_i}{L_i}\right)^{-\alpha} = \alpha\beta \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha},$$

$$(1-\alpha) A_i \left(\frac{N_i}{L_i}\right)^{-\alpha} = (1-\alpha) A_i L_i \left(\frac{N_i}{L_i}\right)^{1-\alpha} + \alpha\beta \sum_{j=1}^{n} s_{ij} A_j \left(\frac{N_j}{L_j}\right)^{1-\alpha}.$$

Since $\mathbf{s}$ is symmetric, this system of equations becomes:

$$(1-\alpha) A_0 \left(\frac{N_0}{L_0}\right)^{-\alpha} = \left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] A_0 \left(\frac{N_0}{L_0}\right)^{1-\alpha} + 2\alpha\beta s_{01} A_1 \left(\frac{N_1}{L_1}\right)^{1-\alpha}$$

$$(1-\alpha) A_1 \left(\frac{N_1}{L_1}\right)^{-\alpha} = \alpha\beta s_{10} A_0 \left(\frac{N_0}{L_0}\right)^{1-\alpha} + \left[(1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})\right] A_1 \left(\frac{N_1}{L_1}\right)^{1-\alpha}$$

Dividing one equation by the other and using $A_i = L_i^{\gamma}$ for $i = 0, \pm 1$, we get:

$$n^{-\alpha} \ell^{\gamma+\alpha} = \frac{\left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} + 2\alpha\beta s_{01}}{\alpha\beta s_{10} \ell^{\gamma} \left(\frac{n}{\ell}\right)^{1-\alpha} + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})} \tag{J.4}$$

Since (J.1) and (J.2) imply

$$n^{-\alpha}\ell^{\gamma+\alpha} = r, \qquad \ell^{\gamma}\left(\frac{n}{\ell}\right)^{1-\alpha} = w, \tag{J.5}$$

we have

$$w^{\alpha}r^{1-\alpha} = \ell^{\gamma} = \left(\frac{L_0}{L_1}\right)^{\gamma}. \tag{J.6}$$

Likewise, combining (J.4) and (J.5), we get:

$$r = \frac{\left[(1-\alpha)L_0 + \alpha\beta s_{00}\right]w + 2\alpha\beta s_{01}}{\alpha\beta s_{10}w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}. \tag{J.7}$$

A sufficient condition for the system (J.6) – (J.7) to have a unique solution $(\overline{w}(\mathbf{s}),\overline{r}(\mathbf{s}))$ is that the graph of the relationship (J.7) between w and r intersects the downward-sloping curve given by (J.6) from below. The RHS of (J.7) is the ratio of two positive linear increasing functions of w. Since the elasticity of a linear increasing function with a positive intercept never exceeds 1, the elasticity of the RHS of (J.7) w.r.t. w is always larger than -1. Restating (J.6) as

$$r = \ell^{\frac{\gamma}{1-\alpha}}w^{-\frac{\alpha}{1-\alpha}}$$

shows that the elasticity of the RHS of this expression w.r.t. w equals $-\alpha/(1-\alpha)$, which is smaller than -1 when $\alpha > 1/2$.

Step 2. Denote by $\left(\overline{\mathbf{W}}(\mathbf{s}),\overline{\mathbf{R}}(\mathbf{s})\right)$ the equilibrium price vector conditional to an arbitrary commuting pattern $\mathbf{s}$ that belongs to a neighborhood of an interior equilibrium commuting pattern $\mathbf{s}^*$, and let $\overline{w}(\mathbf{s})$ and $\overline{r}(\mathbf{s})$ be the corresponding wage ratio and the land-price ratio:

$$\overline{w}(\mathbf{s}) \equiv \frac{\overline{W}_0(\mathbf{s})}{\overline{W}_1(\mathbf{s})} \quad \text{and} \quad \overline{r}(\mathbf{s}) \equiv \frac{\overline{R}_0(\mathbf{s})}{\overline{R}_1(\mathbf{s})}.$$

Consider the following two types of relocations: $0j \to 1j$ (changing place of residence but not the workplace) and $i0 \to i1$ (changing the workplace but not the place of residence). Observe that, in equilibrium, for each individual ν, we have:

$$\frac{V_{0j}^*(\nu)}{V_{1j}^*(\nu)} = \frac{z_{0j}(\nu)}{z_{1j}(\nu)}\left(\overline{r}(\mathbf{s}^*)\right)^{-\beta}, \tag{J.8}$$

$$\frac{V_{i0}^*(\nu)}{V_{i1}^*(\nu)} = \frac{z_{i0}(\nu)}{z_{i1}(\nu)}\overline{w}(\mathbf{s}^*). \tag{J.9}$$

If the individual ν is indifferent between $0j$ and $1j$ for some $j = \{-1,0,1\}$, switching from $0j$ to $1j$ makes this individual strictly worse off if and only if $\overline{r}(\mathbf{s}^*)$ decreases when a small subset of residents (almost indifferent between $0j$ and $1j$) of measure Δ is moved from 0 to 1, i.e.,

$$\frac{\partial\overline{r}(\mathbf{s}^*)}{\partial s_{1j}} - \frac{\partial\overline{r}(\mathbf{s}^*)}{\partial s_{0j}} < 0 \tag{J.10}$$

because (J.8) and (J.10) imply that $V^*_{0j}(\nu)/V^*_{1j}(\nu)$ increases above 1.

Likewise, using (J.9) if ν is an individual indifferent between $i0$ and $i1$ for some $i = \{-1,0,1\}$, switching from $i0$ to $i1$ makes ν strictly worse off if and only if $\overline{w}(\mathbf{s}^*)$ increases when a small subset of workers (almost indifferent between $i0$ and $i1$) of measure Δ is moved from 0 to 1, i.e.,

$$\frac{\partial \overline{w}(\mathbf{s}^*)}{\partial s_{i1}} - \frac{\partial \overline{w}(\mathbf{s}^*)}{\partial s_{i0}} > 0. \tag{J.11}$$

Step 3. We show that the land-price ratio $\overline{r}(\mathbf{s}^*)$ always satisfies the equilibrium condition (J.10). Under a relocation of residents from $0j$ to $1j$ (or, equivalently, from $1j$ to $0j$) for $j = 0{,}1$, the numerator in the RHS of (J.7) decreases pointwise, while the denominator increases pointwise. Therefore, the curve (J.7) shifts downwards in the $(w{,}r)$-plane, while the curve (J.6) remains unchanged. Since (J.7) intersects (J.6) from below, this implies a reduction in $\overline{r}(\mathbf{s})$. Hence, (J.10) holds.

Step 4. It remains to check when (J.11) holds. To this end, we study when the relocation of a $\Delta-$measure subset of workers from $i0$ to $i1$ for $i = 0, \pm 1$ leads to an increase in the relative wage $\overline{w}(\mathbf{s})$. As a result, two cases must be distinguished: (i) a relocation of workers from 00 to 01 and (ii) a relocation of workers from 10 to 11.

Taking the log-differential of (J.6) yields:

$$\alpha d \log w + (1-\alpha) d \log r = \gamma (d \log L_0 - d \log L_1). \tag{J.12}$$

Case 1. Assume that

$$ds_{00} = -\Delta, \qquad ds_{01} = ds_{0,-1} = \Delta/2,$$

$$ds_{ij} = 0 \text{ otherwise.}$$

In this case, (J.12) becomes:

$$\alpha \, d \log w + (1-\alpha) \, d \log r = \gamma \left(\frac{ds_{00}}{L_0} - \frac{ds_{01}}{L_1} \right) = -\gamma \Delta \left(\frac{1}{2L_1} + \frac{1}{L_0} \right)$$

Taking the log-differential of (J.7) yields:

$$d \log r = \frac{d\left[\left((1-\alpha)L_0 + \alpha\beta s_{00}\right) w + 2\alpha\beta s_{01}\right]}{\left[(1-\alpha)L_0 + \alpha\beta s_{00}\right] w + 2\alpha\beta s_{01}} - \frac{d\left[\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})\right]}{\alpha\beta s_{10} w + (1-\alpha)L_1 + \alpha\beta(s_{11} + s_{1,-1})}. \tag{J.13}$$

Since

$$d\left[\left((1-\alpha)L_0 + \alpha\beta s_{00}\right) w + 2\alpha\beta s_{01}\right] = -\Delta \left(1 - \alpha + \alpha\beta\right) w + \alpha\beta\Delta + \left((1-\alpha)L_0 + \alpha\beta s_{00}\right) w d \log w,$$

while

$$d\left[\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})\right]=(1-\alpha)\frac{\Delta}{2}+\alpha\beta s_{10}wd\log w,$$

(J.13) becomes

$$d\log r=\left[\frac{-\left(1-\alpha+\alpha\beta\right)w+\alpha\beta}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}}-\frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]\Delta$$

$$+\left[\frac{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}}-\frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]d\log w$$

Plugging this expression into (J.12), we get:

$$d\log w=\frac{-\gamma\left(\frac{1}{2L_1}+\frac{1}{L_0}\right)+(1-\alpha)\left[\frac{(1-\alpha+\alpha\beta)w-\alpha\beta}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}{\alpha+(1-\alpha)\left[\frac{((1-\alpha)L_0+\alpha\beta s_{00})w}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}}-\frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}\Delta.$$

When $\alpha>1/2$, the denominator in $d\log w$ is always positive because each bracketed term of the denominator is smaller than 1. As a result, the stability condition $d\log w>0$ holds if the numerator is positive:

$$\frac{\left(1-\alpha+\alpha\beta\right)w-\alpha\beta}{\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}>\frac{\gamma}{1-\alpha}\left(\frac{1}{L_0}+\frac{1}{2L_1}\right). \tag{J.14}$$

Case 2. We now assume that

$$ds_{11}=-ds_{10}=\Delta/2,\qquad ds_{-10}=-ds_{-1,-1}=-\Delta/2,$$

$$ds_{ij}=0 \text{ otherwise.}$$

Hence, (J.12) becomes:

$$\alpha d\log w+(1-\alpha)\,d\log r=\gamma\left[\frac{ds_{10}+ds_{-10}}{L_0}-\frac{ds_{11}}{L_1}\right]=-\gamma\Delta\left(\frac{1}{2L_1}+\frac{1}{L_0}\right)$$

Since

$$d\left[\left((1-\alpha)L_0+\alpha\beta s_{00}\right)w+2\alpha\beta s_{01}\right]=-\Delta\left(1-\alpha\right)w+\left((1-\alpha)L_0+\alpha\beta s_{00}\right)wd\log w$$

and

$$d\left[\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})\right]=\alpha\beta\frac{\Delta}{2}w+(1-\alpha)\frac{\Delta}{2}+\alpha\beta s_{10}wd\log w,$$

(J.13) becomes

$$d\log r = \left[\frac{-(1-\alpha)\,w}{((1-\alpha)L_0+\alpha\beta s_{00})\,w+2\alpha\beta s_{01}} - \frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]\Delta$$

$$+\left[\frac{((1-\alpha)L_0+\alpha\beta s_{00})\,w}{((1-\alpha)L_0+\alpha\beta s_{00})\,w+2\alpha\beta s_{01}} - \frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]d\log w.$$

Plugging this expression for $d\log r$ into (J.12), we get:

$$d\log w \;=\frac{-\gamma\left(\frac{1}{2L_1}+\frac{1}{L_0}\right)+(1-\alpha)\left[\frac{(1-\alpha)w}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}{\alpha+(1-\alpha)\left[\frac{((1-\alpha)L_0+\alpha\beta s_{00})w}{((1-\alpha)L_0+\alpha\beta s_{00})w+2\alpha\beta s_{01}}-\frac{\alpha\beta s_{10}w}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})}\right]}\Delta.$$

If $\alpha > 1/2$, the denominator in $d\log w$ is always positive. Hence, the stability condition $d\log w > 0$ becomes:

$$\frac{(1-\alpha)\,w}{((1-\alpha)L_0+\alpha\beta s_{00})\,w+2\alpha\beta s_{01}}+\frac{1}{2}\frac{1-\alpha+\alpha\beta}{\alpha\beta s_{10}w+(1-\alpha)L_1+\alpha\beta(s_{11}+s_{1,-1})} > \frac{\gamma}{1-\alpha}\left(\frac{1}{L_0}+\frac{1}{2L_1}\right). \tag{J.15}$$

When $\alpha > 1/2$, the inequalities (J.14) and (J.15) are necessary and sufficient for an interior equilibrium to be stable.

We now rewrite these two conditions in terms of the variable ρ only. Using Proposition 1 and the equilibrium relationship $\omega^{\frac{1+\varepsilon}{\varepsilon}} = f(\rho)$, as well as $\rho = r^{-\beta\varepsilon}$, $\omega = w^{\varepsilon}$, and $a = \alpha\beta/(1-\alpha)$, (J.14) and (J.15) become

$$\frac{f(\rho)+\frac{1}{2}(1+a)\rho^{-\frac{1}{\beta\varepsilon}}\,[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{((1+a)\rho+2\phi)\,f(\rho)+2a\phi\rho} > \frac{\gamma}{1-\alpha}\left(\frac{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{2(\phi\rho+1+\phi^2)}+\frac{1}{\rho+2\phi}\right), \tag{J.16}$$

$$\frac{(1+a)\,f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}-a\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{((1+a)\rho+2\phi)\,f(\rho)+2a\phi\rho} > \frac{\gamma}{1-\alpha}\left(\frac{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{2(\phi\rho+1+\phi^2)}+\frac{1}{\rho+2\phi}\right), \tag{J.17}$$

Solving the equilibrium condition $f(\rho) = g(\rho;\gamma)$ w.r.t. γ yields

$$\gamma = \frac{\alpha}{1+\varepsilon}\frac{\log\left(\rho^{-b}f(\rho)\right)}{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}\,[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}.$$

Plugging this expression into (J.16) – (J.17), we get:

$$
\begin{aligned}
\Phi(\rho) \equiv \frac{2(1-\alpha)(1+\varepsilon)(\phi\rho+1+\phi^2)}{((1+a)\rho+2\phi)\,f(\rho)+2a\phi\rho} \\
\times \frac{f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}+\frac{a}{2}\rho^{-\frac{1}{\beta\varepsilon}}\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}(\rho+2\phi)+2(\phi\rho+1+\phi^2)} \\
\times \frac{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}{\alpha\log\left(\rho^{-b}f(\rho)\right)} > 1,
\end{aligned}
$$

$$
\begin{aligned}
\Psi(\rho) \equiv \frac{2(1-\alpha)(1+\varepsilon)(\phi\rho+1+\phi^2)}{((1+a)\rho+2\phi)\,f(\rho)+2a\phi\rho} \\
\times \frac{(1+a)\,f(\rho)+\left(\frac{1}{2}\rho^{-\frac{1}{\beta\varepsilon}}-a\right)[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}}{[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}(\rho+2\phi)+2(\phi\rho+1+\phi^2)} \\
\times \frac{\log\left(\frac{\rho+2\phi}{\phi\rho+1+\phi^2}[f(\rho)]^{\frac{\varepsilon}{1+\varepsilon}}\right)}{\alpha\log\left(\rho^{-b}f(\rho)\right)} > 1.
\end{aligned}
$$

Last, we set:

$$\mathbb{F}(\rho) \equiv \min\left\{\Phi(\rho),\Psi(\rho)\right\},$$

which is independent of ρ. Verifying $\mathbb{F}(\rho) > 1$ can be done numerically for any vector of parameters by plotting $\mathbb{F}(\rho)$ as a function of the variable ρ. Q.E.D.

K. Proof of Proposition 11

Step 1. We first extend the main expressions of Section 3.1 to the case of amenities.

Since the indirect utility is given by (36), the share s_{ij} of households choosing the location pair ij:

$$s_{ij} = \frac{\left[O_i D_j W_j/(\tau_{ij}R_i^{\beta})\right]^{\varepsilon}}{\sum_{r\in\mathcal{I}}\sum_{s\in\mathcal{I}}\left[O_r D_s W_s/(\tau_{rs}R_r^{\beta})\right]^{\varepsilon}},$$

which can be rewritten as

$$s_{ij} = \kappa O_i^{\varepsilon} D_j^{\varepsilon} R_i^{-\beta\varepsilon} W_j^{\varepsilon}\tau_{ij}^{-\varepsilon}, \tag{K.1}$$

where

$$\kappa \equiv \frac{1}{\sum_{r\in\mathcal{I}}\sum_{s\in\mathcal{I}}\left[O_r D_s W_s/(\tau_{rs}R_r^{\beta})\right]^{\varepsilon}}. \tag{K.2B}$$

The residential population M_i at i now given by

$$M_i \equiv \sum_{j\in\mathcal{I}} s_{ij} = \kappa O_i^{\varepsilon} R_i^{-\beta\varepsilon}\sum_{j\in\mathcal{I}}(D_j W_j)^{\varepsilon}\tau_{ij}^{-\varepsilon},$$

while the labor force L_j at j is

$$L_j \equiv \sum_{i\in\mathcal{I}} s_{ij} = \kappa(D_jW_j)^{\varepsilon} \sum_{i\in\mathcal{I}} O_i^{\varepsilon} R_i^{-\beta\varepsilon} \tau_{ij}^{-\varepsilon}.$$

Housing market clearing is still given by (10). Substituting (K.1) in (10) gives

$$R_i = \left(\frac{\beta\kappa}{H_i} O_i^{\varepsilon} \sum_{j\in\mathcal{I}} W_j^{1+\varepsilon} D_j^{\varepsilon} \tau_{ij}^{-\varepsilon}\right)^{\frac{1}{1+\beta\varepsilon}}. \tag{K.2}$$

The production function at location j is given by

$$Y_j = A_j L_j^{\alpha} N_j^{1-\alpha},$$

where $0 < \alpha < 1$ and A_j is the location-specific TFP:

$$A_j = T_j L_j^{\gamma}, \tag{K.3}$$

where $\gamma \geq 0$ is the same for all j, while $T_j > 0$ is location-specific exogenous production amenity with $T_{-1} = T_1$.

If location j hosts a positive share of the production sector, the first-order conditions for the production sector yields the equilibrium wage and land rent (13) by (14), while their ratio is still given by (15).

When $\gamma = 0$, we have:

$$R_j = (1-\alpha)T_j \left(\frac{N_j}{L_j}\right)^{-\alpha} = (1-\alpha)T_j^{\frac{1}{1-\alpha}} \left(\frac{W_j}{\alpha}\right)^{-\alpha/(1-\alpha)}. \tag{K.4}$$

Applying the symmetry assumption to the wage and land rent patterns yields (compare to (19)):

$$\begin{aligned} s_{0,0} &= \kappa O_0^{\varepsilon} R_0^{-\beta\varepsilon} (D_0 W_0)^{\varepsilon}, \\ s_{0,1} = s_{0,-1} &= \kappa O_0^{\varepsilon} R_0^{-\beta\varepsilon} (D_1 W_1)^{\varepsilon} \tau^{-\varepsilon}, \\ s_{1,1} = s_{-1,-1} &= \kappa O_1^{\varepsilon} R_1^{-\beta\varepsilon} (D_1 W_1)^{\varepsilon}, \\ s_{1,-1} = s_{-1,1} &= \kappa O_1^{\varepsilon} R_1^{-\beta\varepsilon} (D_1 W_1)^{\varepsilon} \tau^{-2\varepsilon}, \\ s_{1,0} = s_{-1,0} &= \kappa O_1^{\varepsilon} R_1^{-\beta\varepsilon} (D_0 W_0)^{\varepsilon} \tau^{-\varepsilon}. \end{aligned} \tag{K.5}$$

As $\omega \equiv w^{\varepsilon}$ and $\rho \equiv r^{-\beta\varepsilon}$, the first-order condition (K.4) becomes $\rho = \omega^{a}$.

Step 2. We now determine the conditions that generalize (23)-(26) to the case of amenities.

The equilibrium demand for housing and commercial space and the equilibrium supply of workers and residents are:

$$M_0 = \frac{\rho o\,(\omega d + 2\phi)}{\rho o(\omega d + 2\phi) + 2(\phi\omega d + 1 + \phi^2)}, \qquad M_1 = \frac{1 - M_0}{2}, \tag{K.6}$$

$$L_0 = \frac{(\rho o + 2\phi)\omega d}{(\rho o + 2\phi)\omega d + 2(\phi\rho o + 1 + \phi^2)}, \qquad L_1 = \frac{1 - L_0}{2}, \tag{K.7}$$

$$H_0 = \frac{a\rho o\left(1+2\frac{\phi}{d}\omega^{-\frac{1+\varepsilon}{\varepsilon}}\right)}{a\rho o\left(1+2\frac{\phi}{d}\omega^{-\frac{1+\varepsilon}{\varepsilon}}\right)+\rho o+2\phi}, \qquad N_0 = 1 - H_0, \tag{K.8}$$

$$H_1 = \frac{a\left(\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}d+1+\phi^2\right)}{a\left(\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}d+1+\phi^2\right)+\phi\rho o+(1+\phi^2)}, \qquad N_1 = 1 - H_1. \tag{K.9}$$

Using (K.2B), symmetry and (20), we have

$$\kappa = \frac{1}{O_1^{-\varepsilon}R_1^{\beta\varepsilon}(D_1W_1)^{-\varepsilon}\left(2(1+\phi^2)+2\phi\omega d+2\phi\rho o+\rho o\omega d\right)}. \tag{K.11}$$

Using (K.1), (7), symmetry, and (20), we have:

$$M_0 = s_{00} + 2s_{01} = \kappa O_1^{-\varepsilon}R_1^{\beta\varepsilon}(D_1W_1)^{-\varepsilon}\ o\rho(2\phi+\omega d).$$

Substituting for κ from (K.10), we obtain M_0 as a function of ρ and ω. Under symmetry, the residential pattern satisfies $2M_1 + M_0 = 1$, which gives M_1 as a function of ρ and ω.

Using (K.1), (8), symmetry, and (20), we obtain:

$$L_0 = s_{00} + 2s_{10} = \kappa O_1^{-\varepsilon}R_1^{\beta\varepsilon}(D_1W_1)^{-\varepsilon}(2\phi+\rho o)\omega d.$$

Substituting for κ from (K.11), we have L_0 as a function of ρ and ω, so that $L_1 = (1-L_0)/2$ also depend of ρ and ω.

To evaluate the expressions for residential and commercial land, we use (10):

$$R_0H_0 = \beta\kappa\left(O_0^{\varepsilon}R_0^{-\beta\varepsilon}D_0^{\varepsilon}W_0^{1+\varepsilon} + 2\phi O_0^{\varepsilon}R_0^{-\beta\varepsilon}D_1^{\varepsilon}W_1^{1+\varepsilon}\right),$$

$$R_1H_1 = \beta\kappa\left[\phi O_1^{\varepsilon}R_1^{-\beta\varepsilon}D_0^{\varepsilon}W_0^{1+\varepsilon} + (1+\phi^2)O_1^{\varepsilon}R_1^{-\beta\varepsilon}D_1^{\varepsilon}W_1^{1+\varepsilon}\right].$$

Substituting from (K.11) and using (20) leads to

$$H_0 = \beta\frac{W_0}{R_0}\frac{\rho o\omega d+2\phi\rho o\omega^{-\frac{1}{\varepsilon}}}{\rho o\omega d+2\phi\rho o+2\phi\omega d+2(1+\phi^2)},$$

$$H_1 = \beta\frac{W_1}{R_1}\frac{\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}d+(1+\phi^2)}{\rho o\omega d+2\phi\rho o+2\phi\omega d+2(1+\phi^2)}.$$

Using (15) and (9), we get

$$1-N_0 = a\frac{N_0}{L_0}\frac{\rho o\omega d+2\phi\rho o\omega^{-\frac{1}{\varepsilon}}}{\rho o\omega d+2\phi\rho o+2\phi\omega d+2(1+\phi^2)},$$

$$1-N_1 = a\frac{N_1}{L_1}\frac{\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}d+(1+\phi^2)}{\rho o\omega d+2\phi\rho o+2\phi\omega d+2(1+\phi^2)}.$$

Solving for N_0 and N_1 yields:

$$1 = N_0\left[1 + a\frac{1}{L_0}\frac{\rho o\omega d + 2\phi\rho o\omega^{-\frac{1}{\varepsilon}}}{\rho o\omega d + 2\phi\rho o + 2\phi\omega d + 2(1+\phi^2)}\right],$$

$$1 = N_1\left[1 + a\frac{1}{L_1}\frac{\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}d + (1+\phi^2)}{\rho o\omega d + 2\phi\rho o + 2\phi\omega d + 2(1+\phi^2)}\right].$$

Using the expressions L_0 and L_1 from (K.7), we arrive at the commercial land use pattern:

$$N_0 = \frac{\rho o + 2\phi}{(1+a)\rho o + 2\phi + 2a\phi\frac{\rho o}{d}\omega^{-\frac{1+\varepsilon}{\varepsilon}}},$$

$$N_1 = \frac{\phi\rho o + 1 + \phi^2}{\phi\rho o + a\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}d + (1+a)(1+\phi^2)}.$$

Substituting these expressions into the land market clearing conditions, $H_i + N_i = 1$, we find the housing pattern (K.8) and (K.9).

Last, observe that the conditions (K.6)-(K.9) boil down to (23)-(26) when $o = d = 1$. This proves Proposition 1.

Step 3. Finally, we prove Proposition 11.

The land rent equilibrium condition (K.3) at $i = 0{,}1$ leads to

$$R_0 = \left[\frac{\beta\kappa}{H_0}O_0^{\varepsilon}\left(D_0^{\varepsilon}W_0^{1+\varepsilon} + 2\phi D_1^{\varepsilon}W_1^{1+\varepsilon}\right)\right]^{\frac{1}{1+\beta\varepsilon}}, \tag{K.12}$$

$$R_1 = \left[\frac{\beta\kappa}{H_1}O_1^{\varepsilon}\left(\phi D_0^{\varepsilon}W_0^{1+\varepsilon} + (1+\phi^2)D_1^{\varepsilon}W_1^{1+\varepsilon}\right)\right]^{\frac{1}{1+\beta\varepsilon}}. \tag{K.13}$$

Multiplying by R_j both sides of the land market balance condition (9), we get:

$$R_0H_0 + R_0N_0 = R_0H_0 + \frac{1-\alpha}{\alpha}W_0L_0 = R_0, \tag{K.14}$$

$$R_1H_1 + R_1N_1 = R_1H_1 + \frac{1-\alpha}{\alpha}W_1L_1 = R_1. \tag{K.15}$$

Dividing (K.14) over (K.15), we obtain:

$$r = \frac{R_0}{R_1} = \frac{R_0H_0 + \frac{1-\alpha}{\alpha}W_0L_0}{R_1H_1 + \frac{1-\alpha}{\alpha}W_1L_1}. \tag{K.16}$$

It then follows from (K.12) – (K.13) that

$$R_0H_0 = \beta\kappa O_0^{\varepsilon}\left(D_0^{\varepsilon}W_0^{1+\varepsilon}R_0^{-\beta\varepsilon} + 2\phi D_1^{\varepsilon}W_1^{1+\varepsilon}R_0^{-\beta\varepsilon}\right), \tag{K.17}$$

$$R_1H_1 = \beta\kappa O_1^{\varepsilon}\left[\phi D_0^{\varepsilon}W_0^{1+\varepsilon}R_1^{-\beta\varepsilon} + (1+\phi^2)D_1^{\varepsilon}W_1^{1+\varepsilon}R_1^{-\beta\varepsilon}\right]. \tag{K.18}$$

Using (K.16), the labor market balance conditions at $i = 0{,}1$ are given by

$$L_0 = s_{00} + 2s_{10} = \kappa D_0^{\varepsilon} W_0^{\varepsilon} \left(O_0^{\varepsilon} R_0^{-\beta\varepsilon} + 2\phi O_1^{\varepsilon} R_1^{-\beta\varepsilon} \right),$$

$$L_1 = s_{11} + s_{01} + s_{-11} = \kappa D_1^{\varepsilon} W_1^{\varepsilon} \left[(1+\phi^2) O_1^{\varepsilon} R_1^{-\beta\varepsilon} + \phi O_0^{\varepsilon} R_0^{-\beta\varepsilon} \right],$$

so that

$$W_0 L_0 = \kappa \left(O_0^{\varepsilon} R_0^{-\beta\varepsilon} D_0^{\varepsilon} W_0^{1+\varepsilon} + 2\phi O_1^{\varepsilon} R_1^{-\beta\varepsilon} D_0^{\varepsilon} W_0^{1+\varepsilon} \right), \tag{K.19}$$

$$W_1 L_1 = \kappa \left[(1+\phi^2) O_1^{\varepsilon} R_1^{-\beta\varepsilon} D_1^{\varepsilon} W_1^{1+\varepsilon} + \phi O_0^{\varepsilon} R_0^{-\beta\varepsilon} D_1^{\varepsilon} W_1^{1+\varepsilon} \right]. \tag{K.20}$$

Plugging into (K.16) the expressions for $R_j H_j$ ($j = 0{,}1$) given by (K.17)-(K.18) and the expressions for $W_j L_j$ ($j = 0{,}1$) given by (K.19)-(K.20), we get after simplifications:

$$r = \frac{(1+a) o r^{-\beta\varepsilon} w^{1+\varepsilon} d + 2a\phi o r^{-\beta\varepsilon} + 2\phi w^{1+\varepsilon} d}{a\phi w^{1+\varepsilon} d + \phi o r^{-\beta\varepsilon} + (1+a)(1+\phi^2)}. \tag{K.21}$$

Combining (K.4) with (13)-(14), we get:

$$w = t\ell^{\gamma} \left(\frac{n}{\ell} \right)^{1-\alpha} \qquad \text{and} \qquad r = t\ell^{\gamma} \left(\frac{n}{\ell} \right)^{-\alpha}, \tag{K.22}$$

where $t \equiv T_0/T_1$. Dividing (K.19) by (K.20) yields:

$$\ell = \frac{o r^{-\beta\varepsilon} + 2\phi}{\phi o r^{-\beta\varepsilon} + (1+\phi^2)} w^{\varepsilon} d. \tag{K.23}$$

Using (K.22) and (K.23), we get:

$$w^{\alpha} r^{1-\alpha} = t\ell^{\gamma} = t \left(\frac{o r^{-\beta\varepsilon} + 2\phi}{\phi o r^{-\beta\varepsilon} + (1+\phi^2)} w^{\varepsilon} d \right)^{\gamma},$$

or, equivalently,

$$w = t^{\frac{1}{\alpha}} r^{-\frac{1-\alpha}{\alpha}} \left(\frac{o r^{-\beta\varepsilon} + 2\phi}{\phi o r^{-\beta\varepsilon} + (1+\phi^2)} w^{\varepsilon} d \right)^{\frac{\gamma}{\alpha}}. \tag{K.24}$$

Since $\omega > 0$, the conditions (K.21) and (K.24) can be reformulated in terms of $(\rho{,}\omega)$:

$$\rho = \left(\frac{(1+a)\rho o \omega^{\frac{1+\varepsilon}{\varepsilon}} d + 2a\phi\rho o + 2\phi\omega^{\frac{1+\varepsilon}{\varepsilon}} d}{a\phi\omega^{\frac{1+\varepsilon}{\varepsilon}} d + \phi\rho o + (1+a)(1+\phi^2)} \right)^{-\beta\varepsilon}, \tag{K.25}$$

and

$$\omega = t^{\frac{\varepsilon}{\alpha}} \rho^{\frac{1}{a}} \left(\frac{\rho o + 2\phi}{\phi\rho o + (1+\phi^2)} \omega d \right)^{\frac{\gamma\varepsilon}{\alpha}}. \tag{K.26}$$

Solving (K.25) and (K.26) for $\omega^{\frac{1+\varepsilon}{\varepsilon}}$ yields the functions (38) and (39).

Finally, $o = d = 1$ in (K.25) and (K.26) yields:

$$\rho = \left(\frac{(1+a)\rho\omega^{\frac{1+\varepsilon}{\varepsilon}} + 2a\phi\rho + 2\phi\omega^{\frac{1+\varepsilon}{\varepsilon}}}{\phi\rho + a\phi\omega^{\frac{1+\varepsilon}{\varepsilon}} + (1+\phi^2)(1+a)} \right)^{-\beta\varepsilon}, \tag{K.27}$$

and

$$\omega = \rho^{\frac{1}{a}} \left(\omega \frac{\rho + 2\phi}{\phi\rho + 1 + \phi^2} \right)^{\frac{\gamma\varepsilon}{\alpha}}. \tag{K.28}$$

Solving (K.27) – (K.28) for $\omega^{\frac{1+\varepsilon}{\varepsilon}}$ yields the functions (27) and (28). This proves proposition 2. Q.E.D.