ME 104 Lec 3

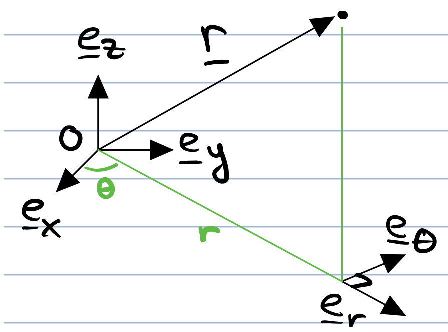
Recall from last time:

Cylindrical Coordinates:

$$r \equiv \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x).$$

$$\Longleftrightarrow$$

$$x = r\cos\theta, \quad y = r\sin\theta.$$

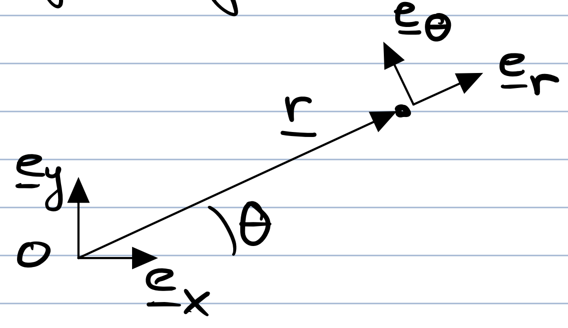


$$\underline{e}_r \equiv (x\,\underline{e}_x + y\,\underline{e}_y)/r = \cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y$$

$$\underline{e}_\theta \equiv (-y\,\underline{e}_x + x\,\underline{e}_y)/r = -\sin\theta\,\underline{e}_x + \cos\theta\,\underline{e}_y$$

$$\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z = r\,\underline{e}_r + z\,\underline{e}_z.$$

In 2D (ignoring $\underline{e}_z$) we have polar coords:



$$\underline{r} = r\,\underline{e}_r$$

In cylindrical (or polar), basis directions depend on the particle's position, thus they can change:

$$\dot{\underline{e}}_r = \frac{d\underline{e}_r}{d\theta}\dot\theta = \dot\theta\,\underline{e}_\theta, \quad \dot{\underline{e}}_\theta = \frac{d\underline{e}_\theta}{d\theta}\dot\theta = -\dot\theta\,\underline{e}_r, \quad \dot{\underline{e}}_z = 0.$$

Given these results we have

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + \dot{z}\,\underline{e}_z$$

$$= \dot{x}\,\underline{e}_x + \dot{y}\,\underline{e}_y + \dot{z}\,\underline{e}_z$$

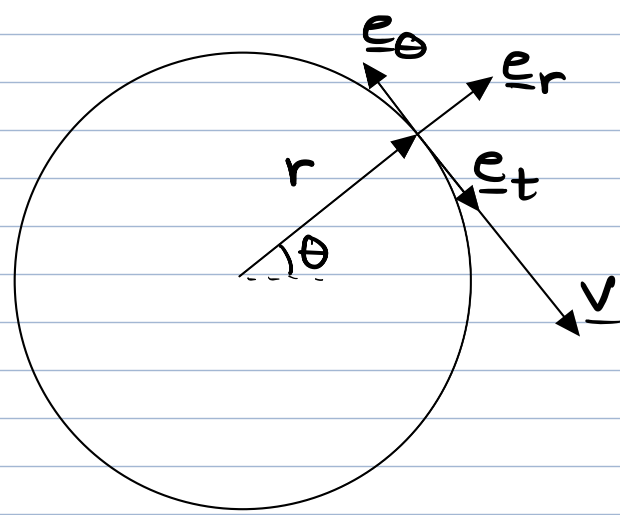
$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta + \ddot{z}\,\underline{e}_z$$

$$= \ddot{x}\,\underline{e}_x + \ddot{y}\,\underline{e}_y + \ddot{z}\,\underline{e}_z$$

Let's revisit circular motion using these new results for the case where the particle speed is not constant. $\Longrightarrow$ $\theta = \theta(t)$.

$$\underline{r}(t) = R_0(\cos\theta(t)\,\underline{e}_x + \sin\theta(t)\,\underline{e}_y) = R_0\,\underline{e}_r .$$

$$r = R_0 = \text{const}, \quad \dot{r} = 0 .$$



$$\underline{v} = \dot{\underline{r}} = r\dot{\theta}\,\underline{e}_\theta = R_0\dot{\theta}\,\underline{e}_\theta \checkmark$$

$$\underline{a} = -r\dot{\theta}^2\,\underline{e}_r + r\ddot{\theta}\,\underline{e}_\theta$$

$$= \underbrace{-R_0\dot{\theta}^2\,\underline{e}_r}_{\text{like before}} + \underbrace{R_0\ddot{\theta}\,\underline{e}_\theta}_{\text{new}}$$

$$= -\frac{\left(\frac{d}{dt}(R_0\theta)\right)^2}{R_0}\,\underline{e}_r + \frac{d}{dt}\Bigg(\underbrace{\frac{d}{dt}(R_0\theta)}_{\substack{=v \text{ if } \dot\theta>0 \\ =-v \text{ if } \dot\theta<0}\Big\} = v\,\mathrm{sign}(\dot\theta)}\Bigg)\,\underline{e}_\theta$$

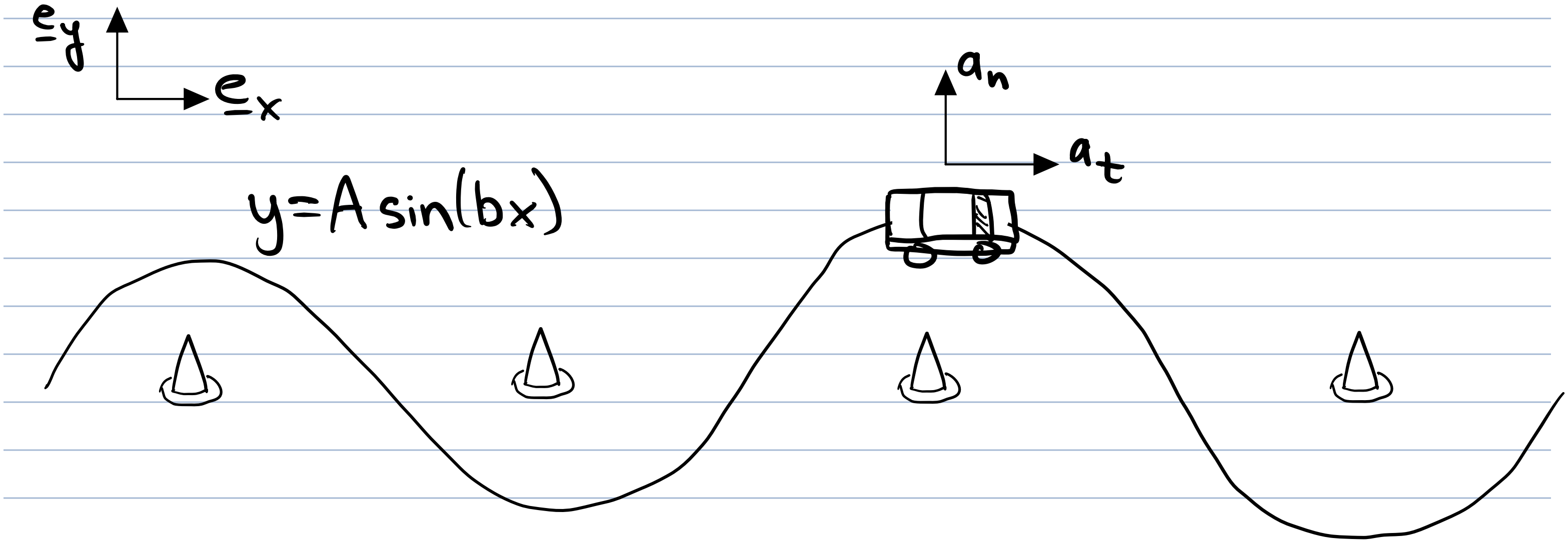
Define $\underline{e}_t \equiv \mathrm{sign}(\dot\theta)\,\underline{e}_\theta$ = "Direction of $\underline{v}$".

$\underline{e}_n \equiv \underline{e}_r$ = "Outward normal direction to the path".

$$\Rightarrow \quad \boxed{\underline{a} = -\frac{v^2}{R_0}\,\underline{e}_n + \dot{v}\,\underline{e}_t}$$

The boxed expression is quite useful when calculating the acceleration of an object at a given speed v and rate-of-speed $\dot{v}$ as it rounds a curve with curvature $\kappa \equiv \frac{1}{R}$ where R is the radius of the circle that "fits in" to the curve at that point.

Example: Car on a slalom track.



At the indicated position, the car has some known v and $\dot{v}$. Find the acceleration $\underline{a}$ at the indicated position.

Ans: $a_t = \dot{v}$, $a_n = -\kappa v^2$, $\underline{e}_t = \underline{e}_x$, $\underline{e}_n = \underline{e}_y$.

For κ, use curvature formula from Math 53:

$$\kappa = \left|\frac{d^2y}{dx^2}\right| \Big/ \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = \frac{|b^2 A \sin(bx)|}{(1 + b^2A^2\cos(bx)^2)^{3/2}}$$

At peak of curve, $x = \pi/2b$.

$$\Rightarrow \kappa = b^2 A \quad \Rightarrow \quad a_n = -b^2 A v^2$$

Thus $\boxed{\underline{a} = \dot{v}\,\underline{e}_x - b^2 A v^2\,\underline{e}_y}$.

Sometimes curves will be defined parametrically like $x = f(q)$, $y = g(q)$ for $0 < q < 1$.

The curvature formula here is:

$$\kappa = |x'y'' - y'x''| / (x'^2 + y'^2)^{3/2}.$$

And now <u>Dynamics</u>:

One-particle dynamics problems are solved in four steps:

① Write out the kinematics: Pick an origin, pick a coord system, then write expressions for $\underline{r}$, $\underline{v}$, and $\underline{a}$.

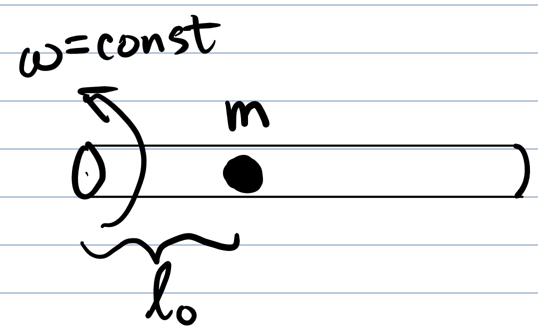
② Draw a <u>free body diagram</u> (FBD):

- Draw the object.
- Draw in force vectors. Hint: forces should appear everywhere the object is <u>touched</u> by something else, and <u>weight</u> should be added if gravity non-negligible.

③ Write out $\Sigma \underline{F} = m\underline{a}$. (Newton's 2nd Law)

④ Perform the analysis (analytic or numeric) to compute $\underline{a}(t) \Rightarrow \underline{v}(t) \Rightarrow \underline{x}(t)$. Often will also want to compute the unknown constraint forces.

Example: A frictionless tube has a ball inside it at rest positioned l_0 from the end. The tube is then spun about the end at $\omega = \text{const}$. Assume linear air drag in tube, $\underline{F}_{drag} = -c_s \underline{v}$, and neglect grav/fric. Compute $\underline{r}(t)$ for the ball as it is flung through the tube, and the contact force between tube and ball, $\underline{F}_{wall}(t)$.



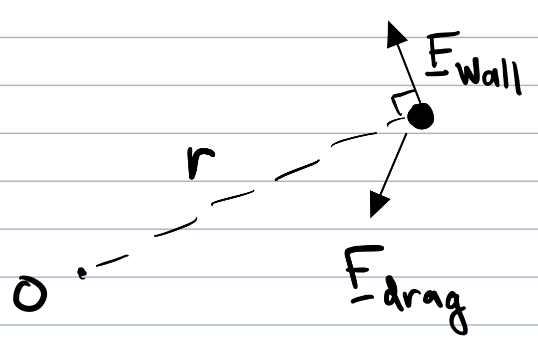
① Kinematics: Use polar with O at left end.

$$\underline{r} = r\underline{e}_r$$

$$\underline{v} = \dot{r}\underline{e}_r + r\underbrace{\dot{\theta}}_{=\omega}\underline{e}_\theta$$

$$\underline{a} = (\ddot{r} - r\underbrace{\dot{\theta}^2}_{\omega^2})\underline{e}_r + (r\underbrace{\ddot{\theta}}_{0} + 2\dot{r}\underbrace{\dot{\theta}}_{\omega})\underline{e}_\theta$$

② FBD: What touches ball? Tube wall & air.



$$\underline{F}_{wall} = \overbrace{F_{wall}}^{\text{unknown}}\underbrace{\underline{e}_\theta}_{\text{normal to wall}}$$

$$\underline{F}_{drag} = -c_s(\dot{r}\underline{e}_r + r\omega\underline{e}_\theta)$$

③ Newton: $\underline{F}_{wall} + \underline{F}_{drag} = m\left[(\ddot{r} - r\omega^2)\,\underline{e}_r + 2\dot{r}\omega\,\underline{e}_\theta\right]$

$$\Rightarrow F_{wall}\,\underline{e}_\theta - c_s(\dot{r}\,\underline{e}_r + r\omega\,\underline{e}_\theta) = m\left[(\ddot{r} - r\omega^2)\,\underline{e}_r + 2\dot{r}\omega\,\underline{e}_\theta\right]$$

④ Solve: Match up $\underline{e}_r$ and $\underline{e}_\theta$ components on both sides of Newton.

$$\underline{e}_r \cdot \Sigma \underline{F} = \underline{e}_r \cdot m\underline{a} \Rightarrow -c_s \dot{r} = m(\ddot{r} - r\omega^2)$$

$$\Rightarrow m\ddot{r} + c_s\dot{r} - m\omega^2 r = 0$$ "linear const coeff ODE".

Suppose $r(t) = \overset{\neq 0}{A} e^{bt}$. Plug in to ODE:

$$mAb^2e^{bt} + c_sAbe^{bt} - m\omega^2Ae^{bt} = 0$$

$$\Rightarrow mb^2 + c_s b - m\omega^2 = 0$$

$$\Rightarrow b = \frac{-c_s + \sqrt{c_s^2 + 4m^2\omega^2}}{2m} \equiv b_p$$

$$\text{OR} = \frac{-c_s - \sqrt{c_s^2 + 4m^2\omega^2}}{2m} \equiv b_m.$$

Gen sol'n: $r(t) = A_1 e^{b_p t} + A_2 e^{b_m t}$.

Use initial conds to solve for A_1, A_2.

$r(0) = l_0 = A_1 + A_2$

$\dot{r}(0) = 0 = b_p A_1 + b_m A_2$ 2 eqs for 2 unks ✓

$$\Rightarrow \quad r(t) = \frac{l_0}{b_m - b_p}\left(b_m e^{b_p t} - b_p e^{b_m t}\right)$$

Thus, $$\boxed{\underline{r}(t) = \frac{l_0}{b_m - b_p}\left(b_m e^{b_p t} - b_p e^{b_m t}\right)\underbrace{\left(\cos\omega t\, \underline{e}_x + \sin\omega t\, \underline{e}_y\right)}_{= \underline{e}_r}}$$

To get F_{wall}, use the $\underline{e}_\theta$ component of Newton's law and the now-known func $r(t)$:

$$\underline{e}_\theta \cdot \sum \underline{F} = \underline{e}_\theta \cdot m\underline{a} \quad \Rightarrow \quad F_{wall} - c_s r\omega = 2m\dot{r}\omega$$

$$\Rightarrow \quad F_{wall} = 2m\dot{r}\omega + c_s r\omega$$

$$= \frac{l_0 \omega}{b_m - b_p}\Big[(2mb_m b_p + c_s b_m)e^{b_p t} - (2mb_m b_p + c_s b_p)e^{b_m t}\Big].$$

Thus, $\underline{F}_{wall} = F_{wall}\, \underline{e}_\theta$

$$= \boxed{\frac{l_0 \omega}{b_m - b_p}\Big[(2mb_m b_p + c_s b_m)e^{b_p t} - (2mb_m b_p + c_s b_p)e^{b_m t}\Big]\left(-\sin\omega t\, \underline{e}_x + \cos\omega t\, \underline{e}_y\right)}$$

The results are clearer to understand if we set $c_s = 0$. Then $b_p = -b_m = \omega$.

$$r(t) = \frac{l_0}{b_m - b_p}\left(b_m e^{b_p t} - b_p e^{b_m t}\right) = l_0 \cosh(\omega t)$$

$$\underline{v}(t) = \dot{r}\,\underline{e}_r + r\omega\,\underline{e}_\theta = l_0\omega \sinh\omega t\,\underline{e}_r + l_0\omega \cosh\omega t\,\underline{e}_\theta .$$

$$F_{wall}(t) = 2m\dot{r}\omega = 2m\,l_0\omega^2 \sinh(\omega t)$$

For $t \gg \omega^{-1}$, $\sinh\omega t \approx \cosh\omega t \approx \frac{1}{2}e^{\omega t}$.

$\Rightarrow$ Ball speed increases exponentially in time!!

$\Rightarrow$ Ball moving outward about as fast as tube is moving laterally ($= r\omega$).

$\Rightarrow$ Wall force grows exponentially in time.

If you've ever passed a track-ball before, these results make sense:

1. The torque on your wrist ($= F_{wall}\, r \sim e^{2\omega t}$) grows really fast (i.e. exponentially).

2. But since the time t_{Tot} the ball spends in the track before exiting is small — $r(t_{Tot}) = L \Rightarrow t_{Tot} \approx \frac{1}{\omega}\ln(2L/l_0)$ — the torque may not actually get too high before exit.