Representation of All Possible Parenthesizations of Matrix-Chain Multiplication in a Directed Acyclic Graphs

Firstly, let's revise some notation. We define G as a directed acyclic graph (DAG) that will represent all possible parenthesizations for $P(i,j)$ for each subproblem $S(i,j)$ where $1 \leq i \leq j \leq n$. The set of vertices for this graph is:

$$V(G) = \{S(i,j) \ : \ 1 \leq i \leq j \leq n\}.$$

For $i < j$, $S(i,j)$ has exactly $2(j-i)$ outgoing edges. For $k = i, \ldots, j-1$ exactly two edges start from $S(i.j)$ and end in $S(i,k)$ and $S(k+1,j)$. This is a *rigid pair* and we label these with index k.

For each vertex $S(i,j)$, we define the set $P_G(i,j)$ of parenthesizations corresponding to $S(i,j)$ in G. For $i < j$, let $K_G(i,j)$ be the set of indexes of rigid pairs outgoing from $S(i,j)$ in G, then

$$P_G(i,j) = \bigcup_{k \in K_G(i,j)} \{(p_1 \times p_2) \ : \ p_1 \in P_G(i,k), \ P_G(k+1,)\}$$