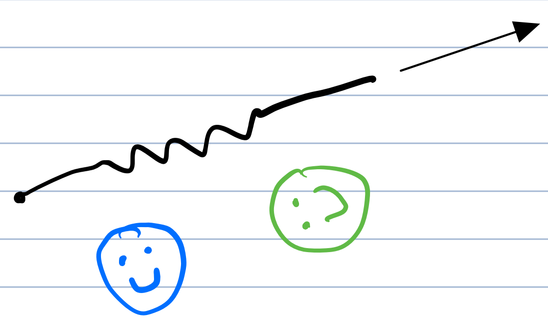
ME 104 Lec 7

Last time: Introduced rotating frames.

Let's clarify a few points.

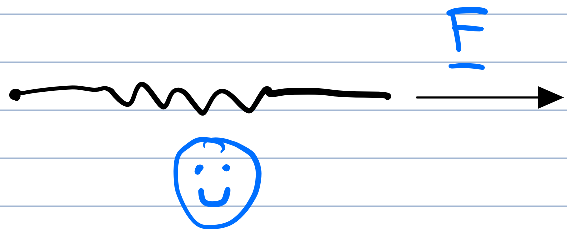
The physical quantities we have dealt with — such as force, position, velocity — are <u>objective quantities</u>. Each observer looking at the same objective quantity will see a clearly defined vector, however, different observers may see a different vector depending on their relative orientation and motion. What makes the quantity "objective" is that we can figure out how the vector changes between any two observers.

Example: Force. Think of a spring...



The blue man and green man are tilted 90° from each other.

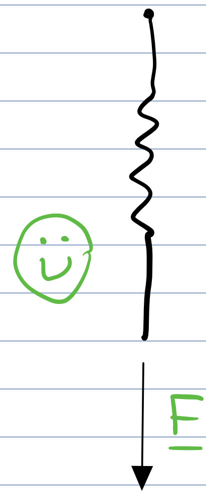
The blue man sees:



He reports that the force points <u>rightward</u>.

The green man sees:

He reports that the force points <u>downward</u>.



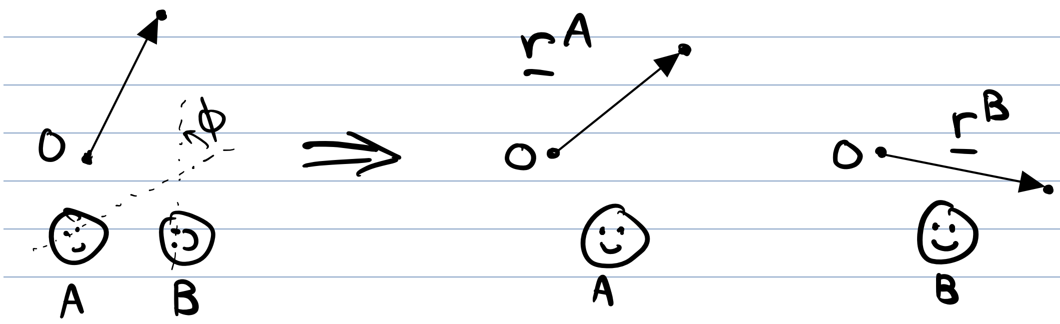
Thus, the same force appears as a different vector in the two frames; i.e. $\underline{F} \neq \underline{F}$.

But because force is objective, I can determine a <u>rule</u> for how the force should look to different observers.

Rule: If observer B is rotated ϕ anti-clockwise from observer A, the force vector seen by observer A is simply the force vector seen by B but rotated ϕ anti-clockwise.

$$\Longrightarrow \boxed{\underline{F}^A = \overset{\text{rotation by } \phi \text{ anti-clock}}{\underline{R}}(\phi)\left[\underline{F}^B\right]}$$

A similar rule relates the position vector from a common origin to a given point.



$$\boxed{\underline{r}^A = \underline{R}(\phi)\left[\underline{r}^B\right]}$$

These rules get more complicated for quantities related to time derivatives of motion like velocity and acceleration. The rate the observer frames <u>move</u> relative to each other can now also affect the rule.

Example: Velocity (in 2D). Last time we showed that if observer B rotates by $\phi(t)$ relative to observer A, and both watch the <u>same</u> moving particle, the velocity each reports obeys the rule:

$$\underline{v}^A = \underline{R}(\phi)\left[\underline{v}^B + \dot{\phi}\,\underline{e}_z \times \underline{r}^B\right]$$

This formula also works in 3D as long as the axis of rotation remains $\underline{e}_z$.

Example: Acceleration in 2D:

$$\underline{a}^A = \underline{R}(\phi)\left[\underline{a}^B + \ddot{\phi}\,\underline{e}_z \times \underline{r}^B - \dot{\phi}^2 \underline{r}^B + 2\dot{\phi}\,\underline{e}_z \times \underline{v}^B\right]$$

In 3D, there is a minor change to the rule:

$$\underline{a}^A = \underline{R}(\phi)\left[\underline{a}^B + \ddot{\phi}\,\underline{e}_z \times \underline{r}^B - \dot{\phi}^2 (\underline{r}^B - z^B \underline{e}_z) + 2\dot{\phi}\,\underline{e}_z \times \underline{v}^B\right]$$

Now, some frames are special; they are called <u>inertial</u>. They are deemed to be moving at constant velocity relative to the universe. Newton's laws <u>only hold</u> in inertial frames.

Suppose frame A is inertial. What are the laws of motion in frame B?

Ans: $\quad \Sigma \underline{F}^A = m\,\underline{a}^A$

$$= \Sigma\left(\underline{R}(\phi)\left[\underline{F}^B\right]\right) \qquad \underline{R}(\phi)\left[\underline{a}^B + \ddot{\phi}\,\underline{e}_z \times \underline{r}^B - \dot{\phi}^2 (\underline{r}^B - z^B \underline{e}_z) + 2\dot{\phi}\,\underline{e}_z \times \underline{v}^B\right]$$

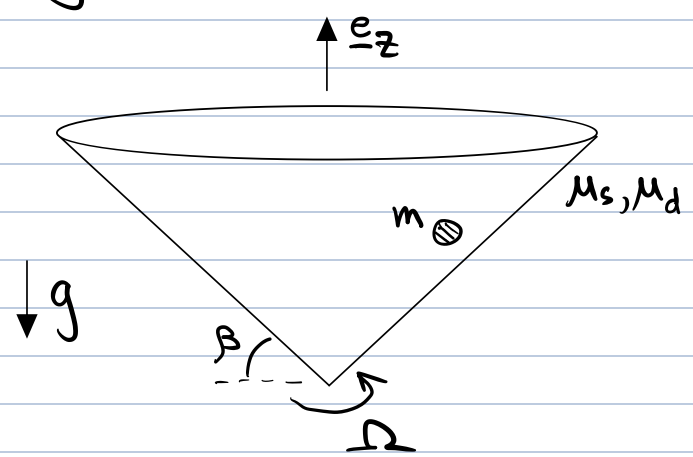
$$= \underline{R}(\phi)\left[\Sigma \underline{F}^B\right]$$

Rotating both sides by $\underline{R}(-\phi)$ now gives:

$$\Sigma \underline{F}^B = m\left(\underline{a}^B + \ddot{\phi}\,\underline{e}_z \times \underline{r}^B - \dot{\phi}^2 (\underline{r}^B - z^B \underline{e}_z) + 2\dot{\phi}\,\underline{e}_z \times \underline{v}^B\right)$$

$$\Rightarrow m\underline{a}^B = \sum \underline{F}^B \underbrace{- m\ddot{\phi}\,\underline{e}_z \times \underline{r}^B}_{\text{Euler force}} \underbrace{+ m\dot{\phi}^2\left(\underline{r}^B - z^B\underline{e}_z\right)}_{\text{Centrifugal force}} \underbrace{- m2\dot{\phi}\,\underline{e}_z \times \underline{v}^B}_{\text{Coriolis force}}$$

Example: A conical frictional surface spins at rate Ω about its axis. Write equations of motion for a mass on the wall when it is moving relative to the wall.



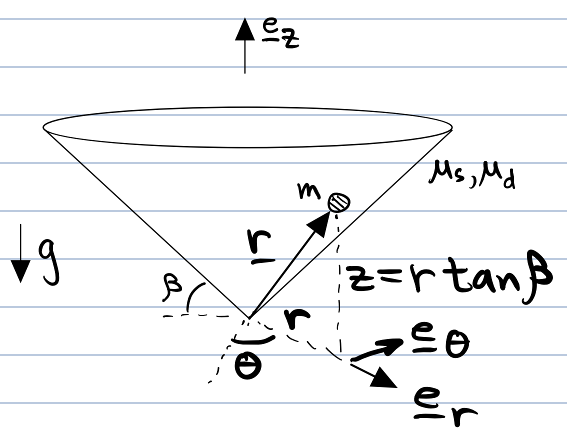
Ans: Solve this in a frame rotating with the cone. The wall now appears stationary.

① Kinematics: Use cylin coords in stat-wall frame.

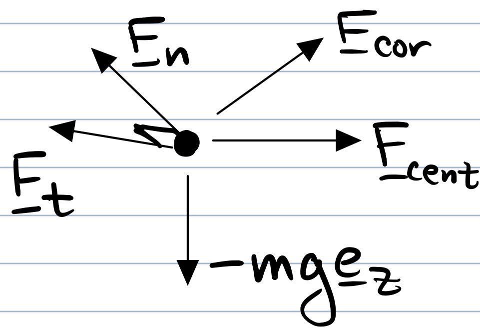
$$\underline{r} = r\underline{e}_r + r\tan\beta\,\underline{e}_z$$

$$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + \dot{r}\tan\beta\,\underline{e}_z$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta + \ddot{r}\tan\beta\,\underline{e}_z .$$



② FBD:



$$\underline{F}_n = F_n \underline{n} = F_n\left(-\sin\beta\, \underline{e}_r + \cos\beta\, \underline{e}_z\right)$$

$$\underline{F}_{cent} = m\Omega^2(\underline{r} - z\underline{e}_z) = m\Omega^2 r\, \underline{e}_r$$

$$\underline{F}_{cor} = -2m\Omega\, \underline{e}_z \times (\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta)$$
$$= -2m\Omega(\dot{r}\underline{e}_\theta - r\dot{\theta}\underline{e}_r)$$

$$\underline{F}_t = -\mu_d F_n\, \underline{v}/|\underline{v}| = -\mu_d F_n \frac{\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + \dot{r}\tan\beta\, \underline{e}_z}{\left(\dot{r}^2 + (r\dot{\theta})^2 + (\dot{r}\tan\beta)^2\right)^{1/2}}$$

③ $\sum \underline{F} = m\underline{a}$: $\underline{F}_n - mg\underline{e}_z + \underline{F}_t + \underline{F}_{cent} + \underline{F}_{corr} = m\underline{a}$

$\underline{e}_z$: $F_n\cos\beta - mg - \mu_d F_n \frac{\dot{r}\tan\beta}{\left(\dot{r}^2 + (r\dot{\theta})^2 + (\dot{r}\tan\beta)^2\right)^{1/2}} + 0 + 0$

$$= m(0 + 0 + \ddot{r}\tan\beta)$$

$\underline{e}_r$: $-F_n\sin\beta + 0 - \mu_d F_n \frac{\dot{r}}{\left(\dot{r}^2 + (r\dot{\theta})^2 + (\dot{r}\tan\beta)^2\right)^{1/2}} + m\Omega^2 r$

$$+ 2m\Omega\dot{\theta} = m\left((\ddot{r} - r\dot{\theta}^2)\underline{e}_r + 0 + 0\right)$$

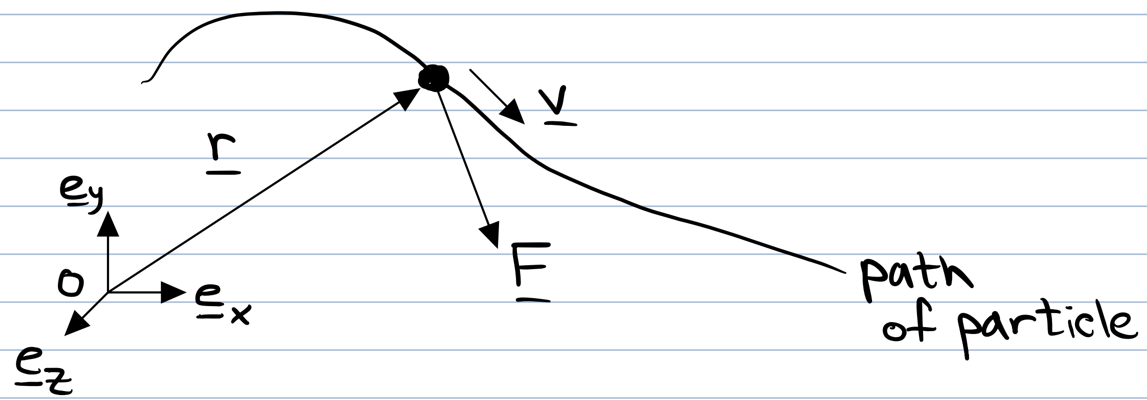
$\underline{e}_\theta$: $0 + 0 - \mu_d F_n \frac{r\dot{\theta}}{\left(\dot{r}^2 + (r\dot{\theta})^2 + (\dot{r}\tan\beta)^2\right)^{1/2}} + 0 - 2m\Omega\dot{r}$

$$= m\left(0 + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) + 0\right)$$

This is 3 eq's for 3 unknowns: $F_n(t)$, $\theta(t)$, and $r(t)$. Can be solved! How?

Use $\underline{e}_\theta$ eq to write F_n in terms of $r, \theta, \dot{r}, \dot{\theta}$. Plug that in to the $\underline{e}_r$ and $\underline{e}_\theta$ eqs and then solve for $r(t)$, $\theta(t)$ numerically.

<u>New topic: Work, energy, and power.</u>



The mechanical <u>power</u> of the force $\underline{F}$ is

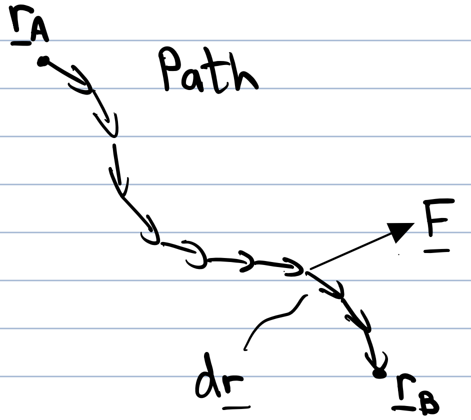
$$\boxed{\text{Power} \equiv \underline{F} \cdot \underline{v}}$$

A force expends positive power on a particle when the particle moves in the overall direction of the force.

Define <u>work</u> as the time integral of power:

$$W_{AB} = \text{"Work done by } \underline{F} \text{ from } t_A \text{ to } t_B\text{"}$$

$$= \int_{t_A}^{t_B} \underline{F} \cdot \underline{v}\, dt = \int_{t_A}^{t_B} \underline{F} \cdot \frac{d\underline{r}}{dt}\, dt = \int_{\substack{\text{Path} \\ \underline{r}(t_A) \to \underline{r}(t_B)}} \underline{F} \cdot d\underline{r}$$



Imagine breaking the path into a chain of little $d\underline{r}$ vectors. At each $d\underline{r}$, draw the force vector there and compute $\underline{F} \cdot d\underline{r}$. Add up all the $(\underline{F} \cdot d\underline{r})$'s and that's the work over the path.

<u>Work-Energy Theorem:</u>

Define the kinetic energy of a particle as: $\boxed{K = \frac{1}{2} m \underline{v} \cdot \underline{v} = \frac{1}{2} m v^2}$. {recall v is speed}

Observe that
$$\dot{K} = \frac{1}{2} m \underline{\dot{v}} \cdot \underline{v} + \frac{1}{2} m \underline{v} \cdot \underline{\dot{v}}$$
$$= (m \underline{\dot{v}}) \cdot \underline{v} = \left(\sum_i \underline{F}^i\right) \cdot \underline{v} = \sum_i \overbrace{(\underline{F}^i \cdot \underline{v})}^{\text{Power of the } i^{\text{th}} \text{ force on the particle}}$$

So the total work done by all forces on a particle is:
$$W_{AB}^{Tot} = \sum_i W_{AB}^i = \sum_i \int_{t_A}^{t_B} \underline{F}^i \cdot \underline{v}\, dt = \int_{t_A}^{t_B} \sum_i (\underline{F}^i \cdot \underline{v})\, dt = \int_{t_A}^{t_B} \dot{K}\, dt$$
$$= \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2.$$

Work-energy theorem: $\boxed{W_{AB}^{Tot} = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2}$.

Why is this useful? Because many forces have a form that let's us simplify the calculation of $W_{AB} = \int_{t_A}^{t_B} \underline{F}^i \cdot \underline{v} \, dt$.