Regional Science and Urban Economics 15 (1985) 325–340. North-Holland

EQUILIBRIUM MODELS WITH LAND

A Criticism and an Alternative*

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Received July 1984, final version received January 1985

Models of economies either of urban areas or with local public goods often involve the use of a continuum of consumers along with the use of a commodity called land; each consumer generally owns a parcel of land of positive area. The purpose of the present study is to show that such models are internally inconsistent (independent of the other assumptions employed) in that only countably many consumers can own parcels of land of non-zero area if land lies in a Euclidean space. This result applies, in particular, to monocentric city models. Moreover, it is shown that the standard justification for the use of economies with an infinity of agents, that they approximate large economies with a finite number of consumers, does not necessarily apply in the case of economies with land and a continuum of consumers. A model where land is represented by subsets of $\mathbb{R}^2$ is presented as an alternative.

> 'Put another way, we are still plagued by the problem mentioned in the introductory chapter, that we are dealing with the user of land as if he were located at a point, while the necessity of equating supply and demand quantities requires that we extend this point so it will have an area.'
>
> Alonso (1964, p. 99)

1. Introduction

Models with land and a continuum of consumers are fundamental theoretical tools used by public finance economists, urban economists and location theorists. Such models are generally employed as approximations to large (but finite) economies in much the same way as continuum models without land are used to approximate large but finite economies without land. For example, Beckmann (1969) and Wheaton (1979) use continuum models with land to explain the residential location of consumers along with the corresponding gradient of rents as distance from the city center varies. Such techniques have been shown to have much empirical content. These models use densities rather than actual plots of land for commodities. The

*This paper is based on chapters 1–3 of the author's Ph.D. dissertation at the University of California, Berkeley. Comments of the dissertation committee consisting of G. Debreu, S. Goldman, and D. Aldous, along with those of B. Allen, J. Brueckner, K. Dunz, R. Guesnerie, T. ten Raa, N. Singh, K. Stahl, J. Thisse, J. Wilson, and an anonymous referee, are gratefully acknowledged. The author bears sole responsibility for any remaining errors.

0166–0462/85/$3.30

purpose of the present study is to show that such densities cannot be interpreted as actual areas of land because there are not enough disjoint subsets in the plane to give positive area to each of a continuum of consumers. As a consequence, the analogy to the standard large economies literature [see Hildenbrand (1974)] breaks down and it is demonstrated that [redacted]

N represents their total measure (in some sense). A contradiction will now be encountered because there is a continuum of consumers inhabiting an area of land in a σ-finite space, so most must own a set of area zero.

The standard justification for eq. (1) is that the integral can be approximated by a discrete sum using rings around the city center as the basis for the sum. That is, if b is the radius of the city, n the number of rings, b/n the width of each ring, and q a function that takes as argument the outer distance of a ring from the city center and that yields as its value the average land holdings of consumers who live in the ring, then the left-hand side of (1) can be approximated by

$$\sum_{i=1}^{n} \frac{\pi\left(i\frac{b}{n}\right)^2 - \pi\left((i-1)\frac{b}{n}\right)^2}{q\left(i\frac{b}{n}\right)}. \tag{2}$$

Consumers are restricted to own land in only one ring, for otherwise double counting is unavoidable. The numerator of each term gives the area of the donut of width b/n with outside at radius $i(b/n)$ from the city center. Dividing this by the average amount of land per person yields the number of consumers whose land holdings lie in the donut. Summing these figures over all donuts gives the total population.

To take limits, it is helpful to rewrite (2) as

$$\sum_{i=1}^{n} \frac{\pi\left(i\frac{b}{n}\right)^2 - \pi\left((i-1)\frac{b}{n}\right)^2}{i\frac{b}{n}-(i-1)\frac{b}{n}} \cdot \frac{1}{q\left(i\frac{b}{n}\right)} \cdot \left[i\frac{b}{n}-(i-1)\frac{b}{n}\right].$$

Letting n tend to infinity, the first part of each term is a derivative with value $2\pi t$ (where t is a parameter), so that if some technical problems are assumed away, (2) converges to the left-hand side of (1). It is shown below that (2) is always infinite, so that its limit (the integral) is undefined.

Suppose that a limit to the following expression exists:

$$\lim_{\substack{k\to\infty \\ n\to\infty}} \sum_{i=1}^{n} \frac{\pi\left(i\frac{b}{n}\right)^2 - \pi\left((i-1)\frac{b}{n}\right)^2}{q\left(i\frac{b}{n}, k\right)}, \tag{3}$$

where k is the number of consumers in the economy, b is fixed (for now), and q, the average land holdings in a donut, is now a function of both the donut and the number of people in the economy. *If b and n are held fixed*, as k gets large, the average land holdings in some donut must go to zero. That is, for fixed b and n, $\exists\, i'$ with $i' \leqq n$ and $\lim_{k\to\infty} q(i'\cdot(b/n), k) = 0$. This is clear because new consumers must be put in some donut, and this reduces q in that donut. Thus, since $\pi(i'(b/n))^2 - \pi((i'-1)(b/n))^2$ is fixed for constant b and n,

$$\lim_{k\to\infty} \frac{\pi\left(i'\dfrac{b}{n}\right)^2 - \pi\left((i'-1)\dfrac{b}{n}\right)^2}{q\left(i'\dfrac{b}{n}, k\right)} = \infty.$$

Now since (3) has a limit, the order in which limits are taken can be interchanged. We rewrite (3) as

$$\lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{n} \frac{\pi\left(i\dfrac{b}{n}\right)^2 - \pi\left((i-1)\dfrac{b}{n}\right)^2}{q\left(i\dfrac{b}{n}, k\right)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \lim_{k\to\infty} \frac{\pi\left(i\dfrac{b}{n}\right)^2 - \pi\left((i-1)\dfrac{b}{n}\right)^2}{q\left(i\dfrac{b}{n}, k\right)}.$$

Plugging $i = i'$ into the expression inside both limits, we have that the i' term in the sum is infinite. Hence the sum itself is infinite, and its limit under n is undefined. This contradicts the assumption that (3) is defined. Hence the expression (3) is undefined *for any b*. Thus, the integral in (1) is undefined. Hence, the density q cannot be interpreted as an area or an average area held by a consumer in the model. This result generalizes to any model with land and a continuum of consumers, even when the amount of land available to the economy is unbounded.

Theorem 1. The following statement is self-contradictory: 'There is a continuum of consumers, each of whom owns a parcel of land of positive area.'

A formal statement and proof of the theorem can be found in the appendix.

Remarks. The intuition behind the result is easy to see. If, as in the monocentric city models, the area of land in the economy is finite and there is an infinity of consumers, the result follows. If, on the other hand, there is a continuum of consumers modelled on the unit interval and land is modelled as $\mathbb{R}^2$, then any partition of $\mathbb{R}^2$ into subsets of positive area can have its elements each assigned a positive integer. Allocating parcel n to agent $1/n$, all of $\mathbb{R}^2$ is exhausted and most agents (e.g., those assigned irrational co-ordinates) are left with no land.

What this means is that even if parcels are mapped to another space, e.g., a space of characteristics in a hedonic model, there is still a one-to-one correspondence between consumers and parcels (by inverting the hedonic map) so that there cannot be a continuum of consumers or, for that matter, a continuum of points in the space of characteristics.

Of course, these same criticisms apply to the other models that postulate monocentric cities, such as Alonso's (1964) bid-rent model [see equation (15) of Wheaton's paper]. They also apply to the optimal cities literature [equations (34), (35), and (36)]. Furthermore, it does not seem that these models can be salvaged in any reasonable way, since the differential equation techniques used to solve them depend on the assumption that there is a measurable (possibly infinite) parcel of land available to a continuum of consumers.

An immediate corollary to the result is that if a continuum model with land is used, consumers cannot be indifferent between sets of zero area (densities) if the model is to be non-trivial. This assumption is not implied by the Hildenbrand (1974) continuum model.

Alternative interpretations of eq. (1) may now be considered. For example, if N is the total number of consumers and each consumer is spread out over his land (uniformly, for example) such that the total mass of the consumer is 1, then Theorem 1 tells us that the number of consumers is countable if the distributions are non-atomic. If any of the distributions is atomic, then (1) cannot be written.

Another alternative is to represent each of a finite number (N) individuals as the aggregate of a continuum of infinitesimal consumers, thus transferring the distribution from the commodity space to the space of agents. However, this implies that the solution to the aggregate maximization problem is the same as the aggregate of the individual maximization problems, which imposes strong conditions on preferences.[1] Furthermore, each of the continuum of individuals must still get utility from owning a set of zero measure.

In summary, the use of a continuum of consumers in a model with land forces most to own parcels with zero area because there is only a countable number of disjoint subsets of positive area in a Euclidean space. Hence, the monocentric city models make no sense if one tries to interpret the densities as actual areas of land. The next question is whether these continuum models are approximations to any reasonable large (but finite) economy.

[1]For example, one might postulate that a continuum of consumers directly represents one large consumer. Such an assumption necessarily involves preference aggregation, and is currently under examination by Thijs ten Raa and the author.

Before tackling the preceding question, it is first necessary to clear up a minor point of confusion in the literature about where the monocentric cities' models stand relative to large economies' models without land. The description of large economies given below is based on Hildenbrand (1974, ch. II.1.1.3). In the standard large economies framework, agents are represented by a measure space $(A, \mathscr{A}, v)$, where A is the set of agents, $\mathscr{A}$ is a collection of subsets of A (a σ-algebra), and v is a countably additive map from $\mathscr{A}$ to the reals that represents the relative size or market power of agents or groups of agents. The uses of measure-theoretic terms such as atomless and density are with regard to the measure v. Allocations can then be defined as measures or densities on $(A, \mathscr{A}, v)$. In such a model with a continuum of consumers, if allocations are densities, then each consumer owns a negligible *fraction* of the total quantity of commodities, but still owns a non-negligible *amount* of such commodities (the value of the density). The market clearing conditions equate mean supply to mean demand.

In contrast, the monocentric city models have densities on the *commodity space* (the plane), not the space of agents $(A, \mathscr{A}, v)$. For example, the density $q(t)$ enters into the utility function of consumers in equation (1) of Wheaton's paper and is therefore a commodity. That is, each consumer's utility is a function of land (q), other goods (x), and commuting distance (t):$u(x, q(t), t)$. Here $q(t)$ must be a density with respect to Lebesgue measure on $\mathbb{R}^1$ since it is integrated against $\mathrm{d}t$ and $2\pi t$ is the density of land in the plane. If consumers at distance t from the center are distributed uniformly around the circle of this radius (as seems to be implicit in a one parameter model), then land holdings are in fact a density in $\mathbb{R}^2$, and are contained in a parcel (a circle) that has area zero in $\mathbb{R}^2$.

Since a commodity bundle is a density on the commodity space, consumers hold negligible *amounts* of land as well as a negligible *fraction* of the total quantity of commodities. These densities have nothing to do with the relative size or market power of agents.

Returning to the approximation question, land is assumed to be represented by L, a measurable subset of some Euclidean space, say $\mathbb{R}^2$. Let $\mathscr{F}$ be the σ-algebra of measurable subsets of L. The key assumption that is now made and maintained for the remainder of the paper is that the consumption set of each consumer is some sub-algebra (or sub-σ-algebra) $\mathscr{B}$ of $\mathscr{F}$. This postulate deserves examination.

Using subsets as commodities instead of other representations of land (quantities, for example) seems as natural as modelling dated commodities in a model with time as having dates on the real line or as integers. It also seems as natural as giving commodities in a model with both time and uncertainty a tree structure. Land seems to have a natural locational attribute, along with a quantity and shape. These qualities may or may not enter into preferences and utilities, but the natural consumption set that allows this entrance is a collection of subsets of $\mathbb{R}^2$. This type of structure is consistent with the models with finitely many consumers and land, such as that of Koopmans and Beckmann (1957), who use a finite collection of subsets as the consumption set of each agent.

Of course, just as one can model land in the same manner as other commodities (as a quantity), one can model other commodities as subsets as well. The crux of the matter about how modelling should proceed for any given commodity is how one interprets a quantity. If one interprets a quantity of land (or another commodity) as, say, an area in $\mathbb{R}^2$ (or any Euclidean space), then this is exactly isomorphic to modelling land as $\mathscr{B}$. This is true even if land is completely homogeneous. For other goods, such as butter, that are viewed as homogeneous, then the interpretation of quantity may end with a scalar, and it might not be necessary to interpret this scalar as an area or subset of a Euclidean space.

Clearly, it is intended that the monocentric city models have an interpretation in $\mathbb{R}^2$, as the term 'distance from the city center' is used in this literature. Consumers are located, and own areas of land, at various distances from the city center. This implies that land is implicitly modelled as subsets of a Euclidean space. For the reasons given above, we restrict attention to potential approximating economies that give the consumption set $\mathscr{B}$ to each of a finite number of consumers.

For the remainder of this paper, 'closeness' of economies refers to the standard notion of Hildenbrand (1974) described formally in the appendix. It means that the population and types (specified by utilities and endowments) of consumers in economies are similar. An implication of closeness is that close economies 'approximate' one another, in the sense that their equilibrium prices, allocations, and comparative statics results are similar. Theorem 2 shows that continuum and finite models are not close unless land holdings in the finite models tend to zero area. If land holdings do not tend to zero area, then it is not necessarily true that the continuum economy is close to or approximates the finite economies.

Theorem 2. Suppose a sequence of economies, each with a finite number of consumers, converges in the sense of Hildenbrand to an economy with a continuum of consumers. That is, utilities, endowments, and the population of consumers are converging. Then land holdings in the sequence of finite economies must tend to zero.

A formal statement and proof of the theorem can be found in the appendix.

Thus, economies that are limits of finite models that involve significant land ownership must allow significant land ownership (i.e., positive area) in the limit; monocentric city models do not allow the ownership of positive areas in the limit. Finite economies converging to monocentric city models cannot allow at least one out of every 100 consumers to own at least one square inch of land, so that monocentric city models are not close (in the Hildenbrand sense) to reasonable finite economies. As a result, equilibrium prices, allocations and comparative statics results of a continuum model are not necessarily close to those of a reasonable finite model, one that we consider to be close to reality. There are, of course, many methods for approximation other than those usually used by mathematical economists. The point of this paper is *only* to show that the standard methods do not work, and that research is needed to find approximation methods that do, in fact, work. Two important points should be kept in mind. First, in what sense are continuum models approximated by finite models? Are the equilibrium prices and allocations, along with comparative statics results, close? To the author, this would only be a coincidence unless consumers have nearly the same endowments and utilities in the economies under consideration. That is, convergence occurs in the usual (Hildenbrand) sense. Second, since the finite economies are supposed to correspond (in some sense) to reality, do they make sense? Does the subsistence level or endowment of a consumer tend to a positive amount, or does it tend to nothing?

In summary, the densities in monocentric city models (and other continuum models with land) cannot be interpreted as areas. Implicitly, this occurs because these densities are defined over the consumption set $\mathbb{R}^2$ rather than over the space of agents, and hence they have a meaning distinct from that in the large economies literature. It can then be demonstrated that if consumers own non-zero areas of land in finite economies, then close continuum economies must also involve ownership of non-zero areas of land, i.e., areas that are not densities. Given these problems, how is it possible to model land in a consistent manner?

3. An alternative

As Debreu (1959) made clear, the attributes that can be specified in the definition of a commodity include location. Thus, in a sense, land or location can be put into a classical general equilibrium model. Problems arise only when examining the assumptions that are needed to demonstrate the existence of an equilibrium. In particular, Schweizer, Varaiya, and Hartwick (1976) point out that the assumption of convexity of preferences makes little sense, as it implies that consumers would desire to own land that is spread out rather than concentrated. There are also non-convexities in production and consumption sets, as detailed by Koopmans and Beckmann (1957). Finally, if land is infinitely divisible, then there is an infinity of commodities (the commodity space is not Euclidean). Thus, the classical general equilibrium framework is inadequate for handling land. Modern variants of the classical framework and their suitability for use in a model of land are discussed below.

At first glance, it may seem that several very general models, such as Bewley (1972), Mas-Colell (1975), or Jones (1984), are directly applicable to the special case of modelling land. However, this is not possible for reasons mainly related to the convexity of preferences, the convexity of consumption sets, and the divisibility of land parcels.

With regard to the Bewley (1972) model of an economy with infinitely many commodities, one could circumvent the objection that land is not divisible or combinable by:

(a) using all possible subsets of the given land as the commodities; combine this with a production technology to assure that divisions of land are feasible (i.e., subsets don't overlap), or
(b) using infinitesimal pieces of land as commodities.

There are several drawbacks to each of these approaches. Mas-Colell (1975) points out that Bewley had time or uncertainty in mind when formulating this model, and that the mathematical context (i.e., L^p spaces) used by Bewley for commodities and prices is inappropriate when modelling a situation where 'closeness' of commodities make sense. Under the (a) approach, it is plain that there are non-convexities in the production set or wherever the feasibility constraint on land is introduced. This is due to a -1, 0, or 1 possibility for a parcel of land in the infinite-dimensional production set; either a parcel is used, remains as is, or is produced through the recombination of land. Bewley's proofs require a convex production set. The (b) approach also requires non-convexities, since each point is either owned by a consumer or not owned by him. In fact, this approach will be the one we take, since the problem associated with it is rectifiable in the right circumstances. However, Mas-Colell's criticism still stands since the model does not handle the proximity of land parcels very well.

There are other approaches that one might take to model land. For example, one could imagine proceeding in the direction of Mas-Colell (1977), who treats indivisibilities in an exchange economy. Ellickson (1979) extended this methodology to production, but problems arise with approximate equilibria in the context of an infinite number of commodities. Furthermore, both of these models involve large economies, and these are hard to deal with in the context of land; recall that models with a continuum of consumers are not well-behaved if land is a commodity. Moreover, the replication of land (or the use of a random economies approach) for the purpose of approximation makes little sense due to the heterogeneity and spatial nature of land. Approximating economies might have holes in the set of land unless the sequence follows a specific pattern. McLean and Muench (1981) avoid this problem by mapping land to new locations, but this technique leads to distortions in distances along with most variables in the economy. Finally, the mathematics would be very complicated or even impossible because land cannot be treated like other commodities since it is not homogeneous and changes to new commodities when split; in other words, new commodities are made from old using a highly non-convex production technology. Ellickson (1979) requires constant returns to scale in production and convex preferences.

It seems that the objections to the use of the preceding models with infinitely many commodities to model land serve to point out that similarities between commodities are not explicitly employed in these models. Two similar, adjacent parcels of land are treated the same, formally, as two dissimilar, non-adjacent parcels. If somehow it were possible to treat similar commodities in almost the same way, the model could be reduced to a finite number of dimensions. The non-convexities might disappear, and agents would have an easier time solving their maximization problems. This suggests very strongly a Lancaster or characteristics approach to the problem, an approach now summarized.

The idea of using characteristics instead of commodities in a model has been discussed for some time. The first paper that explicitly used a characteristic approach was Lancaster (1966), while most of the formal work on models with characteristics is based on Mas-Colell (1975). Commodities are represented by points in a compact space; a point describes the characteristics of a commodity completely. Consumers have positive endowments of goods and preferences that are continuous over characteristics, while prices are also continuous functions of characteristics. There are intrinsic indivisibilities in the commodities in this model, and this leads to the assumption that there is a continuum of traders [see Mas-Colell (1977)]. The results obtained from these hypotheses, combined with a few others, are an existence and a core-equivalence theorem. Ellickson (1977) applies a variant of this model to the housing market, while Jones (1984) supplies conditions for the existence of an equilibrium in a similar model with a finite number of traders.

Some objections to the use of this type of framework for modelling land may be raised along the lines of those raised for Bewley (1972) earlier. However, since the assumptions used in these hedonic models seem even stronger than Bewley's assumptions (especially those pertaining to preferences), the model is even more foreign to the introduction of land. First, the convexity-like and monotonicity-like assumptions on preferences in Jones (1984) and Mas-Colell (1975) do not make sense when land is a commodity. The same reasoning applies here as in the case of the standard general equilibrium framework; it is discussed in Schweizer, Varaiya, and Hartwick

(1976). With regard to monotonicity assumptions, VI (iii)–(v) of Mas-Colell (1975) do not apply to land. In particular, if land parcels are to be the commodities, it might not be possible to have more than one instance of a given parcel type in an economy. If one wishes to restrict consideration to economies with a finite number of traders, one finds that the assumptions in Jones (1984) are even more restrictive. In fact, the assumption of convex preferences is needed. Even though commodities would be parcels rather than locations as in the Schweizer–Varaiya–Hartwick model, the same pathological behavioral assumptions would lead to the ownership of small sites dispersed over a large area.

The final objection to the use of this framework for land has to do with the way characteristics are modelled. The framework attempts to model indivisible, differentiated commodities. In one sense, land is far from indivisible. It can be subdivided and combined, yielding new commodities with characteristics that might not be closely related to those of the original parcel. In essence, commodity bundles are unique, and there is a non-convex production technology for recombining them that includes such factors as whether parcels are adjacent or not. These uniqueness and feasibility constraints are not captured by the model, nor is the market-clearing condition that traders' parcels be disjoint and that their union be the totality of land available. The reason is that land and its characteristics are divisible (in a certain sense), and the model is not designed to handle this. To quote Mas-Colell (1975, p. 265), 'We believe that to encompass differentiated commodities which are available in "infinitesimal" amounts a substantial rethinking of the model is needed'.

Given the problems with models that use an infinite number of consumers, it seems reasonable to start with a model that has only a finite number of consumers. Furthermore, consumers should own subsets of $\mathbb{R}^2$ rather than points. This not only seems more realistic, but avoids the problems discussed in section 2. Consumers can actually own land parcels with positive area.

What follows is a brief summary of some of the results that may be obtained with such a model. It is hoped that they are sufficient to convince the reader that this model does not suffer from the deficiencies of the models discussed previously, that the model is tractable in the sense that theorems can be proved, and that it might be possible to implement the model empirically.

Motivation for the model derives from previous literature such as Koopmans and Beckmann (1957), Shapley and Scarf (1974), and Roth and Postlewaite (1977). The commodity space of each of these models is a finite number of indivisible goods, equal in total quantity to the number of agents in the economy. These goods can be interpreted as indivisible parcels of land that partition the total land available to the economy. Each agent starts with one parcel for an endowment and ends up with one possibly different parcel in equilibrium. Clearly, it is desirable to allow consumers to own more than one parcel and to allow them to divide and recombine parcels in ways that they see fit. That is, the consumption set should consist of unions of elements of some partition of land that is fine. It is not obvious (or even true) that equilibria exist in the latter case, nor that equilibrium prices and allocations do not jump violently as the partition of land becomes finer and finer. If equilibrium prices and allocations are not 'stable' with respect to finer partitions of land, then one must know the precise partition or restriction on the consumption set in order to calculate them. As the results below are implicitly proved by taking limits as the partition of land becomes fine, the demands and equilibria approximate those where the partition of land is not as fine as possible. Of course, conditions on utilities are needed to obtain these results. These conditions guarantee both this sort of stability in taking finer partitions of land and the existence of an equilibrium when parcels can be recombined in virtually any manner.

To be precise, let L be a measurable subset of $\mathbb{R}^2$, $0<m(L)<\infty$, that is to represent land. Anything immobile can be embedded in L. Let $\mathscr{B}$ represent the σ-algebra of measurable subsets of L; $\mathscr{B}$ is the consumption set of each agent, so land can be subdivided and recombined in just about any way. The letter m represents Lebesgue measure (a measure of area) on $\mathbb{R}^2$. Let the number of consumers be n (integer and finite) and let i and j index the consumers. It is assumed that trader i has an initial endowment of land $E_i \in \mathscr{B}$ with $m(E_i)>0$ along with a preference ordering $\succcurlyeq_i$ over $\mathscr{B}$. The E_i partition L; $\bigcup_{i=1}^{n} E_i = L$ and $E_i \cap E_j = \phi \; \forall \, i \neq j$. Preferences take account of objects, such as houses, embedded in L through preferences over $\mathscr{B}$. The following results are summarized from Berliant (1982). Under some fairly weak assumptions, it can be shown that a continuous utility representation U_i of $\succcurlyeq_i$ exists. Under stronger independence assumptions, it can be shown that U_i is of the form $U_i(B)=\int_B h_i(x)\,\mathrm{d}m(x)$ for $B\in\mathscr{B}$, with h_i integrable and $h_i>0$ almost surely. These assumptions yield a marginal utility density that is independent of other points that a consumer might own. They effectively rule out complementarities between parcels, but are useful (at this point) in order to obtain results.

The price space corresponding to the commodity space $\mathscr{B}$ is somewhat problematic, as duality theory does not supply a natural space. It is desirable to have no arbitrage in equilibrium, for otherwise traders would always wish to change their demands. In the context of the model, no arbitrage means that traders cannot put parcels together or take them apart and make a profit. Hence prices should be additive as a function of parcels. If traders are not to make a profit by putting together or taking apart an infinity of parcels, then prices should be countably additive. If, furthermore, a parcel of zero area is to have a zero price, then the Radon–Nikodym theorem yields a price space that is the set of all integrable functions on L; that is, if $B\in\mathscr{B}$, the price of B is $\int_B p(x)\,\mathrm{d}m(x)$ for an integrable p. Thus, the consumer's problem is to maximize $\int_B h_i(x)\,\mathrm{d}m(x)$ subject to $\int_B p(x)\,\mathrm{d}m(x) \leqq \int_{E_i} p(x)\,\mathrm{d}m(x)$ over $B \in \mathscr{B}$. The solution to this problem can be characterized as (approximately) $\{x \in L \mid h_i(x) \geqq r_i p(x)\}$ for a certain $r_i \in \mathbb{R}$. Points with a high marginal utility to price ratio are selected first, and the selection continues until income is exhausted.

The concepts of equilibrium and Pareto Optimum are defined as follows. An *equilibrium* is an assignment of a parcel $B_i \in \mathscr{B}$ to each agent i along with an integrable price density p such that B_i solves the problem of consumer i and such that $\{B_i\}_{i=1}^n$ partitions $L(\bigcup_{i=1}^n B_i = L, B_i \cap B_j = \phi\ \forall\, i \neq j)$. A *Pareto Optimum* is a partition of L, $(B_1, B_2, \ldots, B_n)$, such that for no other partition $(C_1, C_2, \ldots, C_n)$ of L, $C_i \succcurlyeq_i B_i\, \forall\, i$, $1 \leqq i \leqq n$, with strict preference holding for some j. Under the assumption of integral utilities, the existence of an equilibrium and the two welfare theorems can be demonstrated. Furthermore, if utilities are smooth in the sense that each h_i is continuous, then an equilibrium with a continuous price density exists.

Using the characterization of demand above, equilibria can be calculated for specific examples.

In the important case when more general utilities than integrals are allowed, the marginal utility of giving a consumer a point can depend on the other land holdings of the consumer; the proximity of points can enter into preferences. Berliant and Dunz (1983) have shown that the second welfare theorem holds in this context, that equilibrium exists under a weak assumption, and that the first welfare theorem is false. Pareto optima and core allocations might not exist for this model.

In summary, most location models are (mathematically) inconsistent or use undesirable assumptions. Furthermore, they tend to lie outside the mainstream of economic theory. A revision has been proposed, and it is hoped that, at a minimum, a comparison of the advantages and disadvantages of each type of model can now be made.

Appendix

Theorem 1. The following statement is self-contradictory: 'There is a continuum of consumers, each of whom owns a measurable parcel of land of positive area (where area is Lebesgue measure) in a subset of a Euclidean space.'

Proof of Theorem 1. Let E be a subset of a Euclidean space. Therefore E is σ-compact. Hence there is a countable collection of compact sets, call it $\{C_1, C_2, \ldots\}$ such that

$$E = \bigcup_{i=1}^{\infty} C_i.$$

Let m be Lebesgue measure on E. Suppose that there is no C_i such that more than countably many parcels of land intersect it with positive measure. Then the number of parcels is no greater than the number of parcels intersecting C_i with positive measure (at most countably infinite) multiplied by the number of compact sets C_i (at most countably infinite), so that the

It might be argued that the ownership of a density in the consumption set can be pushed back to a density on $(A, \mathscr{A}, \nu)$ by using the map $\nu \cdot s^{-1}$. However, this would still imply that each of a continuum of consumers owns a set of positive area (if the value of the density for a given $a \in A$ is interpreted as area in $\mathbb{R}^2$), since the existence of a density on $(A, \mathscr{A}, \nu)$ implies that the consumption sector is atomless, which implies that there is a continuum of consumers. This, of course, contradicts Theorem 1, so such an argument cannot be made.

A sequence of consumption sectors is a sequence $[s_n; (A_n, \mathscr{A}_n, \nu_n)]_{n=1}^{\infty}$. Such a sequence is said to converge to a consumption sector $[s; (A, \mathscr{A}, \nu)]$ if the sequence $\{s_n\}_{n=1}^{\infty}$ converges to s in distribution, i.e., the sequence $\mu_n \equiv \nu_n \cdot s_n^{-1}$ of preference-endowment distributions converges weakly to the preference-endowment distribution $\mu \equiv \nu \cdot s^{-1}$. This means that the distribution of the consumer characteristics converges, not that the consumer characteristics necessarily converge pointwise. For example, if a continuum economy is postulated at the start, then selecting consumers at random to add to a finite economy will produce a sequence of finite economies converging to the continuum economy [see Hildenbrand (1974, p. 117)].

Theorem 2. Suppose that $[s_n; (A_n, \mathscr{A}_n, \nu_n)]_{n=1}^{\infty}$ is a sequence of simple consumption sectors converging to an atomless limiting consumption sector $[s; (A, \mathscr{A}, \nu)]$. If each consumption sector in the sequence has the property that at least one out of every 100 consumers is endowed with at least one square inch of land, then:

(i) *the limiting consumption sector is* not *one in which all consumers own densities of land (each of a continuum owns a plot of positive area) and*

(ii) *the limiting consumption sector has endowments that do* not *lie in a Euclidean space.*

Proof. To prove (i), note that $\mu_n \equiv \nu_n \cdot s_n^{-1}$ is converging weakly to $\mu \equiv \nu \cdot s^{-1}$. Since area (or Lebesgue measure) is continuous on $\mathscr{B}$, $\mathscr{C} \equiv \{(P, B) \mid P \in \mathscr{P}, B \in \mathscr{B}$, the area of B is greater than or equal to one inch$\}$ is closed in $\mathscr{P} \times \mathscr{B}$. Also, $\mu_n(\mathscr{C}) \geqq 1/200$ for all $n \geqq 100$ by hypothesis, so $\limsup_n \mu_n(\mathscr{C}) \geqq 1/200$. By Hildenbrand [1974, I.D(26)], $\mu(\mathscr{C}) \geqq 1/200$. That is, the measure of consumers owning at least one inch of land in the limit economy is at least 1/200. Since ν is atomless, there is a continuum of consumers in the limiting consumption sector owning at least one square inch of land (not a density of land). Part (ii) follows from an application of Theorem 1 to this situation.

Q.E.D.

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