ADVANCED ENGINEERING MATH

ERWIN KREYSZIG

9th EDITION

CHAPTER 1 : FIRST-ORDER ODE

Chapter 1

example 2

$$y' = ky$$

$$\frac{dy}{y} = k\,dt$$

$$\ln y = kt + c$$

$$\boxed{y = e^{kt+c} = e^{kt}e^{c} = y_0 e^{kt}}$$

example 3

t_0 $\boxed{\text{100 lb salt in 1000 gal } H_2O}$

10 gal/min (each gal contains 5 lb salt) $\rightarrow$ $\boxed{y(t) \text{ salt}}$ $\rightarrow$ 10 gal/min ($10/1000 = 0.01\%$ of $y(t)$)

$$\frac{dy}{dt} = 50 - 0.01y = -0.01(y - 5000)$$

$$\frac{dy}{y-5000} = -0.01\,dt$$

$$\ln|y-5000| = -0.01t + C$$

$$y - 5000 = ce^{-0.01t}$$

$$\boxed{y = 5000 + ce^{-0.01t}}$$

example 3

$$\frac{dT}{dt} = k(T - T_A)$$

$$\frac{dT}{T-T_A} = k\,dt$$

$$\ln|T - T_A| = kt + C$$

$$T - T_A = ce^{kt}$$

$$\boxed{T = T_A + ce^{kt}}$$

example 4

$$\frac{dh}{dt} = -26.56\frac{A}{B}\sqrt{h}$$

$$\frac{dh}{\sqrt{h}} = -26.56\frac{A}{B}\,dt$$

$$2\sqrt{h} = -26.56\frac{A}{B}t + c$$

example 6

$$2xyy' = y^2 - x^2 \quad (1)$$

$$y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y} \quad (2)$$

Using $u = y/x$ (3) into 2

$$y' = \frac{u}{2} - \frac{1}{2u} = \frac{u^2-1}{2u} \quad (4)$$

From (3) $u = y/x \Rightarrow y = ux$

$$y' = u'x + u \quad (5)$$

Using (5) into

$$u'x + u = \frac{u^2-1}{2u}$$

$$u'x = \frac{u^2-1}{2u} - u = \frac{u^2-1-2u^2}{2u}$$

$$u'x = \frac{-u^2-1}{2u}$$

$$\int \frac{2u}{u^2+1}\,du = \int \frac{-1}{x}\,dx + C$$

$$\ln|u^2+1| = -\ln x + \ln c$$

$$u^2+1 = c/x$$

$$\frac{y^2+x^2}{x^2} = \frac{c}{x} \Rightarrow \boxed{x^2+y^2 = cx}$$

Problem set 1.3

2. $y' + (x+2)y^2 = 0$

$$y' = -(x+2)y^2$$

$$\int \frac{1}{y^2}\,dy = -\int (x+2)\,dx + c$$

$$-\frac{1}{y} = -\left(\frac{x^2}{2} + 2x\right) - c^*$$

$$\frac{1}{y} = \frac{x^2}{2} + 2x + c^* = \frac{x^2+4x+2c^*}{2}$$

$$y = \frac{2}{x^2+4x+2c^*}$$

3. $y' = 2\sec 2y$

$$\frac{y'}{\sec 2y} = 2$$

$$\int \frac{1}{\sec 2y}\,dy = \int 2\,dx$$

$$\frac{2}{2}\int \cos 2y\,dy = \int 2\,dx$$

$$\frac{\sin 2y}{2} = 2x + c$$

$$(\sin 2y)' = 2\cos 2y$$

$$\sin 2y = 4x + c^*$$

$$\boxed{y = \frac{\sin^{-1}(4x + c^*)}{2}}$$

4. $y' = (y + 9x)^2$

using $v = y + 9x$ (1)

$y = v - 9x$

$y' = v' - 9$ (2)

substituting (1) & (2) into the main eqn.

$v' - 9 = v^2$

$v' = v^2 + 9$

$$\int \frac{dv}{v^2+9} = \int dx$$

$$\int \frac{dv}{v^2+3^2} = \frac{1}{3}\arctan\left(\frac{v}{3}\right) = x + c$$

$v = 3\tan(3x + c^*)$

Using (1) $\boxed{y = 3\tan(3x + c^*) - 9x}$

5. $yy' + 3x = 0$

$yy' = -3x$

$\int y\,dy = -\int 3x\,dx$

$\frac{y^2}{2} = -\frac{3x^2}{2} + c$

$y^2 = -3x^2 + c^*$

$\boxed{y = \pm\sqrt{c^* - 3x^2}}$

6. $y' = \dfrac{4x^2 + y^2}{xy}$

$y' = \dfrac{4x}{y} + \dfrac{y}{x}$ (1)

Using

$u = y/x$ (2) into (2)

$y' = \dfrac{4}{u} + u$ (3)

Using (2)

$y = ux$ $\quad$ $y' = u'x + u$ (4)

Inserting (4) into (3)

$u'x + \cancel{u} = \dfrac{4}{u} + \cancel{u}$

$u'x = \dfrac{4}{u}$

$uu' = \dfrac{4}{x}$

$\int u\,du = \int \dfrac{4}{x}\,dx$

$$\frac{u^2}{2} = 4\ln|x| + c$$

$$\frac{u^2}{2} = 4\ln|x| + 4\ln|c^*|$$

$$= 4\ln|cx|$$

$$u^2 = 8\ln|cx| \quad 5$$

Inseting the value of u into 5

$$\frac{y^2}{x^2} = 8\ln|cx|$$

$$y^2 = 8x^2\ln|cx|$$

$$\boxed{y = \pm\sqrt{8x^2\ln|cx|}}$$

7. $y' \sin \pi x = y \cos \pi x$

$$\frac{y'}{y} = \frac{\cos \pi x}{\sin \pi x} = \frac{1}{\tan \pi x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{\tan \pi x}$$

$$\ln|y| = \frac{\ln|\sin(\pi x)|}{\pi} + C$$

$$\ln|y| = \ln|\overset{*}{C}\sin(\pi x))^{1/\pi}$$

$$\boxed{y = \overset{**}{C}\sin(\pi x)^{1/\pi}}$$

8. $xy' = \frac{1}{2}y^2 + y$

$$y' = \frac{1}{2}\frac{y^2}{x} + \frac{y}{x}$$

Define

$u = y/x$ (2)

$$y' = \frac{1}{2}uy + u \quad (3)$$

Using (2)

$y = ux \qquad y' = u'x + u$ (4)

Substituting (4) into (3)

$$u'x + \cancel{u} = \frac{1}{2}uy + \cancel{u}$$

$$u' = \frac{1}{2}u^2 \Rightarrow \int \frac{1}{u^2}\,du = \int \frac{1}{2}\,dx$$

$$-\frac{1}{u} = \frac{1}{2}x^2 + c = \frac{x^2 + c^*}{2}$$

$$-u = \frac{2}{x^2 + c^*}$$

$$u = \frac{-2}{x^2 + c^*} \quad (5)$$

Substituting (2) into (5)

$$\frac{y}{x} = \frac{-2}{x^2 + c^*} \Rightarrow \boxed{y = \frac{-2x}{x^2 + c^*}}$$

9. $y' e^{\pi x} = y^2 + 1$

$$\frac{y'}{y^2+1} = e^{-\pi x}$$

$$\int \frac{dy}{y^2+1} = \int e^{-\pi x}\,dx$$

$$\arctan y = -\frac{e^{-\pi x}}{\pi} + c$$

$$\boxed{y = \tan\left(c - \frac{e^{-\pi x}}{\pi}\right)}$$

10. $yy' + 4x = 0 \qquad y(0) = 3$

General soln.

$$yy' = -4x$$

$$\int y\,dy = -\int 4x\,dx$$

$$\frac{y^2}{2} = -4x + C$$

$$y^2 = -8x + c^*$$

$$y = \pm\sqrt{c - 8x}$$

Particular soln.

$$y(0) = 3$$

$$3 = \pm\sqrt{c}$$

$$c = 9$$

$$\boxed{y = \pm\sqrt{9 - 8x}}$$

11. $\frac{dr}{dt} = -2tr \qquad r(0) = r_0$

General soln.

$$\int \frac{dr}{r} = -\int 2t\,dt$$

$$\ln|r| = -2t + C$$

$$r = e^{-2t+C} = c^* e^{-2t}$$

Particular soln.

$$r(0) = r_0$$

$$r_0 = c^* e^{-2(0)} = c^*$$

$$\boxed{r = r_0 e^{-2t}}$$

12. $2xyy' = 3y^2 + x^2 \quad y(1) = 2$

$$2yy' = \frac{3y^2}{x} + x$$

$$2y' = \frac{3y}{x} + \frac{x}{y} \quad (1)$$

Define

$$u = y/x \quad (2)$$

where

$$y = ux$$
$$y' = u'x + u \quad (3)$$

Inserting (2) & (3) into (1)

$$2(u'x + u) = 3u + \frac{1}{u}$$

$$2u'x + 2u = 3u + \frac{1}{u}$$

$$2u'x = u + \frac{1}{u} = \frac{u^2 + 1}{u}$$

$$\int \frac{2u\,du}{u^2+1} = \int \frac{1}{x}\,dx$$

General soln.

$$\ln|u^2+1| = \ln|x| + c$$

$$= \ln|cx|$$

$$u^2+1 = cx \quad (4)$$

Use (5) in eq. (4)

$$\frac{y^2}{x^2} + 1 = cx$$

$$y^2 = (cx-1)x^2 = cx^3 - x^2$$

$$y = \pm x\sqrt{cx-1}$$

Particular soln.

$$y(1) = 2$$

$$2 = \pm 1\sqrt{c-1} = \pm\sqrt{c-1}$$

$$4 = c-1 \Rightarrow c = 5$$

$$\boxed{y = \pm x\sqrt{5x-1}}$$

13 $L\frac{dL}{dt} + RL = 0 \quad L(0) = L_0$

General soln.

$$\frac{dL}{dt} = \frac{-RL}{L}$$

$$\int \frac{dL}{L} = \int -\frac{R}{L}\,dt$$

$$\ln|L| = -\frac{R}{L}t + c$$

$$L = e^{-R/L\,t}e^{c} = c^{*}e^{-R/L\,t}$$

Particular soln.

$$L_0 = c^{*}e^{-R/L(0)} = c^{*}$$

$$\boxed{L = L_0 e^{-R/L\,t}}$$

14. $y' = y/x + \left(2x^3/y\right)\cos(x^2)$

$y\left(\sqrt{\pi/2}\right) = \sqrt{\pi}$

General soln.

Let $u = y/x$ ①

$y = ux$

$y' = u'x + u$ ②

$u'x + \cancel{u} = \cancel{u} + \frac{2x^2\cos(x^2)}{u}$

$u'x = \frac{2x^2}{u}\cos x^2$

$u' = \frac{2x}{u}\cos x^2$

$\int u\,du = \int 2x\cos x^2\,dx$

$\frac{u^2}{2} = \sin x^2 + c$

$$u^2 = 2\sin x^2 + c^*$$

$$u = \pm\sqrt{2\sin x^2 + c^*} \quad (3)$$

Inserting (1) into (3)

$$\frac{y}{x} = \pm\sqrt{2\sin x^2 + c^*}$$

$$y = \pm x\sqrt{2\sin x^2 + c^*}$$

Particular soln.

$$y\left(\sqrt{\pi/2}\right) = \sqrt{\pi}$$

$$\sqrt{\pi} = \pm\sqrt{\pi/2}\sqrt{2\sin \pi/2 + c^*}$$

$$= \pm\sqrt{\pi/2}\sqrt{2 + c^*}$$

$$\sqrt{2} = \pm\sqrt{2 + c^*}$$

$$2 = 2 + c \Rightarrow c = 0 \qquad \boxed{y = \pm x\sqrt{2\sin x^2}}$$

15. $e^{2x}y' = 2(x+2)y^3$

$y(0) = \frac{1}{\sqrt{5}}$

$$\int \frac{1}{y^3}\,dy = \int 2(x+2)e^{-2x}\,dx$$

$$= \int 2xe^{-2x}\,dx + \int 4e^{-2x}\,dx$$