# Probability theory

201. In a hen-house there are 10 hens. Assign outcomes and sample space to the trial "enter the poultry-yard and count the hens that sit on their roosts". (§2.2)

202. Give suitable sample spaces for the following random trials:

(a) Throw a die and count the tosses until one of the six outcomes has appeared twice.

(b) Throw a die and count the tosses until one of the six outcomes has appeared twice in succession. (§2.2)

203. At a sawmill boards are cut into 3-feet lengths, and the remaining pieces smaller than 3 feet are thrown on a heap. Give a suitable sample space for the trial: Choose one of the small parts and note its length (unit: feet). (§2.2)

205. Let $A$, $B$, $C$ and $D$ be four events. Let $E$ be the event that at least three, and $F$ the event that exactly three, of the four events occur. Express $E$ and $F$ in terms of $A$, $B$, $C$ and $D$ (use suitable symbols). (§2.2)

206. The events $A$ and $B$ are mutually exclusive events such that $P(A) = 0.25$, $P(A \cup B) = 0.75$. Determine $P(B)$. (§2.3)

207. Let $A$ and $B$ be two events such that $P(A) = 0.6$, $P(B) = 0.7$ and $P(A \cup B) = 0.8$. Find $P(AB)$. (§2.3)

$\boxed{P(A \cap B)}$

208. From a box containing the letters in the word **PROBABILITY** two letters are taken at random. Using the classical definition of probability, compute the probability of obtaining the letters P and R. (§2.4)

209. Three cards are taken at random without replacement from a deck of cards. Using the classical definition of probability, find the probability that:

(a) all three are hearts;
(b) none is a heart;
(c) all three are aces. (§2.4)

210. From an urn containing three white balls and four black balls, two balls are drawn at random without replacement. Determine the probability that the balls have different colours. (§2.4)

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212. The triplets $A$, $B$ and $C$ play together with seven other children.
   (a) They arrange themselves randomly in a row one behind the other. Find the probability that the triplets are in the front.
   (b) They again line up randomly in a row. Find the probability that the triplets are together.
   (c) They arrange themselves randomly in a circle. Find the probability that the triplets are together. (§2.4)

213. In a lottery there are five tickets left, exactly one of which is a winning ticket. $A$ and $B$ decide to buy one ticket each. $A$ draws first, then $B$. Compute the conditional probability that $B$ obtains the winning ticket, given that $A$ has not obtained it. (§2.5)

214. From a deck of cards four cards are drawn one at a time without replacement.
   (a) If the three first cards are hearts, what is the conditional probability that the fourth is not a heart?
   (b) What is the probability that the three first cards are hearts and the fourth is a spade? (Use the answer to Exercise 209(a).) (§2.5)

215. Three measuring instruments, numbered 1, 2 and 3, function with probabilities 0.9, 0.8 and 0.4, respectively. One instrument is selected at random.

(a) What is the probability that the chosen instrument functions?

220. The events $A$ and $B$ are independent. Prove that $A$ and $B^*$ are independent and that $A^*$ and $B^*$ are also independent. (§2.6)

Note: There will not be proofs on the midterm.

221. The events $A$ and $B$ are independent, and $P(A) = 0.1$, $P(B) = 0.05$. Compute $P(A^*B^*)$. (§2.6)

222. The families $A$, $B$ and $C$ are invited to dinner. The probabilities that they will come are 0.8, 0.6 and 0.9, respectively, and these events are independent. Find the probability that:

(a) all the families come;

(b) no family comes;

(c) at least one family comes. (§2.6)

# One-dimensional RVs

301. A rv $X$ assumes the values $a$ and $b$ ($a < b$) with the same probability. Find the distribution function of $X$ and draw its graph. (§3.3)

303. A rv can assume only the values 3, 4, 7, 8 and 9. It is known that

$p_X(3) = 1/3, \qquad p_X(4) = 1/4, \qquad p_X(7) = 1/6, \qquad p_X(8) = 1/6.$

Compute:

(a) $p_X(9)$;

* (b) $F_X(5)$;

(c) $P(4 \leq X \leq 8)$ and $P(X \geq 8)$. (§3.4)

* This is the CDF. Phi in our notation

304. In a box there are two 10-cent coins and one 50-cent coin. Two of the three coins are chosen at random. Let $X$ be the total value of the two coins.

(a) Which values can $X$ assume?

(b) Find the probability function of $X$. (§3.4)

310. Determine the constant $c$ such that the function

$$f(x) = cx^2 \qquad (0 \le x \le 6)$$

becomes a density function. (§3.6)

311. Determine the constant $c$ such that

$$f(x) = c/\sqrt{x+1} \qquad (-1 < x \leq 1)$$

becomes a density function. Find the probability that a rv with this density function assumes a positive value. (§3.6)

316. A train is scheduled to arrive at a station at 13:03, but is generally somewhat late. The delay may be considered as a rv with density function $f_X(x) = 1/5$ $(0 \le x \le 5)$.

(a) Determine the probability that the train arrives later than 13:06.

(b) Determine the probability that the train arrives between 13:04 and 13:05.

317. Let $X$ be the waiting time in minutes from the opening time of a shop until the first customer arrives. The distribution function of $X$ is given by

$$\Phi_X(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - e^{-0.4x} & \text{if } x \geq 0. \end{cases}$$

Determine the probability that the waiting time is:
(a) at most 3 minutes;
(b) at least 4 minutes;
(c) between 3 and 4 minutes;
(d) at most 3 minutes or at least 4 minutes;
(e) exactly 2.5 minutes. (§3.7)

*321. The rv $X$ has an exponential distribution. Prove that the conditional probability $P(X > t + a | X > a)$ does not depend on $a$. This result implies that if a new unit, say an electric bulb, has a lifetime with an exponential distribution, then the *remaining* lifetime of a unit functioning after time $a$ has the same distribution as the total lifetime of a new unit. Informally, we may say that the exponential distribution has no memory.

Note: There will not be proofs on the midterm.

# Mutlivariate RVs

401. In a certain part of a city, a number of families were surveyed at random regarding the number $X$ of children and the number $Y$ of rooms in their apartment. ($Y$ does not include the kitchen.) Suppose that $(X, Y)$ is a rv with the probability function $p_{X,Y}(j, k)$ given in the following table:

Y rooms

X children

| $j$ \ $k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 0.11 | 0.09 | 0.07 | 0.01 |
| 1 | 0.07 | 0.12 | 0.12 | 0.02 |
| 2 | 0.02 | 0.05 | 0.17 | 0.05 |
| 3 | 0.00 | 0.02 | 0.04 | 0.02 |
| 4 | 0.00 | 0.00 | 0.01 | 0.01 |

(a) Find the probability that a randomly chosen family has at most one child and lives in an apartment with at most three rooms (and kitchen).

(b) If each family consists of two grown-up persons and children, what is the probability that a randomly chosen family lives in an overcrowded apartment? By "overcrowded" we mean that the number of persons/room (apart from the kitchen) exceeds two.

(c) Find the marginal probability functions $p_X(j)$ and $p_Y(k)$ of the number of children and the number of rooms, respectively. (§4.3)

404. The rv's $X$ and $Y$ are independent and have the following probability functions:

| $j$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $p_X(j)$ | 0.10 | 0.20 | 0.40 | 0.20 | 0.10 |

| $k$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $p_Y(k)$ | 0.10 | 0.20 | 0.30 | 0.40 |

(a) Find $P(X = 3, Y = 6)$.
(b) Find $P(X \leq 3, Y \leq 6)$. (§4.5)

406. A person travels first by bus 1 and then by bus 2. The waiting times, $X$ and $Y$, are independent and uniformly distributed over the intervals (0, 10) and (0, 8), respectively (unit: minutes). Compute the probability that the total waiting time is at least 16 minutes.
*Hint*: The rv $(X, Y)$ is uniformly distributed over a rectangle. (§4.5)

# Expectations

602. The rv $X$ has density function

$$f_X(x) = 2x/a^2 \qquad (0 \leq x \leq a).$$

Find $E(X)$. (§6.2)

607. The rv $X$ has density function $f_X(x) = 1/10\,(-5 \leq x \leq 5)$. Find $E[g(X)]$, where

$$g(x) = \begin{cases} -1 & \text{if} \quad x < 0, \\ \phantom{-}2 & \text{if} \quad x \geq 0. \end{cases} \qquad (\S 6.2)$$

611. The rv $X$ has density function $f_X(x) = 2x$ $(0 \leq x \leq 1)$.

(a) Find the mean $m$ and the standard deviation $\sigma$ of $X$.

(b) Find $P(m - 2\sigma < X < m + \sigma)$.

(c) Find $P(m - \sigma < X < m + 2\sigma)$. (§6.3)

702. The rv's $X_1, X_2, X_3, X_4$ are independent with means $2m, 2m, 3m, 2m$, respectively, and common standard deviation $\sigma$. Form the linear combination

$$Y = \tfrac{4}{3}X_1 - \tfrac{1}{2}X_2 - \tfrac{1}{3}X_3 - \tfrac{1}{3}X_4.$$

Find $E(Y)$ and $D(Y)$. (§7.2)

707. When determining the melting-point of cooking fat, the resulting measurement may be regarded as a rv $X$ with standard deviation 2. How many measurements are needed to make the standard deviation of the arithmetic mean of the measurements at most 0.4? (§7.3)

# Normal distribution

802. The rv $X$ is $N(0, 1)$. Determine $x$ such that:

(a) $P(X > x) = 0.001$;

(b) $P(X > x) = 0.999$;

(c) $P(|X| < x) = 0.95$;

(d) $P(X < -x) = 0.10$. (§8.3)

806. Consider a rv $X$ which is $N(180, 5^2)$. Compute the probability that $X \geq 170$ and the probability that $170 \leq X \leq 200$. (§8.4)

815. $X_1, X_2, \ldots, X_n$ are independent and $N(m, 0.2^2)$.

(a) Find the distribution of $\bar{X} - m$.

(b) Find $P(|\bar{X} - m| > 0.2/\sqrt{n})$.

(c) Find $P(|\bar{X} - m| > 0.1)$ if $n = 16$.

(d) We want that $P(|\bar{X} - m| > 0.01)$ is less than 0.001. How large must $n$ be?

(§8.5)

818. The weight (unit: gram) of a randomly chosen pill of a certain type is a rv with mean 0.65 and standard deviation 0.02.

(a) Find the mean and standard deviation of the total weight of 100 pills (whose weights are supposed to be independent).

(b) Use the central limit theorem to determine approximately the probability that 100 pills weigh at most 65.3 grams. (§8.6)

# Point estimation

1206. The rv $X$ has density function $f_X(x) = \theta(1 + x)^{-\theta-1}$ for $x \geq 0$. It is known beforehand that $\theta$ is either 2, 3 or 4. Let 0.2, 0.8 be a random sample of two values from this distribution.

(a) Find the $L$ function for the three possible values of $\theta$.

(b) Determine the $ML$ estimate of $\theta$. (§12.4)

1207. Let $x_1, \ldots, x_n$ be a random sample from a distribution with density function $f_X(x) = \theta x^{\theta - 1}$ for $0 < x < 1$. Find the $ML$ estimate of $\theta$. (§12.4)

Hint: Use log-likelihood