ME104: Extra Notes Engineering Mechanics II

Discussion Week 5 of 15 (9/24/2025)

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Topics:

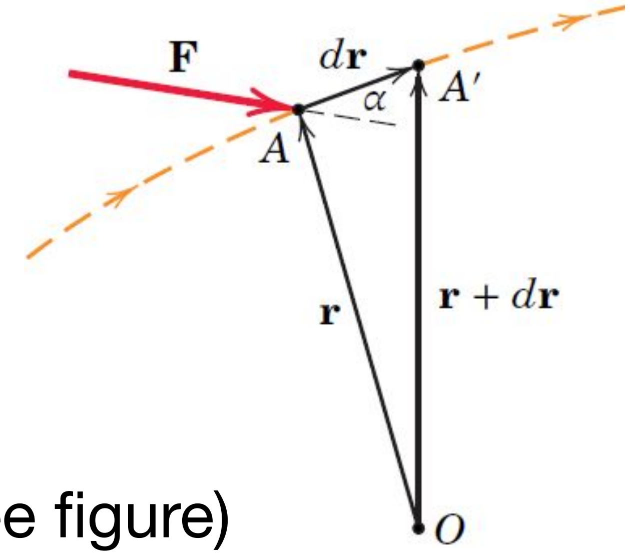
1. Extra notes on work/energy (+ extra examples)
2. Extra practice problem combining work/energy with Newton's law approach (similar to Q1 on PSET 2)
3. Differences between normal-tangential and polar coordinate systems (relevant in general as well as to Q3 and Q4 on PSET 2)
4. Relative motion simple and advanced examples with matlab code attached separately (relevant to Q5 on PSET 2)

Work and Power

Power: $P = \mathbf{F} \cdot \mathbf{v}$ $\quad P = \mathbf{M} \cdot \omega$

Work: $W_{AB} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} P\, dt = \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F} \cdot \mathbf{v}\, dt$

$$= \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F} \cdot d\mathbf{r}$$



Q: What direction is the work acting? (see figure)

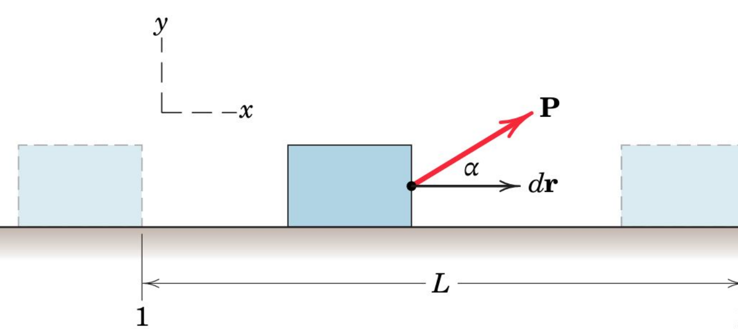
$F_{effective}$ = F cos α with α between path and force

Example: work associated with constant force P

Q: What is direction of d**r**? Which component of P should we take?

$$U_{1\text{-}2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 [(P \cos \alpha)\mathbf{i} + (P \sin \alpha)\mathbf{j}] \cdot dx\, \mathbf{i}$$

$$= \int_{x_1}^{x_2} P \cos \alpha\, dx = P \cos \alpha (x_2 - x_1) \boxed{= PL \cos \alpha}$$



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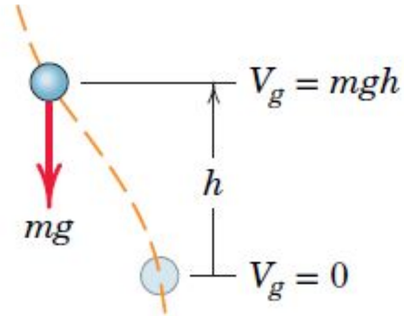
Energy (or more fundamental work equations)

Gravitational Potential Energy: work done against gravity *g* (small altitude variation) to lift particle some *h*

- In practice, consider change in height!

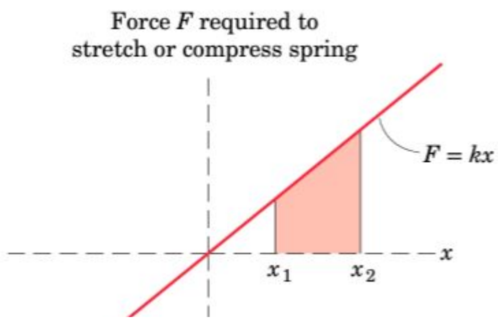
$$\Delta V_g = mg(h_2 - h_1) = mg\Delta h$$



Elastic Potential Energy: work done to deform a spring some amount length x

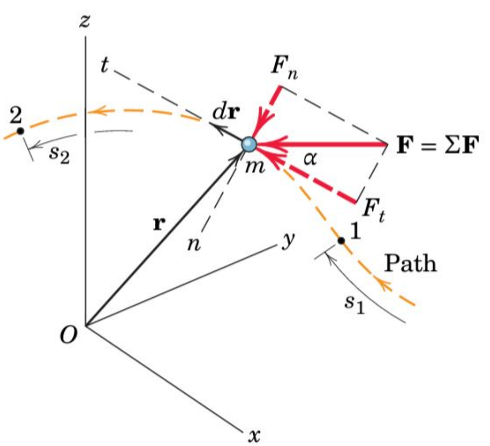
- In practice, consider change in length!

$$\Delta V_e = \frac{1}{2}k({x_2}^2 - {x_1}^2)$$



Kinetic Energy: work done by **F** = m**a** along some curvilinear path with $\mathbf{a} \cdot d\mathbf{r} = a_t ds$ (= v dv)

$$\triangle T = \frac{1}{2}m(v_2^2 - v_1^2)$$



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Work - Energy Theorem

Conservation of energy

- "Before" state: $T_1 + V_1$
- "Between" state: **external** forces performing external $U_{1\rightarrow2}$ work
- "After" state: $T_2 + V_2$

$$\boxed{T_1 + V_1 + U_{1\rightarrow2} = T_2 + V_2}$$

$$\boxed{U_{1\rightarrow2} = \triangle T + \triangle V}$$

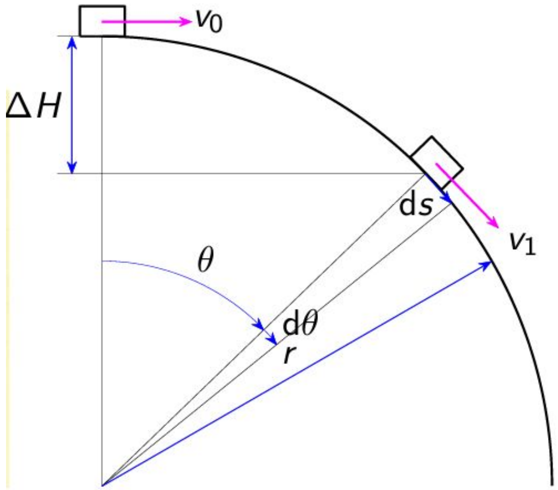
Q: When might it be more useful than going through full equilibrium equations $\sum F = ma$?

Determine velocities, positions, elongations at some moment when trajectory knowledge is not readily available or needed

Q: How can it be combined with $\sum F = ma$?

Examples: initial conditions or evaluation of unknown constants such as traveled distance, velocities

Example: Work and Newton's law



At which angle θ_{max} does the package break free from the belt? Given the belt, with radius R is smooth (μ=0).The initial velocity of the package is v_0.

Q1: What does it mean for a package to break free?

When package sits on the belt firmly it has normal force acting on it, but when it slides off there is no more normal force.

Q2: What kinematics do we have going on here? What coordinate system to choose?

We already know basic kinematics here, its circular motion. Either polar or normal and tangential work here.

Q3: What do we need to do to know when N = 0?

Draw FBD next and see where N appears and how it is related to motion of the packet.

Example: Work and Newton's law

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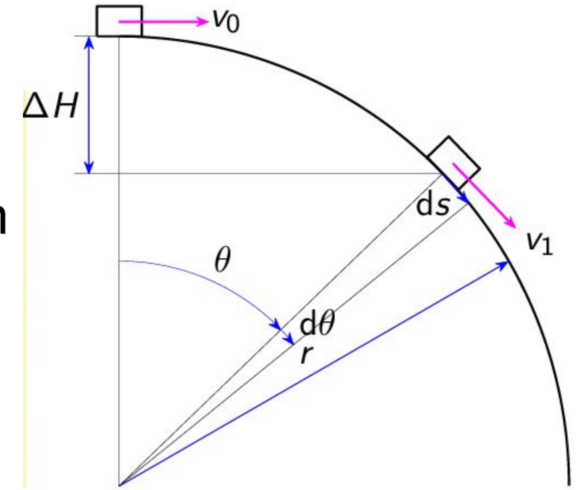
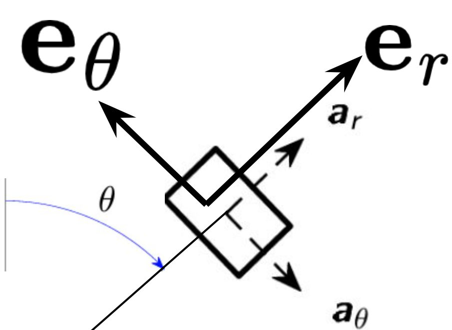


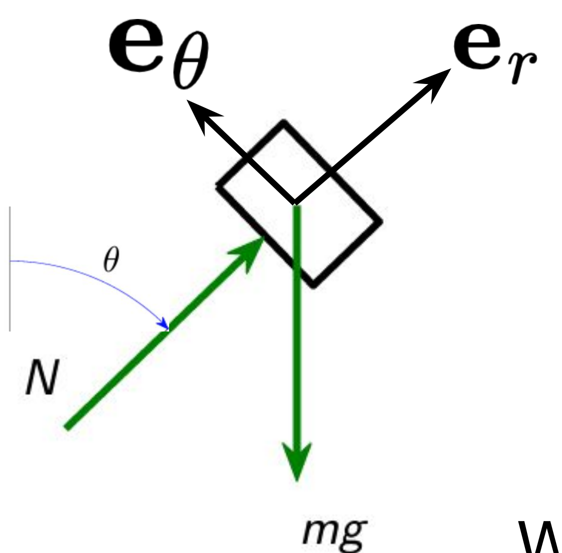
Step 1: Kinematics



$$\mathbf{a}_{polar} = (\ddot{r} - r\omega^2)\mathbf{e}_r + (r\alpha + 2\dot{r}\omega)\mathbf{e}_\theta$$

$$\mathbf{a}_{polar} = (-\frac{v^2}{R})\mathbf{e}_r + a_\theta \mathbf{e}_\theta$$

Step 2: Complete FBD: sufficient to show others and they can write $\sum F = ma$ from it!



Step 3: $\sum F = ma$

$$\sum F_\theta = m\,g\,\sin\theta = m\,a_\theta$$

$$\rightarrow a_\theta = g\,\sin\theta$$

$$\sum F_r = N - m\,g\,\cos\theta = m\,a_r$$

$$= -m\dot{\theta}^2 R = -m\frac{v^2}{R}$$

What is v?

$$\mathbf{v}_{polar} = \dot{r}\mathbf{e}_r + \omega r\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$

Example: Work and Newton's law

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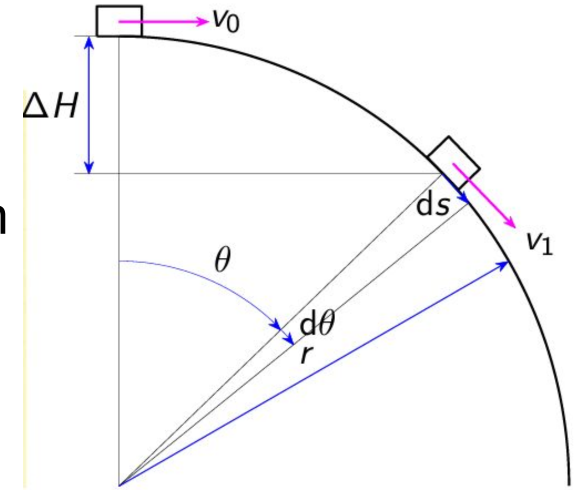

Step 4: Solve - how would you do it using regular way?
(N = 0) Hint use: $a_t$ds (= v dv)

$$\int_{\theta=0}^{\theta_{\max}} g\, R\, \sin\theta \mathrm{d}\theta = g\, R\,(1 - cos\theta_{\max}) = \frac{1}{2}v^2 - \frac{1}{2}v_0^2$$

$$m\, g\; cos\theta_{\max} = m\frac{v^2}{R} \rightarrow v^2 = g\, R\, \cos\theta_{\max}$$

Angle at which package breaks free:

$$2gR\,(1 - cos\theta_{\max}) = g\, R\, \cos\theta_{\max} - v_0^2 \rightarrow$$

$$2g\, R + v_0^2 = 3g\, R\, cos\theta_{\max} \rightarrow \theta_{\max} = \arccos\left(\frac{2}{3} + \frac{v_0^2}{3g\, R}\right)$$

Step 3: ∑F = ma

$$\sum F_\theta = m\, g\, \sin\theta = m\, a_\theta$$

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Example: Work and Newton's law

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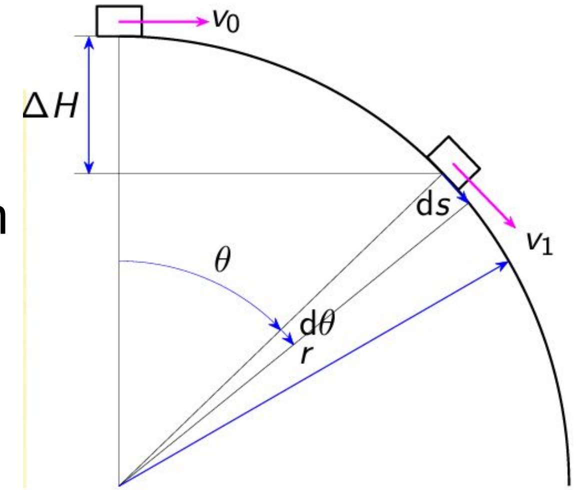

Step 4: Solve - how would you do it using work-energy? (N = 0)

Amount of work delivered is the change between kinetic energy at locations 1 and 2:

$$U_{1-2} = T_2 - T_1$$

Only gravitation is delivering work:

$$m\,g\,\Delta h = \frac{1}{2} m\,v^2 - \frac{1}{2} m\,v_0^2$$
$$\Delta h = R\,(1 - \cos\theta_{\text{max}})$$

Again using equation of motion in radial direction:

$$F_r = N - m\,g\,\cos\theta = m\,a_r = -m\dot{\theta}^2 R = -m\frac{v^2}{R}$$

At θ_{max}: $N = 0$ (package breaks free):

$$\theta_{\text{max}} = \arccos\left(\frac{2}{3} + \frac{v_0^2}{3g\,R}\right)$$

Step 3: $\sum$F = ma

$$\sum F_\theta = m\,g\,\sin\theta = m\,a_\theta$$
$$\rightarrow a_\theta = g\,\sin\theta$$

$$\sum F_r = N - m\,g\,\cos\theta = m\,a_r$$
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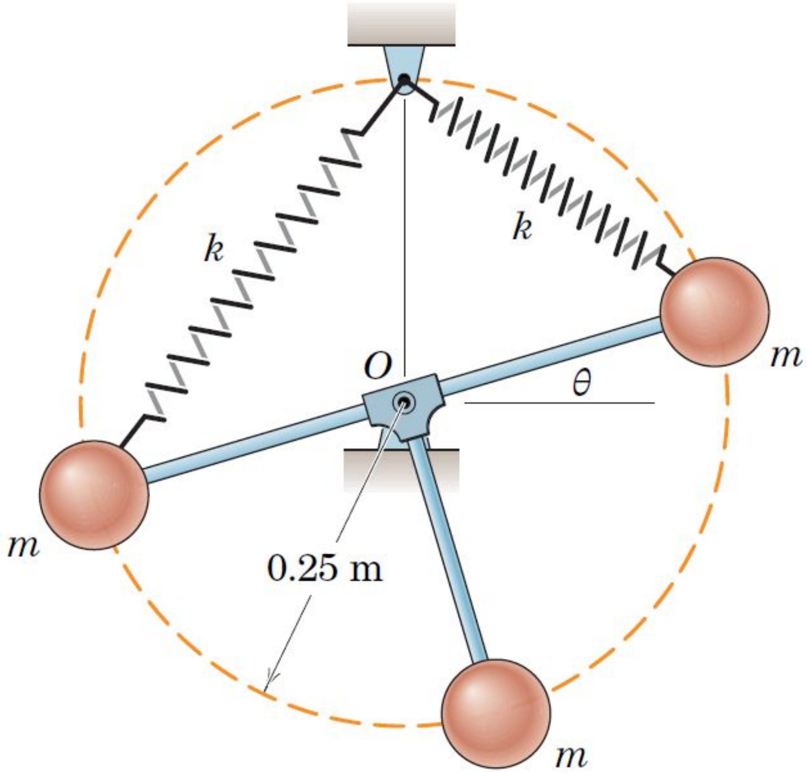
$$\mathbf{v}_{polar} = \dot{r}\mathbf{e}_r + \omega r\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$

Practice Problem 6:

The two springs, each of stiffness $k = 1.2$ kN/m, are of equal length and undeformed when $\theta = 0$. If the mechanism is released from rest in the position $\theta = 20°$, determine its angular velocity $\dot{\theta}$ when $\theta = 0$. The mass m of each sphere is 3 kg. Treat the spheres as particles and neglect the masses of the light rods and springs.



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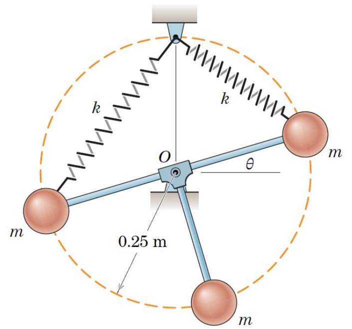
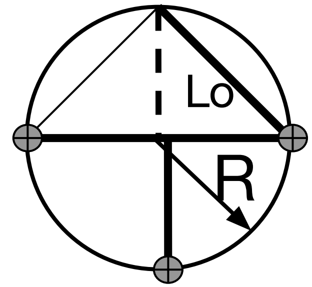
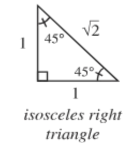


Describe the equilibrium state, before state, state after, and any work it took:

1. Equilibrium state: Undeformed at θ=0. What useful information can it provide about?
   a. Undeformed length of the springs!
   b. To determine relative elongation at at θ.
2. Before/deformed state: what energies are present?
   a. Not kinetic energy: released from rest!
   b. Elastic energy: relative to the equilibrium state
   c. Potential energy: reference point?
3. After: passing through equilibrium after release. What energies are present?
   a. Kinetic energy: angular velocity? $v = \omega R$
   b. No elastic energy
   c. Potential energy

1. Equilibrium state





What is undeformed length Lo?

$$L_0 = R\sqrt{2}$$

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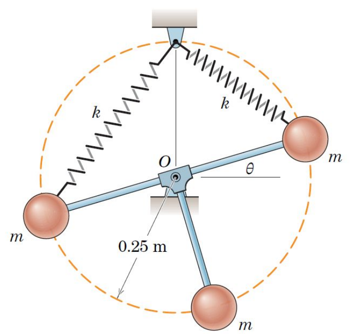

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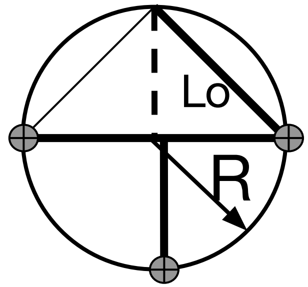
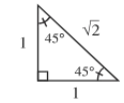
$$v = \omega R$$

1. Equilibrium state





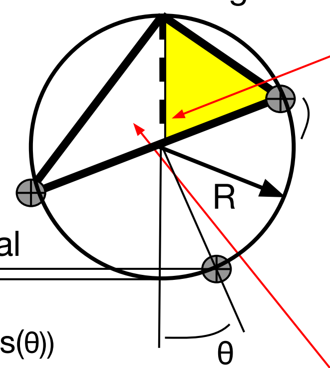
What is undeformed length Lo?

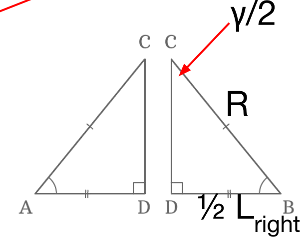
$$L_0 = R\sqrt{2}$$

2. "Before" state

2.1 Calculate L_{right}: Observe shaded triangle

What is the angle here? Let γ = 90 - θ





$$\sin(\gamma/2) = \frac{1/2 L_{right}}{R}$$

$$\boxed{L_{right} = 2R\sin(\gamma/2)}$$

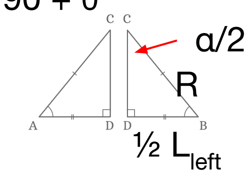
2.4 Get potential energy:

V_1 = mg(R - Rcos(θ))

2.2 Calculate L_{left}: Let α = 90 + θ

$$\sin(\alpha/2) = \frac{1/2 L_{left}}{R}$$

$$\boxed{L_{left} = 2R\sin(\alpha/2)}$$



2.3 Get elongations:

$$L_{left} - L_0 = R(2\sin(\alpha/2) - \sqrt{2})$$
$$L_{right} - L_0 = R(2\sin(\gamma/2) - \sqrt{2})$$

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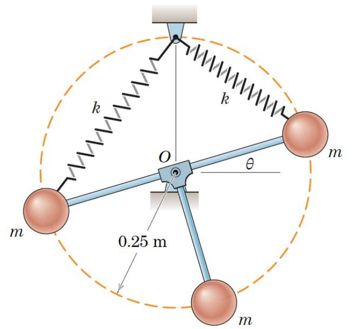

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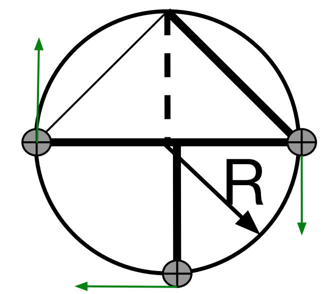
3. Final state 2:

$v = \omega R$



T1 + V1 = T2 + V2

T1 = 0

V1 = PE1 + ½ k $(e_{right})^2$ + ½ k $(e_{left})^2$

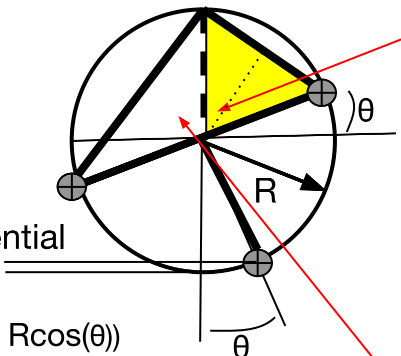
T2 = 3(½ mv^2)= 3(½ $m(wr)^2$)

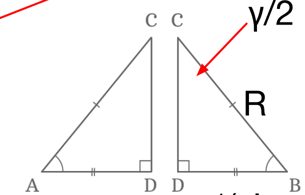
V2 = 0

2. "Before" state 1

2.1 Calculate L_{right}: Observe shaded triangle

What is the angle here? Let γ = 90 - θ





½ L_{right}

$$\sin(\gamma/2) = \frac{1/2 L_{right}}{R}$$

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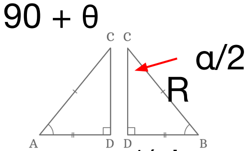

½ L_{left}

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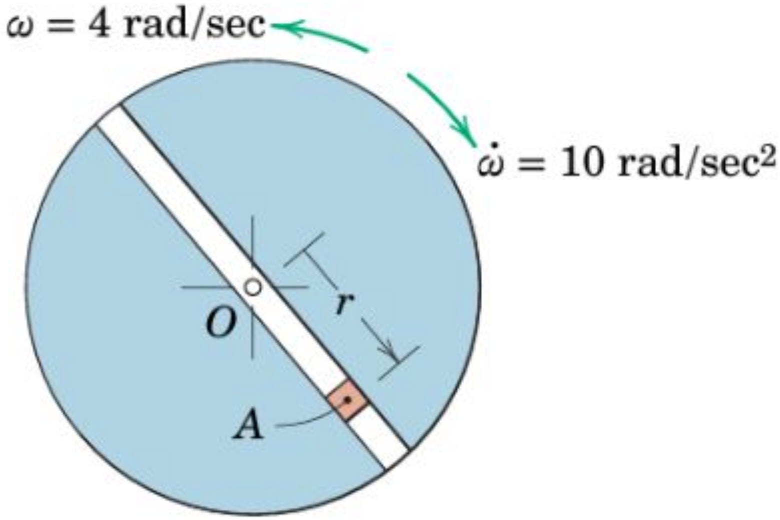
$$L_{left} - L_0 = R(2\sin(\alpha/2) - \sqrt{2})$$
$$L_{right} - L_0 = R(2\sin(\gamma/2) - \sqrt{2})$$

Practice Problem 3

At the instant represented, the disk with the radial slot is rotating about O with a counterclockwise angular velocity of 4 rad/sec which is decreasing at the rate of 10 rad/sec^2. The motion of slider A is separately controlled, and at this instant, $r = 6$ in., $\dot{r} = 5$ in./sec, and $\ddot{r} = 81$ in./sec^2. Determine the absolute velocity and acceleration of A for this position.



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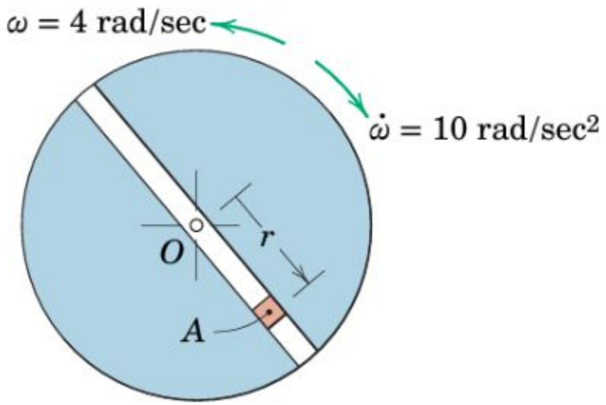


Q1: Do we need a FBD here?

(A) Yes (B) No

Q2: What coordinate system makes sense to use here?

(A) Cartesian (B) Polar

(C) Normal-Tangential (D) Rotating frame

Q3: By convention, which direction does **e_r** point in polar coordinates?

(A) Outward from reference point (B) Inward from reference point

(C) Outward normal the the path (D) Inward normal to the path

Q4: By convention, which direction does **e_theta** point in polar coordinates

(A) Tangent along path (B) Tangent in reverse to path

(C) Clockwise from reference point (D) Counterclockwise from reference point

Review polar versus normal-tangential coordinates: I

Q5: Is **e_r** *always* normal to the surface?

(A) Yes $\boxed{\text{(B) No}}$

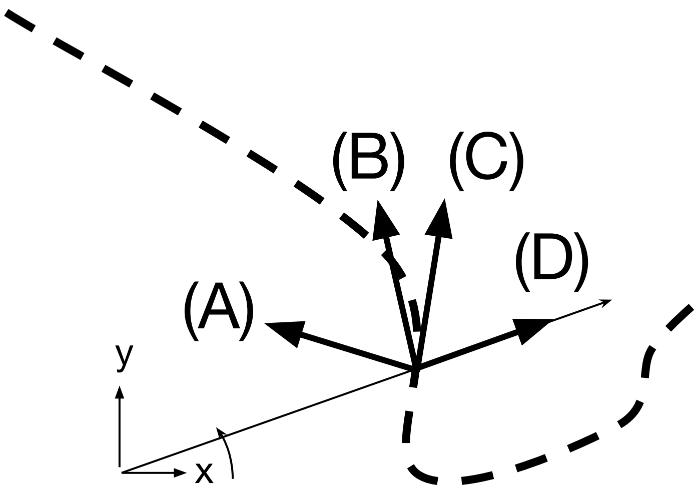
Q6: What is the main difference between **e_r** in polar coordinates and **e_normal** in normal-tangential coordinates? **Reference point, radius change**

Q7: Which arrow represents **e_r**? **D**

Q8: Which arrow represents **e_normal**? **A**

Q9: Which arrow represents **e_theta**? **B**

Q10: Which arrow represents **e_tangent**? **C**



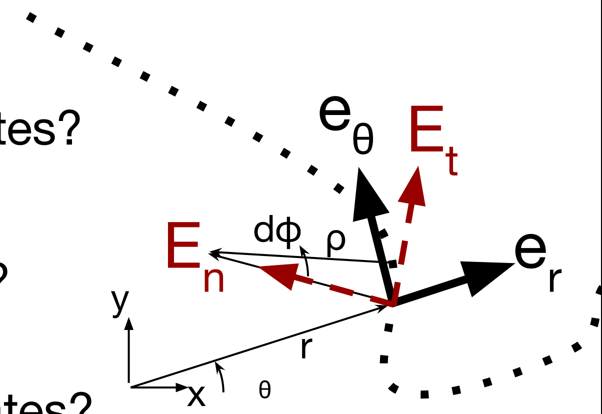
Review polar versus normal-tangential coordinates: II

Q11: What is position formula in polar coordinates?

$$\mathbf{r}_{polar} = r\mathbf{e}_r + z\mathbf{e}_z$$

Q12: What is position formula in normal-tangential coordinates?

$$\mathbf{r}_{N-T} = 0\mathbf{E}_t + 0\mathbf{E}_n + z\mathbf{E}_z$$



Q13: What is distance traveled formula in polar coordinates?

$$s_{polar}(\theta) = \int r(\theta) d\theta$$

Q14: What is distance formula in normal-tangential coordinates?

$$s_{N-T}(\phi) = \int \rho d\phi$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Q15: What is velocity formula in polar coordinates?

$$\mathbf{v}_{polar} = \dot{r}\mathbf{e}_r + \omega r\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\frac{dr}{d\theta} - r\frac{d^2r}{d\theta^2}}$$

Q16: What is velocity formula in N-T coordinate system?

$$\mathbf{v}_{N-T} = \frac{ds}{dt}\mathbf{e}_t + 0\mathbf{e}_n + \dot{z}\mathbf{E}_z$$

Q17: What is acceleration formula in polar coordinates? What direction?

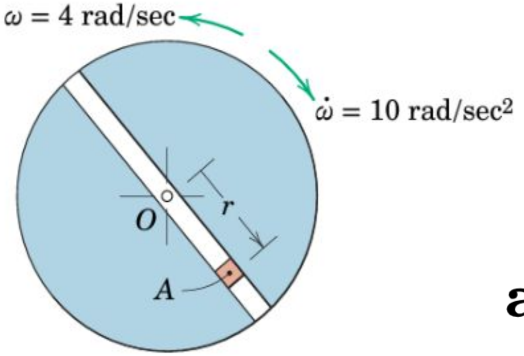
$$\mathbf{a}_{polar} = (\ddot{r} - r\omega^2)\mathbf{e}_r + (r\alpha + 2\dot{r}\omega)\mathbf{e}_\theta + \ddot{z}\mathbf{E}_z$$

Q18: What is acceleration formula in normal-tangential coordinate system? What direction?

$$\mathbf{a}_{N-T} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n + \ddot{z}\mathbf{E}_z$$

Practice Problem 3: polar coordinates approach!

At the instant represented, the disk with the radial slot is rotating about O with a counterclockwise angular velocity of 4 rad/sec which is decreasing at the rate of 10 rad/sec^2. The motion of slider A is separately controlled, and at this instant, $r = 6$ in., $\dot{r} = 5$ in./sec, and $\ddot{r} = 81$ in./sec^2. Determine the absolute velocity and acceleration of A for this position.



Step 1: What we need to find?

Absolute velocity and acceleration

Step 2: What are general equations for the kinematic parameters we need to find?

$$\mathbf{v}_{polar} = \dot{r}\mathbf{e}_r + \omega r\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$

$$\mathbf{a}_{polar} = (\ddot{r} - r\omega^2)\mathbf{e}_r + (r\alpha + 2\dot{r}\omega)\mathbf{e}_\theta + \ddot{z}\mathbf{E}_z$$

Step 3: Determine what we know in the general formulas above?

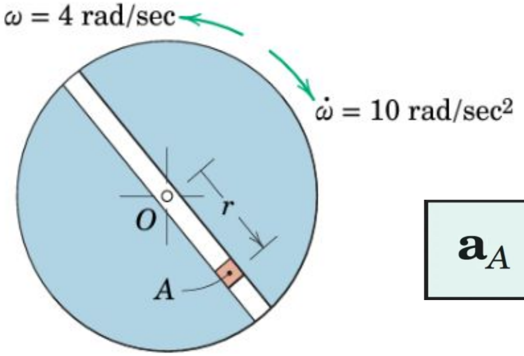
Given: r̈, ṙ, r, ω, ∝ (z = 0, plane)

Step 4: Determine what we don't know in the general formulas above?

Seems like we are good to plug in and move on

Practice Problem 3: rotating-frame coordinates approach!

At the instant represented, the disk with the radial slot is rotating about O with a counterclockwise angular velocity of 4 rad/sec which is decreasing at the rate of 10 rad/sec^2. The motion of slider A is separately controlled, and at this instant, $r = 6$ in., $\dot{r} = 5$ in./sec, and $\ddot{r} = 81$ in./sec^2. Determine the absolute velocity and acceleration of A for this position.



Step 1: What we need to find?

Absolute velocity and acceleration

Step 2: What are general equations for the kinematic parameters we need to find?

$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{rel} + \mathbf{v}_{rel}}$$

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

Step 3: Determine what we know in the general formulas above?

Given: $a_{rel} = \ddot{r}$, $v_{rel} = \dot{r}$, $r_{rel} = r$, ω, ∝, ($z = 0$, plane), $a_B = v_B = 0$ (center is fixed)

Step 4: Determine what we don't know in the general formulas above?

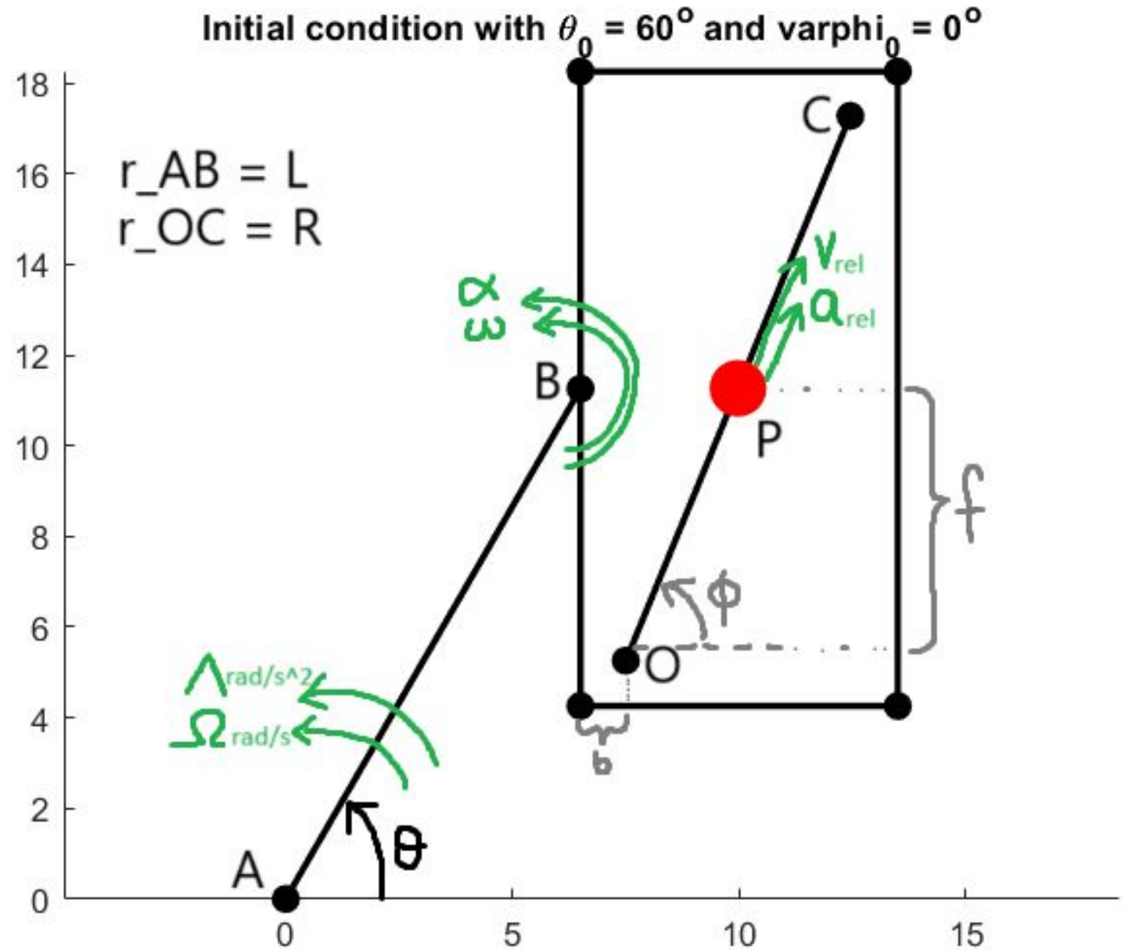
Use: $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ **Seems like we are good to plug in and move on**

$$\vec{a}_A = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \alpha \end{Bmatrix} \times \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} \times \begin{Bmatrix} r \\ 0 \\ 0 \end{Bmatrix} + 2 \begin{Bmatrix} 0 \\ 0 \\ \omega \end{Bmatrix} \times \begin{Bmatrix} \dot{r} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \ddot{r} \\ 0 \\ 0 \end{Bmatrix}$$

$$\vec{a}_A = \begin{Bmatrix} 0 \\ r\alpha \end{Bmatrix} + \begin{Bmatrix} -r\omega^2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 2\dot{r}\omega \end{Bmatrix} + \begin{Bmatrix} \ddot{r} \\ 0 \end{Bmatrix}$$

Compare with: $\mathbf{a}_{polar} = (\ddot{r} - r\omega^2)\mathbf{e}_r + (r\alpha + 2\dot{r}\omega)\mathbf{e}_\theta$

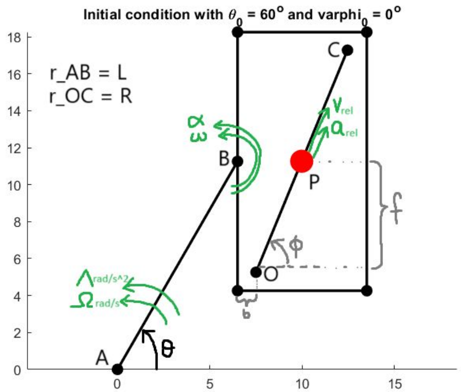
Practice Problem 4: rotating frame demo



Particle P sits in a slot. Platform and arm are rotating, describe resulting motion along the slot. Don't neglect gravity.

Start with sketching relative position, velocity, and acceleration diagrams in order. See if you can derive accelerations yourself!

Practice Problem 4: rotating frame demo



Step 1: What we need to find?

a_{rel} along the slot

Step 2: What are *general* equations for the kinematic parameters we need to find?

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{rel} = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{rel} + \mathbf{v}_{\text{rel}}$$

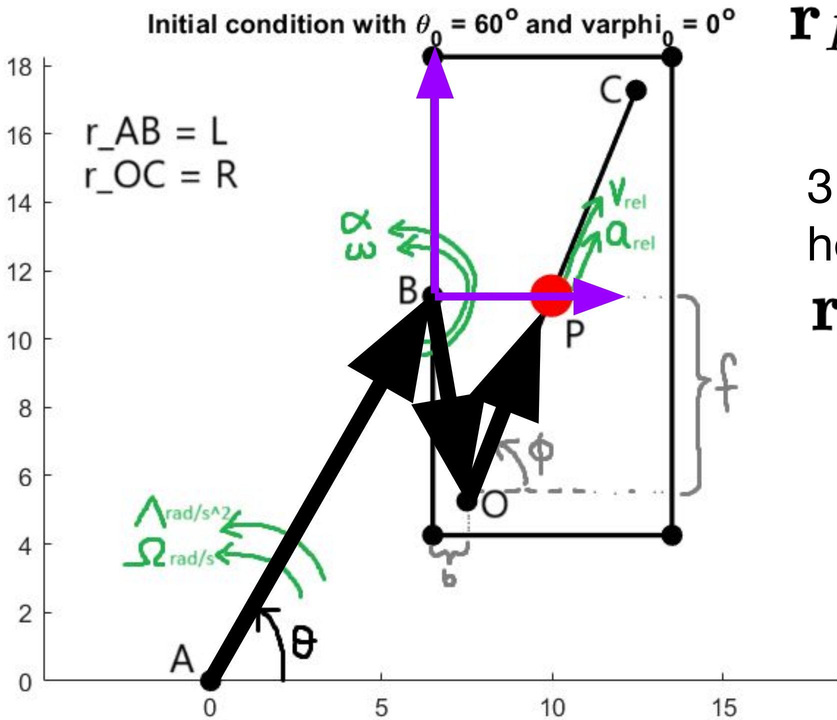
$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

Step 3: Determine what we know in the *general* formulas above?

Position:

3.1 Where can we place rotating frame of reference?

3.2 Using chosen reference frame and vector addition what path can we take to point P?

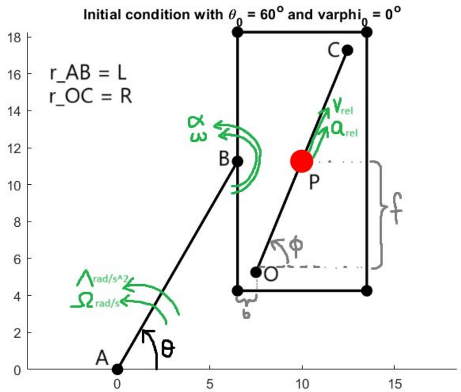


$$\mathbf{r}_P = \mathbf{r}_{AB} + \boxed{\mathbf{r}_{bo} + \mathbf{r}_{op}}$$

3.3 What vector is $\mathbf{r}_{rel}$ here (AB, bo, op)?

$$\mathbf{r}_{op} = \mathbf{r}_{rel}$$

Practice Problem 4: rotating frame demo



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a_{rel} along the slot

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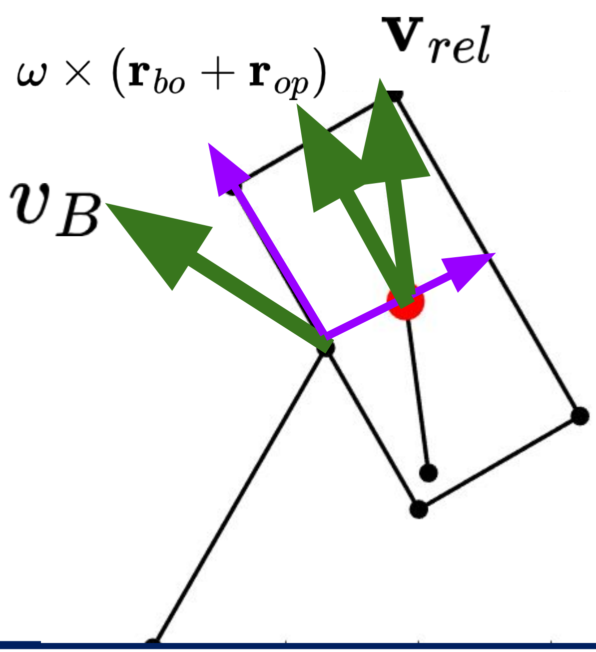
$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

Step 3: Determine what we know in the *general* formulas above?

$$\mathbf{r}_{op} = \mathbf{r}_{rel}$$

$$\mathbf{r}_P = \mathbf{r}_{AB} + \boxed{\mathbf{r}_{bo} + \mathbf{r}_{op}}$$

Velocity:



$$\mathbf{v}_P = \mathbf{v}_B + \boxed{\omega \times (\mathbf{r}_{bo} + \mathbf{r}_{rel}) + \mathbf{v}_{rel}}$$

3.4 Given that arm has angular velocity Ω, what is vB?

$\mathbf{v}_B$ is Ω x $\mathbf{r}_{AB}$

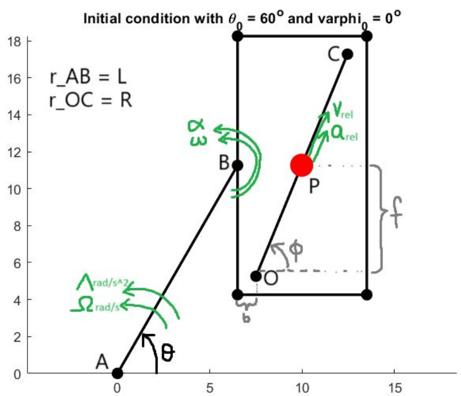
3.5 How does $\mathbf{r}_{bo}$ change?

$\mathbf{r}_{bo}$ is fixed magnitude (only rotates)

3.6 How does $\mathbf{r}_{rel}$ change?

$\mathbf{r}_{rel}$ both rotates and changes in magnitude (slides along the slot)

Practice Problem 4: rotating frame demo



Step 1: What we need to find?

a_{rel} along the slot

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$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{rel} + \mathbf{v}_{\text{rel}}}$$

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}}$$

Step 3: Determine what we know in the *general* formulas above?

$$\mathbf{r}_{op} = \mathbf{r}_{rel}$$

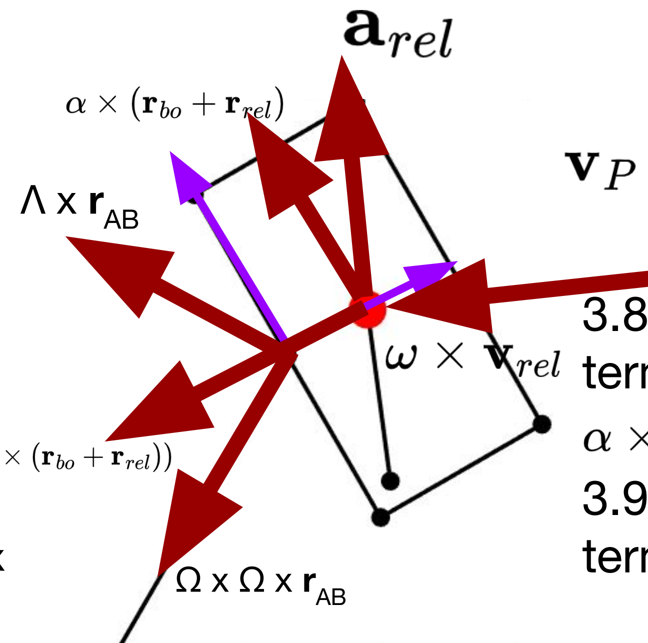
$$\mathbf{r}_P = \mathbf{r}_{AB} + \boxed{\mathbf{r}_{bo} + \mathbf{r}_{op}}$$

$$\mathbf{v}_P = \mathbf{v}_B + \boxed{\omega \times (\mathbf{r}_{bo} + \mathbf{r}_{rel}) + \mathbf{v}_{rel}}$$

Acceleration:

3.7 Given that arm has angular velocity Ω and angular acceleration Λ, what is aB?

a_B is $\Omega \times \Omega \times \mathbf{r}_{AB} + \Lambda \times \mathbf{r}_{AB}$



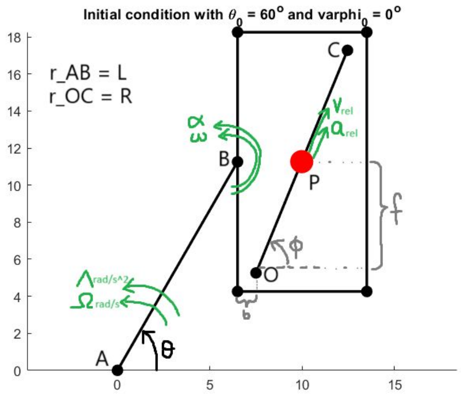
3.8 Chain rule middle term. How many terms? List them. $\omega \times \mathbf{v}_{rel}$

$\alpha \times (\mathbf{r}_{bo} + \mathbf{r}_{rel}) \quad \omega \times (\omega \times (\mathbf{r}_{bo} + \mathbf{r}_{rel}))$

3.9 Chain rule last term. How many terms? List them. $\omega \times \mathbf{v}_{rel} \quad \mathbf{a}_{rel}$

$$\mathbf{a}_P = \mathbf{a}_B + \boxed{\alpha \times (\mathbf{r}_{ob} + \mathbf{r}_{rel}) + \omega \times (\omega \times (\mathbf{r}_{ob} + \mathbf{r}_{rel})) + 2\omega\mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

Practice Problem 4: rotating frame demo



Step 1: What we need to find?

$\mathbf{a}_{rel}$ along the slot

Step 2: What are *general* equations for the kinematic parameters we need to find?

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$$\mathbf{v}_A = \mathbf{v}_B + \omega \times \mathbf{r}_{rel} + \mathbf{v}_{\mathrm{rel}}$$

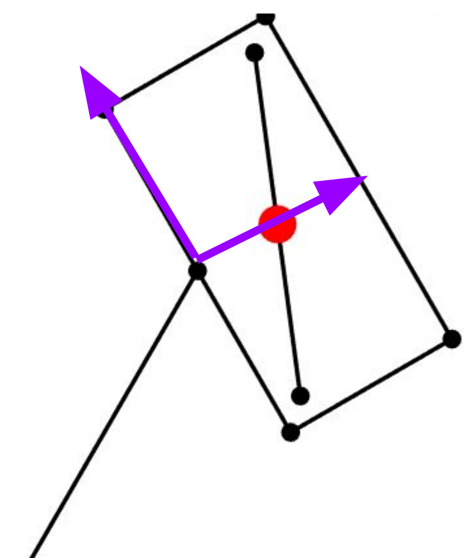
$$\mathbf{a}_A = \mathbf{a}_B + \dot{\omega} \times \mathbf{r}_{rel} + \omega \times (\omega \times \mathbf{r}_{rel}) + 2\omega \times \mathbf{v}_{\mathrm{rel}} + \mathbf{a}_{\mathrm{rel}}$$

Step 4: Determine what we don't know in the *general* formulas above?

$$\mathbf{r}_{op} = \mathbf{r}_{rel}$$

$$\mathbf{r}_P = \mathbf{r}_{AB} + \boxed{\mathbf{r}_{bo} + \mathbf{r}_{op}}$$

4: Look at equations carefully and observe what we need to solve for and how?

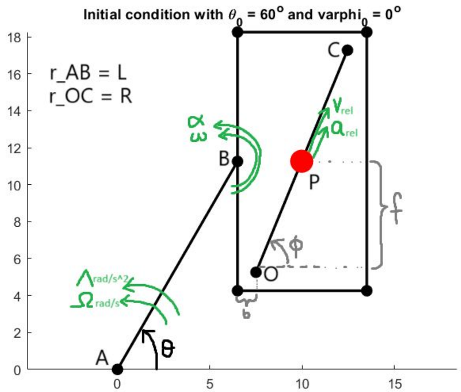


$$\mathbf{v}_P = \mathbf{v}_B + \boxed{\omega \times (\mathbf{r}_{bo} + \mathbf{r}_{rel}) + \mathbf{v}_{rel}}$$

Assuming 2 angular velocities and 2 angular accelerations are known, we don't know $\mathbf{a}_{rel}$, $\mathbf{v}_{rel}$, and $\mathbf{r}_{rel}$: but they are all related! —> FBD in terms of total acceleration —> ODE solver

$$\mathbf{a}_P = \mathbf{a}_B + \boxed{\alpha \times (\mathbf{r}_{ob} + \mathbf{r}_{rel}) + \omega \times (\omega \times (\mathbf{r}_{ob} + \mathbf{r}_{rel})) + 2\omega\mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

Practice Problem 4: rotating frame demo



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a_{rel} along the slot

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$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{\text{rel}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\text{rel}}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}}$$

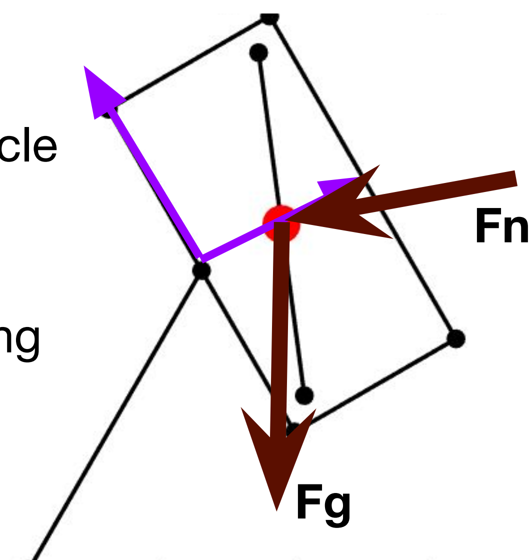
Step 5: Draw a Free Body Diagram (FBD)

5.1 What is touching the particle? Which direction?

- Slot is keeping particle from escaping

- Normal to the wall! Moves with the rotating frame.

$$\mathbf{F}_n = F_n \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix}$$



5.2 Body forces? Which direction?

- Gravity is not neglected

- Does not rotate with reference frame

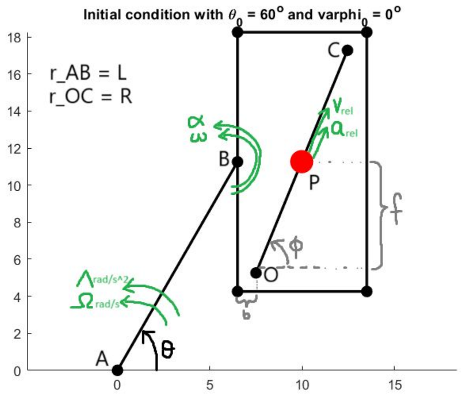
- Project onto **i-j** coordinate system: assuming i-j has rotated some φ angle

$$\mathbf{F}_g = mg \begin{bmatrix} -\sin(\varphi) \\ -\cos(\varphi) \end{bmatrix}$$

5.3 Do we know φ? Yes, func. of time

$$\mathbf{a}_P = \mathbf{a}_B + \boxed{\alpha \times (\mathbf{r}_{ob} + \mathbf{r}_{rel}) + \omega \times (\omega \times (\mathbf{r}_{ob} + \mathbf{r}_{rel})) + 2\omega\mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

Practice Problem 4: rotating frame demo



Step 1: What we need to find?

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Step 6: Assemble equilibrium equation. Comes directly from kinematics and FBD!

$$\sum \mathbf{F} = m\mathbf{a}$$

$$\mathbf{F}_n = F_n \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix}$$

6.1 Vector form:

$$\mathbf{F}_g + \mathbf{F}_n = m\mathbf{a}_p$$

$$\mathbf{F}_g = mg \begin{bmatrix} -\sin(\varphi) \\ -\cos(\varphi) \end{bmatrix}$$

6.2 Decomposed into i-j coordinates: see full notes for that

$$\sum F_i \quad = m\left(a_{rel}c\phi + \alpha(f - r_{rel}s\phi) - 2\omega v_{rel}s\phi - \omega^2(b + r_{rel}c\phi) + \Lambda L s\varphi - \Omega^2 L c\varphi\right) = -F_n s(\phi) - mgs\varphi$$

$$\sum F_j = m(a_{rel}s\phi + \alpha(b + r_{rel}c\phi) + 2\omega v_{rel}c\phi + \omega^2(f - r_{rel}s\phi) + \Lambda L c\varphi + \Omega^2 L s\varphi) = F_n c(\phi) - mgc\varphi$$

6.3 Count number of equations and unknowns: 2 equations, 2 unknowns (F_n and a_{rel})

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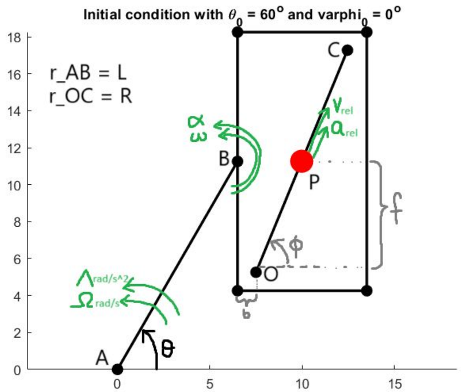
$$\mathbf{a}_P = \mathbf{a}_B + \boxed{\alpha \times (\mathbf{r}_{ob} + \mathbf{r}_{rel}) + \omega \times (\omega \times (\mathbf{r}_{ob} + \mathbf{r}_{rel})) + 2\omega \mathbf{v}_{rel} + \mathbf{a}_{rel}}$$

Practice Problem 4: rotating frame demo



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Step 7: Solve — MATLAB(+ out of scope mentions)

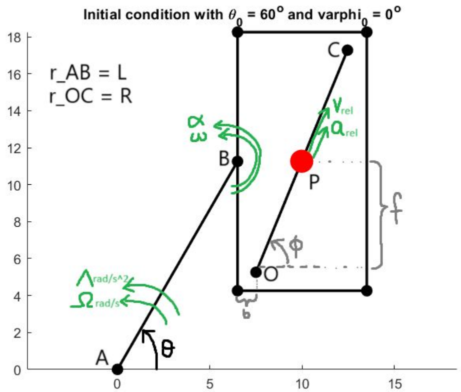
$$\sum F_i = m\left(a_{rel}c\phi + \alpha(f - r_{rel}s\phi) - 2\omega v_{rel}s\phi - \omega^2(b + r_{rel}c\phi) + \Lambda L s\varphi - \Omega^2 L c\varphi\right) = -F_n s(\phi) - mgs\varphi$$

$$\sum F_j = m(a_{rel}s\phi + \alpha(b + r_{rel}c\phi) + 2\omega v_{rel}c\phi + \omega^2(f - r_{rel}s\phi) + \Lambda L c\varphi + \Omega^2 L s\varphi) = F_n c(\phi) - mgc\varphi$$

Equations above hold only for r within 0<r<R. What happens when particle reaches top or bottom?

- Out of scope for today: separate FBDs with Fn along top and bottom, working out show that a_{rel} goes to infinity.
- Computers don't like infinity, hence set a_{rel} to zero in the solver when reaching critical locations and include “events” in ode45 that restart with flipped velocity once particle reaches top or bottom — creates a bounce! (See full notes for details)

Practice Problem 4: rotating frame demo



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Step 7: Solve — MATLAB(+ out of scope mentions)

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$$\sum F_j = m(a_{rel}s\phi + \alpha(b + r_{rel}c\phi) + 2\omega v_{rel}c\phi + \omega^2(f - r_{rel}s\phi) + \Lambda L c\varphi + \Omega^2 L s\varphi) = F_n c(\phi) - mgc\varphi$$

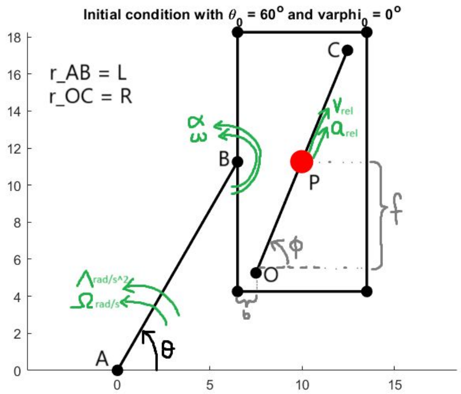
7.1 Need a_{rel} but have F_n: What to do?

Let MATLAB do the work, solve for Fn of one equation and plug into another. Don't do it by hand, let chunks of repeated code/expressions equal to some function!

$$a_{rel} = \frac{H_i \frac{c_j}{c_i} - H_j}{m\left(s\phi - c\phi \frac{c_j}{c_i}\right)}$$

```
if  tol < r < (R - tol)                       (I)
    F_N,i = -F_N sϕ;  ⇒ C_i = -sϕ
    F_N,j =  F_N cϕ;  ⇒ C_j =  cϕ
elseif  r < tol   % bottom                    (II)
    F_N,i = F_N cϕ;  ⇒ C_i = cϕ
    F_N,j = F_N sϕ;  ⇒ C_j = sϕ
else  % r > (R - tol)    (top)                (III)
    F_N,i = - F_N cϕ;  ⇒ C_i = -cϕ
    F_N,j = - F_N sϕ;  ⇒ C_j = -sϕ
end
```

Practice Problem 4: rotating frame demo



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$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{rel}} + \mathbf{a}_{\mathrm{rel}}}$$

Step 7: Solve — MATLAB: acceleration function defined (see full notes for more)

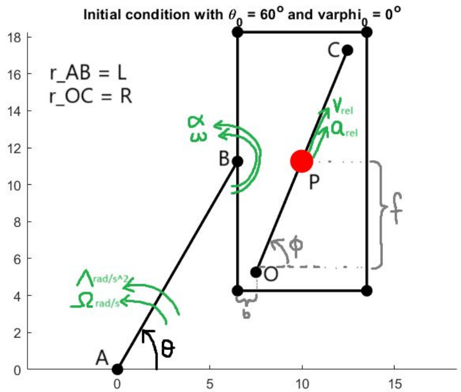
```
function a = a_rel(r, v_rel, varphi, omega, Omega, m, phi, tol, R, alpha, Lambda, f, b, L, g)
H_i = @(r, v_rel, varphi, omega, Omega) m*(alpha*(f - r*sin(phi)) -2*omega*v_rel*sin(phi)    - omega^2*(b +
r*cos(phi))   + Lambda*L*sin(varphi) - Omega^2*L*cos(varphi)   + g*sin(varphi));
H_j = @(r, v_rel, varphi, omega, Omega) m*(alpha*(b + r*cos(phi)) +2*omega*v_rel*cos(phi)    + omega^2*(f -
r*sin(phi))   + Lambda*L*cos(varphi) + Omega^2*L*sin(varphi)   + g*cos(varphi));
if (tol < r) && (r < (R-tol)) % somewhere between the top and bottom
   c_i = -sin(phi);
   c_j = cos(phi);
   numerator = H_i(r, v_rel, varphi, omega, Omega)*c_j/c_i - H_j(r, v_rel, varphi, omega, Omega);
   denominator = m*(sin(phi) - cos(phi)*c_j/c_i);
   a = numerator/denominator;
else % top or bottom, does not matter, explodes there.
   a = 0;
end
end
```

$$a_{rel} = \frac{H_i \frac{c_j}{c_i} - H_j}{m\left(s\phi - c\phi\frac{c_j}{c_i}\right)}$$

$$\sum F_i = m\left(a_{rel}c\phi + \alpha(f - r_{rel}s\phi) - 2\omega v_{rel}s\phi - \omega^2(b + r_{rel}c\phi) + \Lambda L s\varphi - \Omega^2 L c\varphi\right) = -F_n s(\phi) - mgs\varphi$$

$$\sum F_j = m(a_{rel}s\phi + \alpha(b + r_{rel}c\phi) + 2\omega v_{rel}c\phi + \omega^2(f - r_{rel}s\phi) + \Lambda L c\varphi + \Omega^2 L s\varphi) = F_n c(\phi) - mgc\varphi$$

Practice Problem 4: rotating frame demo



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a_{rel} along the slot

Step 2: What are *general* equations for the kinematic parameters we need to find?

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{rel} = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$$

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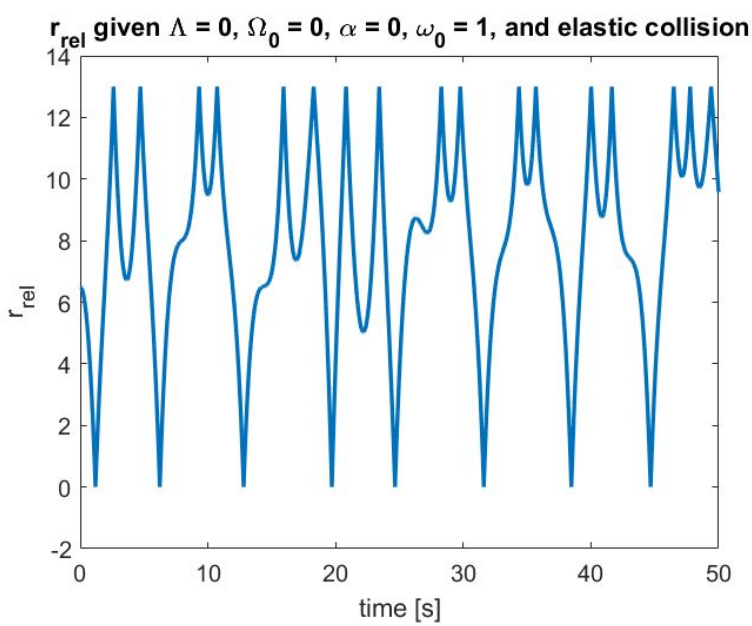
Step 7: Solve — MATLAB: some solutions



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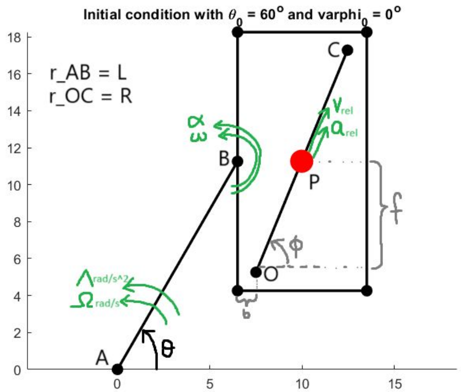

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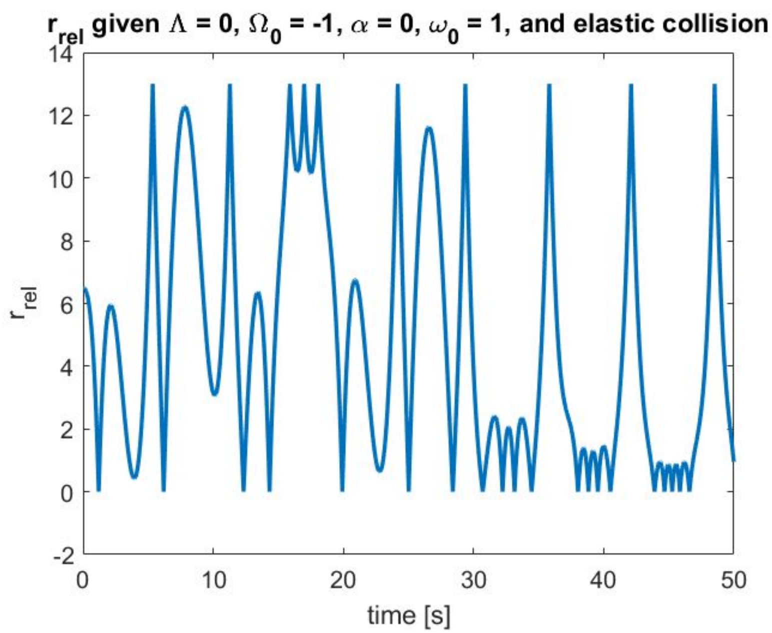

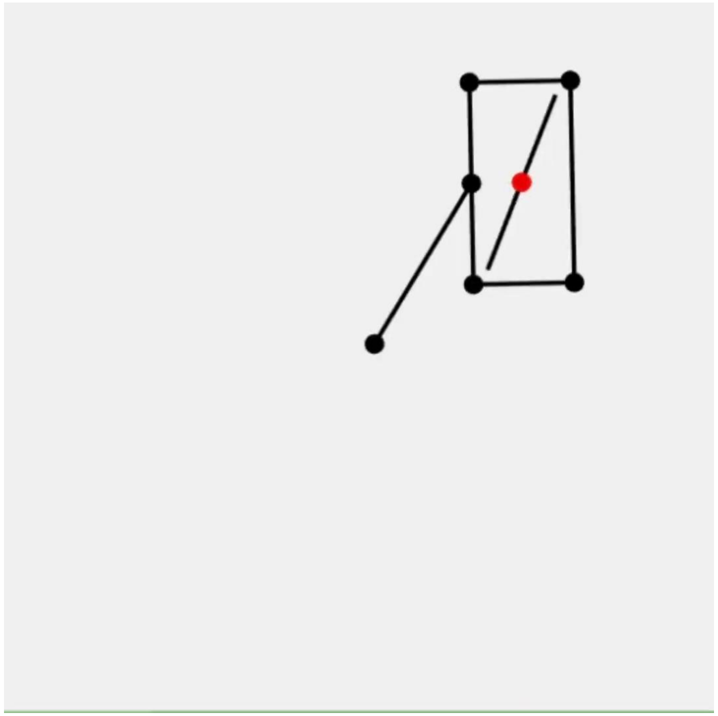

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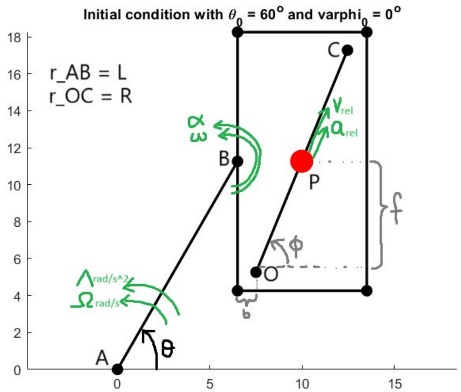

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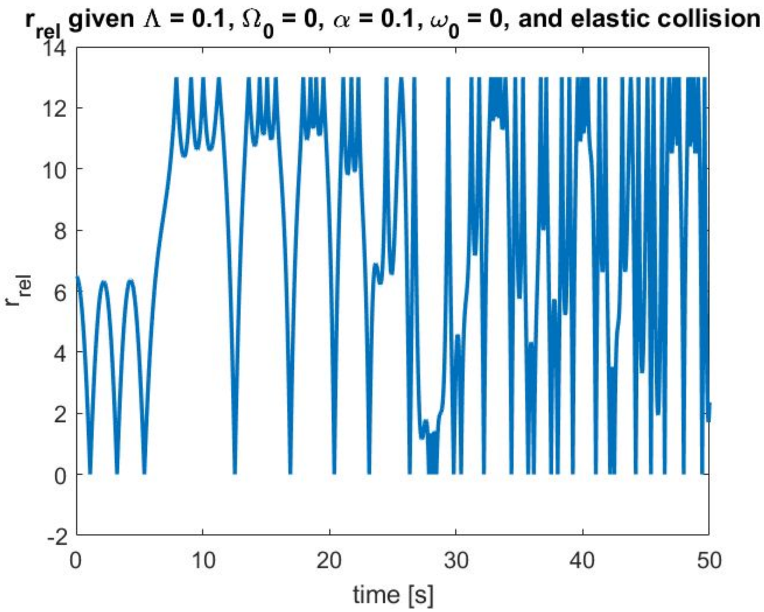

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