1. Show that $\sum_{k=1}^{n} 1/k^2$ is bounded above a constant.

$f(x) = 1/x^2$ is monotonically decreasing, so the upper bound is given by:

$$S(n) = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2}$$
$$\leq 1 + \int_{1}^{\infty} \frac{dx}{x^2} = 2$$

The summation is bounded above by 2.

This bound is comparatively loose, $n = 10\,000$ gives $S(n) \approx 1.644$.

2. Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil$$

Since $\lceil x \rceil < x + 1$, we can write:

$$S(n) = \sum_{k=0}^{\lfloor \lg n \rfloor} \lceil \frac{n}{2^k} \rceil < \sum_{k=0}^{\lfloor \lg n \rfloor} \left(\frac{n}{2^k} + 1 \right) \leq \sum_{k=0}^{\lg n} \left(\frac{n}{2^k} + 1 \right)$$

$$S(n) < n \sum_{k=0}^{\lg n} \left(\frac{1}{2} \right)^k + \sum_{k=0}^{\lg n} 1$$

The first summation in the expression on the left is a geometric series that can be simplified using $\sum_{k=0}^{n} x^k = (x^{n+1} - 1)/(x - 1)$, and which gives $(2^{\lg n+1} - 1)/2^{\lg n} = (2 \cdot 2^{\lg n} - 1)/2^{\lg n} = (2n-1)/n$, while the second summation simply evaluates to $\sum_{k=0}^{\lg n} 1 = \lg n + 1$. Hence:

$$S(n) < n \cdot \left(\frac{2n - 1}{n} \right) + \lg n + 1$$

$$S(n) < 2n + \lg n$$

The linear term n dominates $\lg n$, hence we can write $S(n) = O(n)$. Evaluating $S(n)$ for various values of n, you'll find that the sum adds up to (more or less) $2n$ pretty consistently.

3. Show that the nth harmonic number is $\Omega(\lg n)$ by splitting the summation.

The nth harmonic number is given by:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

We can split the summation as follows:

$$\sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \leq \sum_{k=1}^{n} \frac{1}{k}$$

$$\sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^{i+1}} \leq \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \leq \sum_{k=1}^{n} \frac{1}{k}$$

The leftmost expression can be further simplified as:

$$\sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^{i+1}} = \frac{1}{2} \cdot \sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^i}$$

Since $\sum_{j=0}^{2^i-1} 2^{-i} = 1$, we can write:

$$\sum_{i=0}^{\lfloor \lg n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{2^{i+1}} = \frac{1}{2} \cdot \sum_{i=0}^{\lfloor \lg n \rfloor - 1} (1) = \frac{1}{2} \lfloor \lg n \rfloor$$

So $\frac{1}{2}\lfloor \lg n \rfloor \leq H_n$, and hence $H_n = \Omega(\lfloor \lg n \rfloor)$.

Recall that we say $f(n) = \Omega(g(n))$ if there exist positive constants c and n_0 such that $0 \leq c \cdot g(n) \leq f(n)$ for all $n \geq n_0$. To complete the proof, bear in mind that $\lg n = \log_2 n$, so we have the identity:

$$\lg \frac{n}{2} = \lg n - 1 < \lfloor \lg n \rfloor$$

We seek $c > 0$ such that for sufficiently large n we have:

$$c \cdot \lg n \leq \lg \frac{n}{2} = \lg n - 1$$

Suppose we choose $n_0 = 4$, then:

$$c \cdot \lg n_0 \leq \lg n_0 - 1$$

$$2c \leq 1$$

$$c \leq \frac{1}{2}$$

So any choice of $c \leq 1/2$, for instance $c = 1/4$, will work. When we remember to include the factor of 1/2 from the first part of the proof (i.e., we would select, in this instance, c $= 1/2 \cdot 1/4$), we find that by selecting $c = 1/8$ and $n_0 = 4$, we have for all $n \geq n_0$:

$$0 \leq c \cdot \lg n < \frac{1}{2}\lfloor \lg n \rfloor \leq H_n$$

so we can write $H_n = \Omega(\lg n)$.

4. Approximate $\sum_{k=1}^{n} k^3$ with an integral.

$f(x) = x^3$ is monotonically increasing, so the bounds are given by:

$$\int_0^n x^3 dx \leq \sum_{k=1}^{n} k^3 \leq \int_1^{n+1} x^3 dx$$

$$\frac{1}{4}n^4 \leq \sum_{k=1}^{n} k^3 \leq \frac{1}{4}\left[(n+1)^4 - 1\right]$$

5. Why didn't we use the integral approximation (3.10) directly on $\sum_{k=1}^{n} 1/k$ to obtain an upper bound on the nth harmonic number?

The integral approximation (3.10), for monotonically decreasing functions, is given by:

$$\int_m^{n+1} f(x)dx = \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

In this case $f(k) = 1/k$, so evaluating the upper bound requires computing the value of the integral $\int_0^n dx/x$, which is undefined at the $x = 0$ boundary. To find the upper bound for H_n, we must avoid this boundary:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} \leq 1 + \int_1^n \frac{dx}{x} = 1 + \ln n$$