EC1410-Spring 2022
Problem Set 7 solutions

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Matt Turner

1. Let $Y = f(x) = x^\beta$ describe the production process, where f is the production technology.

   (a) Verify that f is increasing returns to scale if $\beta > 1$.
   Recall that, for any $\alpha > 1$,

   $$f(\alpha x) > \alpha f(x) \text{ is increasing returns to scale.}$$
   $$f(\alpha x) = \alpha f(x) \text{ is constant returns to scale.}$$
   $$f(\alpha x) < \alpha f(x) \text{ is decreasing returns to scale.}$$

   Given our production technology,

   $$\begin{aligned} f(x) &= x^\beta \\ f(\alpha x) &= (\alpha x)^\beta \\ &= \alpha^\beta x^\beta \\ \alpha f(x) &= \alpha x^\beta \\ f(\alpha x) - \alpha f(x) &= \alpha^\beta x^\beta - \alpha x^\beta \\ &= (\alpha^\beta - \alpha) x^\beta \end{aligned}$$

   If $\beta > 1$, then

   $$\begin{aligned} \alpha^\beta &> \alpha \text{ for } \alpha > 1 \\ \alpha^\beta - \alpha &> 0 \\ f(\alpha x) - \alpha f(x) &= (\alpha^\beta - \alpha) x^\beta \\ &> 0 \\ f(\alpha x) &> \alpha f(x) \text{ so we have increasing returns to scale} \end{aligned}$$

   (b) Verify that f is constant returns to scale if $\beta = 1$.
   If $\beta = 1$, then

   $$\begin{aligned} \alpha^\beta &= \alpha^1 = \alpha \\ \alpha^\beta - \alpha &= 0 \\ f(\alpha x) - \alpha f(x) &= (\alpha^\beta - \alpha) x^\beta \\ &= 0 \\ f(\alpha x) &= \alpha f(x) \text{ so we have constant returns to scale} \end{aligned}$$

   (c) Verify that f is decreasing returns to scale if $\beta < 1$.

If $\beta < 1$, then

$$\begin{aligned}
\alpha^\beta &< \alpha \text{ for } \alpha > 1 \\
\alpha^\beta - \alpha &< 0 \\
f(\alpha x) - \alpha f(x) &= (\alpha^\beta - \alpha)x^\beta \\
&< 0 \\
f(\alpha x) &< \alpha f(x) \text{ so we have decreasing returns to scale}
\end{aligned}$$

2. Suppose $f(n_i) = n_i^\alpha$ is decreasing returns to scale in individual labor n_i, and output is given by

$$y_i(n_i) = AN^\sigma f(n_i), \text{ for } N = \sum_i n_i$$

Verify that

$$\frac{\partial y_i}{\partial n_i} \approx AN^\sigma f'(n_i)$$

as n_i gets small.

$$\begin{aligned}
y_i(n_i) &= AN^\sigma f(n_i) \\
&= AN^\sigma n_i^\alpha \\
&= A(\sum_i n_i)^\sigma n_i^\alpha \\
\frac{\partial y_i}{\partial n_i} &= \frac{\partial A(\sum_i n_i)^\sigma n_i^\alpha}{\partial n_i} \\
&= \frac{\partial A(\sum_i n_i)^\sigma}{\partial n_i} n_i^\alpha + A\Big(\sum_i n_i\Big)^\sigma \frac{\partial n_i^\alpha}{\partial n_i} \\
&= A\sigma\Big(\sum_i n_i\Big)^{\sigma-1} 1 * n_i^\alpha + A\Big(\sum_i n_i\Big)^\sigma \alpha n_i^{\alpha-1} \\
&= \sigma AN^{\sigma-1} n_i^\alpha + AN^\sigma f'(n_i)
\end{aligned}$$

Since $f(n_i) = n_i^\alpha$ is decreasing returns to scale in individual labor n_i, based on 1b this means that $\alpha < 1$. So as n_i gets small, n_i^α goes to zero. However,

$$f'(n_i) = \alpha n_i^{\alpha-1} = \frac{\alpha}{n_i^{1-\alpha}}$$

gets large as n_i gets small because the denominator shrinks. So we have:

$$\begin{aligned}
\frac{\partial y_i}{\partial n_i} &= \sigma AN^{\sigma-1} n_i^\alpha + AN^\sigma f'(n_i) \\
n_i^\alpha &\to 0 \text{ as } n_i \to 0 \\
f'(n_i) &\to \infty \text{ as } n_i \to 0 \\
\frac{\partial y_i}{\partial n_i} &\approx AN^\sigma f'(n_i) \text{ as } n_i \text{ gets small}
\end{aligned}$$

3. Consider an economy with two firms (call them Firm 1 and Firm 2) choosing between three locations, A, B and C.

   (a) Create a table with all of the possible combinations of firm/location choice.

Outcome	Location of Firm 1	Location of Firm 2
1	A	A
2	A	B
3	A	C
4	B	A
5	B	B
6	B	C
7	C	A
8	C	B
9	C	C

   (b) Assuming the firms are choosing location randomly, create a new table with the share of outcomes where one firm is in A and the other is in B, where both firms are in A, etc., for each location pair you listed above.

Share of Outcomes	Location of Firm 1	Location of Firm 2
1/9	A	A
2/9	A	B
2/9	A	C
1/9	B	B
2/9	B	C
1/9	C	C

   (c) Define pairwise distance, d_{ij}, to be 1 if the firms are in different locations, and 0 if the firms are in the same location. Add a column for pairwise distance to the table from the previous step.

Share of Outcomes	Location 1	Location 2	Pairwise Distance
1/9	A	A	0
2/9	A	B	1
2/9	A	C	1
1/9	B	B	0
2/9	B	C	1
1/9	C	C	0

   (d) Assuming that the firms choose location at random, plot three histograms about pairwise distances: First, if the firms are not in the same location, what are the relative frequencies of d_{ij} being 0 versus 1? Second, if the firms are in the same location, what are the relative frequencies of d_{ij} being 0 versus 1? Third, compute a weighted sum of these two histograms, weighing each by the relative frequency with which it occurs in your table, to create a "mean" histogram of pairwise distance.

   In the first histogram, $d_{ij} = 1$ with frequency 1. In the second histogram, $d_{ij} = 0$ with frequency 1. Using the table in 3c to provide the weights, in the third histogram $d_{ij} = 0$ should occur with frequency $\frac{1}{3}$ and $d_{ij} = 1$ should occur with frequency $\frac{2}{3}$.

(e) Assume that you have data on 100 industries, each of which has two firms with three possible choices of location. You observe that in 50 industries, the two firms are in the same location, and in the other 50 industries, the firms are in different locations. Do you think the observed location choices are consistent with firms randomly choosing locations? Explain briefly.

If firms randomly chose locations, then based on the histogram in 3d we might expect to see 33 industries with both firms in the same location ($d_{ij} = 0$) and 67 industries with the two firms in different locations ($d_{ij} = 1$). Indeed, this is the most probable outcome - but it is not the only outcome consistent with firms choosing randomly. Consider, for example, that in an individual industry, the firms should locate in the same place $\frac{1}{3}$ of the time if they are choosing location randomly. Then, it is possible that all firms chose location randomly and both firms located in the same place in every industry - though the probability of this occurring by chance is very low, at $\left(\frac{1}{3}\right)^{100}$.

In general, the probability of observing exactly k industries with firms in the same location, and $100 - k$ industries with firms in different locations, is given by the probability mass function of the binomial distribution:

$$\Pr(X = k) = \binom{100}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

But to determine whether the location choices we observed are consistent with firms choosing randomly, we also want to know how frequently we would observe something other than what we do observe, given that the choices are random. That is, if firms chose locations randomly, what is the probability of observing 50 or more industries with both firms in the same location? The cumulative distribution function of the binomial distribution can answer this question. Using an online binomial probability calculator, we find that the probability of observing 50 or more industries with both firms in the same location, $\Pr(X \geq 50)$, is 0.0004. So, seeing 50 or more industries with both firms in the same location, as we did, is quite unlikely if the firms did in fact choose location randomly.

4. Suppose you observe only the part of the Basic Pharmaceuticals graphs (Figure 2 from Duranton and Overman, 2005) for pairwise distances between 88 and 92km. Would you conclude that pharmaceuticals are more agglomerated than would occur by chance? Explain briefly.

   For this specific range of pairwise distances, the realized histogram of pairwise distances (solid line) is within the dashed lines (range of outcomes we would expect if firms chose location randomly). So no, you would instead conclude that the observed agglomeration of pharmaceuticals is consistent with firms choosing location randomly. It is only when looking at the entire range of pairwise distances in the figure that we have enough contextual information to conclude that pharmaceuticals are more agglomerated than would occur by chance.

5. Zipf's law (which we will encounter again later) tells us that the n^{th} largest city in a country is $\frac{1}{n}$ times as large as the largest city.

   (a) How many times would the 32$^{\text{nd}}$ largest city need to double to be the same size as the largest city?

   According to Zipf's law, the 32$^{\text{nd}}$ largest city is $\frac{1}{32}$ the size of the largest city, so it would need to double in size five times (since $2^5 = 32$) to be the same size as the largest city.

   (b) Assume our current best estimate of agglomeration economies is $\sigma = 0.04$. How much more productive would we expect a unit of labor to be in the largest city than in the 32$^{\text{nd}}$ largest city?

   Our estimate of $\sigma = 0.04$ means that doubling city employment increases labor productivity by 4%. Moving from the 32$^{\text{nd}}$ largest city to the largest city means doubling the population five times (here we assume that this means that city employment doubled five times as well). This implies labor productivity increases by

$$(1.04)^5 - 1 \approx 1.22 - 1 = 22\%$$