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ATMOS 6500 Numerical Weather Prediction

Lab 2

LEAPFROG TIME INTEGRATION

a.1) Noting $\mu = c \times (dt / dx)$, if we increase dt while keeping c and dx constant, we will eventually reach a time step that is proportionately too large for the grid spacing and the solution becomes unstable. In this case, for c = 20 ms-1 and dx = 200 km, we see the CFL is no longer met for dt > 10000 s. For dt near 10000 s, the model will 'blow up' after 2300 steps. Fig. 1 illustrates that as we increase dt (keeping all other values constant), this happens in as few as 500 steps for dt ≥ 22500 s.

a.2) Given c = 20 ms-1 and grid spacing dx = 200 km, a time step dt = 3600 s gives a CFL $\mu = 0.36$ and produces a stable computation of kinetic energy. The model does not 'blow up' at any point during our 5000 steps. The calculated total kinetic energy (ek1) at each time step is shown in the bottom right of Fig. 2.

b) The exact solution is represented using very small dt = 10 s. Fig. 3a-d summarized in the table below for dt = 3600 s, dx = 200 km.

	A = 10 ms-1 L = 10 * dx	A = 25 ms-1 L = 10 * dx	A = 10 ms-1 L = 4 * dx	A = 25 ms-1 L = 4 * dx
RMSE	462.27	1081.38	848.91	2860.12
RE	0.02	0.03	0.04	0.06
MAPE	2.47%	3.06%	3.56%	6.18%

UPSTREAM TIME INTEGRATION

a.1) As before, the CFL μ is in excess of 1 for dt ~ 10000 s. However, the upstream method is much more susceptible to numerical instability and appears to 'blow up' even with very low values for μ. For the same dt = 10000, we see exponential growth in kinetic energy after just 400 steps as compared to 2300 for the leapfrog case. Even for very small CFL values μ ~ 0.01 with dt = 100 s, we reach an unstable solution around 80,000 steps.

a.2) Given c = 20 ms-1 and grid spacing dx = 200 km, a time step dt = 10 s gives a CFL of μ = 0.001 and produces a likely stable (tested out to 500,000 steps) computation of kinetic energy. The calculated total kinetic energy (ek1) at each time step is shown in the bottom right of Fig. 4.

b) The exact solution is represented using very small dt = 1 s. Fig. 5a-d summarized in the table below for dt = 10 s, dx = 200 km.

	A = 10 ms-1 L = 10 * dx	A = 25 ms-1 L = 10 * dx	A = 10 ms-1 L = 4 * dx	A = 25 ms-1 L = 4 * dx
RMSE	192.96	1169.13	671.75	4179.94
RE	0.01	0.04	0.03	0.10
MAPE	1.19%	3.84%	3.44%	9.82%

DISCUSSION

It is simple to see the effects that the choice of grid spacing and time step have on computational stability through this exercise. For the examples given above, dx was kept constant but dt was varied, thus increasing/decreasing the CFL (μ) along with it. Note that varying dx while keeping dt constant would yield similar variations in CFL and computational stability, and while this hypothesis was tested for a few cases, results are redundant and not shown here. Following the established criteria of 0 ≤ μ ≤ 1, we see for the leapfrog time integration, computational stability is indeed achieved (with all other constants set to default values). A key difference that is observed in the upstream scheme is that computational instability *can and does* occur for CFL between 0 and 1. As noted in the Pu and Kalnay (2018) text, this can be due to the nonlinearity of the PDEs that are being estimated. In this case, as we decrease dt to very small time steps, the FDE solution becomes bounded once again with very small CFL. The example used above considers dt = 10 s to produce a sufficient CFL value for dx = 200 km. This suggests that other time integration schemes are better suited for large timesteps.

A benefit observed when comparing the error statistics is that the upstream scheme appears to produce a better estimation of the PDE than the leapfrog scheme, though this may be a product of the time steps chosen to approximate, as the leapfrog scheme is stable at larger dt = 3600 s. Conversely, the leapfrog scheme shows it is worthwhile as even for a large dt (as long as CFL is still satisfied), it produces a very good estimation. Even for A = 25 m s-1 and L = 4*dx, the RMSE is large but the estimate is still reasonable, with the MAPE < 10 %. The estimates in the default case A = 10 m s-1 and L = 10*dx are very good in both schemes.

Note that the CFL simply provides the 'bottom line' criteria to be met for computational stability. Even with CFL satisfied, the model can still become unstable as we have seen here. Additionally, just because CFL is satisfied does not necessarily mean that the FDE is a sufficient approximation of the PDE. As we decrease dt, the approximation improves as we approach a FDE solution that nearly mimics the PDE. Furthermore, the choice of scheme will impact both what value of CFL is sufficient ($0 \leq \mu \leq 1$ is not enough for upstream, for example) and the approximation error observed in the FDE solution.

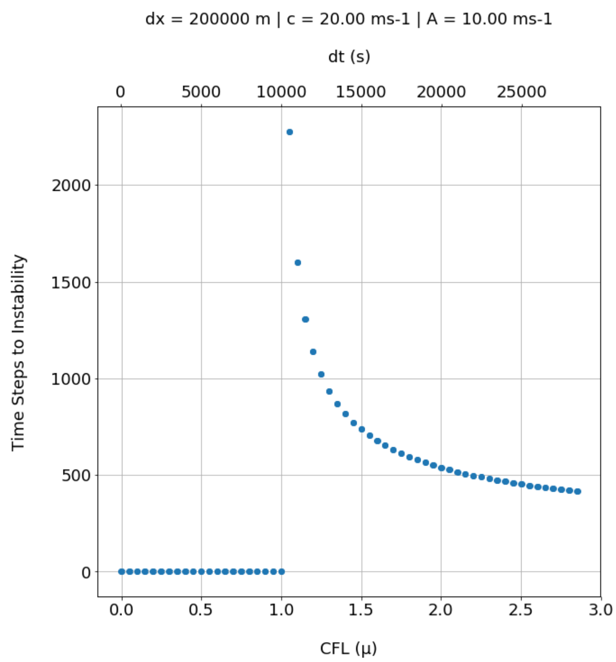

Figure 1 (Leapfrog Scheme)

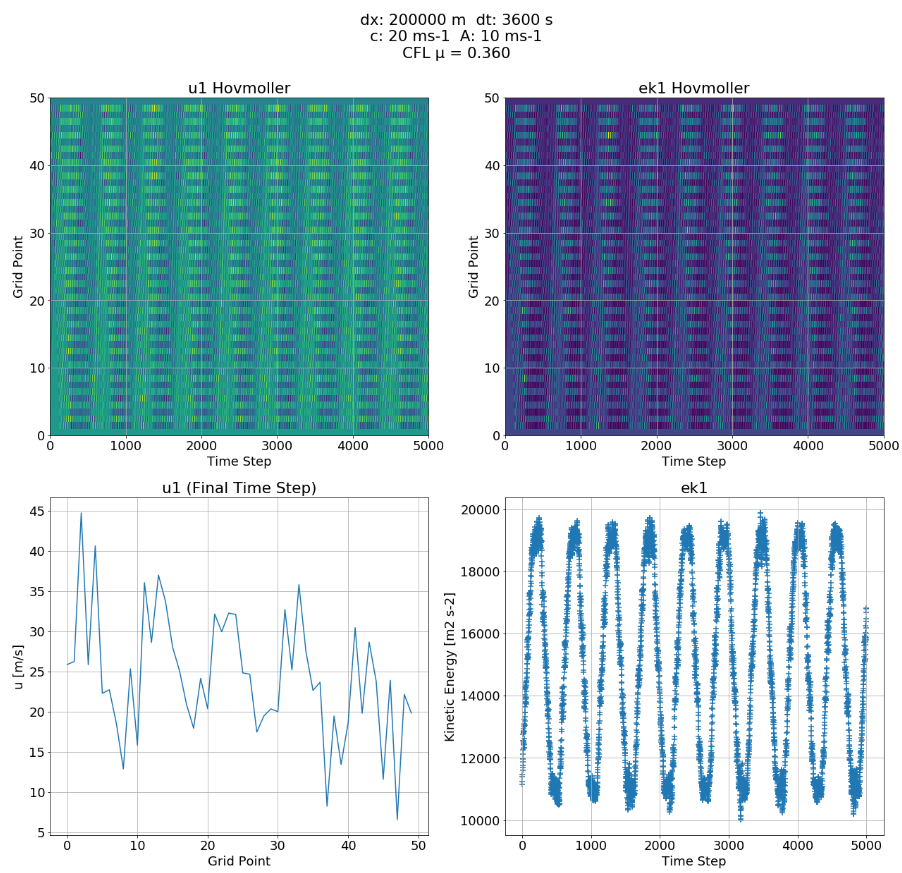

Figure 2 (Leapfrog Scheme)

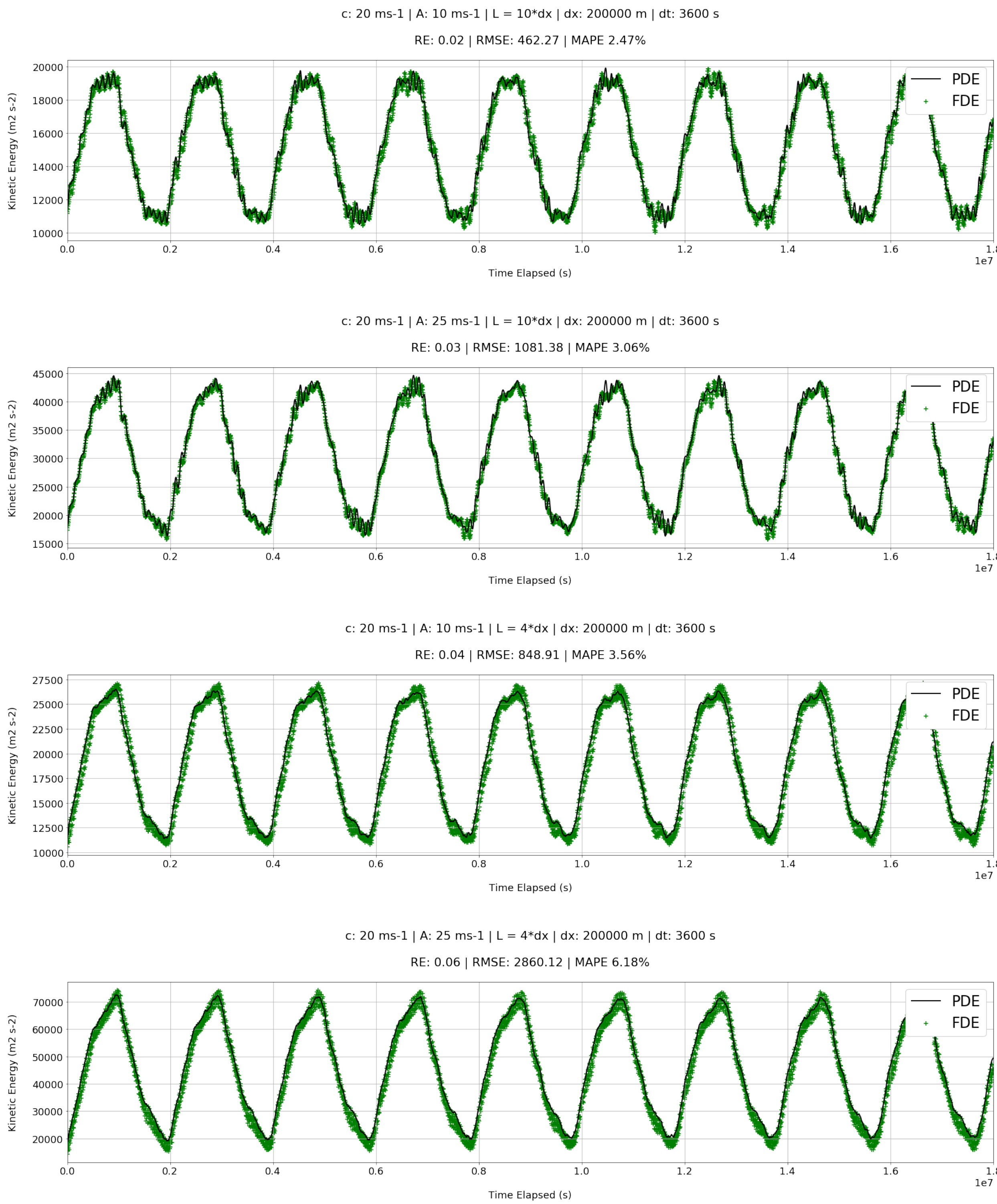

Figure 3 (a-d, top to bottom) (Leapfrog Scheme)

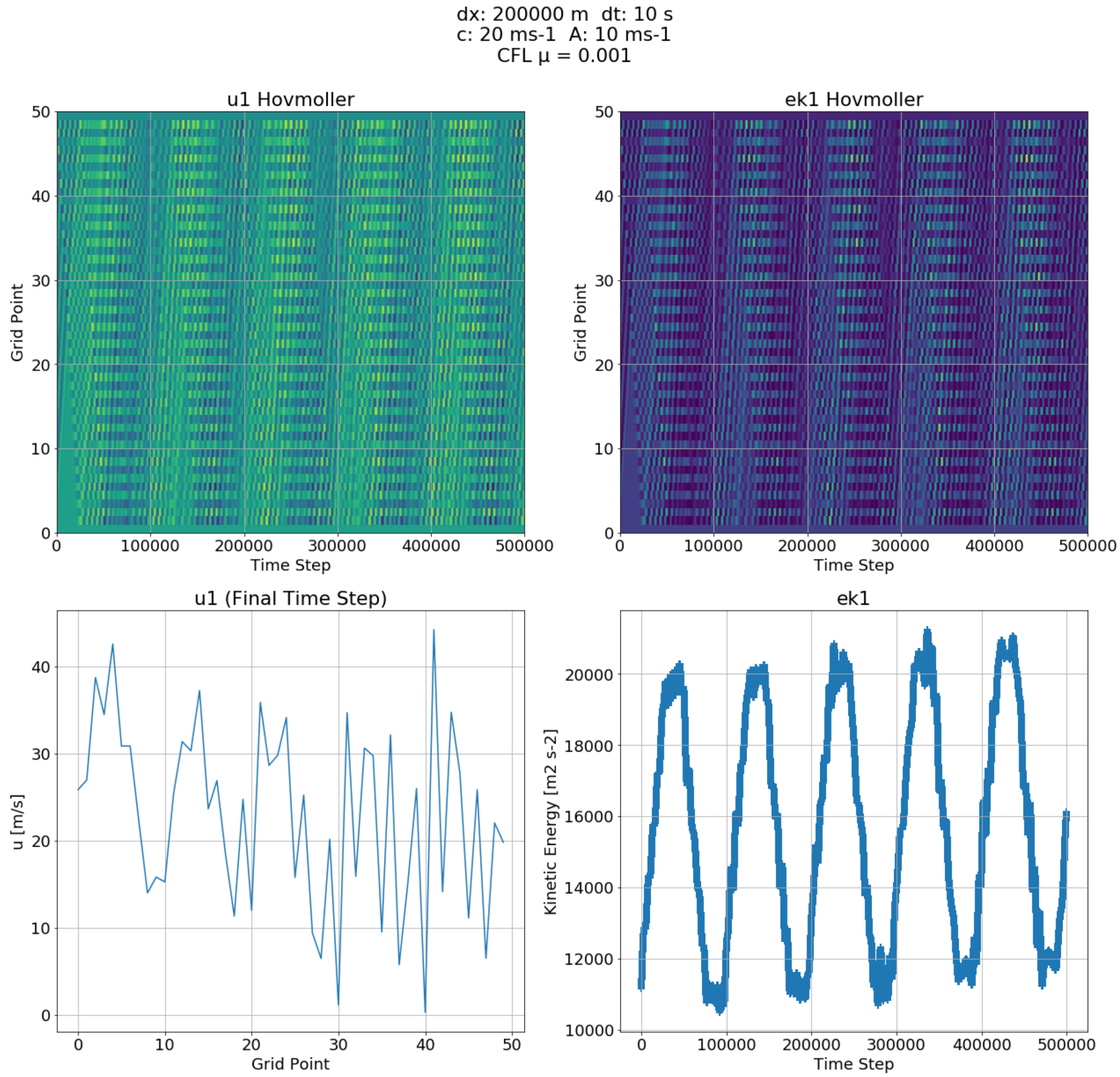

Figure 4 (Upstream Scheme)

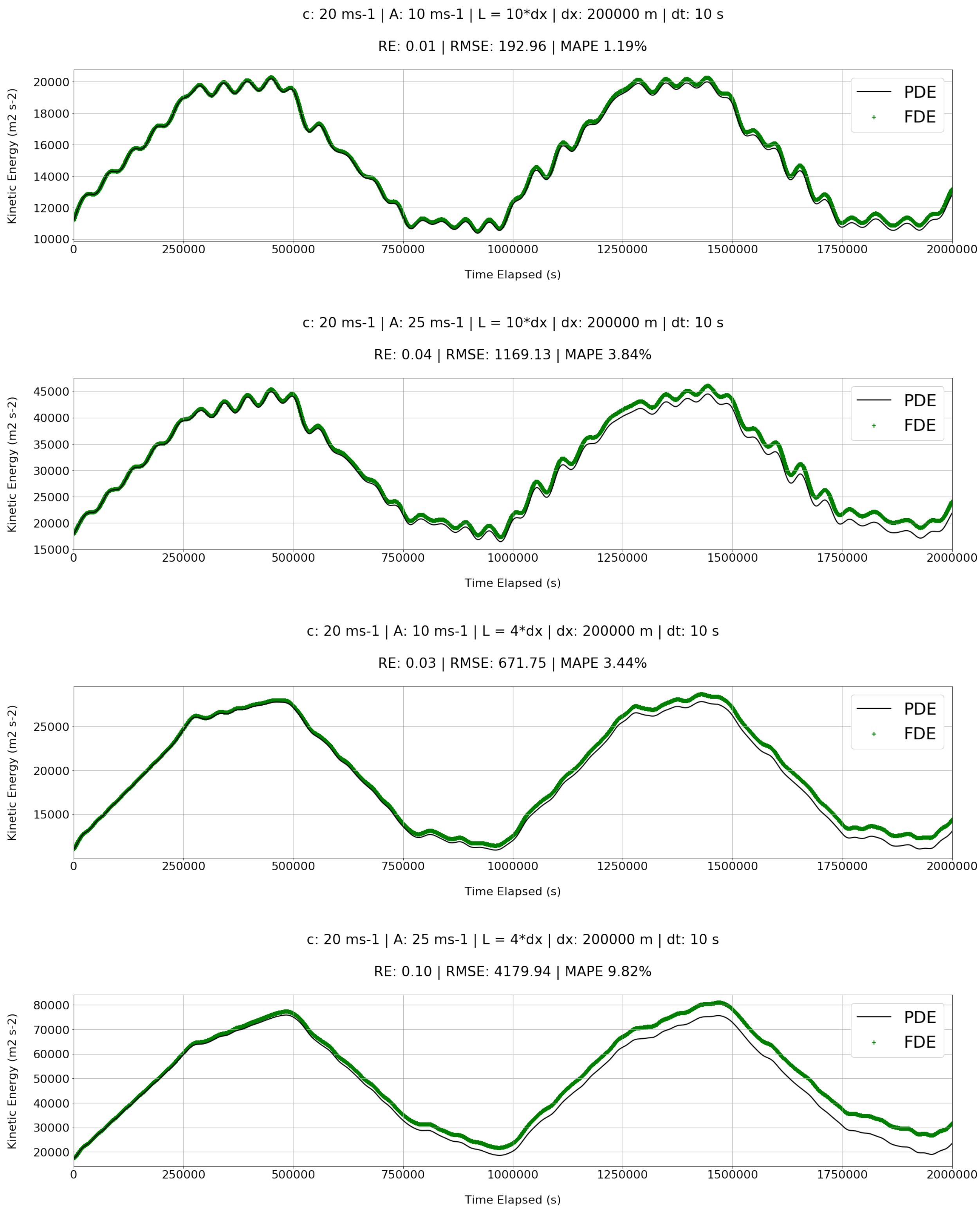

Figure 5 (a-d, top to bottom) (Upstream Scheme)