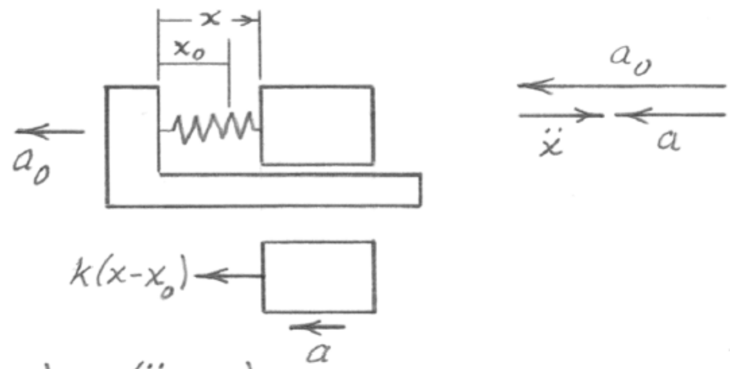
Week 5 Solutions

Monday, September 16, 2024 6:56 PM

Q1



$$\Sigma F_x = ma_x: -k(x-x_0) = m(\ddot{x}-a_0)$$

$$\dot{x}\,d\dot{x} = \ddot{x}\,dx \text{ so } \int_0^{\dot{x}} \dot{x}\,d\dot{x} = \int_{x_0}^{x} \left[a_0 - \frac{k}{m}(x-x_0)\right]dx$$

$$\frac{1}{2}\dot{x}^2 = \left(a_0 - \frac{kx_0}{m}\right)(x-x_0) - \frac{k}{2m}(x^2-x_0^2)$$

$$\frac{d}{dx}\left(\frac{\dot{x}^2}{2}\right) = a_0 + \frac{kx_0}{m} - \frac{kx}{m} = 0 \text{ for max. } \frac{\dot{x}^2}{2} \text{ \& hence max } \dot{x}$$

$$\text{so } \frac{kx}{m} = a_0 + \frac{kx_0}{m}, \quad x = x_0 + \frac{ma_0}{k}$$

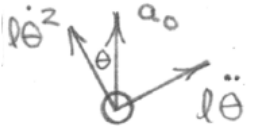
Thus

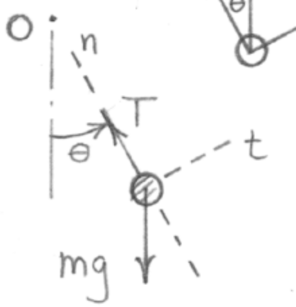
$$(v_{rel})_{max}^2 = \dot{x}_{max}^2 = 2\left(a_0 + \frac{kx_0}{m}\right)\left(x_0 + \frac{ma_0}{k} - x_0\right) - \frac{k}{m}\left(x_0^2 + \frac{2ma_0}{k}x_0 + \frac{m^2a_0^2}{k^2} - x_0^2\right)$$

$$= \frac{ma_0^2}{k}$$

$$\underline{(v_{rel})_{max} = a_0\sqrt{m/k}}$$

Q2





$\sum F_t = ma_t$:

$$-mg\sin\theta = m\left(l\ddot{\theta} + a_0 \sin\theta\right)$$

$$\ddot{\theta} = -\left(\frac{a_0+g}{l}\right)\sin\theta$$

$$\int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = -\int_{\theta_0}^{\theta}\left(\frac{a_0+g}{l}\right)\sin\theta\,d\theta$$

$$\dot{\theta}^2 = 2\left(\frac{a_0+g}{l}\right)\left(\cos\theta - \cos\theta_0\right)$$

$\sum F_n = ma_n$: $\quad T - mg\cos\theta = m\left(l\dot{\theta}^2 + a_0\cos\theta\right)$

$$T = m\left[g\left(3\cos\theta - 2\cos\theta_0\right) + a_0\left(3\cos\theta - 2\cos\theta_0\right)\right]$$

$$= m(g+a_0)\left(3\cos\theta - 2\cos\theta_0\right)$$

When $\theta = 0$,

$$\underline{T_0 = m(g+a_0)(3 - 2\cos\theta_0)}$$

If $\theta_0 = \pi/2$,

$$\underline{T_{\pi/2} = 3m(g+a_0)}$$

Q3

We have motion relative to a rotating path, so that a rotating coordinate system with origin at O is indicated. We attach x-y axes to the disk and use the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Velocity With the origin at O, the term $\mathbf{v}_B$ of Eq. 5/12 disappears and we have

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel} \quad ①$$

The angular velocity as a vector is $\boldsymbol{\omega} = 4\mathbf{k}$ rad/sec, where $\mathbf{k}$ is the unit vector normal to the x-y plane in the $+z$-direction. ② Our relative-velocity equation becomes

$$\mathbf{v}_A = 4\mathbf{k} \times 6\mathbf{i} + 5\mathbf{i} = 24\mathbf{j} + 5\mathbf{i} \text{ in./sec} \qquad \textit{Ans.}$$

in the direction indicated and has the magnitude

$$v_A = \sqrt{(24)^2 + (5)^2} = 24.5 \text{ in./sec} \qquad \textit{Ans.}$$

Acceleration Equation 5/14 written for zero acceleration of the origin of the rotating coordinate system is

$$\mathbf{a}_A = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} + \mathbf{a}_{rel}$$

The terms become

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = 4\mathbf{k} \times (4\mathbf{k} \times 6\mathbf{i}) = 4\mathbf{k} \times 24\mathbf{j} = -96\mathbf{i} \text{ in./sec}^2$$

$$\dot{\boldsymbol{\omega}} \times \mathbf{r} = -10\mathbf{k} \times 6\mathbf{i} = -60\mathbf{j} \text{ in./sec}^2 \quad ③$$

$$2\boldsymbol{\omega} \times \mathbf{v}_{rel} = 2(4\mathbf{k}) \times 5\mathbf{i} = 40\mathbf{j} \text{ in./sec}^2$$

$$\mathbf{a}_{rel} = 81\mathbf{i} \text{ in./sec}^2$$

The total acceleration is, therefore,

$$\mathbf{a}_A = (81 - 96)\mathbf{i} + (40 - 60)\mathbf{j} = -15\mathbf{i} - 20\mathbf{j} \text{ in./sec}^2 \qquad \textit{Ans.}$$

in the direction indicated and has the magnitude

$$a_A = \sqrt{(15)^2 + (20)^2} = 25 \text{ in./sec}^2 \qquad \textit{Ans.}$$

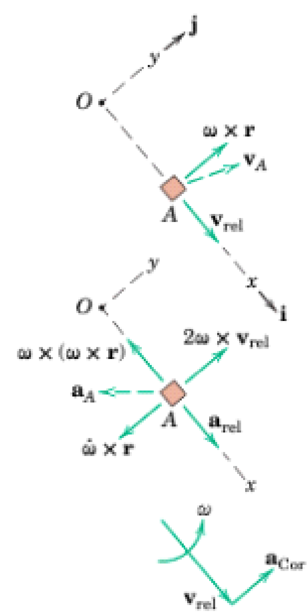
Vector notation is certainly not essential to the solution of this problem. The student should be able to work out the steps with scalar notation just as easily. The correct direction of the Coriolis-acceleration term can always be found by the direction in which the head of the $\mathbf{v}_{rel}$ vector would move if rotated about its tail in the sense of ω as shown.

HELPFUL HINTS

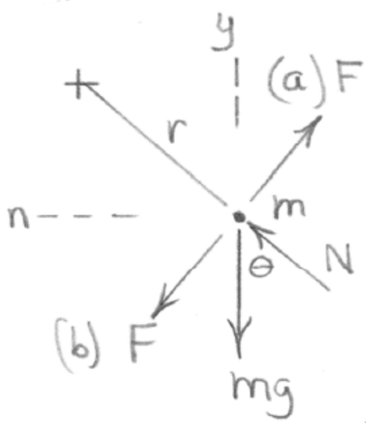
① This equation is the same as $\mathbf{v}_A = \mathbf{v}_P + \mathbf{v}_{A/P}$, where P is a point attached to the disk coincident with A at this instant.

② Note that the x-y-z axes chosen constitute a right-handed system.

③ Be sure to recognize that $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ and $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ represent the normal and tangential components of acceleration of a point P on the disk coincident with A. This description becomes that of Eq. 5/14b.



Q5



(a) Slipping impends down

$\Sigma F_y = 0: \; N\cos\theta - mg + \mu_s N \sin\theta = 0$

$$N = \frac{mg}{\cos\theta + \mu_s \sin\theta}$$

$\Sigma F_n = ma_n: \; N\sin\theta - \mu_s N\cos\theta$

$= mr(1+\sin\theta)\omega^2$

Combine: $$\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta} = \frac{r}{g}(1+\sin\theta)\omega^2$$

With $r = 0.2$ m, $g = 9.81$ m/s^2, $\mu_s = 0.2$, & $\omega = 6 \frac{rad}{s}$

$\theta = 65.8°$

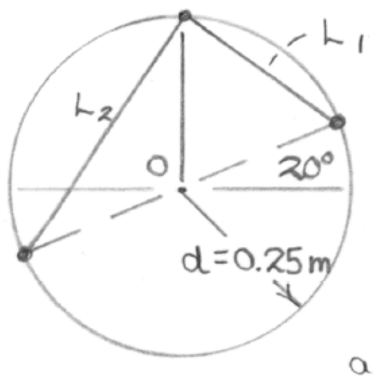
(b) Repeat for $F = \mu_s N$ in direction (b) to

obtain $$\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta} = \frac{r}{g}(1+\sin\theta)\omega^2$$

Numbers : $\theta = 38.7°$

So the range is $\underline{38.7° \leq \theta \leq 65.8°}$.

Q6



$$L_1 = 2d\sin(90°-20°)/2 = 0.287\ m$$

$$\delta_1 = 0.25\sqrt{2} - L_1 = 0.0668\ m$$

$$L_2 = 2d\sin\left(\frac{90° + 20°}{2}\right) = 0.410\ m$$

$$\delta_2 = L_2 - 0.25\sqrt{2} = 0.0560\ m$$

We may ignore the equal and opposite potential energy changes associated with two of the masses.

$T_1 + V_1 = T_2 + V_2$, datum at O.

$$0 - mgd\cos 20° + \tfrac{1}{2}k\delta_1^2 + \tfrac{1}{2}k\delta_2^2 = 3\left(\tfrac{1}{2}md^2\dot{\theta}^2\right) - mgd$$

$$0 - 3(9.81)(0.25)\cos 20° + \tfrac{1}{2}1200(0.0668)^2$$

$$+ \tfrac{1}{2}1200(0.0560)^2 = \tfrac{3}{2}3(0.25)^2\dot{\theta}^2 - 3(9.81)(0.25)$$

Solving $\quad \underline{\dot{\theta} = 4.22\ rad/s}$

Q7

The additional stretch in the spring runs at twice the distance d which the block moves down the incline.

(a)

$T_1 + V_1 + U'_{1-2} = T_2 + V_2$ (datum at original block position

$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 - \mu_k mg\cos\theta\,(d) = \frac{1}{2}mv^2 + \frac{1}{2}k(x_1+2d)^2 - mgd\sin\theta$ *

With $m = 10$ kg, $k = 200$ N/m, $x_1 = 0.025$ m

$\theta = 50°$, $\mu_k = 0.15$, and $v_1 = 0.3$ m/s :

$\underline{v = 0.635 \text{ m/s}}$

(b) Return to Eq. *, but set $v = 0$ & solve for $\underline{d_{max} = 0.1469 \text{ m}}$

Q8

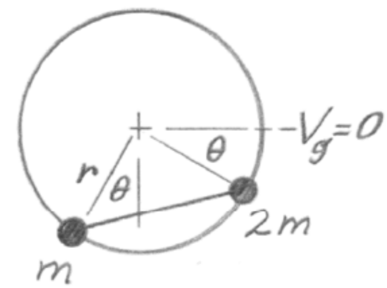
For the unit $U' = \Delta T + \Delta V_g = 0$

$\Delta V_g = (-2mgr\sin\theta - mgr\cos\theta) - (-mgr + 0)$

$= mgr(-2\sin\theta - \cos\theta + 1)$

so $\frac{1}{2}3mv^2 - 0 + mgr(-2\sin\theta - \cos\theta + 1) = 0$

or $v^2/gr = \frac{2}{3}(2\sin\theta + \cos\theta - 1)$



(a) Rod is horiz. when $\theta = 45°$

$v^2/gr = \frac{2}{3}(2\sin 45° + \cos 45° - 1) = 0.748$, $\underline{v_{45°} = 0.865\sqrt{gr}}$

(b) $\frac{d}{d\theta}\left(\frac{v^2}{gr}\right) = \frac{2}{3}(2\cos\theta - \sin\theta) = 0$ for max v^2 & hence max v

$\tan\theta = 2$, $\theta = \tan^{-1} 2 = 63.4°$

so $v_{max}^2/gr = \frac{2}{3}(2\sin 63.4° + \cos 63.4° - 1) = 0.824$

$\underline{v_{max} = 0.908\sqrt{gr}}$

(c) $\theta = \theta_{max}$ when $T = \Delta T = 0$ so $2\sin\theta + \cos\theta - 1 = 0$

$2\sqrt{1 - \cos^2\theta} = 1 - \cos\theta$, $5\cos^2\theta - 2\cos\theta - 3 = 0$

$\cos\theta = 0.2 \pm 0.8 = 1$ or -0.6, $\theta = 0$ or $\underline{\theta_{max} = 126.9°}$