

How storing supply and demand affects price diffusion

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The limit order book is a device for storing supply and demand in financial markets, somewhat like a capacitor is a device for storing charge. We develop a microscopic statistical model of the limit order book under random order flow, using simulation, dimensional analysis, and an analytic treatment based on a master equation. We make testable predictions of the price diffusion rate, the depth of stored demand vs. price, the bid-ask spread, and the price impact function, and show that even under completely random order flow the process of storing supply and demand induces anomalous diffusion and temporal structure in prices.

The random walk model was originally introduced by Bachelier to describe prices, five years before Einstein used it to model Brownian motion [1]. In this paper we take the Bachelier model a level deeper by modeling the microscopic mechanism of price formation. We show how the need to store supply and demand in and of itself induces structure in price changes. Our model allows us to study price diffusion rates, and gives a prediction of a universal functional form for the response of prices to small fluctuations in supply and demand. We are the first to show how the most basic properties of a market, such as the spread, liquidity, and volatility, emerge naturally from properties of order flow. The model makes falsifiable predictions with no free parameters. It differs from standard models in economics in that we assume the agents have zero-intelligence, and their behavior is random [2].

Most modern financial markets operate continuously. The mismatch between buyers and sellers that typically exists at any given instant is solved via an order-based market with two basic kinds of orders. Impatient traders submit market orders, which are requests to buy or sell a given number of shares immediately at the best available price. More patient traders submit limit orders, which also state a limit price, corresponding to the worst allowable price for the transaction. Limit orders often fail to result in an immediate transaction, and are stored in a queue called the limit order book. Buy limit orders are called bids, and sell limit orders are called offers or asks. We will label the best (lowest) offer $a(t)$ and the best (highest) bid $b(t)$. There is typically a non-zero price gap between them, called the spread $s(t) = a(t) - b(t)$.

As market orders arrive they are matched against limit orders of the opposite sign in order of price and arrival time. Because orders are placed for varying numbers of

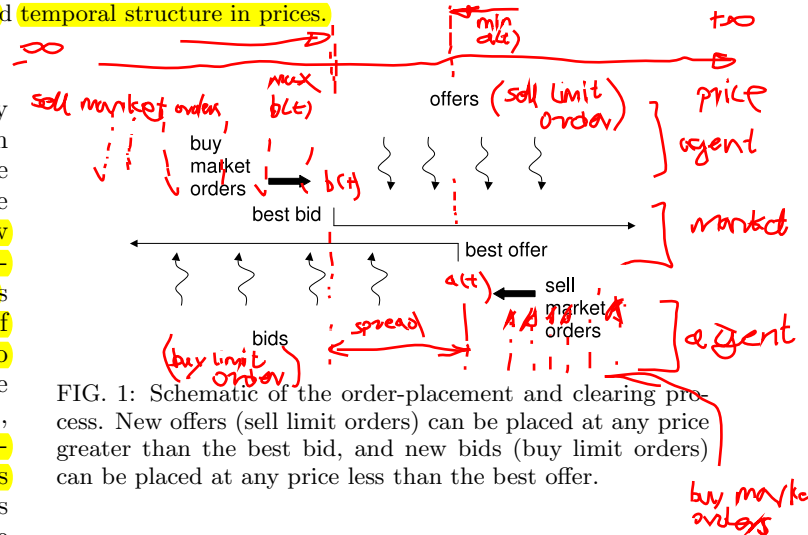


FIG. 1: Schematic of the order-placement and clearing process. New offers (sell limit orders) can be placed at any price greater than the best bid, and new bids (buy limit orders) can be placed at any price less than the best offer.

shares, matching is not necessarily one-to-one. For example, suppose the best offer is for 200 shares at \$60 and the the next best is for 300 shares at \$60.25; a buy market order for 250 shares buys 200 shares at \$60 and 50 shares at \$60.25, moving the best offer $a(t)$ from \$60 to \$60.25. A high density of limit orders results in high liquidity for market orders, i.e., it decreases the average price movement when a market order is placed.

We propose the simple random order placement model shown in Fig. 1. All the order flows are modeled as Poisson processes. We assume that market orders in chunks of σ shares arrive at a rate of μ shares per unit time, with an equal probability for buy and sell orders. Similarly, limit orders in chunks of σ shares arrive at a rate of α shares per unit price and per unit time. Offers are placed with uniform probability at integer multiples of a tick size p_0 in the range $b(t) < p < \infty$, and similarly for bids on $-\infty < p < a(t)$. When a market order arrives it causes a transaction, under the assumption of constant order size, a buy market order removes an offer at price $a(t)$, and a sell market order removes a bid at price $b(t)$. Alternatively, limit orders can be removed spontaneously by being canceled or by expiring. We model this by letting them be removed randomly with constant probability δ per unit time.

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This order placement process is designed to permit an analytic solution. This model builds on previous work modeling limit order behavior [3–10]. While the assumption of limit order placement over an infinite interval is clearly unrealistic [11], it provides a tractable boundary condition for modeling the behavior of the limit order book in the region of interest, near the midpoint price $m(t) = (a(t) + b(t))/2$. It is also justified because limit orders placed far from the midpoint usually expire or are canceled before they are executed. Assuming a constant probability for cancellation is clearly *ad hoc*, but in simulations we find that other assumptions, such as constant duration time, give similar results. For our analytic model we use a constant order size σ . In simulations we also use variable order size, e.g. half-normal distributions with standard deviation $\sqrt{2/\pi}\sigma$. The differences do not affect any of the results reported here. For simplicity we define a limit order as any order that is not executed immediately [12]. This automatically determines the boundary conditions of the order placement process, since an offer with $p < b(t)$ or a bid with $p > a(t)$ would result in an immediate transaction, and thus is effectively the same as a market order. Note these boundary conditions realistically allow limit orders to be placed at prices anywhere inside the current spread.

We seek a distribution over values of the depth profile $N(p, t)$, the density of shares in the order book with price p at time t . For convenience we let the depth be positive for bids and negative for offers. From the symmetry of the order process midpoint prices makes a random walk, with a nonstationary distribution. The key to finding a stationary analytic solution for the average depth is to use comoving coordinates. Without loss of generality, we study the depth of offers, using price coordinates centered at the midpoint $m(t)$ so that $b(t) \equiv -a(t)$. We make two simplifying approximations: Fluctuations about the mean depth at adjacent prices are treated as independent. This allows us to replace the distribution over depth profiles with a simpler probability density over occupation numbers N at each p and t . We then let the bin size $p_0 \rightarrow dp$ be infinitesimal. With finite order flow rates, this will give vanishing probability for the existence of more than one order in any bin as $dp \rightarrow 0$. Let $\pi(N, p, t)$ be the probability that an interval dp centered at price p has N shares at time t , and let $P_+(\Delta p, t)$ be the probability that $m(t)$ increases by Δp , and $P_-(\Delta p, t)$ be the probability that it decreases by Δp . Assume that α , $N\delta$, P_+ , and P_- are small enough so that higher order terms corresponding to simultaneous events can be neglected. Going to continuous time, a general master

equation for π can be written

$$\begin{aligned} \frac{\partial \pi}{\partial t}(N, p) = & \frac{\alpha(p) dp}{\sigma} [\pi(N - \sigma, p) - \pi(N, p)] \\ & + \frac{\delta}{\sigma} [(N + \sigma) \pi(N + \sigma, p) - N \pi(N, p)] \\ & + \frac{\mu(p)}{2\sigma} [\pi(N + \sigma, p) - \pi(N, p)] \\ & + \sum_{\Delta p} P_+(\Delta p) [\pi(N, p - \Delta p) - \pi(N, p)] \\ & + \sum_{\Delta p} P_-(\Delta p) [\pi(N, p + \Delta p) - \pi(N, p)]. \quad (1) \end{aligned}$$

We have not written the variable t , which appears in every term. The α term corresponds to receipt of a limit order, the δ term to spontaneous removal of a limit order, the μ term to receipt of a market order, the P_+ term to an increase in the midpoint $m(t) \rightarrow m(t) + \Delta p$, and the P_- term to a decrease $m(t) \rightarrow m(t) - \Delta p$. These last two terms are particularly important because they imply a diffusion process for depth when viewed in the comoving reference frame, driven by the movement of the reference point of the price coordinate when the midpoint moves. The definition of the best offer gives the boundary condition $\pi(N, p) = 0$ for $p < 0$.

We seek a mean field solution. In the comoving reference frame $\mu(p)$ becomes a function corresponding to the rate at which market orders are executed at price p . One can think of the market order as a “particle” that is created at $b(t)$ and moves to the right until it is “absorbed” at price p . Similarly, the fluctuating boundary conditions affect the limit order placement rate in the comoving frame, so that $\alpha(p)$ also depends on price. An approximate solution for Equation (1) is found by relating the distributions P_+ and P_- self-consistently to mean-field functional forms of $\alpha(p)$ and $\mu(p)$, for which the parameters α and μ furnish boundary conditions. The steady-state mean value $\langle N(p) \rangle$ is then obtained from the generating functional for the moments of π . The derivation and solutions will be presented elsewhere [13].

Figure 2 compares the analytic solutions in nondimensional form to simulation results for the cumulative distribution of the spread and the average depth $\langle N(p) \rangle$. For a wide range of parameters nondimensionalization collapses all the results onto qualitatively similar curves. The simulation results approximately match the mean-field solution. Agreement is good for the spread, but not as good for the average depth, though at other parameter values the opposite is true. In the parameter range shown the asymptotic depth varied by a factor of ten and the width of the transition from the midpoint to the asymptotic region varied by three orders of magnitude. Thus mean-field analysis does a good job of capturing the leading order properties.

The liquidity for executing a market order can be characterized by a price impact function $\Delta p = \phi(\omega, \tau, t)$. Δp is the price shift at time $t + \tau$ caused by a market order ω at time t ; ω is positive for buy orders and negative for sell

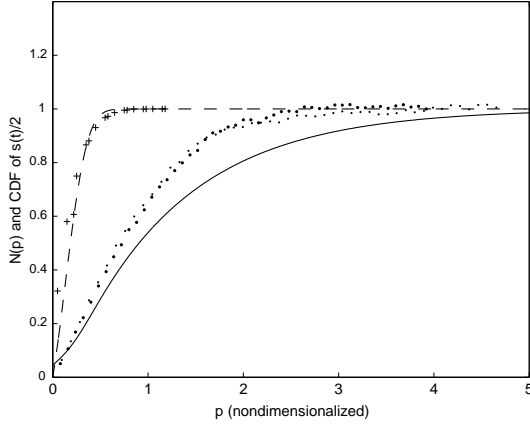


FIG. 2: A comparison of analytic and simulation results. Simulation results for the cumulative distribution of the spread/2 are shown as crosses, and the mean depth of orders $\langle N(p) \rangle$ as circles. We plot the nondimensionalized depth $\hat{N} = \delta N/\alpha$ versus the nondimensionalized price $\hat{p} = 2\alpha p/\mu$, measured relative to the midpoint. In units with the mean order size σ and the tick size p_0 set to one, $\alpha = 0.002$. Two simulation results are shown, with $\delta = 10^{-3}$ and $\mu = 0.1$ in one case, and $\delta = 10^{-4}$ and $\mu = 0.01$ in the other. The curves (dashed for spread, solid for depth) are from the self-consistent solution of Eq. (1)

orders. One of our main results is a derivation of the average price impact function. This is important for practical reasons such as minimizing transaction costs, and also because it is in a sense an inverse demand function, providing a natural starting point for theories of statistical or dynamical properties of markets [14, 15]. Naive arguments predict that ϕ should be a convex function (with increasing derivative) for positive ω . The naive argument goes as follows: Fractional price changes should not depend on the scale of price. Suppose buying a single share raises the price by a factor $a > 1$. If a is constant, buying ω shares in succession should raise it by a^ω . Thus, if buying ω shares all at once affects the price as much as buying them one at a time, price impact should be strongly convex [16]. In contrast, from empirical studies $\phi(\omega)$ for $\omega > 0$ appears to be concave [17–21]. At least for small ω , these studies suggest a function of the form $\phi(\omega) \sim \omega^\beta$. Estimates of β are poor, and some of the results are contradictory, but on balance the current literature seems to indicate that $\beta \sim 0.5$.

Our model predicts this result [22]. The instantaneous price impact $\phi(\omega, 0, t)$ depends on the depth of orders as a function of price. Although the depth at any given time is a discontinuous random function, the average depth $\langle N(p) \rangle$ can be approximated as a smooth function that vanishes at the midpoint. Providing its derivative exists, it can be expanded in a Taylor series, and the leading term can be written $\langle N(p) \rangle \approx 2\lambda p$. We treat the instantaneous depth $N(p, t)$ near the midpoint as a function of this form, with time varying liquidity $\lambda(t)$. The shift in

Quantity	Dimensions	Scaling relation
Asymptotic depth	shares/price	$d \sim \alpha/\delta$
Spread	price	$s \sim \mu/\alpha$
Slope of depth profile	shares/price ²	$\lambda \sim \alpha^2/\mu\delta = d/s$
Volatility ($\tau \rightarrow 0$)	price ² /time	$D_0 \sim \mu^2\delta/\alpha(\delta\sigma/\mu)^{-0.5}$
Volatility ($\tau \rightarrow \infty$)	price ² /time	$D_\infty \sim \mu^2\delta/\alpha(\delta\sigma/\mu)^{0.5}$

TABLE I: Predictions of scaling of market properties as a function of properties of order flow. α is the limit order rate, μ is the market order rate, δ is the spontaneous limit order removal rate, and σ is the order size.

price caused by a market order $\omega > 0$ can be approximated using the continuum transaction condition

$$\omega = \int_0^{\Delta p} N(p, t) dp, \quad (2)$$

which says that the size of the market order equals the number of shares removed from the book. Plugging in the linear approximation and taking time averages yields the expected price impact conditioned on ω ,

$$\langle \Delta p(\omega) \rangle = \langle (\omega/\lambda(t))^{1/2} \rangle. \quad (3)$$

This is confirmed in numerical experiments. Note that the power 1/2 only depends on the assumption of a non-zero derivative at $p = 0$, so the result is generic. In general the time average of $\langle (1/\lambda(t))^{1/2} \rangle \neq 1/\langle \lambda \rangle^{1/2}$, so the magnitude of the price impact cannot be predicted exactly from the stationary solution for the depth.

If we ignore effects caused by finite bin size p_0 and finite order size σ , the scaling behavior of the liquidity and the average spread can be derived from dimensional analysis. The fundamental dimensional quantities are shares, price, and time. In the continuum limit $p_0 \rightarrow 0$ and $\sigma \rightarrow 0$ these are uniquely represented by α , with dimensions of shares/(price \times time), μ , with dimensions of shares/time, and δ , with dimensions of 1/time. The average spread has dimensions of price and is proportional to μ/α ; this comes from a balance between the total order placement rate inside the spread, αs , and the order removal rate μ . The asymptotic depth is the density of shares far away from the midpoint, where market orders are unimportant. It has dimensions of shares/price, and is Poisson distributed with mean α/δ . The liquidity λ depends on the average slope of the depth profile near the midpoint, and has dimensions of shares/price². It is proportional to the ratio of the asymptotic depth to the spread, which implies that it scales as $\alpha^2/\mu\delta$. This is summarized in Table I.

When discreteness is important unique derivations from continuum dimensional analysis are lost and scaling can deviate from the predictions above. In this case the behavior also depends on σ , which has dimensions of shares, and p_0 , which has dimensions of price. Effects due to granularity of orders depend on the nondimensional order size $\hat{\sigma} = \sigma\delta/\mu$, and effects due to finite tick size depend on the nondimensional price $\hat{p} = p_0\alpha/\mu$. (See

figure 2 for an example of how these rescalings result in data collapse.) Simulation results show that there is a well-defined continuum limit with respect to the nondimensional tick size $p_0\alpha/\mu$. The same is not true of the nondimensional granularity parameter $\sigma\delta/\mu$, which does not affect the spread, slope or asymptotic depth, but does affect price diffusion.

The price diffusion rate, or volatility, is a property of central interest. From continuum dimensional analysis the volatility should scale as $\mu^2\delta/\alpha^2$. This also comes from squaring equation (3) before averaging and substituting $\omega = \mu$ and $\lambda = \alpha^2/\mu\delta$. However, this scaling is violated because discreteness is inherent to price diffusion. In Fig. 3 we plot simulation results for the variance of the change in the midpoint price at timescale τ , $\text{Var}(m(t+\tau) - m(t))$. The slope is the diffusion rate. It appears that there are at least two timescales involved, with a faster diffusion rate for short timescales and a slower diffusion rate for long timescales. Such anomalous diffusion is not predicted by mean-field analysis. Simulation results show that the diffusion rate is correctly described by the product of the continuum diffusion rate $\mu^2\delta/\alpha^2$ and a τ -dependent power of the nondimensional granularity parameter $\delta\sigma/\mu$, as summarized in Table I. Why this power is apparently $-1/2$ for short term diffusion and $1/2$ for long-term diffusion remains a mystery to us at this stage. Note that the temporal structure in the diffusion process also implies non-zero autocorrelations of $m(t)$. This corresponds to weak negative autocorrelations in price differences $m(t) - m(t-1)$, that persist for timescales until the variance vs. τ becomes a straight line. The timescale depends on parameters, but is typically the order of 50 market order arrival times. This temporal structure implies that there exists an arbitrage opportunity which, when exploited, would make the structure of the order flow non-random.

Another consequence of this behavior is that the magnitude of the average price impact $\langle\Delta p\rangle = \langle\phi(\omega, \tau)\rangle$ varies with the time horizon τ . Our simulations make it clear that the functional form is $\langle\phi(\omega, \tau)\rangle = f(\tau)\omega^{1/2}$. The function $f(\tau)$ appears to decrease from its initial value, reaching a non-zero value $f(\infty)$ related to the asymptotic diffusion rate. This will be discussed in more detail elsewhere [13].

This model contains several unrealistic assumptions. For real markets order flow rates vary in time, and $\alpha(p)$ (in non-moving coordinates) is not uniform [11]. We expect that the order cancellation rate δ will depend on the elapsed time from order placement. Market participants place orders in response to varying market conditions, and real order flow processes are not unconditionally random. Real markets display super-diffusive behavior of prices at short timescales, and autocorrelations of absolute price changes that decay as a power law, neither of which occur in this model. We are working on enhancements that include more realistic order placement processes grounded on empirical measurements of market data.

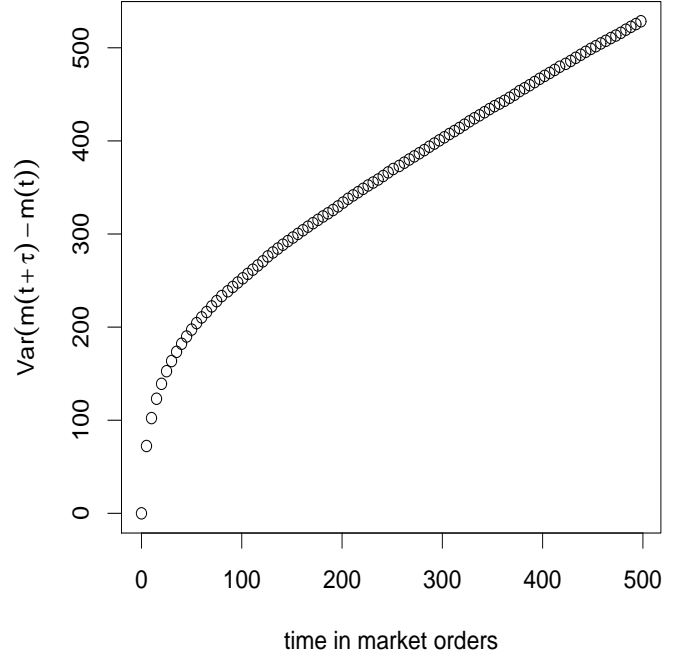


FIG. 3: The variance of the change in the midpoint price on timescale τ . For a pure random walk this would be a line whose slope is the diffusion rate. The fact that the slope is steeper for short times comes from the nontrivial temporal persistence of the order book. Time is measured in terms of the characteristic interval between market orders, σ/μ , with $\alpha = 1$, $\delta = 5 \times 10^{-4}$, $\mu = 0.1$, and $\sigma = 1$.

Despite these reservations, our model makes a valuable contribution that can guide future research. We provide a testable prediction for the price impact function that agrees with the best empirical measurements and solves a long-standing mystery about its functional form. Even though we expect that these predictions are not exact, they provide a good starting point that can guide improvement in the theory. Our model illustrates how the need to store supply and demand gives rise to interesting temporal properties of prices and liquidity even under assumptions of perfectly random order flow. It demonstrates the importance of making realistic models of market mechanisms.

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