1. Let A and B be finite sets, and let $f : A \to B$ be a function. Show that
a. if f is injective, then $|A| \leq |B|$.
b. if f is surjective, then $|A| \geq |B|$.

If f is injective, then each element of the domain maps to a unique element in the codomain. In other words, if $(a, b) \in f$, then we know there is no other $a' \in A$ where $a \neq a'$ such that $(a', b) \in f$. The function f pairs each element of A with exactly one element of B. Hence, we conclude that $|A| \leq |B|$.

If f is surjective, then the range of f is equal to its codomain. In other words, for every $b \in B$, we can find at least one $a \in A$ such that $(a, b) \in f$. The function f pairs each element of B with at least one element in A. Hence, we conclude that $|A| \geq |B|$.

2. Is the function $f(x) = x+1$ bijective when the domain and the codomain are $\mathbb{N}$? Is it bijective when the domain and the codomain are $\mathbb{Z}$?

f is not bijective when the domain and codomain are $\mathbb{N}$. In this case, f is injective, since $a \neq a' \Rightarrow f(a) \neq f(a')$, but f is not surjective. Specifically, as $\mathbb{N} = \{0, 1, 2, ...\}$, there is no $a \in \mathbb{N}$ such that $f(a) = 0$.

f is bijective when the domain and codomain are $\mathbb{Z}$. In this case, f is injective, since $a \neq a' \Rightarrow f(a) \neq f(a')$, and f is also surjective. Specifically $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, so if we are given an arbitrary $b \in \mathbb{Z}$, we can always choose $a = b - 1 \in \mathbb{Z}$ such that $(a, b) = (b - 1, b) \in f$.

3. Give a natural definition for the inverse of a binary relation such that if a relation is in fact a bijective function, its relational inverse is its functional inverse.

[working]

4. Give a bijection from $\mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$.

[working]