Representation of All Possible Parenthesizations of Matrix-Chain Multiplication in a Directed Acyclic Graph

Firstly, let's revise some notation. We define G as a directed acyclic graph (DAG) that will represent all possible parenthesizations for $P(i,j)$ for each subproblem $S(i,j)$ where $1 \leq i \leq j \leq n$. The set of vertices for this graph is:

$$V(G) = \{S(i,j) \ : \ 1 \leq i \leq j \leq n\}.$$

For $i < j$, $S(i,j)$ has exactly $2(j-i)$ outgoing edges. For $k = i, \ldots, j-1$ exactly two edges start from $S(i.j)$ and end in $S(i,k)$ and $S(k+1,j)$. This is a *rigid pair* and we label these with index k.

For each vertex $S(i,j)$, we define the set $P_G(i,j)$ of parenthesizations corresponding to $S(i,j)$ in G. For $i < j$, let $K_G(i,j)$ be the set of indexes of rigid pairs outgoing from $S(i,j)$ in G, then

$$P_G(i,j) = \bigcup_{k \in K_G(i,j)} \{(p_1 \times p_2) \ : \ p_1 \in P_G(i,k), \ P_G(k+1,)\} \tag{1}$$

Let us consider the matrix chain $A = A_1 \times A_2 \times A_3 \times A_4$. We denote the problem of finding the parenthesizations of $A_1 \times \cdots \times A_4$ as $S(1,4)$. The corresponding DAG for this problem can be seen in Figure 1. For $i < j$ we can find the set of indexes $K_G(i,j)$,

$$\begin{aligned}
K_G(1,4) &= \{1,2,3\} \\
K_G(1,3) &= \{1,2\} \\
K_G(2,4) &= \{2,3\} \\
K_G(1,2) &= \{1\} \\
K_G(2,3) &= \{2\} \\
K_G(3,4) &= \{3\}
\end{aligned}$$

These sets can then be used to find all possible parenthesizations $P_G(i,j)$ for all subproblems $S(i,j)$ using 1. When $i = j$, we know that $P_G(i,j) = \{A_i\}$ trivially. Therefore,

$$\begin{aligned}
P_G(1,1) &= \{A_1\} \\
P_G(2,2) &= \{A_2\} \\
P_G(3,3) &= \{A_3\} \\
P_G(4,4) &= \{A_4\}
\end{aligned}$$

We now use 1 to find the parenthesizations for the other subproblems and then use those to find the parenthesizations for $S(1,4)$, the parenthesizations $P_G(1,4)$ can be found using:

$$P_G(1,4) = \{(P_G(1,1) \times P_G(2,4))\} \cup \{(P_G(1,2) \times P_G(3,4))\} \cup \{(P_G(1,3) \times P_G(4.4))\}$$

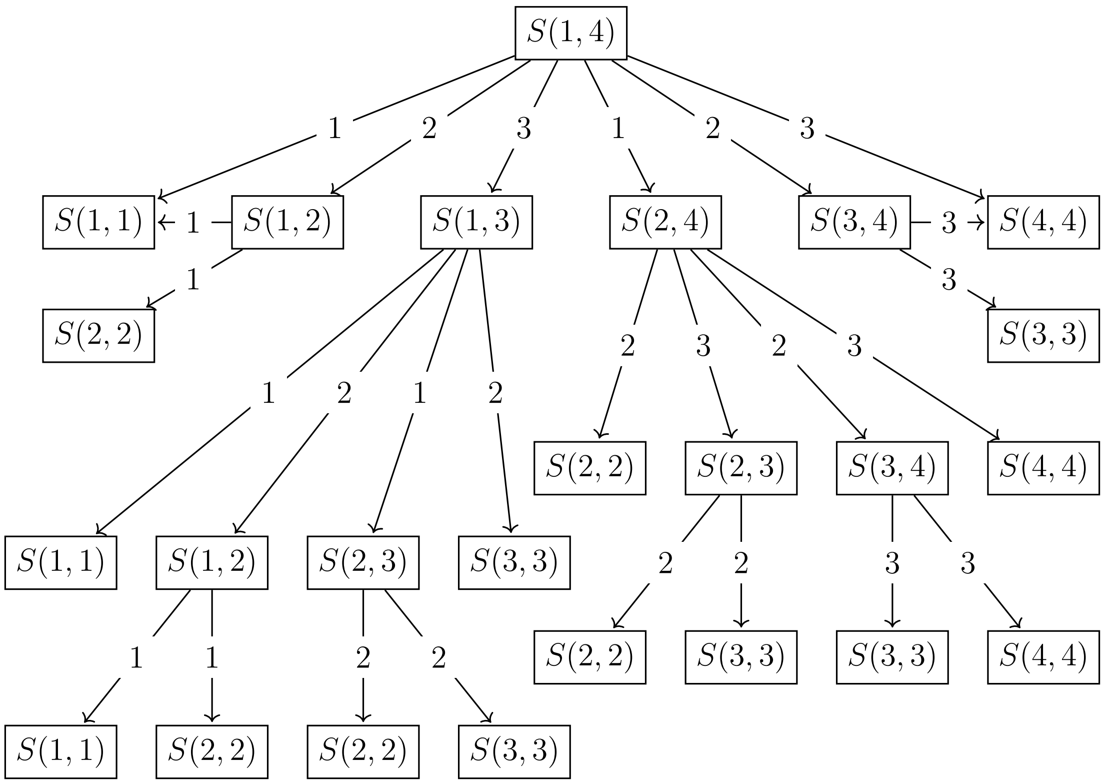

Figure 1: DAG for the matrix chain $A_1 \times \cdots \times A_4$.

Similarly, we can find parenthesizations for the subproblems. We first look at the simpler problems where $i = j - 1$, in this case $|K_G(i,j)| = 1$,

$$\begin{aligned}
P_G(1,2) &= \{P_G(1,1) \times P_G(2,2)\} = \{(A_1 \times A_2)\} \\
P_G(2,3) &= \{P_G(2,2) \times P_G(3,3)\} = \{(A_2 \times A_3)\} \\
P_G(3,4) &= \{P_G(3,3) \times P_G(4,4)\} = \{(A_3 \times A_4)\}
\end{aligned}$$

Next, we take a look at the more complicated cases where $|K_G(i,j)| = 2$,

$$\begin{aligned}
P_G(1,3) &= \{(P_G(1,1) \times P_G(2,3))\} \cup \{(P_G(1,2) \times P_G(3,3))\} \\
&= \{(A_1 \times (A_2 \times A_3)), ((A_1 \times A_2) \times A_3)\} \\
P_G(2,4) &= \{(P_G(2,2) \times P_G(3,4))\} \cup \{(P_G(2,3) \times P_G(4,4))\} \\
&= \{(A_2 \times (A_3 \times A_4)), ((A_2 \times A_3) \times A_4)\}
\end{aligned}$$

We can now use these results to find $P_G(1,4)$,

$$\begin{aligned}
&P_G(1,4) = \{(A_1 \times (A_2 \times (A_3 \times A_4))), (A_1 \times ((A_2 \times A_3) \times A_4)), \\
&\quad ((A_1 \times A_2) \times (A_3 \times A_4)), ((A_1 \times (A_2 \times A_3)) \times A_4), (((A_1 \times A_2) \times A_3) \times A_4)\}
\end{aligned}$$