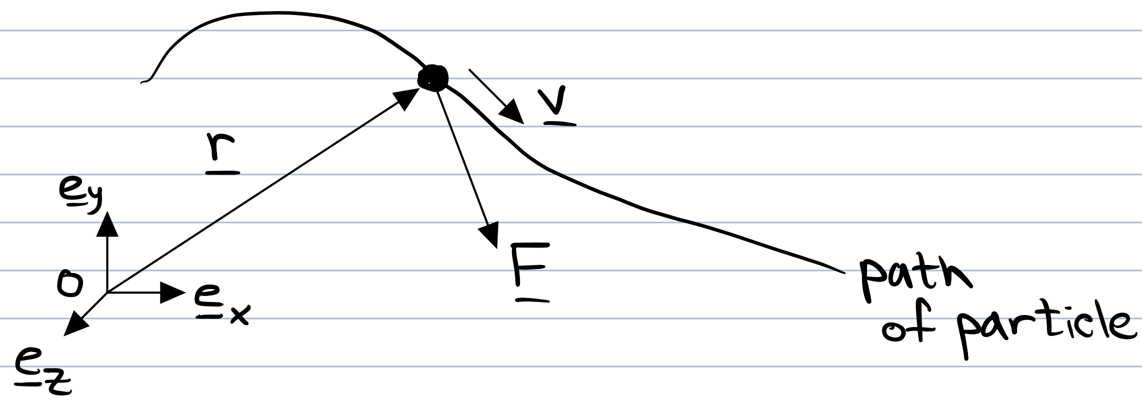
ME 104 Lec 8

Last time: Introduction to work, energy and power.

Recall:

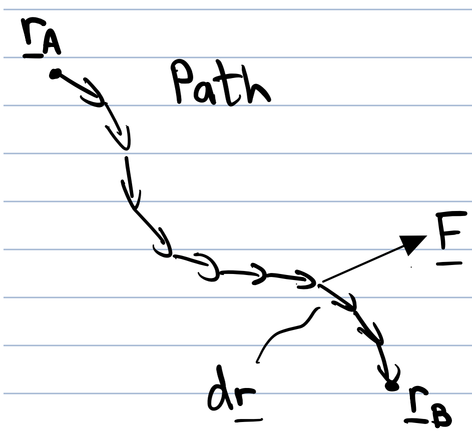


The mechanical <u>power</u> of the force $\underline{F}$ is

$$\boxed{\text{Power} \equiv \underline{F} \cdot \underline{v}}$$

W_{AB} = "Work done by $\underline{F}$ from t_A to t_B"

$$= \int_{t_A}^{t_B} \underline{F} \cdot \underline{v} \, dt = \int_{\substack{\text{Path} \\ \underline{r}(t_A) \to \underline{r}(t_B)}} \underline{F} \cdot d\underline{r}$$



Imagine breaking the path into a chain of little $d\underline{r}$ vectors. At each $d\underline{r}$, draw the force vector there and compute $\underline{F}\cdot d\underline{r}$. Add up all the $(\underline{F}\cdot d\underline{r})$'s and that's the work over the path.

Work-Energy Theorem:

Define the kinetic energy of a particle as: $\boxed{K = \frac{1}{2}m\underline{v}\cdot\underline{v} = \frac{1}{2}mv^2}$. {recall v is speed}

The total work done by all forces on a particle is:

$$W_{AB}^{Tot} = \sum_i W_{AB}^{i} = \sum_i \int_{t_A}^{t_B} \underline{F}^{i}\cdot\underline{v}\,dt = K(t_B) - K(t_A)$$
$$= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 .$$

Work-energy theorem: $\boxed{W_{AB}^{Tot} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2}$.

Why is this useful? Because many forces have a form that let's us simplify the calculation of $W_{AB}^{i} = \int_{t_A}^{t_B} \underline{F}^{i}\cdot\underline{v}\,dt = \int_{\substack{path \\ A\to B}} \underline{F}^{i}\cdot d\underline{r}$.

Forces arising from potential energy:

Suppose a force varies with position according to a formula of the form:

$$\underline{F}(\underline{r}) = -\nabla U(\underline{r}) \equiv -\left[\frac{\partial U}{\partial x}\underline{e}_x + \frac{\partial U}{\partial y}\underline{e}_y + \frac{\partial U}{\partial z}\underline{e}_z\right]$$

where the scalar function $U(\underline{r})$ is called the "potential energy". Such forces are called "conservative". Not all forces are conservative! But many are.

Conservative force examples:

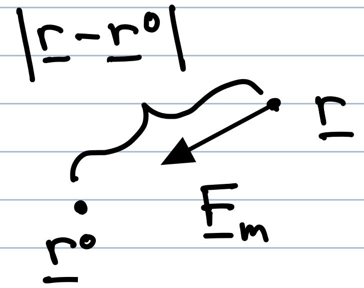
Weight: $\underline{W} = -mg\,\underline{e}_y = -\nabla(mgy). \Rightarrow U_g(\underline{r}) = mgy.$

Spring: $\underline{F}_s(\underline{r}) = -k(|\underline{r}-\underline{r}^{\circ}|-l_0)\dfrac{\underline{r}-\underline{r}}{|\underline{r}-\underline{r}\ |} = -\nabla\left(\frac{k}{2}(|\underline{r}-\underline{r}^{\circ}|-l_0)^2\right).$

$$\Rightarrow U_s(\underline{r}) = \frac{k}{2}(|\underline{r}-\underline{r}^{\circ}|-l_0)^2.$$

Constant force: $\underline{F}_0 = -\nabla(-\underline{F}_0\cdot\underline{r}). \Rightarrow U_0(\underline{r}) = -\underline{F}_0\cdot\underline{r}.$

Power-law force: $\underline{F}_m(\underline{r}) = -C|\underline{r}-\underline{r}^{\circ}|^m \dfrac{\underline{r}-\underline{r}^{\circ}}{|\underline{r}-\underline{r}^{\circ}|}$



$$= -\nabla\, C\cdot\frac{1}{m+1}|\underline{r}-\underline{r}^{\circ}|^{m+1}.$$

$$\Rightarrow U_m(\underline{r}) = \frac{C}{m+1}|\underline{r}-\underline{r}^{\circ}|^{m+1}.$$

Non-conservative (AKA dissipative) forces:

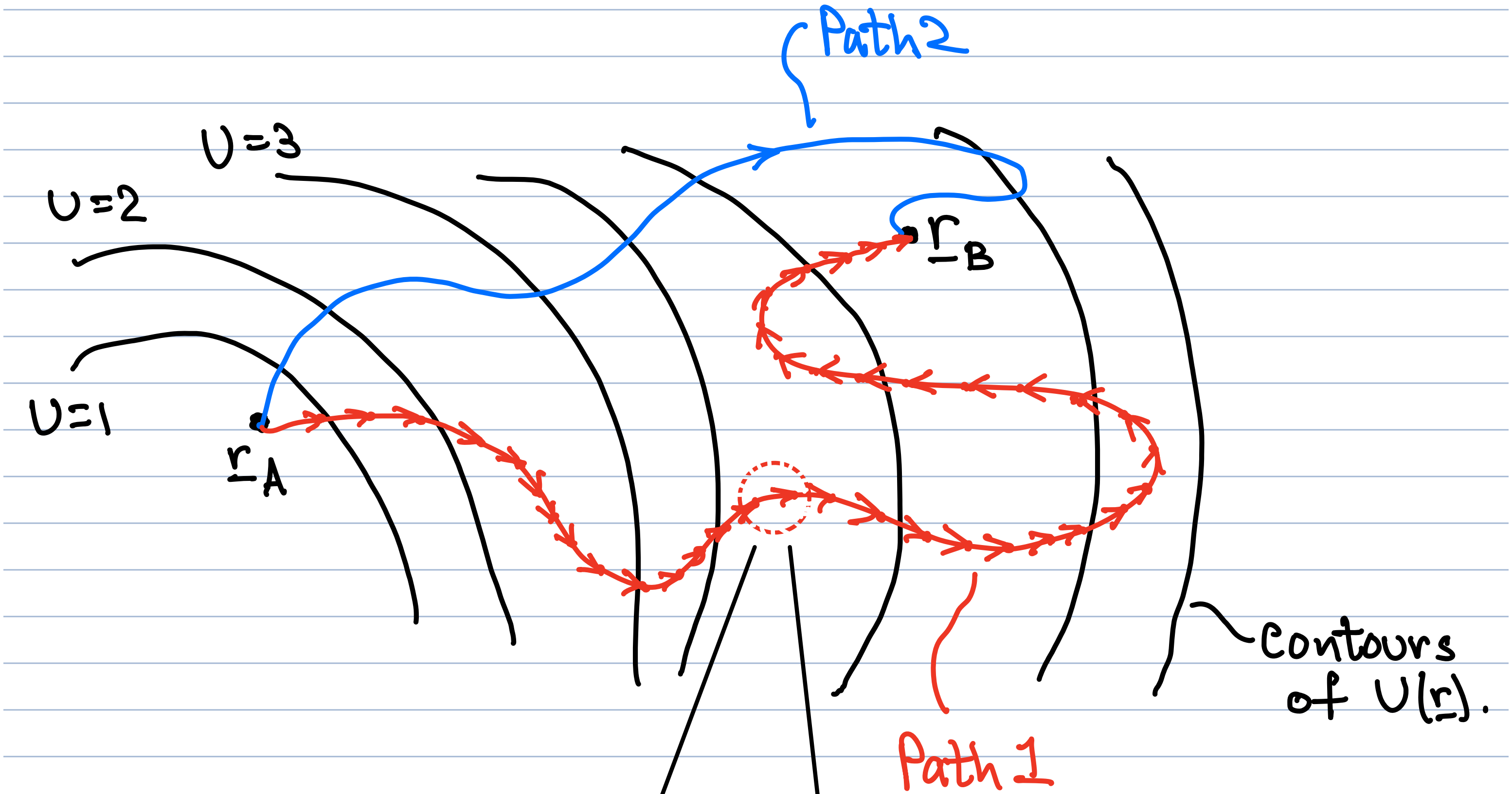
Ex: Friction, fluid drag. If a force makes the system emit sound or heat when its power is non-zero, then the force is dissipative.

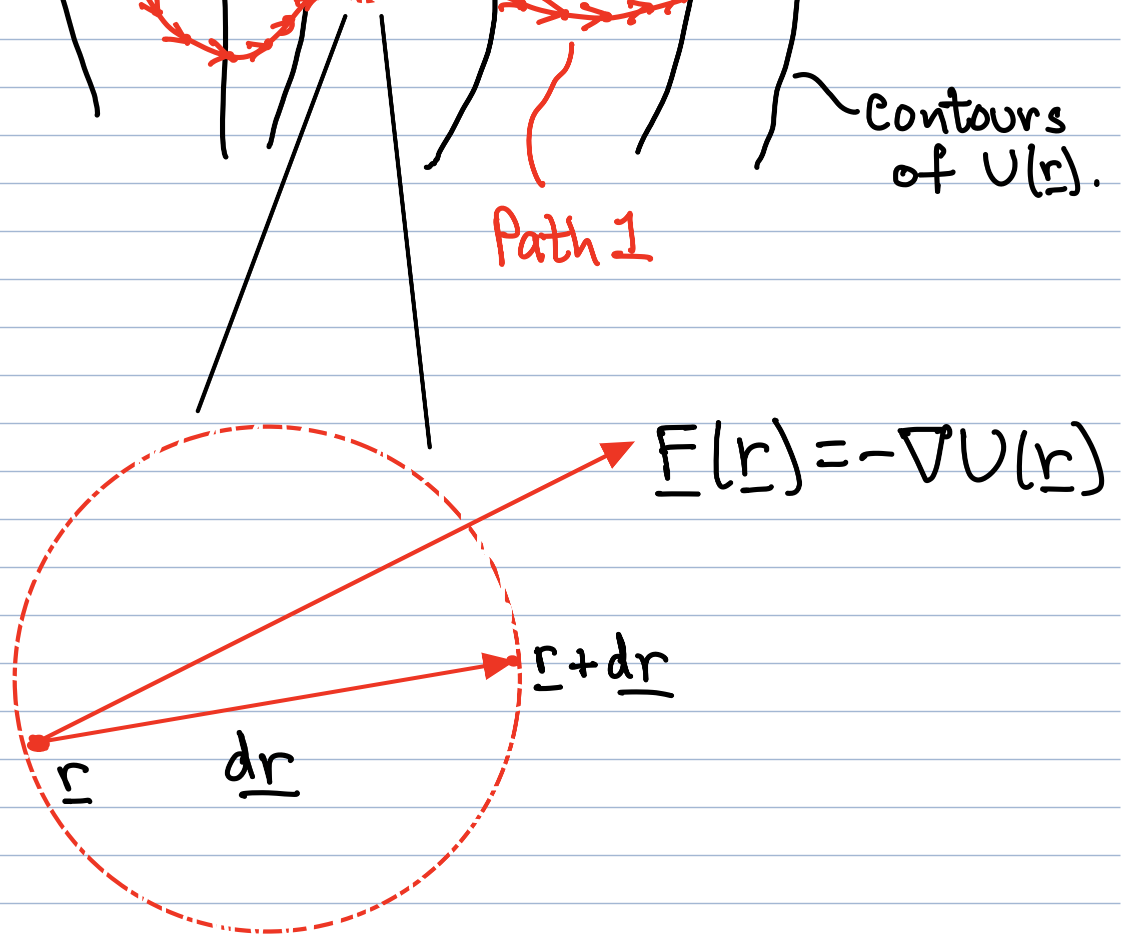
Claim: If $\underline{F} = -\nabla U$, then

$$W_{AB} = \int_{\substack{\text{path} \\ \underline{r}_A \to \underline{r}_B}} \underline{F} \cdot d\underline{r} = -\left(U(\underline{r}_B) - U(\underline{r}_A)\right).$$

Work done by conservative forces depends only on the potential energy of the path's endpoints. The shape of the path has <u>no influence</u>.

Why? Visual proof. Imagine U is like "elevation" out of the page.





Along any chosen $\underline{dr}$ step, the amount of elevation change over that step is:

$$U(\underline{r}+\underline{dr}) - U(\underline{r}) \approx (\nabla U(\underline{r}))\cdot\underline{dr} = -\underline{F}\cdot\underline{dr}.$$

$$\Rightarrow \begin{Bmatrix}\text{Elevation}\\ \text{change}\\ \text{along}\\ \text{path}\end{Bmatrix} = U(\underline{r}_B) - U(\underline{r}_A) = \sum_{\substack{\text{steps}\\ \text{along path}}} \{\text{Change in elevation each step}\}$$

$$= \sum -\underline{F}\cdot d\underline{r}$$

$$\longrightarrow \int_{\text{path}} -\underline{F}\cdot d\underline{r} \quad \text{as } d\underline{r} \to \underline{0}.$$

So path-independence is clear since $\int_{\text{Path 2}} \underline{F}\cdot d\underline{r}$ and $\int_{\text{Path 1}} \underline{F}\cdot d\underline{r}$ both measure just the total change in elevation along two paths starting and ending in the same place.

<u>Energy balance:</u> Divvy up the forces acting on a particle into conservative (c) and non-conservative (nc) forces.

Work-energy theorem says:

$$K_B - K_A = W_{AB}^{Tot} = \int_{\text{path}} \sum \underline{F}\cdot d\underline{r} = \int_{\text{path}} \left(\sum_c \underline{F}\cdot d\underline{r}\right) + \left(\sum_{nc} \underline{F}\cdot d\underline{r}\right)$$

$$= \int_{\text{path}} \sum_c (-\nabla U \cdot d\underline{r}) + \left(\sum_{nc} \underline{F}\cdot d\underline{r}\right)$$

$$= \int_{path} \Big(\nabla \big(\underbrace{\sum_c -U}_{\equiv -U^{Tot}}\big)\Big) \cdot \underline{dr} + \int_{path} \sum_{nc} \underline{F} \cdot \underline{dr}$$

$$= \underbrace{\int_{Path} -\nabla U^{Tot} \cdot \underline{dr}}_{-(U_B^{Tot} - U_A^{Tot})} + \underbrace{\int_{path} \sum_{nc} \underline{F} \cdot \underline{dr}}_{W_{AB}^{nc}}$$

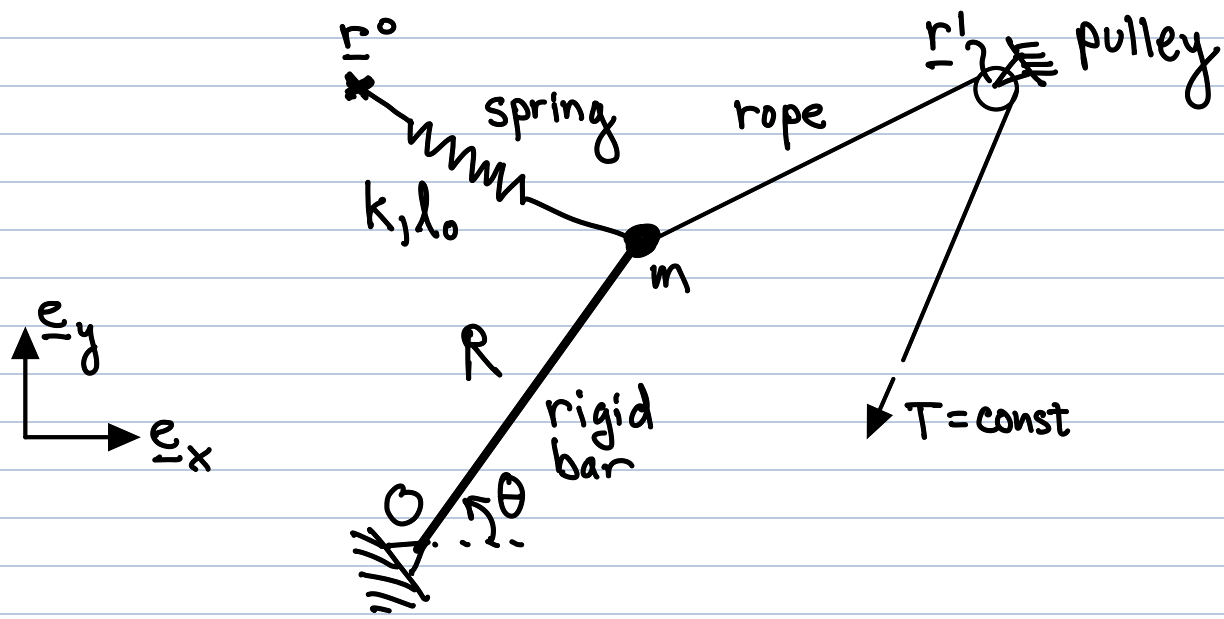
$$\Rightarrow \underbrace{W_{AB}^{nc}}_{\text{Work done by dissipative forces}} = \underbrace{(K_B - K_A)}_{\text{Change in kinetic energy}} + \underbrace{(U_B^{Tot} - U_A^{Tot})}_{\text{Change in total potential energy.}}$$

Or, letting $E = K + U^{Tot}$ = "Total energy",

can just write: $\boxed{W_{AB}^{nc} = E_B - E_A = \Delta E}$.

If all forces on a particle are conservative then $\Delta E = 0$; that is, energy is <u>conserved</u>.

Ex:



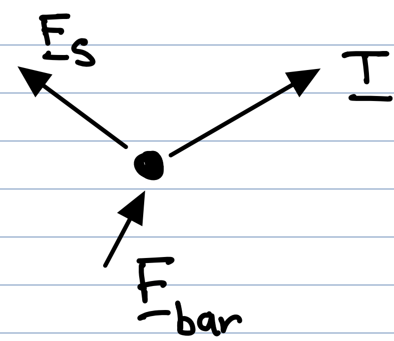
Suppose mass begins stationary at $\theta = \theta_i$. What is $|\dot{\theta}|$ when the mass gets to some θ_f? $\underline{r}^o$ and $\underline{r}_1$ are given and fixed.

① Kinematics:

$$\underline{r} = R\,\underline{e}_r = R(\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y).$$

$$\underline{v} = R\dot{\theta}\,\underline{e}_\theta = R\dot{\theta}(-\sin\theta\,\underline{e}_x + \cos\theta\,\underline{e}_y) \Rightarrow v = R|\dot{\theta}|.$$

② FBD:



$$\underline{F}_s = -k\left(|\underline{r}-\underline{r}^o| - l_o\right)(\underline{r}-\underline{r}^o)/|\underline{r}-\underline{r}^o|$$

$$\underline{T} = T\,\underline{e}_{rope} = T(\underline{r}'-\underline{r})/|\underline{r}'-\underline{r}| \quad (T>0)$$

$$\underline{F}_{bar} = F_{bar}\,\underline{e}_r$$

③ Write law of motion: To use energy balance rather than F=ma, first need to classify forces on FBD:

- $\underline{F}_s$ is conservative. $U_s = \frac{1}{2}(|R\underline{e}_r - \underline{r}^o| - l_0)^2$.
- $\underline{T}$ is a power-law force with $m=0$.
  $\Rightarrow$ Conservative $\Rightarrow$ $U_{rope} = T\,|R\underline{e}_r - \underline{r}'|$.
- $\underline{F}_{bar}$ powerless since $\underline{F}_{bar}\cdot\underline{v}$ must be 0 always. $\Rightarrow$ $\underline{F}_{bar}$ never does work. So $\underline{F}_{bar}$ is non-conservative but powerless.

Thus $U^{Tot} = U_s + U_{rope}$. $E = \frac{1}{2}mv^2 + U^{Tot}$.

Work-energy $\Rightarrow$ $W^{bar} = 0 = \Delta E$.

$$\Rightarrow \quad 0 = \frac{1}{2}mv_f^2 + U^{Tot}(\theta_f) - \left(\frac{1}{2}m\cdot 0^2 + U^{Tot}(\theta_i)\right)$$
$$= \frac{1}{2}mR^2|\dot{\theta}|^2 + U^{Tot}(\theta_f) - U^{Tot}(\theta_i).$$

④ Solve:

$$U_s(\theta) = \frac{k}{2}\left(|R(\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y) - (r_x^o\,\underline{e}_x + r_y^o\,\underline{e}_y)| - l_0\right)^2$$
$$= \frac{k}{2}\left(\sqrt{(R\cos\theta - r_x^o)^2 + (R\sin\theta - r_y^o)^2} - l_0\right)^2.$$

$$U_{rope}(\theta) = T\,|\underline{r} - \underline{r}'| = T\sqrt{(R\cos\theta - r_x')^2 + (R\sin\theta - r_y')^2}.$$

$$\frac{1}{2}mR^2|\dot{\theta}_f|^2 + U_s(\theta_f) + U_{rope}(\theta_f) = U_s(\theta_i) + U_{rope}(\theta_i)$$

$$\Rightarrow \frac{1}{2} m R^2 |\dot{\theta}_f|^2 + \frac{k}{2}\left(\sqrt{(R\cos\theta_f - r_x^0)^2 + (R\sin\theta_f - r_y^0)^2} - l_0\right)^2$$

$$+ T\sqrt{(R\cos\theta_f - r_x^1)^2 + (R\sin\theta_f - r_y^1)^2}$$

$$= \frac{k}{2}\left(\sqrt{(R\cos\theta_i - r_x^0)^2 + (R\sin\theta_i - r_y^0)^2} - l_0\right)^2$$

$$+ T\sqrt{(R\cos\theta_i - r_x^1)^2 + (R\sin\theta_i - r_y^1)^2}$$

$$\Rightarrow \boxed{\begin{aligned} |\dot{\theta}_f| = \sqrt{\frac{2}{mR^2}} \cdot \Bigg[& \frac{k}{2}\left(\sqrt{(R\cos\theta_i - r_x^0)^2 + (R\sin\theta_i - r_y^0)^2} - l_0\right)^2 \\ & + T\sqrt{(R\cos\theta_i - r_x^1)^2 + (R\sin\theta_i - r_y^1)^2} \\ & - T\sqrt{(R\cos\theta_f - r_x^1)^2 + (R\sin\theta_f - r_y^1)^2} \\ & - \frac{k}{2}\left(\sqrt{(R\cos\theta_f - r_x^0)^2 + (R\sin\theta_f - r_y^0)^2} - l_0\right)^2 \Bigg]^{1/2} \end{aligned}}$$