1. Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

By distribution:

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$ and $\sum_{k=1}^{n} a = na$, we can write:

$$= 2 \cdot \left(\frac{1}{2}\right) \cdot n\,(n+1) - n$$

$$= n^2 + n - n = n^2$$

Hence:

$$S(n) = \sum_{k=1}^{n}(2k-1) = n^2$$

2. Show that $\sum_{k=1}^{n} 1/\left(2k-1\right) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

We can define $K(n)$ as:

$$K(n) = \sum_{k=1}^{n} 1/\left(2k-1\right) = 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n-1}$$

so that the harmonic function

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} = \ln(n) + O(1)$$

can be expressed as:

$$H(2n) = \frac{1}{2}H(n) + K(n)$$

$$K(n) = H(2n) - \frac{1}{2}H(n)$$

Substituting the expression from above for $H(n)$ and $H(2n)$, we have:

$$K(n) = \ln(2n) + O(1) - \frac{1}{2}\left\{\ln(n) + O(1)\right\}$$

$$K(n) = \ln(2n) - \frac{1}{2}\ln(n) + \frac{1}{2} \cdot O(1)$$

Since $\ln(2n) = \ln(2) + \ln(n)$, we can write:

$$K(n) = \frac{1}{2}\ln(n) + \frac{1}{2} \cdot O(1) + \ln(2)$$

With asymptotic notation, the expression $\frac{1}{2} \cdot O(1)$ reduces to $O(1)$, and the constant $\ln(2)$ can be absorbed into the $O(1)$ term as well:

$$K(n) = \frac{1}{2}\log(n) + O(1)$$

$$K(n) = \log(\sqrt{n}) + O(1)$$

$$\sum_{k=1}^{n} \frac{1}{2k-1} = \log(\sqrt{n}) + O(1)$$

3. Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

Since $|\frac{k-1}{2^k}| < 1$ for $k > 1$, we can write:

$$S(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$S'(x) = \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$xS'(x) = \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Next we define $F(x)$ as:

$$F(x) = xS'(x) - S(x) = \frac{x}{(1-x)^2} - \frac{1}{1-x}$$

$$F(x) = \sum_{k=0}^{\infty} kx^k - \sum_{k=0}^{\infty} x^k = \frac{2x-1}{(1-x)^2}$$

$$F(x) = \sum_{k=0}^{\infty}(k-1)x^k = \frac{2x-1}{(1-x)^2}$$

We can now write:

$$F(1/2) = \sum_{k=0}^{\infty}(k-1)/2^k = \frac{2(1/2)-1}{1/4} = 0$$
$$\sum_{k=0}^{\infty}(k-1)/2^k = 0$$

4. Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

We can write:

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

Define $S(x)$ as:

$$S(x) = \sum_{k=1}^{\infty}(x^2)^k = \sum_{k=1}^{\infty} x^{2k} = \frac{x^2}{1-x^2}$$

so that

$$S'(x) = \sum_{k=1}^{\infty}(2k)x^{2k-1} = \frac{2x}{(1-x^2)^2}$$
$$xS'(x) = \sum_{k=1}^{\infty}(2k)x^{2k} = \frac{2x^2}{(1-x^2)^2}$$

Now define $F(x)$ as:

$$F(x) = xS'(x) + S(x) = \sum_{k=1}^{\infty}(2k)x^{2k} + \sum_{k=1}^{\infty} x^{2k} = \frac{2x^2}{(1-x^2)^2} + \frac{x^2}{1-x^2}$$
$$F(x) = \sum_{k=1}^{\infty}(2k+1)x^{2k} = \frac{3x^2-x^4}{(1-x^2)^2}$$

5. Use the linearity property of summations to prove that:

$$\sum_{k=1}^{n} O(f_k(m)) = O\left(\sum_{k=1}^{n} f_k(m)\right)$$

By definition, a function $h(m)$ is $O(f(m))$, written $h(m) = O(f(m))$, if we can find positive constants c and m_0 such that $0 \leq h(m) \leq c \cdot f(m)$ of all $m \geq m_0$.

We can therefore expand $\sum_{k=1}^{n} O(f_k(m))$ as follows:

$$\sum_{k=1}^{n} O\left(f_k(m)\right) = O(f_1(m)) + O(f_2(m)) + ... + O(f_n(m))$$

$$= h_1(m) + h_2(m) + ... + h_n(m)$$

where

$$h_k(m) = O(f_k(m))$$

For each $h_k(m)$ we can, by definition, find positive constants c_k and m_k such that $h_k(m) \leq c_k \cdot f_k(m)$ for all $m \geq m_k$.

Let $m_0 = \max\left(m_k\right)$, and define $h(m)$ as:

$$h(m) = \sum_{k=1}^{n} O\left(f_k(m)\right)$$

Provided that $m \geq m_0$, we can now write:

$$h(m) \leq \sum_{k=1}^{n} c_k \cdot f_k(m)$$

Let $c_0 = \max(c_k)$, so that we can write:

$$h(m) \leq \sum_{k=1}^{n} c_k \cdot f_k(m) \leq \sum_{k=1}^{n} c_0 \cdot f_k(m)$$

By the linearity property:

$$h(m) \leq c_0 \cdot \sum_{k=1}^{n} f_k(m)$$

By construction, we have identified positive constants c_0 and m_0 such that the expression above holds true.

Consequently, we can write:

$$h(m) = O\left(\sum_{k=1}^{n} f_k(m)\right)$$

which was to be proved.

6. Prove that:

$$\sum_{k=1}^{\infty} \Omega(f_k(m)) = \Omega\left(\sum_{k=1}^{\infty} f_k(m)\right)$$

We presume that the limit exists, otherwise the problem is ill-posed.

By definition, a function $h(m)$ can be expressed as $h(m) = \Omega(f(m))$ if we can find positive constants c and m_0 such that $0 \leq c \cdot f(m) \leq h(m)$ for all $m \geq m_0$.

We can therefore expand $\sum_{k=1}^{n} \Omega(f_k(m))$ as follows:

$$\sum_{k=1}^{n} \Omega(f_k(m)) = \Omega(f_1(m)) + \Omega(f_2(m)) + ... + \Omega(f_n(m))$$

$$= h_1(m) + h_2(m) + ... + h_n(m)$$

For each $h_k(m)$ we can, by definition, find positive constants c_k and m_k such that $c_k \cdot f_k(m) \leq h_k(m)$ for all $m \geq m_k$.

Let $m_0 = \max(m_k)$, and define $h(m)$ as:

$$h(m) = \lim_{n\to\infty} \sum_{k=1}^{n} \Omega\left(f_k(m)\right) = \sum_{k=1}^{\infty} \Omega\left(f_k(m)\right)$$

Provided that $m \geq m_0$, we can now write:

$$\lim_{n\to\infty} \sum_{k=1}^{n} c_k \cdot f_k(m) \leq h(m)$$

Let $c_0 = \min(c_k)$, so that we can write:

$$\lim_{n\to\infty} \sum_{k=1}^{n} c_0 \cdot f_k(m) \leq \lim_{n\to\infty} \sum_{k=1}^{n} c_k \cdot f_k(m) \leq h(m)$$

By the linearity property:

$$c_0 \cdot \lim_{n\to\infty} \sum_{k=1}^{n} f_k(m) = c_0 \cdot \sum_{k=1}^{\infty} f_k(m) \leq h(m) = \sum_{k=1}^{\infty} \Omega(f_k(m))$$

By construction, we have identified positive constants c_0 and m_0 such that the expression above holds true.

Consequently, we can write:

$$\sum_{k=1}^{\infty} \Omega(f_k(m)) = \Omega\left(\sum_{k=1}^{\infty} f_k(m)\right)$$

which was to be proven.

7. Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

We can factor out the 2, which will contribute 2^n to the final product. Define $P(n)$ as:

$$P(n) = \prod_{k=1}^{n} 4^k = \prod_{k=1}^{n} 2^{2k}$$

$$\lg P(n) = \sum_{k=1}^{n} \lg 2^{2k} = \sum_{k=1}^{n} 2k$$

$$\lg P(n) = n(n+1)$$

$$P(n) = 2^{n(n+1)}$$

Factoring back in 2^n term, we are left with:

$$\prod_{k=1}^{n} 2 \cdot 4^k = 2^{n(n+2)}$$

8. Evaluate the product $\prod_{k=2}^{n} (1 - 1/k^2)$.

Define $P(n)$ as:

$$P(n) = \prod_{k=2}^{n} \left(1 - 1/k^2\right) = \prod_{k=2}^{n} \left(\frac{k^2 - 1}{k^2}\right) = \prod_{k=2}^{n} \left\{\frac{(k+1)(k-1)}{k^2}\right\}$$

$$\lg P(n) = \sum_{k=2}^{n} \{\lg(k+1) + \lg(k-1) - 2\lg k\}$$

$$= \sum_{k=2}^{n} \{\lg(k+1) - \lg k\} + \sum_{k=2}^{n} \{\lg(k-1) - \lg k\}$$

We evaluate each of these two terms separately:

$$
\begin{aligned}
S_1(n) &= \sum_{k=2}^{n} \{\lg(k+1) - \lg k\} \\
&= -\sum_{k=2}^{n} \{\lg k - \lg(k+1)\} \\
&= -\{\ln 2 - \ln(n+1)\} \\
&= \ln\left(\frac{n+1}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
S_2(n) &= \sum_{k=2}^{n} \{\lg(k-1) - \lg k\} \\
&= -\sum_{k=2}^{n} \{\lg k - \lg(k-1)\} \\
&= -\{\ln n - \ln 1\} \\
&= -\ln n
\end{aligned}
$$

so that

$$
\lg P(n) = S_1(n) + S_2(n) = \ln\left(\frac{n+1}{2}\right) - \ln n
$$

$$
\lg P(n) = \ln\left(\frac{n+1}{2n}\right)
$$

$$
P(n) = \frac{n+1}{2n}
$$

so that

$$
\prod_{k=2}^{n} \left(1 - 1/k^2\right) = \frac{n+1}{2n}
$$