Motor Unit Inter-Spike-Interval State-Dependence Near Steady-State Firing

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Abstract

Motor unit firing patterns deviate from renewal statistics as firing rates approach steady-state or maximal discharge levels. This work provides a quantitative treatment of inter-spike-interval (ISI) serial dependence in the Hodgkin–Huxley framework, demonstrating how residual activation of ion channels and slow ionic clearance introduce "state memory" across spikes. When the mean ISI becomes comparable to the recovery time of slow conductances (e.g., calcium-dependent potassium currents), gating variables fail to fully reset, resulting in correlated ISIs. We derive an approximate autoregressive form linking lag-1 ISI correlation to the ratio between ISI duration and the dominant recovery constant, and discuss implications for motor unit physiology and fatigue-related modulation of excitability.

1 Quantitative Overview

It is useful to frame the notion of "state memory" in renewal-type spike trains within the context of Hodgkin–Huxley (HH) dynamics. Consider a simplified HH-type neuron model:

$$C_m \frac{dV}{dt} = -\big(I_{\mathrm{Na}} + I_{\mathrm{K}} + I_{\mathrm{L}} + I_{\mathrm{Ca}} + I_{\mathrm{AHP}}\big) + I_{\mathrm{inj}}, \tag{1}$$

$$I_{\mathrm{Na}} = \bar{g}_{\mathrm{Na}}\, m^3 h\, (V - E_{\mathrm{Na}}), \quad \frac{dh}{dt} = \frac{h_\infty(V) - h}{\tau_h(V)}, \tag{2}$$

and similarly for n- and m-gates, a slow calcium-activated potassium current $I_{\mathrm{AHP}} = \bar{g}_{\mathrm{AHP}}\,[\mathrm{Ca}^{2+}](V - E_K)$, and intracellular calcium evolving as

$$\frac{d[\mathrm{Ca}^{2+}]}{dt} = -\frac{[\mathrm{Ca}^{2+}] - [\mathrm{Ca}^{2+}]_{\mathrm{rest}}}{\tau_{\mathrm{Ca}}} + \Delta_{\mathrm{spike}} \sum_i \delta(t - t_i). \tag{3}$$

In detailed HH formulations, intracellular Ca^{2+} accumulation arises from the current $I_{\mathrm{Ca}}(t) = \bar{g}_{\mathrm{Ca}} m_{\mathrm{Ca}}^p h_{\mathrm{Ca}}^q (V - E_{\mathrm{Ca}})$, which drives the concentration change $\dot{[\mathrm{Ca}^{2+}]} = -\frac{\eta}{2Fd} I_{\mathrm{Ca}}(t)$. Because the spike waveform is brief relative to the decay constant τ_{Ca}, the net integral of this current over each spike can be represented as an instantaneous increment Δ_{spike}. This yields a simplified, analytically tractable "impulse approximation" in which $\Delta_{\mathrm{spike}} \sum_i \delta(t - t_i)$ replaces the detailed Ca^{2+} current waveform while preserving the total charge entry per spike [2, 9, 3].

If the inter-spike interval T_n is short relative to one or more of the slow time constants $\tau_h, \tau_{\mathrm{Ca}}, \tau_{\mathrm{adap}}$, then gating variables and ionic concentrations at the start of the next interval retain a memory of the previous spike (see Table 1 for representative values). The exponential "carryover" relation in Eq. (4) follows directly from solving the first-order recovery equation between spikes and applying the impulsive update at each spike. A full derivation showing the transition from the continuous-time calcium dynamics of Eq. (3) to the discrete recurrence in Eq. (4) is provided in Appendix A.

$$x_{n+1}(0) - x_\infty = \big(x_n(0) - x_\infty\big)\, e^{\frac{-T_n}{\tau_x}}, \tag{4}$$

Symbol	Description	Typical	Refs
$\tau_m(V)$	Na^+ activation	0.1–1.0 ms	[4, 8]
$\tau_h(V)$	Na^+ inactivation	1–10 ms	[4, 5]
$\tau_n(V)$	K^+ activation	2–10 ms	[4, 8]
τ_{Ca}	Ca^{2+} clearance	50–2000 ms	[2, 3]
τ_{AHP}	AHP activation/decay	50–300 ms	[10, 1]
τ_{adap}	Slow M-Current	0.5–5 s	[8, 2]
E_{Na}	Na^+ reversal potential	+50 to +60 mV	[4]
E_K	K^+ reversal potential	−80 to −90 mV	[4]
$\bar{g}_{Na}$	Na^+ conductance	50–120 mS/cm^2	[4]
$\bar{g}_K$	K^+ conductance	30–40 mS/cm^2	[4]
$\bar{g}_{AHP}$	Ca^{2+}-activated K^+ conductance	0.1–1 mS/cm^2	[9, 10]

Table 1: Representative time constants and conductance parameters for Hodgkin–Huxley and motoneuron models. Values compiled from classical and modern sources.

where $x \in \{h, n, [Ca^{2+}]\}$. Note that in this formulation, the "memory" is not a change in Δ_{spike} itself, but rather the residual elevation of the intracellular Ca^{2+} concentration caused by incomplete decay of previous spike-induced increments.

Each spike adds a nearly constant increment Δ_{spike}, representing the integrated Ca^{2+} influx over the action potential, and this increment then decays exponentially with time constant τ_{Ca}. When successive spikes occur before full recovery, the overlapping exponential tails of prior increments carry the system's state forward to the next cycle. Only under conditions where the afterhyperpolarization (AHP) significantly alters the voltage trajectory—for instance, if a spike occurs during a deep AHP or under partial channel inactivation—does the effective Δ_{spike} depart from constancy, reflecting genuine state-dependent modulation of the underlying Ca^{2+} current [2, 9, 3].

Because the subsequent ISI T_{n+1} depends on this initial condition,

$$T_{n+1} \approx F\big(x_{n+1}(0)\big), \tag{5}$$

linearization yields an approximate autoregressive relation:

$$\begin{aligned} T_{n+1} &\approx a_0 + a_1 T_n + \varepsilon_n, \\ a_1 &\approx e^{\frac{-T}{\tau_{slow}}}, \end{aligned} \tag{6}$$

implying a lag-1 correlation

$$\boxed{\rho(\mathrm{ISI}_n, \mathrm{ISI}_{n+1}) \approx a_1 > 0, \quad T \ll \tau_{\mathrm{slow}}.} \tag{7}$$

This mechanism aligns with empirical work linking spike-frequency adaptation, Ca^{2+}-activated K^+ currents, and slow recovery of excitability [9, 11, 8, 12]. Incomplete recovery under high firing rates produces serial dependence which vanishes as recovery is (nearly) complete between spikes.

In the context of motor-unit firing near its physiological upper-rate limit, the fact that $\lambda\, t_0 \lesssim 1$ and multiple slow variables remain active implies tighter state coupling. Thus, one expects an elevated $\rho_{\mathrm{lag}=1}$ in the high-rate regime compared to low or moderate rates.

2 Physiological interpretation

At low rates (long ISIs), ionic concentrations, gating variables, and AHP currents return to baseline; spikes are effectively independent and renewal-like. As firing accelerates, intracellular calcium and Na^+/K^+-pump currents remain elevated between spikes, shifting reversal potentials and maintaining partial K^+ conductance [3, 6, 7]. The next spike therefore begins from a different membrane state, carrying forward residual ionic and conductance effects. Quantitatively, the correlation magnitude reflects the fraction of unrecovered state variables:

$$\rho_{\mathrm{lag}=1} \propto e^{-T/\tau_{\mathrm{slow}}},$$

which increases as the mean ISI shortens or the dominant recovery time constant lengthens (e.g., due to fatigue).

3 Implications for motor-unit firing

In spinal motor units, slow after-hyperpolarization and calcium-activated K^+ currents define the refractory envelope for sustained discharge [5, 7]. Near the steady-state firing rate (SSFR), the neuron's calcium and K^+ conductances do not fully equilibrate between spikes. The observed weakly positive ISI correlations ($\rho \approx 0.2$–0.4) in EMG data likely reflect this slow-memory regime rather than purely stochastic rate modulation. Fatigue-dependent

prolongation of calcium clearance or metabolic recovery would thus reduce k-stage equivalence (effective number of sub-processes) and increase observed ISI correlation, consistent with adaptation-related rate slowing [1, 10].

4 Summary

When the firing rate approaches the physiological limit dictated by the refractory period and slow ionic recovery, HH dynamics predict measurable ISI correlation. These correlations arise naturally from the exponential carryover of gating and ionic states (Equation 4), representing a short-term memory intrinsic to the motoneuron. Empirically, estimating lag-1 ISI correlation may therefore serve as a biomarker of fatigue or altered excitability in human motor-unit recordings.

When the mean ISI becomes shorter than the dominant recovery time constant ($T < \tau_{\mathrm{Ca}}$ or τ_{AHP}), residual ionic activation persists across multiple spikes, producing substantial overlap between successive Ca^{2+} transients. In this strong-overlap regime, lag-1 ISI correlations typically reach $\rho \approx 0.3$–0.6, consistent with high-rate or fatigued motoneuron discharge patterns.

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Appendix A. Derivation of the State-Carryover Relation

Here we show how Eq. (4) arises directly from the calcium and gating dynamics described in Eq. (3). For a slow recovery variable $x(t)$ (e.g., intracellular Ca^{2+}, Na^{+} inactivation, or an adaptation current) obeying

$$\dot{x} = -\frac{x - x_\infty}{\tau_x} + \Delta_x \sum_i \delta(t - t_i),$$

integration between spikes leads to an exponential decay toward x_∞, and instantaneous increments Δ_x at each spike. Explicitly, letting t_n denote the n-th spike time and $T_n = t_{n+1} - t_n$ the corresponding inter-spike interval,

$$x^+_{n+1} - x_\infty = (x^+_n - x_\infty) e^{\frac{-T_n}{\tau_x}} + \Delta_x.$$

Under (i) pre-spike evaluation or (ii) centering around the steady-state fixed point $x^\star = x_\infty + \frac{\Delta_x}{(1-e^{-T/\tau_x})}$ for approximately constant ISI T, this simplifies to the purely multiplicative carryover form

$$x_{n+1} - x_* = e^{\frac{-T_n}{\tau_x}} (x_n - x_*), \quad x_* \in \{x_\infty, x^\star\}.$$

A full, step-by-step derivation is provided in **Appendix A** for completeness.

A Appendix

A.1 From ODE to per-ISI mapping

Starting from Eq. (3),

$$\frac{dC}{dt} = -\frac{C - C_{\text{rest}}}{\tau_{\text{Ca}}} + \Delta_{\text{spike}} \sum_i \delta\big(t - t_i\big), \quad C \equiv [\text{Ca}^{2+}], \tag{8}$$

let t_n be the n-th spike time, and define C_n^+ the concentration immediately after spike n, with T_n defined as the time difference between spike $n+1$ and spike n:

$$C_n^+ \triangleq \lim_{\epsilon \downarrow 0} C(t_n + \epsilon) \tag{9}$$

$$T_n \triangleq t_{n+1} - t_n \tag{10}$$

Between spikes ($t \in (t_n, t_{n+1})$) there is no impulse term, so the linear ODE $\dot{C} = \frac{-(C - C_{\text{rest}})}{\tau_{\text{Ca}}}$ has the integrating-factor solution

$$C(t) = C_{\text{rest}} + \big(C_n^+ - C_{\text{rest}}\big)\, e^{\frac{-(t-t_n)}{\tau_{\text{Ca}}}}. \tag{11}$$

Evaluating just *before* the next spike at $t_{n+1}^- \equiv \lim_{\epsilon \downarrow 0}(t_{n+1} - \epsilon)$,

$$C_{n+1}^- = C_{\text{rest}} + \big(C_n^+ - C_{\text{rest}}\big)\, e^{\frac{-T_n}{\tau_{\text{Ca}}}}. \tag{12}$$

At t_{n+1} the impulse adds Δ_{spike} instantaneously:

$$C_{n+1}^+ \;=\; C_{n+1}^- + \Delta_{\text{spike}} = C_{\text{rest}} + \big(C_n^+ - C_{\text{rest}}\big)\, e^{\frac{-T_n}{\tau_{\text{Ca}}}} + \Delta_{\text{spike}}. \tag{13}$$

Subtracting C_{rest} gives the affine recurrence

$$\boxed{C_{n+1}^+ - C_{\text{rest}} = \big(C_n^+ - C_{\text{rest}}\big)\, e^{\frac{-T_n}{\tau_{\text{Ca}}}} + \Delta_{\text{spike}}.} \tag{14}$$

A.2 Centering on constant ISI

If ISIs are approximately constant $T_n \approx T$, Eq. (14) has a fixed point $C^\star$ satisfying

$$C^\star - C_{\text{rest}} = \big(C^\star - C_{\text{rest}}\big)\, e^{-\frac{T}{\tau_{\text{Ca}}}} + \Delta_{\text{spike}} \tag{15}$$

$$\Longrightarrow C^\star - C_{\text{rest}} = \frac{\Delta_{\text{spike}}}{1 - e^{\frac{-T}{\tau_{\text{Ca}}}}}. \tag{16}$$

Defining deviations $u_n \triangleq C_n^+ - C^\star$, the recurrence becomes a *purely multiplicative* (AR(1)) map:

$$u_{n+1} = u_n e^{\frac{-T_n}{\tau_{\mathrm{Ca}}}} \tag{17}$$

Thus, relative to the steady-state level $C^\star$, the carryover across ISIs is *exactly* exponential with factor $e^{\frac{-T_n}{\tau_{\mathrm{Ca}}}}$.

A.3 Alternative Simplifications

1. **Pre-spike tracking (no jump):** If one tracks the *pre-spike* value C_{n+1}^- instead of C_{n+1}^+, Eq. (12) already has the form

$$C_{n+1}^- - C_{\mathrm{rest}} = \left(C_n^+ - C_{\mathrm{rest}}\right) e^{\frac{-T_n}{\tau_{\mathrm{Ca}}}}, \tag{18}$$

   i.e., no Δ_{spike} term appears because the jump has not yet occurred.

2. **Centered post-spike tracking:** If one tracks post-spike values but *centers* by the fixed point $C^\star$, Eq. (17) gives

$$C_{n+1}^+ - C^\star = e^{\frac{-T_n}{\tau_{\mathrm{Ca}}}} \left(C_n^+ - C^\star\right), \tag{19}$$

   which matches the quoted "state carryover" relation with $x \equiv C$ and $x_\infty \equiv C^\star$.

A.4 Generalization

Any HH-style gating or slow state variable x that obeys $\dot{x} = \frac{\left(x_\infty - x\right)}{\tau_x} + \Delta_x \sum_i \delta(t - t_i)$, with solution between spikes $x(t) = x_\infty + (x_n^+ - x_\infty) e^{\frac{-(t-t_n)}{\tau_x}}$, leads to the exact per-ISI map

$$x_{n+1}^+ - x_\infty = \left(x_n^+ - x_\infty\right) e^{\frac{-T_n}{\tau_x}} + \Delta_x. \tag{20}$$

Under either (i) pre-spike evaluation or (ii) centering around the fixed point

$$x^\star = x_\infty + \frac{\Delta_x}{1 - e^{\frac{-T}{\tau_x}}} \tag{21}$$

for approximately constant ISI T, Eq. (20) reduces to the *pure carryover*

$$\boxed{x_{n+1} - x_* \;=\; e^{\frac{-T_n}{\tau_x}} \left(x_n - x_*\right)} \qquad x_* \in \{x_\infty,\, x^\star\} \tag{22}$$

Link to lag-1 ISI correlation. If the next ISI is a smooth function of the starting state, $T_{n+1} \approx F(x_{n+1})$, a first-order Taylor expansion around the operating point x_* gives

$$\begin{aligned} & T_{n+1} - \bar{T} \\ & \approx F'(x_*)\left(x_{n+1} - x_*\right) \\ & \approx F'(x_*)\, e^{\frac{-T_n}{\tau_x}}\left(x_n - x_*\right) \\ & \propto\ e^{\frac{-T_n}{\tau_x}}\,(T_n - \bar{T}), \end{aligned} \tag{23}$$

so the lag-1 correlation satisfies

$$\rho\big(T_{n+1}, T_n\big)\ \approx\ \mathbb{E}\big[e^{\frac{-T_n}{\tau_x}}\big]\ \approx\ e^{\frac{-\bar{T}}{\tau_x}} \quad \text{for small ISI variance,} \tag{24}$$

which is the quantitative expression used in the main text.