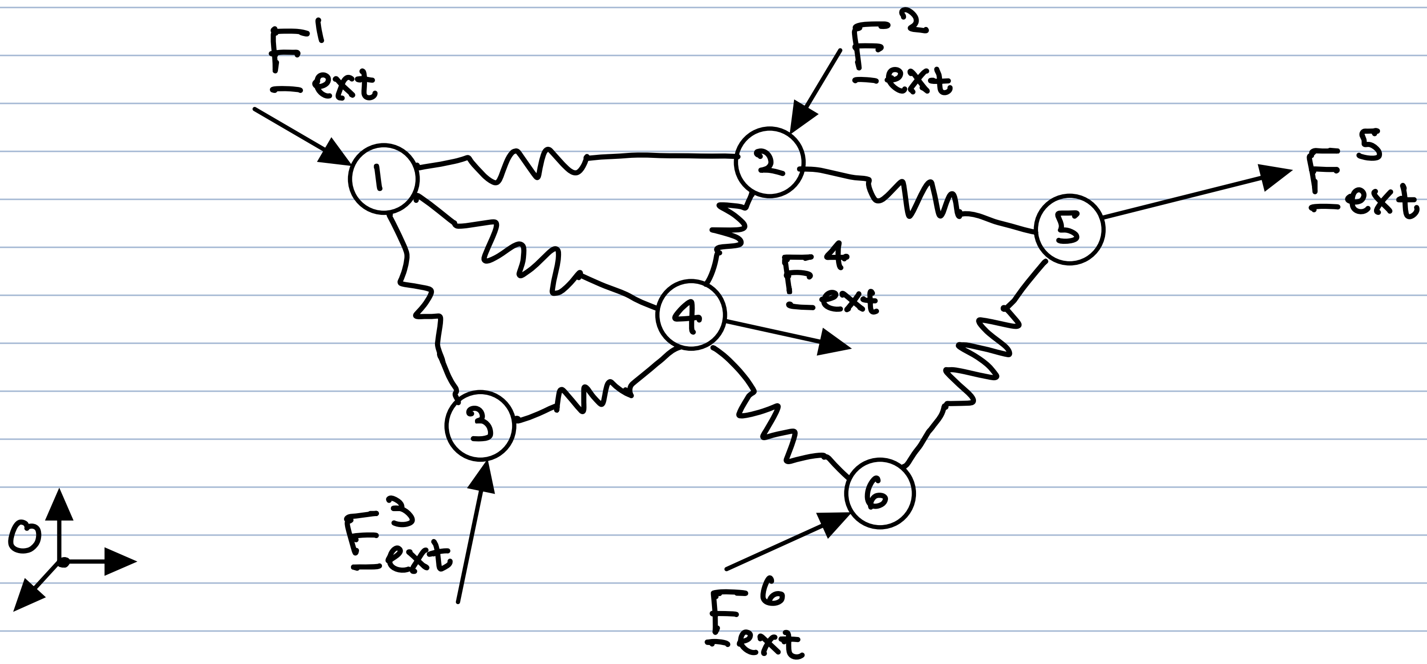
ME 104 Lec 12

Fundamental balance laws for systems of particles:



For the i^{th} particle:

$$\Sigma \underline{F} = m\underline{a}: \qquad \underline{F}^i_{ext} + \underbrace{\sum_j \underline{F}_{ji}}_{\text{Sum over all particles } j \text{ that interact with particle } i.} = m_i \underline{a}_i \, .$$

Define the <u>center of mass</u> of the system as:

$$\boxed{\underline{r}_{CM} = \left(\sum_i m_i \underline{r}_i\right) \Big/ M_{Tot}} \qquad M_{Tot} \equiv \sum_i m_i$$

= Total mass.

Define Momentum $= \underline{P} = \sum_i \underbrace{m_i \underline{v}_i}_{\text{momentum of } i^{th} \text{ particle}} = M_{Tot} \, \underline{v}_{CM}$.

Then the following quantities are all the same:

$$\dot{\underline{p}} = \sum_i m_i \dot{\underline{v}}_i = \sum_i m_i \underline{a}_i = M_{Tot}\, \underline{a}_{CM} = \sum_i \underline{F}^i_{ext} .$$

"Balance of momentum".

$$\Longrightarrow \quad \text{Impulse} = \underline{J} \equiv \int_{t_1}^{t_2} \sum_i \underline{F}^i_{ext}\, dt = \Delta \underline{P} .$$

Moral: Only external forces affect the momentum, or the motion of the CM.

Angular momentum balance:

Define the system's angular momentum as

$$\underline{H} = \sum_i \underbrace{\underline{r}_i \times (m_i \underline{v}_i)}_{\substack{\text{Ang mom of} \\ \text{the ith particle}}}$$

$$\frac{d\underline{H}}{dt} = \sum_i m_i \left(\dot{\underline{r}}_i \times \underline{v}_i + \underline{r}_i \times \dot{\underline{v}}_i \right) = \sum_i m_i \left(\overbrace{\underline{v}_i \times \underline{v}_i}^{=0} + \underline{r}_i \times \underline{a}_i \right)$$

$$= \sum_i \underline{r}_i \times (m_i \underline{a}_i) = \sum_i \underline{r}_i \times \left(\underline{F}^i_{ext} + \sum_j \underline{F}_{ji} \right)$$

$$= \underbrace{\sum_i \underline{r}_i \times \underline{F}^i_{ext}}_{\equiv \underline{M}^i_{ext}} + \sum_i \underline{r}_i \times \Big(\sum_j \underline{F}_{ji}\Big) \qquad (\star)$$

$\underline{M}^i_{ext}$ = "External torque on particle i".

The terms in $\sum_i \underline{r}_i \times \Big(\sum_j \underline{F}_{ji}\Big)$ can be arranged as

$$\underline{r}_1 \times \underline{F}_{21} + \underline{r}_2 \times \underbrace{\underline{F}_{12}}_{-\underline{F}_{21}} + \underline{r}_3 \times \underline{F}_{23} + \underline{r}_2 \times \underbrace{\underline{F}_{32}}_{-\underline{F}_{23}} + \cdots \quad \text{(Newton 3)}$$

$$= (\underline{r}_1 - \underline{r}_2) \times \underline{F}_{21} + (\underline{r}_3 - \underline{r}_2) \times \underline{F}_{23} + \cdots$$

Assume the $\underline{F}_{ij}$ are <u>all central forces</u>. Then by definition, $(\underline{r}_i - \underline{r}_j) \times \underline{F}_{ji} = \underline{0}$ for all i, j.

$$\Rightarrow \sum_i \underline{r}_i \times \sum_j \underline{F}_{ji} = \underline{0}.$$

Thus, $(\star)$ becomes $\boxed{\dot{\underline{H}} = \sum_i \underline{M}^i_{ext}}$. "Balance of angular momentum"

In words, when all internal forces are central, the system's rate of angular momentum balances the total external torque.

If there are no external torques,

$$\dot{\underline{H}} = \underline{0} \Rightarrow \underline{H} = \text{const}.$$ "Ang mom is conserved".

Observe that $\dot{\underline{H}} = \sum_i \underline{M}^i_{ext}$ holds regardless of where we position the origin as long as the origin is <u>fixed</u>.

Sometimes it makes sense to use the CM as a "moving origin" about which to compute ang mom and torque but doing so requires us to re-derive the ang mom bal equation.

Suppose $\underline{H}_{CM} \equiv \sum_i (\underline{r}_i - \underline{r}_{CM}) \times m_i \underline{v}_i = \left\{ \text{"What } \underline{H} \text{ would be if the origin is placed at } \underline{r}_{CM}\text{"} \right\}$

$$\dot{\underline{H}}_{CM} = \sum_i \left[(\underline{v}_i - \underline{v}_{CM}) \times m_i \underline{v}_i + (\underline{r}_i - \underline{r}_{CM}) \times m_i \underline{a}_i \right]$$

$$= \underbrace{\sum_i \left[\underline{v}_i \times m_i \underline{v}_i + \underline{r}_i \times m_i \underline{a}_i \right]}_{\substack{= \sum_i \underline{r}_i \times \underline{F}^i_{ext} \\ \text{from prior derivation}}} - \underline{v}_{CM} \times \underbrace{\sum_i m_i \underline{v}_i}_{M_{Tot} \underline{v}_{CM}} - \underline{r}_{CM} \times \underbrace{\sum_i m_i \underline{a}_i}_{\sum_i \underline{F}^i_{ext}}$$

$$\Rightarrow \dot{\underline{H}}_{CM} = \sum_i \underline{r}_i \times \underline{F}^i_{ext} - \underbrace{\underline{v}_{CM} \times (M_{Tot} \underline{v}_{CM})}_{=0} - \underline{r}_{CM} \times \sum_i \underline{F}^i_{ext}$$

$$\Rightarrow \dot{\underline{H}}_{CM} = \underbrace{\sum_i (\underline{r}_i - \underline{r}_{CM}) \times \underline{F}^i_{ext}}_{\equiv M^i_{ext,CM}} = \underbrace{\sum_i \underline{M}^i_{ext}}_{= \dot{\underline{H}} \text{ (at fixed origin)}} - \underline{r}_{CM} \times \sum_i \underline{F}^i_{ext}.$$

Altogether $\boxed{\dot{\underline{H}}_{CM} = \sum_i \underline{M}^i_{ext,CM}}$ and $\boxed{\dot{\underline{H}} = \dot{\underline{H}}_{CM} + \underline{r}_{CM} \times \sum_i \underline{F}^i_{ext}}$.

Energy balance for a system of particles:

Define the system kinetic energy as

$$K = \sum_i \tfrac{1}{2} m_i v_i^2 = \text{Sum of kinetic energies of each individual particle.}$$

Let's now define the total work done by all forces on all particles as the particle positions move along a path from $(\underline{r}_1^A, \underline{r}_2^A, \ldots) \to (\underline{r}_1^B, \underline{r}_2^B, \ldots)$.

$$W_{AB} \equiv \sum_i W_{AB}^i = \sum_i \int_{t_A}^{t_B} \Big(\underline{F}_{ext}^i + \sum_j \underline{F}_{ji} \Big) \cdot \underline{v}_i \, dt .$$

Since $\Delta\left(\tfrac{1}{2} m_i v_i^2\right) = W_{AB}^i$ from the one-particle derivation (see Lec 8) then we immediately get

$$\Delta K = \Delta\Big(\sum_i \tfrac{1}{2} m_i v_i^2 \Big) = \sum_i W_{AB}^i = W_{AB} .$$

$\boxed{\Delta K = W_{AB}}$ "Work-energy theorem for system".

Potential energy of system:

Suppose $\underline{F}_{ji}$ = "force of j on i" is conservative.

$$\Rightarrow \quad F_{ji} = -\nabla_{\underline{r}_i} U_{ji}(\underbrace{x_i, y_i, z_i}_{\underline{r}_i \text{ comps}}, \underbrace{x_j, y_j, z_j}_{\underline{r}_j \text{ comps}})$$

$$= \frac{\partial U_{ji}}{\partial x_i} \underline{e}_x + \frac{\partial U_{ji}}{\partial y_i} \underline{e}_y + \frac{\partial U_{ji}}{\partial z_i} \underline{e}_z .$$

where $\nabla_{\underline{r}_i}$ holds $\underline{r}_j$ fixed. Since $\underline{F}_{ji} = -\underline{F}_{ij}$, we require that:

$$\underline{F}_{ji} = -\nabla_{\underline{r}_i} U_{ji}(\underline{r}_i, \underline{r}_j) = -\underline{F}_{ij} = \nabla_{\underline{r}_j} U_{ij}(\underline{r}_i, \underline{r}_j).$$

The only way to ensure this is if U_{ij} and U_{ji} satisfy a symmetry rule:

$$U_{ij}(\underline{r}_i - \underline{r}_j) = U_{ij}(\underline{r}_j - \underline{r}_i) = U_{ji}(\underline{r}_i - \underline{r}_j) = U_{ji}(\underline{r}_j - \underline{r}_i)$$

In words:

The potential energy U_{ij} governing the force i applies to j must equal the potential energy U_{ji} governing the force j applies to i.

Further, $U_{ij} = U_{ji}$ only depends on $\underline{r}_i - \underline{r}_j$ and U_{ij} has <u>reflection symmetry</u> in that $U_{ij}(\underline{\xi}) = U_{ij}(-\underline{\xi})$ for all vectors $\underline{\xi}$.