

Representation of All Possible Parenthesizations of Matrix-Chain Multiplication in a Directed Acyclic Graphs

Firstly, let's revise some notation. We define G as a directed acyclic graph (DAG) that will represent all possible parenthesizations for $P(i, j)$ for each subproblem $S(i, j)$ where $1 \leq i \leq j \leq n$. The set of vertices for this graph is:

$$V(G) = \{S(i, j) : 1 \leq i \leq j \leq n\}.$$

For $i < j$, $S(i, j)$ has exactly $2(j - i)$ outgoing edges. For $k = i, \dots, j - 1$ exactly two edges start from $S(i, j)$ and end in $S(i, k)$ and $S(k + 1, j)$. This is a *rigid pair* and we label these with index k .

For each vertex $S(i, j)$, we define the set $P_G(i, j)$ of parenthesizations corresponding to $S(i, j)$ in G . For $i < j$, let $K_G(i, j)$ be the set of indexes of rigid pairs outgoing from $S(i, j)$ in G , then

$$P_G(i, j) = \bigcup_{k \in K_G(i, j)} \{(p_1 \times p_2) : p_1 \in P_G(i, k), P_G(k + 1, j)\}$$

