ME 104 Lec 9

Recall: Two equivalent versions of the work-energy theorem:

① $$\boxed{W_{AB}^{Tot} = \sum_{\substack{\text{All}\\\text{forces}}} \int_{\substack{\text{Path}\\ \underline{r}_A \to \underline{r}_B}} \underline{F}^i \cdot \underline{dr} = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2}\,.$$

② $$\boxed{W_{AB}^{nc} = \sum_{\substack{\text{Non-}\\\text{conservative}\\\text{forces}}} \int_{\substack{\text{Path}\\ \underline{r}_A \to \underline{r}_B}} \underline{F}^{nc} \cdot \underline{dr} = E_B - E_A}$$

where $E = \frac{1}{2} m v^2 + \overbrace{\sum_c U^c}$ → Total potential energy of all conservative forces. $(\underline{F}^c = -\nabla U^c)$

Benefits of using work-energy:

Can determine certain features of the motion without having to

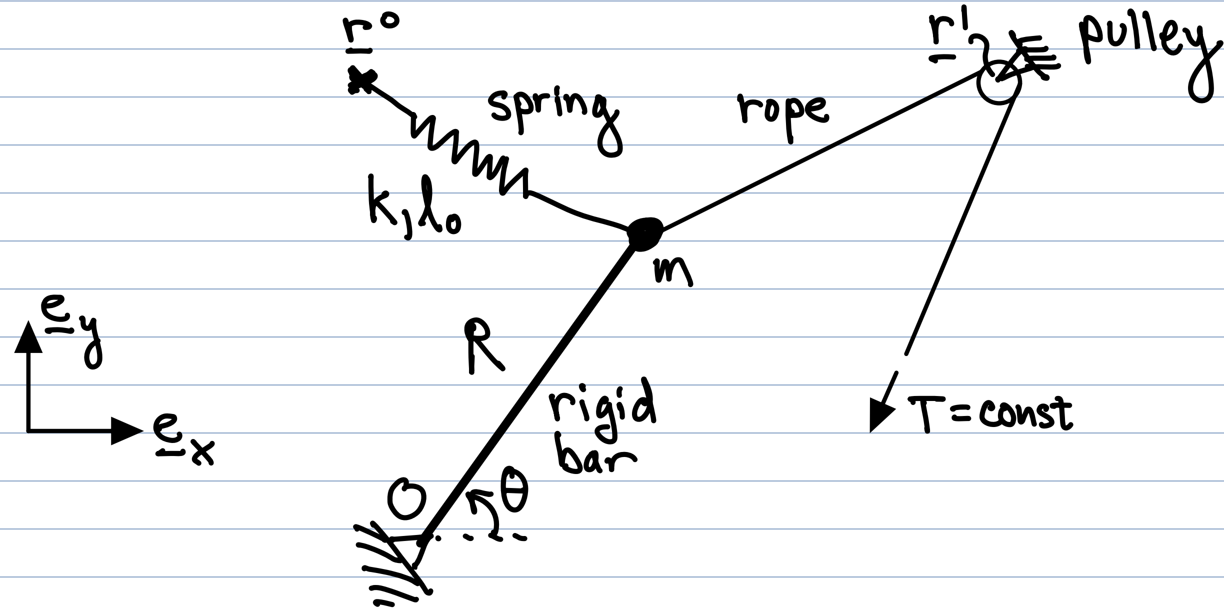
- solve first for (powerless) constraint forces, e.g. normal wall forces or rigid bar forces,
- solve a system of equations,
- solve differential equations.

Issues using energy balance include:

- hard to use if W_{AB}^{nc} is difficult to calculate (best if $W_{AB}^{nc} = 0 = \Delta E$, i.e. energy conserved)
- since it is just one equation it can only tell you one thing about the motion.

Using energy balance, we can often infer unknown forces <u>after the fact</u>.

Example from last time:



We previously calcutated $|\dot{\theta}_f|$ at some θ_f assuming mass started stationary at θ_i. What is $\underline{F}_{bar}$ at θ_f?

Ans: $\sum \underline{F} = m\underline{a}$ gives

$$\left\{F_{bar}\,\underline{e}_r + \underline{F}_s + \underline{F}_{rope} = m\left(-R\dot{\theta}^2\underline{e}_r + R\ddot{\theta}\,\underline{e}_\theta\right)\right\}\cdot\underline{e}_r$$

$$\Rightarrow F_{bar} + \underline{F}_s\cdot\underline{e}_r + \underline{F}_{rope}\cdot\underline{e}_r = -mR\dot{\theta}^2$$

Thus, for $\theta=\theta_f$ and $\dot{\theta}^2 = |\dot{\theta}_f|^2$, we have

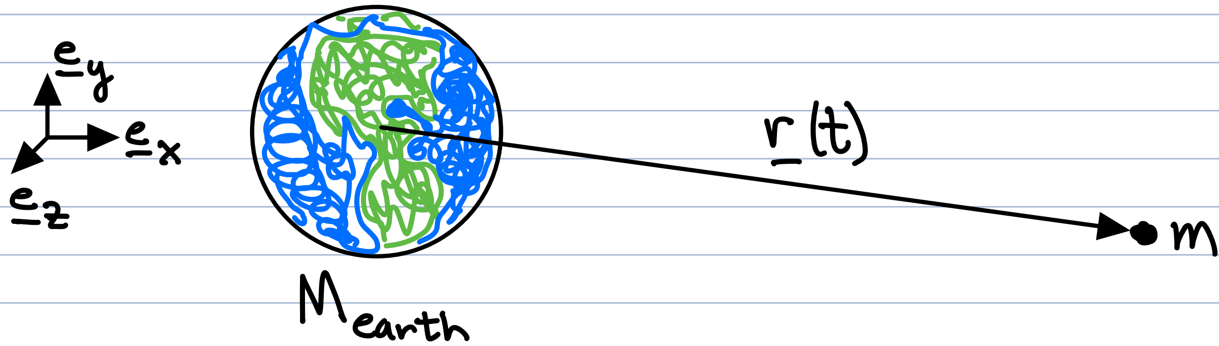
$$\boxed{F_{bar} = -mR|\dot{\theta}_f|^2 - \left[\underline{F}_s(\theta_f, R, \underline{r}^o) + \underline{F}_{rope}(\theta_f, R, \underline{r}')\right] \cdot\left(\cos\theta_f\,\underline{e}_x + \sin\theta_f\,\underline{e}_y\right)}$$

Since we know R, θ_f, $|\dot{\theta}_f|$, $\underline{r}^o$, and $\underline{r}'$ we can compute boxed formula for F_{bar}. ✓

In general, dotting $\Sigma\underline{F} = m\underline{a}$ into the direction normal to a wall let's you compute the normal force knowing only the particle's speed, position, and path curvature.

Orbital mechanics

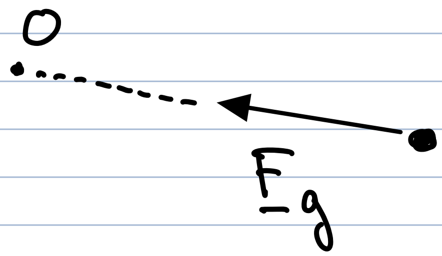
Assume an object interacting with earth's gravity with $m \ll M_{earth}$. $\Rightarrow$ The center of the earth can be assumed stationary. Let the origin be at earth's center.



① $\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z$, $\underline{v} = \dot{x}\,\underline{e}_x + \dot{y}\,\underline{e}_y + \dot{z}\,\underline{e}_z$

$\underline{a} = \ddot{x}\,\underline{e}_x + \ddot{y}\,\underline{e}_y + \ddot{z}\,\underline{e}_z$.

②



$$\underline{F}_g = \frac{G M_{earth} m}{|\underline{r}|^2}\left(\frac{-\underline{r}}{|\underline{r}|}\right)$$

③ Defining $\beta \equiv G M_{earth}$, $\sum \underline{F} = m\underline{a}$ implies

$$\boxed{\underline{a} = -\beta\,\underline{r}/|\underline{r}|^3}$$ "Trajectory equation".

In terms of work-energy, $\underline{F}_g$ is conservative as a power-law force with $U_g = -\beta/|\underline{r}|$. Since no nc forces, energy is conserved so :

$$\boxed{\tfrac{1}{2}mv^2 - \beta m/|\underline{r}| = \text{const}}$$

(4) Calculate the "escape velocity" — the speed that the mass must be launched with from earth's surface to escape earth's pull without any extra propulsion.

$$E(\text{right after launch}) = \tfrac{1}{2} m v_e^2 - \beta m / R_{earth}$$

$$E(\text{at escape}) = \tfrac{1}{2} m \cdot 0^2 - \beta m / |r \to \infty| = 0$$

$$\Rightarrow \tfrac{1}{2} m v_e^2 - \beta m / R_{earth} = 0 \Rightarrow v_e = \sqrt{\frac{2\beta}{R_{earth}}}$$

$$= \boxed{11{,}200 \text{ m/s.}}$$

Need more than just energy balance to learn more trajectory behaviors. Must solve trajectory eq directly or use other conservation laws to augment energy balance.

<u>Angular momentum</u>: Let $\underline{H} = m \underline{r} \times \underline{v}$.

Note that $\dot{\underline{H}} = m \underbrace{\dot{\underline{r}}}_{\underline{v}} \times \underline{v} + m \underline{r} \times \underbrace{\dot{\underline{v}}}_{\underline{a}}$

$$= m \underbrace{\underline{v} \times \underline{v}}_{=\underline{0}} + \underline{r} \times \underbrace{(m\underline{a})}_{= \underline{F}_g = -\beta m \underline{r} / |\underline{r}^3| \propto \underline{r}}$$

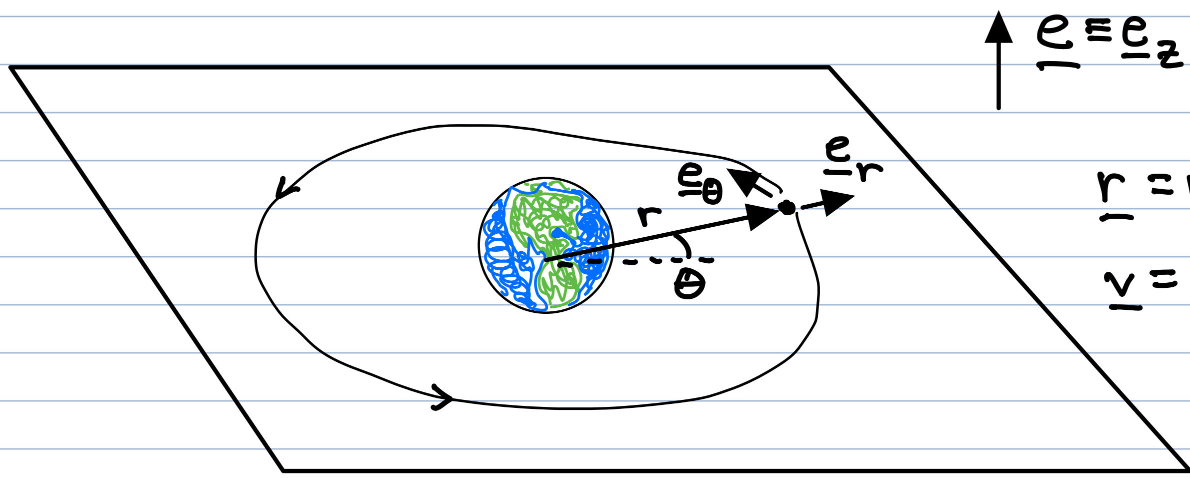
$$= \underline{0}$$

So $\underline{H}$ = "Angular momentum" = $\underline{H}_o$ is conserved.

Generalizing:
$\underline{H}$ is hence conserved whenever a particle is subject to a net force $\parallel \underline{r}$.

$\underline{H} = m\underline{r} \times \underline{v} = \underline{H}_o = H_o \underline{e}$ for some fixed H_o and $\underline{e}$. Fixed $\underline{e}$ implies $\underline{r}$ and $\underline{v}$ must always lie in one fixed plane $\perp$ to $\underline{e}$. But this implies the trajectory of $\underline{r}(t)$ must lie in a single plane. <u>Orbits always lie in one plane!</u>

Knowing this, the problem is now 2D.



$\underline{r} = r\underline{e}_r$

$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$

$$\underline{H} = m(r\underline{e}_r) \times (\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta) = mr^2\dot{\theta}\,\underline{e}_z = H_o\underline{e}_z$$

$\Longrightarrow mr^2\dot{\theta} = H_o$ where $H_o = mr_i^2\dot{\theta}_i$

$\Longrightarrow \dot{\theta} = H_o/mr^2$.

Now conserve energy: $\frac{1}{2}mv^2 - \beta m/r = E_o = \text{const}$

$$\Rightarrow \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \beta m/r = E_o$$

$$\Rightarrow \frac{1}{2}m\left(\dot{r}^2 + \frac{H_o^2}{m^2r^2}\right) - \beta m/r = E_o \longleftarrow \text{ODE for } r(t).$$

Rather than solve for $r(t)$, we can use this to learn the trajectory $r(\theta)$.

$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{H_o}{mr^2}$. Plug in above:

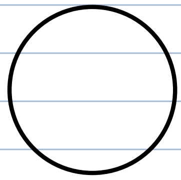
$$\Rightarrow \frac{1}{2}m\left(\left(\frac{dr}{d\theta}\right)^2\left(\frac{H_o}{mr^2}\right)^2 + \frac{H_o^2}{m^2r^2}\right) - \beta m/r = E_o$$

$$\Rightarrow \frac{dr}{d\theta} = \left(\frac{2}{m}\left(E_o + \beta m/r\right) - \frac{H_o^2}{m^2r^2}\right)\frac{m^2r^4}{H_o^2}$$

Solve by separation of variables. We get:

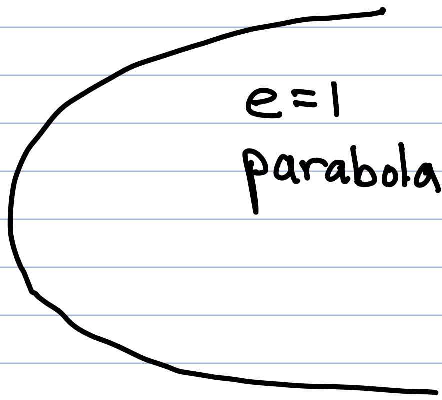
$$\boxed{\begin{aligned} r(\theta) &= \frac{p}{1+e\cdot\cos(\theta-\phi)} \\ p &= \text{"trajectory parameter"} = \frac{H_o^2}{m^2\beta} \\ e &= \text{"eccentricity"} = \left(1+\frac{2E_oH_o^2}{m^3\beta^2}\right)^{1/2} \\ \phi &= \text{"offset angle"} = \theta_i - \cos^{-1}\left(\frac{p-r_i}{er_i}\right) \end{aligned}}$$

What are these shapes? <u>Conic sections!!</u>

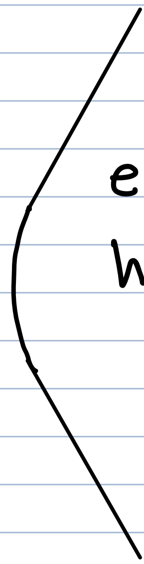


$e=0$
circle

$0<e<1$
ellipse



$e=1$
parabola



$e>1$
hyperbola