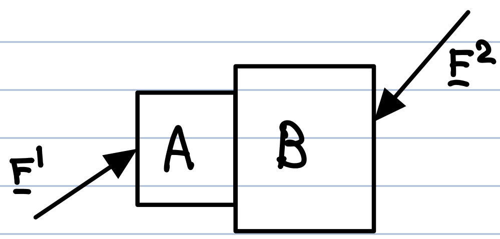
ME 104 Lec 11

Last time: Intro to systems of particles.
Two particles interact through equal and opposite forces (Newton 3).

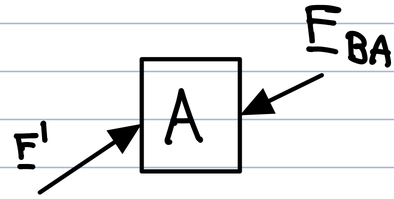
A pair of interaction forces are <u>central</u> if $\underline{F}_{AB} = -\underline{F}_{BA}$ is parallel to $\underline{r}_A - \underline{r}_B$ (a line connecting particle $\underline{A}$ to particle $\underline{B}$).

Contact:

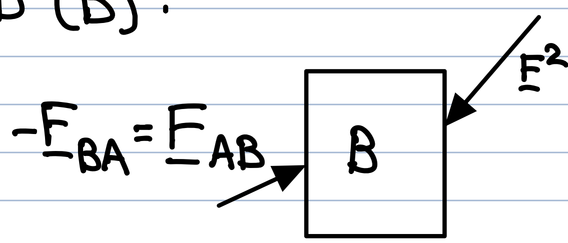


Due to friction, contact forces are not generally central.

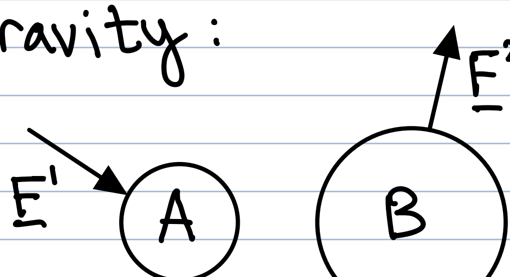
FBD (A):



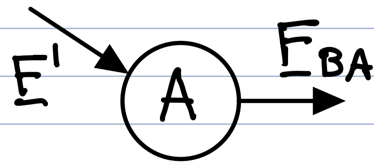
FBD (B):



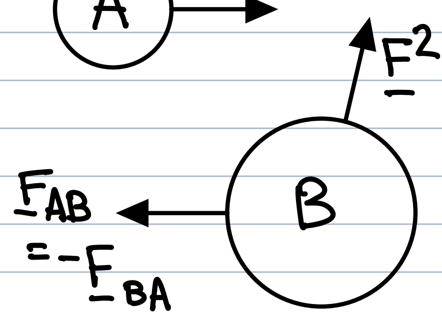
Gravity:



FBD (A):



FBD (B):

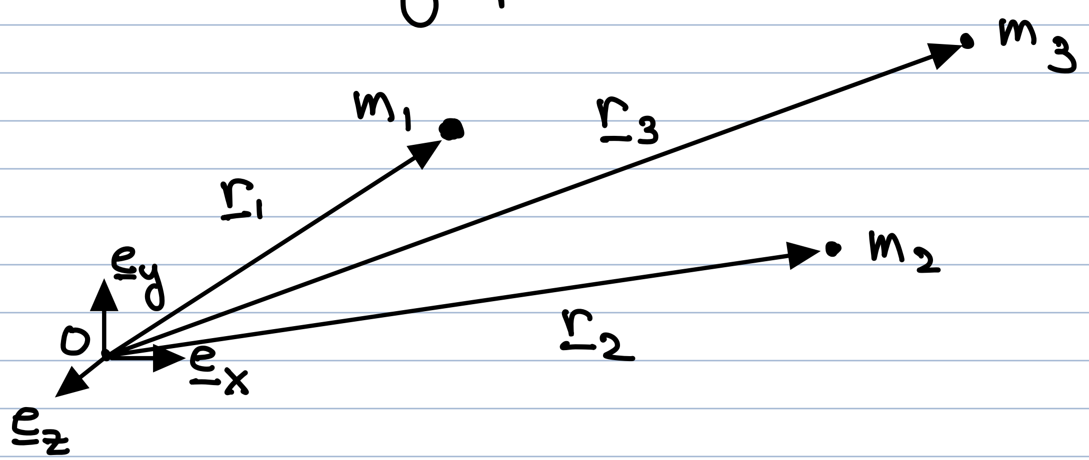


Gravitational forces are central.

$$\underline{F}_g^{ij} \equiv \text{Force of grav from particle } i \text{ on } j$$
$$= \frac{G m_i m_j}{|\underline{r}_i - \underline{r}_j|^3} (\underline{r}_i - \underline{r}_j) \qquad \{\text{points toward } i\}$$

$$\underline{F}_g^{ij} = -\underline{F}_g^{ji} \parallel \underline{r}_i - \underline{r}_j \checkmark$$

Ex: Three-body problem.

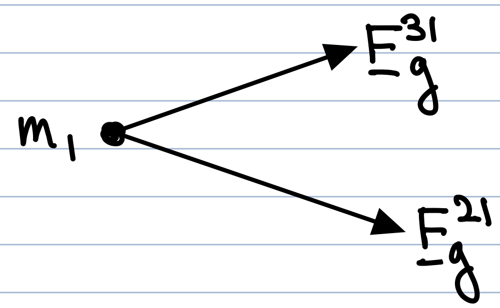


Suppose the three masses interact via gravity. Write equations of motion for the 3 particles and solve numerically.

Ans:

Particle 1:

FBD:

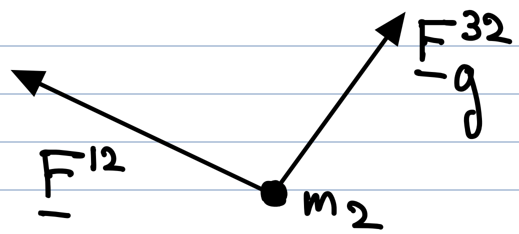


$$\Sigma \underline{F} = m\underline{a} \implies \underline{F}_g^{21} + \underline{F}_g^{31} = m_1 \underline{a}_1 = m_1 \dot{\underline{v}}_1$$

$$\Rightarrow \quad \dot{\underline{v}}_1 = G\left[\frac{m_2}{|\underline{r}_2-\underline{r}_1|^3}(\underline{r}_2-\underline{r}_1) + \frac{m_3}{|\underline{r}_3-\underline{r}_1|^3}(\underline{r}_3-\underline{r}_1)\right]$$

Particle 2:

FBD:

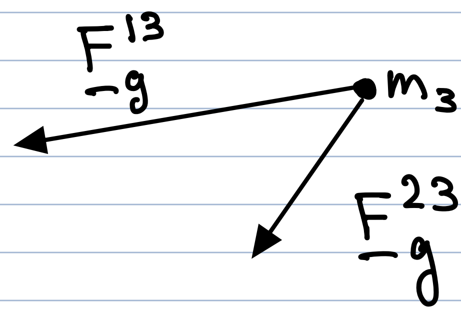


$$\sum \underline{F} = m\underline{a} \quad \Rightarrow \quad \underline{F}_g^{12} + \underline{F}_g^{32} = m_2\underline{a}_2 = m_2\dot{\underline{v}}_2$$

$$\Rightarrow \quad \dot{\underline{v}}_2 = G\left[\frac{m_1}{|\underline{r}_1-\underline{r}_2|^3}(\underline{r}_1-\underline{r}_2) + \frac{m_3}{|\underline{r}_3-\underline{r}_2|^3}(\underline{r}_3-\underline{r}_2)\right]$$

Particle 3:

FBD:



$$\sum \underline{F} = m\underline{a} \quad \Rightarrow \quad \underline{F}_g^{13} + \underline{F}_g^{23} = m_3\underline{a}_3 = m_3\dot{\underline{v}}_3$$

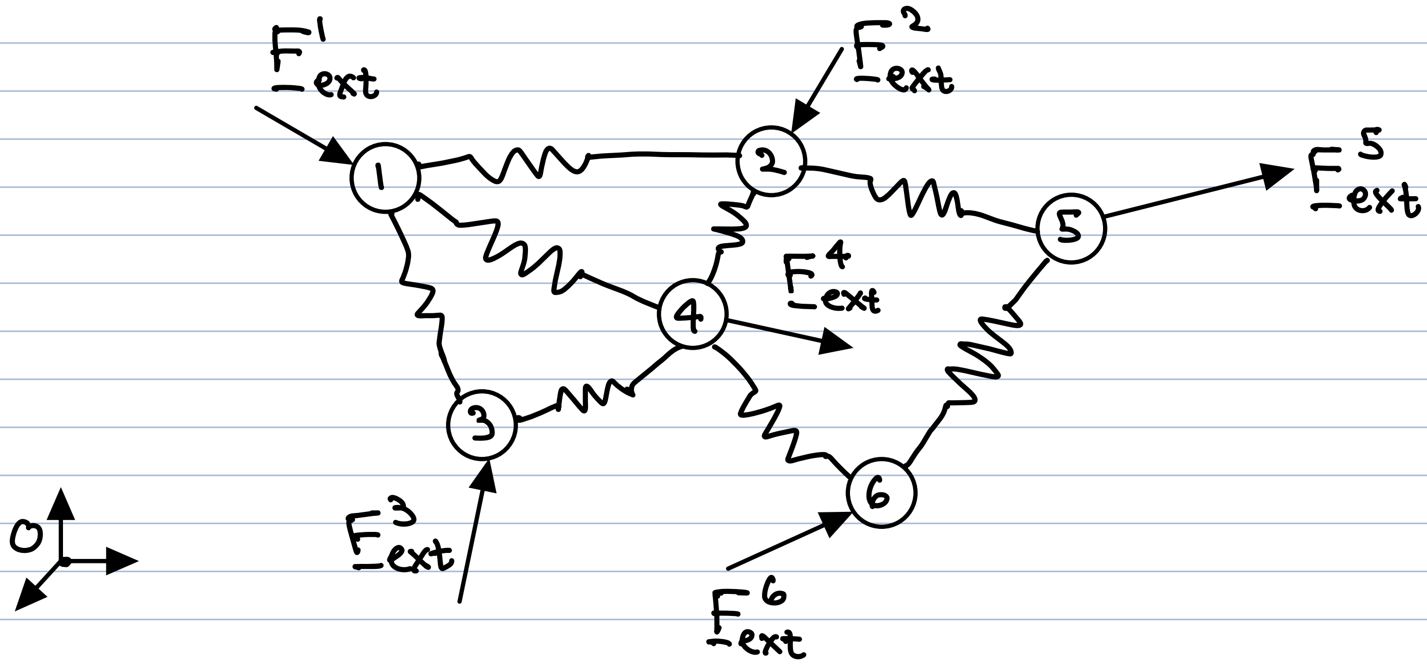
$$\Rightarrow \quad \dot{\underline{v}}_3 = G\left[\frac{m_1}{|\underline{r}_1-\underline{r}_3|^3}(\underline{r}_1-\underline{r}_3) + \frac{m_2}{|\underline{r}_2-\underline{r}_3|^3}(\underline{r}_2-\underline{r}_3)\right]$$

Solve numerically:

$$[\underline{U}] = \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \\ \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{bmatrix}; \quad [\dot{\underline{U}}] = \begin{bmatrix} \dot{\underline{v}}_1 \\ \dot{\underline{v}}_2 \\ \dot{\underline{v}}_3 \\ \dot{\underline{r}}_1 \\ \dot{\underline{r}}_2 \\ \dot{\underline{r}}_3 \end{bmatrix} = \begin{bmatrix} G\left[\frac{m_2}{|\underline{r}_2-\underline{r}_1|^3}(\underline{r}_2-\underline{r}_1) + \frac{m_3}{|\underline{r}_3-\underline{r}_1|^3}(\underline{r}_3-\underline{r}_1)\right] \\ G\left[\frac{m_1}{|\underline{r}_1-\underline{r}_2|^3}(\underline{r}_1-\underline{r}_2) + \frac{m_3}{|\underline{r}_3-\underline{r}_2|^3}(\underline{r}_3-\underline{r}_2)\right] \\ G\left[\frac{m_1}{|\underline{r}_1-\underline{r}_3|^3}(\underline{r}_1-\underline{r}_3) + \frac{m_2}{|\underline{r}_2-\underline{r}_3|^3}(\underline{r}_2-\underline{r}_3)\right] \\ \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \end{bmatrix} = f([\underline{U}])$$

Show Matlab demo.

Fundamental balance laws for systems of particles:



Consider a system of interacting particles. On top of interaction forces, each particle may also have a net external force from other sources (e.g. weight, etc.).

$\Sigma \underline{F} = m\underline{a}$ applies to each particle.

Ex: (particle 5) $\sum \underline{F} = m\underline{a}$: $\underline{F}^5_{ext} + \underline{F}_{25} + \underline{F}_{65} = m_5 \underline{a}_5$.

More gererally, for the ith particle

$$\sum \underline{F} = m\underline{a}: \quad \underline{F}^i_{ext} + \underbrace{\sum_j \underline{F}_{ji}}_{\text{Sum over all particles } j \text{ that interact with particle } i.} = m_i \underline{a}_i \ .$$

Observe that:

$$m_1 \underline{a}_1 + m_2 \underline{a}_2 + \cdots = \sum_i m_i \underline{a}_i = \sum_i \left(\underline{F}^i_{ext} + \sum_j \underline{F}_{ji} \right)$$

$$= \left(\sum_i \underline{F}^i_{ext} \right) + \underbrace{\left(\sum_i \sum_j \underline{F}_{ji} \right)}_{\text{Sum of all interaction forces in the system}}$$

Since interaction forces always come in pairs that are equal and opposite,

$$\sum_i \sum_j \underline{F}_{ji} = \underbrace{\underline{F}_{12} + \underline{F}_{21}}_{=\underline{0}} + \underbrace{\underline{F}_{13} + \underline{F}_{31}}_{=\underline{0}} + \cdots\cdots = \underline{0} \ .$$

So $\sum_i m_i \underline{a}_i = \sum_i \underline{F}^i_{ext}$. ☆

Define the <u>center of mass</u> of the system as:

$$\boxed{\underline{r}_{CM} = \left(\sum_i m_i \underline{r}_i \right) \Big/ M_{Tot}} \qquad M_{Tot} \equiv \sum_i m_i$$

$= \text{Total mass.}$

$$\underline{v}_{CM} = \dot{\underline{r}}_{CM} = \frac{d}{dt}\left(\sum_i m_i \underline{r}_i\right)/M_{Tot} = \left(\sum_i m_i \dot{\underline{r}}_i\right)/M_{Tot}$$

assuming the masses are const.

$$\underline{a}_{CM} = \dot{\underline{v}}_{CM} = \left(\sum_i m_i \ddot{\underline{r}}_i\right)/M_{Tot} = \left(\sum_i m_i \underline{a}_i\right)/M_{Tot}. \quad \star\star$$

Combining $\star$ and $\star\star$ gives

$$\boxed{\sum_i \underline{F}^i_{ext} = M_{Tot}\, \underline{a}_{CM}}$$

The system's center of mass moves like a <u>single particle</u> subject only to the <u>external forces</u>. Cool.

<u>Momentum balance:</u>

Define: $\text{Momentum} = \underline{P} = \sum_i \underbrace{m_i \underline{v}_i}_{\text{momentum of } i^{th} \text{ particle}} = M_{Tot}\, \underline{v}_{CM}.$

$$\dot{\underline{P}} = \sum_i m_i \dot{\underline{v}}_i = \sum_i m_i \underline{a}_i \Rightarrow \boxed{\dot{\underline{P}} = \sum_i \underline{F}^i_{ext}}.$$

Called "balance of (linear) momentum".

A major consequence of the above can

be observed by integrating both sides over a time interval:

$$\int_{t_1}^{t_2} \sum_i \underline{F}^i_{ext}(t)\, dt = \underline{P}(t_2) - \underline{P}(t_1) .$$

Define Impulse $= \underline{J} = \int_{t_1}^{t_2} \sum_i \underline{F}^i_{ext}(t)\, dt$.

Thus, we have $\boxed{\underline{J} = \underline{P}(t_2) - \underline{P}(t_1) = \Delta \underline{P}}$.

If there are no external forces, $\underline{J} = \underline{0}$ and we obtain $\Delta \underline{P} = \underline{0}$, "momentum is conserved".

Angular momentum balance:

Define the system's angular momentum as

$$\underline{H} = \sum_i \underbrace{\underline{r}_i \times (m_i \underline{v}_i)}_{\substack{\text{Ang mom of} \\ \text{the ith particle}}}$$

$$\frac{d\underline{H}}{dt} = \sum_i m_i \left(\dot{\underline{r}}_i \times \underline{v}_i + \underline{r}_i \times \dot{\underline{v}}_i \right) = \sum_i m_i \left(\overbrace{\underline{v}_i \times \underline{v}_i}^{=0} + \underline{r}_i \times \underline{a}_i \right)$$

$$= \sum_i \underline{r}_i \times (m_i \underline{a}_i) = \sum_i \underline{r}_i \times \left(\underline{F}^i_{ext} + \sum_j \underline{F}_{ji} \right)$$

$$= \sum_i \underbrace{\underline{r}_i \times \underline{F}^i_{ext}}_{\equiv \underline{M}^i_{ext}} + \sum_i \underline{r}_i \times \left(\sum_j \underline{F}_{ji}\right). \quad ☆$$

$\underline{M}^i_{ext}$ = "External torque on particle i".

The terms in $\sum_i \underline{r}_i \times \left(\sum_j \underline{F}_{ji}\right)$ can be arranged as

$$\underline{r}_1 \times \underline{F}_{21} + \underline{r}_2 \times \underbrace{\underline{F}_{12}}_{-\underline{F}_{21}} + \underline{r}_3 \times \underline{F}_{23} + \underline{r}_2 \times \underbrace{\underline{F}_{32}}_{-\underline{F}_{23}} + \cdots \quad \text{(by Newton 3)}$$

$$= (\underline{r}_1 - \underline{r}_2) \times \underline{F}_{21} + (\underline{r}_3 - \underline{r}_2) \times \underline{F}_{23} + \cdots$$

Assume the $\underline{F}_{ji}$ are all central forces. Then by definition, $(\underline{r}_i - \underline{r}_j) \times \underline{F}_{ji} = \underline{0}$ for all i, j.

$$\Rightarrow \sum_i \underline{r}_i \times \sum_j \underline{F}_{ji} = \underline{0}.$$

To be continued...