第五章 习题解答

1. $E(X) = -1\times 0.1 + 0\times 0.3 + 1\times 0.4 + 2\times 0.2 = 0.7$

$E(X^2) = (-1)^2\times 0.1 + 0^2\times 0.3 + 1^2\times 0.4 + 2^2\times 0.2 = 1.3$

$E(X-1)^2 = (-1-1)^2\times 0.1 + (0-1)^2\times 0.3 + (1-1)^2\times 0.4 + (2-1)^2\times 0.2 = 0.9$

2. $P\{X=k\} = (0.2)^{k-1}\times 0.8 \quad k=1,2,\cdots$ ← 分布律

$\therefore E(X) = \sum_{k=1}^{\infty} k\times(0.2)^{k-1}\times 0.8 = 0.8\sum_{k=1}^{\infty} k\times(0.2)^{k-1} = 0.8\times\frac{1}{(1-0.2)^2} = 1.25$

$\sum_{k=1}^{\infty} kx^{k-1} = \left(\sum_{k=1}^{\infty} x^k\right)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$

3. 设 $X_i = \begin{cases} 1 & A\text{在第}i\text{次试验中发生} \\ 0 & A\text{在第}i\text{次试验中不发生} \end{cases}$ 则 X_i 的分布律为

X_i	1	0
p_k	p	$1-p$

$\therefore E(X_i) = p \quad i=1,2,\cdots$

~~$X = X_1+E$~~ $X = X_1+X_2+\cdots+X_n \Rightarrow E(X) = E(X_1)+\cdots+E(X_n) = np$.

或利用 $X\sim B(n,p)$. $E(X) = np$.

4. X的分布律为

X	1	2	3	4
p	$\frac{37}{64}$	$\frac{19}{64}$	$\frac{7}{64}$	$\frac{1}{64}$

$P\{X=1\} = P\{1\text{号盒子有球}\} = \frac{C_3^1 3^2 + C_3^2\cdot 3 + C_3^3}{4^3} = \frac{37}{64}$

其余情形类似可求

$\therefore E(X) = \frac{37}{64}\times 1 + \frac{19}{64}\times 2 + \frac{7}{64}\times 3 + \frac{1}{64}\times 4 = \frac{25}{16}$

5. 随机变量X的分布律为

X	1	2	3	4
p	$\frac{3}{32}$	$\frac{45}{64}$	$\frac{3}{16}$	$\frac{1}{64}$

$P\{X=1\} = P\{\text{每个盒子内只有1球}\} = \frac{A_4^4}{4^4} = \frac{3}{32}$

$P\{X=3\} = P\{\text{某盒内有三球，另一球随意}\} = \frac{C_4^1 C_4^3\cdot C_3^1}{4^4}$

$P\{X=4\} = P\{4\text{球在同一盒子}\} = \frac{C_4^1}{4^4} = \frac{1}{64}$

$\therefore E(X) = 1\times\frac{3}{32} + 2\times\frac{45}{64} + 3\times\frac{3}{16} + 4\times\frac{1}{64}$

$= \frac{68}{32} = \frac{17}{8}$

6. (1) $P\{X=k\}=C_{k-1}^{1}p\cdot(1-p)^{k-2}\cdot p=(k-1)p^2(1-p)^{k-2},\quad k=2,3,\cdots$

(前k-1次中有1次成功，第k次试验成功.)

(2) $P\{X=2n\}=(2n-1)p^2(1-p)^{2n-2},\quad n=1,2,\cdots$

$P\{恰好偶数\}=\sum_{n=1}^{\infty}P\{X=2n\}=\sum_{n=1}^{\infty}(2n-1)p^2(1-p)^{2n-2}=p^2\sum_{n=1}^{\infty}(2n-1)(1-p)^{2n-2}$

$\sum_{n=1}^{\infty}(2n-1)x^{2n-2}=\left(\sum_{n=1}^{\infty}x^{2n-1}\right)'=\left(\frac{x}{1-x^2}\right)'=\frac{1+x^2}{(1-x^2)^2}\ \underline{\underline{x=1-p}}\ =\frac{1+(1-p)^2}{(2p-p^2)^2}p^2=\frac{1+(1-p)^2}{(2-p)^2}$

(3) $E(X)=\sum_{k=2}^{\infty}k\cdot(k-1)p^2(1-p)^{k-2}=p^2\sum_{k=2}^{\infty}k(k-1)(1-p)^{k-2}$

$\sum_{k=2}^{\infty}k(k-1)x^{k-2}=\left(\sum_{k=2}^{\infty}x^k\right)''=\left(\frac{x^2}{1-x}\right)''=\left(-x+1+\frac{1}{1-x}\right)''=\frac{2}{(1-x)^3}$

$\therefore E(X)=p^2\cdot\frac{2}{p^3}=\frac{2}{p}$

7. 设 $X_i=\begin{cases}1 & 第i个孩子得到铅笔\\ 0 & 未得到\end{cases}$ X为得到铅笔的孩子的总数，则 $X=\sum_{i=1}^{80}X_i$

$P\{X_i=0\}=\frac{(80-1)^{100}}{80^{100}}=\left(\frac{79}{80}\right)^{100}\approx0.2843$. $P\{X_i=1\}=0.7157\Rightarrow E(X_i)=0.7157$.

$\therefore E(X)=80\times0.7157=57.256$ 即平均有57个孩子得到铅笔.

8. 由 $\sum_{n=0}^{\infty}P\{X=n\}=\sum_{n=0}^{\infty}\frac{ab^n}{n!}=a\sum_{n=0}^{\infty}\frac{b^n}{n!}=ae^b=1$

$E(X)=\sum_{n=0}^{\infty}n\cdot\frac{ab^n}{n!}=a\sum_{n=0}^{\infty}\frac{b^n}{(n-1)!}=ab\sum_{n=0}^{\infty}\frac{b^{n-1}}{(n-1)!}=abe^b=\mu$

$\therefore b=\mu,\ a=e^{-\mu}$

9. $\because \sum_{k=1}^{\infty}(-1)^k\frac{2^k}{k}\cdot\frac{1}{2^k}=\sum_{k=1}^{\infty}(-1)^k\frac{1}{k}$ 是条件收敛的.

不满足定义中绝对收敛的要求 $\therefore$ E(X)不存在.

10. (1) $E(Y)=E(2X)=\int_{-\infty}^{+\infty}2xf(x)dx=2\int_0^2x\cdot\frac{x}{2}dx=\frac{8}{3}$

(2) $E(Y)=E(X^2+1)=\int_0^2(x^2+1)\frac{x}{2}dx=\left[\frac{x^4}{8}+\frac{x^2}{4}\right]_0^2=3$

11. 设球的直径为d 则 $f(d)=\begin{cases}\frac{1}{b-a} & a\le d\le b\\ 0 & 其他\end{cases}$

$V=\frac{4}{3}\pi\left(\frac{d}{2}\right)^3=\frac{1}{6}\pi d^3$

$E(V)=\int_a^b\frac{1}{6}\pi d^3\frac{1}{b-a}dd=\frac{\pi}{6}\frac{1}{b-a}\frac{d^4}{4}\Big|_a^b$

$=\frac{\pi}{24}\frac{b^4-a^4}{b-a}=\frac{\pi(a^2+b^2)(a+b)}{24}$

12. $E(X_1)=\int_{-\infty}^{+\infty}xf_1(x)=\int_0^{+\infty}x\cdot 2e^{-2x}dx=-\int_0^{+\infty}x\,de^{-2x}$

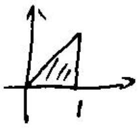
$=-\left[xe^{-2x}\right]_0^{+\infty}+\int_0^{+\infty}e^{-2x}dx=0+\left[-\frac{1}{2}e^{-2x}\right]_0^{+\infty}=\frac{1}{2}$

同理 $E(X_2)=\frac{1}{4}$ $E(X_2^2)=\frac{1}{8}$ （指数分布）

$\therefore E(X_1+X_2)=E(X_1)+E(X_2)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

$E(2X_1-3X_2^2)=2E(X_1)-3E(X_2^2)=2\cdot\frac{1}{2}-3\cdot\frac{1}{8}=\frac{5}{8}$

13. $E(X+Y)=\iint_D(x+y)f(x,y)\,dxdy=\int_0^1dx\int_0^x(x+y)2\,dy$



$=\int_0^1(2x^2+x^2)dx=\frac{2}{3}+\frac{1}{3}=1$

$E(XY)=\iint_D xyf(x,y)dxdy=\int_0^1dx\int_0^x 2xy\,dy=\int_0^1x^3dx=\frac{1}{4}$

14. 设 $X_i=\begin{cases}1 & \text{第}i\text{只球放入第}i\text{只盒子}\\ 0 & \text{未放入}\end{cases}$ 则 $P\{X_i=1\}=\frac{1}{n}$

$\therefore E(X_i)=1\cdot\frac{1}{n}+0\cdot P\{X=0\}=\frac{1}{n}$

X为配对数 则 $X=X_1+X_2+\cdots+X_n$ $E(X)=n\cdot\frac{1}{n}=1$

15. $E(X)=\int_{-\infty}^{+\infty}xf(x)dx=\frac{1}{2\lambda}\left[\int_{-\infty}^{\mu}xe^{-\frac{\mu-x}{\lambda}}dx+\int_{\mu}^{+\infty}xe^{-\frac{x-\mu}{\lambda}}dx\right]$

$=\frac{1}{2\lambda}e^{-\frac{\mu}{\lambda}}\int_{-\infty}^{\mu}xe^{\frac{x}{\lambda}}dx+\frac{1}{2\lambda}e^{\frac{\mu}{\lambda}}\int_{\mu}^{+\infty}xe^{-\frac{x}{\lambda}}dx$

$=\frac{1}{2}e^{-\frac{\mu}{\lambda}}\int_{-\infty}^{\mu}x\,de^{\frac{x}{\lambda}}-\frac{1}{2}e^{\frac{\mu}{\lambda}}\int_{\mu}^{+\infty}x\,de^{-\frac{x}{\lambda}}$

$=\frac{1}{2}e^{-\frac{\mu}{\lambda}}\left(xe^{\frac{x}{\lambda}}\Big|_{-\infty}^{\mu}-\int_{-\infty}^{\mu}e^{\frac{x}{\lambda}}dx\right)-\frac{1}{2}e^{\frac{\mu}{\lambda}}\left(xe^{-\frac{x}{\lambda}}\Big|_{\mu}^{+\infty}-\int_{\mu}^{+\infty}e^{-\frac{x}{\lambda}}dx\right)$

$=\frac{1}{2}e^{-\frac{\mu}{\lambda}}\left(\mu e^{\frac{\mu}{\lambda}}-\lambda e^{\frac{x}{\lambda}}\Big|_{-\infty}^{\mu}\right)-\frac{1}{2}e^{\frac{\mu}{\lambda}}\left(-\mu e^{-\frac{\mu}{\lambda}}+\lambda e^{-\frac{x}{\lambda}}\Big|_{\mu}^{+\infty}\right)$

$=\frac{1}{2}(\mu-\lambda)-\frac{1}{2}(-\mu-\lambda)=\mu$

由 $E(X)$ 的计算过程

$$\boxed{\begin{aligned}\int_{-\infty}^{\mu}xe^{\frac{x}{\lambda}}dx&=\lambda e^{\frac{\mu}{\lambda}}(\mu-\lambda)\\ \int_{\mu}^{+\infty}xe^{-\frac{x}{\lambda}}dx&=\lambda e^{-\frac{\mu}{\lambda}}(\mu+\lambda)\end{aligned}}$$

$\cancel{D(X)}$ $E(X^2)=\frac{1}{2\lambda}\left(\int_{-\infty}^{\mu}x^2e^{-\frac{\mu-x}{\lambda}}dx+\int_{\mu}^{+\infty}x^2e^{-\frac{x-\mu}{\lambda}}dx\right)$

$=\frac{1}{2}e^{-\frac{\mu}{\lambda}}\int_{-\infty}^{\mu}x^2de^{\frac{x}{\lambda}}-\frac{1}{2}e^{\frac{\mu}{\lambda}}\int_{\mu}^{+\infty}x^2de^{-\frac{x}{\lambda}}$

$=\frac{1}{2}e^{-\frac{\mu}{\lambda}}\left(x^2e^{\frac{x}{\lambda}}\Big|_{-\infty}^{\mu}-2\int_{-\infty}^{\mu}xe^{\frac{x}{\lambda}}dx\right)-\frac{1}{2}e^{\frac{\mu}{\lambda}}\left(x^2e^{-\frac{x}{\lambda}}\Big|_{\mu}^{+\infty}-2\int_{\mu}^{+\infty}xe^{-\frac{x}{\lambda}}dx\right)$

$=\frac{\mu^2}{2}-\lambda(\mu-\lambda)+\frac{\mu^2}{2}+\lambda(\mu+\lambda)=\mu^2+2\lambda^2$ $\therefore D(X)=2\lambda^2$

16. $E(X)=\int_0^{+\infty}\frac{\beta}{\Gamma(\alpha)}\beta^{\alpha-1}x^{\alpha}e^{-\beta x}dx$

$$=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{+\infty}x^{\alpha}e^{-\beta x}dx \xlongequal{t=\beta x} \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{+\infty}\left(\frac{t}{\beta}\right)^{\alpha}e^{-t}\frac{dt}{\beta}$$

$$=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{1}{\beta^{\alpha+1}}\int_0^{+\infty}t^{\alpha}e^{-t}dt=\frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha)}{\beta\Gamma(\alpha)}=\frac{\alpha}{\beta}$$

注（$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx$　$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ ← 分部积分可证）

$$E(X^2)=\int_0^{+\infty}\frac{\beta}{\Gamma(\alpha)}\beta^{\alpha-1}x^{\alpha+1}e^{-\beta x}dx=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{+\infty}x^{\alpha+1}e^{-\beta x}dx$$

$$\xlongequal{t=\beta x}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{+\infty}\frac{t^{\alpha+1}}{\beta^{\alpha+1}}e^{-t}\frac{dt}{\beta}=\frac{\Gamma(\alpha+2)}{\beta^2\Gamma(\alpha)}=\frac{(\alpha+1)\alpha\Gamma(\alpha)}{\beta^2\Gamma(\alpha)}=\frac{\alpha^2+\alpha}{\beta^2}$$

$\therefore D(X)=\frac{\alpha}{\beta^2}$

上述换元稍复杂．改进如下

$$E(X)=\int_0^{+\infty}\frac{\beta x}{\Gamma(\alpha)}(\beta x)^{\alpha-1}e^{-\beta x}dx\xlongequal{t=\beta x}\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}t\cdot t^{\alpha-1}e^{-t}\frac{dt}{\beta}$$

$$=\frac{1}{\Gamma(\alpha)}\frac{1}{\beta}\int_0^{+\infty}t^{\alpha}e^{-t}dt=\frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha)}=\frac{\alpha}{\beta}$$

$E(X^2)$ 类似可换．

17. $\because E(X-k)^2=E(X^2-2kX+k^2)=E(X^2)-2kE(X)+k^2$

$\quad D(X)=E(X^2)-E^2(X)$

$$=D(X)+E^2(X)-2kE(X)+k^2$$

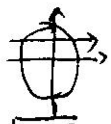
$$=(E(X)-k)^2+D(X)$$

$\therefore k=E(X)$ 时 $E(X-k)^2$ 最小．最小值为 $D(X)$．

18. 由 P109，例1．$E(X)=E(Y)=E(XY)=0$

$\therefore Cov(X,Y)=E(XY)-E(X)E(Y)=0$

$\therefore \rho=0$ 即 X与Y不相关．

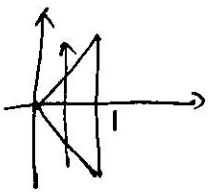


X的边沿密度 $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\begin{cases}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\frac{2}{\pi}\sqrt{1-x^2} & -1\le x\le 1\\ 0 & \text{其他}\end{cases}$

同理 $f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\begin{cases}\frac{2}{\pi}\sqrt{1-y^2} & -1\le y\le 1\\ 0 & \text{其他}\end{cases}$

$\therefore f(x,y)\ne f_X(x)\cdot f_Y(y)$，说明 X，Y 不是相互独立的．

19. $E(X)=\iint\limits_D x f(x,y)\,dxdy = \int_0^1 dx\int_{-x}^{x} x\,dy = \int_0^1 2x^2\,dx = \frac{2}{3}$



$E(Y)=\iint\limits_D y f(x,y)\,dxdy = \int_0^1 dx\int_{-x}^{x} y\,dy = 0$

$E(XY)=\iint\limits_D xy f(x,y)\,dxdy = 0$

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

20. $\because f(x_i) = \frac{1}{3\sqrt{2\pi}} \exp\left(-\frac{(x_i+2)^2}{18}\right) \quad -\infty < x_i < +\infty$.

$\therefore X_{1,2} \sim N(-2, 3^2)$ 即 $\mu_1=\mu_2=-2 \quad \sigma_1^2=\sigma_2^2=9$

$\therefore D(aX_1 - bX_2) = a^2D(X_1) + b^2D(X_2) = (a^2+b^2)9$.

$$\begin{aligned} E(aX_1^2 - bX_2^2) &= aE(X_1^2) - bE(X_2^2) \\ &= a(D(X_1)+E^2(X_1)) - b(D(X_2)+E^2(X_2)) \\ &= 13(a-b) \end{aligned}$$

21. 由 $\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}$ 得 $Cov(X,Y) = \sqrt{D(X)}\sqrt{D(Y)}\,\rho_{XY} = 5\times6\times0.4 = 12$

$\therefore D(X+Y) = D(X)+D(Y)+2Cov(X,Y) = 25+36+24 = 85$

$D(X-Y) = D(X)+D(Y)-2Cov(X,Y) = 25+36-24 = 37$.

22. $E(X) = E(X_1+X_2+X_3) = E(X_1)+E(X_2)+E(X_3) = 0$ 同理 $E(Y)=0$

$E(X_iX_j) \overset{i\neq j}{=\!=} E(X_i)E(X_j) = 0 \qquad E(X_i^2) = D(X_i)+E^2(X_i) = \sigma^2$

$\therefore E(XY) = E[(X_1+X_2+X_3)(X_2+X_3+X_4)] = E(X_2^2)+E(X_3^2) = 2\sigma^2$

$D(X) = D(X_1+X_2+X_3) = 3\sigma^2 \qquad D(Y) = 3\sigma^2$

$\therefore \rho_{XY} = \frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}} = \frac{E(XY)-E(X)E(Y)}{\sqrt{D(X)}\sqrt{D(Y)}} = \frac{2\sigma^2-0}{3\sigma^2} = \frac{2}{3}$

24. 证明: $E(U) = aE(X)+b \qquad E(V) = cE(Y)+d$

$D(U) = a^2D(X) \qquad D(V) = c^2D(Y)$

$E(UV) = E[(aX+b)(cY+d)] = acE(XY)+adE(X)+bcE(Y)+bd$

$$\begin{aligned} \rho_{UV} = \frac{E(UV)-E(U)E(V)}{\sqrt{D(U)}\sqrt{D(V)}} &= \frac{acE(XY)+adE(X)+bcE(Y)+bd-[aE(X)+b][cE(Y)+d]}{ac\sqrt{D(X)}\sqrt{D(Y)}} \\ &= \frac{acE(XY)-acE(X)E(Y)}{ac\sqrt{D(X)}\sqrt{D(Y)}} = \frac{E(XY)-E(X)E(Y)}{\sqrt{D(X)}\sqrt{D(Y)}} = \rho_{XY}. \end{aligned}$$

23. (1) 由 $\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}f(x,y)dy=1$ 得 $a\int_0^{\frac{\pi}{2}}dx\int_0^{\frac{\pi}{2}}\sin(x+y)dy=1$ ⑥

$$左 = a\int_0^{\frac{\pi}{2}}[-\cos(x+y)]_0^{\frac{\pi}{2}}dx = a\int_0^{\frac{\pi}{2}}(\cos x-\cos(\tfrac{\pi}{2}+x))dx$$

$$= a[\sin x-\sin(\tfrac{\pi}{2}+x)]_0^{\frac{\pi}{2}} = 2a \qquad \therefore a=\frac{1}{2}$$

(2) $E(X)=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}xf(x,y)dy = \frac{1}{2}\int_0^{\frac{\pi}{2}}dx\int_0^{\frac{\pi}{2}}x\sin(x+y)dy$

$$\cancel{=\frac{1}{2}\int_0^{\frac{\pi}{2}}(-x)d\cos(x+y)} = \frac{1}{2}\int_0^{\frac{\pi}{2}}x[-\cos(x+y)]_0^{\frac{\pi}{2}}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}x(\cos x-\cos(\tfrac{\pi}{2}+x))dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}x(\cos x+\sin x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}x\,d(\sin x-\cos x)$$

$$= \frac{1}{2}\left(x[\sin x-\cos x]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(\sin x-\cos x)dx\right) = \frac{\pi}{4}$$

同理 $E(Y)=\frac{\pi}{4}$

$E(X^2)=\frac{1}{2}\int_0^{\frac{\pi}{2}}dx\int_0^{\frac{\pi}{2}}x^2\sin(x+y)dy = \frac{1}{2}\int_0^{\frac{\pi}{2}}x^2[-\cos(x+y)]_0^{\frac{\pi}{2}}dx$ (同上)

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}x^2(\cos x+\sin x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}x^2d(\sin x-\cos x)$$

$$= \frac{1}{2}\left(x^2(\sin x-\cos x)\Big|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x(\sin x-\cos x)dx\right)$$

$$= \frac{1}{2}\left(\frac{\pi^2}{4}+2\int_0^{\frac{\pi}{2}}x\,d(\sin x+\cos x)\right) = \frac{\pi^2}{8}+x(\cos x+\sin x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(\cos x+\sin x)dx$$

$$= \frac{\pi^2}{8}+\frac{\pi}{2}-2 \qquad 同理\ E(Y^2)=\frac{\pi^2}{8}+\frac{\pi}{2}-2$$

$$\therefore D(X)=D(Y)=\frac{\pi^2}{8}+\frac{\pi}{2}-2-\left(\frac{\pi}{4}\right)^2=\frac{\pi^2}{16}+\frac{\pi}{2}-2$$

(3) $E(XY)=\frac{1}{2}\int_0^{\frac{\pi}{2}}dx\int_0^{\frac{\pi}{2}}xy\sin(x+y)dy=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\left(-\int_0^{\frac{\pi}{2}}y\,d\cos(x+y)\right)dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}x\left(-y\cos(x+y)\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos(x+y)dy\right)dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}x\left(-\frac{\pi}{2}\cos(\tfrac{\pi}{2}+x)+\sin(\tfrac{\pi}{2}+x)-\sin x\right)dx \qquad \left[\cos(\tfrac{\pi}{2}+x)=-\sin x\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-1\right)\int_0^{\frac{\pi}{2}}x\sin x\,dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}x\cos x\,dx$$

$$= \left(\frac{\pi}{2}-1\right)$$

$$\therefore Cov(X,Y)=\frac{\pi}{2}-1-\left(\frac{\pi}{4}\right)^2=\frac{\pi}{2}-1-\frac{\pi^2}{16}$$

$$\rho_{XY}=\frac{\frac{\pi}{2}-1-\frac{\pi^2}{16}}{\frac{\pi^2}{16}+\frac{\pi}{2}-2}=\frac{8\pi-16-\pi^2}{\pi^2+8\pi-32}$$