EC1410-Spring 2022
Problem Set 1 solutions

(Updated 9 February 2022)

Matt Turner

1. In this problem, we will work through an example of the monocentric city model. Assume we have a linear, open city. Let w=3, $\bar{l} = 1$, p_c = 1, $\overline{R}$ = 0.5, $\overline{u}$ = 0, and $A = 1$. Let $u(c) = \ln(c-1)$.

   (a) Set up the household's problem. Assume we are in a spatial equilibrium, so everyone is optimizing and no one wants to move. Call consumption in this equilibrium c^*. What is $u(Ac^*)$ equal to?
   The household's problem is
   $$\max_{x,c} u(Ac) \text{ such that } w = c + R(x)\bar{l} + 2t|x|$$
   In an open city equilibrium, $u(Ac^*) = \overline{u} = 0$ for everyone.

   (b) Find c^*.
   $$\begin{aligned} u(Ac^*) &= \ln(Ac^* - 1) = 0 \\ Ac^* - 1 &= e^0 = 1 \\ Ac^* &= 2 \\ c^* &= 2 \end{aligned}$$

   (c) Using the constraint from the household's problem, find an expression for $\overline{x}$ in terms of $w, c^*, \overline{R}, \bar{l}$ and t.
   The household's equilibrium constraint is
   $$w = c^* + R^*(x)\bar{l} + 2t|x|$$
   Plugging in constants and using $c^* = 2$, we get
   $$R^*(x) + 2t|x| = 1$$
   At the edge of the city,
   $$\begin{aligned} R^*(\overline{x}) &= \overline{R} = 0.5 \\ R^*(\overline{x}) + 2t\overline{x} &= 0.5 + 2t\overline{x} \\ 0.5 + 2t\overline{x} &= 1 \\ \overline{x} &= \frac{1}{4t} \end{aligned}$$

   (d) Use the assumption that there is one unit of land at each x to derive an expression for N^* in terms of $\overline{x}$ and $\bar{l}$.
   Given that there is one unit of land at each x, there are $2\overline{x}$ units of land total. Each household uses $\bar{l} = 1$ units of land. Therefore the population must be
   $$N^* = \frac{2\overline{x}}{\bar{l}} = 2\overline{x} = 2\frac{1}{4t} = \frac{1}{2t}$$

(e) Use the household's equilibrium budget constraint and the equilibrium extent of the city to solve for the equilibrium rent gradient, $R^*(x)$.

From part 1c, within the city limits:

$$R^*(x) + 2t|x| = 1$$
$$R^*(x) = 1 - 2t|x|$$

Outside of the city limits, $R^*(x) = \overline{R}$. In other words,

$$R^*(x) = \begin{cases} 1 - 2t|x| & \text{for } |x| \leq \overline{x} \\ \overline{R} & \text{for } |x| \geq \overline{x} \end{cases}$$

(f) Take derivatives of your expressions for $\overline{x}, N^*$, and $R^*(x)$ with respect to t. How do the city extent, population, and equilibrium rent gradient change as transportation costs increase? Provide some intuition.

$$\overline{x} = \frac{1}{4t}$$
$$\frac{\partial \overline{x}}{\partial t} = -\frac{1}{4t^2} < 0$$
$$N^* = \frac{1}{2t}$$
$$\frac{\partial N^*}{\partial t} = -\frac{1}{2t^2} < 0$$

We will focus on $R^*(x)$ inside the city. Outside the city, $R^*(x) = \overline{R}$, which does not depend on t (note, though, that because $\overline{x}$ changes when t changes, the rent at a given x may change because x goes from being inside to outside the city, or vice versa).

$$R^*(x) = 1 - 2t|x|$$
$$\frac{\partial R^*(x)}{\partial t} = -2|x| < 0$$

The city extent, population, and equilibrium rent gradient all decrease as transportation costs increase. Intuitively, it gets more costly to commute from a fixed distance away, so the maximum distance people are able to commute and still have utility $\overline{u}$ decreases when transportation costs increase. As the city shrinks in size, since there is one unit of land at each x and each household consumes $\overline{l}$ units of land, the population must decrease as well. For any given distance from the city center, commuting costs are now higher, so to sustain consumption c^* (which is still affordable at $x = 0$), rental costs must decrease.

(g) Assume that transportation costs increase from $t_0 = 1$ to $t_1 = 2$. What is the boundary of the city now? What is $R^*(0)$? Use these three points to draw a picture of how the rent gradient changes when t increases. Please label $R^*(0), \overline{R}$ and $\bar{x}$.

From 1c, we know that at the city boundary, $\overline{x} = \frac{1}{4t}$. So with $t_0 = 1, \overline{x} = 0.25$. With $t_1 = 2$, $\overline{x} = 0.125$. Therefore, our rental gradient $R^*(x)$ should now intersect with the horizontal line
$\overline{R} = 0.5$ at $x = 0.125$ and $x = -0.125$.
At the city center, from 1e, we have

$$R^*(x) = 1 - 2t|x|$$
$$R^*(0) = 1$$

(h) How would total land rent within the boundaries of the city change if we go from $t_0 = 1$ to $t_1 = 2$?
For $t_0 = 1$, the boundary of the city is $\overline{x} = 0.25$. You could manually compute the area if you drew a picture, but using an integral might be easier (otherwise, it is easy to forget that there is a rectangle of height $\overline{R}$ underneath the triangle created by the rent gradients).

$$\begin{aligned}
\text{Total land rent } &= \int_0^{0.25} 1 - 2x\,dx + \int_{-0.25}^{0} 1 + 2x\,dx \\
&= 2\int_0^{0.25} 1 - 2x\,dx \\
&= 2\left[x - x^2\right]_0^{0.25} \\
&= 2(0.25 - 0.25^2 - 0) \\
&= \frac{3}{8}
\end{aligned}$$

For $t_1 = 2$, the boundary of the city is $\overline{x} = 0.125$.

$$\begin{aligned}
\text{Total land rent } &= \int_0^{0.125} 1 - 4x\,dx + \int_{-0.125}^{0} 1 + 4x\,dx \\
&= 2\int_0^{0.125} 1 - 4x\,dx \\
&= 2(x - 2x^2]_0^{0.125} \\
&= 2(01.25 - 2 * 0.125^2 - 0) \\
&= \frac{3}{16}
\end{aligned}$$

Aggregate land rent decreases as transportation costs increase.

2. In this problem, we will analyze property taxes in the monocentric city model.

(a) Assume we have an open, linear city with property tax rate τ_0. $R_0(x)$ is the land rent in this city. Set up the household's problem (you don't need to solve it).
The household's problem is:

$$\max{}_{x,c}\, u(c) \text{ such that } w = c + (1 + \tau_0)R_0(x)\bar{l} + 2t|x|$$

(b) Assume the tax rate increases from τ_0 to τ_1, where $1+\tau_1 = (1.10)(1+\tau_0)$. Set up the household's problem with this new tax rate.

The household's problem is now:

$$\max_{x,c} u(c) \text{ such that } w = c + (1+\tau_1)R_1(x)\bar{l} + 2t|x|$$

(c) Using what you know about c^* in an open city equilibrium, solve for $R_1(x)$ in terms of $R_0(x)$. How did rental prices change when the property tax increased? How does the sum of rent and property taxes change?

In an open city equilibrium, $u(c^*) = \overline{u}$ everywhere, so $c^* = u^{-1}(\overline{u})$ in both tax regimes. That means $w - c^*$ is the same in both cases, so we can equate:

$$\begin{aligned}
(1+\tau_0)R_0(x)\bar{l} + 2t|x| &= (1+\tau_1)R_1(x)\bar{l} + 2t|x| \\
R_1(x) &= \frac{(1+\tau_0)R_0(x)\bar{l}}{(1+\tau_1)\bar{l}} \\
R_1(x) &= R_0(x)\frac{1+\tau_0}{1+\tau_1} \\
R_1(x) &= \frac{R_0(x)}{1.1}
\end{aligned}$$

Rental prices decreased by slightly less than 10% when the property tax rate increased.

The sum of rent and property taxes are:

$$\begin{aligned}
R_1(x) + \tau_1 R_1(x) &= R_1(x)(1+\tau_1) \\
R_0(x) + \tau_0 R_0(x) &= R_0(x)(1+\tau_0)
\end{aligned}$$

Combining what we have recently shown, we know that:

$$\begin{aligned}
R_1(x) &= R_0(x)\frac{1+\tau_0}{1+\tau_1} \\
R_1(x)(1+\tau_1) &= R_0(x)(1+\tau_0)
\end{aligned}$$

Therefore the sum of rent and property taxes stays the same after the tax rate increase.

(d) Suppose landlords are responsible for paying the property tax. What does this suggest about the relationship between what tenants pay and property taxes?

Because changes to the tax rate cannot change the consumption level, c^* that all households receive, the function of a property tax is to divide the land rent between the landlord and the tax collector. Changes to the tax rate cannot change the sum of what the tenant pays the landlord and what the tax collector gets. If the landlord is nominally responsible for paying the property tax, then the total rent the landlord collects (before paying the tax collector) does not vary with the tax rate.

3. In this problem, we will examine rental gradients in practice. Using Zillow or some similar real estate website, pick a radial road out from the center of Providence (for

example, along Angell Street from Kennedy Plaza) and plot the prices of at least 15 similar properties as distance to the center increases. What do you find? You can do this for another city if you would like.

Answers may vary.