The Predictive Power of Zero Intelligence in Financial Markets

J. Doyne Farmer,[1,*] Paolo Patelli,[1,2] and Ilija I. Zovko[1,3]

[1]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA*
[2]*Sant'Anna School of Advanced Studies, Piazza Martiri della Libertà 33, Pisa, Italy*
[3]*CENDEF, University of Amsterdam, Roetersstraat 11, Amsterdam, The Netherlands*

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Standard models in economics stress the role of intelligent agents who maximize utility. However, there may be situations where, for some purposes, constraints imposed by market institutions dominate intelligent agent behavior. We use data from the London Stock Exchange to test a simple model in which zero intelligence agents place orders to trade at random. The model treats the statistical mechanics of order placement, price formation, and the accumulation of revealed supply and demand within the context of the continuous double auction, and yields simple laws relating order arrival rates to statistical properties of the market. We test the validity of these laws in explaining the cross-sectional variation for eleven stocks. The model explains 96% of the variance of the bid-ask spread, and 76% of the variance of the price diffusion rate, with only one free parameter. We also study the market impact function, describing the response of quoted prices to the arrival of new orders. The non-dimensional coordinates dictated by the model approximately collapse data from different stocks onto a single curve. This work is important from a practical point of view because it demonstrates the existence of simple laws relating prices to order flows, and in a broader context, because it suggests that there are circumstances where institutions are more important than strategic considerations.

Contents

I. INTRODUCTION

This work has goals at two levels. At the immediate level, its goal is to investigate the possibility of simple laws relating the flow of trading orders into a market to statistical properties of prices. The laws that we propose and investigate are not temporal predictions, but rather relations restricting the possible values that the underlying variables can take at any given point in time. The ideal gas law provides a simple physical analogy that illustrates both the limited scope and the potential utility of such laws. In our case, the goal is to relate properties of the order flow, such as market order placement rate, limit order placement rate, and cancellation rate, to properties of the market such as the gap between the best prices for buying and selling, or the variability of prices. In addition, we present some results that are related to the nature of supply and demand functions.

At a broader level, this work is interesting because of the nature of the model we test, which makes the simple assumption that agents place orders to buy or sell at random [4, 5]. This is in constrast to standard models in economics, which typically devote considerable effort to modeling the strategic behavior and expectations of agents. No one would dispute that this is important. However, there may be some circumstances where other factors may be more important. For example, Becker [2] showed that a budget constraint is sufficient to guarantee the proper slope of supply and demand curves, and Gode and Sunder [3] demonstrated that if one replaces the students in a standard classroom economics experiment by zero-intelligence agents, the zero-intelligence agents per-

*Electronic address: jdf@santafe.edu

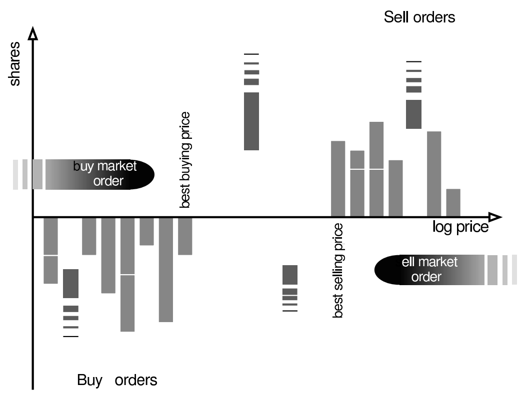

FIG. 1: A random process model of the continuous double auction. Stored limit orders are shown stacked along the price axis, with sell orders (supply) stacked above the axis at higher prices and buy orders (demand) stacked below the axis at lower prices. New sell limit orders are visualized as randomly falling down, and new buy orders as randomly "falling up". New sell orders can be placed anywhere above the best buying price, and new buy orders anywhere below the best selling price. Limit orders can be removed spontaneously (e.g. because the agent changes her mind or the order expires) or they can be removed by market orders of the opposite type. This can result in changes in the best prices, which in turn alters the boundaries of the order placement process. It is this feedback between order placement and price formation that makes this model interesting, and its predictions non-trivial.

form surprisingly well. The model we test here builds on earlier work in financial economics [6, 7, 8, 9] and physics [10, 11, 12, 13, 14]. (See also interesting subseqent work [15, 16]). We show here that in some circumstances the zero-intelligence approach can make surprisingly good quantitative predictions.

A. Continuous double auction

The model of Daniels et al. [4] assumes a continuous double auction, which is the most widely used method of price formation in modern financial markets [5]. There are two fundamental kinds of trading orders: Impatient traders submit *market orders*, which are requests to buy or sell a desired number of shares immediately at the best available price. More patient traders submit *limit orders*, which include the worst allowable price for the transaction. Limit orders may fail to result in an immediate transaction, in which case they are stored in a queue called the *limit order book*, illustrated in Fig. 1. As each buy order arrives it is transacted against accumulated sell limit orders that have a lower selling price, in priority of price and arrival time. Similarly for sell orders. The lowest selling price offered in the book at any point in time is called the *best ask*, $a(t)$, and the highest buying price the *best bid*, $b(t)$.

B. Review of model

The model that we test here [4, 5] assumes that two types of zero intelligence agents place and cancel orders randomly, as shown in Fig. 1. Impatient agents place market orders of size σ, which arrive at a rate μ *shares per time*. Patient agents place limit orders of the same size σ, which arrive with a constant rate density α *shares per price per time*. These agents may be thought of as liquidity demanders and suppliers. Queued limit orders are canceled at a constant rate δ, with dimensions of *1/time*. Prices change in discrete increments called *ticks*, of size *dp*. To keep the model as simple as possible, there are equal rates for buying and selling, and order placement and cancellation are Poisson processes. All of these processes are independent except for coupling through their boundary conditions: Buy limit orders arrive with a constant density α over the semi-infinite interval $-\infty < p < a(t)$, where p is the logarithm of the price, and sell limit orders arrive with constant density α on the semi-infinite interval $b(t) < p < \infty$. As a result of the random order arrival processes, $a(t)$ and $b(t)$ each make random walks, but because of coupling of the buying and selling processes the bid-ask *spread* $s(t) \equiv a(t) - b(t)$ is a stationary random variable.

As new orders arrive they may alter the best prices $a(t)$ and $b(t)$, which in turn changes the boundary conditions for subsequent limit order placement. For example, the arrival of a buy limit order inside the spread will alter the best bid $b(t)$, which immediately alters the boundary condition for sell limit order placement. It is this feedback between order placement and price diffusion that makes this model interesting, and despite its apparent simplicity, quite difficult to understand analytically. This model has been analyzed using both simulation and two different mean field theories [5].

One of the virtues of this model is that it gives simple scaling laws relating the parameters of the model to fundamental properties such as the average bid-ask spread, and the price diffusion rate. The mean value of the spread predicted based on a mean field theory analysis of the model in the limit $dp \to 0$ is

$$\hat{s} = (\mu/\alpha) f(\sigma\delta/\mu). \qquad (1)$$

The nondimensional ratio $\epsilon = \sigma\delta/\mu$ is the ratio of removal by cancellation to removal by market orders, and plays an important role in determining the properties of the model. $f(\epsilon)$ is a relatively slowly varying, monotonically increasing non-dimensional function that can be approximated as $f(\epsilon) = 0.28 + 1.86\epsilon^{3/4}$.

Another prediction of the model is of the price diffusion rate, which drives the volatility of prices and is the primary determinant of financial risk. If we assume that prices make a random walk, then the diffusion rate measures the size and frequency of its increments. The variance V of an uncorrellated normal random walk after time t grows as $V(t) = Dt$, where D is the diffusion rate. We choose to measure the price diffusion rate rather than the volatility because it is a stationary quantity that provides a more fundamental description of the volatility process. This is the main free parameter in the Bachelier model [1], and while its value is essential for risk estimation and derivative pricing there is very little understanding of what determines it. Numerical experiments indicate that the short term price diffusion rate predicted by the model is

$$\hat{D} = k\mu^{5/2}\delta^{1/2}\sigma^{-1/2}\alpha^{-2}, \tag{2}$$

where k is a constant.

The model was constructed to be simple enough to be analytically tractable, and so makes many strong assumptions. For example, it assumes that the rates for buying and selling are equal, the sizes of limit orders and market orders are the same, that limit order deposition is uniform on semi-infinite intervals, and that rates of order submission are unaffected by changes in price. Many of these assumptions are economically unreasonable in the presence of intelligent agents, but the reader should bear in mind that the only market participants in the model are zero-intelligence "noise" traders, who can be thought of as random liquidity suppliers and demanders[1]. While intelligent agents are clearly essential for many purposes, such as determining the levels of prices, what we suggest here is that for other purposes their presence is not essential. We would like to emphasize that the construction of the model and all the predictions derived from it were made prior to looking at the data.

II. TESTING THE SCALING LAWS

A. Data

We test this model with data from the electronic open limit order book of the London Stock Exchange (SETS), which includes about half of the total volume on the exchange. We used data from eleven stocks for August 1st 1998 - April 30th 2000, which includes 434 trading days and a total of roughly six million events. For all these stocks the number of total events exceeds 300,000 and was never less than 80 on any given day (where an event corresponds to an order placement or cancellation). Orders placed during the opening auction are removed to accomodate the fact that the model only applies for the continuous auction. See the Supplementary Material Section A 2 for more details.

[1] A "liquidity demander" is someone who needs to make a transaction quickly. In the sense used here, a noise trader is someone who wants to make transactions for reasons unrelated to this particular market, and so is insensitive to price.

B. Testing procedure

¿From the point of view of the model, the order flow rates μ, α, and δ, and the mean order size σ are all free parameters. In analyzing the model we find scaling relations connecting these parameters to the average spread and the price diffusion rate, as given in Equations 1 and 2. We test the model by testing the validity of these relations, taking advantage of the fact that different stocks have different average values of these parameters. For each stock we measure the average market order arrival rate μ, limit order rate density α, cancellation rate δ, and order size σ, where the averages are taken across the full time period. We then measure the average spread and volatility and compare them to the predictions of the model.

A problem occurs in measuring α and δ due to the simplifying assumption of a uniform distribution of prices for order flow and cancellation. In the real data order placement and cancellation are concentrated near the best prices [15, 20]. We cope with this by making the assumption that order placement is uniform inside a price window around the best prices, and zero outside this window. We choose the price window to correspond to roughly 60% of limit orders at the best prices, and compute α by dividing the number of shares of limit orders placed inside the price window by the size of the price window. We do this for each day and compute the average value of α for each stock. We compute δ as the inverse of the average cancellation time for orders cancelled inside the same price window. See the Supplementary Material Section A 3 for details.

The scaling laws that we describe here do not make temporal predictions, but rather are restrictions of state variables. The ideal gas law, $PV = RT$, provides a good analogy. It predicts that pressure P, volume V, and temperature T are constrained – any two of them determines the third. Similarly, here we are testing two relations relating properties of orders to properties of prices. We are not attempting to predict the temporal behavior of the order flows, only trying to see whether the restrictions between order flows and prices are valid.

We would like to emphasize that in testing the model we are not treating the order flow rates and order size as free parameters in the regressions. Instead, we are testing the predictions of the model based on order flow rates against the measured values in the same period. The only free parameters are in the specification of the price interval as described above (which was done more or less arbitrarily).

C. Spread

To test Equation 1, we measure the average spread $\bar{s}$ across the full time period for each stock, and compare to the predicted average spread $\hat{s}$ based on order flows. Spread is measured as the daily average of

$\log b(t) - \log a(t)$. The spread is measured after each event, with each event given equal weight. The opening auction is excluded.

To test our hypothesis that the predicted and actual values coincide, we perform a regression of the form $\log \bar{s} = A \log \hat{s} + B$. We used logarithms because the spread is positive and the log of the spread is approximately normally distributed. We use the free parameters A and B for hypothesis testing. Based on the model we predict that the comparison should yield a straight line with $A = 1$ and $B = 0$, but because of the degree of freedom in choosing the price interval as described above, the value of B is somewhat arbitrary.

The least squares regression, shown together with the data comparing the predictions to the actual values in Fig. 2, gives $A = 0.99 \pm 0.10$ and $B = 0.06 \pm 0.29$. We thus strongly reject the null hypothesis that $A = 0$, indicating that the predictions are far better than random. More importantly, we are unable to reject the null hypothesis that $A = 1$. In fact, we are also unable to reject $B = 0$, but this is probably largely a matter of luck in our choice of the price interval. The regression has $R^2 = 0.96$, so the model explains most of the variance. Note that because of long-memory effects and cross-correlations between stocks the errors in the regression are larger than they would be for IID data (see the discussion in the Supplemenary Material Section A 5).

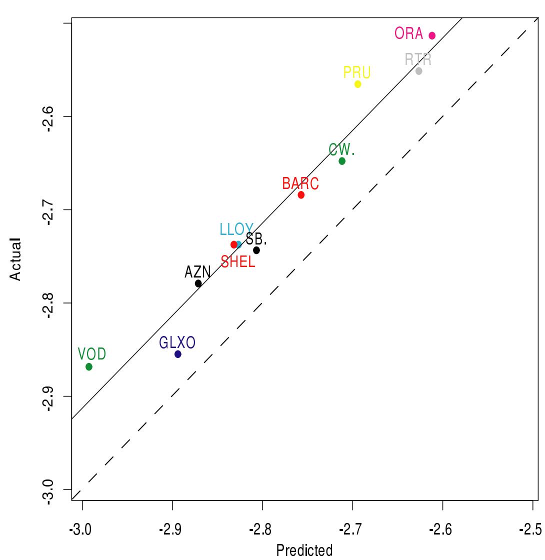

FIG. 2: Regressions of predicted values based on order flow parameters vs. actual values for the log spread. The dots show the average predicted and actual value for each stock averaged over the full 21 month time period. The solid line is a regression; the dashed line is the diagonal, representing the model's prediction without any adjustment

D. Price diffusion rate

As for the spread, we compare the predicted price diffusion rate based on order flows to the actual price diffusion rate $\bar{D}_i$ for each stock averaged over the 21 month period, and regress the logarithm of the predicted vs. actual values, as shown in Fig. 3.

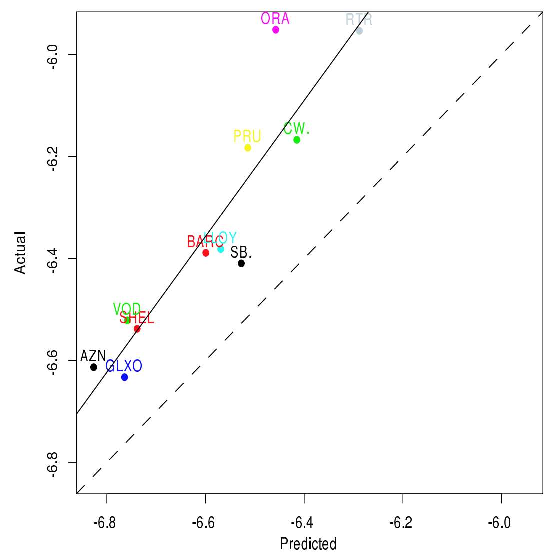

FIG. 3: Regressions of predicted values based on order flow parameters vs. actual values for the logarithm of the price diffusion rate. The dots show the average predicted and actual value for each stock averaged over the full 21 month time period. The solid line is a regression; the dashed line is the diagonal, representing the model's prediction without any adjustment of slope or intercept.

The regression gives $A = 1.33 \pm 0.25$ and $B = 2.43 \pm 1.75$. Thus, we again strongly reject the null hypothesis that $A = 0$. We are still unable to reject the null hypothesis that $A = 1$ with 95% confidence, though there is some suggestion that the scaling of the model and the actual values are not quite the same. (This could happen if, for example, the scaling exponent predicted by the model of one or more of the order flow rates is wrong; however this suggests that it is at least quite close). Although the results are not as good as for the spread, $R^2 = 0.76$, so the model still explains most of the variance.

III. AVERAGE MARKET IMPACT

Market impact is practically important because it is the dominant source of transaction costs for large traders, and conceptually important because it provides a convenient probe of the revealed supply and demand functions in the limit order book. When a market order of size ω arrives, if sufficiently large, it will remove all the stored limit orders at the best bid or ask, causing a change in the midpoint price $m(t) \equiv (a(t) + b(t))/2$. The average market impact function ϕ is the average logarithmic midpoint price shift Δp conditioned on order size, $\phi(\omega) = E[\Delta p|\omega]$.

A long-standing mystery about market impact is that it is highly concave [15, 17, 18, 24, 25, 26, 27, 28]. This is unexpected since simple arguments would suggest that because of the multiplicative nature of returns, market impact should grow at least linearly [5]. The model we are testing predicts a concave average market impact function, with the concavity becoming more pronounced for small values of $\epsilon = \sigma\delta/\mu$. However, these predictions are not in good detailed agreement with the data, in that the model predicts a larger variation with ϵ than what is actually observed. However, the model is still useful for understanding market impact, as described below.

A. Collapse in non-dimensional coordinates

A surprising regularity of the average market impact function is uncovered by simply plotting the data in non-dimensional coordinates, as shown in Fig. 4. See the Supplementary Material Section A 1 for a discussion of how the nondimensional coordinates are derived from the model. Each market order ω_i causes a possible change Δp_i in the midquote price. If we bin together events with similar ω and plot the mean order size as a function of the mean price impact Δp, we typically see highly variable behavior for different stocks, as shown in Fig. 4(b). We have also explored other ways of renormalizing the order size, such as taking the ratio of each order's size to the daily or full-sample mean, but they give similar behavior, as shown in the Supplementary Material Section A 7.

Plotting the data in non-dimensional units tells a simpler story. This involves normalizing the price shift and order size by appropriate dimensional scale factors based on the daily order flow rates. In particular, $\Delta p \to \Delta p \alpha_t/\mu_t$ and $\omega \to \omega\delta_t/\mu_t$, where α_t, μ_t, and δ_t are the average order flow rates for day t. The data collapses onto roughly a single curve, as shown in Fig. 4(a). The variations from stock to stock are quite small; on average the corresponding bins for each stock deviate from each other by about 8%, roughly the size of the statistical sampling error. We have made an extensive analysis, but due to problems caused by the long-memory property of these time series and cross correlations between stocks, it remains unclear whether these differences are statistically significant. In contrast, using standard coordinates the differences are highly statistically significant. This collapse illustrates that the non-dimensional coordinates dictated by the model provide substantial explanatory power: We can understand how the average market impact varies from stock to stock by a simple transformation of coordinates. Plotting in double logarithmic scale shows that the curve of the collapse is roughly a power law of the form $\omega^{0.25}$ (see Supplementary Material, Section A 7). This provides a more fundamental explanation for the empirically constructed collapse of average market impact for the New York Stock Exchange found earlier [17].

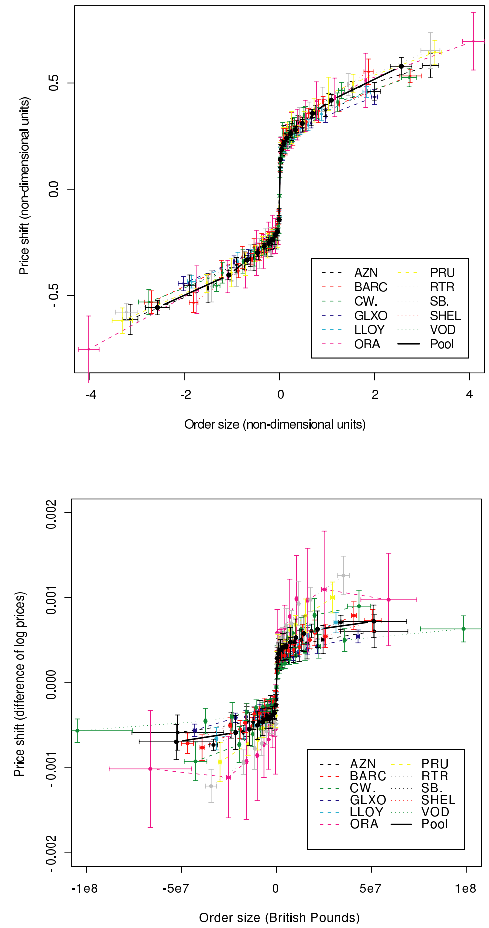

FIG. 4: The average market impact as a function of the mean order size. In (a) the price differences and order sizes for each transaction are normalized by the non-dimensional coordinates dictated by the model, computed on a daily basis. Most of the stocks collapse extremely well onto a single curve; there are a few that deviate, but the deviations are sufficiently small that given the long-memory nature of the data and the cross-correlations between stocks, it is difficult to determine whether these deviations are statistically significant. This means that we understand the behavior of the market impact as it varies from stock to stock by a simple transformation of coordinates. In (b), for comparison we plot the order size in units of British pounds against the average logarithmic price shift.

IV. CONCLUSIONS

The model we have presented here does a good job of predicting the average spread, and a decent job of predicting the price diffusion rate. Also, by simply plotting the data in non-dimensional coordinates we get a better understanding of the regularities of market impact. These results are remarkable because the underlying model completely drops agent rationality, instead focusing all its attention on the problem of understanding the constraints imposed by the continuous double auction.

The approach taken here can be viewed as a divide and conquer strategy. Rather than attempting to explain the properties of the market from fundamental assumptions about utility maximization by individual agents, we divide the problem into two parts. The first and much easier problem, addressed here, is that of understanding the characteristics of the market given the order flows. The second (and harder) problem, which remains to be investigated, is that of explaining why order flow varies as it does. Explaining order flow involves behavioral and/or strategic issues that are likely to be much more difficult to understand.

The model that we test succeeds in part because it takes explicit advantage of information that is available in a continuous double auction, that is not available in a standard Walrasian auction. By measuring the rate of market order placement vs. limit order placement, and the rate of order cancellation, we are able to measure how patient or impatient traders are. A higher ratio of market orders to limit orders, or a higher rate of cancellation implies a less patient, and therefore more volatile market, with larger spreads. The model makes this quantitative. The agreement with the model indicates that the degree of patience is an important determinant of market behavior. This is potentially compatible with either a rationality-based explanation in terms of information arrival, or a behavioral-based explanation driven by emotional response, but in either case it suggests that patience is a key factor.

This is part of a broader research program that might be characterized as the "low-intelligence" approach to economics: We begin with zero-intelligence agents to get a good benchmark of the effect of market institutions, and once this benchmark is well-understood, add a little intelligence, moving toward market efficiency. We thus start from zero rationality and work our way up, in contrast to the canonical approach of starting from perfect rationality and working down. Follow-up research will examine the effects of adding bounded rationality. See Ref. [30].

These results have several practical implications. For market practitioners, understanding the spread and the market impact function is very useful for estimating transaction costs and for developing algorithms that minimize their effect. For regulators they suggest that it may be possible to make prices less volatile and lower transaction costs by creating incentives for limit orders and disincentives for market orders. These scaling laws might be used to detect anomalies, e.g. a higher than expected spread might be due to improper market maker behavior.

The model we test here was constructed before looking at the data [4, 5], and was designed to be as simple as possible for analytic analysis. A more realistic (but necessarily more complicated) model would more closely mimic the properties of real order flows, which are price dependent and strongly correlated both in time and across price levels, or might incorporate elements of the strategic interactions of agents. An improved model would hopefully be able to capture more features of the data than those we have studied here. We know there are ways in which the current model is inappropriate, e.g., predicts unrealistically strong negative autocorrelations in prices, allowing arbitrage opportunities that do not exist in the real market. Nonetheless, as we have shown above, this extremely simple model does a good job of explaining some important properties of markets, such as transaction costs, price diffusion and market impact. It does this by focusing on the way order placement and price formation interact to alter the accumulation of stored supply and demand. For the phenomena studied here this appears to be the dominant effect. We do not mean to claim that market participants are unintelligent: Indeed, one of the virtues of this model is that it provides a benchmark to separate properties that are driven by the statistical mechanics of the market institution from those that are driven by conditional strategic behavior. It is surprising that such a simple model can explain anything at all about a system as complex as a market.

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APPENDIX A: SUPPLEMENTARY MATERIAL

1. Additional background information on the model

One of the virtues of this model is that we can make approximate predictions of several of its properties with almost no work using dimensional analysis. This also greatly simplifies the analysis and understanding of the model, and is particularly useful for understanding market impact.

There are three fundamental dimensional quantities describing everything in this model: *shares*, *price*, and *time*. There are five parameters defined in the model. When the dimensional constraints between the parameters are taken into account, this leaves only two independent degrees of freedom. It turns out that the order flow rates μ, α, and δ are more important than the discreteness parameters σ and dp, in the sense that the properties of the model are much more sensitive to variations in the order flow rates than they are to variations in σ or dp. It therefore natural to construct non-dimensional units based on the order flow parameters alone. There are unique combinations of the three order flow rates with units of *shares*, *price*, and *time*. This gives characteristic scales for price, shares, and time, that are unique up to a constant. In particular, the characteristic number of shares $N_c = \mu/\delta$, the characteristic price interval $p_c = \mu/\alpha$, and the characteristic timescale $t_c = 1/\delta$.

These characteristic scales can be used to define non-dimensional coordinates based on the order flow rates. These are $\hat{p} = p/p_c$ for price, $\hat{N} = N/N_c$ for shares, and $\hat{t} = t/t_c$ for time. The use of non-dimensional coordinates has the great advantage that it reduces the number of degrees of freedom from five to two, and many quantities are much more well-behaved and easily understood when plotted in non-dimensional coordinates than they are otherwise.

The remaining two degrees of freedom are naturally discussed in terms of non-dimensional versions of the discreteness parameters. A non-dimensional scale parameter based on order size is constructed by dividing the typical order size σ (with dimensions of *shares*) by the characteristic number of shares N_c. This gives the non-dimensional parameter $\epsilon \equiv \sigma/N_c = \delta\sigma/\mu$, which characterizes the granularity of the order flow. A non-dimensional scale parameter based on tick size is constructed by dividing the tick size dp by the characteristic price, i.e. $dp/p_c = \alpha dp/\mu$. The usefulness of this is that the properties of the model only depend on the two non-dimensional parameters, ϵ and dp/p_c: Any variations of the parameters μ, α, and δ that keep these two non-dimensional parameters constant gives exactly the same market properties. One of the interesting results that emerges from analysis of the model is that the effect of the granularity parameter ϵ is generally much more important than the tick size dp/p_c. For a more detailed discussion, see reference [5].

While $a(t)$ and $b(t)$ make random walks, the increments of their random walks are strongly anti-correlated. This is a good example of how the properties of this model are not simple to understand. One might naively think that under IID Poisson order flow, price increments should also be IID. However, due to the coupling of boundary conditions for the buy market order/sell limit order process to those of the sell market order/buy limit order process, this is not the case. Because of the fact that supply and demand tend to build as one moves away from the center of the book, price reversals are more common than price changes in the same direction. As a result, the price increments generated by this model are more anti-correlated than those of real price series. This has an interesting consequence: If we add the assumption of market efficiency, and assume that real price increments must be white, it implies that real order flow should be positively autocorrelated in order to compensate for the anticorrelations induced by the continuous double auction. This has indeed been observed to be the case [21, 22].

This is of course also a criticism of the model, since it implies a lack of arbitrage efficiency. However, we wish to stress that we make no claims that this model explains everything about the market; just that it explains a few things fairly well.

2. The London Stock Exchange (LSE) data set

The London Stock Exchange is composed of two parts, the electronic open limit order book, and the non-electronic upstairs market, which is used to facilitate large block trades. During the time period of our dataset 40% to 50% of total volume was routed through the electronic order book and the rest through the upstairs market. It is believed that the limit order book is the dominant price formation mechanism of the London Stock Exchange: about 75% of upstairs trades happen between the current best prices in the order book [19]. Our analysis involves only the data from the electronic order book. We chose this data set to study because we have a complete record of every action taken by every participating institution, allowing us to measure the order flows and cancellations and estimate all of the necessary parameters of our model.

We used data from the time period August 1st 1998 - April 30th 2000, which includes a total of 434 trading days and roughly six million events. We chose 11 stocks each having the property that the number of total number of events exceeds 300,000 and was never less than 80 on any given day. Some statistics about the order flow for each stock are given in table I.

The trading day of the LSE starts at 7:50 with a

stock ticker	num. events (1000s)	average (per day)	limit (1000s)	market (1000s)	deletions (1000s)	eff. limit (shares)	eff. market (shares)	# days
AZN	608	1405	292	128	188	4,967	4,921	429
BARC	571	1318	271	128	172	7,370	6,406	433
CW.	511	1184	244	134	134	12,671	11,151	432
GLXO	814	1885	390	200	225	8,927	6,573	434
LLOY	644	1485	302	184	159	13,846	11,376	434
ORA	314	884	153	57	104	12,097	11,690	432
PRU	422	978	201	94	127	9,502	8,597	354
RTR	408	951	195	100	112	16,433	9,965	431
SB.	665	1526	319	176	170	13,589	12,157	426
SHEL	592	1367	277	159	156	44,165	30,133	429
VOD	940	2161	437	296	207	89,550	71,121	434

TABLE I: Summary statistics for stocks in the dataset. Fields from left to right: stock ticker symbol, total number of events (effective market orders + effective limit orders + order cancellations) in thousands, average number of events in a trading day, number of effective limit orders in thousands, number of effective market orders in thousands, number of order deletions in thousands, average limit order size in shares, average market order size in shares, number of trading days in the sample.

roughly 10 minute long opening auction period (during the later part of the dataset the auction end time varies randomly by 30 seconds). During this time orders accumulate without transactions; then a clearing price for the opening auction is calculated, and all opening transactions take place at this price. Following the opening at 8:00 the market runs continuously, with orders matched according to price and time priority, until the market closes at 16:30. In the earlier part of the dataset, until September 22nd 1999, the market opening hour was 9:00. During the period we study there have been some minor modifications of the opening auction mechanism, but since we discard the opening auction data anyway this is not relevant.

Some stocks in our sample (VOD for example) have stock price splits and tick price changes during the period of our sample. We take splits into account by transforming stock sizes and prices to pre-split values. In any case, since all measured quantities are in logarithmic units, of the form $\log(p_1) - \log(p_2)$, the absolute price scale drops out. Our theory predicts that the tick size should change some of the quantities of interest, such as the bid-ask spread, but the predicted changes are small enough in comparison with the effect of other parameters that we simply ignore them (and base our predictions on the limit where the tick size is zero). Since granularity is much more important than tick size, this seems to be a good approximation.

3. Measurement of model parameters

Our goal is to compare the predictions of the model with real data. The parameters of the model are stated in terms of order arrival rates, cancellation rate, order size, and tick size. We choose an appropriate time interval and measure the parameters over that interval, and then compare to the properties of the market over that same interval.

Reconstructing the limit order book on a moment-by-moment basis makes it clear that the properties of the market tend to be relatively stationary during each day, changing more dramatically at the beginning and at the end of day. It is therefore natural to measure each parameter for each stock on each day. Since the model does not take the opening auction into account, we simply neglect orders leading up to the opening auction, and base all our measurements on the remaining part of the trading day, when the auction is continuous. Averaging daily parameters, rather than computing the parameters directly across the whole period, has the important advantage in computing volatility, of neglecting the effect of overnight price movements, which our model does not attempt to explain.

In order to treat simply and in a unified manner the diverse types of orders traders can submit in a real market (for example, crossing limit orders, market orders with limiting price, 'fill-or-kill, execute & eliminate) we use redefinitions based on whether an order results in an immediate transaction, in which case we call it an *effective market order*, or whether it leaves a limit order sitting in the book, in which case we call it an *effective limit order*. Marketable limit orders (also called crossing limit orders) are limit orders that cross the opposing best price, and so result in at least a partial transaction. The portion of the order that results in an immediate transaction is counted as a effective market order, while the non-transacted part (if any) is counted as a effective limit order. Orders that do not result in a transaction and do not leave a limit order in the book, such as for example, failed fill-or-kill orders, are ignored altogether. These have no affect on prices, and in any case, make up only a very small fraction of the order flow, typically less than 1%. Note that we drop the term "effective", so that e.g. "market order" means "effective market order".

A limit order can be removed from the book for many reasons, e.g. because the agent changes her mind, because a time specified when the order was placed has been reached, or because of the institutionally-mandated 30 day limit on order duration. We will lump all of these together, and simply refer to them as "cancellations".

Our measure of time is based on the number of events, i.e., the time elapsed during a given period is just the total number of events, including effective market order placements, effective limit order placements, and cancellations. We call this *event time*. Price intervals are computed as the difference in the logarithm of prices, which is consistent with the model, in which all price intervals are assumed to be logarithmic in order to assure prices are always positive.

We measure the average value of the five parameters of the model, μ, α, δ, σ, and dp for each day. This has the advantage that it allows us to skip over the opening auction, but is not essential for this analysis. μ, σ, and dp are straightforward to measure, but there are problems in measuring α and δ that must be understood in order to properly interpret our results.

The parameter μ_t, which characterizes the average market order arrival rate on day t, is straightforward to measure. It is just the ratio of the number of shares of effective market orders (for both buy and sell orders) to the number of events during the trading day. Similarly, σ_t is the average limit order size[2] in shares for that day.

Measuring the cancellation rate δ_t and the limit order rate density α_t is more complicated, due to the highly simplified assumptions we have made for the model. In contrast to our assumption of a constant density for placement of limit orders across the entire logarithmic price axis, real limit order placement is highly concentrated near the best prices (roughly 2/3 of all orders are placed at inside of the best prices), with a density that falls off as a power law as a function of the distance Δ from the best prices [15, 20]. In addition, we have assumed a constant cancellation rate, whereas in reality orders placed near the best prices tend to be cancelled much faster than orders placed far from the best prices. We cope with these problems as described below.

In order to estimate the limit order rate density for day t, α_t, we make an empirical estimate of the distribution of the relative price for effective limit order placement on each day. For buy orders we define the relative price as $\Delta = m - p$, where p is the logarithm of the limit price and m is the logarithm of the midquote price. Similarly for sell orders, $\Delta = p - m$. We then somewhat arbitrarily choose Q_t^{lower} as the 2 percentile of the density of Δ corresponding to the limit orders arriving on day t, and Q_t^{upper} as the 60 percentile of Δ. Assuming constant density within this range, we calculate α_t as $\alpha_t = L/(Q_t^{\text{upper}} - Q_t^{\text{lower}})$ where L is the total number of shares of effective limit orders within the price interval $(Q_t^{\text{lower}}, Q_t^{\text{upper}})$ on day t. These choices are made in a compromise to include as much data as possible for statistical stability, but not so much as to include orders that are unlikely to ever be executed, and therefore unlikely to have any effect on prices.

Similarly, to cope with the fact that in reality the average cancellation rate δ decreases [15] with the relative price Δ, whereas in the model δ is assumed to be constant, we base our estimate for δ only on canceled limit orders within the range of the same relative price boundaries $(Q_t^{\text{lower}}, Q_t^{\text{upper}})$ defined above. We do this to be consistent in our choice of which orders are assumed to contribute significantly to price formation (orders closer to the best prices contribute more than orders that are further away). We then measure δ_t, the cancellation rate on day t, as the inverse of the average lifetime of a canceled limit order in the above price range. Lifetime is measured in terms of number of events happening between the introduction of the order and its subsequent cancellation. Some simple diagnostics of the parameter estimates are presented in Fig. 5.

4. Measuring the price diffusion rate

The measurement of the price diffusion rate requires some discussion. We measure the intraday price diffusion by computing the variance $V(\tau)$ of $m(i-\tau) - m(i)$, averaged over different intraday events i. Here an event is anything that changes the midpoint price m. If we assume that the events are asymptotically IID, then the estimated slope of the variance plot is the diffusion rate D_t for day t. To compute this we regress $V(\tau)$ against τ, using the assumption $V(\tau) = D_t\tau$. We use an ordinary least squares regression to estimate D_t, weighting each value of τ by the square root of the number of independent observations. An example of this procedure is given in Fig. 6.

One must bear in mind that the price diffusion rate from day to day has substantial correlations, as illustrated in Fig. 7.

5. Estimating the errors for the regressions

The error bars presented in the text are based on a bootstrapping method. We are driven to use this method for two reasons: First, the spread, price diffusion rates, and parameters are highly cross-correlated between stocks, and second, because order flow variables, spread, and price diffusion rates all have slowly decaying positive autocorrelation functions. Indeed, it has recently been shown that order sign, order volume and liquidity as reflected by volume at the best price, are long-memory

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[2] The model assumes that the average size of limit orders and market orders is the same. For the real data this is not strictly true, though as seen in Table I, it is a good approximation to within about 20%. For the purposes of the analysis we use the limit order size as the measure because for theoretical reasons we think this is more important than the market order size, but because the two are approximately the same, this will not make a significant difference in the results.

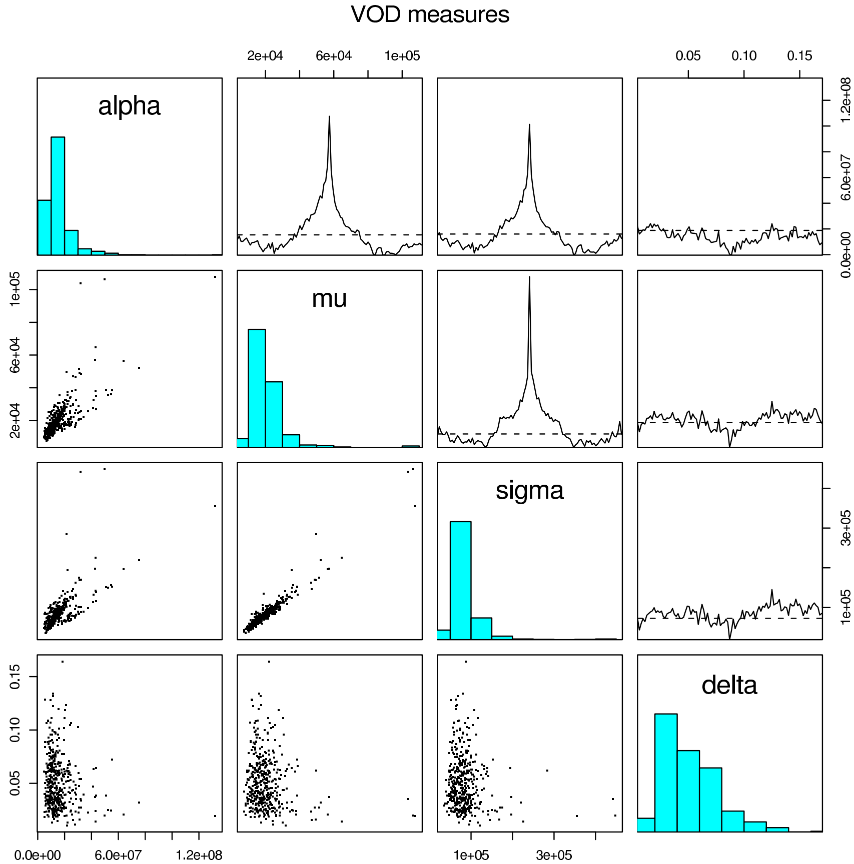

FIG. 5: Density estimations and cross correlations for Vodafone between the four model parameter measures. On the diagonal we present the histogram of the corresponding parameter. Upper off-diagonal plots are the time cross correlation. We see that δ is uncorrelated with other measures, while the other three are quite correlated although without any noticeable lead-lag effects. The lower off-diagonal plots are scatter plots between the parameters. μ and α are particularly strongly correlated; fortunately, for the prediction of the spread their hgratio is the most important quantity, and this correlation largely cancels out.

processes [21, 22]. These effects complicate the statistical analysis, and make the assignment of error bars difficult.

The method we use is inspired by the variance plot method described in Beran [23], Section 4.4. We divide the sample into blocks, apply the regression to each block, and then study the scaling of the deviation in the results as the blocks are made longer to coincide with the full sample. We divide the N daily data points for each stock into m disjoint blocks, each containing n adjacent days, so that $n \approx N/m$. We use the same partition for each stock, so that corresponding blocks for each stock are contemporaneous. We perform an independent regression on each of the m blocks, and calculate the mean M_m and standard deviation σ_m of the m slope parameters A_i and intercept parameters B_i, $i = 1, \ldots, m$. We then vary m and study the scaling as shown in Figs. 8 and 9.

Figs. 8(a) and (b) illustrate this procedure for the spread, and Figs. 9(a) and (b) illustrate this for the price diffusion rate. Similarly, panels (c) and (d) in each figure show the mean and standard deviation for the intercept and slope as a function of the number of bins. As expected, the standard deviations of the estimates decreases as n increases. The logarithm of the standard deviation for the intercept and slope as a function of $\log n$ is shown in panels (e) and (f). For IID normally distributed data we expect a line with slope $\gamma = -1/2$; instead we observe $\gamma > -1/2$. For example for the spread $\gamma \approx -0.19$. $|\gamma| < 1/2$ is an indication that this is a long memory process; see the discussion in Section (A 7).

This method can be used to extrapolate the error for $m = 1$, i.e. the full sample. This is illustrated in panels (e) and (f) in each figure. The inaccuracy in these error bars is evident in the unevenness of the scaling. This is particularly true for the price diffusion rate. To get a feeling for the accuracy of the error bars, we estimate the standard deviation for the scaling regression assuming standard error, and repeat the extrapolation for the one standard deviation positive and negative deviations of the regression lines, as shown in panels (e) and (f) of

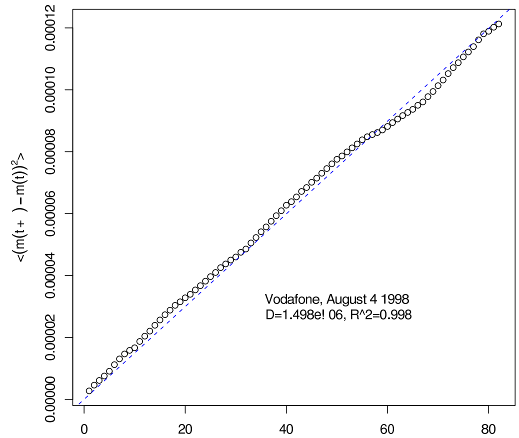

FIG. 6: Illustration of the procedure for measuring the price diffusion rate for Vodafone (VOD) on August 4th, 1998. On the x axis we plot the time τ in units of ticks, and on the y axis the variance of mid-price diffusion $V(\tau)$. According to the hypothesis that mid-price diffusion is an uncorrelated Gaussian random walk, the plot should obey $V(\tau) = D\tau$. To cope with the fact that points with larger values of τ have fewer independent intervals and are less statistically significant, we use a weighted regression to compute the slope D.

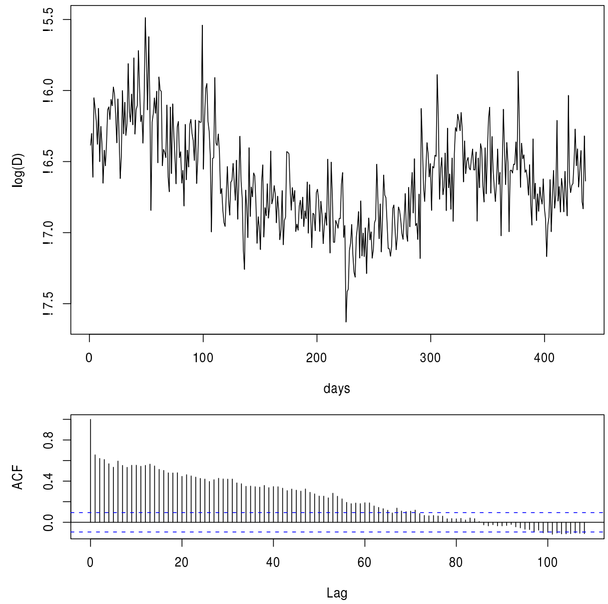

FIG. 7: Time series (top) and autocorrelation function (bottom) for daily price diffusion rate D_t for Vodafone. Because of long-memory effects and the short length of the series, the long-lag coefficients are poorly determined; the figure is just to demonstrate that the correlations are quite large.

Figs. 8 and 9 The results are summarized in Table II.

One of the effects that is evident in Figs. 8(c-d) and 9(c-d) is that the slope coefficients tend to decrease as m increases. We believe this is due to the autocorrelation bias discussed in Section (A 6).

6. Longitudinal vs. cross-sectional tests

It is possible to test this model either longitudinally (across different time intervals for a given stock) or cross-sectionally (across different stocks over the same time period). We have applied tests of both types, but due to the very strong autocorrelations of the order flow rates, spread, and price diffusion rates, there are difficulties in getting a clean test of the model longitudinally. In this section we discuss these problems, and discuss some of our results on the longitudinal tests.

A priori we would expect to do a better job making cross-sectional rather than longitudinal predictions. Indeed, it is not clear that this model should predict anything at all about longitudinal variations. To see why, imagine that the assumptions of the model are satisfied perfectly, and suppose that the five parameters of the order flow process (μ, α, etc.) for a given stock are fixed in time. Then the only daily variations we would observe in testing the model would be due to sample errors in the estimation process. Even though the assumptions are satisfied perfectly, we would find no correlation between predicted and actual values. To observe such a correlation requires real variations in the parameters of the order flow process. There are also possible problems with relaxation times: If a parameter is suddenly changed, according to the model it takes the system time to reach a new steady state behavior. There are two characteristic times in the model: σ/μ, which is the characteristic time for removal of limit orders by market orders, and $1/\delta$, which is is the characteristic time for spontaneous removal of limit orders. For the data here it appears that σ/μ is typically less than a minute, whereas $1/\delta$ ranges from a few minutes to a few hours. Thus, $1/\delta$ is the slowest relaxation time, and in some cases at least it is potentially problematic for a daily analysis. In addition, there is the very significant problem that real order flows are strongly autocorrelated, discussed below.

Cross-sectionally, in contrast, we expect *a priori* that different stocks should have different parameters. There are likely to be larger variations in the parameters between stocks than in the parameters for a given stock at different times. In addition, for a cross-sectional analysis there are no problems with relaxation times, and in any case averaging over longer periods of time reduces the sampling error. Thus cross-sectional analysis is expected to be more promising and more reliable.

As noted, for the daily analysis, and even for cross-sectional analysis over long periods of time, there are problems caused by the long range autocorrelations of real order flow, spreads, and price diffusion rates. Autocorrelations can remain strongly positive on the order of 50 days. This creates problems in performing the regres-

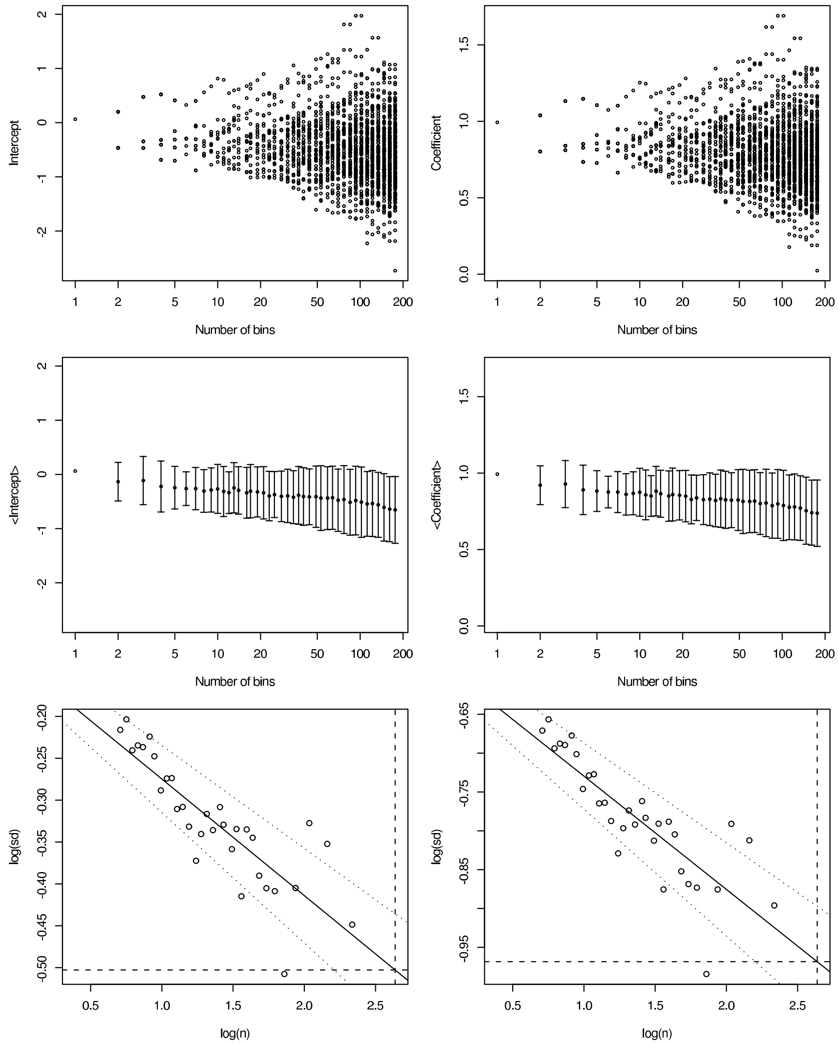

FIG. 8: Subsample analysis of regression of predicted vs. actual spread. To get a better feeling for the true errors in this estimation (as opposed to standard errors which are certainly too small), we divide the data into subsamples (using the same temporal period for each stock) and apply the regression to each subsample. (a) (top left) shows the results for the intercept, and (b) (top right) shows the results for the slope. In both cases we see that progressing from right to left, as the subsamples increase in size, the estimates become tighter. (c) and (d) (next row) shows the mean and standard deviation for the intercept and slope. We observe a systematic tendency for the mean to increase as the number of bins decreases. (e) and (f) show the logarithm of the standard deviations of the estimates against $\log n$, the number of each points in the subsample. The line is a regression based on binnings ranging from $m = N$ to $m = 10$ (lower values of m tend to produce unreliable standard deviations). The estimated error bar is obtained by extrapolating to $n = N$. To test the accuracy of the error bar, the dashed lines are one standard deviation variations on the regression, whose intercepts with the $n = N$ vertical line produce high and low estimates.

sion, and can result in a systematic bias in the estimated parameters. It causes severe systematic biases and interpretation problems for a daily analysis.

To produce estimates of the average values of the parameters and of the price diffusion and spread across the full 21 month period for the cross-sectional regressions, we have used the event-weighted average of the daily values. The alternative would have been to repeat the measurements as done for the daily data on a 21 month rather than a daily time-scale. However, this latter approach would run into problems because of the opening auction, which is not treated by our model. There are price changes driven by the orders received during the opening auction, and if we measured price diffusion across the full period we would be including these as well as the intra-day price movements. As a simple solution

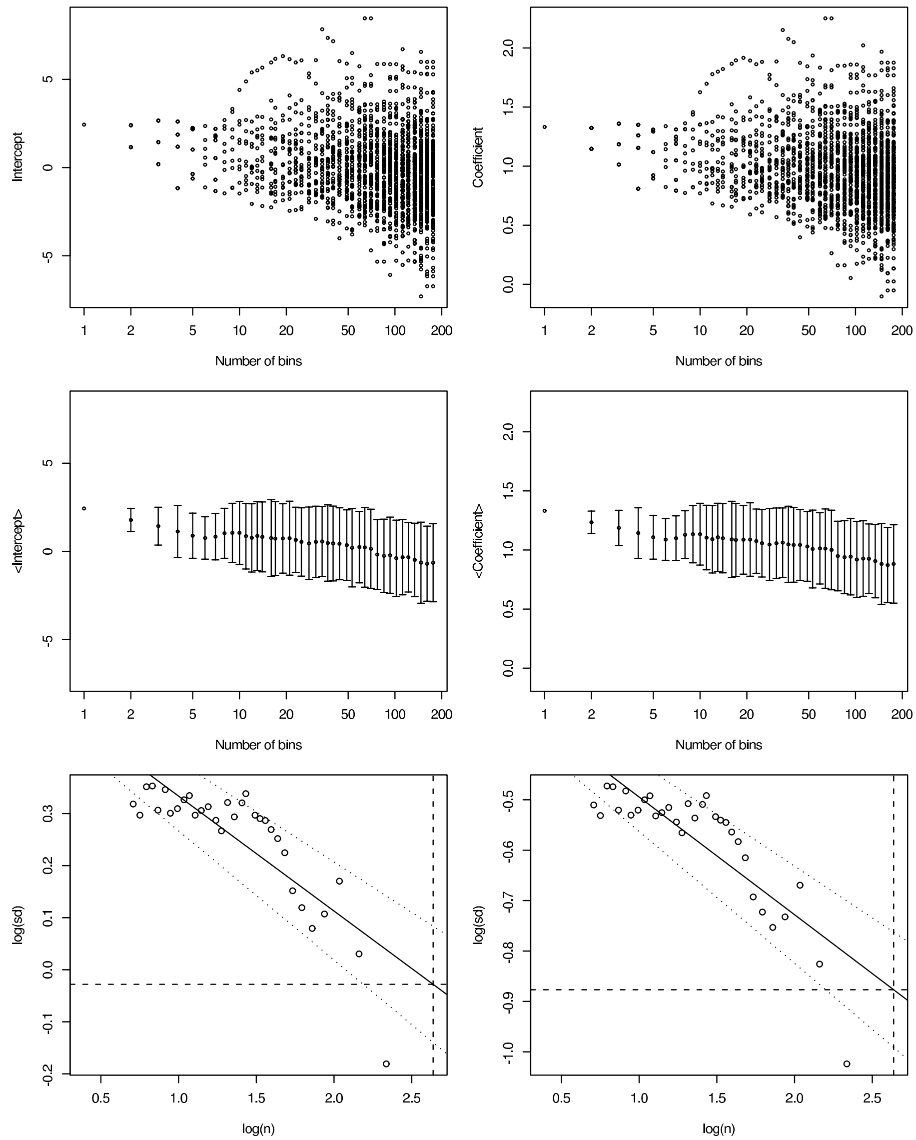

FIG. 9: Subsample analysis of regression of predicted vs. actual price diffusion (see Fig. 6), similar to the previous figure for the spread. The scaling of the errors is much less regular than it is for the spread, so the error bars are less accurate.

regression	estimated	standard	bootstrap	low	high
spread intercept	0.06	0.21	0.29	0.25	0.33
spread slope	0.99	0.08	0.10	0.09	0.11
diffusion intercept	2.43	1.22	1.76	1.57	1.97
diffusion slope	1.33	0.19	0.25	0.23	0.29

TABLE II: A summary of the bootstrap error analysis described in the text. The columns are (left to right) the estimated value of the parameter, the standard error from the cross sectional regression in Fig. 6, the one standard deviation error bar estimated by the bootstrapping method, and the one standard deviation low and high values for the extrapolation, as shown in Figs. 8(e-f) and 9(e-f).

to this problem we use an event-weighted 21 month average of daily values to compute values for each of the order flow parameters, and then make predictions for each stock based on the average values. The weighting is done by the number of events in a day, which for simple quantities such as the market impact rate reduces to something that is equivalent to applying the analysis over the full period. Similarly, to get the 21 month average of the spread and price diffusion we simply compute an event-weighted average of their daily values. We have tried several variations on this procedure and the differences appear to be inconsequential.

When we perform longitudinal regressions at a daily time-scale we get values for the slope coefficient of the regressions that are less than one, often by a statistically significant amount. We believe this is caused by the strong autocorrelation. For example, consider a time series process of the form

$$y_t = ax_t + \rho y_{t-1} + n_t \tag{A1}$$

where n_t is an IID noise process. In case x_t are i.i.d., regressing y_t against x_t will result in coefficients that are systematically too small, due to the fact that the y_{t-1} term damps the response of y_t to changes in x_t. Of course, one can fix this in the simple example above by simply including y_{t-1} in the regression [31]. For the real data, however, the autocorrelation structure is more complicated – indeed we believe it is a long-memory process – which is not well modeled by an AR process in the above form. Without finding a proper characterization of the autocorrelation structure, we are likely to make errors in estimating the dependence of the predicted and actual values. This is borne out in the error analysis presented in Section (A 5), where we see that as we break the data into shorter subsamples, the estimated slope coefficients systematically decrease for the spread and the price diffusion.

If we fit a function of the form $\phi(\omega) = K\omega^\beta$ to the market impact curve, we get $\beta = 0.26 \pm 0.02$ for buy orders and $\beta = 0.23 \pm 0.02$ for sell orders, as shown in Fig. 10. The functional form of the market impact we observe here is not in agreement with a recent theory by Gabaix et al. [18], which predicts $\beta = 0.5$. While the error bars given are standard errors, and are certainly too optimistic, it is nontheless quite clear that the data are inconsistent with $\beta = 1/2$, as discussed in Ref. [29]. This relates to an interesting debate: The theory for average market impact put forth by Gabaix et al. follows traditional thinking in economics, and postulates that agents optimize their behavior to maximize profits, while the theory we test here assumes that they behave randomly, and that the form of the average market impact function is dictated by the statistical mechanics of price formation.

7. Market impact

The market impact function is closely related to the more familiar notions of supply and demand. We have chosen to measure average market impact in this paper rather than average relative supply and demand for reasons of convenience. Measuring the average relative supply and demand requires reconstructing the limit order book at each instant, which is both time consuming and error prone. The average market impact function, in contrast, can be measured based on a time series of orders and best bid and ask prices.

At any instant in time the stored queue of sell limit orders reveals the quantity available for sale at each price, thus showing the supply, and the stored buy orders similarly show the revealed demand. The price shift caused by a market order of a given size depends on the stored supply or demand through a moment expansion [5]. Thus, the collapse of the market impact function reflects a corresponding property of supply and demand. Normally one would assume that supply and demand are functions of human production and desire; the results we have presented here suggest that on a short timescale in financial markets their form is dictated by the dynamical interaction of order accumulation, removal by market orders and cancellation, and price diffusion.

8. Alternative market impact collapse plots

We have demonstrated a good collapse of the market impact using nondimensional units. However, in deciding what "good" means, one should compare this to the best alternatives available. We compare to three such alternatives. In figure 11, the top left pane shows the collapse when using non-dimensional units derived from the model (repeated from the main text). The top right plot shows the average market impact when we instead normalise the order size by its sample mean. Order size is measured in units of shares and market impact is in log price difference. The bottom left attempts to take into account daily variations of trading volume, normalising the order size by the average order size for that stock on that day. In the bottom right we use trade price to normalise the order sizes which are now in monetary units (British Pounds). We visually see that none of the alternative rescalings comes close to the collapse we obtain when using non-dimensional units; because of the much greater dispersion, the error bars in each case are much larger.

9. Error analysis for market impact

Assigning error bars to the average market impact is difficult because the absolute price changes Δp have a slowly decaying positive autocorrelation function. This may be a long-memory process, although this is not as

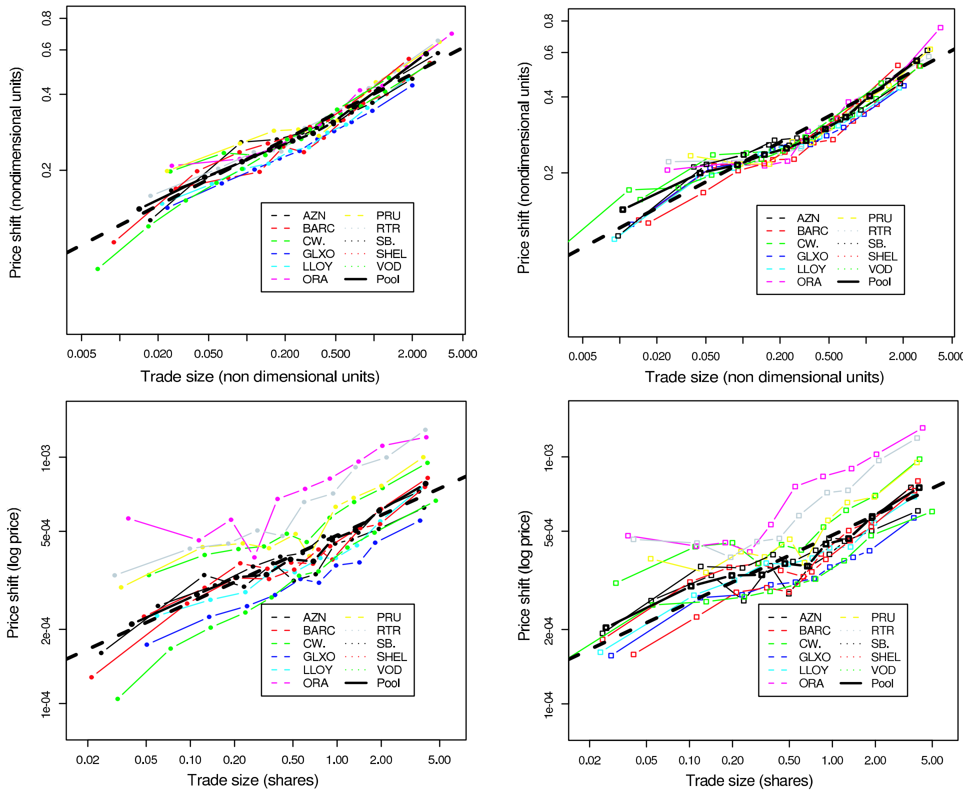

FIG. 10: The average market impact vs. order size plotted on log-log scale. The upper left and right panels show buy and sell orders in non-dimensional coordinates; the fitted line has slope $\beta = 0.26 \pm 0.02$ for buy orders and $\beta = 0.23 \pm 0.02$ for sell orders. In contrast, the lower panels show the same thing in dimensional units, using British pounds to measure order size. Though the exponents are similar, the scatter between different stocks is much greater.

obvious as it is for other properties of the market, such as the volume and sign of orders [21, 22]. The *signed* price changes Δp have an autocorrelation function that rapidly decays to zero, but to compute market impact we sort the values into bins, and all the values in the bin have the same sign. One might have supposed that because the points entering a given bin are not sequential in time, the correlation would be sufficiently low that this might not be a problem. However, the autocorrelation is sufficiently strong that its effect is still significant, particularly for smaller market impacts, and must be taken into account.

To cope with this we assign error bars to each bin using the variance plot method described in, for example, Beran [23], Section 4.4. This is a more straightforward version of the method discussed in Section (A 5). The sample of size $N = 434$ is divided into m subsamples of n points adjacent in time. We compute the mean for each subsample, vary n, and compute the standard deviation of the means across the $m = N/n$ subsamples. We then make use of theorem 2.2 from Beran [23] that states that the error in the n sample mean of a long-memory process is $\hat{e} = \sigma n^{-\gamma}$, where γ is a positive coefficient related to the Hurst exponent and σ is the standard deviation. By plotting the standard deviation of the m estimated intercepts as a function of n we estimate γ and extrapolate to $n =$ sample length to get an estimate of the error in the full sample mean. An example of an error scaling plot for one of the bins of the market impact is given in Fig. 12.

A central question about Fig. 4 is whether the data for different stocks collapse onto a single curve, or whether there are statistically significant idiosyncratic variations from stock to stock. From the results presented in Fig. 4 this is not completely clear. Most of the stocks collapse onto the curve for the pooled data (or the pooled data set with themselves removed). There are a few that appear to make statistically significant variations, at least if we assume that the mean value of the bins for different order size levels are independent. However, they are most definitely *not* independent, and this non-independence is difficult to model. In any case, the variations are always fairly small, not much larger than the error bars. Thus the collapse gives at least a good approximate understanding of the market impact, even if there are some small idiosyncratic variations it does not capture.

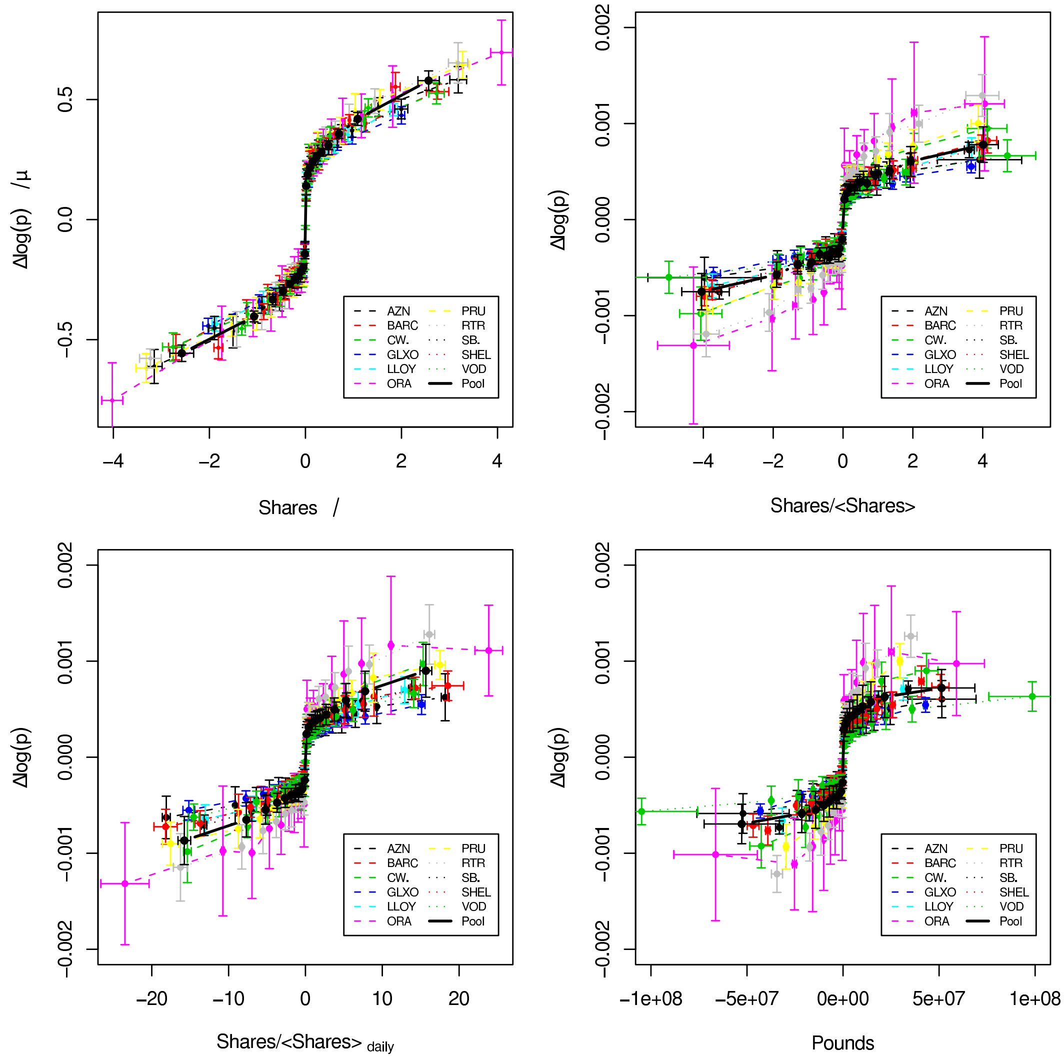

FIG. 11: Market impact collapse under 4 kinds of axis rescaling. In each case we plot a normalised version of the order size on the horizontal axis vs. a (possibly normalised) average market impact $\log(p_{t+1}) - \log(p_t)$ on the vertical axis. (a) (top left) collapse using non-dimensional units based on the model; (b) (top right) order size is normalised by its mean value for the sample. (c) (bottom right) order size is normalised the average daily volume. (d) (bottom right) Order size is multiplied by the current best midpoint price, making the horizontal axis the monetary value of the trade.

10. Extending the model

In the interest of full disclosure, and as a stimulus for future work, in this section we detail the ways in which the current model does not accurately match the data, and sketch possible improvements. This model was intended to describe a few average statistical properties of the market, some of which it describes very well. However, there are several aspects that it does not describe well, such as the scale-free power law properties. This would require a more sophisticated model of order flow, including a more realistic model of price dependence in order placement and cancellations [15, 20], long-memory properties [21, 22] and the relationship of the different components of the order flow to each other. This is a much harder problem, and is likely to require a more complicated model. While this would have some advantages, it would also have some disadvantages.

Some market properties that might profit from such an improved model are detailed below.

- *Price diffusion.* The variance of real prices obeys the relationship $\sigma^2(\tau) = D\tau^{2H}$ to a good approximation for all values of τ, with H close to and typically a little greater than 0.5. In contrast, under Poisson order flow, due to the dynamics of the double continuous auction price formation process, prices make a strongly anti-correlated random walk, so that the function $\sigma^2(\tau)$ is nonlinear. Asymptotically $H = 0.5$, but for shorter times $H < 0.5$. Alternatively, one can character-

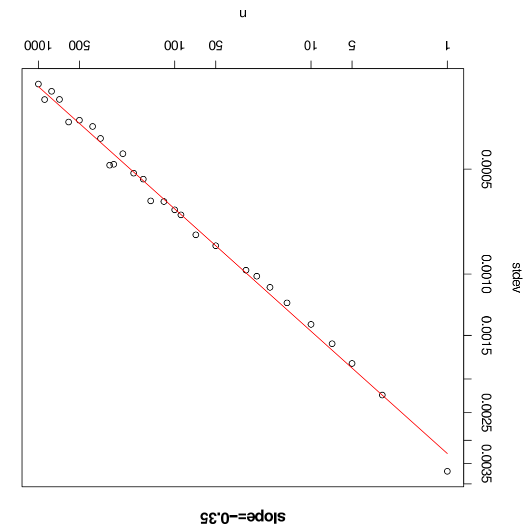

FIG. 12: The variance plot procedure used to determine error bars for mean market impact conditional on order size. The horizontal axis n denotes the number of points in the m different samples, and the vertical axis is the standard deviation of the m sample means. We estimate the error of the full sample mean by extrapolating n to the full sample length.

ize this in terms of a timescale-dependent diffusion rate $D(\tau)$, so that the variance of prices increases as $\sigma^2(\tau) = D(\tau)\tau$. Refs. [4, 5] showed that the limits $\tau \to 0$ and $\tau \to \infty$ obey well-defined scaling relationships in terms of the parameters of the model. In particular, $D(0) \sim \mu^2\delta/\alpha^2\epsilon^{-1/2}$, and $D(\infty) \sim \mu^2\delta/\alpha^2\epsilon^{1/2}$. Interestingly, and for reasons we do not fully understand, the prediction $D(0)$ does a good job of matching the real data, as we have shown here, while $D(\infty)$ does a poor job. Note that it is very interesting that the double continuous auction produces anti-correlations in prices, even with no correlation in order flow. One can turn this around: Given that prices are uncorrelated, there must be correlations in order flow. And indeed this is observed to be the case [21, 22].

- *Market efficiency.* The question of market efficiency is closely related to price diffusion. The anti-correlations mentioned above imply a market inefficiency. We are investigating the addition of “low-intelligence” agents to correct this problem.

- *Correlations in spread and price diffusion.* We have already discussed in Section (A 6) the problems that the autocorrelations in spread and price diffusion create for comparing the theory to the model on a daily scale.

- *Lack of dependence on granularity parameter.* In Section (A 7) we discuss the fact that the model predicts more variation with the granularity parameter than we observe. Apparently the Poisson-based non-dimensional coordinates work even better than one would expect. This suggests that there is some underlying simplicity in the real data that we have not fully captured in the model.

Although in this paper we are stressing the fact that we can make a useful theory out of zero-intelligence agents, we are certainly not trying to claim that intelligence doesn’t play an important role in what financial agents do. Indeed, one of the virtues of this model is that it provides a benchmark to separate properties that are driven by the statistical mechanics of the market institution from those that are driven by conditional intelligent behavior.