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Spatial Competition with a Land Market: Hotelling and Von Thunen Unified

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We introduce into the standard spatial competition model the consumption of land by households, and study the spatial competition under the influence of a land market. In contrast to the standard assumption of a fixed, given distribution of households, we introduce the possibility of households' relocation in reaction to firms' location decisions. Thus, the spatial distribution of households is treated as endogenous, and a land market is introduced on which households compete for land-use. Households consume simultaneously land and firms' output. Accordingly, the demand of each household becomes in turn endogenous as it depends on the income left after the land rent is paid. Not surprisingly, the results obtained within this more general framework prove to be very different from the standard results. For example, in the 2- and 3-firm cases, the optimal configuration of firms is a Nash equilibrium when transport costs are high enough or the amount of vacant land is large enough. The existence property is restored in the 3-firm case when the transport costs are high enough. The introduction of vacant land causes a discontinuous change in the set of equilibrium configurations.

1. INTRODUCTION

The model of spatial competition has been devised to study the oligopolistic process generated by a market operating with dispersed firms and households. In the simplest case, households purchase a single commodity from the nearest firms, and firms compete on locations so as to maximize profits. An equilibrium configuration is obtained when, given the locations of the other firms, no firm can make higher profits at an alternative location. Conditions for the existence of such an equilibrium are well-known and are discussed in many places (see, e.g. Eaton and Lipsey (1975)).

In all the contributions to spatial competition we are aware of (see Jaskold-Gabszewicz and Thisse (1986) for a recent survey), the spatial distribution of households is considered as exogeneous (usually, a uniform distribution is assumed). This is a reasonable assumption when the model is interpreted within the framework of product differentiation. Indeed, households' locations correspond to ideal points in a Lancasterian space of characteristics. Hence, since these points reflect households' preferences, they can be taken as given and fixed. By contrast, in the geographical interpretation of the model, households' locations are normally variable. In response to firms' location decisions, households may decide to re-establish themselves into a new residential pattern. In this case, the model should be generalized to allow for an endogeneous distribution of households over space. For this distribution to be nondegenerate, a *land market* must then be introduced in which households are competing for land-use (see Fujita (1986) for a recent survey of land-use theory).

Furthermore, in the standard model of spatial competition, the demand of a household for the firms' output is also given *a priori* (typically, the individual demand is supposed to be perfectly inelastic). However, when households compete in the land market, the demand of a household becomes in turn endogeneous, as it now depends on the income left after the land rent is paid. Given that the residential pattern depends on firms' locations, the demand of a household turns out to be an *endogeneous* function of firms' locations. (Notice that this is different from the model of spatial competition in which households are *given* an elastic demand (see, e.g. Smithies (1941)). Here the demand function is determined within the model).

In this paper, an attempt is made to study the process of spatial competition when households are assumed to consume *simultaneously* land and firms' output. This brings into the picture some "general equilibrium" ingredients in that firms' and households' locations are *interdependent* (see also Fujita (1985)). Not surprisingly, the results obtained within this more general framework prove to be very different from the standard results.

We limit ourselves to the particular, albeit meaningful, cases of 2- and 3-firms competing on location (while prices are taken as parametric). In the standard model, it is well known that (i) in the 2-firm case, a unique location equilibrium is obtained when the two firms are clustered at the market center; and (ii) in the 3-firm case, there exist no location equilibria. In the model with a land market, the main results are as follows: (a) the set of location equilibria is substantially enlarged, in that new and very different configurations may emerge at equilibrium; (b) in the 2-firm case, the agglomeration is no longer an equilibrium when there is vacant land; (c) in the 3-firm case, the existence of a location equilibrium is restored when transport costs are high enough or when there is enough vacant land; (d) asymmetric configurations may occur in the 3-firm case; (e) the optimal configuration of firms is a location equilibrium if the transport rate is sufficiently large or if the amount of excess land is large enough; and (f) the introduction of vacant land causes a discontinuous change in the set of location equilibria.

These results are in sharp contrast to those obtained in the standard case. The main reason is that, because of the existence of a land market, the individual consumption of firms' output now changes endogeneously with firms' locations. Quite often, this new effect proves to be sufficient to overwhelm the awkward implications of the market area effect emphasized by Hotelling and his successors. Moreover, it turns out that the vacant and nonvacant land cases yield different results. This is so because, in the nonvacant land case, the individual demand for firms' output is affected only through the income effect generated by the land rent. On the other hand, in the vacant land case, households' locations also change, which adds a new dimension to the individual consumption effect.

The paper is organized as follows. The model is presented in Section 2. Sections 3 and 4 deal with the 2- and 3-firm cases respectively. Each of these sections is subdivided into two subsections devoted to the nonvacant land and vacant land cases. Section 5 discusses a reformulation of the model under an alternative land ownership assumption. Finally, Section 6 contains our conclusions.

2. THE MODEL

Imagine a long-narrow residential land of length l and width 1. Since the width of the land is sufficiently small, the residential land is treated as one-dimensional, and represented by the interval $X=[0, l]$. Location and distance from the origin are identically denoted by $x \in X$. Idential N households reside on X. Since N is sufficiently large, the distribution of households over the land can be treated in terms of density. The entire land is owned by absentee landlords.[1]

The utility function of each household is given by $U(z, s)$, where z represents the amount of the composite consumer good, and s the consumption of land, or the lot size of the house. The composite good z is supplied by m firms (or supermarkets), $i=1, 2, \ldots, m$, who sell it at the same, fixed f.o.b. price p.[2] All firms are owned by absentee shareholders. Define $I=\{1, 2, \ldots, m\}$ and $I_{-i}=\{1, 2, \ldots, i-1, i+1, \ldots, m\}$. *Let* $x_i \in X$ be the location of firm $i \in I$; denote $\underset{\sim}{x}=(x_1, x_2, \ldots, x_m)$ and $\underset{\sim}{x}_{-i}=(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. Finally, for simplicity, it is assumed that firms do not use land.

We assume that firms and consumers choose their location one at a time in a *two-stage process.* In the first stage, firms choose their location according to rules that will be described below. In the second stage, consumers choose their location taking the configuration of firms $\underset{\sim}{x}$ as given.

The outcome of the second stage is a residential equilibrium under the configuration $\underset{\sim}{x}$. Let us describe it in detail. The income of each household is exogenously given by Y which is spent on the composite good (bought from the nearest firm) and land. For each $x \in X$, let us define the *minimum travel distance* as

$$r(x) \equiv r(x \mid \underset{\sim}{x}) = \min_{i \in I} |x_i - x|. \tag{1}$$

Then, for a household locating at x, the budget constraint is given by $(p+ar(x))z+R(x)s=Y$, where $a>0$ is the transport cost per unit of z per unit of distance, and $R(x)$ is the land rent per unit of land at x. Assuming that each household can freely choose its location and lot size, the residential choice behaviour of each household can be expressed as

$$\max_{x,z,s} U(z, s), \text{ s.t. } (p+ar(x))z+R(x)s=Y.$$

For simplicity, we assume that the lot size of each house is fixed at one unit $(s=1)$.[3] Then, since the maximization of U is equivalent to maximization of z, the residential choice of each household can be restated as

$$\max_x z \equiv (Y-R(x))/(p+ar(x)). \tag{2}$$

Given $\underset{\sim}{x}$, define the bid rent function $\Psi(x, z) \equiv \Psi(x, z \mid \underset{\sim}{x})$ of each household as

$$\Psi(x, z) = Y-(p+ar(x))z. \tag{3}$$

Let $n(x)$ be a density distribution of households over X, where $n(x) \in \{0, 1\}$. The support X_+ of distribution $n(x)$ is defined by

$$X_+ = \{x \in X;\, n(x)=1\}. \tag{4}$$

At a residential equilibrium, all households attain the same maximum utility, and hence they consume the same amount z^* of the composite consumer good. For this to be possible, the land rent curve $R(x)$ must satisfy the relations $R(x)=\max\{\Psi(x, z^*), 0\}$ and $R(x)=\Psi(x, z^*)$ at each $x \in X_+$. Accordingly $(z^*, n^*(x), R^*(x))$ is a *residential equilibrium* under $\underset{\sim}{x}$ if and only if the following conditions hold:

$$R^*(x) = \max\{\Psi(x, z^*), 0\}, \quad \text{at each } x \in X, \tag{5}$$

$$n^*(x) \in \{0, 1\} \quad \text{at each } x \in X, \tag{6}$$

$$R^*(x) = \Psi(x, z^*) \quad \text{if } x \in X_+, \tag{7}$$

$$n^*(x) = 1 \quad \text{if } R^*(x) > 0, \tag{8}$$

$$\bar{X}_+ = N, \tag{9}$$

where $\bar{X}_+$ is the length of X_+. Condition (8) means that no vacant land is left in the area with positive land rent. Condition (9) represents the population constraint.

In order to assure the existence of a residential equilibrium, we always assume that $l \geqq N$. Given $\underset{\sim}{x}$, we define the *maximum travel distance* $\delta \equiv \delta(\underset{\sim}{x})$ as

$$\delta = \sup_{x \in X_+} r(x). \tag{10}$$

We normalize the land rent configuration so that[4]

$$R(x) = 0 \quad \text{if } r(x) = \delta. \tag{11}$$

We also adopt the convention,

$$n(x) = 1 \quad \text{if } r(x) = \delta. \tag{12}$$

Then equilibrium conditions (5)-(9) can be rewritten as follows:

$$z^* = Y/(p + a\delta), \tag{13}$$

$$R^*(x) = z^* \max\{a\delta - ar(x), 0\}, \tag{14}$$

$$n^*(x) = 1 \quad \text{if } r(x) \leqq \delta,$$
$$= 0 \quad \text{if } r(x) > \delta, \tag{15}$$

$$\bar{X}_+ = N. \tag{16}$$

It is not difficult to see that under any $\underset{\sim}{x}$, relations (13)-(16) determine a unique residential equilibrium. In equilibrium, individual demand for the composite good, i.e. z^* as given by (13), depends on the configuration $\underset{\sim}{x}$ through δ.

Now consider the first stage. Firms choose their location anticipating consumers' reaction to a change in the configuration of firms. The outcome of the first stage is a location equilibrium for the n firms. Let $\underset{\sim}{x} = (x_1, x_2, \ldots, x_m)$ be a location configuration of firms and $(z^*, n^*(x), R^*(x)) \equiv (z^*(\underset{\sim}{x}), n^*(x|\underset{\sim}{x}), R^*(x|\underset{\sim}{x}))$ be the residential equilibrium under $\underset{\sim}{x}$. The *market area* $M_i \equiv M_i(\underset{\sim}{x})$ of firm $i \in I$ is defined as

$$M_i = \{x \in X_+; |x_i - x| = r(x)\}. \tag{17}$$

Let $m(x) \equiv m(x|\underset{\sim}{x})$ be the number of firms at $x \in X$. Then, the *total sales* of firm $i \in I$ are as follows:[5]

$$Q_i(x_i|\underset{\sim}{x}_{-i}) = z^* \bar{M}_i / m(x_i), \tag{18}$$

where $\bar{M}_i$ is the length of M_i. Clearly, the sales of firm i depend not only upon firm i's position in $\underset{\sim}{x}$ but also upon consumers' land use X_+ and consumption level z^*.

We assume that all firms have the same, constant marginal cost $c < p$. Each firm chooses its location so as to maximize its profit assuming that locations of other firms are unchanged. Since the profit per unit of good is a constant $p - c$, the payoff function of firm i can be chosen as $Q_i(x_i|\underset{\sim}{x}_{-i})$. We say that $\underset{\sim}{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ is a *location equilibrium* if and only if

$$Q_i(x_i^*|\underset{\sim}{x}_i^*) = \max_{x_i \in X} Q_i(x_i|\underset{\sim}{x}_i^*), \quad \text{for all } i \in I. \tag{19}$$

In game-theoretic words, this means that a location equilibrium is a Nash equilibrium of a noncooperative game whose players are firms, payoffs are sales and strategies are locations. Without loss of generality, we take

$$x_1^* \leqq x_2^* \leqq \cdots \leqq x_m^*. \tag{20}$$

Notice that function $Q_i(x_i \mid \underset{\sim}{x}_{-i})$ is differentiable at x_i only if $m(x_i) = 1$. In this case, we have

$$\frac{\partial Q_i}{\partial x_i} \equiv \frac{\partial Q_i(x_i \mid \underset{\sim}{x}_{-i})}{\partial x_i} = \frac{\partial z^*}{\partial x_i} \bar{M}_i + z^* \frac{\partial \bar{M}_i}{\partial x_i}, \quad \text{for all } i \in I. \tag{21}$$

This expression shows that the impact on sales of a change in the firm's location results from the combination of two different effects: (i) a *consumption effect*, in which $\partial z^*/\partial x_i$ is different from zero whenever the change in x_i affects the maximum travel distance δ; a *market area effect* similar to that studied in the standard model of spatial competition. As $\partial z^*/\partial x_i$ can be negative, the sign of $\partial Q_i/\partial x_i$ is *a priori* undetermined.

Let us now summarize our equilibrium concept. Given a configuration $\underset{\sim}{x}$ of firms, consumers choose their location at the corresponding residential equilibrium, which is of the competitive type. With respect to firms, consumers are the followers of a Stackelberg game in which firms are the leaders. Finally, firms choose their location at the Nash equilibrium of a noncooperative game whose players are the firms. The separation into two stages is dictated by the fact that each household's location decision has a negligible impact on the market solution, while each firm's location decision turns out to be a strategic choice.

Finally, let us consider the optimal configurations for the above economy. As the lot size is exogeneously given, the specification of a utility level common to all households is equivalent to the specification of a consumption level z. Given z, the production and transport cost corresponding to the firm configuration $\underset{\sim}{x}$ and the household distribution $n(x)$ is

$$C(n(x), \underset{\sim}{x}, z) = cNz + \int_X zn(x)ar(x \mid \underset{\sim}{x})\,dx. \tag{22}$$

Given z and $\underset{\sim}{x}$, is is easy to see that the solution to the problem

$$\begin{aligned} &\min_{n(x)} C(n(x), \underset{\sim}{x}, z) \\ &\text{s.t. } n(x) \in \{0, 1\}, \forall x \in X \text{ and } \int_X n(x)\,dx = N \end{aligned} \tag{23}$$

is the household distribution $n^*(x \mid \underset{\sim}{x})$ in the residential equilibrium under $\underset{\sim}{x}$ (defined by (13)-(15)). A configuration of firms is said to be *optimal* if and only if it is a solution of the following problem:

$$\begin{aligned} &\min_{\underset{\sim}{x}} C(n^*(x \mid \underset{\sim}{x}), \underset{\sim}{x}, z) \\ &\text{s.t. } \underset{\sim}{x} \in X^m. \end{aligned} \tag{24}$$

Clearly, this solution is independent of the level of consumption z.

3. THE 2-FIRM CASE

3.1. *Nonvacant land*

Let us assume that $l = N$ so that no vacant land exists. Since households are evenly distributed over X in any residential equilibrium, our problem resembles the standard

Hotelling problem with fixed prices. However, in our model, the total quantity demanded by consumers is no longer constant but depends on the firms' locations through the level z^* of the individual demands. As a result, the profit functions are different and the two problems are not equivalent.

As there is no vacant land, it turns out that $\bar{M}_1=(x_1+x_2)/2$ and $\bar{M}_2=l-(x_1+x_2)/2$. Therefore, (21) can be rewritten as

$$\frac{\partial Q_1}{\partial x_1}=\frac{\partial z^*}{\partial x_1}\cdot\left(\frac{x_1+x_2}{2}\right)+\frac{z^*}{2} \tag{25}$$

and

$$\frac{\partial Q_2}{\partial x_2}=\frac{\partial z^*}{\partial x_2}\cdot\left(l-\frac{x_1+x_2}{2}\right)-\frac{z^*}{2} \tag{26}$$

with z^* given by (13). The following result is then proved in Appendix 1.

Theorem 1. *Assume that $m=2$ and $l=N$. Then,*

(i) *the central agglomeration $(l/2, l/2)$ is a location equilibrium;*

(ii) *any configuration $(l-x_2^*, x_2^*)$ such that* $\max\{l/2, p/a\}\leqq x_2^*\leqq 3l/4$ *is a location equilibrium;*

(iii) *there exists no location equilibrium other than those considered in* (i) *and* (ii).

The following comments are in order. First, as in the standard model, *the clustering of the two firms at the market centre is a location equilibrium.* Intuitively, the argument is as follows. Given $x_1<x_2=l/2$, an increase in x_1 does not change the equilibrium consumption z^* since it does not change the maximum travel distance δ which remains equal to $l/2$ (see Figure 2(a)). But market size $\bar{M}_1$ increases with x_1. As a result, given (21), when firm 2 is at the centre it is profitable for firm 1 to locate at $l/2$. Second, unlike the standard model, *dispersed equilibria may emerge.* Suppose $l/2<x_2\leqq 3l/4$. Then, any inward movement of firm 1 away from $l-x_2$ leads to an increase in the maximum travel distance δ which, in turn, causes a decrease in the equilibrium consumption z^* (see Figure 2(b)). Given (13), when the transport rate a is sufficiently high, the decrease in Q_1, resulting from the decrease in z^* dominates the increase in Q_1, resulting from the increase in market size $\bar{M}_1$ (see (21)). Thus, firm 1 has no incentive to move toward firm 2 so that the symmetric arrangement $(l-x_2, x_2)$ is a location equilibrium. Third, and last, among the possible dispersed equilibria is the configuration given by the first and third quartiles, i.e. $(l/4, 3l/4)$. This implies that, *when $p/a\leqq 3l/4$, the optimal configuration can be sustained as a location equilibrium.* Such a result is in sharp contrast with the standard location model whose solution is always inefficient. Intuitively, the reason is as follows. For a high enough transport rate, we know that the negative impact on z^* becomes dominant. One possible way of reducing this effect is then to minimize the maximum travel distance, i.e. to have the firms locate at the efficient locations.

The set of possible equilibria is depicted in Figure 1. The shaded and dotted areas represent the possible values for x_1 and x_2 respectively together with the corresponding possible values for p/a. Notice that $x_1=x_2=l/2$ is a location equilibrium under any value of p/a.

The change in land rent curve corresponding to a movement in x_1 is illustrated in Figures 2(a) and 2(b) for $x_2=l/2$ and $x_2=3l/4$.

When the transport rate is high, there are infinitely many location equilibria. Some of them have interesting properties, as shown below.

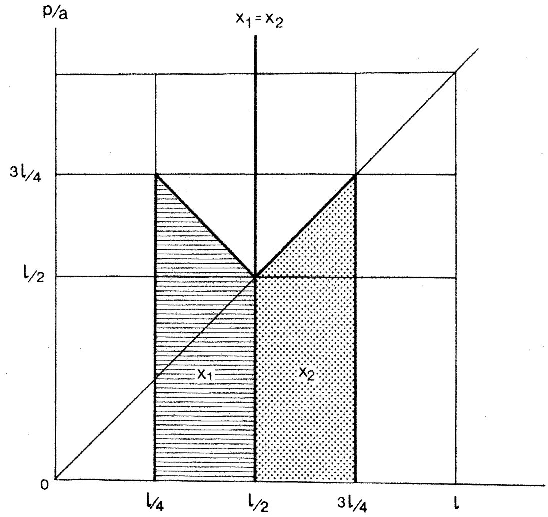

FIGURE 1
Equilibrium locations ($m=2$, $l=N$)

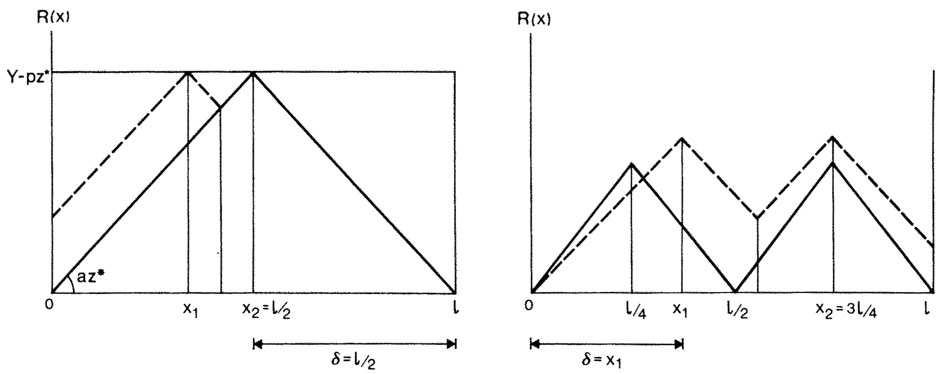

FIGURE 2
(a) Land rent curve for $x_2 = l/2$, (b) Land rent curve for $x_2 = 3l/4$

Theorem 2. *Assume that $m=2$ and $l=N$. If $p/a \leqq 3l/4$, then*

(i) *both the equilibrium utility and profit levels are maximized at the location equilibrium $(l/4, 3l/4)$;*

(ii) *the total equilibrium land rent is maximized at some unique location equilibrium $(l-x_2^*, x_2^*)$ such that $l/2 < x_2^* < 3l/4$.*

Proof. From Theorem 1, it follows that $\delta = l - x_2^*$ for any location equilibrium. Thus, given (13), z^* is maximum and hence the equilibrium utility level is maximized at $x_2^* = 3l/4$. Since $Q_1 = z^* l/2$ under each location equilibrium, the profit of each firm is maximized when z^* is maximum, i.e. when $x_2^* = 3l/4$. Finally, statement (ii) can be shown by a straightforward calculation. ||

3.2. *Vacant land*

We assume that $l = N + \varepsilon$ with $\varepsilon > 0$ so that there is vacant land in any residential equilibrium. This means that the household distribution is no longer uniform over X and depends on the locations chosen by the two firms.

The first major change from the nonvacant land case is as follows: the (central) agglomeration is no longer a location equilibrium. Suppose, indeed, that the two firms are located together at point A in Figure 3. Then, there exists vacant land on both sides of X. The maximum travel distance δ equals $N/2$ and z^* is the corresponding consumption level. Assume now that firm 1 moves to point B in Figure 3, while firm 2 remains at A. Then, at the new residential equilibrium, some households previously located on the right of A are now situated on the left of B in a way such that both firms have the same market area. However, since the maximum travel distance δ' is smaller than $N/2$, it follows from (13) that the new equilibrium consumption z' is larger than z^*. Therefore $Q_1(B|A) > Q_1(A|A)$ and, hence, A is not the optimal location for firm 1 when firm 2 is established at A.

More generally, we have the following result (see Appendix 2 for a proof):

Lemma 1. *Assume $m = 2$ and $l > N$. If (x_1^*, x_2^*) is a location equilibrium, then $R^*(0) = 0$, $R^*(l) = 0$ and $R^*((x_1^* + x_2^*)/2) = 0$.*

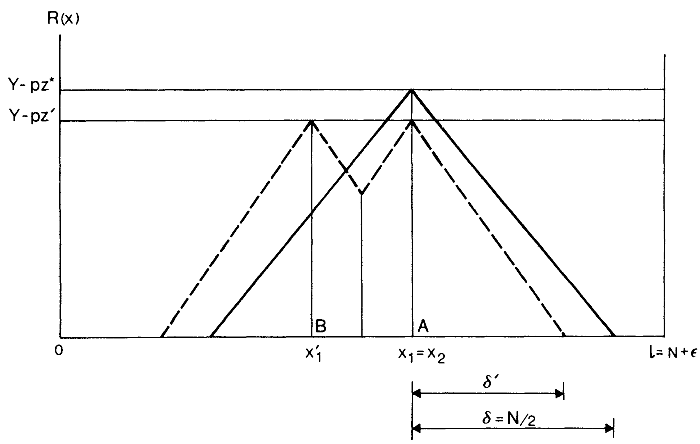

FIGURE 3

With vacant land, locating together is not an equilibrium

From this, we can conclude that, if it exists, a location equilibrium must be such that $x_1^* \geqq N/4$, $x_2^* \leqq l - N/4$ and $x_2^* - x_1^* \geqq N/2$. That is, *if vacant land exists*, however small it is, *then the only possible equilibrium configurations are those in which the two firms are located at the centre of their own market areas and capture the same number of customers.* An illustration is provided in Figure 4 where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$.

In consequence, we have:

Theorem 3. *Assume $m = 2$ and $l > N$. Then any location equilibrium is optimal.*

Because of the existence of vacant land consumers' locations are variable. Hence, if there exists a location equilibrium, both firms are always able to choose locations which

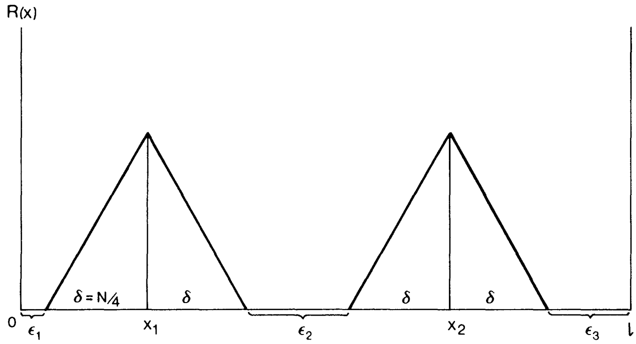

FIGURE 4
A possible equilibrium configuration ($m=2$, $\varepsilon>0$)

minimize the maximum travel distance δ without reducing their market areas. This implies that the equilibrium consumption z^* and, therefore, firms' volume of sales are maximized at the corresponding locations which are also optimal.

What about now the existence of a location equilibrium? The next result gives a complete characterization of the existence conditions (see Appendix 3 for a proof):

Theorem 4. *Assume $m=2$ and $l>N$. Then (x_1^*, x_2^*) is a location equilibrium if and only if*

$$x_1^* \in \left[\max\left\{N/4, \left(N-\frac{N}{2-(N/2)(a/p)}\right)f(p/a)\right\}, N/4+\varepsilon\right],$$

$$x_2^* \in \left[3N/4, \min\left\{3n/4+\varepsilon, l-\left(N-\frac{N}{2-(N/2)(a/p)}\right)f(p/a)\right\}\right],$$

and $x_2^*-x_1^* \geqq N/2$, where $f(p/a)=1$ if $p/a>3N/4$ and $f(p/a)=0$ otherwise.

Define $\hat{k}=(3N/4)(1-2\varepsilon/3N)/(1-2\varepsilon/N)(>3N/4)$, which is the solution of $N-N/(2-(N/2)k)=N/4+\varepsilon/2$ (refer to Figure 5). Theorem 4 then implies:

(i) if $\varepsilon \geqq N/2$, then there are infinitely many location equilibria;

(ii) if $N/2>\varepsilon>0$, then there exists no location equilibrium when $p/a>\hat{k}$, a unique location equilibrium when $p/a=\hat{k}$ and an infinity of location equilibria when $p/a<\hat{k}$.[6]

Equilibrium locations are depicted in Figures 5 and 6. Consider first Figure 5 coresponding to the case $N/2>\varepsilon$. For each value of p/a, the distance between curve AB and the left vertical line represents the minimum size of firm 1's hinterland such that firm 2 cannot increase its sales by approaching firm 1. The distance between curve CD and the right vertical line can be similarly interpreted. Furthermore, curves AB and $A'B'$ are horizontally parallel with distance $N/2$. The same is true with curves CD and $C'D'$. Let us now draw a horizontal line with height $p/a \leqq \hat{k}$. On this horizontal line, if x_1 belongs to the shaded area, x_2 to the dotted area and $x_2-x_1 \geqq N/2$, then (x_1, x_2) is a location equilibrium. From Figure 5, we can see that, when ε is very small, the equilibrium configurations are such that firms 1 and 2 are located close to the first and third quartiles respectively. This shows that *the introduction of any small amount of excess land leads to a "catastrophic" change in equilibrium configurations* (contrast with Theorem 1). Furthermore, as ε approaches $N/2$, $\hat{k}$ tends to infinity and an equilibrium always exists. Figure

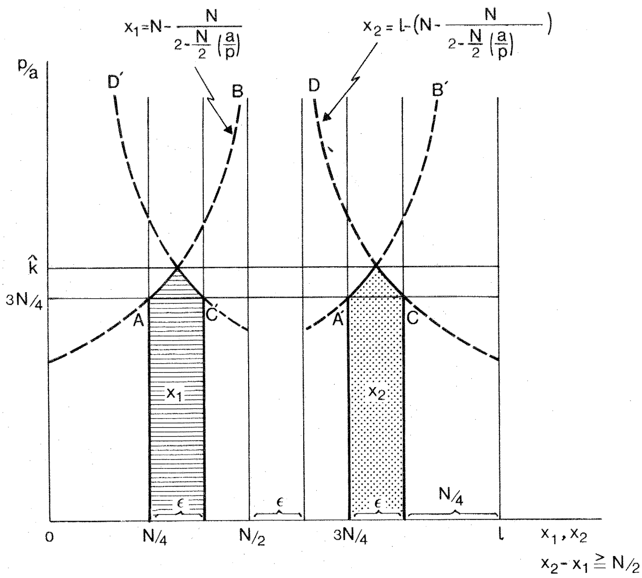

FIGURE 5

Equilibrium locations ($m=2, 0<\varepsilon<N/2$)

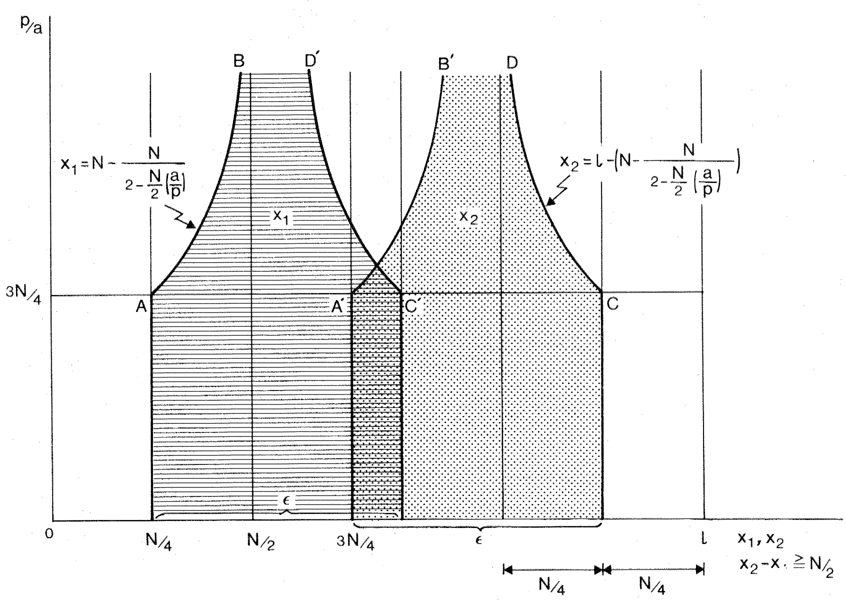

FIGURE 6

Equilibrium locations ($m=2, \varepsilon \geqq N/2$)

6 depicts the equilibrium configurations when $\varepsilon \geqq N/2$. For any value of p/a, there are now infinitely many location equilibria.

In view of the above, for a location equilibrium to exist, it must be that there is enough vacant land or that the transport rate is large enough. On the contrary, when both ε and a are small enough, there is no location equilibrium. This is so because it is always profitable for a firm to approach its competitor. The only possible configuration is then a central agglomeration, as in Figure 3. But we already know that such a configuration is not an equilibrium. In contrast to the standard Hotelling model, *the existence of a 2-firm location equilibrium is therefore not guaranteed when there is vacant land.*

4. THE 3-FIRM CASE

4.1. *Nonvacant land*

There exists no location equilibrium in the standard model with three firms: peripheral firms tend to sandwich the central firm which finds itself with a vanishing market. As a result, this firm leapfrogs one of its competitors, thus generating instability. As shown by the next result, the introduction of a land market leads to very different conclusions (see Appendix 4 for a proof).

Thoorem 5. *Assume that $m=3$ and $l=N$. Then,*

(i) *(x_1^*, x_2^*, x_3^*) such that $x_1^*=x_2^*$ or $x_2^*=x_3^*$ is a location equilibrium if and only if $x_1^*=l-x_3^*=l/4$ and $p/a \leqq 3l/4$;*

(ii) *(x_1^*, x_2^*, x_3^*) such that $x_1^*<x_2^*<x_3^*$ is a location equilibrium if and only if $l/6 \leqq x_1^* = l-x_3^* \leqq l/4$ and* $\max\{l-3x_1^*, p/a\} \leqq x_2^* \leqq \min\{3x_1^*, l-p/a\}$.

Thus, *a location equilibrium with three firms exists if and only if $p/a \leqq 3l/4$*. This is so because, when $a>4p/3l$, the increase in the market area generated by an inward movement of a peripheral firm cannot compensate the corresponding decrease in the consumption level. The opposite is true when $a<p/3l$ so that no equilibrium exists for exactly the same reasons as those outlined in the standard model.

Furthermore, we see that different types of equilibrium configuration may emerge. In all of them, the peripheral firms are symmetrically located inside the first and fifth sextiles but outside the first and third quartiles. On the other hand, the central firm may occupy quite different positions in the market. Either this firm is located together with one of the peripheral firms or it is established at an intermediate point which may not be $l/2$. In the former case, we have an asymmetric configuration with two centres of different sizes. In the latter case, firms are isolated but, except when they locate at $l/6$, $l/2$, $5l/6$, they do not equally share the market.

Based on Theorem 5, Figure 7(a), 7(b) and 7(c) show how equilibrium configurations change with parameter p/a and the size of peripheral firms' hinterland. More precisely, the shaded area and the heavy lines in Figure 7(a) represents the equilibrium locations of firm 2 and the corresponding admissible values of p/a when $x_1^*=l/4$ (and hence $x_3^*=3l/4$). The same is done in Figure 7(b) when firm 1 is located between $l/6$ and $l/4$ (and hence firm 3 is located symmetrically between $3l/4$ and $5l/6$). Finally, in Figure 7(c), we see that if $p/a \leqq l/2$ and if $x_1^*=l-x_3^*=l/6$, then (x_1^*, x_2^*, x_3^*) is a location equilibrium if and only if $x_2^*=l/2$.

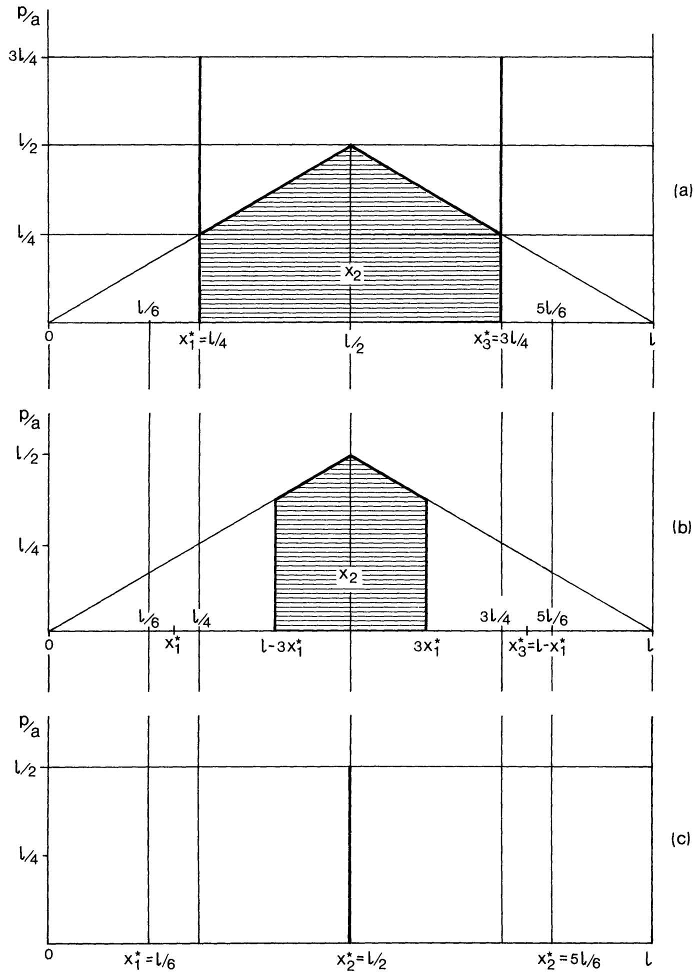

FIGURE 7
Examples of equilibrium configuration ($m=3$, $l=N$)

Notice that the configuration represented in Figure 7(c) is the optimal configuration. In consequence, we can state:

Theorem 6. *Assume $m=3$ and $l=N$. If $p/a \leqq N/2$, then the optimal configuration is a location equilibrium.*

The reason for this result is similar to that discussed after Theorem 1. However, here also, there exist location equilibria that are not optimal.

4.2. *Vacant land*

In the case of two firms with vacant land, every location equilibrium has been shown to be optimal. Unfortunately, this result does not carry over to the case of three firms.

The following lemma replaces Lemma 1 in the 3-firm case. (The proof is similar to that of Lemma 1 and is therefore omitted.)

Lemma 2. *Assume* $m=3$ *and* $l>N$. *If* (x_1^*, x_2^*, x_3^*) *is a location equilibrium, then* (i) $R^*(0)=0$ *and* $R^*(l)=0$; (ii) $R^*((x_1^*+x_2^*)/2)>0$ *implies* $R^*((x_2^*+x_3^*)/2)=0$, *and* $\delta(x^*)=x_1^*=(x_3^*-x_2^*)/2$; (iii) $R^*((x_2^*+x_3^*)/2)>0$ *implies* $R^*((x_1^*+x_2^*)/2)=0$ *and* $\delta(x^*)=l-x_3^*=(x_2^*-x_1^*)/2$.

Thus, $R^*((x_1^*+x_2^*)/2)$ and $R^*((x_2^*+x_3^*)/2)$ may not be both equal to zero. From this lemma, we can see that, if it exists, a location equilibrium must be either (i) $x_1^*\geqq N/6$, $x_2^*-x_1^*\geqq N/3$, $x_3^*-x_2^*\geqq N/3$ and $x_3^*\leqq l-N/6$, or (ii) $x_1^*=\delta(\underline{x}^*)>N/6$, $x_2^*-x_1^*<2\delta(\underline{x}^*)$, $x_3^*-x_2^*=2\delta(\underline{x}^*)$ and $x_3^*=l-\varepsilon-\delta(\underline{x}^*)$ or, symmetrically, $x_3^*=l-\delta(\underline{x}^*)$, $x_3^*-x_2^*<2\delta(\underline{x}^*)$, $x_2^*-x_1^*=2\delta(\underline{x}^*)$ and $x_1^*=\varepsilon+\delta(\underline{x}^*)$. These configurations are illustrated in Figures 8(a) and 8(b). More specifically, Figure 8(a) represents an optimal configuration with $\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4=\varepsilon$, whereas Figure 8($b$) depicts a nonoptimal configuration.

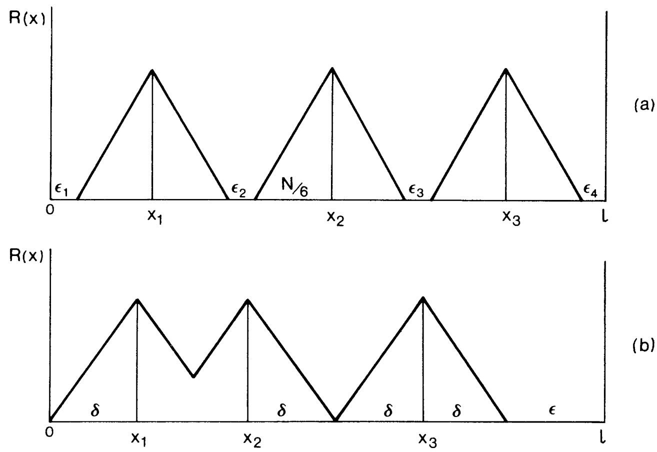

FIGURE 8

(a) An optimal equilibrium configuration, (b) A nonoptimal equilibrium configuration ($m=3, 0<\varepsilon<N/3$)

Our next result provides a complete characterization of the existence conditions of an optimal location equilibrium. (The proof resembles that of Theorem 4 and is not given.)

Theorem 7. *Assume* $m=3$ *and* $l>N$. *Then, an optimal configuration* (x_1^*, x_2^*, x_3^*) *is a location equilibrium if and only if*

$$x_2^*\in\left[\max\left\{N/2, \left(N-\frac{N}{3-(N/2)(a/p)}\right)g(p/a)\right\},\right.$$

$$\left.\min\left\{N/2+\varepsilon, l-\left(N-\frac{N}{3-(N/2)(a/p)}\right)g(p/a)\right\}\right],$$

$$x_2^*-x_1^*\geqq N/3 \quad \textit{and} \quad x_3^*-x_2^*\geqq N/3,$$

where $g(p/a)=1$ *if* $p/a>N/2$ *and* $g(p/a)=0$ *otherwise.*

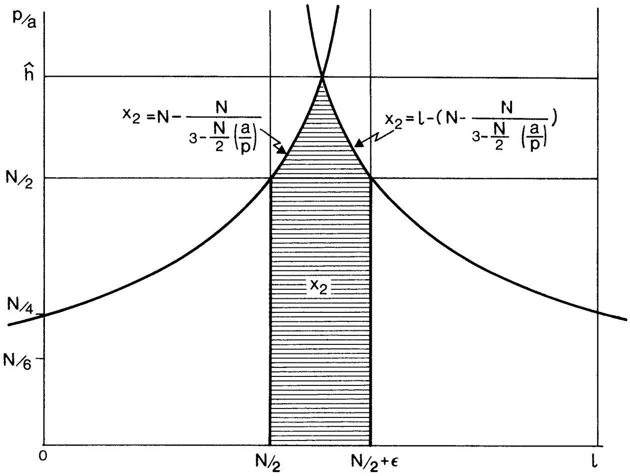

FIGURE 9

Range of x_2 in optimal equilibrium configurations ($m=3, 0<\varepsilon<N/3$)

Define $\hat{h}=(N/2)(1-\varepsilon/N)/(1-3\varepsilon/N)(>N/2)$, which is the solution of $N-N/(3-(N/2)h)=N/2+\varepsilon/2$ (refer to Figure 9). Theorem 7 then implies:

(i) if $\varepsilon \geqq N/3$, there are infinitely many optimal location equilibria;

(ii) if $N/3>\varepsilon>0$, then there exists no optimal location equilibrium when $p/a>\hat{h}$, a unique optimal location equilibrium when $p/a=\hat{h}$ and an infinity of optimal location equilibria when $p/a<\hat{h}$.

Figure 9 depicts the range of x_2-values in optimal equilibria in the case where $\varepsilon<N/3$. As ε approaches $N/3$, $\hat{h}$ tends to infinity and we see that an optimal location equilibrium always exists.

It remains to deal with nonoptimal equilibria. The following result provides sufficient conditions for the existence and the nonexistence of such equilibria (see Appendix 5 for a proof).

Theorem 8. *Assume $m=3$ and $l>N$. Then,*

(i) *if $\varepsilon<N/3$ and $p/a<N/2$, there exist an infinity of nonoptimal location equilibria*;

(ii) *if $\varepsilon \geqq N/3$, nonoptimal location equilibrium does not exist.*

Thus, a nonoptimal configuration may be an equilibrium only when the amount of excess land is not too large. In addition, the transport rate must be sufficiently large. Such equilibrium configurations are highly *asymmetric* in the sense that two firms compete on the same submarket while the third firm has a market independent of the others.

Combining Theorems 7 and 8, we can conclude as follows:

(i) *If $\varepsilon \geqq N/3$, then, under any value of p/a, there are infinitely many location equilibria, all of which are optimal.*

(ii) *If $0<\varepsilon<N/3$ and $p/a \leqq N/2$, then there exist both optimal and nonoptimal location equilibria.*

5. THE CASE OF PUBLIC OWNERSHIP

One may wonder about the robustness of our results to changes in the assumption of absentee landlords and firms' owners. In this section, we show that our most surprising result, that is, the optimal configuration is a location equilibrium when the transport rate is large enough, remains valid in the alternative case of public land ownership in which the total rent is equally shared among households.[7] More specifically, the budget constraint of a household at $x \in X$ is now given as

$$z(p+ar(x))+R(x)=Y+TR/N \tag{27}$$

where $TR \equiv \int_X R(x)dx$ is the total rent.

Let $\underset{\sim}{x}$ be a configuration of firms. Instead of (3), we now define the bid rent function of a household at x as

$$\Psi(x, z)=Y+TR/N-(p-ar(x))z. \tag{28}$$

Then, the residential equilibrium $(z^*, n^*(n), R^*(x))$ under $\underset{\sim}{x}$ can be defined, as before, by relations (5)-(9). Let $\delta \equiv \delta(\underset{\sim}{x})$ as in (10) and keep conventions (11) and (12). Then

$$Y+TR/N=(p+a\delta)z^*. \tag{29}$$

As in Section 2, the equilibrium land rent curve is given by

$$R^*(x)=\max\{\Psi(x, z^*), 0\}=z^* \max\{a\delta-ar(x), 0\}.$$

Let $g \equiv g(\underset{\sim}{x})=\int_X \max\{a\delta-ar(x), 0\}dx$. Then, the total land rent is given by $TR=z^*g$. Thus, from (29), the equilibrium consumption is given by $z^*=Y/(p+a\delta-g/N)$. Therefore, equilibrium conditions can be restated as follows:

$$z^*=Y/(p+a\delta-g/N), \tag{30}$$

$$g=\int_X \max\{a\delta-ar(x), 0\}dx, \tag{31}$$

$$R(x)=z^* \max\{a\delta-ar(x), 0\}, \tag{32}$$

$$\begin{aligned} n(x)&=1 \quad \text{if } r(x) \leqq \delta, \\ &=0 \quad \text{if } r(x)>\delta, \end{aligned} \tag{33}$$

$$\bar{X}_+=N. \tag{34}$$

Let us now show that the optimal configuration is a location equilibrium when the transport rate is large enough. We limit ourselves to the case $m=2$ and $l=N$. Since $l=N$, we have

$$a\delta N-g=\int_X ar(x)dx=aNr_m(\underset{\sim}{x}) \tag{35}$$

where $r_m(\underset{\sim}{x})$ is the average travel distance associated with $\underset{\sim}{x}$ so that (30) becomes

$$z^*(\underset{\sim}{x})=Y/(p+ar_m(\underset{\sim}{x})). \tag{36}$$

Let us fix $x_2=3l/4$. It is then easy to see that, for $x_1<l/4$, $\partial z^*/\partial x_1>0$ and $\partial \bar{M}_1/\partial x_1>0$, and hence

$$\partial Q_1(x_1|3l/4)/\partial x_1>0 \quad \text{for } x_1<l/4. \tag{37}$$

Similarly, we can show that

$$\partial Q_1(x_1|3l/4)/\partial x_1 < 0 \quad \text{for } 3l/4 < x_1. \tag{38}$$

Furthermore, for $l/4 < x_1 < 3l/4$,

$$\partial Q_1/\partial x_1 = \frac{x_1 + 3l/4}{2} \cdot \frac{\partial z^*}{\partial x_1} + \frac{z^*}{2}.$$

A simple manipulation then shows that

$$\partial Q_1(x_1|3l/4)/\partial x_1 < 0 \quad \text{if and only if } p/a < l/8. \tag{39}$$

Finally,

$$Q_1(l/4|3l/4) > Q_1(3l/4|3l/4) \tag{40}$$

since $z^*(l/4, 3l/4) > z^*(3l/4, 3l/4)$. Consequently, if $a > 8p/l$, $x_1 = l/4$ is the best location of firm 1 given $x_2 = 3l/4$ and hence, by symmetry, $(l/4, 3l/4)$ is a location equilibrium.

6. CONCLUSIONS

In this paper we have demonstrated that the introduction of a land market into the spatial competition model leads to quite substantial modifications of location equilibria.

It remains to be discussed how our results are affected by the assumptions of given mill prices and fixed land consumption. Modern theories of spatial competition have relaxed the assumption of given mill prices in order to allow firms to choose both price and location. However, the question of the nonexistence of a price-location equilibrium has concerned several authors. More specifically, it can be shown that a price-location equilibrium exists only under restrictive conditions (see Jaskold-Gabszewicz and Thisse (1986) for a detailed discussion). We do not believe that the introduction of a land market into the price-location model can markedly contribute to the solution of this problem, and this precisely for the reasons encountered in the standard model. Rather, it seems to us that a promising line of research to deal with price competition in our model is to suppose that firms sell differentiated products and consumers desire variety, as in Dixit and Stiglitz (1977). Hopefully, consumers' reactions to firms' price cuts are smoothed and a price-location equilibrium exists (see Ben-Akiva, de Palma and Thisse (1985) for a study of Hotelling's model with differentiated products).

The assumption of inelastic demand for land is a severe restriction which discards much potential richness of the analysis. Introducing variable lot size into our model drastically complicates the mathematics. However, a preliminary analysis undertaken for a Cobb-Douglas type utility function suggests that many of the most important results of this paper can be extended to the case of variable lot size. (Notice that the vacant land case requires the introduction of a positive agriculture land rent; otherwise all land is consumed by the households.) A detailed analysis is postponed for future research.

The paper has two interesting implications. First, our results can be given an interpretation in terms of urban organization. In urban economics, it is standard to start with the assumption of a monocentric configuration, that is, a city with a pre-specified centre around which consumers distribute themselves. Clearly, from the theoretical viewpoint, the existence of such a centre should not be imposed *a priori* but, instead, should be explained within a more general framework accounting for the possible existence of several centres. We propose to retain the spatial competition process as one possible explanation (among many others) of the number and the location of centres. More precisely, the equilibrium locations of firms can be used to describe the spatial structure of the city (one or several centres? how many and where?), whereas the equilibrium distribution of households permits one to determine the size of the city. For instance, in the 2-firm case without vacant land, we may observe a city with a single centre or a city with two centres. In the 3-firm case, the city may have two or three centres. When two centres arise, we have seen that one centre is larger than the other, which implies some kind of hierarchy within the city.

On the other hand, the model with vacant land opens the door to a reinterpretation in terms of system of cities. In both the 2- and 3-firm cases, a system of two or three cities of equal size can emerge. However, when three firms are in business, we may also observe a completely different pattern, i.e. a large city with two centres and a small monocentric city, another example of urban hierarchy.

The above multiplicity of possible equilibria should not cause concern. Rather, it generates the appealing implication that the precise nature of urban configurations in a particular economy will be dependent on the history of that economy.

Second, the analysis of the 3-firm case reveals that the nonexistence of a location equilibrium observed in the standard location model can vanish when the problem is embedded in a more general setting. This is interesting because this suggests that nonexistence results established in partial equilibrium models do not necessarily extend to general equilibrium models.

APPENDIX 1

Proof of Theorem 1

Lemma 1.1. $(l/2, l/2)$ is a location equilibrium.

Proof. We have the following sequence of implications: $x_1 < x_2^* = l/2 \Rightarrow \delta = x_2^* \Rightarrow \partial z^*/\partial x_1 = 0$ (from (13)) $\Rightarrow \partial Q_1/\partial z_1 > 0$ (from (25)). Thus, given $x_2^* = l/2$, the best location for firm 1 is $x_1^* = l/2$. By symmetry, given $x_1^* = l/2$, the best location for firm 2 is $x_2^* = l/2$. The lemma then follows. ‖

Lemma 1.2. *If* (x_1^*, x_2^*) *is a location equilibrium, then* $x_1^* = l - x_2^* \geqq l/4$.

Proof. First, we have $x_1^* < x_2^*$ and

$$x_1^* + x_2^* < l \Rightarrow \delta = \max\{l - x_2^*, (x_2^* - x_1^*)/2\}$$
$$\Rightarrow \partial z^*/\partial x_1 \geqq 0 \Rightarrow \partial Q_1/\partial x_1 > 0 \Rightarrow (x_1^*, x_2^*)$$

is not a location equilibrium. Similarly: $x_1^* < x_2^*$ and $x_1^* + x_2^* > l \Rightarrow \partial Q_1/\partial x_1 < 0 \Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium. In consequence, if (x_1^*, x_2^*) is a location equilibrium, it must be that $x_1^* + x_2^* = l$. Furthermore, $x_1^* = l - x_2^* < l/4 \Rightarrow \delta = (x_2^* - x_1^*)/2 \Rightarrow \partial z^*/\partial x_1 \geqq 0 \Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium. ‖

Lemma 1.3. *If* $\max\{l/2, p/a\} \leqq x_2^* \leqq 3l/4$ *and* $x_1^* = l - x_2^*$, *then* (x_1^*, x_2^*) *is a location equilibrium.*

Proof. Let us consider the following steps:

(i) $x_1 < x_2$ and $x_1 < l - x_2 \Rightarrow \partial Q_1/\partial x_1 > 0$ (as can be shown from the proof of Lemma 1.2). Hence, on $[0, l - x_2]$, $Q_1(x_1 | x_2)$ reaches its maximum at $l - x_2$.

(ii) $$l - x_2 < x_1 < x_2 \leqq 3l/4 \Rightarrow \delta = x_1 \Rightarrow Q_1(x_1|x_2) = Y(x_1 + x_2)/2(p + ax_1)$$
$$\Rightarrow \partial Q_1/\partial x_1 = Y(p - ax_2)/2(p + ax_1)^2 \Rightarrow \partial Q_1/\partial x_1 \leqq 0$$

if and only if $x_2 \geqq p/a$. Hence, given x_2 such that $l/2 < x_2 \leqq 3l/4$, $Q_1(x_1|x_2)$ is maximized at $l - x_2$ on $[l - x_2, x_2]$ if and only if $x_2 \geqq p/a$.

(iii) $x_2 > l/2 \Rightarrow \sup\{Q_1(x_1|x_2); x_1 \in [0, x_2)]\} \geqq \sup\{Q_1(x_1|x_2); x_1 \in [x_2, l]\}$. Accordingly, given x_2^* such that $\max\{l/2, p/a\} \leqq x_2^* \leqq 3l/4$, $x_1^* = l - x_2^*$ is the best location of firm 1. By symmetry, given x_1^* such that $l/4 \leqq x_1^* \leqq \max\{l/2, p/a\}$, $x_2^* = l - x_1^*$ is firm 2's best location. ||

Theorem 1 immediately follows from Lemmas 1.1, 1.2 and 1.3. ||

APPENDIX 2

Proof of Lemma 1

Assume that the lemma does not hold. Then, four cases are conceivable:

(i) $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) = 0$, and $R^*(l) = 0$ (or: $R^*(0) = 0$, $R^*((x_1^* + x_2^*)/2) = 0$, $R^*(l) > 0$);

(ii) $R^*(0) = 0$, $R^*((x_1^* + x_2^*)/2) > 0$, and $R^*(l) = 0$;

(iii) $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) > 0$, and $R^*(l) = 0$ (or: $R^*(0) = 0$, $R^*((x_1^* + x_2^*)/2) > 0$, and $R^*(l) > 0$);

(iv) $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) > 0$, $R^*(l) > 0$.

In case (iv), we have $\bar{X}_+ = l > N$ which violates the equilibrium condition (9). Hence, case (iv) cannot happen in equilibrium. Let us now examine the first three cases one by one.

Case (i). When $R^*((x_1^* + x_2^*)/2) = 0$, two sub-cases may arise: $(x_2^* - x_1^*)/2 > \delta$ or $(x_2^* - x_1^*)/2 = \delta$. In the first one, $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) = 0$, $R^*(l) = 0$ and $(x_2^* - x_1^*)/2 > \delta \Rightarrow x_1^* < \delta = (N - x_1^*)/3$ and $\bar{M}_1 = x_1^* + (N - x_1^*)/3 \Rightarrow \partial z^*/\partial x_1 > 0$ and $\partial \bar{M}_1/\partial x_1 > 0 \Rightarrow \partial Q_1/\partial x_1 > 0 \Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium. In the second one, $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) = 0$, $R^*(l) = 0$ and $(x_2^* - x_1^*)/2 = \delta \Rightarrow x_1^* < \delta = N - x_2^*$ and $\bar{M}_1 = (x_1^* + x_2^*)/2 \Rightarrow \partial z^*/\partial x_1 = 0$ and $\partial \bar{M}_1/\partial x_1 > 0 \Rightarrow \partial Q_1/\partial x_1 > 0 \Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium.

Case (ii). $R^*(0) = 0$, $R^*((x_1^* + x_2^*)/2) > 0$ and $R^*(l) = 0 \Rightarrow \delta = (N - (x_2^* - x_1^*))/2$ and $\bar{M}_2 = \delta + (x_2^* - x_1^*)/2 \Rightarrow \partial z^*/\partial x_2 > 0$ and $\partial \bar{M}_2/\partial x_2 > 0 \Rightarrow \partial Q_2/\partial x_2 > 0 \Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium.

Case (iii). When $R^*(0) > 0$ and $R^*((x_1^* + x_2^*)/2) > 0$, two sub-cases may arise: $x_1^* < x_2^*$ or $x_1^* = x_2^*$. In the first one, $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) > 0$, $R^*(l) = 0$ and $x_1^* < x_2^* \Rightarrow x_1^* < \delta = N - x_2^*$ and $\bar{M}_1 = (x_1^* + x_2^*)/2 \Rightarrow \partial z^*/\partial x_1 = 0$ and $\partial \bar{M}_1/\partial x_1 > 0 \Rightarrow \partial Q_1/\partial x_1 > 0 \Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium. In the second one, $R^*(0) > 0$, $R^*((x_1^* + x_2^*)/2) > 0$, $R^*(l) = 0$ and $x_1^* = x_2^* \Rightarrow x_1^* < \delta = N - x_2^*$ and $\bar{M}_1 = N/2 \Rightarrow Q_1(x_1^*|x_2^*) = z^* N/2 < z^*(N - x_2^* - \eta) = Q_1(x_2^* + \eta/x_2^*)$ with $\eta > 0$ arbitrarily small $\Rightarrow (x_1^*, x_2^*)$ is not a location equilibrium. ||

APPENDIX 3

Proof of Theorem 4

We prove the theorem in four steps.

Step 1. If (x_1, x_2) is a location pair that satisfies $N/4 \leqq x_1$, $x_2 \leqq l - N/4$ and $x_2 - x_1 \geqq N/2$, then $\delta = N/4 = \min_{x,y} \delta(x, y)$. Hence we have $z^* = Y/(p + aN/4) = \max_{x,y} z^*(x, y)$.

Step 2. Let (x_1, x_2) be a location pair such that $N/2 \leqq x_1$, $l-x_2 \geqq N/4$ and $x_2 - x_1 \geqq N/2$. From this, it follows that $\bar{M}_2(x_1, x_2) = N/2 = \max_y \bar{M}_2(x_1, y)$. Using step 1, we then have $Q_2(x_1|x_1) = \max_y Q_2(y|x_1)$.

Step 3. Let (x_1, x_2) be a location pair such that $N/4 \leqq x_1 < N/2$, $l - x_2 \geqq N/4$ and $x_2 - x_1 \geqq N/2$. It is then easy to see that $\bar{M}_2(x_1, x_2) = N/2 = \max\{\bar{M}_2(x_1, y);\ y \in [0, x_1] U [N - x_1, l]\}$. Hence, using step 1, we obtain $Q_2(x_2|x_1) = \max\{Q_2(y|x_1);\ y \in [0, x_1] U [N - x_1, l]\}$. Consequently, $Q_2(x_2|x_1) = \max_y Q_2(y|x_1)$ iff $Q_2(x_2|x_1) = \sup\{Q_2(y|x_1);\ x_1 < y < N - x_1\}$. Consider now the interval $x_1 < y < N - x_1$. We have $\delta = N - y$ and $\bar{M}_2 = N - (x_1 + y)/2$. Hence $Q_2(y|x) = Y(N - (x_1 + y)/2)(p + a(N - y))^{-1}$ and $\partial Q_2(y|x_1)/\partial y = (Y/2)(a(N - x_1) - p)(p + a(N - y))^{-2}$. Therefore

$$\sup\{Q_2(y|x_1);\ x_1 < y < N - x_1\} = Q_2(N - x_1|x_1), \qquad \text{if } x_1 \leqq N - p/a$$

$$\sup\{Q_2(y|x_1);\ x_1 < y < N - x_1\} = \lim_{y \to x_{1+}} Q_2(y|x_1)$$

$$= \frac{Y}{p + a(N - x_1)}(N - x_1), \quad \text{if } x_1 > N - p/a.$$

A simple calculation then shows that, for $N - p/a < x_1 < N/2$, $Q_2(x_2|x_1) \geqq \lim_{y \to x_{1+}} Q_2(y|x_1)$ if and only if $x_1 \geqq N - N/(2 - (N/2)(a/p))$. Accordingly, given $N/4 \leqq x_1 < N/2$, $l - x_2 \geqq N/4$ and $x_2 - x_1 \geqq N/2$, $Q_2(x_2|x_1) = \max_y Q_2(y|x_1)$ if and only if one of the following two conditions holds: (i) $x_1 \leqq N - p/a$, or (ii) $x_1 > N - p/a$ and $x_1 \geqq N - N/(2 - (N/2)(a/p))$. It is easy to see that $N - N/(2 - (N/2)(a/p)) > N - p/a$ if and only if $p/a > 3N/4$. Similarly, $x_1 \geqq N/4$ is consistent with $x_1 \leqq N - p/a$ if and only if $p/a \leqq 3N/4$. Hence, when $p/a > 3N/4$, condition (i) never holds while condition (ii) can be restated as $x_1 \geqq N - N/(2 - (N/2)(a/p))$. Furthermore, when $p/a \leqq 3N/4$, we have $x_1 \leqq N - p/a$ or $x_1 \geqq N - p/a$. In the latter case, we also have $x_1 > N - N/(2 - (N/2)(a/p))$ since now $N - p/a \geqq N - N/(2 - (N/2)(a/p))$. Consequently, conditions (i) and (ii) are always satisfied. In conclusion, conditions (i) and (ii) can be replaced by

$$\left(N - \frac{N}{2 - (N/2)(a/p)}\right) f(p/a) \leqq x_1$$

where $f(p/a) = 1$ if $p/a > 3N/4$ and $f(p/a) = 0$ otherwise.

Step 4. Notice first that, given $x_2 - x_1 \geqq N/2$, $l - x_2 \geqq N/4$ implies that $x_1 \leqq N/4 + \varepsilon$. Hence, combining this with the results of steps 2 and 3 and with Lemma 1, we see that $Q_2(x_2|x_1) = \max_y Q_2(y|x_1)$ if and only if

$$x_1 \in \left[\max\left\{N/4, \left(N - \frac{N}{2 - (N/2)(a/p)}\right) f(p/a)\right\}, N/4 + \varepsilon\right]$$

and

$$x_2 - x_1 \geqq N/2.$$

By symmetry, we obtain that $Q_1(x_1|x_2) = \max_y Q_1(y|x_2)$ if and only if

$$x_2 \in \left[3N/4, \min\left\{3N/4 + \varepsilon, l - \left(N - \frac{N}{2 - (N/2)(a/p)}\right) f(p/a)\right\}\right]$$

and

$$x_2 - x_1 \geqq N/2.$$

Theorem 4 then follows. ‖

APPENDIX 4

Proof of Theorem 5

Lemma 4.1. *If* (x_1^*, x_2^*, x_3^*) *is a location equilibrium, then* $x_1^* = l - x_3^*$.

Proof. Suppose that $x_1^* > l - x_3^*$. We then have

$$Q_2(x_2 = x_1^* \mid x_1^*, x_3^*) = (Y/(p+a\delta))(x_1^* + x_3^*)/4 > (Y/(p+a\delta))\left(l - \frac{x_1^* + x_3^*}{2}\right)\Big/ 2$$

$$= Q_2(x_2 = x_3^* \mid x_1^*, x_3^*)$$

where $\delta \equiv \delta(x_1, x_2 = x_1, x_3) = \delta(x_1, x_2 = x_3, x_3)$. Hence, x_2^* must be such that $x_1^* \leqq x_2^* < x_3^*$. Next, $x_1^* > l - x_3^*$ and $x_2^* \neq x_3^* \Rightarrow \delta(\underline{x}^*) \geqq x_1^* > l - x_3^*$ and $\bar{M}_3 = l - (x_2^* + x_3^*)/2 \Rightarrow \partial z^*/\partial x_3 = 0$ and $\partial \bar{M}_3/\partial x_3 = -\frac{1}{2} < 0 \Rightarrow \partial Q_3/\partial x_3 < 0 \Rightarrow (x_1^*, x_2^*, x_3^*)$ is not a location equilibrium.

A similar argument applies to $x_3^* < l - x_1^*$. ‖

Lemma 4.2. *If* (x_1^*, x_2^*, x_3^*) *is a location equilibrium, then* $l/6 \leqq x_1^* \leqq l/4$.

Proof. Let $x_1^* < x_2^* \leqq x_3^*$. Suppose that $x_1^* < l/6$. Then, $x_1^* < l/6$ and $x_1^* \neq x_2^* \Rightarrow \delta(\underline{x}^*) > x_1^*$ and $\bar{M}_1 = (x_1^* + x_2^*)/2 \Rightarrow \partial z^*/\partial x_1 = 0$ and $\partial \bar{M}_1/\partial x_1 = \frac{1}{2} > 0 \Rightarrow \partial Q_1/\partial x_1 > 0 \Rightarrow (x_1^*, x_2^*, x_3^*)$ is not a location equilibrium. Let us now assume that $x_1^* > l/4$. Then, $x_1^* > l/4$ and $x_1^* < x_2^* \leqq x_3^* \Rightarrow \delta(\underline{x}^*) = x_1^*$ and $\bar{M}_2 < l/4 \Rightarrow Q_2(x_2^* \mid x_1^*, x_3^*) = z^* \bar{M}_2 < z^*(x_1^* - \eta) = Q_2(x_1^* - \eta \mid x_1^*, x_3^*)$ with $0 < \eta < x_1^* - l/4 \Rightarrow (x_1^*, x_2^*, x_3^*)$ is not a location equilibrium.

A similar argument applies to $x_1^* \leqq x_2^* < x_3^*$. ‖

Lemma 4.3. *If* (x_1^*, x_2^*, x_3^*) *is a location equilibrium, then* $\delta = x_1^*$.

Proof. As there is no vacant land, it must be that $x_1^* \leqq \delta$. Suppose that $x_1^* < \delta$. Then, $x_1^* \neq x_2^*$ and $x_1^* < \delta \Rightarrow Q_1 = (Y/(p+a\delta))(x_1^* + x_2^*)/2 \Rightarrow \partial Q_1/\partial x_1 > 0 \Rightarrow (x_1^*, x_2^*, x_3^*)$ is not a location equilibrium. Similarly, $x_2^* \neq x_3^*$ and $l - x_3^* = x_1^* < \delta \Rightarrow Q_3 = (Y/(p+a\delta))(l - (x_2^* + x_3^*)/2) \Rightarrow \partial Q_3/\partial x_3 < 0 \Rightarrow (x_1^*, x_2^*, x_3^*)$ is not a location equilibrium. ‖

Lemma 4.4. *Let* $x_1 \leqq x_2 \leqq x_3 = l - x_1$. *Then* $x_1 = \delta(\underline{x})$ *if and only if* $l - 3x_1 \leqq x_2 \leqq 3x_1$.

Proof. We have:

$$\begin{aligned} \delta(\underline{x}) = x_1 = l - x_3 &\Leftrightarrow \max\{(x_2 - x_1)/2, (x_3 - x_2)/2\} \leqq x_1 \\ &\Leftrightarrow \max\{(x_2 - x_1)/2, (l - x_1 - x_2)/2\} \leqq x_1 \\ &\Leftrightarrow l - 3x_1 \leqq x_2 \leqq 3x_1. \end{aligned}$$

‖

Lemma 4.5. *Let* $x_1^* = x_2^*$, *or* $x_2^* = x_3^*$. *Then,* (x_1^*, x_2^*, x_3^*) *is a location equilibrium if and only if* $x_1^* = l - x_3^* = l/4$ *and* $p/a \leqq 3l/4$.

Proof. (i) Let (x_1^*, x_2^*, x_3^*) be a location equilibrium. Then, $x_1^* = x_2^*$ and $x_1^* = \delta$ (by Lemma 4.3)$\Rightarrow z^*(x_1^* - \eta) = Q_2(x_1^* - \eta | x_1^*, x_3^*) \leqq Q_2(x_1^* | x_1^*, x_3^*) = z^*(l - (x_1^* + x_3^*)/2)/2$ where $\eta > 0$ and $z^* = Y/(p + ax_1^*) \Rightarrow x_1^* \leqq (l - (x_1^* + x_3^*)/2)/2 \Rightarrow x_1^* \leqq l/4$ (by Lemma 4.1). Similarly, $x_1^* = x_2^*$ and

$$x_1^* = \delta \Rightarrow z^*(x_3^* - x_1^*)/2 = Q_2\left(\frac{x_1^* + x_3^*}{2} \middle| x_1^*, x_2^*\right) \leqq Q_2(x_1^* | x_1^*, x_2^*)$$
$$= z^*(l - (x_1^* + x_3^*)/2)/2 \Rightarrow (x_3^* - x_1^*)/2 \leqq (l - (x_1^* + x_3^*)/2)/2 \Rightarrow x_1^* \geqq l/4.$$

Thus, it must be that $l/4 = x_1^* (= l - x_3^*)$. By symmetry, $x_2^* = x_3^*$ implies that $l/4 = l - x_3^* (= x_1^*)$.

(ii) Next,

$$Q_1(x_1 | 3l/4, 3l/4) = (Y/(p + ax_1))(x_1 + 3l/4)/2$$

for

$$l/4 \leqq x_1 < 3l/4 \Rightarrow \partial Q_1/\partial x_1 = (Y/2(p + ax_1)^2)(p - 3la/4)$$
$$\Rightarrow \partial Q_1/\partial x_1 \leqq 0 \quad \text{for } l/4 < x_1 < 3l/4 \text{ if and only if } p/a \leqq 3l/4.$$

From this, we can easily see that

$$Q_1(l/4 | 3l/4, 3l/4) = \max_{x_1} Q_1(x_1 | 3l/4, 3l/4)$$

if and only if $p/a \leqq 3l/4$. Furthermore, it is straightforward to show

$$Q_2(3l/4 | l/4, 3l/4) = \max_{x_2} Q_2(x_2 | l/4, 3l/4)$$

and

$$Q_3(3l/4 | l/4, 3l/4) = \max_{x_3} Q_3(x_3 | l/4, 3l/4).$$

Hence, $(x_1 = l/4, x_2 = 3l/4, x_3 = 3l/4)$ is a location equilibrium if and only if $p/a \leqq 3l/4$. By symmetry, we see that $(x_1 = l/4, x_2 = x_3 = 3l/4)$ is a location equilibrium if and only if $p/a \leqq 3l/4$. From (i) and (ii), we can conclude as the lemma. ||

Lemma 4.6. *Let $x_1^* < x_2^* < x_3^*$. Then, (x_1^*, x_2^*, x_3^*) is a location equilibrium if and only if $l/6 \leqq x_1^* = l - x_3^* \leqq l/4$ and $\max\{l - 3x_1^*, p/a\} \leqq x_2^* \leqq \min\{3x_1^*, l - p/a\}$.*

Proof. (i) Let (x_1^*, x_2^*, x_3^*) be a location equilibrium such that $x_1^* < x_2^* < x_3^*$. By Lemmas 4.1, 4.2 and 4.4, we get $l/6 \leqq x_1^* = l - x_3^* \leqq l/4$ and $l - 3x_1^* \leqq x_2^* \leqq 3x_1^*$. Next, $x_1^* < x_2^* < x_3^*$ and $\delta = x_1^*$ (by Lemma 4.3)$\Rightarrow Q_1(x_1 | x_2^*, x_3^*) = (Y/(p + ax_1))(x_1 + x_2^*)/2$ for $x_1^* \leqq x_1 < x_2^* \Rightarrow \partial Q_1/\partial x_1 = (Y/2(p + ax_1)^2)(p - ax_2^*) \Rightarrow \partial Q_1/\partial x_1 \leqq 0$ for $x_1^* \leqq x_1 < x_2^*$ if and only if $p/a \leqq x_2^*$. Therefore, $Q_1(x_1^* | x_2^*, x_3^*) \geqq Q_1(x_1 | x_2^*, x_3^*)$ implies $p/a \leqq x_2^*$.

A similar argument shows that $x_2^* \leqq l - p/a$,

(ii) Let (x_1, x_2, x_3) be a triple such that $x_1 < x_2 < x_3$, $l/6 \leqq x_1 = l - x_3 \leqq l/4$ and $\max\{l - 3x_1, p/a\} \leqq x_2 \leqq \min\{3x_1, l - p/a\}$. Then, Lemma 4.4 implies $\delta(\underset{\sim}{x}) = x_1$. Hence $Q_1(y | x_2, x_3) = (Y/(p + ay))(y + x_2)/2$ for $x_1 \leqq y \leqq x_2 \Rightarrow \partial Q_1/\partial y = (y/2(p + ay)^2)(p - ax_2) \leqq 0$ since $p/a \leqq x_2$. From this, we can easily see that $Q_1(x_1 | x_2, x_3) = \max_y Q_1(y | x_1, x_2)$. By a similar argument, it can be shown that $Q_3(x_3 | x_1, x_2) = \max_y Q_3(y | x_1, x_2)$. Finally, we have $Q_2(x_2 | x_1, x_3) = (Y/(p + a\delta)) \times (x_3 - x_1)/2 = Q_2(y | x_1, x_3)$ for $x_1 < y < x_3$; $Q_2(x_2 | x_1, x_3) > Q_2(y | x_1, x_3) = (Y/(p + a\delta))y$ for $0 \leqq y < x_1$ since $(x_3 - x_1)/2 = (l - 2x_1)/2 \geqq x_1$; $Q_2(x_2 | x_1, x_3) \geqq Q_2(x_1 | x_1, x_3) = (Y/(p + a\delta))(x_1 + x_3)/4$. Thus $Q_2(x_2 | x_1, x_3) = \max_y Q_2(y | x_1, x_3)$. ||

Theorem 5 immediately follows from Lemmas 4.5 and 4.6 ||

APPENDIX 5

Proof of Theorem 8

(i) Let (x_1, x_2, x_3) be a triple such that $x_1=\delta>N/6$, $x_2-x_1<2\delta$, $x_3-x_2=2\delta$ and $x_3=l-\varepsilon-\delta(*)$ (as depicted in Figure 8(*b*)). If $\varepsilon<N/3$, then we can always choose x_1 and x_2 that satisfy $x_1+x_2\geqq 2\varepsilon$. First, we have:

$$x_1<y<x_2 \Rightarrow \bar{M}_1(y, x_2, x_3)=\frac{y+x_2}{2}$$

and

$$z^*(y, x_2, x_3)=y/(p+ay) \Rightarrow Q_1(y\,|\,x_2, x_3)=\frac{y+x_2}{2}\cdot\frac{y}{p-ay} \Rightarrow \partial Q_1/\partial y \leqq 0$$

if and only if $x_2 \geqq p/a$. Furthermore,

$$Q_1(x_1\,|\,x_2, x_3)=\frac{x_1+x_2}{2}\cdot\frac{y}{p+a\delta}>\varepsilon\frac{y}{p+a\delta}\geqq Q_1(y\,|\,x_2, x_3) \quad \text{for } y \geqq x_3.$$

It is then easy to see that $Q_1(x_1\,|\,x_2, x_3)=\max_y Q_1(y\,|\,x_2, x_3)$ if and only if $x_2 \geqq p/a$.

Second, we have:

$$x_2<y<x_3 \Rightarrow \bar{M}_3(x_1, x_2, y)=\left(N-\frac{x_2+y}{2}\right)$$

and

$$z^*(x_1, x_2, y)\frac{Y}{p+a(N-y)} \Rightarrow Q_3(x_3\,|\,x_1, x_2)=\left(N-\frac{x_2+y}{2}\right)\cdot\frac{y}{p+a(N-y)} \Rightarrow \partial Q_3/\partial y \geqq 0$$

if and only if $N-x_2 \geqq p/a$. It is then straightforward to show that $Q_3(x_3\,|\,x_1, x_2)=\max_y Q_3(y\,|\,x_1, x_2)$ if and only if $N-x_2 \geqslant p/a$. Third, it follows from $x_1+x_2 \geqq 2\varepsilon$ that $Q_2(x_2\,|\,x_1, x_3)=\max Q_2(y\,|\,x_1, x_3)$. Consequently, given conditions $(*)$, (x_1, x_2, x_3) is a nonoptimal location equilibrium if and only if $p/a \leqq \min\{x_2, N-x_2\}$. When $p/a<N/2$, it is always possible to find $x_1<x_2<x_3$ that satisfy all the desired inequalities.

(ii) We know from Lemma 2 that any nonoptimal location equilibrium (up to a symmetry) must be such that $x_1=\delta>N/6$, $x_2-x_1<2\delta$, $x_3-x_2=2\delta$ and $x_3=l-\varepsilon-\delta$. Then

$$Q_1(x_1\,|\,x_2, x_3)=\frac{x_1+x_2}{2}\cdot\frac{Y}{p+a\delta}.$$

Now,

$$Q_1(l-x_1\,|\,x_2, x_3)=\frac{2x_1+\varepsilon}{2}\cdot\frac{y}{p+a\delta}.$$

Consequently, $Q_1(l-x_1\,|\,x_2, x_3)>Q_1(x_1\,|\,x_2, x_3)$ if and only if $\varepsilon>x_2-x_1$. As $x_2-x_1<N-4\delta$ and $\delta>N/6$, we see that $x_2-x_1<N/3$. If $\varepsilon \geqq N/3$, it then follows that $Q_1(l-x_1\,|\,x_2, x_3)>Q_1(x_1\,|\,x_2, x_3)$ and, hence, (x_1, x_2, x_3) is not a location equilibrium. ||

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NOTES

1. The alternative case of public land ownership is considered in Section 5.
2. This assumption is discussed in the conclusions.
3. This assumption and its possible relaxation are discussed in Section 6.
4. This normalization is necessary only when $l = N$ (i.e. no vacant land exists). It is not difficult to see that if $l > N$, then condition (11) is satisfied in any residential equilibrium.
5. Equation (18) implies that if more than one firm locate at the same point, they equally share the total sales in their common market area.
6. Notice that the infinity of equilibria in (i) and (ii) should not confuse the reader. All these equilibria are essentially of the same nature. This is to be contrasted with the nonvacant land case in which equilibria of different natures may exist.
7. We expect the same result to hold when firm's profits are equally shared among consumers.

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