

1. Suppose we are comparing implementations of insertion sort and merge sort on the same machine. For inputs of size  $n$ , insertion sort runs in  $8n^2$  steps, while merge sort runs in  $64n \lg n$  steps. For which values of  $n$  does insertion sort beat merge sort? How might one rewrite the merge sort procedure to make it even faster on small inputs?

We wish to find  $n_0$  such that for  $n > n_0$ , the following inequality holds:

$$8n^2 < 64n \lg n$$

Defining  $f(n) = 64n \lg n - 8n^2$ , this is equivalent to finding values of  $n$  for which  $f(n) > 0$ . Newton's method is a useful numerical technique for calculating the roots of transcendental functions like this one. Starting with an initial guess  $x_0$ , the subsequent (hopefully better) guess is given by:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In this case,  $f(n) = 64n \lg n - 8n^2$  and  $f'(n) = 64 \lg n + 64 - 16n$ . The following Python script calculates a numerical approximation for the roots of this function using Newton's method:

```

1 import math
2
3 def newton(F, dF, x):
4     """ Return next iterative approximation. """
5     return x - F(x)/dF(x)
6
7 def F(x):
8     """ Value of 64x*lg(x) - 8x^2 """
9     return 64*x*math.log(x) - 8*math.pow(x,2)
10
11 def dF(x):
12     """ Value of derivative of 64x*lg(x) - 8x^2. """
13     return 64*math.log(x) + 64 - 16*x
14
15 def find_root(F, dF, initial_guess, tolerance):
16     """ Find root using Newton's method to specified tolerance,
17     using initial guess. """
18     def guess(x0):
19         x1 = newton(F, dF, x0)
20         diff = abs(x0-x1)
21         return (x1, diff)
22
23     # Iterate until we obtain a result within tolerance.
24     (x1, diff) = guess(initial_guess)
25     while (diff > tolerance):

```

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25         (x1, diff) = guess(x1)
26     return x1
27
28 # Find the root
29 print find_root(F, dF, 20, 0.01)

```

The script returns 26.0934 as an approximate root, which indeed is valid:

$$8 * 26 * 26 = 5,408 < 5,421.47 \approx 64 * 26 * \lg(26)$$

$$8 * 27 * 27 = 5,832 > 5,695.21 \approx 64 * 27 * \lg(27)$$

The answer to the question is that for inputs  $n < 27$ , insertion sort will run faster than merge sort.

[Working]

2. What is the smallest value of  $n$  such that an algorithm whose running time is  $100n^2$  runs faster than an algorithm whose running time is  $2^n$  on the same machine?

We wish to find  $n_0$  such that for  $n > n_0$  the following inequality holds:

$$100n^2 < 2^n$$

Defining  $f(n) = 2^n - 100n^2$ , this is equivalent to finding values of  $n$  for which  $f(n) > 0$ . Newton's method is a useful numerical technique for calculating the roots of transcendental functions like this one. Starting with an initial guess  $x_0$ , the subsequent (hopefully better) guess is given by:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In this case,  $f(n) = 2^n - 100n^2$  and  $f'(n) = (\ln 2) \cdot 2^n - 200n$ . The following Python script calculates a numerical approximation for the roots of this function using Newton's method:

```

1 import math
2
3 def newton(F, dF, x):
4     """ Return next iterative approximation. """
5     return x - F(x)/dF(x)
6
7 def F(x):
8     """ Value of 2^x - 100x^2. """
9     return math.pow(2,x) - 100*math.pow(x,2)
10

```

```

11 def dF(x):
12     """ Value of derivative of 2^x - 100x^2. """
13     return math.log(2)*math.pow(2,x) - 200*x
14
15 def find_root(F, dF, initial_guess, tolerance):
16     """ Find root using Newton's method. """
17     def guess(x0):
18         x1 = newton(F, dF, x0)
19         diff = abs(x0-x1)
20         return (x1, diff)
21
22     # Iterate until we obtain a result within tolerance
23     (x1, diff) = guess(initial_guess)
24     while (diff > tolerance):
25         (x1, diff) = guess(x1)
26     return x1
27
28 # Find the root with initial guess of 20
29 print find_root(F, dF, 20, 0.01)

```

The script returns 14.3247 as an approximate root, which indeed is valid:

$$100 * 14 * 14 = 19,600 > 16,384 = 2^{14}$$

$$100 * 15 * 15 = 22,500 < 32,768 = 2^{15}$$

The answer to the question is that for  $n = 15$  and higher the  $T(n) = 100n^2$  procedure will outperform the  $T(n) = 2^n$  procedure.

The problem illustrates that an exponential procedure grows much more rapidly than a quadratic function, and that even for  $n = 15$  a very bad quadratic procedure will greatly outperform an exponential one.