5. Prove the identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

for $0 < k \leq n$.

We specify $k > 0$ so that $\binom{n-1}{k-1}$ is not ill-defined.

On the left-hand side we have:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

On the right-hand side we have:

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$$

Multiplying this expression by $\frac{n}{k}$, we have:

$$\frac{n}{k}\binom{n-1}{k-1} = \frac{n(n-1)!}{k(k-1)!(n-k)!}$$

and since $n! = n(n-1)!$ and $k! = k(k-1)!$, we can write:

$$\frac{n}{k}\binom{n-1}{k-1} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

which was to be proved.

6. Prove the identity:

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

for $0 \leq k < n$.

We specify $0 < n - k$ so that $\frac{n}{n-k}$ is not ill-defined.

The expression on the right can be re-written as:

$$\frac{n}{n-k}\binom{n-1}{k} = \frac{n(n-1)!}{k!(n-k)(n-k-1)!}$$

$$\frac{n}{n-k}\binom{n-1}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

which was to be proved.

7. To choose k objects from n, you can make one of the objects distinguished and consider whether the distinguished object is chosen. Use this approach to prove that:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

If we select k objects from a set of n objects, the distinguished object will either be chosen, or it will not be chosen. If the object is not chosen, there are $\binom{n-1}{k}$ ways to select the k objects from the remaining set of $n-1$ objects. If the object is chosen, there are $\binom{n-1}{k-1}$ ways to select the remaining $k-1$ objects from the remaining set of $n-1$ objects. Adding these two numbers together, we arrive at the result.

The result can also be easily obtained analytically:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-k)(n-1)! + k(n-1)!}{k!(n-k)!}$$

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{n(n-1)!}{k!(n-k)!}$$

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{n!}{k!(n-k)!}$$

8. Using the result of Exercise 7, make a table for $n = 0, 1, ..., 6$ and $0 \leq k \leq n$ of the binomial coefficients $\binom{n}{k}$ with $\binom{0}{0}$ at the top, $\binom{1}{0}$ and $\binom{1}{1}$ on the next line, and so forth. Such a table of binomial coefficients is called ***Pascal's triangle***.

$$\binom{0}{0}$$

$$\binom{1}{0}\binom{1}{1}$$

$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$

$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$

$$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$$

$$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$$

$$\binom{6}{0}\binom{6}{1}\binom{6}{2}\binom{6}{3}\binom{6}{4}\binom{6}{5}\binom{6}{6}$$

Bearing in mind that $\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1$, we know that the "edges" of this triangle will be 1. Filling in the rest of the triangle using the expression $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$:

$$1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

9. Prove that:

$$\sum_{i=1}^{n} i = \binom{n+1}{2}$$

The result $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ can be established by induction.

For the case $n = 1$:

$$\sum_{i=1}^{1} i = 1$$

and

$$\frac{1}{2}(1)(2) = 1$$

which establishes the result.

Suppose next that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ holds for n, and consider:

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{1}{2}n(n+1) + (n+1)$$

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)+2(n+1)}{2}$$
$$\sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+2)$$

which establishes the result.

It remains to evaluate the right-hand side of the expression:

$$\binom{n+1}{2} = \frac{(n+1)!}{2!(n-1)!} = \frac{1}{2}n(n+1)$$

which was to be proved.

14. By differentiating the entropy function $H(\lambda)$, show that it achieves its maximum value at $\lambda = 1/2$. What is $H(1/2)$?

The entropy function is given by:

$$H(\lambda) = -\lambda \lg \lambda - (1-\lambda)\lg(1-\lambda)$$

which we can rewrite and differentiate as follows:

$$H(\lambda) = (-1)\left[\lambda \lg \lambda + (1-\lambda)\lg(1-\lambda)\right]$$
$$H'(\lambda) = (-1)\left[\left(\lg \lambda + \frac{1}{\ln 2}\right) + \left(-\lg(1-\lambda) - \frac{1}{\ln 2}\right)\right]$$
$$H'(\lambda) = -\lg\left(\frac{\lambda}{1-\lambda}\right) = \lg\left(\frac{1-\lambda}{\lambda}\right)$$

Setting this equal to zero to obtain the maximum, we have:

$$\lg\left(\frac{1-\lambda}{\lambda}\right) = 0$$
$$\frac{1-\lambda}{\lambda} = 1$$
$$1-\lambda = \lambda$$
$$1 = 2\lambda$$
$$\lambda = 1/2$$

The value of $H(1/2)$ is:

$$H(1/2) = -\frac{1}{2}\lg\left(\frac{1}{2}\right) - \left(1 - \frac{1}{2}\right)\lg\left(1 - \frac{1}{2}\right)$$

$$H(1/2) = -\frac{1}{2}\lg\left(\frac{1}{2}\right) - \frac{1}{2}\lg\left(\frac{1}{2}\right) = -\lg\left(\frac{1}{2}\right)$$

$$H(1/2) = 1$$

[add a plot to 14]
[working]