EC1410 Topic #6

The Roback Model, the Value of Amenities, the Quality of Life and income taxes in a spatial equilibrium

Matthew A. Turner
Brown University
Spring 2022

(Updated December 8, 2021)

Outline

Amenities, Productivity, Wages and Rents

- Can we use cross-city differences in wages and rents to think about the value of changes to a city's attractiveness and productivity? For example, we expect that climate change will affect the attractiveness, and maybe the productivity of cities differently. Can infer these values from cross-city differences in rent, wages, and climate?
- The monocentric city model can address such questions, but it has an important shortcoming for this purpose.
- The Rosen-Roback model **?** is a cousin of the monocentric city model that is particularly useful for comparing one city to another. It is the second workhorse model of urban economics.

Amenities and productivity in the moncentric city model.

Consider the standard monocentric city model, with two minor modifications

- Each city has and amenity value A_c such that $u(A_c c) = \overline{u}$.
- Each city has a 'productivity value', or a 'productive amenity' A_y such that $w = w(A_y)$ and $w' > 0$.

To understand A_y suppose

- production is constant returns to scale (so city size doesn't matter)
- Production depends only on labor.
- The labor market is perfectly competitive

Amenities and productivities are not choice variables for the household, so the household's problem is largely unchanged from the standard monocentric city model,

$$\max_{c,x} u(A_c c)$$
$$\text{s.t. } w(A_y) = c + R(x)\overline{\ell} + 2t|x|$$

In spatial equilibrium, we must have

$$u(A_c c) = \overline{u}$$
$$\implies c^* = \frac{u^{-1}(\overline{u})}{A_c}$$

Thus, spatial equilibrium implies,

$$w(A_y) = \frac{u^{-1}(\overline{u})}{A_c} + R(x)\bar{\ell} + 2t|x|$$

In spatial equilibrium, $A_c c^*$ *cannot* change. If it goes up, people move into the city, driving rents up, if it goes down, people move out, driving rents down.

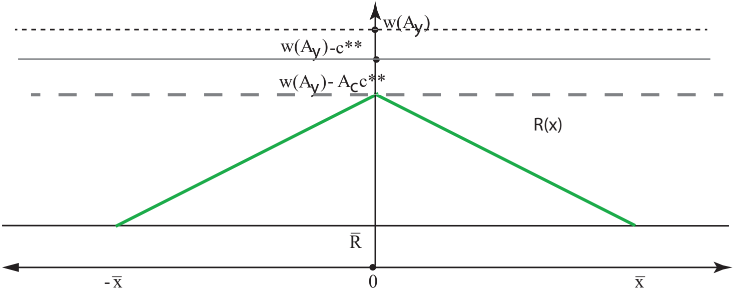
- What happens if A_y increases? To keep consumption constant, this must be offset by a matching increase in rents.
- What happens if A_c increases? In this case, c^* goes down. This means that rents must go up or wages must go down to preserve the spatial equilibrium condition.
- But wages are fixed by perfect competition. So, rents must go up.

...and now we can see the basic logic that motivates the Roback model.

- If we change some attribute of a city, e.g., climate, public transit, pollution, and we see wages and rents change, then the change must have affected productivity.
- If we change an attribute of a city and only rents change, then the changes must have been an amenity.
- Note that in this model, high rents are GOOD. They indicate a place is either very productive or very nice.
- However, if (e.g.) we see wages go up and rents go up, it's hard to tell if the city become more productive, or if it became more productive AND nicer. This is the problem **?** helps us solve.

- Note that, if we use aggregate land rent as a measure of welfare, we can look at how aggregate land rent changes in response to our policy, and use that to value the policy, and we don't really need to do anything else. But aggregate land rent can be hard to observe.

Monocentric/Roback model I



Note that (1) gap between $R(0)$ and w is filled partly by consumption, and partly by amenities (2) wages depend on city level productivity.

Roback Model I

The Roback model exploits the same basic intuition we just saw, but

- Commuting within a city is free and is dropped from the model.
- Households choose how much land to consume.
- Firms consume land.
- Firms and households compete for land in the city.

By abstracting from commuting, Roback gives us a more realistic description of the land market, and arrives at theory for understanding how changes in city attributes affect land and labor markets, and a method for inferring the value of city attributes from observable quantities.

Reading **?** requires a familiarity with multivariate calculus beyond the scope of this course. In order to understand this model, we're going to work through an example based on Cobb-Douglas production and utility functions that will (hopefully) be familiar to anyone who has taken intermediate micro.

The particular example we'll work assumes a city level amenity, A, that increases both the utility of residents and productivity of firms. This is something like 'days of sunshine'. One can also imagine 'amenities' that have opposite effects on firms and households. To accommodate this, one could replace A with $1/A$ in either the production or utility function given below.

The set up for households in the Roback model is similar to what we use for the monocentric city,

$$
\begin{aligned}
A \in [A_1, A_2] &\sim \text{Amenity level} \\
c &\sim \text{consumption} \\
\ell_c &\sim \text{residential land} \\
w &\sim \text{ wage/income} \\
r &\sim \text{ land rent} \\
\overline{u} &\sim \text{reservation utility}
\end{aligned}
$$

Households solve

$$
\begin{aligned}
&\max_{c,\ell_c} U(c, \ell_c, A) = Ac^{1-\alpha}\ell_c^{\alpha} \\
&\text{s.t. } w = c + r\ell_c
\end{aligned}
$$

and in spatial equilibrium, we have

$$U(c, \ell_c, A) = \overline{u}$$

To solve the household problem, solve the constraint for c and substitute back into the objective to get an unconstrained maximization problem in one variable,

$$\max_{c,\ell_c} A(w - r\ell_c)^{1-\alpha}\ell_c^{\alpha}$$

Differentiate and set the first order condition equal to zero,

$$\alpha A(w - r\ell_c)^{1-\alpha}\ell_c^{\alpha-1} + (1-\alpha)(-r)A(w - r\ell_c)^{-\alpha}\ell_c^{\alpha} = 0$$

rearranging and simplifying, we get the demand for residential land,

$$\ell_c = \alpha w / r$$

Substituting into the constraint, we can solve for the demand for consumption,

$$
\begin{aligned}
w &= c + r(\alpha w / r) \\
\implies c &= (1-\alpha) w
\end{aligned}
$$

We can now substitute both demand functions back into the utility function to get the indirect utility function. That is, utility as a function of prices and income rather than as a function of land and consumption,

$$
\begin{aligned}
V(w, r, A) &= U(((1-\alpha) w), (\alpha w / r), A) \\
&= A((1-\alpha) w)^{1-\alpha} (\alpha w / r)^{\alpha} \\
&= \alpha^{\alpha} (1-\alpha)^{1-\alpha} A \frac{w}{r^{\alpha}}
\end{aligned}
$$

In a spatial equilibrium, we must have

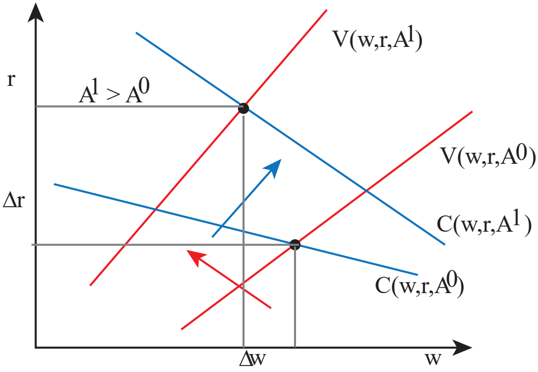
$$
\begin{aligned}
\bar{u} &= U(c, \ell_c, A) \\
\implies \bar{u} &= V(w, r, A) \\
\implies \bar{u} &= \alpha^{\alpha}(1-\alpha)^{1-\alpha} A \frac{w}{r^{\alpha}} \\
\implies r &= \left[\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha} A}{\bar{u}} \right] w^{1/\alpha}
\end{aligned}
$$

Using this last expression, we can evaluate the rate at which r changes with w in order to keep utility constant.

$$
\begin{aligned}
\frac{dr}{dw} &= \left[\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha} A}{\bar{u}} \right] \frac{1}{\alpha} w^{\frac{1-\alpha}{\alpha}} \\
&> 0
\end{aligned}
$$

N.B.: Finding the indirect utility function for Cobb-Douglas preferences, what we just did, is a standard exercise in an intermediate micro class. We didn't change anything about the standard problem except the variable names.
Three comments about indifference curves as rent and wages change,

- $\frac{dr}{dw} > 0$
- As A increases, the slope of the indifference curve increases and the curve gets steeper.
- As A increases, these indifference curves shift up and to the left. More amenities must be offset by a combination of lower wages and higher rent in order to keep utility constant.



The blue lines illustrate indifference curves in (w, r) space, and how they change as A increases. Higher wages must be offset by higher rents to stay on an isoquant. People accept lower wages and higher rents as amenities increase.

We now need to know how firms respond to prices wages and amenities. For this we need a little more notation,

$$N \in [A_1, A_2] \sim \text{labor/city population}$$
$$\ell_p \sim \text{commercial land}$$

Firms make the production good c, and as we implicitly assumed for the household problem, we have $p_c = 1$.

There is free entry of firms, so firm profits are driven to zero

Finally, we assume that production in constant returns to scale. That is, doubling inputs exactly double outputs.

The firm wants to choose inputs to produce outputs as cheaply as possible. That is, to produce Y units as cheaply as possible, the firm solves,

$$\begin{aligned} \min_{N,\ell_p} C(w.r, Y) =& wN + r\ell_p \\ & \text{s.t. } AN^{1-\beta}\ell_p^{\beta} = Y \end{aligned}$$

Because we have restricted attention to constant returns to scale, if we find the way to produce one unit as cheaply as possible, we can just multiply by Y to get the cost for Y units. So, we'll instead solve for the 'unit cost function',

$$\min_{N,\ell_p} C(w.r,1) = wN + r\ell_p$$
$$\text{s.t. } AN^{1-\beta}\ell_p^{\beta} = 1$$

To solve this, first solve the constraint for ℓ_p,

$$\ell_p = A^{-1/\beta}N^{\beta-1/\beta}$$

Substituting back into the firm's problem gives an unconstrained minimization problem in one variable,

$$\min_N wN + rA^{-1/\beta}N^{\frac{\beta-1}{\beta}}$$

We can solve by differentiating, setting the FOC to zero, and then isolating the cost minimizing level of employment,

$$0 = w + rA^{-1/\beta} \frac{\beta - 1}{\beta} N^{-1/\beta}$$
$$\implies N^* = \left[\frac{w}{r} \frac{\beta}{1-\beta} \right]^{-\beta} A^{-1}.$$

Substituting this back into the production function/constraint, we have

$$\ell_p^* = A^{-1/\beta} \left[\left[\frac{w}{r} \frac{\beta}{1-\beta} \right]^{-\beta} A^{-1} \right]^{1-\beta}$$
$$\implies = \frac{1}{A} \left[\frac{w\beta}{r(1-\beta)} \right]^{1-\beta}$$

If we substitute the two factor demands back into the cost function, we get (skiping a lot of algebra)

$$
\begin{aligned}
C(w, r, S) &= wN^* + r\ell_p^* \\
&= w\frac{1}{A}\left[\frac{w}{r}\frac{\beta}{1-\beta}\right]^{-\beta} + r\frac{1}{A}\left[\frac{w\beta}{r(1-\beta)}\right]^{1-\beta} \\
&= \frac{1}{\beta^\beta(1-\beta)^{(1-\beta)}}\frac{w^{1-\beta}r^\beta}{A}
\end{aligned}
$$

$C(w, r, S)$ is the cost minimizing way to make one unit of output at given input costs. With free entry of firms, profits must be driven to zero, and so we must have $C(w, r, S) = p_c = 1$ in any equilibrium.

This is the equation that tells us how production responds to wages, rents and amenities.

Therefore, in spatial equilibrium we have

$$
\begin{aligned}
1 &= C(w, r, S) \\
\implies 1 &= \frac{1}{\beta^{\beta}(1-\beta)^{(1-\beta)}} \frac{w^{1-\beta} r^{\beta}}{A} \\
\implies r^{-\beta} &= \frac{1}{\beta^{\beta}(1-\beta)^{(1-\beta)}} \frac{w^{1-\beta}}{A} \\
\implies r &= \left(\frac{1}{\beta^{\beta}(1-\beta)^{(1-\beta)}} \frac{1}{A} \right)^{\frac{-1}{\beta}} w^{\frac{\beta-1}{\beta}}
\end{aligned}
$$

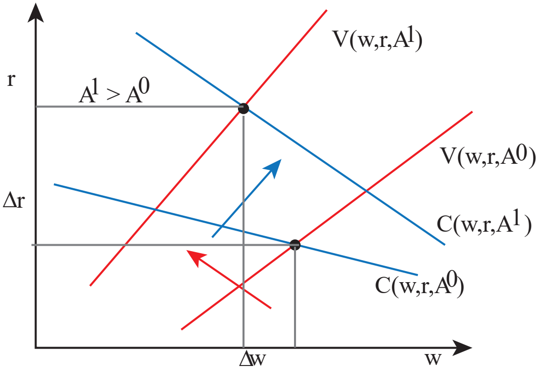
Differentiating this last expression with respect to w, we get that Therefore, we have

$$\frac{dr}{dw} = \left(\frac{1}{\beta^{\beta}(1-\beta)^{(1-\beta)}}\frac{1}{A}\right)^{\frac{-1}{\beta}} \frac{\beta - 1}{\beta} w^{\frac{-1}{\beta}}$$
$$< 0$$

This allows us to establish the following comparative statics.

- With $0 < \beta < 1, \frac{dr}{dw}$ is negative.
- $\frac{dr}{dw}$ is increasing in A for all w.
- As $\beta \to 0, \frac{dr}{dw} \to \infty$. That is, as land becomes less important in production, equilibrium rents need to change more to offset a given change in wages. This seems intuitive.
- As A increases, productivity increases. To keep costs constant, we require a combination of higher wages and higher rent, so the isocost curve must shift up and to the right.

Equilibrium (1) I



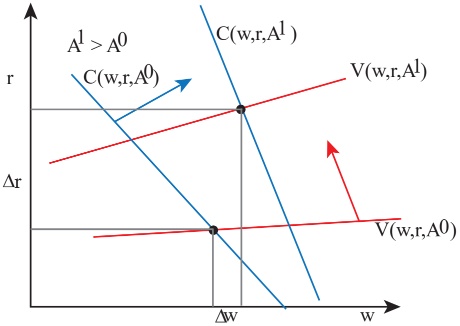
Equilibrium (1) II

Red lines are isocost lines. To keep costs constant, rents must decrease as wages increase. Firms can pay hihger wages and rents as productivity increases.

Equilibria occur at intersections of isocost and indifference curves.

In this example, as A increases, wages decrease and rent increases. Why? The increase in amenities draws in more people. The resulting increase in rents causes firms to substitute away from land. This lowers the marginal product of labor, in spite of the fact that the increase in A increases productivity.

Equilibrium (2) I



It doesn't have to work out this way. Here, as A increases, we have an increase in wages and rents. Why? In this case, the increase in productivity from increasing amenities more than compensates for

Equilibrium (2) II

the decrease in labor productivity caused by the increasing scarcity of land.

If *A* is bad for firms and households, then the arrows in the pictures point the other way.

What happens if *A* is good for Households, but bad for firms?

A little technical aside I

All of this hardware gives us two handy ways to think about the 'value of A'. Because A is often place specific attributes that don't have an explicit market price, e.g., 'the absence of crime' or 'days of sunshine', this is handy.

Before we state these results, we need two more tricks.

- We can derive a 'price' from V by using Roy's identity (more-or-less). That is, define

$$p_A \equiv \frac{\frac{\partial V}{\partial A}}{\frac{\partial V}{\partial w}}$$

  In words, the 'price' of A is the ratio of the marginal utility of A to the marginal utility of income. This is a standard trick, and

A little technical aside II

is just slightly different than Roy's identity (For Roy's identity we would differentiate by the price of *A*, not *A*).

- If x is a function of A, then recalling the derivative of a logarithm, we have

$$\frac{1}{A}\frac{dx}{dA} = \frac{d}{dA}\ln x(A)$$

Roback Theorem

We can now state the first part of Roback's main theorem:

$$p_A \equiv \frac{\frac{\partial V}{\partial A}}{\frac{\partial V}{\partial w}} = \ell_c \frac{dr}{dA} - \frac{dw}{dA}$$

Dividing both sides by w, we can restate this result as

$$\begin{aligned} \frac{p_A}{w} &= \frac{\ell_c r}{w} \frac{1}{r} \frac{dr}{dA} - \frac{1}{r} \frac{dw}{dA} \\ &= \frac{\ell_c r}{w} \frac{d \ln R}{dA} - \frac{d \ln w}{dA} \end{aligned}$$

To see why this is a big deal, consider data describing rents, wages and amenities for a set of cities $i = 1, .., K$. If we perform the regressions,

$$r_i = B_0, + B_1 A_i + \epsilon_i$$
$$w_i = C_0, + C_1 A_i + \mu_i$$

then $\frac{dr}{dA} = B_1$ and $\frac{dw}{dA} = C_1$. If we use these estimates in the Roback theorem, we have

$$p_A = \ell_c B_1 - C_1$$

Recalling that ℓ_c is household land consumption, this means we can calculate the value of an amenity, like climate, crime or good schools from easily available data. This is an important and widely used tool for this purpose. If you want to know whether you should direct public dollars to crime reduction or better parks, this is a pretty reasonable place to start.

Note how intuitive this is. The value of a marginal increase in the amenity A to a household reflects the change in wage and the total change housing expenditure required to obtain it. The result is 'marginal' in the sense that we are considering small enough changes in A the household does not reoptimize its choice of ℓ_c. (If you have studied the envelope theorem, you may recognize an application of it in this result).

The second way of stating the result gives and expression for $\frac{p_A}{w}$. This measures the importance of amenities, in real terms, and is widely used as an index for 'quality of life' in a city. More on this shortly.

We can write this index in terms of the share of income used for housing, $\frac{\ell_c r}{w}$, $\frac{d \ln R}{dA}$, and $\frac{d \ln w}{dA}$. We can calculate from our other regression coefficients, e.g., $\frac{d \ln R}{dA} = \frac{1}{A} B_1$ or estimate them directly, e.g.

$$\ln r_i = \widetilde{B}_0, + \widetilde{B}_1 A_i + \epsilon_i,$$

... if we have done everything right, we should get $\widetilde{B}_1 = \frac{1}{A} B_1$.

The second part of the main Roback Theorem applies about the same logic to firms, to arrive at estimates of the effect of amenities on costs. These results are, We can now state the first part of Roback's main theorem.

$$\frac{dC}{dA} = \frac{N}{Y}\frac{dr}{dA} - \frac{\ell_p}{Y}\frac{dw}{dA}$$

Here, N is total population of the city, which is assumed equal to its labor supply. Like the result for p_A, this lets us calculate the marginal effect of A on unit costs of A from things we can observe easily. This result is used less widely than the result for households, but if you want check a statement like ‘better public transit improves productivity’, this might be a good place to start.

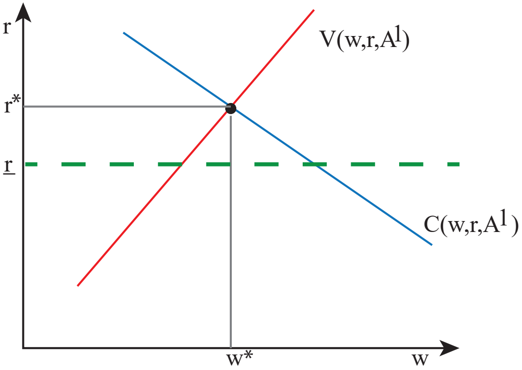
Roback model – comments

Two comments about the Roback model:

First, **?** generalizes her model to allow households to consumer housing instead of land. This requires the addition of a construction sector where free entry drives profits to zero and construction is constant returns to scale. Conceptually, this generalization is very similar to the case we have treated, but results in a system of three equations, one for firms, one for households, and one for the construction sector. This more complicated formulation is often used as the basis for empirical work.

Second, if you read the paper carefully, you will notice that the paper does not describe the land supply. Unlike the monocentric city model, where land is available if you will pay a little more commuting costs, their is no description of how much land is available, and at what cost.

To see how this can create a problem, suppose that land is supplied perfectly elastically at price $\underline{r}$, much like agricultural land in the monocentric city model. If we then try to choose rents, wages to satisfy firm and household first order conditions and the supply condition for land, we'll run into trouble.



Here, we cannot satisfy the land supply condition and both first order conditions. This appears to be an important, but not widely noted, shortcoming of this framework

Application #1 Climate

There have been several papers that use the Roback framework, or something similar to it, to try to value climate. For example, **?** or **?**.

The basic quality of life index from Roback is,

$$\frac{p_A}{w} = \frac{\ell_c r}{w}\frac{d \ln R}{dA} - \frac{d \ln w}{dA}$$

where we estimate

$$r_i = B_0, + B_1 A_i + \epsilon_i$$
$$w_i = C_0, + C_1 A_i + \mu_i$$

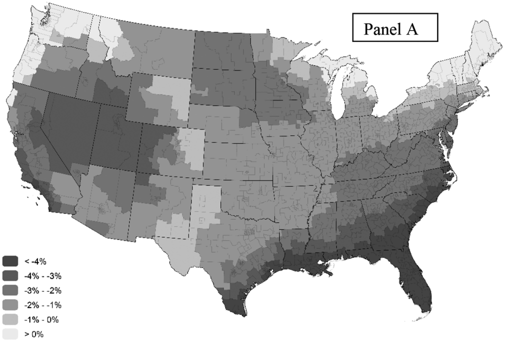
to get estimates of $\frac{d \ln R}{dA}$ and $\frac{d \ln w}{dA}$.

To use this framework to think about the value of climate change, **?** makes several changes to this set-up. Of these, some are technical and I'll skip them. Three of them are more practical and deserve comment,

- The share of expenditure on housing is typically around 25%. For the purpose of the Roback type exercise, it probably makes sense to think of the 'income share of housing' as the 'income share of housing and all other goods not traded across cities'. This increases the weight placed on the $\frac{d \ln R}{dA}$ term in the index.
- Most income is taxed, and tax rates vary across cities. Given this, it probably makes sense to base the analysis on the after tax wage. Practically, this means scaling down the $\frac{d \ln w}{dA}$ term by $1 - \tau$, where τ is the relevant tax rate on income.

- In addition to this, ‘climate’ is complicated, and if we want to describe it, we’ll need a vector, e.g. spring, summer, fall, winter temperature and rainfall. Up until now, we’ve been thinking of A as a scalar. We’ll need to evaluate each of these climate variables, and then consider the total value of a hypothetical change in all of them.

? conduct this exercise. They use wage, and real estate data from the 2000 US census, and a change in climate consistent with ‘business as usual’ in 2100, that is, no effort at carbon reduction. This leads them to produce the following map.



This seems pretty sensible. The biggest declines in the quality of life are in hot places, and some places that are too cold get better.

Notice that applying the Roback framework in this way does not ‘conserve people’. When we change climate in a place, the

Roback model is going to require a change in labor forces, and hence in population. When we apply a hypothetical climate change to the whole country, there is nothing to guaranty that the aggregate change in population sums to zero. **?** discusses a technique for addressing this problem, but it's pretty hard.

Application #2 Income taxes I

Thinking about income taxes in the context of the Roback model raises a number interesting issues.

- People require higher wages to live in worse places and accept lower wages to live in nicer places. If income is taxed, this creates and implicit subsidy for amenities. In a spatial equilibrium, this is going to shift people away from places with bad weather.
- People receive higher wages to go work in more productive places. If income is taxed, in a spatial equilibrium, this is going to tend to push people into less productive places.

Application #2 Income taxes II

- Increasing marginal tax rates make both of these issues more complicated. Moreover, places that pay high taxes, on average, receive less money back from the federal government than they pay in, and conversely.
- The ‘Home Mortgage Tax deduction’ allows people to pay mortgage interest with before tax dollars. This benefits people who live in places where amenitity values are capitalized into housing prices, and tends to benefit people with higher incomes, those who live in productive places.

Application #2 Income taxes III

In all, this suggests that we should worry about how INCOME taxes are capitalized into local wages and real estate prices, but it is hard to guess how all these effects will net out. **?** considers this issue and finds that the tax system is having important implications on how people are organized across the landscape.

Here is his main table of results,

TABLE 2
Tax Differentials and Their Effects on Prices and Employment, 2000

	Tax Payment Rank	Federal Tax Differential: From Wage (1)	Federal Tax Differential: From Deduct (2)	Federal Tax Differential: Total Federal (3)	State Tax Differential (4)	Total Tax Differential (5)	Total Tax Differential Effects: Wage (6)	Total Tax Differential Effects: Housing Cost (7)	Total Tax Differential Effects: Land Rent (8)	Total Tax Differential Effects: Employment (9)
Metro area:										
San Francisco, CA	1	.068	−.023	.045	.004	.048	.015	−.103	−.480	−.288
New York, NY	2	.054	−.013	.041	.002	.043	.013	−.093	−.434	−.261
Detroit, MI	3	.035	−.003	.032	.004	.036	.011	−.076	−.355	−.213
Hartford, CT	4	.039	−.005	.035	.001	.035	.011	−.076	−.352	−.211
Chicago, IL	6	.035	−.007	.029	.003	.031	.009	−.067	−.310	−.186
Washington, DC	7	.034	−.005	.029	.002	.031	.009	−.066	−.306	−.183
Philadelphia, PA	8	.030	−.002	.028	.002	.030	.009	−.065	−.303	−.182
Boston, MA	9	.035	−.011	.025	.001	.026	.008	−.056	−.259	−.155
Minneapolis, MN	10	.023	−.002	.021	.005	.026	.008	−.055	−.256	−.154
Los Angeles, CA	15	.033	−.012	.021	.000	.021	.006	−.045	−.209	−.126
Jacksonville, FL	114	−.019	.003	−.016	.000	−.015	−.005	.033	.153	.092
Oklahoma City, OK	160	−.032	.006	−.026	.002	−.024	−.007	.051	.239	.143
Norfolk, VA	188	−.028	.002	−.026	−.004	−.030	−.009	.064	.300	.180
Tucson, AZ	195	−.029	.000	−.029	−.003	−.032	−.010	.069	.322	.193
Killeen, TX	241	−.060	.007	−.053	−.005	−.058	−.018	.125	.582	.349