

EC1340-Fall 2025

Problem Set 9 solutions

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Matt Turner

1. (a) Firm 1 solves

$$\begin{aligned} & \max py_1 - 3y_1^2 \\ \implies & 5 = 6y_1 \\ \implies & y_1^* = 5/6 \end{aligned}$$

Firm 2 solves

$$\begin{aligned} & \max py_2 - y_2^2 \\ \implies & 5 = 2y_2 \\ \implies & y_2^* = 5/2 \end{aligned}$$

Thus, total output is $5/6 + 5/2 = 10/3$

- (b) From (a) $y_1^*(p) = p/6$ and $y_2^*(p) = p/2$. If we charge a tax τ then total output is $y_1^*(p - \tau) + y_2^*(p - \tau)$. Thus, we want to choose τ to solve

$$\begin{aligned} & \frac{p - \tau}{6} + \frac{p - \tau}{2} = 1 \\ \implies & \tau = p - 3/2 \end{aligned}$$

Thus if $p = 5$, we have $\tau = 7/2$. For this value of τ , we have $y_1^* = 1/4$ and $y_2^* = 3/4$.

- (c) Total production costs when each firm produces one half unit are $c_1(1/2) + c_2(1/2) = 1$.

$$\begin{aligned} \pi_1^* &= 5 \times \frac{1}{2} - \frac{3}{4} = \frac{7}{4} \\ \pi_2^* &= 5 \times \frac{1}{2} - \frac{1}{4} = \frac{9}{4} \end{aligned}$$

- (d) Firms prefer taxes or quota according to which gives them higher profits. We just found profits under quotas. We need to find profits under taxes.

From (b), $y_1^*(p - \tau) = 1/4$ and $y_2^*(p - \tau) = 3/4$.

\implies

$$\begin{aligned} \pi_1^* &= (5 - 7/2) \times \frac{1}{4} - 3 \times (1/4)^2 = \frac{3}{16} \\ \pi_2^* &= (5 - 7/2) \times \frac{3}{4} - (3/4)^2 = \frac{9}{16}. \end{aligned}$$

Comparing these profits with those we calculated in (c) we see that firms prefer quotas.

(e) Under tradable quota, each firm chooses y_i to solve

$$\begin{aligned}
& \max y_i - c_i(y_i) + \left(\frac{1}{2} - y_i\right)p_Q \\
& \implies \\
& \max (p - p_Q)y_i - c_i(y_i) + \frac{1}{2}p_Q \\
& \implies \\
& (p - p_Q) = c'_i(y_i)
\end{aligned}$$

The third term in the first line is the firm's net from selling or buying quota. The first order condition is exactly what we got for taxes in parts (a) and (b).

Since the firms need one unit of quota per unit of output, if the market for quota clears, we have $y_1^*(p - p_Q) + y_2^*(p - p_Q) = 1$. This is exactly the expression we solved earlier to get τ , so we have $p_Q = 7/2$.

Profits are also the same as under taxes, except that firms are better off by the value of their quota endowment. This value is $p_Q \times 1/2 = 7/4$.

Thus, with a tradable quota system, firm profits are $\tilde{\pi}_1 = 3/16 + 7/4 = 31/16$ and $\tilde{\pi}_2 = 9/16 + 7/4 = 37/16$

(f) Firms will choose non-tradable quotas over tradable quotas if it is more more profitable.

Thus, the government can charge each firm any amount up to the difference in profits between the two regimes. These differences are $1/16$ for firm 2 and $3/16$ for firm 1. Thus the government can charge any amount up to $1/16$ for the $1/2$ unit of endowment of tradable quota. At any price above $1/16$, firm 2 prefers non-tradable quota.

Let $B(y) = 2y - \frac{1}{2}y^2$ where $\pi = \begin{cases} 1 & \text{with } pr = 1/2 \\ 0 & \text{with } pr = 1/2 \end{cases}$
 $C(y) = \pi y + \frac{1}{2}y^2$

$$\begin{aligned} (1) \max_y E[B(y) - C(y)] &= \max_y E[2y - \frac{1}{2}y^2 - (\pi y + \frac{1}{2}y^2)] \\ &= \max_y 2y - \frac{1}{2}y^2 - (E(\pi)y + \frac{1}{2}y^2) \\ &\quad \left\{ \text{Note: } E(\pi) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2} \right\} \\ &= \max_y 2y - \frac{1}{2}y^2 - (\frac{1}{2}y + \frac{1}{2}y^2) \\ &= \max_y \frac{3}{2}y - y^2 \end{aligned}$$

FOC: $\frac{3}{2} - 2y = 0 \Rightarrow \boxed{y^* = \frac{3}{4}}$

$$\begin{aligned} (2) \text{ Expected welfare: } E[B(y^*) - C(y^*)] \\ &= \frac{3}{2}y^* - (y^*)^2 \\ &= \frac{3}{2}\left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 \\ &= \boxed{\frac{9}{16}} \end{aligned}$$

(3) Firm's response function:

$$\max_y py - C(y) = \max_y py - (\pi y + \frac{1}{2}y^2)$$

FOC: $p - \pi - y = 0 \Rightarrow \boxed{\hat{y}(p) = p - \pi}$

p. 2

(4) Planner chooses \hat{p} to solve $\max_p E[B(\hat{y}(p)) - C(\hat{y}(p))]$

$$= \max_p E \left[2(p-z) - \frac{1}{2}(p-z)^2 - (z(p-z) + \frac{1}{2}(p-z)^2) \right]$$

$$= \max_p E \left[(2-z)(p-z) - (p-z)^2 \right]$$

$$= \max_p E \left[2p - 2z - zp + z^2 - p^2 + 2pz - z^2 \right]$$

$$= \max_p E \left[2p - 2z + pz - p^2 \right]$$

$$= \max_p 2p - 2E[z] + pE[z] - p^2$$

{recall: $E[z] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$ }

$$= \max_p 2p - 2 \cdot \frac{1}{2} + \frac{1}{2}p - p^2 \quad \leftarrow \text{--- --- --- } \textcircled{A}$$

FOC: $\frac{5}{2} - 2p = 0 \Rightarrow \boxed{\hat{p} = \frac{5}{4}}$

(5) recall: $\hat{y}(p) = p - z$

So $\hat{y}(5/4) = \begin{cases} 5/4 - 0 & \text{with } pr = 1/2 \quad (z=0) \\ 5/4 - 1 & \text{with } pr = 1/2 \quad (z=1) \end{cases}$

$$\boxed{\hat{y}(5/4) = \begin{cases} 5/4 & \text{with } pr = 1/2 \\ 1/4 & \text{with } pr = 1/2 \end{cases}}$$

(6) Expected social welfare: $E[B(\hat{y}(\hat{p})) - C(\hat{y}(\hat{p}))]$

(from \textcircled{A}) $= 2\hat{p} + \frac{1}{2}\hat{p} - 1 - \hat{p}^2 \quad \leftarrow \text{--- --- ---}$

$$= \frac{5}{2} \left(\frac{5}{4} \right) - 1 - \left(\frac{5}{4} \right)^2$$

$$= \frac{25}{8} - 1 - \frac{25}{16}$$

$$= \frac{25}{16} - 1$$

$$\boxed{= \frac{9}{16}}$$

