Solutions for 1-D acoustic fields in ducts with an axial temperature gradient

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1 Introduction

The presence of mean axial temperature in a duct (caused, for example, by heat transfer to or from the walls) affects the propagation of sound waves and the stability of small amplitude disturbances in a duct. Understanding the behaviour of this acoustic field is of considerable scienctific and practical interest.

A physical description of the effect of a mean temperature gradient upon wave propagation in a duct can be obtained by assuming that the gas in the duct consists of infinitesimally thin gas layers, each at a different(constant) temperature, that are in contact with one another. In this case, propagation of sound from one layer to another is accompanied by wave transmission and reflection, which modifies the wave structure in the duct. For the analysis, The one-dimensional wave equation for a constant area duct with an arbitrary axial temperature profile is derived for a perfect, inviscid and non-heat-conducting gas. The analysis neglects the effects of mean flow, and therefore the solutions obtained are valid only for mean Mach numbers less than 0.1.

Numerical analysis is done using the Fourth order Runge-Kutta scheme to obtain the the Pressure and Velocity profiles and the eigenvalues of a closed/open duct with a linear temperature gradient.

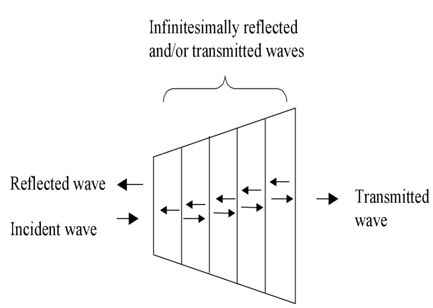

Figure 1: Acoustic pressure waves in a duct with axial temperature gradient

2 Derivation of the wave equation

The wave equation is derived for a constant area duct with a mean axial temperature gradient. Assuming a perfect, inviscid and non-heat conducting gas, the one-dimensional momentum, energy and state equations can be expressed in the following form:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \tag{1}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0, \tag{2}$$

$$p = \rho RT \tag{3}$$

Expressing each of the dependent variables as the sums of steady and time dependent, small amplitude solutions:

$$u = \bar{u}(x) + u'(x,t), \quad p(x,t) = \bar{p}(x) + p'(x,t), \quad \rho(x,t) = \bar{\rho}(x) + \rho'(x,t), \tag{4}$$

and substituting these expressions into the conservation equations, yields the system of steady linearized equations. Assuming that the mean flow Mach number is smaller than 0.1, the solution of the steady momentum equation shows that the mean pressure $\bar{p}$ is constant in the duct.

The wave equation is derived from the first order acoustic momentum and energy equations:

$$\frac{\partial u'}{\partial t} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0, \tag{5}$$

$$\frac{\partial p'}{\partial t} + \gamma\bar{p}\frac{\partial u'}{\partial x} = 0. \tag{6}$$

Differentiating the momentum equation with respect to x and the energy equation with respect to t and eliminating the cross-derivative term yields the wave equation with variable coefficients:

$$\frac{\partial^2 p'}{\partial x^2} - \frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dx}\frac{\partial p'}{\partial x} - \frac{\bar{\rho}}{\gamma\bar{p}}\frac{\partial^2 p'}{\partial t^2} = 0. \tag{7}$$

Differentiating the steady equation of state and recalling that the steady duct pressure is constant, yield the following relationship between the steady temperature and density:

$$\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dx} + \frac{1}{\bar{T}}\frac{d\bar{T}}{dx} = 0. \tag{8}$$

Using equation (8), equation (7) can be reduced to:

$$\frac{\partial^2 p'}{\partial x^2} + \frac{1}{\bar{T}}\frac{d\bar{T}}{dx}\frac{\partial p'}{\partial x} - \frac{1}{\gamma R\bar{T}}\frac{\partial^2 p'}{\partial t^2} = 0. \tag{9}$$

Assuming that the solution has periodic time dependence (i.e., $p'(x,t) = P'(x)e^{iwt}$), equation (9) reduces to the following second order ordinary differential equation for the complex amplitude $P'(x)$:

$$\frac{d^2P'}{dx^2} + \frac{1}{\bar{T}}\frac{d\bar{T}}{dx}\frac{dP'}{dx} + \frac{\omega^2}{\gamma R\bar{T}}P' = 0. \tag{10}$$

Using the derived solution for the acoustic pressure and acoustic momentum equation (i.e., the equation (5)), the following expression for acoustic velocity can also be obtained:

$$U'(x) = -\frac{1}{iw\bar{\rho}}\frac{dP'}{dx} \tag{11}$$

3 Numerical method: Explanation of the Fourth Order Runge-Kutta integration scheme

The Runge-Kutta method is used to find approximate values of y for given values of x, of an ordinary differential equation of the form $\frac{dy}{dx}$ when the initial value of y i.e., y(0) is given.

$$\frac{dy(t)}{dx} = y'(t) = f(y(t),t), \quad \text{with} \quad y(t_0) = y_0 \tag{12}$$

The basic forward Euler method uses the information of the slope or derivative of y at a given time step to extrapolate the solution to the next time-step. $y(t_0 + h) \approx y(t_0) + f(y(t_0), t_0)h$

The RK4 method uses information of the slope at more than one point to extrapolate the solution to a future time step. Only first-order ordinary differential equations can be solved by using this method.

The below algorithm is used to compute the next value.

Starting from some known initial condition $y(t_0) = y_0$, We will use the following slope approximations to estimate the slope at some time t_0 (assuming we only have an approximation to $y(t_0)$ which we call $y*(t_0)$).

$$k_1 = f(y*(t_0), t_0) \tag{13}$$

$$k_2 = f(y*(t_0) + k_1\frac{h}{2}, t_0 + \frac{h}{2} \tag{14}$$

$$k_3 = f(y*(t_0) + k_2\frac{h}{2}, t_0 + \frac{h}{2} \tag{15}$$

$$k_4 = f(y*(t_0) + k_3h, t_0 + h) \tag{16}$$

Each of these slope estimates can be described verbally.

- k_1 is the slope at the beginning of the time step.

- If we use the slope k_1 to step halfway through the time step, then k_2 is an estimate of the slope at the midpoint
- If we use the slope k_2 to step halfway through the time step then k_3 is another estimate of the slope at the midpoint.
- Finally, we use the slope k_3 to step all the way across the time step to $t_0 + h$, and k_4 is an estimate of the slope at the endpoint.

We then use a weighted sum of these slopes to get our final estimate of $y*(t_0+h)$

$$y*(t_0+h) = y*(t_0) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}h \tag{17}$$

$$= y*(t_0) + (\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4)h \tag{18}$$

$$= y*(t_0) + mh \tag{19}$$

where m is a weighted average slope approximation

4 Acoustic behavior of a duct with a linear temperature distribution

The derived Equation (10) has variable coefficients. Solutions are obtained for the case of a linear mean temperature profile

$$\bar{T} = T_0 + mx \tag{20}$$

where T_0 and m are constants that describe the temperature at x = 0 and the temperature gradient, respectively.

To demonstrate the effect of a linear temperature gradient upon the acoustic properties of a duct, the acoustic characteristics of a duct closed at one end and open at the other (i.e., a quarter-wave tube) are investigated. The equations are numerically integrated using the fourth order Runge-Kutta method described. The solution obtained satisfies the boundary conditions U' = 0 and P=2000 at the closed end of the duct (i.e., at $x = 0$, where $\bar{T} = T_1$, and $P' = 0$ at the open end of the duct (i.e., at $x =$ L where $\bar{T} = T_2$).

The dependence of the eigenfrequency ω upon the properties of the mean temperature distribution in the duct was investigated by solving the equation for different values of T_1, the temperature at the closed end, and m, the temperature gradient, for fixed values of $T_2 = 300K$ at $L = 4m$. The calculated eigenfrequencies are given in Table 1. It is shown in Table 1 that when a linear temperature gradient is present in the duct, the higher harmonics are no longer integral multiples of the fundamental frequency as is the case in a duct with uniform temperature. The effect of temperature gradient upon distributions of the amplitudes of the acoustic pressure and velocity are also investigated. The results show that when a mean temperature gradient is present in the duct, the pressure and velocity nodes and anti-nodes are unevenly spaced.

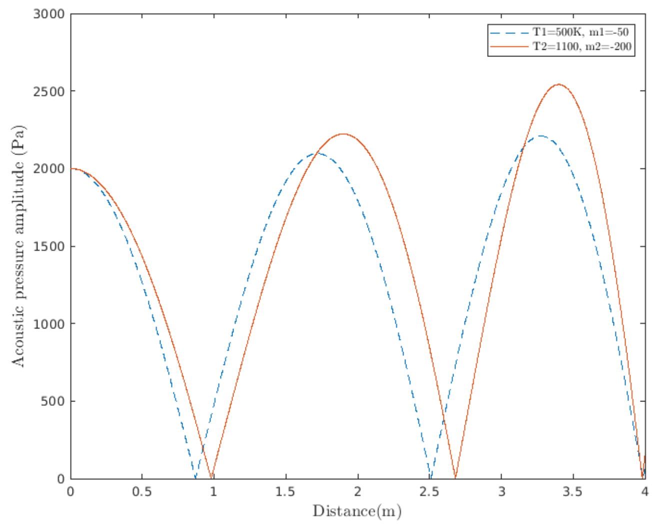

Figure 2: The variation of acoustic pressure amplitude with axial distance in a duct closed at one end and open at the other, for different linear mean temperature profiles.

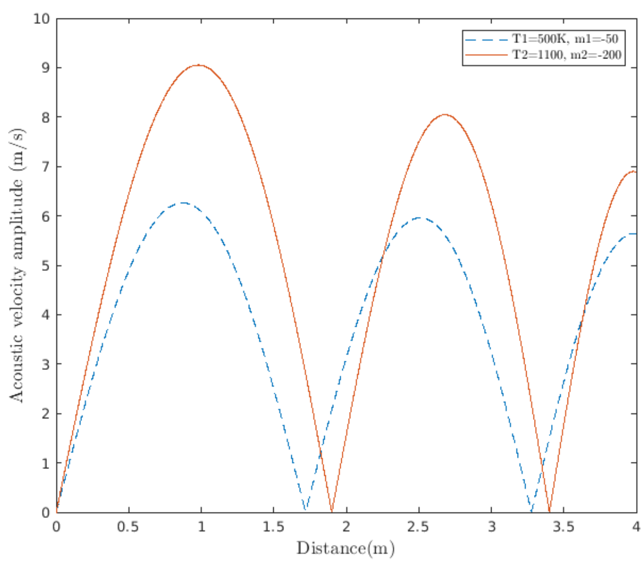

Figure 3: The variation of acoustic velocity amplitude with axial distance in a duct closed at one end and open at the other, for different linear mean temperature profiles.

$T_1(K)$	$m(\frac{K}{m})$	First (Hz)	Second (Hz)	Third (Hz)	Fourth (Hz)	Fifth (Hz)
500	-50	23.60	74.22	124.49	173.18	223.96
700	-100	25.06	81.38	135.64	192.19	247.11
900	-150	26.45	88.05	146.76	208.65	264.17
1100	-200	27.45	93.4	158.46	221.86	285.24

Table 1: The dependence of eigenvalues of a closed/open duct with a linear mean temperature gradient upon the temperature T_1 at the closed end

5 Code

```
function xout = rk4singlestep(fun,dt,tk,xk)
f1 = fun(tk,xk);
f2 = fun(tk+dt/2, xk+(dt/2)*f1);
f3 = fun(tk+dt/2, xk+(dt/2)*f2);
f4 = fun(tk+dt, xk+dt*f3);
xout = xk+(dt/6)*(f1+2*f2+2*f3+f4);
```

Listing 1: Function defining the RK4

```
function dp = temp_pressure(x,p,w,T,m,gamma,R)
dp = [p(2), (-1/T)*m*p(2)-w^2*p(1)/(gamma*R*T)];
```

Listing 2: Function defining the system equations

```
T0=500;
m=-50;
n=3;
h=0.001;
x = 0:h:4;
gamma=1.4;
R=287;
rhobar=1.4;
L=4;
index=[];
P=[];
fr=[];
U1=[];
T=T0;
for i=1:40
    disp(i)
    c=sqrt(gamma*R*T);
    Tx=T0+m*x;
    f=(2*n-1)*c/(4*L);
    w=2*pi*f;
    p0 = [2000,0];
    p = p0;
    for j=2:length(x)
        fun = @(x,p) temp_pressure(x, p, w,Tx(j),m,gamma,R);
        p(j,:) =  rk4singlestep(fun,h,x(j),p(j-1,:));
    end
    P=[P p(:,1)];
    fr=[fr f];
    T=T0+m*i/10;
```

```
    Tx=T0+m*x;
    rhox=1e5./(R*Tx);
end
p1=p;
U1=[U1 -p(:,2)'./(1i*rhox*w)];
```

Listing 3: Main code

6 References

1. Sujith, R. I., G. A. Waldherr, and B. T. Zinn. "An exact solution for one-dimensional acoustic fields in ducts with an axial temperature gradient." Journal of Sound and Vibration 184.3 (1995): 389-402.

2. https://lpsa.swarthmore.edu/NumInt/NumIntFourth.html