### Probability of events

Axioms:

$$P(e) \geq 0 \qquad \forall e \in \mathcal{E}$$
$$P(\Omega) = 1$$
$$P(\cup_{i=1}^{n} e_i) = \sum_{i=1}^{n} P(e_i)$$

$e_i$ are disjoint events.

Theorems:

$$P(\{\}) = 0$$
$$e_1 \subseteq e_2 \;\Rightarrow\; P(e_1) \leq P(e_2) \qquad \forall e_1, e_2 \in \mathcal{E}$$
$$P(\Omega \backslash e) = 1 - P(e) \qquad \forall e \in \mathcal{E}$$
$$P(e) \in [0,1] \qquad \forall e \in \mathcal{E}$$
$$P(e_1 \cup e_2) = P(e_1) + P(e_2) - P(e_1 \cap e_2) \qquad \forall e_1, e_2 \in \mathcal{E}$$

### Probability density functions:

Definition: $P(e) = \int_e p(\omega) d\omega$

Requirements: $p(\omega) \geq 0 \qquad \forall\, \omega \in \Omega$

$$\int_\Omega p(\omega) d\omega = 1$$

### Cumulative distribution function:

Definition: $\Phi_Y(y) \;=\; P(Y < y) \;=\; \int_{-\infty}^{y} p_Y(\xi)\, d\xi$

### Expected value:

$$E[Y] = \int_{\Omega_Y} y\, p_Y(y)\, dy$$
$$E\left[\sum_{i=1}^{N} \alpha_i Y_i\right] = \sum_{i=1}^{N} \alpha_i E[Y_i]$$
$$E[g(Y)] = \int_{\Omega_Y} g(y)\, p_Y(y) dy$$

### Variance:

$$Var[Y] = E[(Y - E[Y])^2] = E[Y^2] - E[Y]^2$$
$$Var\left[\sum_{i=1}^{N} \alpha_i Y_i\right] = \sum_{i=1}^{N} \alpha_i^2\, Var[Y_i] + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \alpha_i \alpha_j Cov(Y_i, Y_j)$$

### Standard deviation: $\sigma_Y = \sqrt{Var[Y]}$

### Multivariate random variables:

$$Y = (Y^1, Y^2, \dots, Y^D)$$
$$E[Y] = (E[Y^1], E[Y^2], \dots, E[Y^D])$$
$$Var[Y] \;=\; E\left[(Y - E[Y])^T (Y - E[Y])\right]$$
$$Cov[Y^i, Y^j] = \sigma_{i,j}^2 = E[(Y^i - E[Y^i])(Y^j - E[Y^j])]$$

### Marginal distribution:

$$p_X(x) = \int_{\Omega_Y} p_{XY}(x,y)\, dy$$

### Independence:

$$P(A \cap B) = P(A)P(B)$$
$$P(A|B) = P(A)$$
$$p_{XY}(x,y) = p_X(x) p_Y(y)$$
$$p_{Y|X=x}(y) = p_Y(y)$$
$$p_{X|Y=y}(x) = p_X(x)$$

### Conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$
$$p_{Y|X=x}(y) = \frac{p_{XY}(x,y)}{p_X(x)} \qquad \forall y \in \Omega_Y$$

### Correlation coefficient: $\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

**Bernoulli**: $Y \sim \mathcal{B}(\alpha)$

$$p_Y(y) = \begin{cases} 1-\alpha & \text{if } y = 0 \\ \alpha & \text{if } y = 1 \end{cases}$$

$$E[Y] = \alpha$$

$$Var[Y] = \alpha(1-\alpha)$$

**Exponential:** $Y \sim \mathcal{E}(\lambda)$

$$p_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \lambda^{-1}$$

$$Var[Y] = \lambda^{-2}$$

**Discrete uniform:** $Y \sim \mathcal{U}(a,b)$

$$p_Y(y) = \begin{cases} \frac{1}{b-a+1} & \text{if } y \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \frac{a+b}{2}$$

$$Var[Y] = \frac{(b-a+1)^2-1}{12}$$

**Continuous uniform**: $Y \sim \mathcal{U}(a,b)$

$$p_Y(y) = \begin{cases} \frac{1}{b-a} & \text{if } y \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \frac{a+b}{2}$$

$$Var[Y] = \frac{(b-a)^2}{12}$$

**Gaussian:** $Y \sim \mathcal{N}(\mu, \sigma^2)$

$$p_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right)$$

$$E[Y] = \mu$$

$$Var[Y] = \sigma^2$$

**Point estimators:**

$$\text{Bias}[\hat{\Theta}_N] = E[\hat{\Theta}_N] - \theta$$

$$\text{Var}[\hat{\Theta}_N] = E\left[\left(\hat{\Theta}_N - E[\hat{\Theta}_N]\right)^2\right]$$

**Mean squared error:**

$$\begin{aligned} \text{MSE}[\hat{\Theta}_N] &= E[(\hat{\Theta}_N - \theta)^2] \\ &= Var[\hat{\Theta}_N] + \left(\text{Bias}[\hat{\Theta}_N]\right)^2 \end{aligned}$$

**Sample mean:**

$$\bar{Y}_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$$

**Unbiased sample variance:**

$$S_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y}_N)^2$$

**Biased sample variance:**

$$\tilde{S}_N^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y}_N)^2$$

**Likelihood:** $\mathcal{L}(\underline{\theta}; \mathcal{D}) = \prod_{i=1}^{N} p_Y(y_i; \underline{\theta})$

**Confidence intervals:**

$$\rho = \frac{\sigma_Y}{\sqrt{N}}\left|\Phi_N^{-1}\left(\frac{1-\gamma}{2}\right)\right| \quad \text{or} \quad \rho = \frac{\hat{\sigma}_N}{\sqrt{N}}\left|\Phi_{t(\nu)}^{-1}\left(\frac{1-\gamma}{2}\right)\right|$$

**Z and t statistics:**

$$Z = \frac{\bar{Y}_N - \mu_Y}{\sigma_Y/\sqrt{N}} \qquad t = \frac{\bar{Y}_N - \mu_Y}{S_N/\sqrt{N}}$$