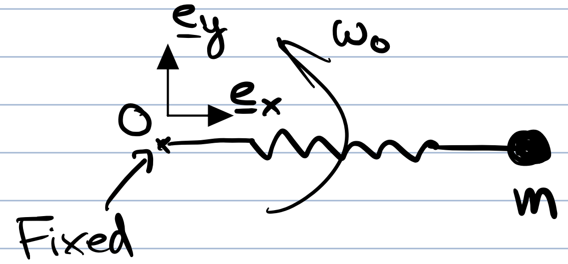
ME 104 Lec 5

Let's finish up example from last time.

Ex:



What's the frequency of small oscillations of the spring if m is perturbed outward a bit from a stable spin of ω_0?

Left off at step 4, with the following two coupled relations due to $\Sigma \underline{F} = m\underline{a}$:

$$-k(r-l_0) = m(\ddot{r} - r\dot{\theta}^2) \quad \& \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

If we define $H = mr^2\dot{\theta}$, then

$$\dot{H} = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = mr\underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{=0 \text{ from } 2^{nd} \text{ eqn above.}}$$

$\Rightarrow H = \text{const} =$ "Angular momentum".

$\Rightarrow H = H_0 = mr_0^2\omega_0$. What is r_0?

Initially, mass was in a stable circular path about O.

$$\Rightarrow \quad r = r_0 = \text{const}, \; \ddot{r} = 0. \quad \Rightarrow \quad -k(r_0 - l_0) = (-r_0 \omega^2)$$

$$\Rightarrow \quad r_0 = k l_0 \big/ [k - m\omega_0^2].$$

So $H = H_0 \Rightarrow m r^2 \dot{\theta} = m \omega_0 r_0^2$

$$\Rightarrow \quad \dot{\theta} = \left(\frac{r_0}{r}\right)^2 \omega_0 .$$

Plugging this into the first ODE gives

$$-k(r - l_0) = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Rightarrow \quad -k(r - l_0) = m\left(\ddot{r} - r\left(\frac{r_0}{r}\right)^4 \omega_0^2\right)$$

$$\Rightarrow \quad \underline{m\ddot{r} + kr - \frac{m}{r^3} r_0^4 \omega_0^2 = k l_0}$$

A single ODE in r only. Decoupled ☺

This ODE does not have an analytical sol'n. <u>But</u> since we're only interested in a <u>small</u> perturbation from a circular orbit, we can seek out an approximate solution of the form

$$r(t) = r_0 + \varepsilon \tilde{r}(t) \qquad \text{for } \varepsilon \ll r_0$$

Plugging this in to the underlined ODE

gives:

$$m\varepsilon\ddot{\tilde{r}} + k(r_0+\varepsilon\tilde{r}) - \frac{m}{(r_0+\varepsilon\tilde{r})^3} r_0^4 \omega_0^2 = k l_0$$

{Note: Taylor says $\frac{1}{(r_0+\varepsilon\tilde{r})^3} \approx \left.\frac{1}{(r_0+\varepsilon\tilde{r})^3}\right|_{\varepsilon=0} + \varepsilon \cdot \left.\frac{d}{d\varepsilon}\frac{1}{(r_0+\varepsilon\tilde{r})^3}\right|_{\varepsilon=0}$

$$= \frac{1}{r_0^3} - 3\varepsilon\frac{\tilde{r}}{r_0^4}.\}$$

Thus, $m\varepsilon\ddot{\tilde{r}} + k(r_0+\varepsilon\tilde{r}) - m r_0^4\omega_0^2\left(\frac{1}{r_0^3} - 3\varepsilon\frac{\tilde{r}}{r_0^4}\right) = k l_0$

$$\Rightarrow \underbrace{k r_0 - k l_0 - m r_0 \omega_0^2}_{=0 \text{ by definition of } r_0.} + \varepsilon\left(m\ddot{\tilde{r}} + k\tilde{r} + 3m\omega_0^2\tilde{r}\right) = 0$$

$$\Rightarrow \ddot{\tilde{r}} = -\left(k/m + 3\omega_0^2\right)\tilde{r} \Rightarrow \tilde{r}(t) = A\cos\left(\sqrt{k/m + 3\omega_0^2}\, t\right) + B\sin\left(\sqrt{k/m+3\omega_0^2}\, t\right)$$

$\Rightarrow$ Period of oscillation $= 2\pi \Big/ \sqrt{\frac{k}{m} + 3\omega_0^2}$

$$\Rightarrow \boxed{\text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{k}{m}+3\omega_0^2}}$$

Check: If $\omega_0 = 0$, freq is $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ which is the "usual" answer for the natural freq of a spring.

Friction: $\underline{v}_s \equiv \underline{v} - \underline{v}_{wall}$.

If $|\underline{v}_s| > 0$, then $\underline{F}_t = -\mu F_n \underline{v}_s / |\underline{v}_s|$.

If $|\underline{v}| = 0$, then $|\underline{F}_t| \leq \mu F_n$.

Often, we model μ to be a function of $|\underline{v}_s|$:

$\mu = \mu_s$ if $|\underline{v}_s| = 0$ "Static friction"

$\mu = \mu_d$ if $|\underline{v}_s| > 0$ "Dynamic friction"

where $\mu_s > \mu_d$.

BUT

This is an incomplete description of friction though, because it does not determine $\underline{F}_t$ when $|\underline{v}_s| = 0$, nor does it give the direction of $\underline{F}_t$ the instant sliding first begins.

Add two reasonable assumptions to resolve this:

1. If $\underline{v}_s = \underline{0}$ and the value of $\underline{F}_t$ that would make $\dot{\underline{v}}_s = \underline{0}$ satisfies $|\underline{F}_t| \leq \mu_s F_n$, then $\underline{F}_t$ takes this value.

In math: Define $\underline{F}^*$ as sum of all forces except for $\underline{F}_t$. $\underline{F}_t^* \equiv \underline{F}^* - (\underline{F}^* \cdot \underline{e}_n)\underline{e}_n$ is the projection of $\underline{F}^*$ onto the plane.

$$\sum \underline{F} = m\underline{\ddot{x}} = m(\underline{\dot{v}}_{wall} + \underline{\dot{v}}_s)$$

$$\Rightarrow \underbrace{\sum \underline{F} - (\underline{e}_n \cdot \sum \underline{F})\underline{e}_n}_{= \underline{F}_t + \underline{F}_t^*} = m\left((\underline{\dot{v}}_{wall} + \underline{\dot{v}}_s) - (\underline{e}_n \cdot (\underline{\dot{v}}_{wall} + \underline{\dot{v}}_s))\underline{e}_n\right)$$

Hence, the force needed for $\underline{\dot{v}}_s = \underline{0}$ is:

$$\underline{F}_t = \underbrace{m[\underline{\dot{v}}_{wall} - (\underline{e}_n \cdot \underline{\dot{v}}_{wall})\underline{e}_n] - \underline{F}_t^*}_{\equiv \underline{F}_{trial}}$$

If $|\underline{F}_{trial}| \leq \mu_s F_n$, accept it as correct.

$$\boxed{\text{If } |\underline{v}_s| = 0 \text{ and } |\underline{F}_{trial}| \leq \mu_s F_n \text{ then } \underline{F}_t = \underline{F}_{trial}.}$$

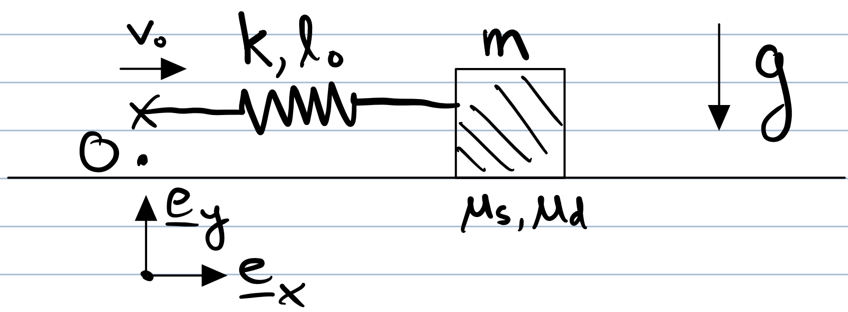
(2) If $\underline{v}_s = \underline{0}$ and the value of $\underline{F}_t$ that would make $\underline{\dot{v}}_s = \underline{0}$ satisfies $|\underline{F}_t| > \mu_s F_n$, then $|\underline{F}_t| = \mu_s F_n$ and $\underline{F}_t$ points in the direction that minimizes $\underline{\dot{v}}_s$. Sliding starts.

This is also equivalent to saying we pick $\underline{F}_t$ by scaling down $\underline{F}_{trial}$ to a length of $\mu_s F_n$.

In math:

$$\boxed{\begin{array}{l}\text{If } |\underline{v}_s| = 0 \text{ and } |\underline{F}_{trial}| > \mu_s F_n, \\ \text{then } \underline{F}_t = \mu_s F_n \left(\underline{F}_{trial} / |\underline{F}_{trial}| \right).\end{array}}$$

Example:



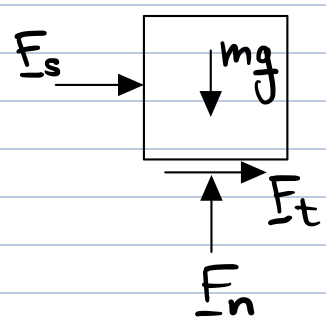
V_0 = Constant velocity of back of spring.

If the block starts at rest and spring begins unloaded, how does block move?

① Use cartesian with O at initial location of spring's back end.

$$\underline{r} = x\,\underline{e}_x, \quad \underline{v} = \dot{x}\,\underline{e}_x, \quad \underline{a} = \ddot{x}\,\underline{e}_x$$

② FBD of block:



(3) $\sum \underline{F} = m\underline{a}$: $\underline{F}_s + \underline{F}_t + \underline{F}_n - mg\,\underline{e}_y = m\ddot{x}\,\underline{e}_x$

$$\underline{F}_s = -k\Big(\big|\overbrace{x\underline{e}_x}^{\underline{r}} - \overbrace{v_0 t\,\underline{e}_x}^{\underline{r}^A}\big| - l_0\Big)\underbrace{(x\underline{e}_x - v_0 t\,\underline{e}_x)/|x\underline{e}_x - v_0 t\,\underline{e}_x|}_{=\operatorname{sign}(x - v_0 t)\,\underline{e}_x}$$

$\underline{F}_n = F_n\,\underline{e}_y$, $\underline{F}_t = F_t\,\underline{e}_x$.

Balance $\underline{e}_y$: $F_n - mg = 0 \implies F_n = mg$.

Balance $\underline{e}_x$: $-k\big(|x - v_0 t| - l_0\big)\operatorname{sign}(x - v_0 t) + F_t = m\ddot{x}$

Here $\underline{v}_{wall} = \underline{0}$, so $\underline{v}_s = \dot{x}\,\underline{e}_x$ and $\underline{F}_{trial} = -\underline{F}_t^* = -\underline{F}_s$.

$$\implies \underline{F}_t = \begin{cases} -\underline{F}_s & \text{if } \dot{x} = 0 \ \& \ |\underline{F}_s| \le \mu_s mg \\ \mu_s mg\,(-\underline{F}_s/|-\underline{F}_s|) & \text{if } \dot{x} = 0 \ \& \ |\underline{F}_s| > \mu_s mg \\ -\mu_d mg\,\operatorname{sign}(\dot{x})\,\underline{e}_x & \text{if } \dot{x} \ne 0. \end{cases}$$

With this, we can now numerically solve for $x(t)$. Matlab demo...