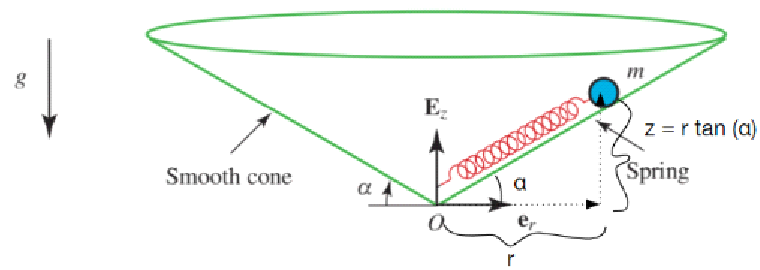
Week 6 Solutions

Monday, September 30, 2024 8:04 PM

Q1

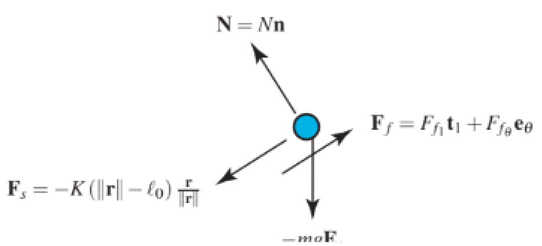


Step 1: Kinematics

$$\mathbf{r} = r\mathbf{e}_r + r\tan(\alpha)\mathbf{E}_z,$$
$$\mathbf{v} = \dot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + r\dot{\theta}\mathbf{e}_\theta.$$
$$\mathbf{a} = \ddot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r.$$

$$\mathbf{H}_O = \mathbf{r}\times m\mathbf{v} = \frac{mr^2\dot{\theta}}{\cos(\alpha)}\left(\cos(\alpha)\mathbf{E}_z - \sin(\alpha)\mathbf{e}_r\right)$$

Step 2: FBD



Total vector force: $\mathbf{F} = N\mathbf{n} - mg\mathbf{E}_z - K\left(\|\mathbf{r}\| - \ell_0\right)\dfrac{\mathbf{r}}{\|\mathbf{r}\|}$

Step 3: Conservation of angular momentum:
if **r** × m**a** (m**a** = **F**) is zero in $\mathbf{E}_z$ direction

$$\mathbf{r}\times\mathbf{F} = \mathbf{r}\times N\mathbf{n} - \mathbf{r}\times mg\mathbf{E}_z + \mathbf{r}\times\left(-K\left(\|\mathbf{r}\| - \ell_0\right)\frac{\mathbf{r}}{\|\mathbf{r}\|}\right)$$
$$= \left(mgr - \frac{Nr}{\cos(\alpha)}\right)\mathbf{e}_\theta.$$

Consequently, $\mathbf{H}_O\cdot\mathbf{E}_z$ is conserved:

$$mr^2\dot{\theta} = \text{constant}$$

Q2

The forces on the particle are its weight and the normal reaction exerted by the smooth surface of the bowl. Neither force exerts a moment about the axis *O-O*, so that angular momentum is conserved about that axis. Thus,

$[(H_O)_1 = (H_O)_2]$ $\qquad mv_0r_0 = mvr\cos\theta$ ①

Also, energy is conserved so that $E_1 = E_2$. Thus

$[T_1 + V_1 = T_2 + V_2]$ $\qquad \frac{1}{2}mv_0{}^2 + mgh = \frac{1}{2}mv^2 + 0$

$$v = \sqrt{v_0{}^2 + 2gh}$$

Eliminating v and substituting $r^2 = r_0{}^2 - h^2$ give

$$v_0r_0 = \sqrt{v_0{}^2 + 2gh}\sqrt{r_0{}^2 - h^2}\cos\theta$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{1 + \frac{2gh}{v_0{}^2}}\sqrt{1 - \frac{h^2}{r_0{}^2}}}$$

Q3

System angular momentum conserved during impact: $\cancel{r+}\ H_{O_1} = H_{O_2}$:

$$0.050(300)(0.4\cos 20^\circ) - 3.2(0.2)^2 6 - 3.2(0.4)^2 6$$

$$= (0.050 + 3.2)(0.4)^2 \omega' + 3.2(0.2)^2 \omega'$$

$$\underline{\omega' = 2.77 \text{ rad/s} \quad (\text{CCW})}$$

Energy considerations after impact:

$T' + V' = \cancel{T}^{\,0} + V$, choose datum @ O:

$$\tfrac{1}{2}(0.05 + 3.2)[0.4(2.77)]^2 + \tfrac{1}{2}(3.2)[0.2(2.77)]^2$$

$$+[3.2(0.2) - (3.2 + 0.05)(0.4)]9.81 = 0 +$$

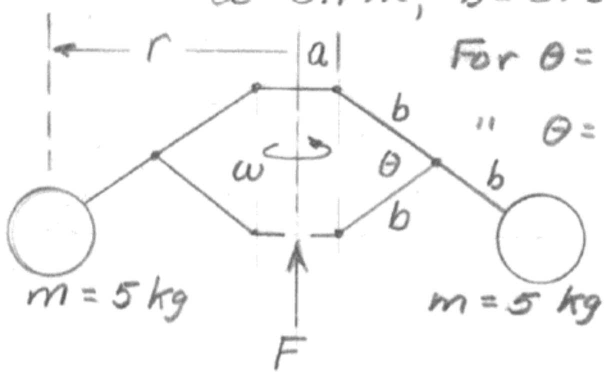
$$[3.2(0.2) - (3.2 + 0.05)(0.4)]9.81 \cos\theta$$

$$\underline{\theta = 52.1^\circ}$$

Q4

$\omega_0 = 40(2\pi)/60 = 4.19$ rad/s

$a = 0.1$ m, $b = 0.3$ m



For $\theta = 90°$, $r_0 = 0.1 + 2(0.3)\cos 45° = 0.524$ m

" $\theta = 60°$, $r = 0.1 + 2(0.3)\cos 30° = 0.620$ m

$\Delta H = 0;\ 2mr_0^2\omega_0 - 2mr^2\omega = 0$

$$\omega = \frac{r_0^2}{r^2}\omega_0 = \left(\frac{0.524}{0.620}\right)^2(4.19)$$

$= \underline{3.00 \text{ rad/s}}$

(or $\frac{3.00}{2\pi}60 = 28.6$ rev/min)

$$U = \Delta T + \Delta V_g = 2\left(\tfrac{1}{2}m\right)\left(r^2\omega^2 - r_0^2\omega_0^2\right) + 2mg\,\Delta h$$

where $\Delta h = 2b(\sin 45° - \sin 30°)$

$= 2(0.3)(0.7071 - 0.5) = 0.1243$ m

$$U = 5\left([0.620 \times 3.00]^2 - [0.524 \times 4.19]^2\right) + 2(5)(9.81)(0.1243)$$

$= -6.850 + 12.190 = \underline{5.34 \text{ J}}$

Q5

$$T_1 + U_{1-2} = T_2 \quad \begin{cases} ① : \text{launch} \\ ② : \text{max. altitude} \end{cases}$$

$$\frac{1}{2} m v_0^2 + mgR^2 \left[\frac{1}{3R/2} - \frac{1}{R} \right] = 0$$

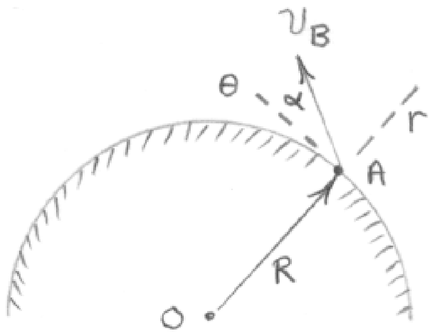
$$v_0 = \sqrt{\frac{2gR}{3}} = \sqrt{\frac{2(9.825)(6371 \cdot 10^3)}{3}}$$

$$= \underline{6460 \text{ m/s}}$$

Work expression used:

$$U_{1-2} = mgR^2 \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

Q6



$v_\theta = v\cos\alpha = 2000\cos 30°$
$= 1732$ m/s

$v_r = v\sin\alpha = 2000\sin 30°$
$= 1000$ m/s

$$v^2 = 2gR^2\left(\frac{1}{r} - \frac{1}{2a}\right)$$

Using conditions at B:
$a = 3.2906 \times 10^6$ m

$$T_B = \tfrac{1}{2} m v_B^2 = \tfrac{1}{2} m (2000)^2 = 2\times10^6 \text{ m}$$

$$V_B = \frac{-mgR^2}{r} = \frac{-m(9.825)(6.371\times10^6)^2}{6.371\times10^6}$$

$$= -6.2595 \times 10^7 \text{ m}$$

$$E = T_B + V_B = -6.0595 \times 10^7 \text{ m}$$

$$h = r v_\theta = 6.371(10^6)(1732) = 1.1035 \times 10^{10}$$

Now use $e = \sqrt{1 + \frac{2Eh^2}{mg^2R^4}}$ to get $e = 0.9525$

Finally, $r_{max} = a(1+e) = 3.2906(10^6)(1 + 0.9525)$
$= 6.4249 \times 10^6$ m

$h_{max} = r_{max} - R = 53\,900$ m or $\underline{53.9 \text{ km}}$