1. Prove that the subset relation $\subseteq$ on all subsets of $\mathbb{Z}$ is a partial order but not a total order.

The subset relation $\subseteq$ on all subsets of $\mathbb{Z}$ is defined such that for any $A, B \subset \mathbb{Z}$, we have $A \subseteq B$ if and only if $x \in A \Rightarrow x \in B$.

A partial order is a binary relation R that is (a) reflexive; (b) anti-symmetric; and (c) transitive:

1. ***Reflexive:*** A binary relation $R \subseteq A \times A$ is *reflexive* if $a\,R\,a$ for all $a \in A$. To show that the subset relation $\subseteq$ is reflexive, we must show that for all $A \subset \mathbb{Z}$, we have $A \subseteq A$. This is clearly the case since $x \in A \Rightarrow x \in A$.

2. ***Anti-Symmetric:*** A binary relation $R \subseteq A \times A$ is *anti-symmetric* if $a\,R\,b$ and $b\,R\,a$ imply $a = b$. The subset relation $\subseteq$ is anti-symmetric as per the definition of set equality. We say that two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$. Thus the definition of set equality coincides with the definition of anti-symmetry in this case. So clearly, if we have $A \subseteq B$ and $B \subseteq A$ for $A, B \subset \mathbb{Z}$, we must have $A = B$ as per the definition of set equality.

   More formally, $A \subseteq B$ implies that $x \in A \Rightarrow x \in B$, and likewise $B \subseteq A$ implies that $x \in B \Rightarrow x \in A$. Hence, if $A \subseteq B$ and $B \subseteq A$ with $A, B \subset \mathbb{Z}$, then $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$ and so the two sets contain the same elements, and we say that $A = B$.

3. ***Transitive:*** A binary relation $R \subseteq A \times A$ is *transitive* if $a\,R\,b$ and $b\,R\,c$ imply $a\,R\,c$ for all $a, b, c \in A$. Suppose we know that $A \subseteq B$ and $B \subseteq C$ with $A, B, C, \subset \mathbb{Z}$, so that $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in C$. Hence if $x \in A$ we can conclude that $x \in C$, and we can write $A \subseteq C$, so the subset relation $\subseteq$ is transitive. This establishes the subset relation $\subseteq$ on all subsets of $\mathbb{Z}$ as a partial order.

The partial order $\subseteq$ on all subsets of $\mathbb{Z}$ would be a total order if for every $A, B \subset \mathbb{Z}$, we have $A \subseteq B$ or $B \subseteq A$. Clearly this is not the case since we could choose $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, and clearly both $A \not\subseteq B$ and $B \not\subseteq A$. Consequently, the subset relation $\subseteq$ on all subsets $\mathbb{Z}$ is not a total order.

2. Show that for any positive integer n, the relation "equivalent modulo n" is an equivalence relation on the integers. (We say that $a \equiv b \mod n$ if there exists an integer q such that $a - b = qn$). Into what equivalence classes does this relation partition the integers?

We have the relation $R = \{(a,b) : a, b \in \mathbb{Z} \text{ and } a - b = qn \text{ where } q, n \in \mathbb{Z}\}$. To show that R is an equivalence relation we must show that it is (a) reflexive; (b) symmetric; and (c) transitive:

1. ***Reflexive:*** A binary relation $R \subseteq A \times A$ is *reflexive* if $a\,R\,a$ for all $a \in R$. Clearly, (a, a) since $a - a = 0$ and $0 \in \mathbb{Z}$, so R is reflexive.

2. ***Symmetric:*** A binary relation $R \subseteq A \times A$ is *symmetric* if $a\,R\,b$ implies $b\,R\,a$ for all $a, b \in A$. Suppose that (a, b). Then we can write $a - b = qn$ where $q, n \in \mathbb{Z}$. Multiplying this expression by -1, we have $b - a = (-q)n$, with $-q \in \mathbb{Z}$. Hence (b, a), and R is symmetric.

3. ***Transitive:*** A binary relation $R \subseteq A \times A$ is *transitive* if $a\,R\,b$ and $b\,R\,c$ imply $a\,R\,c$. Suppose that we have (a, b) and (b, c). Then we can write $a - b = qn$ and $b - c = rn$ for $q, r, n \in \mathbb{Z}$. Combining these two expressions we get $a - c = (q + r)n$, with $q + r \in \mathbb{Z}$. Hence (a, b) and (b, c) imply (a, c), and R is transitive.

For any $a \in \mathbb{Z}$, the equivalence class $[a] = \{b \in \mathbb{Z} : a\,R\,b\}$ will look like $[a] = \{..., a - 2n, a - n, a, a + n, a + 2n, ...\}$, with the set $\mathbb{Z}$ being partitioned into n equivalence classes.

For example, choosing $n = 3$, we have the following equivalence classes:

$$[0] = \{..., -6, -3, 0, 3, 6, ...\}$$
$$[1] = \{..., -5, -2, 1, 4, 7, ...\}$$
$$[2] = \{..., -4, -1, 2, 5, 8, ...\}$$

3. Give examples of relations that are

   (a) reflexive and symmetric but not transitive;

   (b) reflexive and transitive but not symmetric;

   (c) symmetric and transitive but not reflexive;

(a) reflexive and symmetric but not transitive;

The relation R on the reals $\mathbb{R}$ defined by $R = \{(x, y) : |x - y| < 2\}$ is reflexive and symmetric, but not transitive:

(a) Reflexive: For any $x \in \mathbb{R}$, we have (x, x) since $|x - x| = 0 < 2$.

(b) Symmetric: For any $x, y \in \mathbb{R}$, (x, y) implies that $|x-y| < 2$. Since $|x-y| = |y-x|$, we have that $|x-y| < 2$ implies $|y-x| < 2$. We conclude that (x, y) implies (y, x) and that R is symmetric.

(c) Transitive: Consider that $(9, 8) \in R$ since $|9-8| = 1 < 2$, and that $(8, 7) \in R$ since $|8-7| = 1 < 2$. However, $(9, 7) \notin R$ since $|9-7| = 2 \not< 2$. Hence (a, b) and (b, c) does not imply that (a, c), and consequently R is not transitive.

(b) reflexive and transitive but not symmetric;

The relation $\leq$ on the integers $\mathbb{Z}$ is reflexive and transitive, but not symmetric:

(a) Reflexive: For any integer $a \in \mathbb{Z}$, we have $a \leq a$.

(b) Symmetric: For any $a, b \in \mathbb{Z}$, $a \leq b$ does not necessarily imply that $b \leq a$. For example, $1 \leq 3$, but $3 \not\leq 1$.

(c) Transitive: For any $a, b, c \in \mathbb{Z}$, if $a \leq b$ and $b \leq c$ then $a \leq c$.

(c) symmetric and transitive but not reflexive;

The relation R on the integers $\mathbb{Z}$ defined by $R = \{(1,1), (1,2), (2,1), (2,2)\}$ is symmetric and transitive, but not reflexive.

(a) Reflexive: A binary relation $R \subseteq A \times A$ is reflexive if $a\,R\,a$ for all $a \in A$. In this case, R is not reflexive since $3 \in \mathbb{Z}$ but $(3, 3) \notin R$.

(b) Symmetric: A binary relation $R \subseteq A \times A$ is symmetric if $a\,R\,b$ implies $b\,R\,a$ for all $a, b \in A$. Consider the following cases:

i. $(1,1) \Rightarrow (1,1)$ since $(1,1) \in R$;
ii. $(1,2) \Rightarrow (2,1)$ since $(2,1) \in R$;
iii. $(2,1) \Rightarrow (1,2)$ since $(1,2) \in R$;
iv. $(2,2) \Rightarrow (2,2)$ since $(2,2) \in R$.

There is no other ordered pair (a, b) with $a, b \in \mathbb{Z}$ where $(a, b) \in R$. Based on the four cases enumerated above, we conclude that R is symmetric.

(c) Transitive: A binary relation $R \subseteq A \times A$ is transitive if $a\,R\,b$ and $b\,R\,c$ imply $a\,R\,c$ for all $a, b, c \in A$. Consider the following cases:

i. $(1,1) \wedge (1,1) \Rightarrow (1,1)$ since $(1,1) \in R$;
ii. $(1,1) \wedge (1,2) \Rightarrow (1,2)$ since $(1,2) \in R$;
iii. $(1,2) \wedge (2,1) \Rightarrow (1,1)$ since $(1,1) \in R$;

iv. $(1,2) \wedge (2,2) \Rightarrow (1,2)$ since $(1,2) \in R$;

v. $(2,1) \wedge (1,1) \Rightarrow (2,1)$ since $(2,1) \in R$;

vi. $(2,1) \wedge (1,2) \Rightarrow (2,2)$ since $(2,2) \in R$;

vii. $(2,2) \wedge (2,1) \Rightarrow (2,1)$ since $(2,1) \in R$;

viii. $(2,2) \wedge (2,2) \Rightarrow (2,2)$ since $(2,2) \in R$.

There is no other combination of ordered pairs $(a,b),(b,c)$ with $a,b,c \in \mathbb{Z}$ where $(a,b),(b,c) \in R$. Based on the eight cases enumerated above, we conclude that R is transitive.

4. Let S be a finite set, and let R be an equivalence relation on $S \times S$. Show that if in addition R is antisymmetric, then the equivalence classes of S with respect to R are singletons.

From the definition of equivalence class, we have $[a] = \{b \in S : a\,R\,b\}$. Because R is an equivalence relation, we know that R is reflexive, symmetric and transitive. We proceed as follows:

1. Since R is reflexive, we have $a \in [a]$;
2. Suppose further that $b \in [a]$, so that $a\,R\,b$;
3. Since R is symmetric, we have $a\,R\,b \Rightarrow b\,R\,a$.

Since $a\,R\,b$ and $b\,R\,a$, and since R is anti-symmetric, we conclude that $a = b$. Hence, the equivalence classes of S with respect to R are singletons.

5. Professor Narcissus claims that if a relation R is symmetric and transitive, then it is also reflexive. He offers the following proof. By symmetry, $a\,R\,b$ implies $b\,R\,a$. Transitivity, therefore, implies $a\,R\,a$. Is the professor correct?

No, the professor is not correct. In Exercise 5.3(c) above, we have given an example of a relation that is symmetric and transitive, but not reflexive.

The error is that in order for a relation to be reflexive, we must have $a\,R\,a$ for *all* $a \in A$, while in order for it to be symmetric, we need not have $a\,R\,b$ for *every* pair of elements $a,b \in A$. Rather, symmetry implies that *if* $a\,R\,b$, *then* we have $b\,R\,a$, but there may be numerous ordered pairs (a,b) for which $(a,b) \notin R$ even though R is symmetric.

As an example, the $=$ relation on the integers $\mathbb{Z}$ is an equivalence relation, so we know that it is reflexive, symmetric and transitive. Because it is reflexive, we have $a = a$ for *all* $a \in \mathbb{Z}$. Moreover, since it is symmetric, we know that *if* we have $a = b$ for $a, b \in \mathbb{Z}$, *then* it necessarily follows that $b = a$. However, there are numerous ordered pairs (a, b) with $a, b \in \mathbb{Z}$ for which $(a, b) \notin R$, for example $(3, 5)$ since $3 \neq 5$.

The Professor's error lies in assuming that just because he found an ordered pair $(a, b) \in R$ for which symmetry holds, that this implies that reflexivity must hold for *all* $a \in A$, which is not necessarily the case.