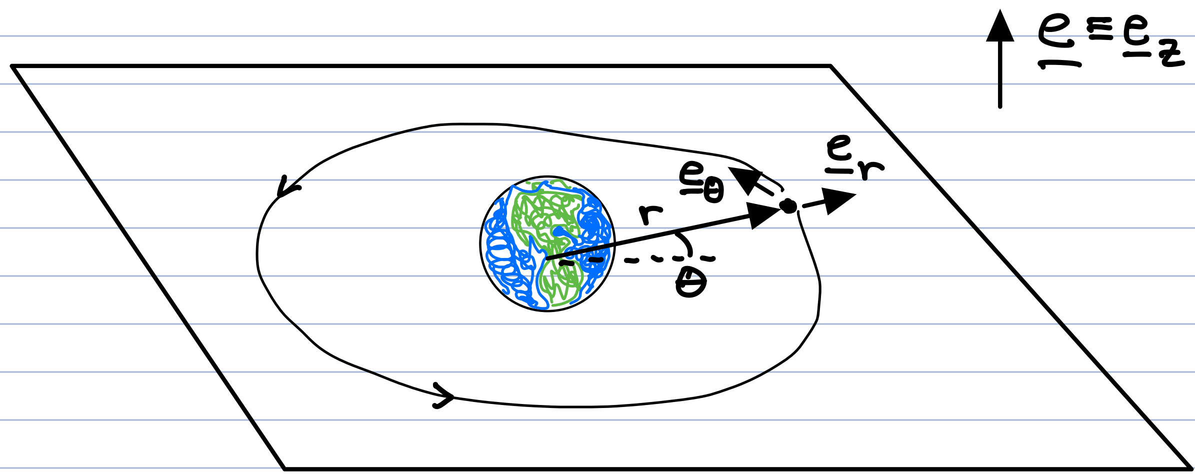
ME 104 Lec 10

Last time: Intro to orbital mechanics.



Assuming $m \ll M$, we find, for $\beta \equiv GM$,

- Trajectory equation: $\underline{a} = \ddot{\underline{r}} = -\beta\, \underline{r}/|\underline{r}|^3$
- Energy conservation: $\frac{1}{2}mv^2 - \beta m/|\underline{r}| = E_0 = \text{const}$
- Angular momentum conservation: $m\,\underline{r} \times \underline{v} = \underline{H}_0 = \text{const}$

$\underline{H}_0$ fixed means $\underline{H}_0/|\underline{H}_0|$ fixed $\Longrightarrow$ trajectory must lie in a single plane $\perp$ to $\underline{H}_0$.

Let $\underline{e}_z \equiv \underline{H}_0/|\underline{H}_0|$. $\quad \underline{r} = r\underline{e}_r \quad\quad H_0 = |\underline{H}_0| = mr^2\dot{\theta}$

$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta \quad\quad = \text{const.}$

Combining energy and ang mom cons, and defining the trajectory as $r = \hat{r}(\theta)$, we find:

$$\boxed{\begin{gathered} \hat{r}(\theta) = \frac{p}{1 + e\cdot\cos(\theta - \phi)} \\ p = \text{"trajectory parameter"} = \frac{H_0^2}{m^2\beta} \\ e = \text{"eccentricity"} = \left(1 + \frac{2E_0H_0^2}{m^3\beta^2}\right)^{1/2} \\ \phi = \text{"offset angle"} = \theta_i - \cos^{-1}\left(\frac{p - r_i}{e r_i}\right) \end{gathered}}$$

$e = 0 \Rightarrow$ circular orbit, $0 < e < 1 \Rightarrow$ elliptical orbit

$e = 1 \Rightarrow$ parabolic path, $e > 1 \Rightarrow$ hyperbolic path.

Orbital geometry: In elliptical orbit

"periapsis" $= r_p = \hat{r}(\theta = \phi) = p/(1+e)$

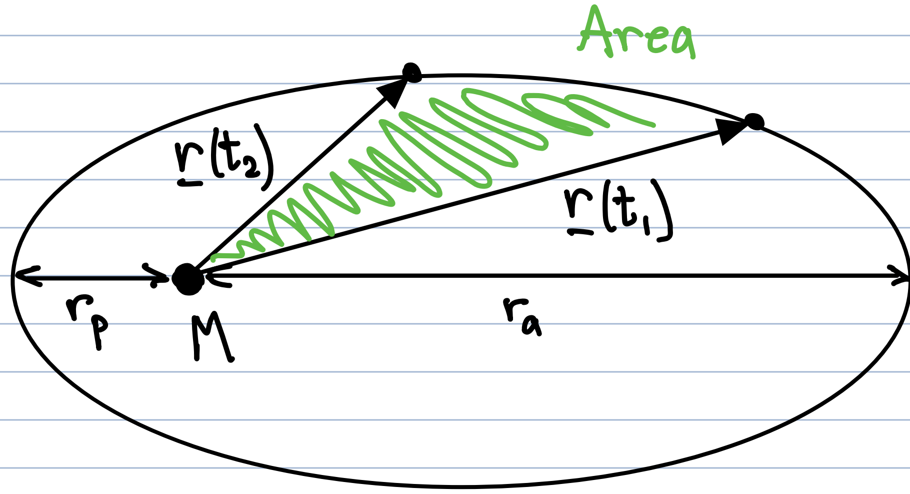
$=$ closest distance between M and m.

"apoapsis" $= r_a = \hat{r}(\theta = \phi + \pi) = p/(1-e)$

$=$ farthest distance between M and m.

Then we see that $e = (r_a - r_p)/(r_a + r_p)$.

<u>Kepler's Laws:</u>



{Area swept out between t_1 and t_2}

$$= \int_{\theta(t_1)}^{\theta(t_2)} \frac{1}{2}\hat{r}(\theta)^2\, d\theta = \int_{t_1}^{t_2} \frac{1}{2}\hat{r}(\theta(t))^2 \frac{d\theta}{dt}\, dt \qquad \left\{ \begin{aligned} H_0 &= \text{const} \\ &= m r^2 \dot{\theta} \end{aligned} \right\}$$

$$= \int_{t_1}^{t_2} \frac{1}{2m} H_0\, dt = \frac{H_0}{2m}(t_2 - t_1)$$

$\Longrightarrow$ $\boxed{\text{“Equal area, equal time”.}}$ Kepler's 2nd law.

What's the orbital period?

{Area swept out during a full orbit}

$$= \{\text{Area of entire ellipse}\} = \int_0^{2\pi} \frac{1}{2} \underbrace{\left[\frac{p}{1 - e\cdot\cos(\theta - \phi)}\right]^2}_{r^2} d\theta$$

$$= \pi p^2/(1-e)^{3/2} \qquad \text{From } p \equiv \frac{H_0^2}{m^2 \beta}$$

$$= (H_0/2m)\cdot \Delta T_{orbit} = \frac{1}{2}\cdot\sqrt{\beta p}\cdot \Delta T_{orbit}$$

$$\Longrightarrow \boxed{\Delta T_{orbit} = \frac{2\pi}{\sqrt{\beta}} p^{3/2}/(1-e)^{3/2} = \frac{2\pi}{\sqrt{\beta}} r_a^{3/2}}$$

Kepler's 3rd law.

In the case of a circular orbit, $r_a = r_p = R$.

$$\Rightarrow \Delta T_{orbit} = \frac{2\pi}{\sqrt{\beta}} R^{3/2} .$$

$$\{\text{Speed of } m \text{ in an orbit with radius } R\} = \frac{2\pi R}{\Delta T_{orbit}}$$

$$= (2\pi R) \Big/ \left(\frac{2\pi}{\sqrt{\beta}} R^{3/2}\right) = \sqrt{\beta / R} = \sqrt{GM/R} .$$

Setting $R = R_{earth}$ we get the least speed needed to shoot a bullet and make it orbit earth:

$\boxed{v_{orbit} = \sqrt{GM/R_{earth}}}$. Recalling the escape speed is $v_e = \sqrt{2GM/R_{earth}}$, we note that $v_e = \sqrt{2}\, v_{orbit}$.

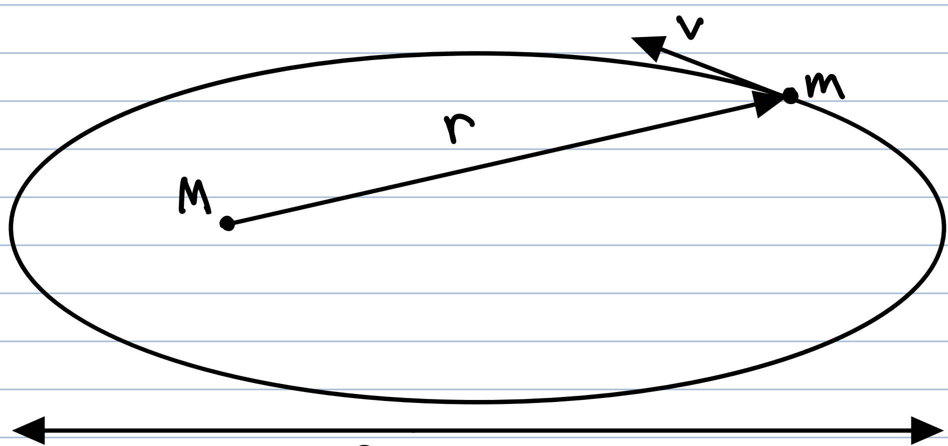
$\Rightarrow$ The kinetic energy one must impart to escape earth is <u>twice</u> that needed to orbit earth.

<u>Vis-viva equation:</u> For an elliptical orbit at periapsis or at apoapsis, $\underline{v} \perp \underline{r}$.

$$\Rightarrow H_0 = |m \underline{r} \times \underline{v}| = m r_p v_p = m r_a v_a \Rightarrow v_p = \frac{r_a}{r_p} v_a .$$

Energy cons: $E_0 = \frac{1}{2} m v_p^2 - \beta m / r_p = \underbrace{\frac{1}{2} m v_a^2}_{E_0 + \beta m / r_a} \left(\frac{r_a}{r_p}\right)^2 - \beta m / r_p$

$$\Rightarrow E_0 = -\frac{\beta m}{r_a + r_p} = -\frac{\beta m}{2a}$$ for a = semi-major axis of ellipse.



$2a = r_a + r_p$

So the total energy in any elliptical orbit is given only from the "max width" of the ellipse. Plugging this back into energy balance and equating to energy at arbitrary point on path:

$$E_0 = -\frac{\beta m}{2a} = \frac{1}{2} m v^2 - \beta m / r$$

$$\Rightarrow \boxed{v = \sqrt{\beta(2/r - 1/a)}}$$ "Vis-viva equation"

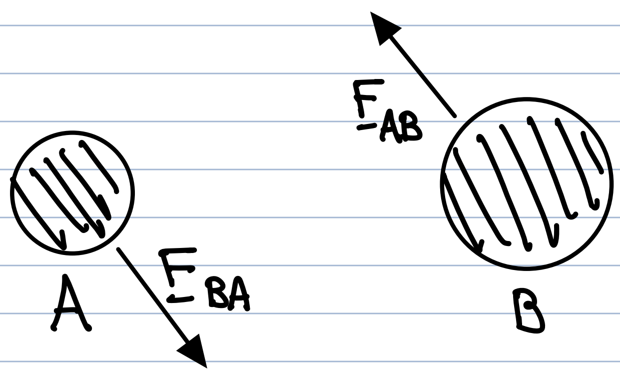
This is very useful for orbital maneuvering because it tells you how the orbit changes when speed is boosted.

New topic: <u>Systems of particles</u>

So far we have focused on the dynamics

of one particle. Let us now study how these concepts apply to systems of many interacting particles.

Key bridge: <u>Newton's 3rd Law</u>. "The law of action and reaction."



Suppose particles A and B interact with each other through contact or by some other force like gravity, charge, etc. Newton's third law says the force A exerts on B, $\underline{F}_{AB}$, equals $-\underline{F}_{BA}$.

{Exception: Magnetic forces can violate Newton's third.}

Can you think of a case where $\underline{F}_{AB} = -\underline{F}_{BA}$ is not $\parallel$ to $(\underline{r}_A - \underline{r}_B)$? Frictional contact.

Gravity and electric forces are "central forces" where $\underline{F}_{AB} = -\underline{F}_{BA} \parallel \underline{r}_A - \underline{r}_B$.

$\sum \underline{F} = m\underline{a}$ still holds for each particle and determines how each moves.