Cumulative Urban Growth

and Urban Density Functions

DAVID HARRISON, JR. and JOHN F. KAIN[1]

Department of Economics, Harvard University,
Cambridge, Massachusetts 02138

INTRODUCTION

Economists have long been interested in explaining urban spatial structure, i.e., the location and density of residential and nonresidential activity in urban areas and their spatial linkages. Because of the widespread availability of information on the level of population by geographic subareas for large numbers of cities at different points in time, much of their attention has focused on population density gradients, that is, the functional relationship between population density and distance from the center of the city. Colin Clark provided the first systematic empirical analysis of these density gradients and suggested the use of the negative exponential function to describe the decline in population densities with distance from the urban center.[2] Numerous authors have attempted to provide theoretical explanations of these empirical regularities, and Edwin Mills and Richard Muth have both made some effort to use density functions to test their models of urban spatial structure.[3]

---

[1] The authors wish to thank an anonymous referee for his perceptive and most helpful evaluation of our paper. His careful reading and useful suggestions greatly improved the paper.

[2] Colin Clark, Urban population densities, *J. Roy. Statistical Soc.*, Ser. A, 114 375-386 (1951).

[3] William Alonso, *Location and Land Use* (Cambridge: Harvard University Press, 1964); Martin Beckman, "On the Distribution of Urban Rent and Residential Density," *J. Econ. Theory* 1, 60-67 (1969), pp. 60-67; Aldo Montesano, "A Restatement of Beckman's Model on the Distribution of Urban Rent and Residential Density", *J. Econ. Theory* 4, (1972), pp. 329-354; John F. Kain, "The Journey to Work as a Determinant of Residential Location", *Papers and Proceedings of the Regional Science Association*, Vol. IX (1962), pp. 137-161; Edwin S. Mills, "An Agregative Model of Resource Allocation in a Metropolitan Area", *Amer. Econ. Rev.* (1967), pp. 197-211; Edwin S. Mills, "Urban Density Functions", *Urban Studies*, Vol. 7, No. 1 (February 1970). Richard Muth, *Cities and Housing* (Chicago: The University of Chicago Press, 1969); Robert M. Solow, "Congestion, Density and the Use of Land in Transportation", *Swedish Journal of Economics* (1972), pp. 161-73; Lowdon Wingo, Jr., *Transportation and Urban Land* (Washington, D.C.: Resources for the Future, Inc., 1961).

All of these economic models of urban spatial structure explain observed differences in the intensity of land use by distance from the center by using a framework which explains the gradients as resulting from equilibrium adjustments to changes in the level of employment in the urban core, commuting costs, and family incomes. We propose an alternative model which emphasizes the durability of residential and nonresidential capital and the "disequilibrium" nature of urban growth.[4] This alternative model depicts urban growth as a layering process and urban spatial structure at any point of time as the result of a cumulative process spanning decades. Current levels of population, commuting costs, transport costs, and other factor prices determine the density of development during this period, but the density of past and future development depends on the level of these variables during those time periods.[5] In this paper we also demonstrate how this model of cumulative growth can be used to generate plausible density functions for U.S. metropolitan areas. We thereby illustrate that an alternative model, markedly different in its assumptions and implications, provides at least as good an explanation of these widely studied phenomena as traditional theories of urban spatial structure.

THE DENSITY OF RESIDENTIAL DEVELOPMENT

Our model of urban spatial structure represents urban densities at any point in time as resulting from cumulative urban development spanning decades. The density of a particular urban area at a point in time, as Eq. (1) illustrates, is then the sum of the density of development in each time period weighted by the amount of growth during each time period.

$$D_t = \frac{d_0 u_0 + d_1 u_1 + \cdots + d_t u_t}{u_0 + u_1 + \cdots + u_t} = \sum_{i=0}^{t} d_i u_i \Big/ \sum_{i=0}^{t} u_i , \qquad (1)$$

[4]This model was developed in an earlier paper. See David Harrison, Jr. and John F. Kain, "An Historical Model of Urban Form," Harvard Program on Regional and Urban Economics, Discussion Paper No. 63. November 1970.

[5]This simple cumulative model is clearly an oversimplification. The process of urban development is more appropriately described as a stock-adjustment process, with the density of current development affected both by the nature of past development and the prevailing levels of the underlying determinants. Edwin Mills has proposed and tested a stock adjustment model of this kind. See Edwin Mills, "Urban Density Functions," *Urban Studies*. Still, available data do not permit satisfactory estimates of such a stock adjustment model, and the simpler cumulative model appears to have high explanatory power and to offer more insight about the process of urban growth than the alternative long run equilibrium framework.

where

D_t average net residential density of the metropolitan area at time period t

d_i net residential density of the dwelling units added to the area in time period i (number of dwelling units added minus demolitions and conversions $\div$ change in land devoted to residential uses).

u_i number of dwelling units added to the area in time period i

Equation (1) shifts the emphasis from total residential density at one point in time to the density of each of the various increments to the housing stock added in successive time periods. The major difference between our model and the dominant equilibrium models consists of our hypothesis that current spatial structure can be explained more adequately as the aggregation of historical patterns of development rather than as an equilibrium adjustment to current conditions.[6] To explain current differences in residential density among major urban areas, we need to consider the forces which have affected the density of residential development over time.

Many factors influence the density of residential development in each time period: the price of residential land, the price of non-land factors, the preferences of consumers for relatively more land (i.e., lower residential density), the per family incomes of likely purchasers, and the transportation costs incurred in locating at various positions within the metropolitan area are among the most important. The theoretical influences of these factors on urban densities are discussed in the major equilibrium models. Equation (2) depicts a model to explain incremental net residential density.

$$d_t = F(x_t^{i}, x_t^{2}, \cdots, x_t^{n}), \tag{2}$$

where d_t is the net residential density of the increment of the housing stock added to the metropolitan area in time period t, and the x_t^{i}'s are the value of the above-mentioned forces in time period t. The combined effect of these forces on the decisions of those who produce and consume dwelling units determines the density of residential development in that time period.

[6] Although Muth does not consider historical development patterns in the theoretical model he presents in *Cities and Housing*, he does recognize the probable importance of history in the development of real cities. Therefore, he includes the proportion of dwelling units built prior to 1920 as an independent variable in two of his statistical analyses of urban density gradients. First, he includes the variable in his regression analysis of inter-urban variations in density function parameters as a taste variable to test the hypothesis that households have an aversion to living in the central city because of the age of its dwelling units (pp. 151-56). Then Muth uses the same variable to explain intra-urban variations in density in his analysis of the determinants of gross density in six selected cities. (pp. 192-95). While this second use of the proportion built prior to 1920 is in the spirit of our analysis, a single cross-section measure of age cannot reflect the complicated timing of urban growth which has characterized the physical development of large U.S. metropolitan areas. Therefore, it should come as no surprise that this part of his statistical analysis yields rather disappointing results. Neither of these *ad hoc* approaches provides a satisfactory representation of the effects of timing on urban spatial structure.

The values of these explanatory variables will change over time, and, with them, the residential density of successive increments to the housing stock. Total residential density will change gradually in response to these changes in the determinants of net residential density. The reason for the gradual nature of changes in urban form is apparent from the definition of net residential density at time t as depicted by Eq. (1). Urban form changes primarily as a result of changes in the density of incremental development; and changes in the X_t's only affect incremental density. Since incremental development is only a fraction of existing development, the impact of changes in the X_t's on urban structure from one time period to the next is relatively small.

This does not mean that there is no scope for changing the density of existing development. Urban densities can be modified by the conversion, merger or demolition of existing units. However, changes of this kind historically have had only a limited impact on urban form.[7] It appears that the stocks of residential and nonresidential structures are so durable that it is usually too costly to alter them in this way. As a result urban densities have been modified principally through new construction.

The preceding discussion suggests the principal differences in urban form among U.S. urban areas are due to differences in the timing of their development. Boston and Los Angeles are good illustrations of this principle. Critics adversely compare postwar development in Los Angeles and Phoenix with that in Boston and other northeastern cities. Yet, recent growth in Boston has been more scattered and of lower density than that in Los Angeles. What gives the contrary impression is the fact that recent low density development is

---

[7]The 1960 U.S. Census of Housing report on the Components of Inventory Change, 1950-59, reported that merged units accounted for only 1.3% and converted units 3.0% of the 1959 SMSA housing inventory. In contrast, units added by new construction during the 10-year period accounted for 28.0% of the 1959 stock. Moreover, the number of units added by conversion almost exactly cancelled out the number of units subtracted by merger, leaving the net effect of merger and conversion a relatively insignificant aspect of inventory change.

Demolition of existing units and their replacement at the same site or elsewhere by units of more suitable density have been more important ways of modifying the density of urban development, although only 3.8% of the 1950 SMSA housing inventory was reported as demolished in 1959. The effect of demolitions on incremental density has likely changed over time. In the first part of the century most demolition probably involved razing lower density structures to permit their replacement by higher density ones. Since World War II, the record of demolition has been more mixed. In some cases lower density structures have been replaced by higher density ones. On net, however, post-war demolition has probably worked to reduce average net residential densities in most urban areas, as high to medium structures in central cities were typically "replaced" by much lower density structures in the suburbs. Thus, in the decade from 1950 to 1959, only 38.0% of the units demolished in SMSA's were single family structures, compared to 80.3% single-family for the units constructed during the period.

See U.S. Bureau of the Census, *U.S. Census of Housing, 1960*, Vol. IV, Components of Inventory Change, Part 1A, No. 1, Government Printing Office, Washington, D. C., 1962.

quantitatively so much more important in Los Angeles than in Boston. In the rapidly growing Los Angeles metropolitan area (SMSA), dwelling units constructed between 1950 and 1960 accounted for almost 40% of the total number of dwelling units existing in 1960. (See Fig. 1.) In the Boston metropolitan area, buildings erected between 1950 and 1960 accounted for only 16% of the total dwelling units in the Boston SMSA in 1960. Not all of the differences in L.A.'s and Boston's spatial structure can be explained by their different time profiles, but much can.
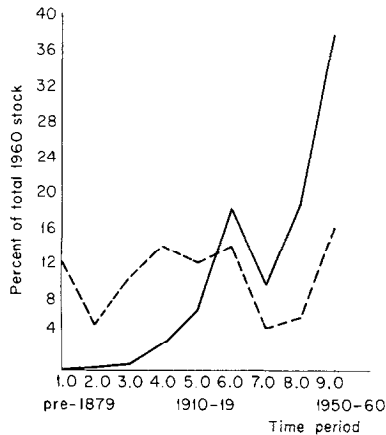

Fig. 1. Percent of the total housing stock in Boston (– – –) and Los Angeles (–), 1960 by time period in decades.

EMPIRICAL TESTS

To test our model of cummulative urban growth, we estimated a series of econometric models designed to explain incremental residential density in 83 metropolitan areas over a 90 year period. These analyses require a consistent measure of the density of urban development for a large number of urban areas and over a long period of time. Net residential density (dwelling units or population per residential area) is probably the best single measure of urban form, but the required estimates of net residential density do not exist for the large numbers of cities and long time periods required by the analysis. Therefore, we used the percentage of dwelling units built in each time period made up of single-family detached units (subsequently referred to as S^*) as a proxy for this more desirable measure of incremental density.[8]

For most of the 90-year period included in the analysis S^* is a reasonably adequate surrogate for net residential density. Until World War II most of the

[8] There are two major difficulties involved in using S^* to characterize net residential density. First, the measure fails to consider the structure-type composition of the remaining dwelling units. Two metropolitan areas might have the same value of S^* in a given time period (or a single metropolitan area might have the same percentage in two time periods), yet the residential density of the two increments might differ as a result of differences in the

variation in net residential density among areas and over time is due to differences in the fraction of units that are single family. In the decades since World War II, when S^* approaches 100 percent in many areas, differences in average lot size become more important as a determinant of net residential density. In spite of its shortcomings, S^* is used as a measure of net residential density because Bureau of the Census data are available which permit us to estimate this proxy measure for 83 metropolitan areas and as far back as 1879.

Data problems are not limited to the dependent variable; data on transport costs, per capita incomes, and other explanatory variables were even less available. Because of these data constraints, we estimated several statistical models for different subsamples and years. The simplest of these, hereafter referred to as the basic model, contains only two variables: a time trend, used to represent secular changes in transportation costs, incomes, tastes and the like; and metropolitan size at the beginning of the period (the total number of dwelling units), used to represent the differences in land costs among metropolitan areas. All the factors represented by the time trend are assumed to have a uniform effect on incremental net residential densities in all metropolitan areas.[9] Since most of these secular changes, i.e., higher incomes and improvements in transportation, have fostered a lower density of urban development, the value of S^* should increase over time to the present and the coefficient of the time trend should therefore be positive. All economic models of urban spatial structure indicate that land rents will be higher in larger cities.[10]

(Footnote 8 – *Continued*)

mix of multi-unit structures. For example, in one city the remaining units might consist primarily of apartment houses, and in another they might be primarily duplexes and three-family structures. S^* will be biased to the extent that differences in density over time and among different metropolitan areas are accounted for by differences in composition among the remaining structure types.

The second major bias results from variations in average lot size for each structure type over time and among metropolitan areas. By using S^* we assume that the amount of residential land per dwelling unit is constant over time and between metropolitan areas for single-family-detached and other structures. While lot size per unit probably does not vary much for multiple units, there is evidence that the lot size of single-family units differs significantly between different metropolitan areas at the same point in time and over time within the same metropolitan areas. S^* is therefore additionally biased to the extent that differences in density are accounted for by differences in the average lot size of single-family dwelling units.

[9] Of course, these variables vary among metropolitan areas as well; but this cross-section variation is relatively small compared to the changes in their magnitudes over time.

[10] Because most new development occurs on vacant land, the price of vacant land rather than occupied land is of primary importance in influencing the density of new development. The theoretical link between city size and land rents was developed to explain developed land rents. However, undeveloped land prices are likely to be closely correlated to developed land prices because of greater speculative demand in large cities.

Since higher land rents encourage higher density development, the coefficient of city size should be negative.

Estimates of incremental net residential density (S^*) and city size (H) were obtained from the 1940, 1950 and 1960 census of housing for 83 metropolitan areas and 13 time periods from pre-1879 to 1955-1960. The time trend variable, (T) was constructed by arbitrarity assigning the pre-1879 decade a value of 1.0, the 1880-1889 decade a value of 2.0, the first half of the decade of the 50's (1950-1954) a value of 8.5, and the second half of the decade (1955-1960) a value of 9.0.

1.0	pre-1879	6.0	1925-29
2.0	1880-89	6.5	1930-34
3.0	1890-99	7.0	1935-39
4.0	1900-09	7.5	1940-44
5.0	1910-19	8.0	1945-49
5.5	1920-24	8.5	1950-54
		9.0	1955-60

All of the values of S^* and H for periods prior to 1940 are based on a cross tabulation of structure type by year built from the 1940 census of housing.[11]

Two functional forms of this simple model were tested; the first, using the total number of dwelling units at the beginning of the period (H_{ij}), is portrayed by Eq. (3); the second, using the logarithm of H_{ij}, is summarized by Eq. (4). The t-test statistics are given in parentheses.

$$S_{ij}^* = 17.06 + 8.34\,T_i - 0.015\,H_{ij} \qquad R^2 = 0.678 \qquad (3)$$

$$(16.3) \quad (47.4) \quad (-12.1)$$

$$S_{ij}^* = 25.66 + 10.58\,T_i - 5.99\,\text{Log}\,H_{ij} \quad R^2 = 0.714 \qquad (4)$$

$$(23.8) \quad (46.4) \quad (-17.3)$$

where

S_{ij}^* $\dfrac{\text{number of single family detached units added}}{\text{total dwelling units built}} \times 100$

T_i time period
H_{ij} total dwelling units (in 1000's) at the beginning of the period
Log H log total dwelling units (in 1000's) at the beginning of the period
i time period
j urban area

---

[11] Our earlier paper presents a more detailed discussion of how these data are used in the analysis. Eleven observations were eliminated because fewer than 100 dwelling units were reported for that area in that time period. The regressions are thus based on 1068 observations.

The linear estimates, Eq. (3), indicate that the percentage of single family units (S^*) added to urban areas has increased by over 8% each decade, when the size of the area is held constant. With time period held constant, the percentage is decreased on the average by 1.5% for each increase of 100,000 dwelling units. The equation using the log of city size suggests an even more pronounced secular trend toward lower residential densities. The explained variance is 68% in the linear formulation and 71% in the logarithmic formulation. Despite their simplicity, Eqs. (3) and (4) are powerful predictors of the incremental density of metropolitan development during each time period. The hypothesis that these equations were designed to test—namely that differences in net residential density are due largely to the interaction of the timing of development and size of the metropolitan area—is strongly supported by these results.

ALTERNATIVE FORMULATIONS

While the basic model appears to be useful for predicting the density of residential development, the functional forms used are quite restrictive. They require both that those forces influencing urban form over time affect S^* equally in each decade and that the effect of city size on S^* be the same in every time period. In our earlier paper other formulations of the model were estimated in which both these restrictions were relaxed: a brief discussion of these results is given below.[12] While these alternative formulations are conceptually preferable to the basic model, both the spirit of the analysis and the empirical results are not markedly different.

The technological and socioeconomic forces represented by a time trend in the basic model have undoubtedly varied in a more complex manner than can be captured by a simple linear relationship. The trend toward lower density has probably been more rapid during some time periods than others. An estimate of the combined effects of those forces in each time period can be obtained by using time-period dummies. The form of the model is identical to that used in Eqs. (3) and (4) except that the linear trend variable is replaced by twelve separate dummy variables representing each time period.

This formulation also permits us to test some *a priori* expectations as to the relative importance of various time periods in forming the long-term secular trend toward lower density. An obvious example is concerned with the effect of the automobile on urban spatial structure. The invention and widespread adoption of the automobile is perhaps the most significant, and certainly the most frequently discussed, change in intra-urban transportation that has occurred during the past century. Although mechanically powered vehicles

[12] Our earlier paper presents a more detailed discussion of these alternative formulations. That paper also reported the results of our efforts to use data on per capita income and per capita automobile registration to replace the time period proxy. Since these formulations are not used in this paper as data for estimating density functions, the results are not discussed here.

appeared in the United States in the early 1880's, the automobile did not become an important means of transportation until the second decade of the twentieth century. There were less than half a million cars registered in 1910; by 1919 the figure had jumped to nearly 7 million—almost a fifteenfold increase in ten years. Unless there were countervailing forces, this rapid change should be evident as a noticeable increase in the slope of the nonlinear time trend.

Measurement error in the dependent variable is the second major factor which we anticipated might influence the estimated time trend. As noted previously, the percent of all dwellings added during a period that are single-family is not a completely satisfactory measure of incremental net residential density. In particular it does not allow for differences in lot size over time and among areas. Yet several published studies and our own research indicate average lot size does differ among metropolitan areas and over time.[13] In earlier periods, changes in residential density resulted primarily from a shift from multi-family to single-family structures. However, as the percentage of single-family dwelling units approaches 100, it becomes less sensitive as a measure of change in net residential density. While the trend toward lower density residential development may still be strong, our model will show a levelling off in S^* in the later time periods.

Thus, because of the presumed influence of the introduction of the automobile and the measurement errors in S^* for recent periods, we anticipated that the use of dummy variables would reveal a nonlinear low density time trend, having at least two major inflection points. The slope should increase in the 1910-1919 period and should later show a tapering off, probably by the post-World War II period.

Equations (5) and (6) illustrate the results for linear and semilogarithmic specifications using time dummies.

$$S_{ij}^* = 41.8 - 6.0 \log H_{ij} + 2.6\, d_2 + 10.2\, d_3 + 20.9\, d_4$$

$$+ 34.0\, d_5 + 46.3\, d_{5.5} + 51.2\, d_{6.0} + 59.2\, d_{6.5}$$

$$+ 65.1\, d_{7.0} + 56.4\, d_{7.5} + 68.8\, d_8 + 73.8\, d_{8.5}$$

$$+ 75.4\, d_{9.0} \qquad R^2 = .755 \qquad\qquad (5)$$

(t ratios of all coefficients but $d_2 > 5.0$; for d_2, $t = 1.4$.)

---

[13]For example, Sherman Maisel determined that in San Francisco-Oakland Bay Area average lot size has tended to increase in the postwar period, and he suggested that this trend might accelerate in the future. Sherman Maisel, "Background Information on Costs of Land for Single Family Housing", in Appendix to the *Report on Housing in California* (San Francisco, April 1963), pp. 221-82.

$$S_{ij}{}^* = 34.8 - 0.016\,H_{ij} - 2.0\,d_2 + 1.1\,d_3 + 8.0\,d_4$$
$$+ 18.7\,d_5 + 29.9\,d_{5.5} + 34.1\,d_{6.0} + 41.9\,d_{6.5}$$
$$+ 47.5\,d_{7.0} + 38.4\,d_{7.5} + 50.3\,d_8 + 54.8\,d_{8.5}$$
$$+ 56.1\,d_{9.0} \qquad R^2 = .726. \tag{6}$$

(t ratios of all coefficients exceed 4.0 except for those for d_2 and d_3; for d_2 and d_3, $t < 1.5$.)

The two inflection points in the 1910-1920 period and after 1945 are evident in Fig. 2, which compares the linear trend from Eq. (4) with the nonlinear trend
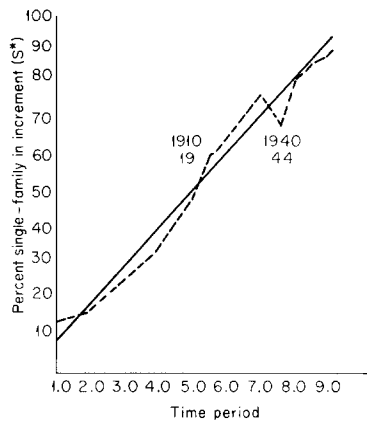
Fig. 2. A comparison of the two formulations: time trends for areas of 100,000 dwelling units basic model trend (—); dummy formulation trend (— — —).

obtained from Eq. (5). Both graphs are drawn for areas of 100,000 dwelling units.[14] While these inflection points are not necessarily due to the introduction of the automobile and the growing inadequacy of the dependent variable, the graphical results of the dummy variable formulation are very suggestive.[15]

While the dummy variable specification permits the effect of time-related forces to vary, it still requires that the effect of city size on S^* be the same in every time period. The logical next step in the analysis is to relax this

[14] The choice of 100,000 dwelling units is completely arbitrary. Only the intercepts of the graphs are affected by the level of the city size variable.

[15] In addition to the inflection point corresponding to the introduction of the automobile and the postwar topping out, Fig. 2 illustrates a number of other historical phenomena that are discussed in our earlier paper. For example, the effects of controls during World War II are clearly evident from the sharp decline in S^* during 1940-1944.

assumption and allow city size (H_{ij}) to affect S^* differently from one time period to another. This is done by estimating cross-section regressions. The form of these regressions is given by Eqs. (7) and (8); the empirical results for each time period are summarized in Table 1.

TABLE 1
Cross Section Models of Net Residential Density

Time Period	Constant	Log H	R^2		Constant	H	R^2
Pre-1879	39.2	−3.84	0.24		37.1	−0.22	0.15
1880-89	40.7	−4.15	0.21		35.1	−0.13	0.17
1890-99	49.0	−4.92	0.21		38.0	−0.070	0.17
1900-09	66.4	−7.08	0.27		44.8	−0.045	0.17
1910-19	84.6	−8.21	0.24		55.3	−0.034	0.16
1920-24	98.9	−8.58	0.28		66.2	−0.027	0.18
1925-29	110.3	−9.95	0.35		70.3	−0.024	0.24
1930-34	109.5	−7.90	0.35		77.4	−0.020	0.28
1935-39	105.5	−5.72	0.22		82.0	−0.014	0.20
1940-44	97.8	−5.93	0.12		72.9	−0.014	0.11
1945-49	114.3	−6.78	0.27		84.8	−0.014	0.27
1950-54	111.8	−5.26	0.32		88.0	−0.0096	0.31
1955-60	115.4	−5.66	0.40		89.2	−0.010	0.46
Overall R^2			0.74				0.76
Explained by Stratification			0.68				0.68

$$S_j^* = A + A^1 H_j \quad \text{for each time period} \tag{7}$$

$$S_j^* = B + B^1 \log H_j \quad \text{for each time period.} \tag{8}$$

The intercept values of both equations clearly show the strong secular trend toward lower densities. The only departures are 1940-1944 in both models and 1950-1954 in the log H model. In the log model, the values of the coefficients of log H appear to fall into three time-period groups. In the three periods before 1900, the coefficient of log H varied between 4 and 5; in the period 1900-1935 the values ranged between 7 and 10, and in the most recent period 1935-1960 they again fell into the narrow range of 6-7.

The much lower R^2's obtained for the individual cross-section regressions shown in Table 1 indicate that most of the explanatory power of the pooled regression is provided by the trend variables. Equations (7) and (8) are not, however, poorer representations of the determinants of net residential density. The appropriate comparison with the pooled regressions is not the R^2's for the individual equations, but the R^2's of all the equations considered together,

including the variance explained by stratifications. The overall explained variance of Eq. (7) is 74%, and for Eq. (8) it is 76%. Of this total explained variance, 68% in both equations is accounted for by the time-period stratification. Again this illustrates that the time-series variance of S^* is much larger than the cross-section variance.

INCREMENTAL DENSITY AND URBAN DENSITY FUNCTIONS

The incremental density equations provide good explanations of the average density of urban development in United States metropolitan areas. If it is assumed that urban areas grow at their periphery, these equations can be used to obtain urban density gradients.[16] A theoretical rationale for peripheral growth is provided by the importance of the central business district as the major employment center, the desire by workers to minimize commuting costs, and the external economies of urban agglomeration for most firms.

Real world exceptions to the assumption of uniform peripheral growth are commonplace, particularly in recent decades when the boundaries of many metropolitan regions have expanded to encompass previously independent cities. In addition, growth may be greater in some directions than in others because of the presence of large secondary employment centers, topographical and water barriers, and the like. Still, a simple model of urban growth and development that assumes regular peripheral growth is of considerable theoretical and empirical interest.

To estimate a gross density gradient from our incremental density model, we need only: (1) to determine the size of each metropolitan area during each time period, (2) to estimate incremental density, S^*, during each time period using one of the incremental density equations, i.e., Eqs. (3)-(8); (3) to convert the estimated percent single family, S^*, to an estimate of gross population density, GD, the measure used in most previous analyses; (4) to estimate the number of square miles of new development in each area during each time period, and (5) to specify the number of degrees, i.e., radians, around the urban center that are available for development in each urban area. These data allow us to estimate the average gross density of development for each period and the width of the band of development during each period.

The figures shown in Table 2 illustrate these computations for Chicago. The values of S^* in Table 2 are obtained by substituting the time period and total dwelling units in Eq. 5 (the dummy variable version). To estimate incremental gross population density, GD, from S^*, we had to develop a conversion formula. This formula, Eq. (9), was obtained by regressing gross population density on the percent of dwelling units that were single family detached for 40 large cities in 1960. Estimates obtained from 1950 data yielded virtually identical results.

[16] This "concentric circle" model of urban growth is hardly novel, having been suggested by Burgess many years ago. Robert E. Park and Ernest W. Burgess (eds.), *The City* (Chicago: University of Chicago Press, 1925).

TABLE 2
Illustrative Computations of Density Gradients: Chicago

Period	Total DU's	S^*	Incremental Gross Density (GD)	Area of Development (M)	Distance midpt. (r)
Pre-1879	31,002	21.2	15,505	6.8	1.0
1880-89	108,968	16.2	16,494	16.1	2.9
1890-99	284,434	18.1	16,123	37.0	4.9
1900-09	507,712	25.3	14,676	51.7	7.1
1910-19	760,001	36.0	12,514	68.5	9.3
1920-24	923,026	47.1	10,281	53.9	11.2
1925-29	1,177,005	50.5	9,584	90.1	12.9
1930-34	1,208,880	58.4	7,996	13.6	14.1
1935-39	1,239,479	64.1	6,850	15.2	14.4
1940-44	1,302,386	55.2	8,653	24.7	14.8
1945-49	1,404,885	67.0	6,260	55.7	15.6
1950-54	1,597,998	71.3	5,403	121.5	17.2
1955-60	1,846,028	72.0	5,252	160.6	19.5

$$GD = 19,740 - 201 \, SF \qquad R^2 = 0.59 \qquad (9)$$

$$(12.1) \quad (7.4)$$

where

　　GD　gross density (persons/sq. mile)
　　SF　percentage single family detached.

Square miles of new development in each time period (M_i) is equal to population growth during the period, (P_i), divided by the estimate of incremental gross density.

$$M_i = P_i/GD_i . \qquad (10)$$

Incremental population growth is obtained by multiplying the increase in total dwelling units during the period by 3.4, an estimate of the average population per dwelling unit.

Estimates of the distance midpoint, the last column in Table 2, are derived from Eqs. (11)-(13). The total area of development at time period t, is, of course, the sum of the square miles of new development from period 1 to period t, i.e., Eq. (11).

$$A_t = \sum_{i=1}^{t} M_i . \qquad (11)$$

The radius of development up to time period t, b_t Eq. (12), may be computed

from the formula for the area of a circle and the estimated number of degrees available for development, θ.

$$b_t = (A_t 360/\pi\theta)^{1/2}. \tag{12}$$

Estimates of θ were made by inspection of maps of each urban area. For Chicago, θ equals $190°$. The midpoint radius of development (r_t) is then easily calculated from the boundaries of development in time periods t and $t-1$.

$$r_t = b_{t-1} + (b_t - b_{t-1})/2. \tag{13}$$

Gross population density functions for 1960 then are obtained by regressing the estimated incremental gross density on the estimated midpoint distance for all thirteen time periods. A scatter diagram of these for Chicago is shown in Fig. 3. In contrast to the theoretical models developed by Muth and Mills, our formulation implies no particular functional form for the relationship between distance and gross density. But to make our estimated density functions comparable to those reported by earlier authors, we use the negative exponential form of the density function shown by Eq. (14),

$$GD(d) = D_0 e^{-D_1 d}, \tag{14}$$

where $GD(d)$ = gross population density at distance d, and D_0 and D_1 are the function parameters which are to be estimated.[17] For Chicago, these
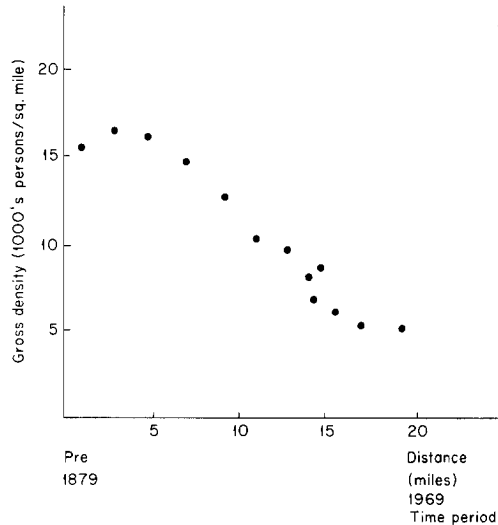

Fig. 3. Scatter diagram of estimated gross density by distance and time period: Chicago, pre 1879-1960.

[17] The parameters were estimated by ordinary least squares using the natural log of gross density as the dependent variable, following Muth's estimation procedure.

parameters, computed from the data shown in Figure 3 and Table 2, are 20.9 and 0.070.

The procedures described above were used to estimate density gradients for all 47 urban areas included in the analysis. Values of D_0 and D_1 in 1960 for several of these urban areas are shown in Table 3. Estimates for all 47 cities are shown in Appendix Table A1. Tables 3 and A1 include four density functions for each metropolitan area in 1960: Functions based on estimates of S^* obtained from the dummy variable model [Eq. (5)], the basic model [Eq. (4)], the cross section model (Table 1), and actual values of S^*.

The parameters of the density functions for a particular urban area are not very sensitive to which equation specification is used to predict residential density, but large differences in the estimates of D_0 and D are obtained among urban areas. Intercept estimates based on values of S_t^* and r_t obtained from the dummy variable data range from a low of 10,000 persons per square mile in Wichita to a high of 27,400 persons per square mile in Albany. The slope coefficients range from a low of 0.06 in Los Angeles to a high of 0.79 in Utica. If these density functions are interpreted as measures of urban form, our model clearly predicts very different urban structures among urban areas in 1960 and differences, moreover, that seem to conform rather closely to generally perceived differences among areas.

TABLE 3

Estimated Density Functions for Selected Urban Areas, 1960[a]

$$GD = D_0 e^{-D_1 r}$$

City	Dummy Variables		Basic Model		Cross Section		Actual S^*	
	D_0	D_1	D_0	D_1	D_0	D_1	D_0	D_1
Albany	25	.47	30	.51	27	.47	31	.53
Birmingham	11	.17	13	.22	12	.18	11	.15
Cleveland	20	.15	20	.16	20	.15	22	.18
Dallas	11	.17	11	.18	11	.18	14	.22
Denver	14	.25	15	.27	15	.26	13	.23
Hartford	21	.42	22	.45	18	.42	27	.52
Los Angeles	14	.06	14	.06	13	.06	11	.05
Milwaukee	19	.19	20	.20	20	.19	18	.16
Seattle	12	.14	13	.14	13	.14	12	.14
Utica	27	.79	30	.85	26	.78	26	.72
Wichita	10	.27	12	.41	11	.37	12	.38

[a]Estimated by $\log GD = \log D_0 - D_1 r + e$.

COMPARISONS WITH OTHER ESTIMATES

Muth has published estimates of density functions for 46 United States cities in 1950, computed from samples of 25 census tracts in each city. Some of Muth's estimates have been updated to 1960 by James Barr. Mills has published estimates for 18 United States cities for a number of postwar years, computed using an ingenious, if somewhat heroic estimating technique. In Table 4, we compare our estimates of density functions (calculated from the dummy variable model) with those obtained by Muth and Mills for 11 urban areas that appear in all three studies. The estimates obtained by Barr for 6 of these urban areas are also reported.[18]

From Table 4 it is evident that our estimates of the parameters of the density functions for these 11 cities are of the same order of magnitude as those obtained by Muth, Mills, and Barr. Indeed, the similarities for a number of cities are striking. At the same time, there are noticeable differences in the mean values. Our mean estimates of the intercept for common samples of cities are considerably smaller than those obtained by Mills, Muth or Barr. Our estimates of the gradient parameter are also generally smaller than those obtained by Muth and Mills, although the mean value of our estimates is identical to Barr's for the sample of 22 common cities. But not too much should be made of these aggregate sample comparisons. Differences in mean values may mask significant relationships between our estimates and those of Mills, Muth and Barr.

In Eqs. (15a)-(17b) we perform a more direct comparison of our estimates of the intercepts and gradients of the density functions for these 11 cities with those obtained by Mills, Muth and Barr. In Eq. (15a) we regress our estimate of D_0 on Mills' estimate of D_0 and in Eq. (15b) we regress our estimate of D_1 on Mills' estimate of D_1. Similarly, Eqs. (16a) and (16b) and (17a) and (17b) present regressions of our estimates of D_0 and D_1 on those obtained by Muth and Barr. The t-ratios for the individual regression coefficients are shown in parentheses.

H-K on Mills (11 cities)

$$D_0^{Mills} = 5.5 + 1.16 D_0^{H\text{-}K} \qquad R^2 = .43 \qquad (15a)$$

$$(.7) \quad (2.6)$$

---

[18] These other density function estimates were obtained from the following sources: Muth, *op. cit.*, p. 142; Mills, *op. cit.*, p. 10; and James Barr, "Transportation Costs, Rent, and Intraurban Location," Working Paper, Washington University, St. Louis, November 10, 1970, pp. 44-5. We wish to thank James Barr for making his unpublished estimates available to us.

None of these three other estimates of metropolitan density functions can be considered "correct" in any final sense. Muth's and Barr's estimates are based on samples of 25 central city census tracts. The technique used by Mills to estimate density functions is ingenious, but it is based on two sample points for each city.

$$D_1^{\text{Mills}} = .093 + 1.23D_1^{H\text{-}K} \qquad R^2 = .65 \tag{15b}$$

$$(1.1) \quad (4.1)$$

H-K on Muth (11 cities)

$$D_0^{\text{Muth}} = -8.1 + 2.09D_0^{H\text{-}K} \qquad R^2 = .27 \tag{16a}$$

$$(.4) \quad (1.8)$$

$$D_1^{\text{Muth}} = .292 + .263D_1^{H\text{-}K} \qquad R^2 = .03 \tag{16b}$$

$$(2.0) \quad (.5)$$

TABLE 4
Comparison of Muth, Barr, Mills and Harrison-Kain
Estimated Density Functions

City	D_0				D_1			
	Muth (1950)	Barr (1960)	Mills (1958)	H-K (1960)	Muth (1950)	Barr (1960)	Mills (1958)	H-K (1960)
Baltimore	69	46	37	23	.52	.39	.36	.22
Columbus	10	7	35	16	.19	.11	.58	.34
Denver	17		20	14	.33		.38	.25
Houston	14		12	11	.28		.24	.14
Milwaukee	61	48	38	19	.44	.38	.32	.19
Pittsburgh	17		22	23	.09		.24	.18
Rochester	43	31	24	23	.64	.41	.47	.45
Sacramento	15		17	11	.36		.48	.29
San Diego	18	11	13	11	.39	.15	.21	.14
Toledo	6	4	32	20	.20	.14	.67	.43
Wichita	19		20	10	.53		.63	.27
Mean	26	24	24	17	.36	.26	.42	.26
S.D.	20	18	9	5	.16	.13	.14	.11

	Muth (32 cities) H-K				Mills (12 cities) H-K			
	D_0	D_1	D_0	D_1	D_0	D_1	D_0	D_1
Mean	25	.39	16	.28	22	.37	16	.26
S.D.	17	.27	5	.16	7	.13	5	.10

	Barr (22 cities) H-K			
	D_0	D_1	D_0	D_1
Mean	20	-.29	17	-.29
S.D.	15	.25	5	.18

H-K on Barr (6 cities)

$$D_0^{Barr} = -20.0 + 2.39 D_0^{H\text{-}K} \qquad R^2 = .31 \qquad\qquad (17a)$$

$$(.6) \quad (1.3)$$

$$D_1^{Barr} = .281 - .059 D_1^{H\text{-}K} \qquad R^2 = .02 \qquad\qquad (17b)$$

$$(1.7) \quad (.2)$$

Muth on Mills (11 cities)

$$D_0^{Mills} = 18.2 + .242 D_0^{Muth} \qquad R^2 = .30 \qquad\qquad (18a)$$

$$(4.4) \quad (2.0)$$

$$D_1^{Mills} = .386 + .083 D_1^{Muth} \qquad R^2 = .01. \qquad\qquad (18b)$$

$$(3.0) \quad (.3)$$

These results indicate that, for these 11 cities at least, our density function parameters are quite closely related to those obtained by Mills, but much less closely related to those obtained by Muth and Barr. Indeed, our density gradient estimates are essentially uncorrelated with those reported by Muth and Barr for these cities. It should be added, however, that the density gradients obtained by Mills for these same 11 cities are even less closely related to Muth's estimates, as revealed by Eq. (18b).

If we limit the analysis to two-way comparisons, larger numbers of observations are available for a comparison between our estimates and those obtained by Muth for 1950 and by Barr for 1960. Because only 12 cities are included in both our sample and that used by Mills, a two-way comparison of these cities adds little. Equations (19a) and (19b) present simple regressions of our intercept and slope coefficients on Muth's estimates for a sample of 32 urban areas. Similar results are reported for the 22 cities common to ours and Barr's sample in Eqs. (20a) and (20b). In Eqs. (21a) and (21b) we report regressions of our intercept and slope estimates on Muth's 1950 estimates for this 22 cities sample.

H-K on Muth (32 cities)

$$D_0^{Muth} = 14.4 + .677 D_0^{H\text{-}K} \qquad R^2 = .04 \qquad\qquad (19a)$$

$$(1.3) \quad (1.0)$$

$$D_1^{Muth} = .033 + 1.26 D_1^{H\text{-}K} \qquad R^2 = .53 \qquad\qquad (19b)$$

$$(.5) \quad (5.8)$$

H-K on Barr (22 cities)

$$D_0^{\text{Barr}} = -13.6 + 1.98 D_0^{H\text{-}K} \qquad R^2 = .41 \qquad (20a)$$

$$(1.5) \quad (3.8)$$

$$D_1^{\text{Barr}} = .018 + 1.26 D_1^{H\text{-}K} \qquad R^2 = .60 \qquad (20b)$$

$$(.3) \quad (5.4)$$

H-K on Muth (22 cities)

$$D_0^{\text{Muth}} = -15.8 + 2.68 D_0^{H\text{-}K} \qquad R^2 = .47 \qquad (21a)$$

$$(1.4) \quad (4.2)$$

$$D_1^{\text{Muth}} = -.050 + 1.32 D_1^{H\text{-}K} \qquad R^2 = .63. \qquad (21b)$$

$$(.6) \quad (5.8)$$

Use of the larger sample yields much more favorable results, particularly for the 22 cities in the Barr sample. Our estimates of both the intercept and the gradient are very highly correlated with those obtained by Barr in 1960 and Muth in 1950 for these 22 cities. Our gradient estimates are also highly correlated with those estimated by Muth for the 32 city sample, although the intercepts are not closely related.

The analyses summarized by Eqs. (16a)-(21b) reveal a strong relationship between our estimates and those reported by Mills, Muth and Barr. While substantial differences exist between our estimates and theirs for individual cities, much of the difference appears to be systematic. These differences may reflect differences in method, definitions, or data sources. For example, our estimates of gross population density in each ring of new development are partially based upon estimates of gross population density for the entire city [see Eq. (9)]. These estimates include both residential and nonresidential land. Muth and Barr, by contrast, use only predominately residential tracts to calculate their density function estimates. This difference in method presumably would cause our estimates of D_0 and D_1 to be systematically lower than those obtained by the Muth procedure. It is possible that the lower means of D_0 and D_1 we obtain as compared to Mills, Muth and Barr are attributable to systematic measurement differences of this kind. When regression equation results are used to adjust our density function for these systematic differences, our estimates are quite similar to those reported by Mills, Muth and Barr.

CHANGE IN DENSITY FUNCTIONS OVER TIME

Muth's long run comparative static approach to urban spatial structure does not lend itself to considering changes in urban spatial structure over time in a very realistic way. His formulation implicitly assumes that urban spatial

structure will adjust instantaneously to changes in the values of the forces he indentifies as the principal determinants of residential density gradients. Mill's stock-adjustment model also makes the gross density function depend on current conditions in each urban area, but it adds the caveat that the adjustments "required" by changes in the underlying forces only occur after a time lag.

To estimate the parameters of his stock adjustment model, Mills computes estimates of population density functions in 18 U.S. metropolitan areas for four post World War II years and for six urban areas as far back as 1910. Our model can also be used to compute density functions at different points in time. Comparisons of the density functions obtained by our model with the density functions estimated by Mills are shown in Table 5 for five of the six urban areas for which Mills' estimates go back to 1910. The two sets of estimates are quite similar for the later periods, but quite different for the early periods. Our estimates and those computed by Mills imply quite different spatial structures for these five urban areas in early time periods.

The differences in urban spatial structure predicted by Mills and by us for early periods can be illustrated by Denver. Figure 4 presents a graphical representation of the density function obtained from our model for Denver in 1910. The parameters of this density function are $D_0 = 14,500$ and $D_1 =$

TABLE 5
Historical Changes in Density Functions, Mills and Harrison-Kain
Models: 1910, 1920, 1930, 1950, 1960

|	1910		1920		1930		1950		1960	
	D_0	D_1	D_0	D_1	D_0	D_1	D_0	D_1	D_0	D_1
Baltimore										
Mills	111	.97	80	.75	68	.64	51	.48	37	.36
H-K	18	.08	20	.14	23	.21	25	.25	23	.22
Denver										
Mills	28	.87	35	.87	36	.83	28	.59	20	.38
H-K	13	.01	14	.11	15	.24	17	.32	14	.25
Milwaukee										
Mills	109	.88	114	.81	74	.56	58	.47	39	.32
H-K	15	.04	16	.08	17	.14	19	.17	19	.19
Rochester										
Mills	82	1.44	96	1.37	58	.96	40	.73	24	.47
H-K	16	.12	17	.21	19	.32	23	.45	23	.45
Toledo										
Mills	41	1.13	86	1.43	56	1.01	41	.83	32	.67
H-K	14	.08	15	.17	17	.30	19	.41	20	.43
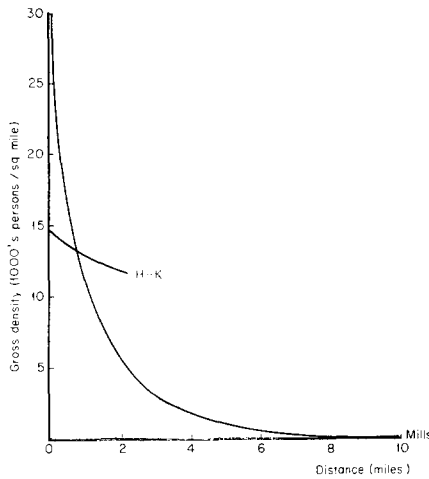
Fig. 4. Estimated density functions for Denver in 1910, Harrison-Kain and Mills.

—.103.[19] Our model suggests that Denver's early growth was low-to-moderate density and that the boundary of major development in 1910 extended to slightly more than 2 miles from the city center. Some development no doubt existed beyond the 2 mile boundary in 1910, either in scattered suburban units or in small mining or agricultural communities largely independent of, or even in competition with, Denver; but our model implies that this fringe development was quantitatively very unimportant.

Mills' estimated density function for 1910 is also graphed on Fig. 4. Mills' estimates are obtained using information on the distance of the Denver city boundary from the center of the city, the population of Denver central city, and the population of the total Denver urbanized area. These data for each decade from 1910 to 1960 are shown in Table 6.[20] The parameter estimates are derived by substituting the city and area population estimates into formulas obtained by integrating a negative exponential function to k, the city boundary, and infinity, the boundary of the urban area.

The density function Mills computes for Denver in 1910 is very different from the one we obtain using our model of cumulative urban growth. Our model suggests the density function in 1910 is discontinuous, consisting of 2 segments. One segment describes a relatively homogeneous, low-to-moderate density developed area extending about 2 miles from the center of the city. Beyond this

[19]The parameters of this function were estimated omitting the pre-1880 period observation, and thus are based on 3 observations. The pre-1879 observation is somewhat suspect in young cities such as Denver which had relatively little development by 1880. Enclusion of this earliest observation has almost no effect on the 1960 estimates (the estimated gradient only changes from —.25 to —.26). But the 1910 function is probably more accurately estimated with the early observation omitted.

[20]We wish to thank Ed Mills for providing us with the data he used to estimate his Denver density functions.

TABLE 6
Denver Development (1910-1960) Mills Estimates[a]

Year	CC boundary (mi)	CC pop (1000)	Met pop (1000)	Sub pop (1000)	Percent Sub	Density function Estimates	
						D_0	D_1
1910	4.29	213	239	36	15	28.9	−.87
1920	4.29	255	287	32	11	34.9	−.87
1930	4.29	287	330	43	13	36.3	−.83
1940	4.29	454	541	87	16	35.3	−.76
1950	4.61	416	564	158	28	29.2	−.57
1960	4.61	456	929	473	51	19.2	−.36

[a]Some of the population figures used by Mills appear to be inaccurate. The largest discrepancy is the Denver city population for 1940, which is reported as 322,000 in the 1940 Census. The other major discrepancy is for 1960, where the 1960 Census reports a Denver city population of 494,000. The other city figures correspond closely to the census figures. Since the metropolitan area boundary is not clearly defined, it is hard to determine the accuracy of the metropolitan population estimates. The 1940 figure appears suspect however. Whether these uncertainties in the basic data significantly affect Mills' density function estimates is difficult to determine. In our discussion of Mills' results we refer to the data used by Mills rather than the corrected figures.

developed area the model suggests a very low density fringe, which we portray as having zero density in Fig. 4. Since the city boundary of Denver in 1910 is about 4 miles from the center, our predictions imply a great deal of vacant land within the city limits. Mills, in contrast, describes Denver's development in 1910 in terms of a continuous density function which has a much higher value at the center and which declines regularly to a zero value at infinity. As shown in Fig. 4, Mills predicts an extremely dramatic decline in density with distance for Denver in 1910.

Both of these characterizations of Denver's spatial structure in 1910 cannot be correct. Unfortunately no solid evidence is available to determine which view is more accurate. Data were not reported for small geographic areas, such as census tracts, in 1910, so a procedure such as that used by Muth for 1950 cannot be used to derive an independent estimate. Some rather impressionistic early information is available for Denver, however, which seems to support our characterization of Denver in 1910. For example, a map prepared by Clason Map Company in 1912 shows the boundary of developed area well within the Denver city limits.[21] Moreover, photographs of Denver around 1910 depict the residential areas surrounding the city center as relatively homogeneous low-to-moderate density areas.[22]

[21] See Denver Municipal Facts, Vol. IV, No. 9, March 2, 1912, p. 8-9.
[22] See East Denver Park District, Information Concerning the Issue of Bonds of the City and County of Denver, 1912.

    While the density functions we obtain from our model differ markedly from those computed by Mills in early periods, the estimates of Denver urban spatial structure become quite similar in recent decades. Our estimates of the nature of Denver's development in the 13 time periods from pre-1880 to 1960 as well as our estimates of the cumulative population and boundary of development at the end of each period are shown in Table 7. The growing similarity over time of the two estimates is consistent with our characterization of the growth process, although the trend certainly does not constitute proof of the accuracy of our early estimates. As Denver urban development expanded in the decades after 1910, the discrepancy diminishes between our discontinuous density function estimates and Mills' continuous estimate.

    In the later time periods, when our model projects Denver's new development as low density development extending well beyond the city limits, the difference between our discontinuous estimate and Mills' continuous estimate virtually disappears. The 1960 density function estimate obtained from our model of cumulative urban growth and the estimate obtained by Mills are graphed in Fig. 5. While Mills' estimated gradient for 1910 was almost 9 times ours, the Mills estimate for 1960 is only 50% larger. Muth's estimates for 1950 are midway between our two estimates for 1960.[23]
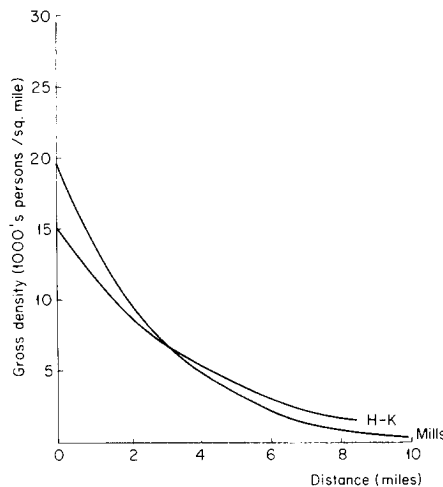

Fig. 5. Estimated density functions for Denver in 1960, Harrison-Kain and Mills.

[23]Actually Muth's analysis for Denver seems to confirm our estimates; his 1950 estimate is virtually identical to our estimate for 1950. Unfortunately Barr did not include Denver in his analysis. Therefore, we have no 1960 estimate using the Muth technique.

In addition to the similarity of the estimated density functions in later time periods, our model and Mills' estimates suggest very similar patterns of post-war suburban growth. Included in Table 7 are estimates of the percent suburban population in the Denver SMSA as estimated by our model for each year.[24] According to our model, suburban growth in Denver was truly a post-war phenomenon. Since our model does not project the boundary of development beyond the Denver city limits until the 1945-1949 period, we predict no suburban population prior to that time. But, by 1960 we predict a suburban population that is as large as the total population of the city of Denver. It should be emphasized, however, that the low density development in Denver began long before the post-war period. The post-war period in Denver was different primarily because the added development took place in the areas beyond the largely arbitrary city boundaries.

The data used by Mills to estimate his density functions also includes information on the percent of the population living outside the city of Denver in each time period. The pattern of suburbanization revealed by his data is strikingly similar to our predictions. In the four pre-war decades, Mills estimates that between 11 and 16% of the Denver urban population lived outside the city limits. While it is difficult to determine whether these persons were legitimate suburbanites, tied to the city economy, or residents of outlying areas largely independent of the city of Denver, the decline in their numbers between 1910 and 1920 suggests that many of them may have been employed in mining or agricultural pursuits. In any event, the population located outside the city of Denver was a small fraction of the areas population until 1950, when the suburban percentage almost doubled. By 1960, the suburban percentage reported by Mills was almost identical to the percentage estimated by our model for 1960.

ELABORATIONS OF THE MODEL

While it is difficult to assess the accuracy of our estimated density gradients, it is clear that the use of additional data would enable us to improve our estimates. Within the framework of our model, additional data on two important relationships would be especially valuable. First, it would be desirable to obtain a more accurate specification of the relationship between S^* and the gross and net residential densities of new development in each period; and second, it would be useful to be able to account for changes in population density in

---

[24] This percentage suffers from several inadequacies as a measure of urban spatial structure. The measure is not very useful in comparing various urban areas since it depends so heavily on the happenstance of central city boundaries. For one city, the measure may be misleading as a reflection of differences over time because of "suburban style" areas within the central city and concentrated employment "city" areas in the suburbs. Still the percent of the total urban population living outside the central city is a useful indication of changes over time in the relative numerical importance of central city and suburban districts.

TABLE 7
Denver Development (Pre-1879-1960) Harrison-Kain Estimates

Time period	Actual S^* (%)	Est S^* (%)	Est GD (1000)	Boundary of dev (mi)	Total pop (1000)	Sub pop (1000)	Percent Sub	Density function Estimates	
								D_0	D_1
pre-1879	35	39	11.9	.4	5	0	0		
1880-89	32	32	13.4	.8	29	0	0		
1890-99	38	33	13.2	1.4	83	0	0		
1900-09	46	39	11.9	2.1	174	0	0	14.5	-.10
1910-19	61	50	9.6	2.5	233	0	0	14.6	-.11
1920-24	74	62	7.3	2.8	276	0	0		
1925-29	72	66	6.5	3.2	324	0	0	15.1	-.24
1930-34	68	73	5.0	3.4	345	0	0		
1935-39	74	79	3.9	3.8	375	0	0		
1940-44	71	69	5.8	4.1	418	0	0		
1945-49	77	80	3.6	5.0	517	50	10	17.2	-.32
1950-54	84	84	2.9	6.8	708	240	34		
1955-60	81	84	2.9	8.4	925	457	49	14.3	-.25

built-up areas. Determination of these relationships requires more historical information on the nature of urban growth in each time period. While these data are not available for all urban areas, we have collected sufficient information for Chicago to illustrate how more detailed information on these relationships could be used to improve our density function estimates.

Regressions of S^* on gross population density in 40 large central cities in 1950 and 1960 indicated that the relationship between the percent single family detached and gross population density was quite stable for the decade. Over the eight decades considered in our analysis, however, significant changes could have occurred in this relationship. At least four major influences could have caused a shift in the relationship between GD and the incremental percentage of single family units, S^*. These are: (1) changes in average lot size for various structure types; (2) changes in the mix of multi-unit structures, (3) changes in the proportion of residential to nonresidential land, and (4) changes in population per dwelling unit.

Equation (22) indicates how these various influences affect gross population density.

$$GD = \frac{27{,}878{,}400 \qquad (\beta) \qquad (\gamma)}{\alpha^1 H^1 + \alpha^2 H^2 + \cdots + \alpha^n H^n} \tag{22}$$

where

> GD gross population density (persons/square mile)
> H^i dwelling units of the ith structure type added/total dwelling units added.
> α^i residential land per dwelling unit for the ith structure type (square feet)
> 27,878,400 square feet/square mile
> β family size (population/dwelling unit)
> γ acres of residential land added/total land added

The information required to estimate gross population density from Eq. (22) is unavailable for most urban areas. But for Chicago, sufficient information can be pieced together to support some useful analysis. For early time periods the analysis must draw on largely nonquantitative studies, but they support crude estimates of changes in the structural parameters.[25] Since considerable uncertainty remains about the true value of the parameters of Eq. (22), we compute several estimates of incremental gross density for each period based

---

[25] Studies prepared by Chicago planning agencies, particularly the 1943 "Master Plan of Residential Land Use of Chicago", are the major references for our estimates. Homer Hoyt's 1933 study, *One Hundred Years of Land Values in Chicago*, was an additional reference for early development.

upon different assumptions about trends in these parameters. Estimates of Chicago's density function in 1920, 1930, 1945, 1950, and 1960, based on a variety of assumptions about the true values of these parameters, are shown in Table 8.

The first set of density functions in Table 8 assume constant values of the α's, the H's, β and γ, and are comparable to those estimated for all urban areas in 1960. The second set of density functions allow the mix of multi-family units (the H's) and average lot sizes (the α's) to change over time.[26] Persons per

TABLE 8
Historical Changes in Chicago Density Function, for
Model Elaborations, 1920, 1930, 1945, 1950 and 1960

	Intercept (D_0)				
	1920	1930	1945	1950	1960
Dummy Variable Model	17	19	20	20	21
Lot Sizes vary	28	28	36	39	38
Land Use Proportions and lot sizes vary	28	28	36	40	40
Family size and lot sizes vary	28	28	36	32	28
All above vary	28	28	36	34	29

	Slope (D_1)				
	1920	1930	1945	1950	1960
Dummy Variable Model	.03	.05	.06	.06	.07
Lot Sizes vary	.05	.05	.11	.13	.12
Land Use Proportions and lot sizes vary	.05	.05	.11	.13	.13
Family size and lot sizes vary	.05	.05	.11	.11	.10
All above vary	.05	.05	.11	.12	.11

[26]Estimates of average lot sizes in each time period were derived from data on lot sizes by structure types obtained from a number of local Chicago sources (Appendix Table A-2), while the mix of units by structure type in each time period were derived from the 1940, 1950 and 1960 censuses of Housing (Appendix Table A-3).

dwelling unit, β, and net residential land/gross urban land, γ, were set at 3.3 and 0.33 respectively for these calculations. These numbers are rough estimates based on various censuses and land use surveys.

The third set of density gradients in Table 6 assumes γ is equal to 0.33 for periods up to 1944, but declines to 0.25 for periods after 1945. This shift in γ is consistent with a more scattered pattern of urban development in the post war period. The post-war value of γ was selected rather arbitrarily to illustrate the sensitivity of the density gradients to changes in this parameter. In fact, although there is a widespread criticism about the scattered and "wasteful" character of recent development, there is little evidence that development following World War II is markedly less compact than development in earlier periods.[27]

Changes over time in average family size (population per dwelling unit) also affect gross density. Although no data are available on the average size of households occupying new units in each period, the relative constancy of family size for the urban area as a whole over the 80-year period considered in the study suggests that the size of families occupying new units has not varied a great deal over time. But even if size of families occupying new units has remained relatively consistent over time, the size of families living in the oldest structures may have changed considerably. Quantification of these changes would enable us to account for changes in population densities within built-up areas and would provide a second major means of improving our estimated density gradients.

To analyze these changes in gross population density, it is useful to distinguish between structure density (dwelling units per square mile) and population density (persons per square mile). Spatial patterns of population density are determined largely, but not entirely, by structure density. Thus far, our models of cumulative urban development assume that neither the dwelling unit nor the population densities of built-up areas change as new areas are developed. Because residential structures are so durable, the assumption of unchanging dwelling unit density is quite plausible. Of course, conversions and mergers of existing dwelling units occur as conditions change and some older structures are demolished and are replaced by new residential or nonresidential structures, and these processes alter the density of older areas somewhat. But most changes in the gross population density of built-up areas over time result from changes in the size of families who occupy old units. While old, centrally located units often housed "typical families" (i.e., 3.3 persons per dwelling unit) when they

[27]Time series land use data for the city of Chicago indicates that residential development accounted for a smaller fraction of developed land in 1900 than for land developed after 1900. (See Table A-4) Since 1923, however, the share of developed land devoted to residential purposes in Chicago has been quite constant, ranging from 32 percent in 1941 to 36 percent in 1936. The relatively low share of developed land accounted for by residential use in the early surveys reflects the large share of developed land devoted to roads and highways during these periods. Of course, these land use figures exclude vacant land and indicate nothing about the density of development in suburban areas.

were built, they have come increasingly to house young married couples, retired or childless couples, and single people. Racial discrimination insures that many old, centrally located units house relatively large, poor black families; but overall family size in built-up areas has decreased.

Unfortunately the census does not provide cross classifications of family size by the year dwelling units were built. But since the age of the housing stock of most census tracts is quite uniform, changes in census tract population provide an indication of changes in family size for existing units. Because we are concerned primarily with population declines in centrally located, built-up areas, our empirical analysis is limited to the 935 census tracts in the city of Chicago. From the 1940, 1950, and 1960 censuses, we obtained census tract statistics on total population, total dwelling units, occupied dwelling units by year built, and median persons per dwelling unit.[28] The distance from the Loop was measured to the nearest tenth of a mile. The census tract information on structures by year built allow us to consider two other important issues: (1) how valid is the assumption of regular peripheral growth used to obtain density functions from our model of incremental density?, and (2) how much change in structure densities occurs within built-up areas?

The data on the number of dwelling units built during each time period by distance from the Loop, displayed in Table 9 indicate that the assumption of regular peripheral growth corresponds reasonably well to the process of urban growth in Chicago. Since only city census tracts were used in the analysis, recent suburban growth is not represented. Most of the close in residential development in Chicago during the decade 1950-1960, is probably accounted for by urban renewal and public housing programs. Edward Hearle and John Niedercorn have shown that the urban renewal process usually produces lower densities.[29]

The analysis of changes in numbers of dwelling units was much less conclusive. We planned to use the distributions of dwelling units by year built to estimate the net changes in the housing stock by age for each census tract. These net changes would reflect the combined effect of mergers, demolitions, and other transformations of the housing stock. Unfortunately, the results obtained were generally quite implausible. They indicated an increase over time in the number of dwellings at most distances. (See Appendix Table A5.) The estimated increases between 1950 to 1960 were especially large, a result that is inconsistent with other information. It is probable that a change in the census reporting unit from a "dwelling unit" to a "housing unit" is responsible for these unanticipated results. The later definition includes single rooms without kitchen facilities while the former does not. In spite of these definitional problems, the analysis supports the general impression that transformation of the stock had only a modest effect on dwelling unit density over the period studied.

[28] In some instances data for a number of census tracts had to be combined to make the data comparable for all three census years.

[29] E.F.R. Hearle and J. H. Niedercorn, *The Impact of Urban Renewal on Land Use*, The RAND Corporation, RM-4186-RC, June 1964.

HARRISON AND KAIN

TABLE 9
Percentage of Housing Units Added in Each Time
Period by Distance from the Loop, City of Chicago[a]

Distance (mi)	Total Units in 1960 (1000's)	Year Built (percentage)					
		Pre-1920	1920-30	1930-40	1940-50	1950-60	All Years
0-1	41	68	15	2	1	14	100
1-2	156	83	8	2	1	6	100
2-3	253	78	16	2	1	3	100
3-4	264	65	28	2	1	4	100
4-5	231	36	42	6	4	12	100
5-6	138	16	37	6	12	29	100
6-7	49	25	21	7	22	25	100
7-8	41	36	17	5	20	22	100
City Total	1,173	56	26	4	4	10	100
Suburban Total	920	20	16	6	14	44	100

[a]U.S. Bureau of the Census, U.S. Census of Population: 1950 Vol. III, Census Tract Statistics, Chicago, Ill.; and U.S. Bureau of the Census, U.S. Census of Population and Housing: 1960 Census Tracts. Final Report PHC (1) – 26, Chicago, Ill. U.S. Bureau of the Census, U.S. Census of Housing: 1950 Vol. I, General Characteristics, Part 3: Idaho – Massachusetts. Table 13-20, U.S. Government Printing Office, Washington, D.C. 1953; U.S. Bureau of the Census. U.S. Census of Housing, 1960 Vol. 1, States and Small Areas. Part 3: Delaware - Indiana. Table 15-14, U.S. Government Printing Office, Washington, D.C., 1963.

The analysis of census tract data was carried out primarily to determine whether the size of families occupying existing units changed in a systematic way. In particular, we suspected that central area population and, thereby, gross population densities declined in the post war period because of a decline in the average size of families in areas of older housing. This pattern is evident from Table 10, which gives average population per dwelling unit by distance from the Loop in 1940, 1950 and 1960. The final density functions in Table 8 incorporate these changes in family size.

CONCLUSION

Regular declines in gross population density with distance from the center of urban areas have been observed and analyzed for many years. But rigorous theoretical explanations of these phenomena have been provided only recently. In general, these theoretical explanations have been obtained from long run equilibrium models which derive the form of these density functions from equilibrium values of employment locations, commuting costs, and family

TABLE 10
Average Population Per Dwelling Unit by Distance
from the Loop, City of Chicago 1940, 1950 and 1960[a]

Distance \ Year	1940	1950	1960
0-1	2.41	2.45	2.05
1-2	3.07	2.93	2.70
2-3	3.20	2.93	2.81
3-4	3.22	2.89	2.60
4-5	3.13	2.89	2.80
5-6	3.35	3.28	3.02
6-7	3.17	3.32	3.15
7-8	3.64	3.38	3.15
City of Chicago	3.6	2.9	2.6
Suburban Chicago	3.9	3.4	3.3

[a] Same as Table 9 plus, U.S. Bureau of the Census, U.S. Census of Housing: 1940 Vol. II General Characteristics Reports by States Part 2, Illinois; U.S. Bureau of the Census, Census of Housing: 1950 Nonform Housing Characteristics, Vol II, Part 3.

incomes. In this paper, we present an alternative theoretical explanation of these empirical regularities which emphasizes the disequilibrium nature of urban growth and the durability of residential and nonresidential capital.

Our model views urban growth as a cumulative process and explains current residential densities as a weighted average of the density of new development over several time periods. In this framework, current levels of explanatory variables, such as commuting costs and family incomes, determine the density of development during this period, but the character of past and future development depends on the levels of these variables during those time periods. Analyses of data for 82 urban areas indicate that a simple econometric model provides good predictions of the density of residential development in each urban area for discrete time periods between 1880 and 1960 and of variations in current residential densities among these areas.

If it is assumed that urban growth occurs in a regular way at the periphery of built up areas, this simple model can be used to compute density functions for sample cities. For the analyses presented in this paper we estimated density functions for 47 of the 82 urban areas used in our analysis of intermetropolitan variations in residential density. Estimates of both the intercepts and gradients of the density functions varied considerably among the 47 cities included in the analysis, and the differences corresponded to generally perceived differences

among cities. Edwin Mills, Richard Muth and James Barr have computed density functions using different estimation techniques. Comparison of our estimates with comparable ones obtained by Mills for 11 urban areas, by Muth for 32 urban areas and by Barr for 22 urban areas indicates a fairly strong degree of correspondence between our estimates and theirs. These results provide additional evidence of the usefulness of our approach.

The methods used to compute density functions in our analysis involve a number of unrealistic assumptions. The actual historical process of development in each urban area is certainly far more complex than this simple model implies. There are no theoretical impediments to incorporating more realistic assumptions; the constraint is empirical. No consistent data exist on changes in lot size, rates of demolitions, urban redevelopment, and population shifts for our sample metropolitan areas.

To evaluate how these omitted variables may affect our results, we collected additional historical data for Chicago and used these data to evaluate changes in Chicago's density functions under a variety of assumptions. While these more complex analyses changed the estimated intercepts and gradients for Chicago, the changes were not dramatic. More detailed historical information on development would certainly be revealing, but we doubt if these data would change our findings in any fundamental way.

APPENDIX

TABLE A-1
Estimated Density Functions, 47 S.M.S.A.'s
1960, Alternative Models

	Dummy Variable		Basic Model		Cross Section		Actual $S*$	
	D_0	D_1	D_0	D_1	D_0	D_1	D_0	D_1
Albany	27.4	.473	30.0	.510	27.2	.468	30.9	.530
Atlanta	13.0	.221	13.7	.237	13.3	.227	16.3	.274
Baltimore	22.7	.223	24.1	.234	22.9	.222	25.3	.185
Birmingham	11.4	.168	12.8	.217	12.0	.175	11.0	.149
Bridgeport	18.1	.478	20.2	.522	17.7	.472	15.4	.264
Chattanooga	13.2	.461	14.5	.516	13.2	.456	10.6	.400
Chicago	20.9	.069	21.2	.070	20.9	.069	26.4	.102
Cleveland	19.5	.150	20.1	.155	19.6	.151	21.5	.184
Columbus	16.3	.344	17.6	.375	16.3	.344	12.8	.264
Dallas	10.7	.169	11.2	.183	11.1	.177	13.5	.221
Davenport	20.6	.639	23.9	.709	20.1	.628	13.3	.486
Dayton	15.5	.349	16.4	.370	15.3	.346	13.4	.328
Denver	14.4	.254	15.4	.272	14.6	.256	13.2	.227
Des Moines	16.6	.545	18.6	.605	16.5	.940	11.9	.443
Detroit	16.1	.076	16.4	.078	16.4	.077	16.1	.108
Ft. Worth	10.1	.244	10.7	.269	10.4	.251	11.0	.314
Grand Rapids	18.3	.526	20.3	.577	18.3	.525	10.3	.399
Harrisburg	18.6	.602	20.9	.657	20.2	.600	26.6	.677
Hartford	20.5	.419	22.0	.448	18.2	.415	27.4	.519
Houston	10.6	.144	11.0	.500	10.9	.150	12.1	.192
Indianapolis	17.1	.318	18.1	.337	17.2	.319	15.4	.326
Johnstown	20.0	.756	21.5	.855	18.0	.753	16.7	.640
Knoxville	12.1	.425	13.4	.470	11.9	.421	10.6	.418
Los Angeles	13.5	.056	13.9	.058	13.0	.055	11.2	.053
Memphis	13.7	.295	14.5	.317	14.0	.300	17.7	.344
Milwaukee	19.3	.186	20.4	.199	19.5	.190	17.6	.158
Nashville	14.9	.396	16.2	.438	14.7	.393	15.4	.378
New Haven	25.3	.554	28.0	.598	24.8	.546	29.4	.624
New Orleans	20.2	.267	21.5	.282	20.2	.266	26.9	.292
Oklahoma City	10.1	.251	10.0	.243	12.4	.279	10.4	.270
Peoria	16.0	.544	18.3	.619	15.5	.532	9.5	.382
Pittsburgh	22.5	.183	23.2	.188	22.4	.182	22.8	.172
Portland	13.8	.213	20.9	.657	14.1	.218	10.4	.195
Reading	24.8	.785	28.9	.869	24.1	.769	31.9	.838
Richmond	16.2	.428	17.7	.468	15.9	.424	18.5	.470
Rochester	22.9	.447	24.8	.475	22.9	.444	15.3	.377
Sacramento	10.6	.292	11.3	.324	10.4	.290	3.1	.173
Salt Lake City	14.6	.435	15.3	.468	14.5	.433	12.8	.392
San Antonio	12.2	.261	12.9	.284	12.4	.265	12.6	.260
San Diego	10.6	.141	11.0	.151	10.7	.143	9.9	.118
Seattle	12.4	.135	13.0	.144	13.2	.144	11.8	.139
Spokane	13.2	.436	15.3	.529	14.9	.507	9.8	.364

TABLE 1 (Continued)

Estimated Density Functions, 47 S.M.S.A.'s

1960, Alternative Models

	Dummy Variable		Basic Model		Cross Section		Actual S^*	
	D_0	D_1	D_0	D_1	D_0	D_1	D_0	D_1
Syracuse	19.5	.516	21.4	.556	19.2	.510	19.0	.512
Tacoma	13.6	.381	14.9	.421	13.7	.383	9.2	.254
Toledo	20.0	.427	21.9	.479	20.0	.426	12.5	.309
Utica	26.9	.793	30.0	.847	26.1	.780	26.1	.718
Wichita	10.0	.274	11.7	.406	10.8	.369	11.9	.379

TABLE A-2
Average Lot Size (sq. ft.) by Structure Type by Time Period, Chicago[a]

		Structure — Type				
Time Period	1 Family Detached	(1 Family Attached) 2 Family	3-4 Family	5-9 Family	10-19 Family	20 or more
Pre-1879	3,125	1,562	900	445	205	125
1880-89	3,125	1,562	900	445	205	125
1890-99	3,125	1,562	900	445	205	125
1900-09	3,125	1,562	900	445	205	125
1910-19	3,750	1,875	1,070	535	250	150
1920-24	3,750	1,875	1,070	535	250	150
1925-29	3,750	1,875	1,070	535	250	150
1930-34	5,000	2,500	1,430	715	333	200
1935-39	5,625	2,815	1,600	800	375	225
1940-44	5,625	2,815	1,600	800	375	225
1945-49	6,750	3,375	1,930	965	450	270
1950-54	7,425	3,710	2,120	1,060	495	300
1955-60	8,100	4,050	2,315	1,160	540	325

[a]The following city of Chicago documents were used to estimate "typical lots" in each time period: "Master Plan of Residential Land Use of Chicago", The Chicago Plan Commission, Chicago, 1943; "Planning the Region of Chicago", The Chicago Regional Planning Association, Chicago, 1956, "The Comprehensive Plan of Chicago" Vol. 1, Analysis of City Systems: Residential Areas, Recreation and Park Land, Education, Public Safety and Health, Department of Development and Planning, Chicago, 1967. The other source was Homer Hoyt, *One Hundred Years of Land Values in Chicago*, The University of Chicago Press, Chicago, 1933.

The estimates of lot sizes for multi-family units was obtained by dividing the "typical lot" figure by the number of units in the structure type. For example the 715 sq. ft./unit figure for 5-9 units structures in the pre-1879 period was obtained by dividing 5,000 sq. ft. by 7 units.

TABLE A-3

Fraction of Units by Structure Type by Time Period, Chicago[a]

Structure – Type

Time Period	Total Units Added (in 1000's)	1 Family Detached	1 Family Attached, 2 Family	3-4 Family	5-9 Family	10-19 Family	20 + Family
Pre-1879	31	.16	.22	.32	.18	.07	.05
1880-89	78	.12	.24	.33	.19	.06	.06
1890-99	175	.15	.26	.28	.15	.07	.09
1900-09	223	.18	.28	.24	.15	.07	.08
1910-19	252	.24	.28	.19	.15	.06	.08
1920-24	163	.34	.19	.13	.11	.07	.16
1925-29	254	.33	.13	.11	.10	.08	.25
1930-34	32	.49	.10	.10	.08	.07	.16
1935-39	30	.80	.06	.04	.06	.02	.02
1940-44	63	.75	.13	.02	.05	.05	
1945-49	102	.81	.09	.04	.02	.04	
1950-54	190	.83	.06	.02	.02		.07
1955-60	240	.82	.03	.03	.05		.07

[a]U.S. Bureau of the Census, U.S. Census of Population: 1940, Characteristics by Structure Type, Illinois Table C-1 U.S. Bureau of the Census, U.S. Census of Population: 1950, Vol. II, Part 2; and U.S. Bureau of the Census, U.S. Census of Population and Housing: 1960, Vol. II, Part 2, Table 43 – A-6.

TABLE A-4
Historical Changes in Utilized Land by
Various Uses; Chicago (1850-1961)[a]

		1850	1870	1890	1923	1941	1956	1961
Utilized Land	Acres	2,458	12,669	48,008	91,514	104,289	116,736	128,832
	Percent	100	100	100	100	100	100	100
Residential	Acres	465	3,481	11,008	31,004	33,228	41,600	45,184
	Percent	19	28	23	34	32	36	35
Industrial	Acres	185	2,703	10,647	16,577	19,696	19,712	19,712
	Percent	8	21	22	18	19	17	15
Commercial	Acres	33	535	1,722	6,317	6,359	8,704	8,960
	Percent	1	4	4	7	6	7	7
Road & Highway	Acres	1,630	4,725	20,721	29,624	33,852	33,216	34,496
	Percent	66	37	43	32	32	28	27
Other Public & Semi Public	Acres	145	1,235	3,890	7,686	11,154	13,504	20,480
	Percent	6	10	8	9	11	12	16

[a]"Master Plan of Residential Land Use of Chicago", The Chicago Plan Commission, Chicago, 1943; and John M. Niedercorn and Edward F. R. Hearle, "Recent Land Use Trends in Forty-eight large American cities", Rand Memorandum RM3664-FF, Santa, Monica, Calif, April, 1963.

TABLE A-5
Dwelling units built before 1940 and between
1950-1960 in the city of Chicago. Estimates from the
1940, 1950 and 1960 census by distance from the Loop (in 1,000's)[a]

Distance	Pre-1940 Units				1940-50 Units	
	1940 census	1950 census	1960 census		1950 census	1960 census
0-1	35	35	45		0.2	0.6
1-2	141	145	132		1.7	2.5
2-3	236	243	249		2.1	3.9
3-4	252	252	272		2.6	3.8
4-5	192	193	201		9.9	10.5
5-6	81	81	84		15.7	14.7
6-7	25	26	26		10.8	9.9
7-8	23	24	24		7.9	7.3
City Total	985	999	1,033		50.9	53.2
Suburban Total	291	391	428		130.5	129.6

[a]Same as Table 7 and U.S. Bureau of the Census, U.S. Census of Housing: 1940, Housing Characteristics by Type of Structure, Illinois, Table B-1, C-1, U.S. Government Printing Office, Washington, D.C. 1945.