EC1340 Topic #7

Discounting, or how to compare present and future consumption

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Mathematics of discounting

- Let r denote the real return to capital, and I_t the value of an investment at time t. Then

$$
\begin{aligned}
I_1 &= (1+r)I_0 \\
I_2 &= (1+r)I_1 \\
&= (1+r)^2 I_0 \\
&\vdots \\
I_t &= (1+r)^t I_0
\end{aligned}
$$

- Conversely, the value of an investment required at 0 to yield I_t at t is, obtained by solving the equations above for I_0.
- So, to get I_t, save $I_0 = (1+r)^{-t} I_t$ at $t = 0$.

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- At t you'll have $(1+r)^t I_0 = (1+r)^t((1+r)^{-t} I_t) = I_t$.
- Say that $(1+r)^{-t} I_t$ is the 'discounted present value of I_t at $t = 0$'. r is called the 'discount rate'.
- Because it's easier, we sometimes write $\delta = \frac{1}{1+r}$. δ is the 'discount factor'

We want to compare sequences of utility along different mitigation/growth/warming paths.
Consider two,

$$U_1 = (u_{1t})_{t=0}^{\infty} \equiv (u_{11}, u_{12}, \ldots)$$
$$U_2 = (u_{2t})_{t=0}^{\infty} \equiv (u_{21}, u_{22}, \ldots)$$

How can we compare these two paths?

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If $u_{1t} > u_{2t}$ for all t, it's easy. Chose U_1.

If we don't get lucky, we might try comparing undiscounted sums.

That is, choose U_1 if

$$\sum_{t=0}^{\infty} u_{1t} > \sum_{t=0}^{\infty} u_{2t}$$

$$\Leftrightarrow$$

$$\sum_{t=0}^{\infty} (u_{1t} - u_{2t}) > 0$$

Using undiscounted sums gives rise to three problems.

Problem 1: What does $\sum_{t=0}^{\infty} u_{1t}$ mean? Infinity is not really a number. Let's define it as $\lim_{k\to\infty} \sum_{t=0}^{k} u_{1t}$.

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Problem 2: Suppose

$$(u_{1t})_{t=0}^{\infty} = (1, -1, 1, -1, 1, ...)$$
$$(u_{2t})_{t=0}^{\infty} = (0, 0, 0, ...)$$

Then

$$(u_{1t} - u_{2t})_{t=0}^{\infty} = (1, -1, 1, -1, 1, ...).$$

This means that $\sum_{t=0}^{k}(u_{1t} - u_{2t})$ is 1 for k even or zero, and 0 otherwise.

It follows that for this pair of sequences, $\lim_{k\to\infty} \sum_{t=0}^{k}(u_{1t} - u_{2t})$ does not exist.

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Problem 3: More substantively, undiscounted sums give rise to 'tyranny of the future'.

Consider a pair of utility paths;

$$
\begin{aligned}
(u_{1t})_{t=0}^{\infty} &= \begin{cases} -1000 & \text{if} \quad t \leq 999 \\ 1 & \text{if} \quad t \geq 1000 \end{cases} \\
(u_{2t})_{t=0}^{\infty} &= 0 \text{ for all } t
\end{aligned}
$$

Then $\sum_{t=0}^{\infty}(u_{1t} - u_{2t}) = \infty$. Since this sum is positive, we have to choose U_1 over U_2.
More generally, with this decision rule, we accept any finite cost for any finite number of generations in order to secure any benefit, however small, for perpetuity, a 'tyranny of the future'.

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Using discounted present value to choose utility/consumptions streams

To resolve these problems, rank paths according to their discounted present value.

$$\begin{aligned}\text{Discounted PV of } U_1 &\equiv \sum_{t=0}^{\infty} \delta^t u_{1t} \\ &= u_{10} + \delta u_{11} + \delta^2 u_{12} + \delta^3 u_{13} + \ldots\end{aligned}$$

Unlike undiscounted sums, this sum almost always converges, so discounted PV almost always exists.

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Problem 1: How to evaluate constant utility paths like $\sum_{t=0}^{k}\delta^t u$ and $\sum_{t=0}^{\infty}\delta^t u$?
Let $V=\sum_{t=0}^{k}\delta^t u$. Then,

$$\begin{aligned}
(1-\delta)V &= \sum_{t=0}^{k}\delta^t u-\delta\sum_{t=0}^{k}\delta^t u \\
&= u(1+\delta+\delta^2+...+\delta^k)-\delta u(1+\delta+\delta^2+...+\delta^k) \\
&= u(1+\delta+\delta^2+...+\delta^k)-u(\delta+\delta^2+...+\delta^k+\delta^{k+1}) \\
&= u(1-\delta^{k+1})
\end{aligned}$$

$$\Longrightarrow$$

$$V=u\frac{1-\delta^{k+1}}{1-\delta}$$

Thus, $\sum_{t=0}^{k}\delta^t u=u\frac{1-\delta^{k+1}}{1-\delta}$. It follows that $\sum_{t=0}^{\infty}\delta^t u=u\frac{1}{1-\delta}$.
Since $\delta=\frac{1}{1+r}$, we have $\sum_{t=0}^{\infty}(\frac{1}{1+r})^t u=u\frac{1+r}{r}=u(1+\frac{1}{r})$.

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Problem 2: Tyranny of the present.
Consider two utility paths:

$$
\begin{aligned}
(u_{1t})_{t=0}^{\infty} &= \begin{cases} 1 & \text{if} \quad t \leq 99 \\ -k & \text{if} \quad t \geq 100 \end{cases} \\
(u_{2t})_{t=0}^{\infty} &= 0 \text{ for all } t
\end{aligned}
$$

Choose $(u_{1t})_{t=0}^{\infty}$ if $\sum_{t=0}^{\infty} \delta^t (u_{1t} - u_{2t}) > 0$.
Evaluate this discounted sum,

$$
\begin{aligned}
\sum_{t=0}^{\infty} \delta^t (u_{1t} - u_{2t}) &= \sum_{t=0}^{99} \delta^t + \sum_{t=100}^{\infty} -k\delta^t \\
&= \sum_{t=0}^{99} \delta^t - k\delta^{100} \sum_{t=0}^{\infty} \delta^t \\
&= \frac{1-\delta^{100}}{1-\delta} - k\delta^{100} \frac{1}{1-\delta}
\end{aligned}
$$

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This is positive if.

$$\frac{1-\delta^{100}}{1-\delta} - k\delta^{100}\frac{1}{1-\delta} > 0$$

$$1-\delta^{100} - k\delta^{100} > 0$$

$$1-\delta^{100}(1+k) > 0$$

$$\frac{1}{\delta^{100}} - (1+k) > 0$$

$$\frac{1}{\delta^{100}} - 1 > k$$

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Recalling that $\delta = \frac{1}{1+r}$,

$$\frac{1}{\left(\frac{1}{1+r}\right)^{100}} - 1 > k$$
$$(1+r)^{100} - 1 > k$$

That is, we need r 'big enough' that we don't care about the penalty that happens in the far future.

If $r = 0.013$, choose U_1 if $k < 2.64$. If $r = 0.055$ then choose U_1 if $k < 210.47$.

If k is the cost of warming 100 years from now, and 1 is the benefit from not pursuing mitigation, then our willingness to undertake mitigation (path U_2) depends crucially on the discount rate.

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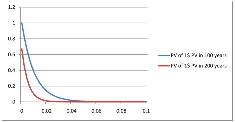
- Another way to see the importance of the discount rate is to calculate the discounted present value of one dollar in 100 and 200 years, $\left(\frac{1}{1+r}\right)^{100}$ and $\left(\frac{1}{1+r}\right)^{200}$ as r varies ..



- Nordhaus bases his analysis on $r = 5.5\%$. Stern bases his on $r = 1.3\%$.
- This difference results in about a 100 fold difference in the present value that they assign to 1 dollar of harm in 100 years.
- The choice of discount rate is as important to our analysis as is resolution of uncertainty about the contemporaneous damages associated with 3 degrees of warming.
- What discount rate should we use? NOBODY KNOWS!

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Here are some discount rates we observe:

Table 4.1. *Estimated returns on financial assets and direct investment*

Asset	Period	Real return (%)
High-income industrial countries		
Equities	1960–84 (a)	5.4
Bonds	1960–84 (a)	1.6
Nonresidential capital	1975–90 (b)	15.1
Govt. short-term bonds	1960–90 (c)	0.3
U.S.		
Equities	1925–92 (a)	6.5
All private capital, pretax	1963–85 (d)	5.7
Corporate capital, posttax	1963–85 (e)	5.7
Real estate	1960–84 (a)	5.5
Farmland	1947–84 (a)	5.5
Treasury bills	1926–86 (c)	0.3
Developing countries		
Primary education	various (f)	26
Higher education	various (f)	13

Sources: Quoted in Nordhaus, 1994: (a) Ibbotson and Brinson, 1987, updated by Nordhaus, 1994; (b) UNDP, 1992, Table 4, results for G-7 countries; (c) Cline, 1992; (d) Stockfisch, 1982; (e) Brainard *et al.*, 1991; (f) Psacharopoulos, 1985.

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Nordhaus is in the middle of this range, Stern is at the bottom.

Nordhaus 2008, fig 3.2, from Arrow et al 1995

A very interesting recent study Schmelzing (2020) documents a ling run decline in interest rates. Note that recent rates are closer to Stern than Nordhaus.

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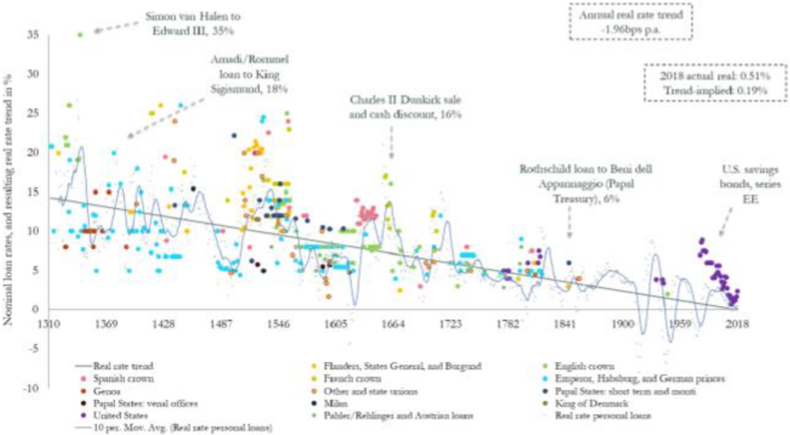

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Foundations of the discount rate

- Given the importance of the discount rate, we should take the time to understand where it comes from.
- Here is the BDICE model, but without the global warming part,

$$\begin{aligned} &\max_{s,c_1} u(c_1, c_2) \\ &\text{s.t. } W = c_1 + I \\ &\qquad c_2 = (1+r)I \end{aligned}$$

- $r > 0$ is the rate of return to capital. $s > 0$ is savings.

This is just a problem about optimal saving.

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- To investigate the determinants of the discount rate, assume 'Constant Relative Risk Aversion' (CRRA) utility function

$$u(c_1, c_2) = \frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{c_2^{1-\alpha}}{1-\alpha}$$

- $\rho > 0$ is 'the pure rate of time preference'. As ρ increases, the utility from future consumption decreases. ρ reflects impatience.
- $\alpha \in (0, 1]$ measures inequality or risk aversion.
- Nordhaus and Stern both base their analysis on CRRA utility because (1) it's easy to work with (2) provides a transparent description of preferences with risk aversion and decreasing marginal utility of consumption.

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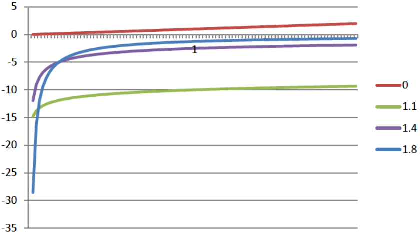
- Here is what $\frac{c_1^{1-\alpha}}{1-\alpha}$ looks like for different values of α. Note that people usually use $\alpha > 1$, so $1 - \alpha < 0$.



- If $\alpha = 0$, then U is linear (so maximize U at a 'corner' where consumption in one period is zero)
- U becomes 'more concave' as α approaches one. This makes our agent more averse to risk within a period and inequality in consumption across periods.

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Aside: Expected utility and risk aversion

Consider a consumer with utility u with $u'' < 0$ who faces a gamble $x = (x_1, x_2, p, 1-p)$. Then,

- The expected value of x, $E(x) = px_1 + (1-p)x_2$.
- The utility of certain payoff x_1 is $u(x_1)$. The utility of certain payoff $E(x)$ is $u(E(x))$.
- The expected utility of gamble x is $E(u(x)) = pu(x_1) + (1-p)u(x_2)$.

Say that a consumer is 'risk averse' if $u(E(x)) > E(u(x))$. That is, if they prefer the expected value of a gamble to the gamble.

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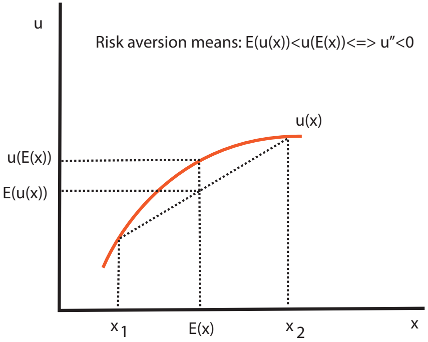

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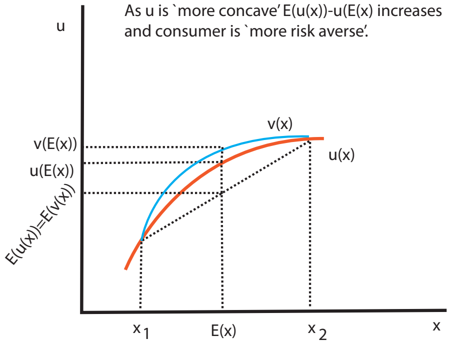

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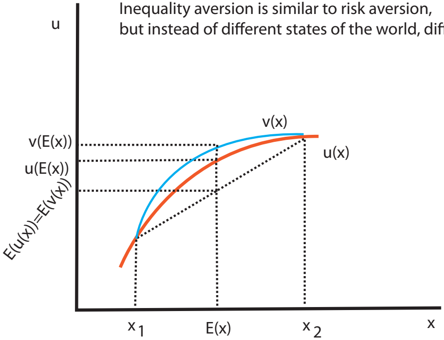
Inequality aversion is similar to risk aversion,
but instead of different states of the world, different days/years



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- Putting it all together,

$$\max_{s,c_1} \frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{c_2^{1-\alpha}}{1-\alpha}$$
$$\text{s.t. } W = c_1 + I$$
$$c_2 = (1+r)I$$

- Now, reorganize the two constraints to get

$$\max_{s,c_1} \frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{c_2^{1-\alpha}}{1-\alpha}$$
$$\text{s.t. } c_2 = (1+r)(W-c_1)$$

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- Next, substitute our constraint into the utility function,

$$\max_{c_1} \frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{((1+r)(W-c_1))^{1-\alpha}}{1-\alpha}$$

- The first order condition for this (complicated) unconstrained maximization problem in one variable is

$$\begin{aligned}
0 &= (1-\alpha)\left(\frac{c_1^{-\alpha}}{1-\alpha}\right) + \\
&\qquad \frac{1}{1+\rho}(1-\alpha)\frac{((1+r)(W-c_1))^{-\alpha}(-(1+r))}{1-\alpha} \\
0 &= c_1^{-\alpha} + \frac{-(1+r)}{1+\rho}((1+r)(W-c_1))^{-\alpha}
\end{aligned}$$

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$$c_1^{-\alpha} + \frac{-(1+r)}{1+\rho} c_2^{-\alpha} = 0$$

$$\left(\frac{c_1}{c_2}\right)^{-\alpha} = \frac{(1+r)}{1+\rho}$$

$$\left(\frac{c_2}{c_1}\right)^{\alpha} (1+\rho) = (1+r)$$

- If we let g be the 'rate of consumption growth', $g = \frac{c_2}{c_1} - 1$, then this is

$$(1+g)^{\alpha}(1+\rho) = 1+r$$

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- Recall the that, for x small

$$\begin{aligned}\ln(1+x) &\approx \ln(1) + x\frac{d}{dx}\ln(x)|_{x=1}\\ &= \ln(1) + x\frac{1}{1}\\ &\approx x\end{aligned}$$

- Taking logs and using this approximation, we have

$$\begin{aligned}\ln((1+g)^{\alpha}(1+\rho)) &= \ln(1+r)\\ \alpha\ln(1+g) + \ln(1+\rho)) &= \ln(1+r)\\ \alpha g + \rho &= r\end{aligned}$$

This equation appears in Nordhaus on p173, and something close in Stern on p 183-5.

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- This gets us a little closer to knowing what the discount rate should be.
- That is, in equilibrium, the rate of return to capital (which is the discount rate) is going to be the sum of the pure rate of time preference and the product of the aversion to inequality and the growth rate of consumption.

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- Stern wants $\rho = 0.001$, $\alpha = 1$, $g = 0.013$, so that $r = 1.3\%$
- Nordhaus wants $\rho = 0.02$, $\alpha = 2$, $g \approx 0.015$ so that $r = 5.5\%$
- This is crucial, both because it affects the present value calculations, as we have seen, but because the choice of α affects risk aversion, which we'll turn to soon.

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- Discounting is problematic. So is every other way of comparing present and future consumption.
- We could have a philosophical discussion of this problem for the rest of the term and (1) not resolve anything or (2) learn anything about global warming.
- In order to organize the global warming problem in a way that treats costs and benefits consistently across time, we have to something. Discounting is it.
- If you are interested in an alternative, read the optional Solow paper on the website.

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Social cost of current emissions

With discounting, we can relate current and future consumption. This lets us finish our calculation of the cost of emissions.

- Let Y_0 denote current world gdp. With consumption growth g of 1.5%/year $(\approx g)$, $Y_{100} = (1.015)^{100} Y_0 = 4.4 Y_0$
- Doubling $\mathrm{CO_2}$ concentrations costs $3/100$ of Y_{100} starting in $t = 100$ and each year afterwards.

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- The PV of this stream of costs is:

$$
\begin{aligned}
& \sum_{t=100}^{\infty} \delta^t 0.03 Y_{100} \\
= & \delta^{100} \sum_{t=0}^{\infty} \delta^t 0.03 Y_{100} \\
= & \frac{\delta^{100}}{1-\delta} 0.03 (4.4 Y_0) \\
= & \frac{\delta^{100}}{1-\delta} 0.13 Y_0
\end{aligned}
$$

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- Dividing by Gt of C needed to get to CO_2 concentration of 560ppm, gives

$$\begin{aligned}
\text{Cost per Gt } C &= \frac{1}{1064}\frac{\delta^{100}}{1-\delta}0.13Y_0 \\
&= \frac{0.13}{1064}\frac{\delta^{100}}{1-\delta}(7.7\times 10^{13}) \\
&= 9.4\times 10^{-4}\times 10^{13} \\
&= (9.4\times 10^{9})\frac{\delta^{100}}{1-\delta}
\end{aligned}$$

- Dividing by 10^9 gives cost per ton of C emissions of $9.4\times\frac{\delta^{100}}{1-\delta}$
- Evaluating at $r = 0.013$ and 0.055 we have the cost per ton of C emissions in current dollars of about 200$ 2010 and 0.8$ 2010.

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This calculation is very sensitive to the rate of growth of gdp and to the choice of discount rate. These are the factors that enter exponentially. Others (just) enter multiplicatively. This is why the results in Dell et al. are so important.

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Uncertainty I

We would like to incorporate uncertainty into our model of savings, and ultimately into our model of global warming.
Recall, our simple model of savings is:

$$\max_{s,c_1} \frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{c_2^{1-\alpha}}{1-\alpha}$$
$$\text{s.t. } W = c_1 + I$$
$$c_2 = (1+r)I$$

and recall that the CRRA function looks like...

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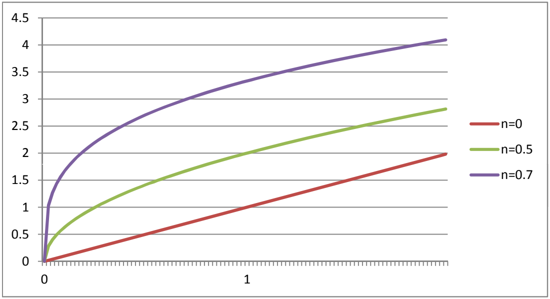

so that our consumer is more risk averse as $\alpha \to 1$. To incorporate

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uncertainty about the return to savings (the benefits of mitigation) into our model, we must first describe this uncertainty.

To do this, let x be a random variable that affects the returns to savings, with $x = (1-\epsilon, 1+\epsilon, \frac{1}{2}, \frac{1}{2})$, so that $E(x) = 1$ and write our consumer's problem as

$$\max_{s,c_1} E\left(\frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{c_2^{1-\alpha}}{1-\alpha}\right)$$

$$\text{s.t. } W = c_1 + I$$

$$c_2 = x(1+r)I$$

This is 'like' global warming. We are uncertain about the impact that mitigation, 'savings', has on future consumption.

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Combining the constraints this becomes

$$\max_{c_1, c_2} E\left(\frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{c_2^{1-\alpha}}{1-\alpha}\right)$$
$$\text{s.t. } c_2 = x(1+r)(W-c_1)$$

Substituting the constraint into the objective, we have

$$\max_{c_1} E\left(\frac{c_1^{1-\alpha}}{1-\alpha} + \frac{1}{1+\rho}\frac{(x(1+r)(W-c_1))^{1-\alpha}}{1-\alpha}\right)$$

Recall that for random variable x and constants a and b,

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- $E(a) = a$
- $E(ax) = aE(x)$
- $E(a + x) = a + E(x)$
- $E(a + bx) = a + bE(x)$

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Using the basic facts about E,

$$\max_{c_1} E\left(\frac{c_1^{1-\alpha}}{1-\alpha}+\frac{1}{1+\rho}\frac{(x(1+r)(W-c_1))^{1-\alpha}}{1-\alpha}\right)$$

becomes

$$\max_{c_1} \frac{c_1^{1-\alpha}}{1-\alpha}+\frac{((1+r)(W-c_1))^{1-\alpha}}{(1+\rho)(1-\alpha)}E\left[(x)^{1-\alpha}\right]$$

or

$$\max_{c_1} \frac{c_1^{1-\alpha}}{1-\alpha}+\frac{E\left[(x)^{1-\alpha}\right]}{(1+\rho)}\frac{(c_2)^{1-\alpha}}{(1-\alpha)}$$

Given that $x=(1-\epsilon, 1+\epsilon, \frac{1}{2}, \frac{1}{2})$, uncertainty increases as ϵ

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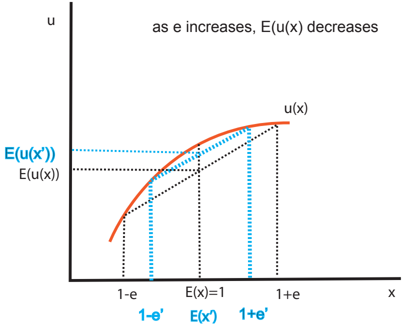
increases. Therefore, if we want to know how behavior changes as 'uncertainty increases', look at what happens as ϵ increases.



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Since $(x)^{1-\alpha}$ is a concave function, it follows that $E((x)^{1-\alpha})$ decreases as ϵ increases.

This means that in

$$\max_{c_1} \frac{c_1^{1-\alpha}}{1-\alpha} + \frac{E\left[(x)^{1-\alpha}\right]}{(1+\rho)} \frac{(c_2)^{1-\alpha}}{(1-\alpha)}$$

an increase in uncertainty ‘increases the price’ of future consumption, or ‘increases impatience’.
Thus, if we are risk averse ($\alpha > 0$), more uncertainty leads us to save less. In the context of our problem, the return to mitigation is lower as we are more uncertain about its affects.
This suggests that Stern’s insistence that we engage in MORE mitigation because we have a lot of uncertainty is one we should regard with suspicion.
Notice that if we are risk averse uncertainty also makes us poorer, which is our next topic.

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Uncertainty II

Since uncertainty makes us poorer, shouldn't we be willing to pay to avoid it?

Yes. To see this analytically, we need a model in which we can consider the way that savings responds to changes in how uncertain are future costs.

Let x be a random variable

$$\begin{aligned} x &= (1-(\epsilon - s), 1+(\epsilon - s), \frac{1}{2}, \frac{1}{2}) \\ &= (1-\epsilon + s, 1+\epsilon - s, \frac{1}{2}, \frac{1}{2}). \end{aligned}$$

So that uncertainty increases as ϵ increases and decreases as s increases (for $s \leq \epsilon$).

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Consider the following simpler variant of our savings problem,

$$\max_{s,c_1} E\left(c_1 + \frac{c_2^{1-\alpha}}{1-\alpha}\right)$$
$$\text{s.t. } W = c_1 + I$$
$$c_2 = xc$$
$$s \leq \epsilon$$

We would like to know whether optimal savings increases with uncertainty. That is, will we save more as uncertainty about future income increases? (This is what Stern wants us to do).
To do this, we want to evaluate $ds/d\epsilon$. This is complicated. Substituting from the constraint and using basic facts about

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expectations,

$$
\begin{aligned}
&\max_{s,c_1} E\left(c_1 + \frac{c_2^{1-\alpha}}{1-\alpha}\right)\\
&\text{s.t. } W = c_1 + I\\
&\qquad c_2 = xc\\
\implies &\max_{s} E\left(W - s + \frac{x^{1-\alpha}c^{1-\alpha}}{1-\alpha}\right)\\
\implies &\max_{s} W - s + \frac{c^{1-\alpha}}{1-\alpha}E\left(x^{1-\alpha}\right)\\
\implies &\max_{s,c_1} W - s + \frac{c^{1-\alpha}}{1-\alpha}\left(\frac{1}{2}(1-\epsilon+s)^{1-\alpha} + \frac{1}{2}(1+\epsilon-s)^{1-\alpha}\right)
\end{aligned}
$$

The first order necessary condition for

$$\max_{s} W - s + \frac{c^{1-\alpha}}{1-\alpha}\left(\frac{1}{2}(1-\epsilon+s)^{1-\alpha} + \frac{1}{2}(1+\epsilon-s)^{1-\alpha}\right)$$

is,

$$0 = \frac{d(.)}{ds}$$

$$0 = -1 + \frac{c^{1-\alpha}}{2(1-\alpha)}\left((1-\alpha)(1-\epsilon+s)^{-\alpha} + (1-\alpha)(1+\epsilon-s)^{-\alpha}(-1\right.$$

$$0 = -1 + \frac{c^{1-\alpha}}{2}\left((1-\epsilon+s)^{-\alpha} - (1+\epsilon-s)^{-\alpha}\right)$$

Technical aside: Implicit differentiation

Say we have a function, $y = f(x)$ which is 'implicitly' defined by $F(x, y) = 0$. Then $\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$, if $\partial F/\partial y \neq 0$.
For example, if $F(x, y) = y - x^2 = 0$

$$\frac{dy}{dx} = -\frac{\partial(y - x^2)/\partial x}{\partial(y - x^2)/\partial y}$$
$$\frac{dy}{dx} = -\frac{-2x}{1}$$
$$\frac{dy}{dx} = 2x$$

But, if we solve F for y, we get $y = x^2$ and hence that $\frac{dy}{dx} = 2x$.
Implicit differentiation is a trick for finding derivatives when it's hard to solve for $y = f(x)$ explicitly, a trick we really need...

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Our first order condition is,

$$0 = -1 + \frac{c^{1-\alpha}}{2}\left((1-\epsilon+s)^{-\alpha} - (1+\epsilon-s)^{-\alpha}\right)$$

This condition describes the optimal choice of s for given ϵ. We'd like to know if utility maximizing savings goes up as uncertainty goes up, that is, if $\frac{ds}{d\epsilon} > 0$

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To differentiate implicitly, note that since

$$0 = F(s, \epsilon) = -1 + \frac{c^{1-\alpha}}{2} \left((1 - \epsilon + s)^{-\alpha} - (1 + \epsilon - s)^{-\alpha} \right)$$

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we have

$$
\begin{aligned}
\frac{\partial F}{\partial \epsilon} &= \frac{c^{1-\alpha}}{2}\left(-\alpha(1-\epsilon+s)^{-\alpha-1}(-1) - -\alpha(1+\epsilon-s)^{-\alpha-1}\right) \\
&= \frac{c^{1-\alpha}}{2}\left(\alpha(1-\epsilon+s)^{-\alpha-1} + \alpha(1+\epsilon-s)^{-\alpha-1}\right) \\
&> 0 \\
\frac{\partial F}{\partial s} &= \frac{c^{1-\alpha}}{2}\left(-\alpha(1-\epsilon+s)^{-\alpha-1} - -\alpha(1+\epsilon-s)^{-\alpha-1}(-1)\right) \\
&= \frac{c^{1-\alpha}}{2}\left(-\alpha(1-\epsilon+s)^{-\alpha-1} - \alpha(1+\epsilon-s)^{-\alpha-1}\right) \\
&< 0
\end{aligned}
$$

So we have

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$$
\begin{aligned}
\frac{ds}{d\epsilon} &= \frac{-\frac{\partial F}{\partial \epsilon}}{\frac{\partial F}{\partial s}} \\
&= -1 \times \frac{(-)}{(+)} \\
&> 0
\end{aligned}
$$

Thus, $\frac{ds}{d\epsilon} > 0$ and we optimally save more when doing so can reduce future uncertainty.
That is, Stern is half right.

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Conclusion

- The last step in calculating the cost of carbon emissions was to relate future costs to current costs. We solve this problem with discounting and present value calculations.
- The details of this calculation are crucial, and there is not satisfactory way to choose a discount rate. At the end of the day, I suspect that this problem will be resolved by choosing a price of future consumption, e.g., 1$ today for 10$ in 100 years, an backing out the implied discount rate.
- Uncertainty about future damages reduces willingness to save by reducing the value of future consumption.
- Increasing uncertainty about future damages increases the willingness to save to reduce uncertainty.
- It is not clear whether the total effect of uncertainty should be to increase or decrease savings (hence mitigation).

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