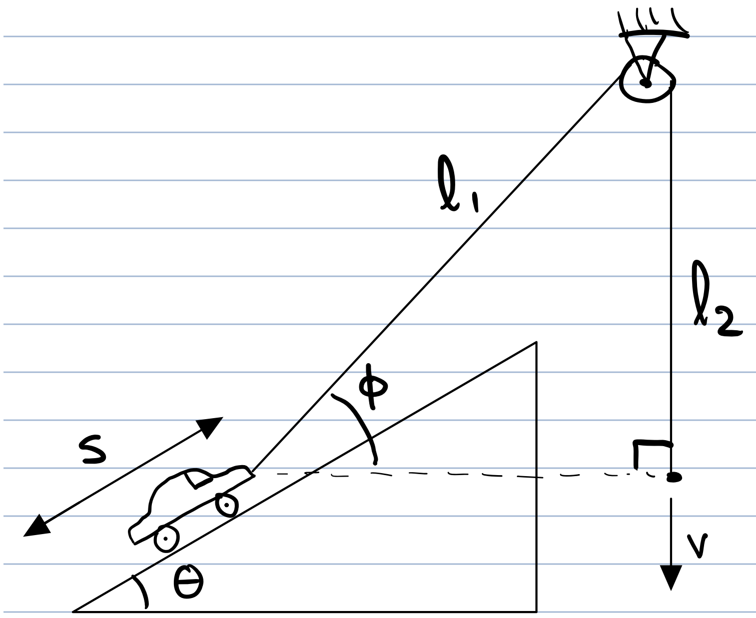
ME 104 Lec 2

Kinematics practice problem:



In terms of $l_1, l_2, \theta, \phi, v$, how fast is the car moving up the slope at this instant if the rope is being tugged down at speed v?

Ans: Use related rates. Write true relationships relating the geometric lengths, then take d/dt.

$$l_1 \sin\phi = l_2 \xRightarrow{d/dt} \boxed{\dot{l}_1 \sin\phi + l_1 \cos\phi\, \dot{\phi} = \dot{l}_2}$$

$$l_1 + l_2 = \text{const} \Longrightarrow \boxed{\dot{l}_1 + \dot{l}_2 = 0}$$

$$s\cos\theta + l_1\cos\phi = \text{const} \Rightarrow \boxed{\dot{s}\cos\theta + \dot{l}_1\cos\phi - l_1\sin\phi\,\dot{\phi} = 0}$$

$$\boxed{\dot{l}_2 = v}$$

Use boxed eqs to solve for $\dot{s}$.

One way is to write as a matrix eq and solve.

$$\begin{bmatrix} \sin\phi & -1 & l_1\cos\phi & 0 \\ 1 & 1 & 0 & 0 \\ \cos\phi & 0 & -l_1\sin\phi & \cos\theta \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{\phi} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ v \end{bmatrix}$$

Basic Analytic Solution Methods:

How do we solve for $x(t)$ in the cases below, for $x_0 = x(0)$ and $v_0 = v(0)$?

① $\underline{a = f(t)}$: $a = \dot{v} = \ddot{x}$.

$$v(t) = v_0 + \int_0^t f(t)\,dt$$

$$\Rightarrow x(t) = x_0 + \int_0^t v(t)\,dt.$$

② $\underline{a = g(v)}$:

$$\dot{v} = \frac{dv}{dt} = g(v) \Rightarrow dt = \frac{dv}{g(v)}$$

$$\Rightarrow \int_0^t dt = t = \int_{v_0}^{v} dv/g(v).$$

This gives $t = t(v)$ which can be inverted to get $v = v(t)$. Then $x = x_0 + \int_0^t v(t)\,dt$.

③ $\underline{a = h(x)}$: Notice that $a = \frac{d}{dt}\left(\tilde{v}(x(t))\right) = \frac{d\tilde{v}}{dx}\underbrace{\frac{dx}{dt}}_{=v}$

$$\Rightarrow \frac{d\tilde{v}}{dx} v = h(x) \Rightarrow v\,dv = h(x)\,dx$$

$$\Rightarrow \int_{v(0)}^{v(t)} v\,dv = \tfrac{1}{2}v(t)^2 - \tfrac{1}{2}v(0)^2 = \int_{x_0}^{x} h(x)\,dx.$$

$$\Rightarrow v = \pm\sqrt{v(0)^2 + 2\int_{x_0}^{x} h(x)\,dx} \equiv \tilde{v}(x).$$

Once we have $\tilde{v}(x)$, we write $\tilde{v}(x) = \frac{dx}{dt}$

$$\Rightarrow dt = \frac{dx}{\tilde{v}(x)} \Rightarrow t = \int_{x_0}^{x} dx/\tilde{v}(x) = t(x).$$

Invert $t(x)$ to get $x(t)$.

Circular motion:

Assume ω const > 0.

$$\underline{r}(t) = R_0 \left(\cos(\omega t)\, \underline{e}_x + \sin(\omega t)\, \underline{e}_y \right).$$

Check: $$|\underline{r}| = \sqrt{R_0^2 \underbrace{\left(\cos(\omega t)^2 + \sin(\omega t)^2 \right)}_{=1}} = R_0 = \text{const.}$$

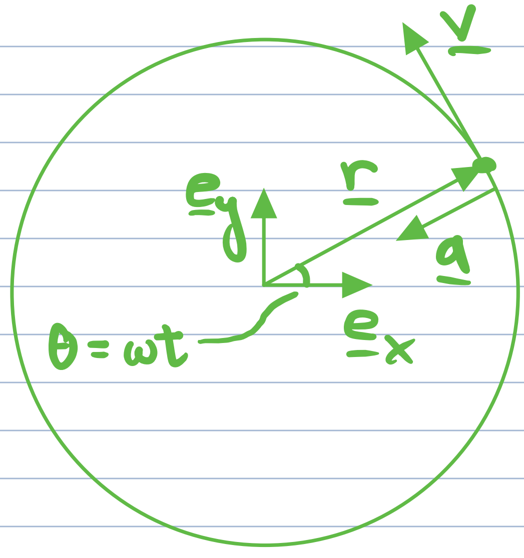
$$\underline{v} = \dot{\underline{r}} = R_0 \omega \left(-\sin(\omega t)\, \underline{e}_x + \cos(\omega t)\, \underline{e}_y \right)$$

Check: $$v = |\underline{v}| = \sqrt{(R_0 \omega)^2 \underbrace{\left(\sin^2(\omega t) + \cos^2(\omega t) \right)}_{=1}}$$

$$= R_0 \omega \quad \checkmark$$

$$\underline{a} = \ddot{\underline{r}} = \dot{\underline{v}} = -R_0 \omega^2 \left(\cos(\omega t)\, \underline{e}_x + \sin(\omega t)\, \underline{e}_y \right)$$

$$= -\omega^2 \underline{r} \quad \Longrightarrow$$ acceleration points inward.

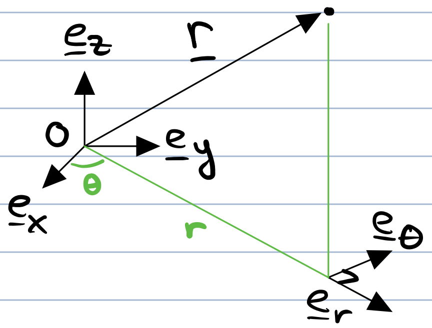


Cylindrical Coordinates:

$$r \equiv \sqrt{x^2+y^2}\,, \quad \theta = \tan^{-1}(y/x).$$

$$\Longleftrightarrow$$

$$x = r\cos\theta\,, \quad y = r\sin\theta\,.$$



$$\underline{e}_r \equiv (x\,\underline{e}_x + y\,\underline{e}_y)/r = \cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y$$
$$\underline{e}_\theta \equiv (-y\,\underline{e}_x + x\,\underline{e}_y)/r = -\sin\theta\,\underline{e}_x + \cos\theta\,\underline{e}_y$$

$$\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z = r\,\underline{e}_r + z\,\underline{e}_z\,.$$

We can use $\{\underline{e}_r, \underline{e}_\theta, \underline{e}_z\}$ instead of $\{\underline{e}_x, \underline{e}_y, \underline{e}_z\}$ as the basis of our coordinate system. This is called the <u>cylindrical basis</u>; in 2D $\{\underline{e}_r, \underline{e}_\theta\}$ is called the <u>polar basis</u>. Unlike cartesian, these bases are <u>not</u> fixed. The basis is determined by the particle's current position through its θ value.

$$\frac{d\underline{e}_r}{d\theta} = -\sin\theta\,\underline{e}_x + \cos\theta\,\underline{e}_y = \underline{e}_\theta$$

$$\frac{d\underline{e}_\theta}{d\theta} = -\cos\theta\,\underline{e}_x - \sin\theta\,\underline{e}_y = -\underline{e}_r$$

$$\frac{d\underline{e}_z}{d\theta} = 0\,.$$

$$\Rightarrow \quad \dot{\underline{e}}_r = \frac{d\underline{e}_r}{d\theta}\dot{\theta} = \boxed{\dot{\theta}\,\underline{e}_\theta}\,, \quad \dot{\underline{e}}_\theta = \frac{d\underline{e}_\theta}{d\theta}\dot{\theta} = \boxed{-\dot{\theta}\,\underline{e}_r}\,,$$

$$\dot{\underline{e}}_z = 0\,.$$

Given these results we have

$$\underline{v} = \dot{\underline{r}} = \overbrace{\dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r}^{\text{chain rule}} + \dot{z}\,\underline{e}_z$$

$$= \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + \dot{z}\,\underline{e}_z$$

$$= \dot{x}\,\underline{e}_x + \dot{y}\,\underline{e}_y + \dot{z}\,\underline{e}_z \leftarrow$$ (the same velocity vector represented two different ways using different coordinate systems).

$$\underline{a} = \dot{\underline{v}} = \frac{d}{dt}\left(\dot{r}\,\underline{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\,\underline{e}_\theta\right) + \dot{z}\,\underline{e}_z$$

$$= \ddot{r}\,\underline{e}_r + \dot{r}\underbrace{\dot{\underline{e}}_r}_{=\dot{\theta}\underline{e}_\theta} + \dot{r}\dot{\theta}\,\underline{e}_\theta + r\ddot{\theta}\,\underline{e}_\theta + r\dot{\theta}\underbrace{\dot{\underline{e}}_\theta}_{=-\dot{\theta}\underline{e}_r} + \ddot{z}\,\underline{e}_z$$

$$= \boxed{\left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta + \ddot{z}\,\underline{e}_z}\,.$$

$$= \ddot{x}\,\underline{e}_x + \ddot{y}\,\underline{e}_y + \ddot{z}\,\underline{e}_z\,.$$

Let's revisit circular motion using these new results for the case where the particle speed is not constant. $\Longrightarrow \theta = \theta(t)$.