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Urban Development with Lags

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This paper specifies a model of urban development under uncertainty when there is a lag between the decision to begin a project and its completion. These lags have three important effects. First, the deterrent effect of uncertainty on development is smaller with lags than without them. Second, in some cases, development occurs later under uncertainty than with certainty. Third, lags may lead to leapfrog development, when distant land is developed prior to nearby land. © 1996 Academic Press, Inc.

I. INTRODUCTION

The conversion of land from agricultural to urban use is a fundamental aspect of urban growth, and it has attracted significant attention from economists. Arnott and Lewis [4] show that a landowner should develop when the value of land in urban use exceeds the value of land in the best alternate use plus the cost of conversion. Capozza and Helsley [8] show that the optimal development strategy is considerably more complicated in the presence of irreversibility and uncertainty. In this case, the developer should delay development until urban value equals the value of land in the best alternate use plus conversion cost plus an option value term. The option term is the value of the flexibility that is enjoyed by delaying development. By delaying, it is possible to develop immediately if the

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market is strong; if the market is weak, an irreversible and regrettable decision has been avoided.[1]

This paper adds an important aspect of reality to the model of urban development: lags between the moment when an irreversible decision is undertaken and when the development's first revenues are received. These lags may be quite long. For example, Wheaton [28, p. 284] reports that the lag between issuing a construction permit and the completion of an office building is between 18 and 24 months. This understates the true lag by a considerable amount, since it does not include the time spent buying and servicing land and meeting land use regulations prior to the issuance of the construction permit.

The purpose of this paper is to show that the presence of a lag has significant effects on the decision to develop. First, the deterrent effect of uncertainty on development is smaller with lags than without them. Second, in some situations, an increase in uncertainty may lead to earlier development. Therefore, even with uncertainty, it may be optimal to develop before the current net present value of the completed project is positive. Third, lags and uncertainty together can lead to so-called leapfrog development, where distant land is developed prior to the development of land that is closer to the city center.

It is difficult to make the intuition behind our results clear before presenting them formally, but it is worth trying. Suppose that urban rents have reached the point where the net present value of development is zero. The Capozza−Helsley result is that a developer will choose to wait in this situation. The benefit of waiting is enjoyed in bad states of nature; in these states, a committed developer will regret the project, while an uncommitted developer can choose not to develop. This benefit of waiting rises with uncertainty because the probability of states of nature bad enough to regret development also rises with uncertainty. The opportunity cost of waiting is the foregone profit during the delay. Without lags, this does not depend on uncertainty since it is possible to develop immediately if a good state should occur.

One might suppose that lags simply increase the uncertainty under which development occurs, but this is not so. Adding lags changes the developer's choice by making the opportunity cost of waiting depend on uncertainty. With a lag of two years, for example, the foregone returns from waiting are uncertain; they depend on the random path followed by rent over the two year period. When the developer can abandon a project, even at a cost, these returns are bounded below. Consequently, the developer's profits over this period are a convex function of rents. This

[1] Titman [24] and Clarke and Reed [9] present similar analyses of real options in models without space.

means that expected profits increase with uncertainty, and therefore that the opportunity cost of waiting rises with uncertainty. Since both the benefit and the cost of waiting rise with uncertainty, the effect of uncertainty on development is ambiguous in theory. We show in Section IV that it is possible that uncertainty may hasten development. In any case, uncertainty has a weaker effect on development than without lags because without lags uncertainty has no effect on the opportunity cost of waiting. Both the ability to abandon and lags are essential to generating this result. Without abandonment, the developer must endure the extremely bad states that a high level of uncertainty may generate. Without lags, there is no reason to develop today to be in the market tomorrow.[2]

The intuition behind our leapfrogging result is parallel. Suppose that the city's equilibrium rent function is a decreasing function of distance to the city center and that rents at the city center follow geometric Brownian motion, perhaps because income at the city center follows such a stochastic process. In this case, the rent at some positive distance from the city center does not follow the same stochastic process as rents at the city center. Geometric Brownian motion implies that rents at the center are expected to vary by some percentage each period. Since rents are lower away from the center, the percentage change in rents is greater. Therefore, the percentage variance of the stochastic process governing rents increases with distance to the city center. Since it is possible that in the presence of lags uncertainty can encourage development, this means that it is possible that distant landowners, whose rents are highly uncertain, may choose to wait.

Leapfrogging is not the only real-world anomaly that our analysis can help explain. The result that uncertainty does not necessarily delay development in the presence of lags is consistent with the overbuilding that characterizes many housing markets. The case of Houston is a good example. Smith and Tesarek [23, pp. 397–398] note that "[d]uring the three-year period between 1982 and 1984 the city added 164,000 units (16%) to the stock of housing, 50% more than could have been absorbed, even if the Houston economy had maintained its high rate of growth." They go on to observe that (p. 398) "[b]y 1985, 18.4% of the entire housing stock (single- and multi-family) within the metropolitan area was vacant, approximately 220,000 units. This was 117,000 more vacant units than would normally be expected from a city the size of Houston. Of those 117,000 excess vacancies, 85% were generated by overbuilding as opposed to net outmigration." By showing that uncertainty is less of a deterrent to

[2] The result that abandonment alone does not reverse the uncertainty delays investment result is found in the aspatial investment models of Dixit [11] and Pindyck [21], among others.

development with lags, our analysis provides part of an explanation for this phenomenon.[3]

The paper is organized as follows. Section II presents a model of urban development with irreversibility, uncertainty, and lags. Section III solves for the equilibrium development rule. Section IV carries out simulations on the effects of uncertainty on development and shows that uncertainty and lags together can lead to leapfrogging. Section V concludes. The technical details of some of our proofs are presented in Appendixes A–E.

II. A MODEL OF URBAN DEVELOPMENT WITH IRREVERSIBILITY, UNCERTAINTY, AND LAGS

A. A Simple Monocentric Model

Consider a monocentric city where locations are differentiated only by their distance z from the central business district (CBD), where all employment is located. The city is populated by identical households who derive utility from the consumption of housing, h, and other goods, x. We assume that households rent one unit of housing inelastically and that utility is linear in the amount of numeraire good consumed: $v = x$.

A household located z miles from the CBD incurs commuting costs tz and earns income y. The budget constraint of a typical household is therefore

$$y = x + tz + r(z), \tag{II.1}$$

where $r(z)$ is the rent paid for housing at location z.

In equilibrium, rents adjust so that the households earn identical levels of utility at all occupied locations. This implies that the slope of the equilibrium rent function equals $-t$. Suppose that the city is open, so households enjoy an exogenous utility level v^*. These assumptions imply a linear equilibrium rent function for housing:

$$r(z) = y - tz - v^*. \tag{II.2}$$

The equilibrium rent function for housing increases in y and decreases in v^*.[4]

[3] See Grenadier [16] for an analysis that focuses directly on the probability of overbuilding.

[4] Although the open city assumption is realistic, it is not innocuous. We use this assumption to specify the payoffs from development. This, in turn, allows us to solve for the equilibrium timing of development. If we instead considered a closed city, whose population was fixed, then equilibrium rents would depend on the timing of development. In this case, it would be necessary to solve for equilibrium rents and development rules simultaneously. This approach is taken in Wheaton's [25, 26] closed city models. Not surprisingly, these models do not consider urban development under uncertainty. Instead, they overcome the complications of the closed city specification by supposing perfect foresight or myopia.

B. Irreversible Development with Lags

We suppose that land not in urban use is used in agriculture. All land used in agriculture earns rents equal to R regardless of location. Land may be converted to housing using 1 unit of capital and 1 unit of land in fixed proportions. Assume that the price of capital is k. The marginal cost to landlords of renting out housing is w per unit of time. This development is at least partially irreversible. The only way to exit the market is to abandon at cost $l \geq 0$.[5]

The most important new assumption in our model is that there are development lags. We suppose that after deciding to develop, a period of h years elapses before the developer receives any revenue from his project. For tractability, we assume that the developer continues to receive agricultural rents during construction. The implications of this assumption are discussed further below.

C. Uncertainty

We incorporate uncertainty by supposing that the income level, y, follows geometric Brownian motion,

$$\frac{dy}{y} = \mu\, dt + \sigma\, dz, \tag{II.3}$$

where μ is the expected growth rate and dz is the increment to a standard Wiener process. With this process, percentage changes in income are distributed normally and income is distributed lognormally.[6] It would be possible to analyze uncertain systemwide utility levels in a parallel fashion.

D. The Developer's Choice Problem

A risk-neutral owner of undeveloped land chooses whether or not to develop, given the current level of income. This amount to choosing an optimal development rule of the form: develop whenever $y > y_{\mathrm{H}}$. An owner of developed land chooses whether or not to abandon, again given the current income level. This amounts to choosing an optimal abandonment rule: abandon whenever $y < y_{\mathrm{L}}$.

This model solves for the optimal development rule as a function of income. Other models (e.g., Capozza and Helsley [8]) have solved for the

[5] Assuming $l < 0$ makes development more reversible and strengthens our results. See Bar-Ilan and Strange [5] for more discussion of this point.

[6] The expected value of income t in the future is $y_0 e^{\mu t}$, where y_0 is the current level of income. The variance of a Wiener process increases in proportion to time, and the standard deviation increases in proportion to the square root of time. Thus, t in the future, the standard deviation of $\log y$ is $t^{1/2}\sigma$. See Dixit and Pindyck [12] for more on stochastic processes.

optimal development rule as a function of rents, exploiting the dependence of rents on fundamentals. Capozza and Helsley suppose that income follows arithmetic Brownian motion. This, with linear bid−rent curves, implies that rents also follow arithmetic Brownian motion, and it is possible for them to solve for an optimum development rule as a function of rents.

This is not possible in our model. Because bid−rent curves are linear, rents at all locations rise or fall by the same absolute amount in response to a movement in income. This means that the percentage variance of the stochastic process determining rent rises with distance to the CBD. Thus, it is not possible for us to suppose that rents follow geometric Brownian motion and solve for the development rule as a function of rents. It also means that the assumption that rents at all locations follow geometric Brownian motion cannot be derived from the standard monocentric model without making some additional assumptions.[7]

E. Summary

The model closest to ours is Capozza and Helsley. As noted in the Introduction, they find that uncertainty delays development. They also find that uncertainty cannot cause leapfrogging. There are three major differences between our model and theirs. First, we assume that income follows geometric Brownian motion, while they assume that income follows an arithmetic process. Second, we impose a lower bound on the developer's profits by allowing abandonment at a cost. Third, we allow for development lags. The next two sections show that these differences have important effects on the responses of developers to uncertainty.[8]

Another paper that relates to ours is Fujita [15]. He characterizes the equilibrium in a closed city when population growth follows a stochastic process bounded below by 0. Since in his model population never declines and there are no lags, rents always rise and there is no possibility of a developer regetting the decision to develop. Our work is less closely related to the early models of urban dynamics (Anas [1], Arnott [3],

[7] It is more common to suppose that uncertainty is represented by geometric rather than arithmetic Brownian motion. There are two reasons for this. First, some feel it is more reasonable to suppose that incomes change by constant percentage increments than to suppose that incomes change by constant absolute amounts. Second, with arithmetic Brownian motion, it is possible for incomes to take on negative values.

[8] Our model also has some features in common with financial papers on real options. Clarke and Reed [9] and Titman [24] focus on the relationship of uncertainty to the construction decision. These models include neither space nor lags, however. Grenadier [16] considers the relationship between lags and the probability of overbuilding, but does not consider space and does not present results on trigger prices.

Brueckner and von Rabenau [7], Fujita [14], Wheaton [25, 26], for example), where development occurs under myopia or perfect foresight. See Miyao [19] or Wheaton [27] for surveys.

III. OPTIMAL DEVELOPMENT WITH UNCERTAINTY AND LAGS

The solution for the optimal development policy is a four stage process. First, we solve for functions giving the value of urban and agricultural land and land in conversion as functions of the current level of income. Second, we employ these value functions to solve for levels of income that trigger development and abandonment. Third, having solved for the triggers, we present analytical results on the optimal development and abandonment rules. Fourth, we employ simulations to evaluate the effects of lags on development. The first three stages are covered in this section. The simulations are presented in Section IV.

A. The Value of Agricultural Land

Let $V_a(y)$ denote the value of agricultural land when income is y. $V_a(y)$ satisfies the ordinary differential equation typical in investment problems with geometric Brownian motion,[9]

$$\frac{\sigma^2}{2} y^2 V_a''(y) + \mu y V_a'(y) + R - \rho V_a(y) = 0, \tag{III.1}$$

subject to the boundary condition $\lim_{y \to 0} V_a(y) = R/\rho$, where ρ is the discount rate. The boundary condition implies that when income is very low, agricultural land will not be converted in the foreseeable future, and therefore its value is the capitalized value of the agricultural rents. The first two terms in (III.1) are the expected capital gain from holding agricultural land. The third is the agricultural rent. The fourth is the normal rate of return on an investment. The intuition behind (III.1) is that holding agricultural land must provide returns equal the normal rate of return on an investment.

The solution to the homogeneous part of the differential equation (III.1) is $Ay^{\alpha} + By^{\beta}$, where A and B are parameters to be determined and α and β are the roots of the characteristic equation

$$\frac{\sigma^2}{2} \xi(\xi - 1) + \mu\xi - \rho = 0. \tag{III.2}$$

[9] See Appendix A for details. (III.1) appears in Dixit [10, 11] and Pindyck [21] and Dixit and Pindyck [12].

The solution for $\alpha < 0$ and $\beta > 0$ is

$$\alpha = \frac{(1-m) - \left[(1-m)^2 + 4r\right]}{2}, \tag{III.3}$$

$$\beta = \frac{(1-m) + \left[(1-m)^2 + 4r\right]^{1/2}}{2}, \tag{III.4}$$

where

$$m \equiv 2\mu/\sigma^2, \qquad r \equiv 2\rho/\sigma^2. \tag{III.5}$$

A particular solution to the nonhomogeneous part of (III.1) is R/ρ. The boundary condition $\lim_{y \to 0} V_a(y) = R/\rho$ implies $A = 0$. The solution is (III.1) and the boundary condition is thus $V_a(y) = R/\rho + By^{\beta}$. Let y_H denote the income level that triggers development. We solve for y_H below. Let $V_c(y, \theta)$ denote the value of land under conversion when conversion will be complete in θ years. By construction

$$V_a(y) = \begin{cases} \dfrac{R}{\rho} + By^{\beta} & \text{for } y \le y_H \\ V_c(y, h) - k & \text{for } y \ge y_H, \end{cases} \tag{III.6}$$

where h is the length of the lag. Intuitively, when income is below the trigger value, the value of agricultural land is the capitalized value of agricultural rents plus the value of the option to develop, By^{β}. When income is above the trigger value, agricultural land is worth the value of land in conversion, where conversion will be complete in h years.

B. The Value of Urban Land

Let $V_u(y)$ denote the value of urban land when income is y. It satisfies[10]

$$\frac{\sigma^2}{2} y^2 V_u''(y) + \mu y V_u'(y) - \rho V_u(y) = w + tz + v^* - y, \tag{III.7}$$

with boundary condition $\lim_{y \to \infty} V_u(y) = \lim_{y \to \infty} y/(\rho - \mu) - (w + tz + v^*)/\rho$. The boundary condition implies that when y is very large, the urban land will not be abandoned. In this case, the instantaneous profit rate is given by $y - (w + tz + v^*)$, with y expected to grow at rate μ. The solution for $V_u(y)$ is of the form $V_u(y) = Ay^{\alpha} + y/(\rho - \mu) - (w + tz + v^*)/\rho$, where the coefficient B of the term By^{β} is zero because of the boundary condition. A is a parameter to be determined.

[10] See Appendix B for details.

Let y_L denote the income level that triggers abandonment. We solve for y_L below. By construction $V_u(y)$ is given by

$$V_u(y) = \begin{cases} Ay^{\alpha} + \dfrac{y}{\rho - \mu} - \dfrac{w + tz + v^*}{\rho} & \text{for } y \geq y_L \\ V_a(y) - l & \text{for } y \leq y_L. \end{cases} \tag{III.8}$$

Intuitively, the value of urban land equals the value of the option to abandon, Ay^{α}, plus the capitalized value of urban rents when income is above the abandonment trigger. When income is below the abandonment trigger, urban land is worth the same as agricultural land, less the costs of abandonment.

C. The Value of Land in Conversion

Before determining the value of land in conversion, it is necessary to present a lemma. We have assumed above that the capital costs of construction are completely sunk when the project is begun. We have also assumed that abandonment is costly. Marginal costs begin being paid only on completion of the project. Since agricultural rents are received during conversion, there is no benefit to abandoning during conversion. Since delaying the costs of abandonment is desirable, the developer will never abandon prior to completion. Waiting until the project is complete has a further benefit should incomes rise enough to justify remaining active.[11, 12]

As noted above, $V_c(y, \theta)$ represents the value of land that will become urban in θ years. Consider a project that will be complete in θ years. Waiting an additional dt has expected value $E_t[V_c(y(t + dt), \theta - dt)e^{-\rho dt}]$, where $y(t + dt)$ denotes income at time $t + dt$ and E_t denotes an expectation taken at time t. It must be the case that

$$V_c(y(t), \theta) = e^{-\rho dt} E_t\big[V_c(y(t + dt), \theta - dt)\big] + R\,dt. \tag{III.9}$$

[11] Developers might choose to abandon during construction with negative abandonment costs. However, assuming negative abandonment costs would make investment more reversible. When investment is more reversible, the developer has less incentive to delay in the presence of uncertainty, since the loss associated with bad states of nature is smaller. Thus, relaxing our assumption of positive abandonment costs would strengthen our results.

[12] The result that there is no abandonment during conversion requires a number of strong assumptions, but it is necessary for our approach to the solution. Without it, we would no longer be able to derive the solution as a system of algebraic equations. The strongest is that agricultural rents are received during conversion. However, since the numerical analysis of lags in Section IV are for $R = 0$, our results do not hinge on the assumption that agricultural rents are received throughout conversion.

Using Ito's Lemma, (III.9) yields the partial differential equation

$$\frac{1}{2}\sigma^2 y^2 \frac{\partial^2 V_c(y,\theta)}{\partial y^2} + \mu y \frac{\partial V_c(y,\theta)}{\partial y} - \rho V_c(y,\theta) - \frac{\partial V_c(y,\theta)}{\partial \theta} = 0. \tag{III.10}$$

The boundary conditions are

$$\lim_{y\to 0} V_c(y,\theta) = \frac{R}{\rho} - le^{-\rho\theta}, \tag{III.11}$$

$$\lim_{y\to\infty} V_c(y,\theta) = \frac{R}{\rho}(1-e^{-\rho\theta}) + \frac{ye^{-(\rho-\mu)\theta}}{\rho-\mu} - \frac{(w+tz+v^*)e^{-\rho\theta}}{\rho}, \tag{III.12}$$

$$\lim_{\theta\to 0} V_c(y,\theta) = \begin{cases} V_u(y) & \text{for } y \geq y_L \\ V_a(y) - l & \text{for } y \leq y_L. \end{cases} \tag{III.13}$$

Because y follows a geometric process, when income is very close to zero it will not rise above the trigger level for abandonment, y_L, during the construction period and above the trigger for development, y_H, over any finite horizon; this is expressed in the boundary condition (III.11), where the right side is the present value of receiving agricultural rents forever and abandoning the building when it is completed in θ years. Similarly, a very high level of income will not fall below the abandonment trigger y_L, as in (III.12). Condition (III.13) states that near the end of construction, the developer will either go ahead with development or abandon, depending on whether income is above or below y_L.

Given Eqs. (III.6) and (III.8), the boundary condition (III.13) can be rewritten as

$$\lim_{\theta\to 0} V_c(y,\theta) = \begin{cases} Ay^\alpha + \dfrac{y}{\rho-\mu} - \dfrac{w+tz+v^*}{\rho} & \text{for } y \geq y_L \\ \dfrac{R}{\rho} + By^\beta - l & \text{for } y \leq y_L. \end{cases} \tag{III.13$'$}$$

The solution $V_c(y,\theta)$ of the differential equation (III.10), subject to the boundary conditions (III.11), (III.12), and (III.13′) can be found as follows.

Since the developer will not abandon during the construction period, $V_c(y, \theta)$ is

$$V_c(y(t), \theta) = e^{-\rho\theta} E_t\left[V_c(y(t+\theta), 0)\right] + \frac{R}{\rho}(1 - e^{-\rho\theta}). \quad \text{(III.14)}$$

Substituting in (III.6) for $V_a(y)$ and (III.8) for $V_u(y)$ yields

$$\begin{aligned} &E_t\left[V_c(y(t+\theta), 0)\right] \\ &\quad = \int_{y_L}^{\infty} \left(A(y(t+\theta))^{\alpha} + \frac{y(t+\theta)}{\rho - \mu} - \frac{w + tz + v^*}{\rho} \right) \\ &\qquad \times f(y(t+\theta))\, dy(t+\theta) \\ &\qquad + \int_0^{y_L} \left(\frac{R}{\rho} + B(y(t+\theta))^{\beta} - l \right) f(y(t+\theta))\, dy(t+\theta), \end{aligned} \quad \text{(III.15)}$$

where $f(y(t+\theta))$ denotes the probability density function of the income at time $t + \theta$ when the project will be completed. From now on, $y(t+\theta)$, $y(t)$, and $E_t[\ ]$ will be denoted by $y(\theta)$, y, and $E[\ ]$, respectively. For the geometric process (II.3), income $y(\theta)$, given current income y, is distributed lognormally. This implies (see Appendix C for details) that $V_c(y, \theta)$ satisfies

$$\begin{aligned} V_c(y, \theta) = &\frac{R}{\rho}(1 - e^{-\rho\theta}) + (1 - \Phi(u - \alpha\sigma)) A y^{\alpha} \\ &+ (1 - \Phi(u - \sigma)) \frac{y e^{-(\rho - \mu)\theta}}{\rho - \mu} \\ &- (1 - \Phi(u)) \frac{(w + tz + v^*) e^{-\rho\theta}}{\rho} + \Phi(u - \beta\sigma) B y^{\beta} \\ &+ \Phi(u) \left(\frac{R}{\rho} - l e^{-\rho\theta} \right), \end{aligned} \quad \text{(III.16)}$$

where $\Phi(\ )$ is the cumulative distribution function of the standard normal distribution and u is defined as

$$u = u(y, \theta) = \frac{\log y_L - \log y - (\mu - \sigma^2/2)\theta}{\sigma\sqrt{\theta}}. \quad \text{(III.17)}$$

The arguments of the function u are suppressed to economize on notation. Equation (III.19) is by construction the solution to the differential equation (III.10) with boundary conditions (III.11), (III.12), and (III.13′) as can be readily verified.

D. Solution

We have identified functions $V_a(y)$ and $V_u(y)$ giving the values of agricultural and urban land as a function of income. We have also derived a function $V_c(y, \theta)$ giving the value of land in conversion as a function of income and the time until conversion is complete. To complete the solution of the developer's problem, we must solve for the parameters A and B and the triggers y_H and y_L. There is extensive literature on investment under stochastic returns, and we can draw on that literature here. Dixit [10, 11], Pindyck [21], and Dixit and Pindyck [12] have shown that A, B, y_H, and y_L must solve

$$V_a(y_H) = V_c(y_H, h) - k, \tag{III.18}$$

$$V_a'(y_H) = \frac{\partial V_c}{\partial y}(y_H, h), \tag{III.19}$$

$$V_u(y_L) = V_a(y_L) - l, \tag{III.20}$$

$$V_u'(y_L) = V_a'(y_L). \tag{III.21}$$

Equation (III.18) is called the value matching condition at y_H. Clearly, development is not optimal unless the value of land in conversion and the value of agricultural land are equal at the development trigger y_H. Equation (III.19) is called the smooth pasting condition at y_H. Intuitively, unless the slopes of the two value functions are equal at y_H, the development rule is suboptimal. Equations (III.20) and (III.21) are the corresponding value matching and smooth pasting conditions at y_L.

Using (III.6), (III.8), and (III.16) to substitute for $V_a(y)$, $V_u(y)$, and $V_c(y, \theta)$, the system (III.18)–(III.21) becomes

$$\begin{aligned} By_H^{\beta} = {} & \frac{R}{\rho}(1 - e^{-\rho\theta}) + \big(1 - \Phi(u_H - \alpha\sigma)\big)Ay_H^{\alpha} \\ & + \big(1 - \Phi(u_H - \sigma)\big)\frac{y_H e^{-(\rho-\mu)h}}{\rho - \mu} \\ & - \big(1 - \Phi(u_H)\big)\frac{(w + tz + v^*)e^{-\rho h}}{\rho} + \Phi(u_H - \beta\sigma)By_H^{\beta} \\ & + \Phi(u_H)\left(\frac{R}{\rho} - l\right)e^{-\rho h} - k, \end{aligned} \tag{III.22}$$

$$\beta B y_{\mathrm{H}}^{\beta-1} = \left(1 - \Phi(u_{\mathrm{H}} - \alpha\sigma)\right)\alpha A y_{\mathrm{H}}^{\alpha-1} + A y_{\mathrm{H}}^{\alpha} \frac{\phi(u_{\mathrm{H}} - \alpha\sigma)}{y_{\mathrm{H}}\sigma\sqrt{h}}$$

$$+ \frac{e^{-(\rho-\mu)h}}{\rho-\mu}\left(1 - \Phi(u_{\mathrm{H}} - \sigma)\right) + \frac{y_{\mathrm{H}} e^{-(\rho-\mu)h}}{\rho-\mu} \frac{\phi(u_{\mathrm{H}} - \sigma)}{y_{\mathrm{H}}\sigma\sqrt{h}}$$

$$- \frac{\phi(u_{\mathrm{H}})(w + tz + v^*)e^{-\rho h}}{\rho y_{\mathrm{H}}\sigma\sqrt{h}} + \Phi(u_{\mathrm{H}} - \beta\sigma)\beta B y_{\mathrm{H}}^{\beta-1}$$

$$- \frac{\phi(u_{\mathrm{H}} - \beta\sigma)}{y_{\mathrm{H}}\sigma\sqrt{h}} B y_{\mathrm{H}}^{\beta} + \frac{\phi(u_{\mathrm{H}})\left(\dfrac{R}{\rho} - l\right)e^{-\rho h}}{y_{\mathrm{H}}\sigma\sqrt{h}}, \qquad \text{(III.23)}$$

$$A y_{\mathrm{L}}^{\alpha} + \frac{y_{\mathrm{L}}}{\rho-\mu} - \frac{w + tz + y^*}{\rho} = \frac{R}{\rho} + B y_{\mathrm{L}}^{\beta} - l, \qquad \text{(III.24)}$$

$$\alpha A y_{\mathrm{L}}^{\alpha-1} + \frac{1}{\rho-\mu} = \beta B y_{\mathrm{L}}^{\beta-1}, \qquad \text{(III.25)}$$

where $\phi(\ )$ denotes the standard normal probability density function and

$$u_{\mathrm{H}} = u(y_{\mathrm{H}}, h) = \frac{\log y_{\mathrm{L}} - \log y_{\mathrm{H}} - (\mu - \sigma^2/2)h}{\sigma\sqrt{h}}. \qquad \text{(III.26)}$$

Equations (III.22)–(III.25) determine A, B, and, in particular, the triggers y_{L} and y_{H}.

E. *Analytical Results*

Since (III.22)–(III.25) are a system of nonlinear equations, it should not be surprising that it is impossible to find closed form solutions for y_{L}, y_{H}, A, and B. Even so, there are a few properties of the solution that can be demonstrated analytically. First, unlike Capozza–Helsley, the triggers for development vary across space. This can be seen by observing the tz terms in (III.22)–(III.24), and it will be apparent in the numerical simulations in the next section. As noted earlier, when income follows geometric Brownian motion and the equilibrium rent function is linear, rents do not follow geometric Brownian motion. Any shock to income has a larger percentage effect on rents far from the CBD than on rents close to the CBD. In Capozza–Helsley, the triggers do not vary over space because rents follow arithmetic Brownian motion.

Second, when the costs of abandonment are prohibitive, our results are qualitatively similar to Capozza–Helsley. With no abandonment possible, the value of urban land is $V_{\mathrm{u}}(y) = y/(\rho - \mu) - (w + tz + v^*)/\rho$; without the ability to abandon, urban land is worth the expected net present value of rents less costs. Likewise, the value of land in conversion with completion in θ years is $V_c(y, \theta) = e^{-\rho\theta}[e^{\mu\theta}y/(\rho - \mu) - (w + tz + v^*)/\rho]$. $V_a(y)$ remains as in (III.6). The solution for optimal development requires determination of B and y_H. These are given by the smooth pasting and value matching conditions for y_H,

$$\frac{R}{\rho} + By_H^{\beta} = \frac{y_H e^{-(\rho-\mu)h}}{\rho - \mu} - \frac{(w + tz + v^*)e^{-\rho h}}{\rho} - k, \qquad \text{(III.27)}$$

$$\beta B y_H^{\beta-1} = \frac{e^{-(\rho-\mu)h}}{\rho - \mu}. \qquad \text{(III.28)}$$

with β given by (III.4) as before. This implies

$$y_H = \frac{\beta}{\beta - 1}\frac{\rho}{\rho - \mu}e^{(\rho-\mu)h}\left[(w + tz + v^* + R)e^{-\rho h} + \rho k\right]. \qquad \text{(III.29)}$$

In this case, uncertainty affects development through the term $\beta/(\beta - 1)$ which rises with σ (see (III.4)). This means that more uncertainty raises y_H and delays development, the Capozza–Helsley result. Assuming that there are lags and that income follows geometric Brownian motion does not, therefore, reverse the Capozza–Helsley result unless abandonment is possible.

Third, lags impose holding costs on developers, decreasing the incentive to develop. The cost of the project is paid up-front, while revenues are not received until h years later. This has the effect of reducing the value of the revenue stream by $e^{-\rho h}$.

Fourth, in our model, abandonment proceeds from the edge of the city toward its center. This is contrary to empirical observation (for example, Salins [22]). This result holds because our model does not feature neighborhood amenities or deterioration of housing over time.

In order to analyze the system (III.22)–(III.25) further, it is necessary to carry out numerical simulations of the solution. The next section describes such simulations.

IV. NUMERICAL SIMULATIONS

The main objective of this paper is to provide an understanding of the effects of lags on urban development. This section takes the additional step of numerically calibrating the model to study the effects of lags on the uncertainty–development relationship and on urban spatial structure. In simulating for the solution to (III.22)–(III.25) we employ the following parameters: $l = 0$, $R = 0$, $\rho = 0.045$, $\mu = 0.4$, $\sigma = 0.15$, $w = 0.1$, $v^* = 0.9$,

$t = 0.005$, and $k = 1$.[13] We consider lags of between zero and five years.[14] As noted above, lags reduce the value of development by delaying its returns. In this section, our goal is to assess the interaction between lags and uncertainty in the process of urban development. To control for the loss of value associated with the delay of the returns, we multiply k by $e^{-\rho h}$. Of course, the developer is committed once construction is begun. It is useful to begin by considering urban development without lags.

A. *Urban Development without Lags*

Table 1 shows the solutions to (III.22)–(III.25) when $h = 0$. y_{L} and y_{H} are the triggers for development under uncertainty, with $\sigma = 0.15$, while Y_{L} and Y_{H} are the triggers for development without uncertainty.[15]

Most of the results of this table are familiar. As in monocentric models of cities, a higher level of income is required to generate development at more distant locations. As in real option models, the trigger for development under uncertainty exceeds the trigger for development without uncertainty. In the language of Capozza and Helsley, because of uncertainty agricultural land has an option value in addition to its use value. Likewise, the trigger for abandonment under uncertainty is below the trigger for abandonment without uncertainty. The two latter results imply that uncertainty increases the range of values under which an inactive developer would not build but an active developer would not abandon.

The only unfamiliar result in Table 1 concerns the relationship between the option value of agricultural land and location. At the central business district, $y_{\mathrm{H}} = 1.256$ and $Y_{\mathrm{H}} = 1.045$, a difference of 0.211. At 10 miles from the central business district, the difference is 0.219. At 20 miles, the difference is 0.226. This implies that the option value rises with distance. This is contrary to the analytical results of Capozza–Helsley, who find an option value that does not vary with location. The source of the difference is that we employ geometric Brownian motion on income, while they apply arithmetic Brownian motion. With a linear bid–rent curve, arithmetic Brownian motion of income implies arithmetic Brownian motion of rents, with the same degree of uncertainty at all locations. With geometric Brownian motion, a shock to income results in a larger percentage loss of

[13] k is a normalization for the cost of sunk structural capital in the housing, $w = 0.1$ incorporates the annual cost of maintaining a house, while v^* is the monetized utility available elsewhere. t is the annual per mile roundtrip commuting cost. $\mu = .04$ reflects a growing city. $\rho = .045$ is the discount rate; of course, $\rho > \mu$ is necessary for convergence. Finally, $R = 0$ and $l = 0$ for convenience.

[14] There are some technicalities required to apply our solution when $h = 0$. The details are given in Appendix D.

[15] The solution for the triggers under certainty is standard (e.g., Dixit and Pindyck [12]). See Appendix E for the details.

TABLE 1
Development Triggers without Lags

z	y_L	y_H	Y_L	Y_H
0	0.731	1.256	0.768	1.045
2	0.739	1.268	0.777	1.055
4	0.747	1.279	0.785	1.065
6	0.756	1.291	0.794	1.075
8	0.764	1.302	0.802	1.085
10	0.772	1.314	0.811	1.095
12	0.780	1.325	0.820	1.105
14	0.788	1.337	0.828	1.115
16	0.797	1.348	0.837	1.125
18	0.805	1.360	0.846	1.135
20	0.813	1.371	0.854	1.145
22	0.821	1.383	0.863	1.155
24	0.830	1.394	0.872	1.165
26	0.838	1.406	0.880	1.175
28	0.846	1.417	0.889	1.185
30	0.854	1.429	0.898	1.195
32	0.863	1.440	0.907	1.205
34	0.871	1.452	0.915	1.215
36	0.879	1.463	0.924	1.225
38	0.887	1.475	0.933	1.235
40	0.896	1.486	0.941	1.245

Note. This table reports solutions of the system (III.22)–(III.25) for the levels of income that trigger development and abandonment when $h = 0$. y_H and y_L are the triggers for $\sigma = 0.15$, while Y_H and Y_L are the triggers under certainty. z is the distance in miles from the CBD.

rent at more distant locations. Thus, uncertainty rises with distance and so do the option values. This suggests that in the absence of lags, landowners at the edge of a city will exhibit more caution than will central landowners.

B. Lags and the Relationship of Uncertainty to Development

To see the effects of lags on development, begin by supposing a two year lag. Table 2 reports the triggers when $h = 2$. The results here are distinctly different from the familiar results in Table 1. At all locations, the value of waiting to develop is smaller. At the CBD, for example, the difference between y_H and Y_H is 0.163, approximately three-fourths of the difference with $h = 0$. With $h = 2$, both y_H and y_L are smaller than with $h = 0$. y_H is smaller because it takes time to build, while y_L is smaller because it takes time to rebuild. However, the effect on y_H is stronger. At the CBD, with $h = 0$, the difference between y_H and y_L is 0.525. At $h = 2$ the

TABLE 2

Development Triggers with 2 Year Lags

z	y_L	y_H	Y_L	Y_H
0	0.700	1.127	0.768	0.965
2	0.707	1.137	0.777	0.974
4	0.715	1.147	0.785	0.983
6	0.723	1.157	0.794	0.992
8	0.731	1.167	0.802	1.002
10	0.738	1.178	0.811	1.011
12	0.746	1.188	0.820	1.020
14	0.754	1.198	0.828	1.029
16	0.762	1.208	0.837	1.039
18	0.770	1.218	0.846	1.048
20	0.777	1.228	0.854	1.057
22	0.785	1.238	0.863	1.066
24	0.793	1.248	0.872	1.075
26	0.801	1.258	0.880	1.085
28	0.809	1.268	0.889	1.094
30	0.816	1.278	0.898	1.103
32	0.824	1.287	0.907	1.112
34	0.832	1.297	0.915	1.122
36	0.840	1.307	0.924	1.131
38	0.848	1.317	0.933	1.140
40	0.856	1.327	0.941	1.149

Note. This table reports solutions of the system (III.22)–(III.25) for the levels of income that trigger development and abandonment when $h = 2$. y_H and y_L are the triggers for $\sigma = 0.15$, while Y_H and Y_L are the triggers under certainty. z is the distance in miles from the CBD.

difference is 0.428. Thus, the range of incomes over which it is neither optimal to begin a new development nor to abandon an existing development is smaller without a lag. In other words, there is less inertia when there are lags. There is also somewhat less sensitivity to increases in distance when there are lags. When $h = 0$, the difference between y_H at $z = 0$ and y_H at $z = 40$ is 0.230. At $h = 2$, the difference is 0.207. Table 2 also shows that the value of waiting to abandon is larger with $h = 2$ than with $h = 0$. With $h = 2$, y_L is 0.700 at the CBD, while with $h = 0$ y_L is 0.731. At 10 miles, the values for y_L are 0.738 and 0.772 at $h = 2$ and $h = 0$, respectively. Since there is no abandonment lag in our model, an increase in uncertainty increases the value of remaining in the market.

To show that the results from Table 2 are not specific to the two year lag, Table 3 presents triggers for a range of lag lengths. z is set equal to 10, while h varies from 0 to 5. The most obvous result here is that longer lags

TABLE 3
Development Triggers at 10 Miles with and without Lags

h	y_L	y_H	Y_L	Y_H
0.0	0.772	1.314	0.811	1.095
0.5	0.763	1.280	0.811	1.073
1.0	0.754	1.247	0.811	1.052
1.5	0.746	1.213	0.811	1.031
2.0	0.738	1.178	0.811	1.011
2.5	0.732	1.141	0.811	0.991
3.0	0.726	1.102	0.811	0.971
3.5	0.720	1.059	0.811	0.952
4.0	0.716	1.004	0.811	0.933
4.5	0.718	0.833	0.811	0.915
5.0	0.724	0.828	0.811	0.897

Note. This table reports solutions of the system (III.22)–(III.25) for the levels of income that trigger development and abandonment at $z = 10$. y_H and y_L are the triggers for $\sigma = 0.15$, while Y_H and Y_L are the triggers under certainty. h is the length of the lag in years.

imply smaller option values.[16] Thus, with longer lags, uncertainty is less of a deterrent to development. For $h = 5$, this result becomes even more extreme; $y_H = 0.828$, while $Y_H = 0.897$. The trigger under uncertainty is smaller than the trigger under certainty. This implies that projects with negative expected net present value after completion will be undertaken in the presence of lags.

In sum, Tables 2 and 3 together show that with lags uncertainty is less of a deterrent to development. In some situations, development may even occur sooner under uncertainty. These results are novel, and it is worthwhile describing the economic intuition behind them.

We begin by considering the effect of uncertainty on development without lags. It is useful to compare the position of a developer who has already built with one who has not. Suppose that incomes were to rise to the point that development exactly breaks even, earning the developer an expected net present value of zero. If incomes should rise further, raising rents, the two developers' positions would be symmetrical. The developer who has built would earn high profits, but so would the developer who has not. This is because the developer who has not yet built could do so immediately. If incomes were to fall, however, the two positions would no longer be symmetrical. The developer who has built would be committed,

[16] Note that y_L does not depend on h. This can be seen by substituting $ke^{-\rho h}$ for k in (E.4) and (E.5).

and would either suffer operating losses or exit the market, paying the costs of abandonment and sacrificing the irreversible investment in conversion. The developer who has not built would not lose money. This asymmetry is the key to the effects of uncertainty on development. Since uncertainty increases the probability of low incomes, it increases both the probability and the cost of the low income states of nature. Because of this, a developer would not develop as soon as incomes rose enough that the net present value of the project was zero. The developer would instead wait until the undesirable low-income states became sufficiently unlikely.

The ability to abandon has no qualitative effect on this analysis of development without lags. It is true that the ability to abandon would reduce the cost of being committed should incomes fall. Nonetheless, as long as the development is at least partially irreversible, the asymmetry between the committed and uncommitted developers remains, and development is delayed by the presence of uncertainty. See Dixit and Pindyck [12] for a discussion of investment with the possibility of abandonment.

The situation is more complicated with lags. It is again useful to compare the positions of a developer who has built and another who has not. As above, the two positions are not symmetrical when income falls, and this asymmetry favors the uncommitted developer. However, when incomes rise the two developer's positions are no longer symmetrical. In this case, only the developer who has built enjoys the benefits of high rents. The developer who has not built does not. Since uncertainty increases the likelihgood of high incomes, more uncertainty raises both the probability and the benefit of high income states of nature. This result depends crucially on the ability to abandon. If the developer cannot abandon, then the increased likelihood of high incomes is balanced with the increased likelihood of low incomes. Abandonment truncates losses in low-income states, and this means that an increase in uncertainty increases the value of being in the market. It is easy to see that as lag length becomes short, the value of being in the market becomes independent of uncertainty. But when lags are long, uncertainty affects the value of being in the market and the developer must compare the traditional value of waiting with the value of being in the market. This means that uncertainty has less of a deterrent effect on development with lags than without them.

Another way to understand the effects of lags is to consider the mathematical intuition behind the result. By (III.16), the value of land in conversion, $V_c(y, \theta)$, is a convex function of income, y.[17] By Jensen's inequality, the expected value of a convex function increases with variance. Consequently, uncertainty increases the expected value of being in the market.

[17] This is implied by the parameter values $A > 0$, $B > 0$, $\alpha < 0$, and $\beta > 1$.

C. Lags and Urban Spatial Structure

Table 4 reports income triggers for $h = 4$. This sort of lag might be found in building a concrete condominium high-rise. In this case, the deterrent effect on development is much smaller than with two year lags or with no lags at all. At the CBD, the difference between y_H and Y_H is 0.079, while it was 0.163 at $h = 2$, and 0.211 at $h = 0$. At 10 miles, the differences are 0.219, 0.167, and 0.071 for lags of 0, 2, and 4 years, respectively. Likewise, the value of waiting to abandon is a larger number than it was with $h = 0$ or 2. At the CBD, $y_L = 0.678$. At 10 and 20 miles, y_L is 0.716 and 0.755. With $h = 2$, y_L was 0.700 at the CBD, 0.738 at 10 miles, and 0.777 at 20 miles. With longer lags in rebuilding, the developer is willing to suffer losses before abandoning.

TABLE 4
Development Triggers with 4 Year Lags

z	y_L	y_H	Y_L	Y_H
0	0.678	0.969	0.768	0.890
2	0.686	0.976	0.777	0.899
4	0.694	0.983	0.785	0.908
6	0.701	0.990	0.794	0.916
8	0.709	0.997	0.802	0.925
10	0.716	1.004	0.811	0.933
12	0.724	1.010	0.820	0.942
14	0.732	1.017	0.828	0.950
16	0.739	1.023	0.837	0.959
18	0.747	1.029	0.846	0.967
20	0.755	1.034	0.854	0.976
22	0.762	1.040	0.863	0.984
24	0.770	1.045	0.872	0.993
26	0.778	1.050	0.880	1.001
28	0.784	0.952	0.889	0.010
30	0.793	0.935	0.898	1.018
32	0.801	0.943	0.907	1.027
34	0.809	0.950	0.915	1.035
36	0.817	0.958	0.924	1.044
38	0.824	0.950	0.933	1.052
40	0.833	0.975	0.941	1.061

Note. This table reports solutions of the system (III.22)–(III.25) for the levels of income that trigger development and abandonment when $h = 4$. y_H and y_L are the triggers for $\sigma = 0.15$, while Y_H and Y_L are the triggers under certainty. z is the distance in miles from the CBD.

There is another result in Table 4 that is not present in Tables 1 and 2. Throughout Tables 1 and 2, the trigger for development rises with distance. This is intuitive; since more distant locations generate lower rents than central locations, it is not surprising that a higher level of income would be required to induce development. However, Table 4 shows that it is possible that the trigger for development might fall with distance. From the CBD out to 26 miles, y_{H} rises. At 26 miles from the CBD, y_{H} is 1.050. At 28 miles it is 0.952. After another drop at 30 miles, y_{H} begins to rise again. The relationship of y_{L} to distance is standard.

It is worth commenting on the source of this anomaly. At greater distances, rents are lower. Thus, at greater distances, the percentage change in rents associated with a percentage change in income is greater. This essentially means that there is more uncertainty at greater distances. Since increases in uncertainty can encourage development in the presence of long lags, this leads to a lower trigger for development.

This result is important because it may provide part of an explanation for leapfrog development. Leapfrogging occurs when distant land is developed while there remains unused interior land. Monocentric models, like the one presented in Section II, suggest that urban growth should be continuous, with the city occupying all land adjacent to its border before moving out to more distant land. Critics of leapfrogging observe that this type of development raises the cost of infrastructure provision and spoils pristine farmland before it is necessary to do so. Supporters of free-markets point out that this kind of development reserves land for future generations in a form that allows easy conversion. This paper has nothing to say about the benefits of leapfrogging. Instead, we are concerned with its causes.

There have been many explanations of leapfrogging. Leapfrogging may be caused by differences in agricultural productivity in different lots, presumably arising from the nonuniform distribution of topsoil and water, among other things. Leapfrogging may also be caused by differences in the costs of conversion, presumably arising from the nonuniform distribution of rocks in the ground. As Ohls and Pines [20] note, leapfrogging can also be caused by heterogeneity in housing demand coupled with irreversibility. There have been many other models that have demonstrated leapfrogging. Braid [6], Fujita [13], Mills [18], Wheaton [25], and many others show that reserving land for future development may be either socially efficient or individually profitable or both. See Braid [6] for a recent literature review. In all of these papers, the motivation for leapfrogging is that there may be a conflict between static and dynamic efficiency. Today's market may not support development that would be efficient over a longer horizon. As in Ohls and Pines, this argument requires heterogeneity and irreversibility.

Capozza and Helsley correctly note that without heterogeneity, uncertainty and irreversibility are not sufficient to cause leapfrogging. To see this in our model, suppose that $h = 0$, so there are no lags. In this case, an increase in uncertainty always delays development. Since there is greater uncertainty at greater distances from the CBD, this means that uncertainty reinforces the tendency that is already present in the monocentric model; proximate locations are developed before distant ones.[18]

The results of Table 4 suggest that uncertainty and lags together can lead to leapfrogging. At any level of income between 0.935 and 1.050, there will be some leapfrogging. As noted above, in some situations an increase in uncertainty can hasten development. This can cause leapfrog development.

V. CONCLUSION

In sum this paper has three important results. First, it is possible that uncertainty may hasten development in the presence of lags. This means that developments may be undertaken that are expected to have negative net present values on completion. Second, it is always the case that development is less sensitive to uncertainty than in the absence of lags. Third, lags may lead to leapfrog development, when distant land is developed prior to nearby land.

These results arise because adding lags to the development decision creates an opportunity cost of waiting that opposes the familiar benefit of waiting in an uncertain environment. The benefit of waiting is enjoyed in bad states of nature; in these states, a committed developer will regret the project, while an uncommitted developer can choose not to develop. The opportunity cost of waiting arises through the interaction between the development lag and the ability to abandon once development is complete. Essentialy, adding development lags makes the developer's profits a convex function of income. This means that expected profits increase with uncertainty, and therefore that the opportunity cost of waiting rises with uncertainty.

APPENDIX A: THE VALUE OF AGRICULTURAL LAND

This appendix provides the details of the derivation of (III.1). Suppose that a landowner holds agricultural land of value $V_a(y)$. This investment must provide normal returns, $\rho V_a(y)$. There are two components to the returns from holding vacant land. First, vacant land provides agricultural

[18] Fujita [15] notes that risk aversion can lead to leapfrogging if it is not possible for landowners to diversify their portfolios in some other way.

returns R. Second, vacant land provides capital gains. This appreciation originates in the possibility of income increases. Higher income will raise rents and the value of both urban and agricultural land, since the latter is more likely to be converted to urban use. The rate of capital gains is $dV_a(y)/dt$. In order to evaluate the differential $dV_a(y)$, we expand $V_a(y)$ in a Taylor Series, taking into account the stochastic process for dy, (II.3). Ignoring terms of higher order than linear in dt gives the expression for capital gains:

$$\frac{\sigma^2}{2} y^2 V_a''(y) + \mu y V_a'(y). \tag{A.1}$$

This procedure of finding the differential of a function of a stochastic variable in continuous time is called Ito's Lemma. It is presented in Dixit and Pindyck [12], among others. The requirement that holding agricultural land yield normal returns gives (III.1) directly.

APPENDIX B: THE VALUE OF URBAN LAND

This appendix provides the details of the derivation of (III.7). As in Appendix A, holding urban land must provide normal returns, $\rho V_u(y)$. There are three components to the returns from holding urban land. First, holding urban land yields rent equal to $y - v^* - tz$. Second, holding urban land imposes costs equal to w. Third, as above, holding urban land provides capital gains. Ito's Lemma implies that the capital gains are

$$\frac{\sigma^2}{2} y^2 V_u''(y) + \mu y V_u'(y). \tag{B.1}$$

The requirement that holding agricultural land yields normal returns gives (III.7) directly.

APPENDIX C: THE VALUE OF LAND IN CONVERSION

This appendix addresses two aspects of the solution for $V_c(y, \theta)$: the derivation of the differential equation (III.10) and the derivation of the solution (III.16). The derivation of (III.10) is exactly parallel to the derivation of (III.7). Holding land in conversion must provide a normal return.

The derivation of (III.16) is more involved. As shown in the text, $V_c(y, \theta)$ is given by (III.14). y is distributed lognormally:

$$\log y(\theta) \sim N\left(\log y + \left(\mu - \frac{\sigma^2}{2}\right)\theta, \sigma^2\theta\right). \tag{C.1}$$

We make use of the following two properties of the lognormal distribution:[19]

(i) If $\psi = \log x \sim N(g, s^2)$, then the rth moment of x around 0 is

$$E(x^r) = e^{rg + r^2 s^2/2}. \tag{C.2}$$

(ii) When ψ is truncated from below at $\log x_0$, then the rth moment of x around 0 is

$$E(x^r) = \frac{1 - \Phi(u - r\sigma)}{1 - \Phi(u)} e^{rg + r^2 s^2/2}, \tag{C.3}$$

where $\Phi(\ )$ denotes the cumulative distribution function of the standard normal distribution as in the text and u is now defined as

$$u = \frac{1}{s}(\log x_0 - g). \tag{C.4}$$

Equation (C.4) implies

$$\int_{y_L}^{\infty} y^r(\theta) f(y(\theta))\, dy(\theta) = (1 - \Phi(u - r\sigma)) E(y^r(\theta)), \tag{C.5}$$

$$\int_0^{y_L} y^r(\theta) f(y(\theta))\, dy(\theta) = \Phi(u - r\sigma) E(y^r(\theta)). \tag{C.6}$$

This gives

$$\begin{aligned} V_c(y, \theta) = \frac{R}{\rho}(1 - e^{-\rho\theta}) + e^{-\rho\theta}\Bigg\{ & A(1 - \Phi(u - \alpha\sigma)) E[y^\alpha(\theta)] \\ & + (1 - \Phi(u - \sigma)) \frac{E[y(\theta)]}{\rho - \mu} - (1 - \Phi(u)) \frac{w + tz + v^*}{\rho} \\ & + B\Phi(u - \beta\sigma) E[y^\beta(\theta)] + \Phi(u)\left(\frac{R}{\rho} - l\right)\Bigg\}. \end{aligned} \tag{C.7}$$

[19] See Aichison and Brown [2] and Johnson and Kotz [17].

Using (C.2) and (C.3) gives

$$V_c(y,\theta) = \frac{R}{\rho}(1-e^{-\rho\theta}) + e^{-\rho\theta}\Bigg\{(1-\Phi(u-\alpha\sigma))Ay^{\alpha}e^{[\alpha(\mu-\sigma^2/2)+\alpha^2\sigma^2/2]\theta}$$

$$-(1-\Phi(u))\frac{w+tz+v^*}{\rho} + (1-\Phi(u-\sigma))\frac{ye^{\mu\theta}}{\rho-\mu}$$

$$+\Phi(u-\beta\sigma)By^{\beta}e^{[\beta(\mu-\sigma^2/2)+\beta^2\sigma^2/2]\theta} + \Phi(u)\left(\frac{R}{\rho}-l\right)\Bigg\}. \quad \text{(C.8)}$$

Since α and β are the roots of the characteristic equation (III.2), $V_c(y,\theta)$ simplifies to (III.16).

APPENDIX D: DEVELOPMENT WITHOUT LAGS

Without lags, A, B, y_L, and y_H are given by

$$V_a(y_H) = V_u(y_H) - k, \quad \text{(D.1)}$$

$$V_a'(y_H) = V_u'(y_H), \quad \text{(D.2)}$$

together with (III.20) and (III.21). These are the same as (III.18)–(III.21) with $V_u(y)$ replacing $V_c(y,\theta)$. Substituting (III.6) and (III.8) into (D.1) and (D.2) gives

$$\frac{R}{\rho} + By_H^{\beta} = \frac{y_H}{\rho-\mu} - \frac{(w+tz+v^*)}{\rho} - k \quad \text{(D.3)}$$

$$\beta By_H^{\beta-1} = \frac{1}{\rho-\mu}. \quad \text{(D.4)}$$

Together with (III.24) and (III.25), (D.3) and (D.4) characterize A, B, y_L, and y_H.

APPENDIX E: DEVELOPMENT UNDER CERTAINTY

This appendix solves for the triggers Y_L and Y_H under certainty. First, suppose that the income is greater than Y_L. Then the value of urban land is

$$V_u(y) = \int_0^{\infty}(ye^{\mu s} - w - tz - v^*)e^{-\rho s}\,ds = \frac{y}{\rho-\mu} - \frac{w+tz+v^*}{\rho}. \quad \text{(E.1)}$$

This gives

$$V_c(y,\theta) = \frac{R}{\rho}(1 - e^{-\rho\theta}) + e^{-\rho\theta}\left(\frac{ye^{\mu\theta}}{\rho - \mu} - \frac{w + tz + v^*}{\rho}\right), \quad \text{(E.2)}$$

since under certainty the value of land under construction is the discounted value of the urban land it will become. The value of agricultural land is

$$\begin{aligned} V_a(y) &= \frac{R}{\rho}(1 - e^{-\rho(T+h)}) + e^{-\rho T}(V_c(Y_H, h) - k) \\ &= \frac{R}{\rho}\left(1 - e^{-\rho h}\left(\frac{y}{Y_H}\right)^{\rho/\mu}\right) \\ &\quad + \left(\frac{Y_H e^{-(\rho-\mu)h}}{\rho - \mu} - \frac{(w + tz + v^*)e^{-\rho h}}{\rho} - k\right)\left(\frac{y}{Y_H}\right)^{\rho/\mu}, \quad \text{(E.3)} \end{aligned}$$

where T is the time until income reaches Y_H, defined by $Y_H = ye^{\mu T}$. Under certainty, the trigger for development maximizes the value of agricultural land, while the trigger for abandonment sets the value of urban land equal to the value of agricultural land minus abandonment cost. With some manipulation, this implies:

$$Y_H = (w + tz + v^* + R)e^{-\mu h} + \rho k e^{(\rho-\mu)h}, \quad \text{(E.4)}$$

$$\mu e^{-(\rho-\mu)h} Y_H^{-(\rho-\mu)/\mu} Y_L^{\rho/\mu} + (\rho - \mu)(w + tz + v^* + R - \rho l) - \rho Y_L = 0. \quad \text{(E.5)}$$

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