Week 1

Tuesday, September 3, 2024 6:29 AM

Simple Pendulum (ODE solver)

2nd-order ODE:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

here $\theta(t)$ and $\omega = \dot{\theta} = \frac{d\theta}{dt}$, $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.

Remember from high school physics, if θ is small, then the equation becomes:

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \Rightarrow \omega^2 = \frac{g}{L}$$

$\Rightarrow$ no ODE solver needed, solved analytically.

However for large θ, we need to solve this 2nd-order ODE using an ODE solver.

First we reduce this 2nd-order ODE as a system of two first order ODEs since ODE solver requires first-order ODEs.

$$\frac{d\theta}{dt} = \omega \quad \text{——— (1)}$$

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta \quad \text{——— (2)}$$

A system of two first-order ODEs (in vector form):

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -\frac{g}{L}\sin\theta\end{bmatrix}$$

In MATLAB,

Let's define :

- $y_1 = \theta$
- $y_2 = \omega = \frac{d\theta}{dt} = \frac{dy_1}{dt}$

The system of ODEs is :

$$\begin{cases} \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = -\frac{g}{L}\sin(y_1) \end{cases}$$

, Take $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\frac{dy}{dt} = \left[y(2) ; -\frac{g}{L}\sin(y(1)) \right]$$

ode45 solver in matlab :

$[t, y] = \text{ode45}(\text{odefun}, \text{tspan}, y_0)$

Initial condition: $y_0 = [\theta_0, \omega_0]$

Time range : $\text{tspan} = [t_0, t_f]$

Problems

① The position vector of the particle is given by :

$$\underline{r}(t) = 3t\,\underline{e}_x + 4t\,\underline{e}_y + 10\,\underline{e}_z$$

In component form: $\underline{r}(t) = (3t, 4t, 10)$

We can also write as :

$x(t) = 3t$, $y(t) = 4t$, $z(t) = 10$

Since $z(t) = 10$ is constant, so the particle moves in a plane parallel to x-y plane at $z = 10$.

Now, express $y(t)$ in terms of $x(t)$:

$$t = \frac{x}{3} = \frac{y}{4} \Rightarrow y(t) = \frac{4}{3} x(t) \rightarrow \text{Eq. of a line}$$

The linear relationship $y = \frac{4}{3}x$ confirms that the particle's path is a straight line on a x-y plane at $z = 10$.

The velocity vector is:

$$\underline{v}(t) = \frac{d}{dt}\underline{r}(t) = \frac{d}{dt}(3t\,\underline{e}_x + 4t\,\underline{e}_y + 10\,\underline{e}_z)$$

$$\underline{v}(t) = 3\,\underline{e}_x + 4\,\underline{e}_y$$

The speed of the particle is:

$$v(t) = \sqrt{\underline{v}(t) \cdot \underline{v}(t)} = \sqrt{(3)^2 + (4)^2}$$

$$\underline{\underline{v(t) = 5 \text{ m/s}}} \quad \text{(constant speed at all times)}.$$

(2) The position vector of the particle is:

$$\underline{r}(t) = 10\cos(n\pi t)\,\underline{e}_x + 10\sin(n\pi t)\,\underline{e}_y$$

The position vector is: $\underline{r}(t) = (10\cos(n\pi t), 10\sin(n\pi t))$

and

$$x(t) = 10\cos(n\pi t), \quad y(t) = 10\sin(n\pi t)$$

Squaring and adding,

$$x(t)^2 + y(t)^2 = (10\cos(n\pi t))^2 + (10\sin(n\pi t))^2$$

$$x(t)^2 + y(t)^2 = (10)^2 \rightarrow \text{Eq. of a circle}$$

This shows that the particle moves in a circle of radius 10 m.

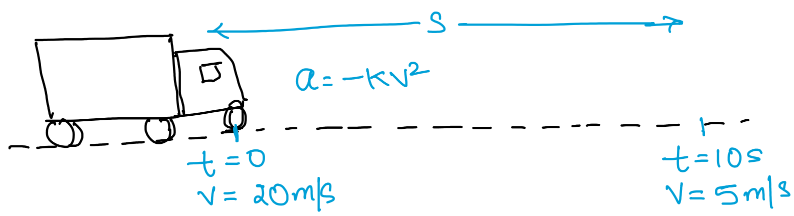
To find the time period, T :

The particle completes a full cycle of 2π in a time period T :

$$\Rightarrow \quad n\pi T = 2\pi$$

$$\Rightarrow \quad T = \frac{2}{n}.$$

(3)



$$\Rightarrow \text{Acceleration, } a = \frac{dv}{dt} \quad \Rightarrow \quad -kv^2 = \frac{dv}{dt}$$

$$\int_{20}^{v} \frac{dv}{v^2} = -k\int_0^t dt$$

$$\left[-\frac{1}{v}\right]_{20}^{v} = -kt \quad \Rightarrow \quad -\frac{1}{v} + \frac{1}{20} = -kt$$

$$v(t) = \frac{20}{1+20kt}.$$

To find k, use $t = 10s$, $v = 5 m/s$:

$$5 = \frac{20}{1+200k} \quad \Rightarrow \quad k = \frac{3}{200} \, m^{-1}$$

$$\Rightarrow \quad v(t) = \frac{200}{10+3t}$$

The distance moved by the truck is :

$$v = \frac{ds}{dt} \quad \Rightarrow \quad \frac{ds}{dt} = \frac{200}{10+3t}$$

$$\int_0^s ds = \int_0^t \frac{200}{10+3t}\,dt$$

$$s = \frac{200}{3}\left[\ln(10+3t) - \ln(10)\right]$$

$$s(t) = \frac{200}{3}\ln\left(\frac{10+3t}{10}\right).$$

After 10 s, the truck travelled a distance of,

$$s = \frac{200}{3}\ln\left(\frac{10+30}{10}\right) = \frac{200}{3}\ln(4)$$

$$\underline{\underline{s = 92.42\ m}}$$