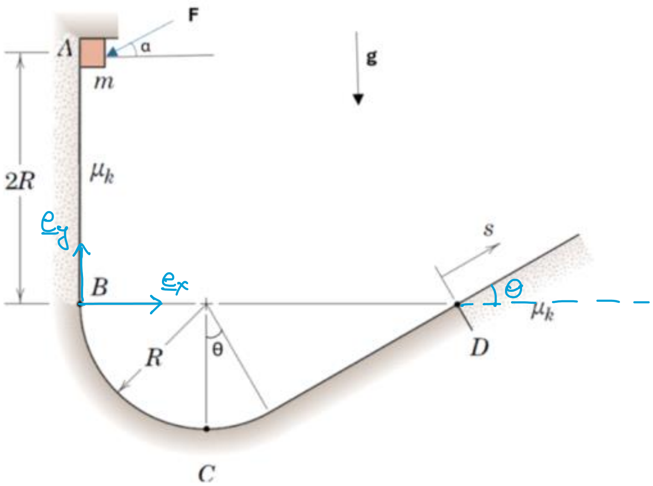
Problem Set 2 Solutions

Tuesday, September 17, 2024 9:30 AM

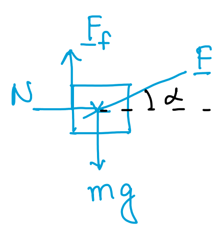
**Q1**




(a)

FBD:

(A–B):




In $\underline{e}_x$: $\quad \Sigma \underline{F} = 0 \quad \Rightarrow \quad N - F\cos\alpha = 0$

$\Rightarrow \quad N = F\cos\alpha$

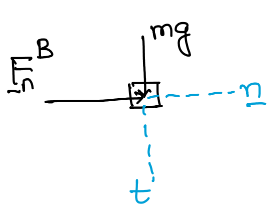
Friction force: $\quad \underline{F}_f = \mu_k N \underline{e}_y = \mu_k F\cos\alpha \, \underline{e}_y$ .

Using Work-energy theorem from A to B:

$$2mgR + 2RF\sin\alpha - 2R\mu_k F\cos\alpha = \frac{1}{2} m v_B^2$$

$$v_B^2 = 4gR + \frac{4RF\sin\alpha}{m} - \frac{4R\mu_k F\cos\alpha}{m}$$

Normal force $F_n^B$ exerted by the track on the slider just after B is:




$$\Sigma \underline{F}_n = m\underline{a}_n^B$$

$$F_n^B = \frac{m v_B^2}{R}$$

$$F_n^B = 4mg + 4F\sin\alpha - 4\mu_k F\cos\alpha$$

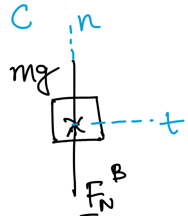
$$\Rightarrow \quad \underline{F_n^B} = 4\left[\underline{mg + F(\sin\alpha - \mu_k\cos\alpha)}\right].$$

(b) Energy conservation for points B and C:

$$\frac{1}{2}mv_B^2 = -mgR + \frac{1}{2}mv_C^2$$

$$v_C^2 = 2gR + v_B^2$$

FBD: At point C




$$\Sigma F_n = ma_n$$

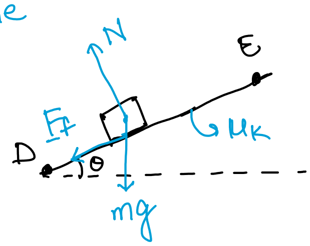
$$F_n^C - mg = m\frac{v_C^2}{R}$$

$$F_n^C = mg + \frac{mv_C^2}{R} = 3mg + \frac{mV_B^2}{R}$$

$$\Rightarrow \quad \underline{F_n^C} = \underline{7mg + 4F(\sin\alpha - \mu_k\cos\alpha)}$$

(c) Let the slider stops at point E after travelling a distance 'S' along the incline,

FBD: on incline




$$F_f = \mu_k N$$
$$= \mu_k mg\cos\theta$$

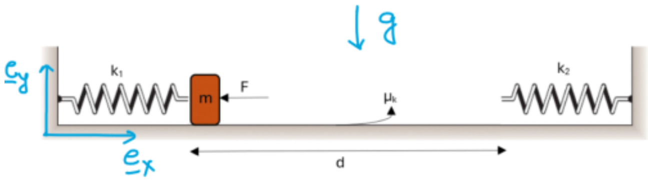
Work-Energy Theorem for B and E:

$$\frac{1}{2}mv_B^2 - mg\sin\theta \cdot S - \mu_k mg\cos\theta \cdot S = 0$$

$$S = \frac{V_B^2}{2g(\sin\theta + \mu_k\cos\theta)}$$

$$\Rightarrow \quad \underline{S} = \frac{2R[mg + F(\sin\alpha - \mu_k\cos\alpha)]}{\underline{mg(\sin\theta + \mu_k\cos\theta)}}.$$

**Q2**



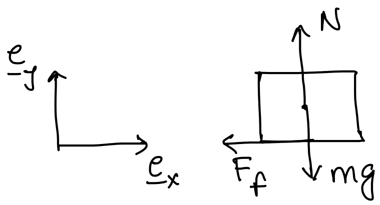

Compression of the first spring due to force F:

$$x_1 = \frac{F}{k_1}$$

Total mechanical energy of the block when it is released:

$$E_{total} = \frac{1}{2} k_1 x_1^2 = \frac{F^2}{2k_1}$$

FBD:




$$N = mg$$

$$F_f = \mu_k N$$

Work done by the friction over one oscillation, i.e. over a distance of '2d' is:

$$W_f = F_f \cdot 2d = 2\mu_k m g d$$

This is the energy lost per oscillation of the block.

Total no. of oscillations (N):

$$E_{total} - N \cdot W_f \leq 0$$

$$N \geq \frac{E_{total}}{W_f} = \frac{F^2/2k_1}{2\mu_k m g d} = \frac{F^2}{4\mu_k k_1 m g d}$$

Using: $m = 1\,kg$, $F = 100N$, $k_1 = 100\,N/m$, $d = 5m$, $\mu_k = 0.05$, $g = 9.81\,m/s^2$

$\underline{\underline{N \geq 10.19}}$ (10 complete oscillations)

**Q3**

→ Solution for Problem # 3: Rotating Arm

<u>Given</u>: • $r_I(\varphi) = R$ & $r_{II}(\varphi) = \frac{2}{3}R\left(1+\frac{\varphi}{\pi}\right)$

• angular acceleration is constant

• time to reach $\varphi = \pi$ is the same for both: $t_{tot}$

• Arm apelies both axial AND "theta" force: $F_r$ & $F_\theta$

→ What is the form of angular acceleration?

$\ddot{\varphi}(t) = \alpha$ (let constant ang. acceleration be $\alpha$)

→ integrate to get angular velocity

$\dot{\varphi}(t) = \alpha t + \cancel{C}$ ($C=0$ bcz start from rest)

→ integrate to get angular acceleration

$\varphi(t) = \frac{1}{2}\alpha t^2 + \cancel{C}_{0}$ ($C=0$ bcz start from $\varphi=0$)

At $t = t_{tot}$, $\varphi = \pi \Rightarrow$ calculate $\alpha$:

$\varphi(t_{tot}) = \pi = \frac{1}{2}\alpha t^2 \Rightarrow \boxed{\alpha = \frac{2\pi}{t_{tot}^2}}$

→ Use polar coordinates in both cases because we ask for axial force!

→ general acceleration equation:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

<u>Case I</u>:

Step 1: Kinematics

→ Compare general form $\vec{a}$ with our specific form:

$r(\varphi) = R$, constant: so $\dot{r} = \ddot{r} = 0$

$\ddot{\theta} = -\ddot{\varphi} = -\alpha = -2\pi/t_{tot}^2$

$\dot{\theta} = -\dot{\varphi} = -\alpha t = -\frac{2\pi}{t_{tot}^2}t$

$\theta = -\varphi = -\frac{1}{2}\alpha t^2 = -\frac{\pi}{t_{tot}^2}t^2$

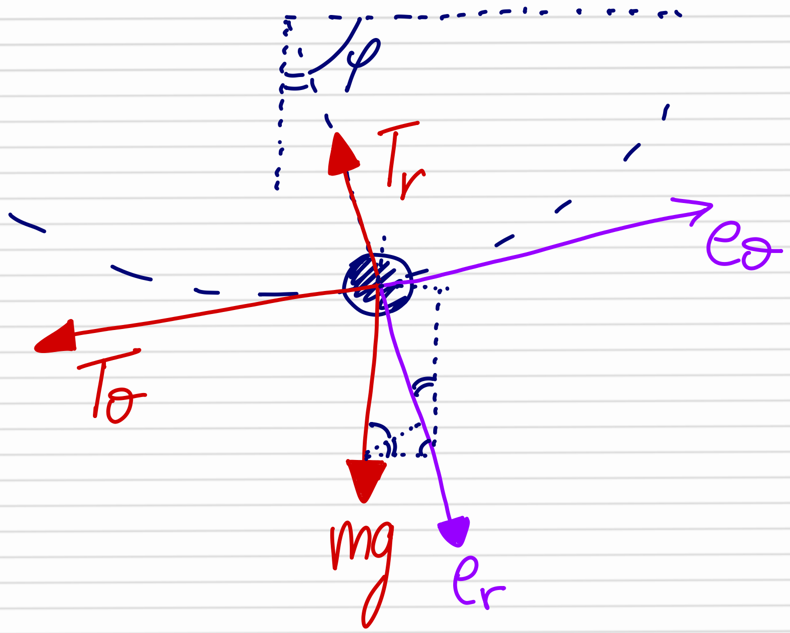
added negative because of clockwise movement in the problem, but $\vec{e}_\theta$ is defined counterclockwise

$$\vec{a}_I = \left(0 - R\left(\frac{-2\pi}{t_{tot}^2}t\right)^2\right)\vec{e}_r + \left(0 + R\left(-2\pi/t_{tot}^2\right)\right)\vec{e}_\theta$$

$$\boxed{\vec{a}_I = -R\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\vec{e}_r - \frac{2\pi R}{t_{tot}^2}\vec{e}_\theta}$$

## Step 2 : Dynamics

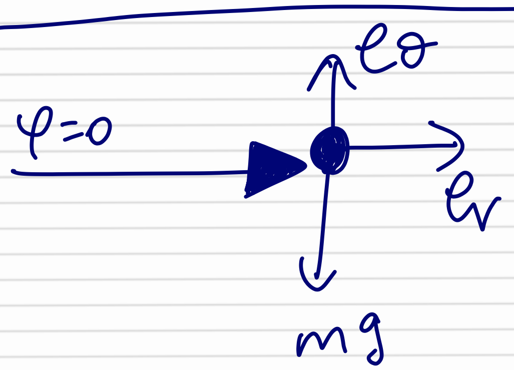
2.1 Complete FBD such that can give it to the person next to you & they can write $\sum F = ma$ without needing to read the problem




## Step 3 $\sum F = ma$ (equilibrium equations)

r-direction: $\sum F_r = ma_r = -T_r + mg\sin(\varphi)$

θ-direction: $\sum F_\theta = ma_\theta = -T_\theta - mg\cos(\varphi)$




∴ Sanity check:

check that mg decomposition makes sense

→ at $\varphi = 0 \Rightarrow \sin(\varphi) = 0$ & $\cos(\varphi) = 1$

substitute in accelerations & $\varphi$

r-direction: $m\left(-R\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\right) = -T_r + mg\sin\left(\frac{\pi}{t_{tot}^2}t^2\right)$

θ-direction: $m\left(-\frac{2\pi R}{t_{tot}^2}\right) = -T_\theta - mg\cos\left(\frac{\pi}{t_{tot}^2}t^2\right)$

$$\boxed{\vec{a}_I = -R\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\vec{e}_r - \frac{2\pi R}{t_{tot}^2}\vec{e}_\theta}$$

Step 4 solve for $T_r$ (asked in the problem)

$$\boxed{T_r^{I}(t) = \frac{mR4\pi^2}{t_{tot}^4} t^2 + mg\sin\left(\frac{\pi}{t_{tot}^2} t^2\right)}$$

Case II: (FBD is the same, only kinematics are different)

Step 1:

$$\left.\begin{aligned} \ddot{\theta} &= -\ddot{\varphi} = -\alpha = -2\pi / t_{tot}^2 \\ \dot{\theta} &= -\dot{\varphi} = -\alpha t = -\frac{2\pi}{t_{tot}^2} t \\ \theta &= -\varphi = -\frac{1}{2}\alpha t^2 = -\frac{\pi}{t_{tot}^2} t^2 \end{aligned}\right\}$$

Same as before since angular acceleration is the same for both cases!

$r(\varphi) = \frac{2}{3}R\left(1 + \frac{\varphi}{\pi}\right)$ : need $\dot{r}$ & $\ddot{r}$ ⇒ use $\varphi(t)$!

$$\Rightarrow \hat{r}(t) = r(\varphi(t)) = \frac{2}{3}R\left(1 + \frac{1}{\pi}\left(\frac{\pi}{t_{tot}^2} t^2\right)\right) = \frac{2}{3}R\left(1 + \frac{t^2}{t_{tot}^2}\right)$$

(drop ^ for ease of writing)

$$\dot{r} = \frac{2}{3}R\left(2\frac{t}{t_{tot}^2}\right) = \frac{4}{3}R\frac{t}{t_{tot}^2}$$

$$\ddot{r} = \frac{4R}{3t_{tot}^2}$$

→ Plug all into general form equation

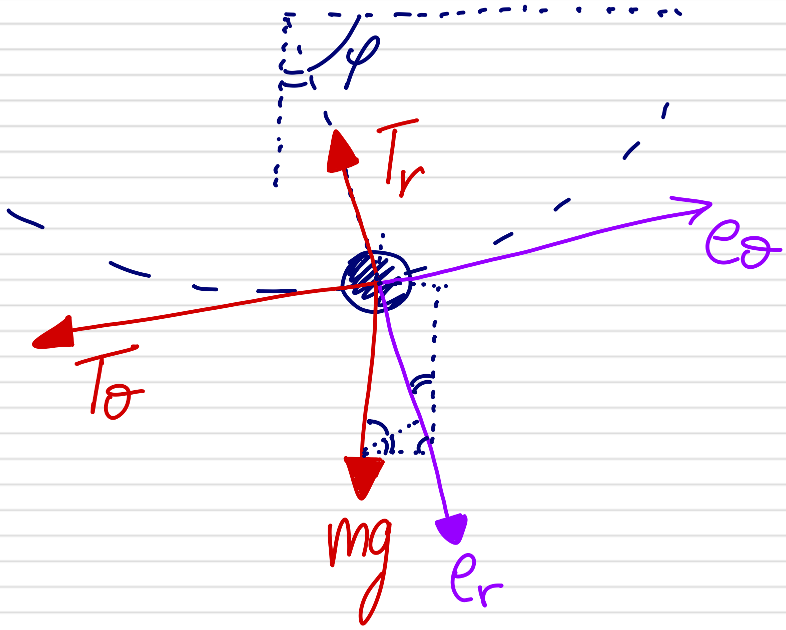
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

$$\vec{a}_{II} = \Bigg(\underbrace{\frac{4R}{3t_{tot}^2}}_{\ddot{r}} - \underbrace{\left(\frac{2}{3}R\left(1 + \frac{t^2}{t_{tot}^2}\right)\right)}_{r}\underbrace{\left(-\frac{2\pi}{t_{tot}^2} t\right)^2}_{\dot{\theta}^2}\Bigg)\vec{e}_r$$

$$+ \Bigg(2\underbrace{\left(\frac{4}{3}R\frac{t}{t_{tot}^2}\right)}_{\dot{r}}\underbrace{\left(-\frac{2\pi}{t_{tot}^2} t\right)}_{\dot{\theta}} + \underbrace{\left(\frac{2}{3}R\left(1 + \frac{t^2}{t_{tot}^2}\right)\right)}_{r}\underbrace{\left(-2\pi / t_{tot}^2\right)}_{\ddot{\theta}}\Bigg)\vec{e}_\theta$$

$$\vec{a}_{II} = \left(\frac{4R}{3t_{tot}^2} - \frac{2}{3}R\left(1 + \frac{t^2}{t_{tot}^2}\right)\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\right)\vec{e}_r + \left(\frac{-16}{3}\frac{\pi R}{t_{tot}^4} t^2 - \frac{4\pi R}{3t_{tot}^2}\left(1 + \frac{t^2}{t_{tot}^2}\right)\right)\vec{e}_\theta$$

Step 2: FBD (same as before in case I)




Step 3: $\sum F = ma$, same $\sum F$ as case I

r-direction: $\sum F_r = ma_r = -T_r + mg\sin(\varphi)$

$\theta$-direction: $\sum F_\theta = ma_\theta = -T_\theta - mg\cos(\varphi)$

$\Longrightarrow$ Accelerations are different, however!

$$\vec{a}_{II} = \left(\frac{4R}{3t_{tot}^2} - \frac{2}{3}R\left(1+\frac{t^2}{t_{tot}^2}\right)\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\right)\vec{e}_r + \left(\frac{-16}{3}\frac{\pi R}{t_{tot}^4}t^2 - \frac{4\pi R}{3\,t_{tot}^2}\left(1+\frac{t^2}{t_{tot}^2}\right)\right)\vec{e}_\theta$$

$$r:\ m\left(\frac{4R}{3t_{tot}^2} - \frac{2}{3}R\left(1+\frac{t^2}{t_{tot}^2}\right)\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\right) = -T_r + mg\sin\left(\frac{\pi}{t_{tot}^2}t^2\right)$$

$$\theta:\ m\left(\frac{-16}{3}\frac{\pi R}{t_{tot}^4}t^2 - \frac{4\pi R}{3\,t_{tot}^2}\left(1+\frac{t^2}{t_{tot}^2}\right)\right) = -T_\theta - mg\cos\left(\frac{\pi}{t_{tot}^2}t^2\right)$$

Step 4: Solve for $T_r$ (since it was asked for)

$$T_r^{II} = -m\left(\frac{4R}{3t_{tot}^2} - \frac{2}{3}R\left(1+\frac{t^2}{t_{tot}^2}\right)\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\right) + mg\sin\left(\frac{\pi}{t_{tot}^2}t^2\right)$$

→ Finally compare by plotting : $T^{I}(t)$ & $T^{II}(t)$ to find which one is larger.

$$T_r^{I}(t) = \frac{mR4\pi^2}{t_{tot}^4} t^2 + mg\sin\left(\frac{\pi}{t_{tot}^2} t^2\right)$$

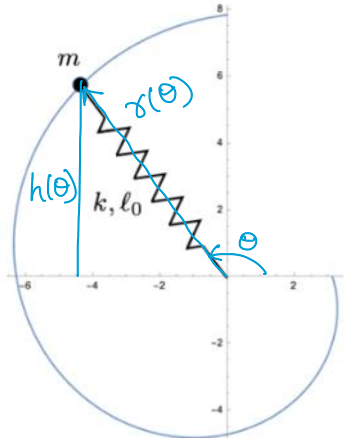
$$T_r^{II} = -m\left(\frac{4R}{3t_{tot}^2} - \frac{2}{3}R\left(1 + \frac{t^2}{t_{tot}^2}\right)\left(\frac{4\pi^2 t^2}{t_{tot}^4}\right)\right) + mg\sin\left(\frac{\pi}{t_{tot}^2} t^2\right)$$

$T_r^{I}$ generally has lower value.

$T_r^{II}$ only really surpasses $T_r^{I}$ when $t_{tot}$ is sufficiently large & $R$ is sufficiently small.

$\boxed{\text{Redesign is not worth it!}}$

**Q4**




$r(\theta) = L_0(3\pi - \theta)$

$h(\theta) = r(\theta)\sin(\pi - \theta)$
$= r(\theta)\sin\theta$

Total energy at any angle $\theta$ will be:

$$E_{total} = \frac{1}{2}mv^2 + mgh(\theta) + \frac{1}{2}k(r(\theta) - l_0)^2$$

Total energy at $\theta = \pi/2$:

At $\theta = \frac{\pi}{2}$, bead starts with rest i.e, $v = 0$,

$$E_{total} = mgh\left(\frac{\pi}{2}\right) + \frac{1}{2}k\left(r\left(\frac{\pi}{2}\right) - l_0\right)^2$$

here, $r\left(\frac{\pi}{2}\right) = L_0\left(3\pi - \frac{\pi}{2}\right) = \frac{5\pi L_0}{2}$, $\sin\left(\frac{\pi}{2}\right) = 1$

$$\Rightarrow E_{total} = \frac{5\pi mgL_0}{2} + \frac{1}{2}k\left(\frac{5\pi L_0}{2} - l_0\right)^2 \quad \text{—— (1)}$$

Energy conservation at $\theta = 2\pi$:

$$E_{total} = \frac{1}{2}mv_{2\pi}^2 + mgh(2\pi) + \frac{1}{2}k\left(r(2\pi) - l_0\right)^2$$

here, $r(2\pi) = L_0(3\pi - 2\pi) = \pi L_0$, $\sin(2\pi) = 0$

$$\Rightarrow E_{total} = \frac{1}{2}mv_{2\pi}^2 + \frac{1}{2}k(\pi L_0 - l_0)^2 \quad \text{—— (2)}$$

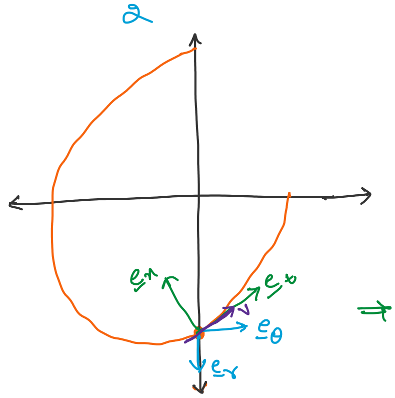
Equating ① & ②:

$$\frac{1}{2} m V_{2\pi}^2 + \frac{1}{2}k(\pi L_0 - l_0)^2 = \frac{5\pi m g L_0}{2} + \frac{1}{2}k\left(\frac{5\pi L_0}{2} - l_0\right)^2$$

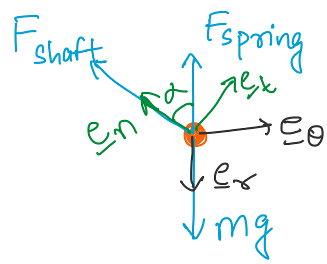
It gives:

$$\underline{\underline{V_{2\pi}}} = \underline{\underline{\sqrt{5\pi g L_0 + \frac{K}{m}\left[\left(\frac{5\pi}{2}L_0 - l_0\right)^2 - (\pi L_0 - l_0)^2\right]}}}$$

FBD: At $\theta = \frac{3\pi}{2}$




$\Rightarrow$




$$\Sigma F_n = m a_n = m \rho V_{3\pi/2}^2$$

$\rho$: curvature

$$F_{spring}\cos\alpha + F_{shaft} - mg\cos\alpha = m\rho V_{3\pi/2}^2$$

To find curvature ($\rho$):

$$\rho(\theta) = \frac{|r^2 + 2r'^2 - r r''|}{(r^2 + r'^2)^{3/2}}, \quad \text{here } r' \equiv \frac{dr}{d\theta}$$

$$\left.\begin{aligned} r(\theta) &= L_0(3\pi - \theta) \\ r'(\theta) &= -L_0 \\ r''(\theta) &= 0 \end{aligned}\right\} \quad \rho(\theta) = \frac{|L_0^2(3\pi-\theta)^2 + 2L_0^2|}{(L_0^2(3\pi-\theta)^2 + L_0^2)^{3/2}}$$

At $\theta = 3\pi/2$,

$$\rho(\theta = 3\pi/2) = \frac{\left| L_0^2 (3\pi - 3\pi/2)^2 + 2L_0^2 \right|}{\left( L_0^2 (3\pi - 3\pi/2)^2 + L_0^2 \right)^{3/2}}$$

$$= \frac{18\pi^2 + 16}{L_0 (9\pi^2 + 4)^{3/2}}$$

To find $\alpha$:

The velocity vector is: $\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta = |\underline{v}|\,\underline{e}_t$

$\underline{e}_n$ is perpendicular to the direction of $\underline{v}$:

$$\underline{v} = \frac{dr}{d\theta} \cdot \dot{\theta}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta = -L_0\dot{\theta}\,\underline{e}_r + L_0(3\pi - \theta)\dot{\theta}\,\underline{e}_\theta$$

Direction of $\underline{v}$: $$\frac{\underline{v}}{|\underline{v}|} = \frac{-L_0\dot{\theta}\,\underline{e}_r + L_0(3\pi - \theta)\dot{\theta}\,\underline{e}_\theta}{\sqrt{(-L_0\dot{\theta})^2 + (L_0(3\pi - \theta))^2}}$$

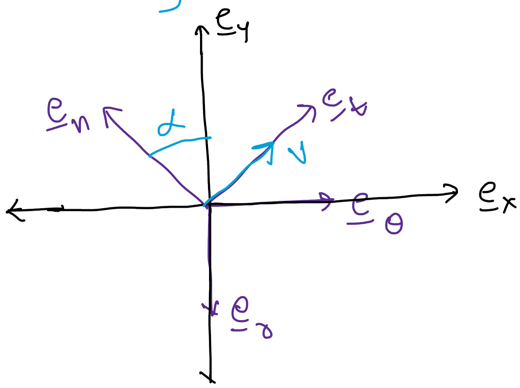
at $\theta = 3\pi/2$:

$$\underline{e}_t = \frac{\underline{v}}{|\underline{v}|} = \frac{-\underline{e}_r + 3\pi/2\,\underline{e}_\theta}{\sqrt{1 + (3\pi/2)^2}} = \frac{-2\underline{e}_r + 3\pi\,\underline{e}_\theta}{\sqrt{4 + 9\pi^2}}$$

then,

$$\underline{e}_n = \frac{d\underline{e}_t}{d\theta} = \frac{1}{\sqrt{4 + 9\pi^2}} \left( -\frac{d\underline{e}_r}{d\theta} + 3\pi \frac{d\underline{e}_\theta}{d\theta} \right)$$

$$= \frac{1}{\sqrt{4 + 9\pi^2}} \left( -2\underline{e}_\theta - 3\pi\,\underline{e}_r \right)$$



# Problem Set 2 Q4

Wednesday, September 25, 2024 2:33 PM

You can also find the $\underline{e}_t - \underline{e}_n$ using the following method:

$$r(\theta) = L_0(3\pi - \theta)$$

The direction of $\underline{e}_t$ is the slope of the curve at $\theta = \frac{3\pi}{2}$, i.e.,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

here, $\frac{dr}{d\theta} = -L_0$

$$\Rightarrow \left.\frac{dx}{d\theta}\right|_{\theta = \frac{3\pi}{2}} = -L_0\cos\left(\frac{3\pi}{2}\right) - L_0\left(3\pi - \frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow \left.\frac{dy}{d\theta}\right|_{\theta=\frac{3\pi}{2}} = -L_0 \sin\left(\frac{3\pi}{2}\right) + \left(3\pi - \frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right)$$

$$= L_0$$

Then, at $\theta = 3\pi/2$,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \left(\frac{dx}{d\theta}\right)^{-1} = \frac{2L_0}{3\pi L_0}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=3\pi/2} = \frac{2}{3\pi}$$

The tangential direction:

$$\underline{e}_t = \frac{3\pi \underline{e}_x + 2\underline{e}_y}{\sqrt{9\pi^2+4}} = \frac{3\pi \underline{e}_\theta - 2\underline{e}_r}{\sqrt{9\pi^2+4}}$$

$$\underline{e}_n = \frac{-2\underline{e}_x + 3\pi \underline{e}_y}{\sqrt{9\pi^2+4}} = \frac{-2\underline{e}_\theta - 3\pi \underline{e}_r}{\sqrt{9\pi^2+4}}$$

$$\tan\alpha = \frac{2}{3\pi} \quad , \quad \cos\alpha = \frac{3\pi}{\sqrt{4+9\pi^2}}$$

$$F_{spring}\cos\alpha + F_{shaft} - mg\cos\alpha = m\rho V_{3\pi/2}^2$$

here,

$$F_{spring} = K\left(r\left(\frac{3\pi}{2}\right) - l_0\right) = K\left(\frac{3\pi}{2}L_0 - l_0\right)$$

Also, total energy at $\theta = \frac{3\pi}{2}$,

$$E_{total} = \frac{1}{2}mV_{3\pi/2}^2 + mg\,h\left(\frac{3\pi}{2}\right) + \frac{1}{2}K\left(r\left(\frac{3\pi}{2}\right) - l_0\right)^2$$

also, $h\left(\frac{3\pi}{2}\right) = r\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right) = -\frac{3\pi L_0}{2}$

$$E_{total} = \frac{1}{2}mV_{3\pi/2}^2 - \frac{3\pi mgL_0}{2} + \frac{1}{2}K\left(\frac{3\pi}{2}L_0 - l_0\right)^2 \quad \text{—④}$$

From energy conservation, equating ① & ④ :

$$\frac{1}{2}mV_{3\pi/2}^2 - \frac{3\pi mgL_0}{2} + \frac{1}{2}K\left(\frac{3\pi L_0}{2} - l_0\right)^2 = \frac{5\pi mgL_0}{2} + \frac{1}{2}K\left(\frac{5\pi L_0}{2} - l_0\right)^2$$

$$V_{3\pi/2}^2 = 8\pi gL_0 + \frac{K}{m}\left[\left(\frac{5\pi L_0}{2} - l_0\right)^2 - \left(\frac{3\pi}{2}L_0 - l_0\right)^2\right]$$

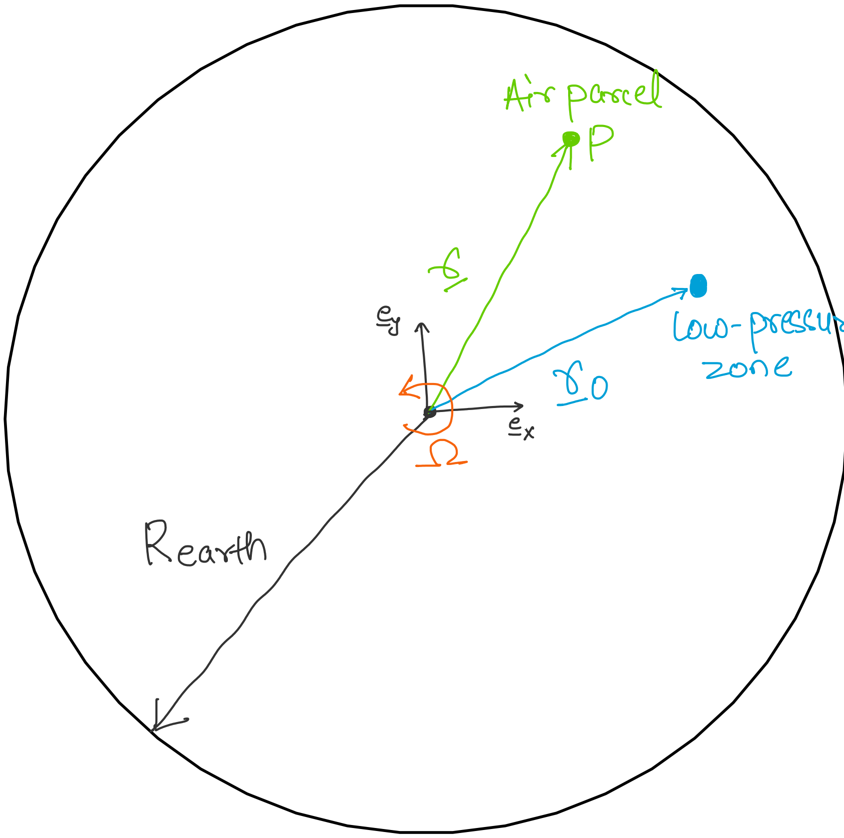
from ③ :

$$F_{shaft} = mg\cos\alpha + m\rho V_{3\pi/2}^2 - K\left(\frac{3\pi L_0}{2} - l_0\right)\cos\alpha$$

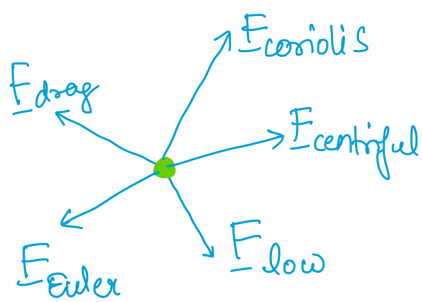
$$= \frac{3\pi mg}{\sqrt{4+9\pi^2}} + \frac{(18\pi^2+16)m}{L_0(9\pi^2+4)^{3/2}}\left(8\pi gL_0 + \frac{K}{m}\left[\left(\frac{5\pi L_0}{2} - l_0\right)^2 - \left(\frac{3\pi}{2}L_0 - l_0\right)^2\right]\right)$$

$$- \frac{3\pi K}{\sqrt{4+9\pi^2}}\left(\frac{3\pi}{2}L_0 - l_0\right).$$

**Q5**




FBD:




Solving this dynamical ~~system~~ in a rotating reference frame:

$$m\frac{d^2\underline{r}}{dt^2} = \underline{F}_{low} + \underline{F}_{drag} + \underline{F}_{coriolis} + \underline{F}_{Euler} + \underline{F}_{centrifugal}$$

$$\Rightarrow \quad m\frac{d^2\underline{r}}{dt^2} = -F_0(\underline{r}-\underline{r}_0) - C_s(\underline{v}-\underline{v}_{earth}) - 2m\dot{\phi}\,\underline{e}_z \times \underline{v} - m\ddot{\phi}\,\underline{e}_z \times \underline{r} + m\dot{\phi}^2\,\underline{r}$$

Since earth rotation rate is constant i.e. $\Omega = \dot{\phi} = 1$ rev/day

so, $\ddot{\phi} = 0$ and $\underline{v}_{earth} = 0$ for this reference frame.

It gives:

$$\frac{d^2 \underline{r}}{dt^2} = -\frac{F_0}{m}(\underline{r} - \underline{r}_0) - \frac{c_s}{m}\underline{V} - 2\Omega \underline{e}_z \times \underline{V} + \Omega^2 \underline{r}$$

here, $\underline{r} = \begin{Bmatrix} x \\ y \end{Bmatrix}$, $\underline{V} = \begin{Bmatrix} V_x \\ V_y \end{Bmatrix}$, $\underline{r}_0 = \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix}$

writing in each direction:

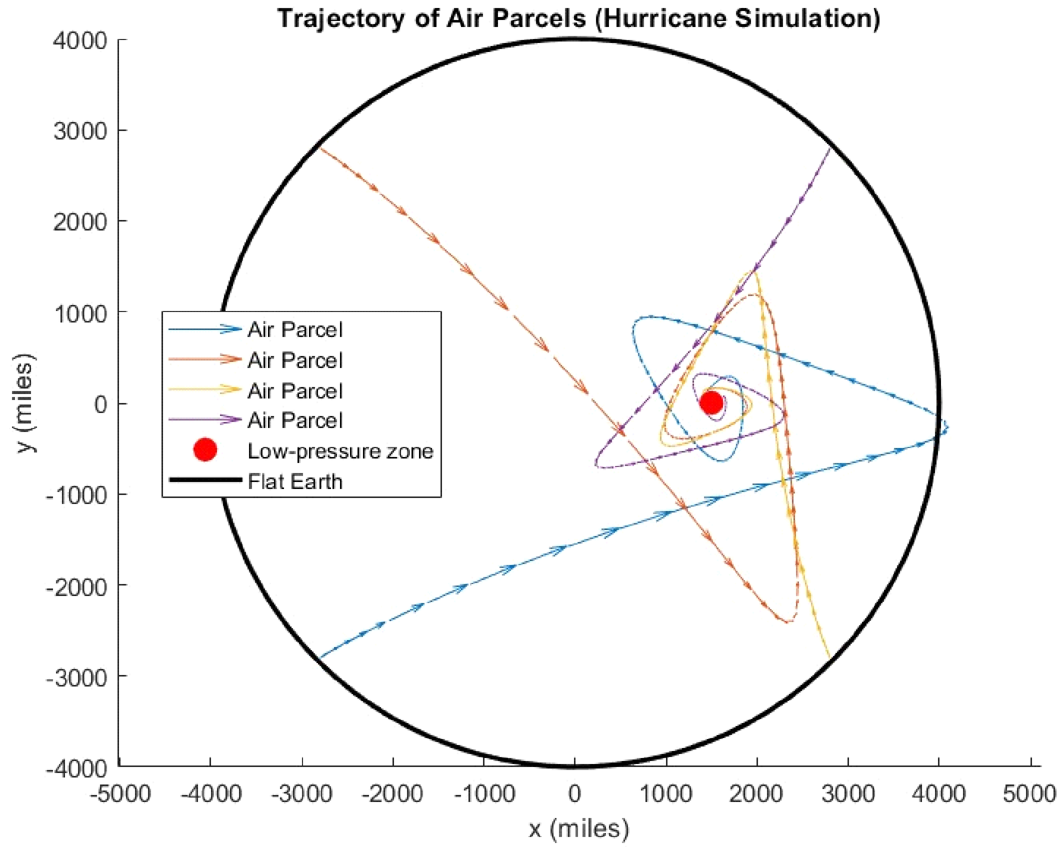
$$a_x = \frac{d^2 x}{d^2 t^2} = -\frac{F_0}{m}[x - x_0] - \frac{c_s}{m}V_x + 2\Omega V_y + \Omega^2 x$$

$$a_y = \frac{d^2 y}{dt^2} = -\frac{F_0}{m}[y - y_0] - \frac{c_s}{m}V_y - 2\Omega V_x + \Omega^2 y$$

We have two 2nd order ODEs, we will reduce them to four 1st order ODEs:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ V_x \\ V_y \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ a_x \\ a_y \end{bmatrix},$$

Run it for few air particles for a time span and initial condition.




Air parcels are spinning counterclockwise. Their rotation vector is in the same direction as that of the earth's. This is because of the coriolis effect or coriolis force

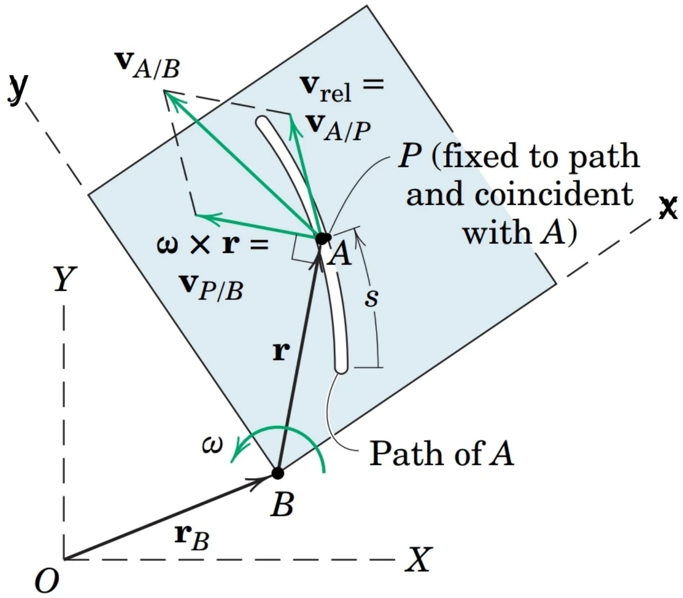
**We have also solved this problem using vector products. See below.**

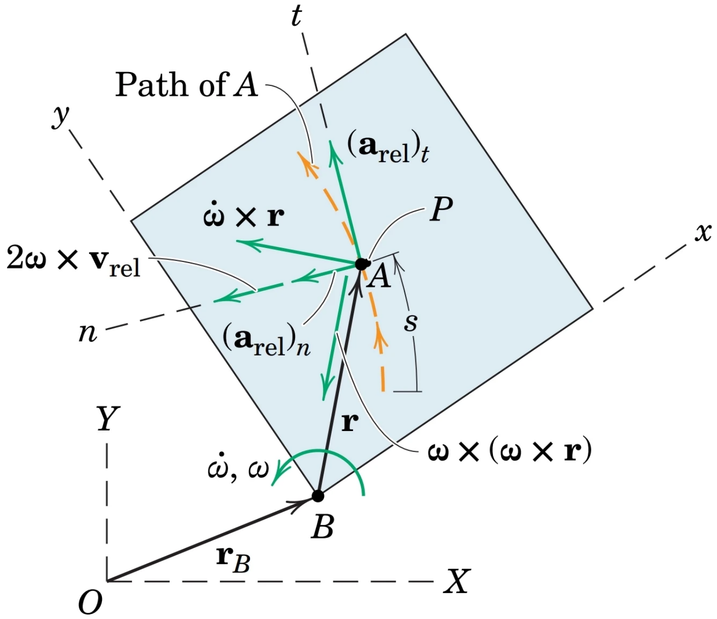
# Q5

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{rel} = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j}) =$$

$$\boxed{\dot{\mathbf{i}} = \omega \times \mathbf{i} \qquad \text{and} \qquad \dot{\mathbf{j}} = \omega \times \mathbf{j}}$$

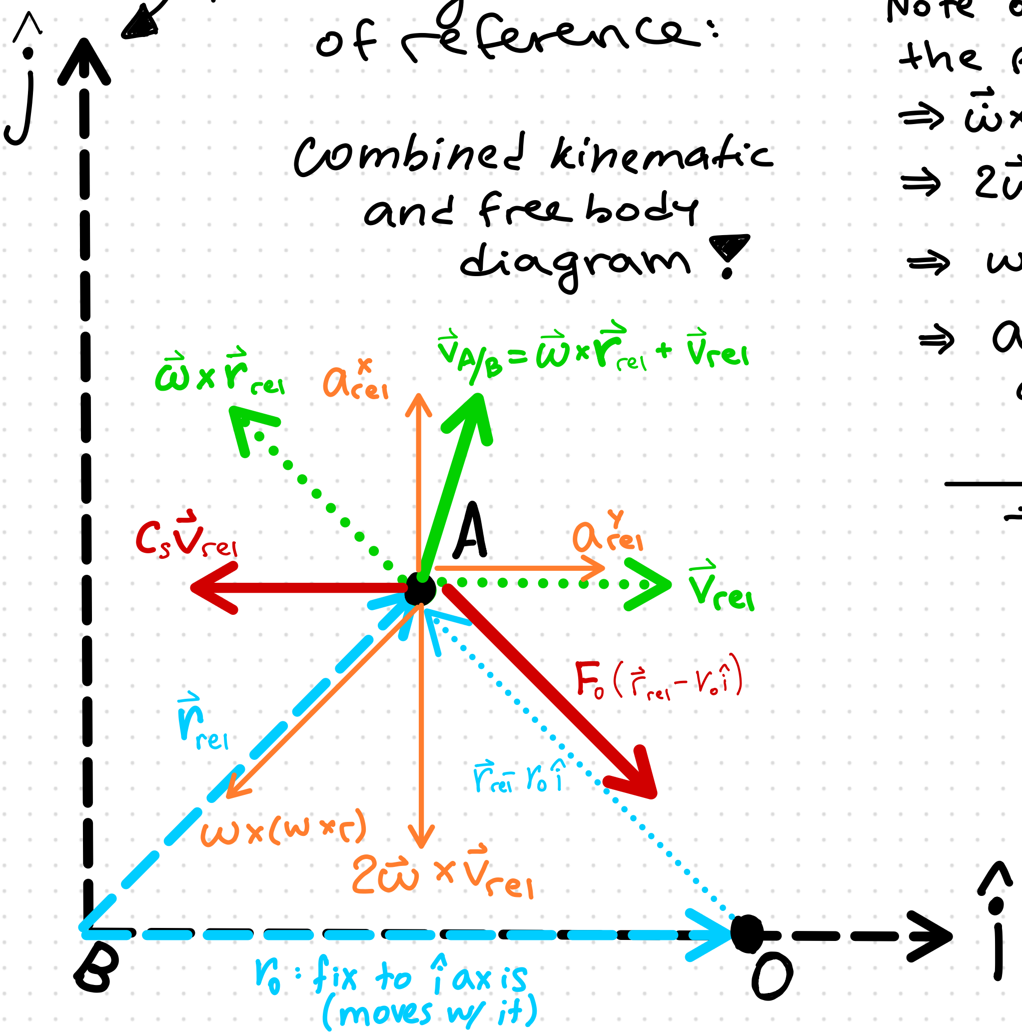
$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \omega \times \mathbf{r}_{rel} + \mathbf{v}_{rel}} \qquad \boxed{\mathbf{a}_A = \mathbf{a}_B + \dot{\omega} \times \mathbf{r}_{rel} + \omega \times (\omega \times \mathbf{r}_{rel}) + 2\omega \times \mathbf{v}_{rel} + \mathbf{a}_{rel}}$$




Rotating frame of reference:

Combined kinematic and free body diagram!




Note on directions of the pseudoforces:

⇒ $\vec{\omega} \times \vec{r}_{rel} \perp r_{rel}$

⇒ $2\vec{\omega} \times \vec{V}_{rel} \perp V_{rel}$

⇒ $\omega \times (\omega \times r) \parallel -\vec{r}_{rel}$

⇒ $a_{rel}$ is general acceleration on the plane

---

→ here $\dot{\omega} = 0$ bcz ω is constant

$a_B = 0$ (center is assumed to be not moving)

$r_B = 0$

$$\sum F = m\left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_{rel}) + 2\vec{\omega} \times \vec{V}_{rel} + \vec{a}_{rel}\right) = -C_s \vec{V}_{rel} - F_0\left(\vec{r}_{rel} - r_0\hat{i}\right)$$

$$m\vec{a}_{rel} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}_{rel}) - 2m\vec{\omega} \times \vec{v}_{rel} - C_s \vec{V}_{rel} - F_0\left(\vec{r}_{rel} - r_0\hat{i}\right)$$

# Matlab implementation:

$$m\begin{bmatrix} a^x_{rel} \\ a^y_{rel} \end{bmatrix} = -m\underbrace{\begin{bmatrix}0\\0\\\omega\end{bmatrix}\times\Bigg(\underbrace{\begin{bmatrix}0\\0\\\omega\end{bmatrix}\times\begin{bmatrix}r^x_{rel}\\r^y_{rel}\\0\end{bmatrix}}_{\begin{bmatrix}-\omega r^y_{rel}\\ \omega r^x_{rel}\\0\end{bmatrix}}\Bigg)}_{\begin{bmatrix}-\omega^2 r^x_{rel}\\ -\omega^2 r^y_{rel}\\0\end{bmatrix}} - 2m\underbrace{\begin{bmatrix}0\\0\\\omega\end{bmatrix}\times\begin{bmatrix}V^x_{rel}\\V^y_{rel}\\0\end{bmatrix}}_{\begin{bmatrix}-\omega V^y_{rel}\\ \omega V^x_{rel}\\0\end{bmatrix}} - C_s\begin{bmatrix}V^x_{rel}\\V^y_{rel}\\0\end{bmatrix} - F_0\left(\begin{bmatrix}r^x_{rel}\\r^y_{rel}\\0\end{bmatrix} - \begin{bmatrix}r_0\\0\\0\end{bmatrix}\right)$$

$$\sum F_x = m a^x_{rel} = -m(-\omega^2 r^x_{rel}) - 2m(-\omega V^y_{rel}) - C_s V^x_{rel} - F_0(r^x_{rel} - r_0)$$

$$\sum F_y = m a^y_{rel} = -m(-\omega^2 r^y_{rel}) - 2m(\omega V^x_{rel}) - C_s V^y_{rel} - F_0(r^y_{rel})$$

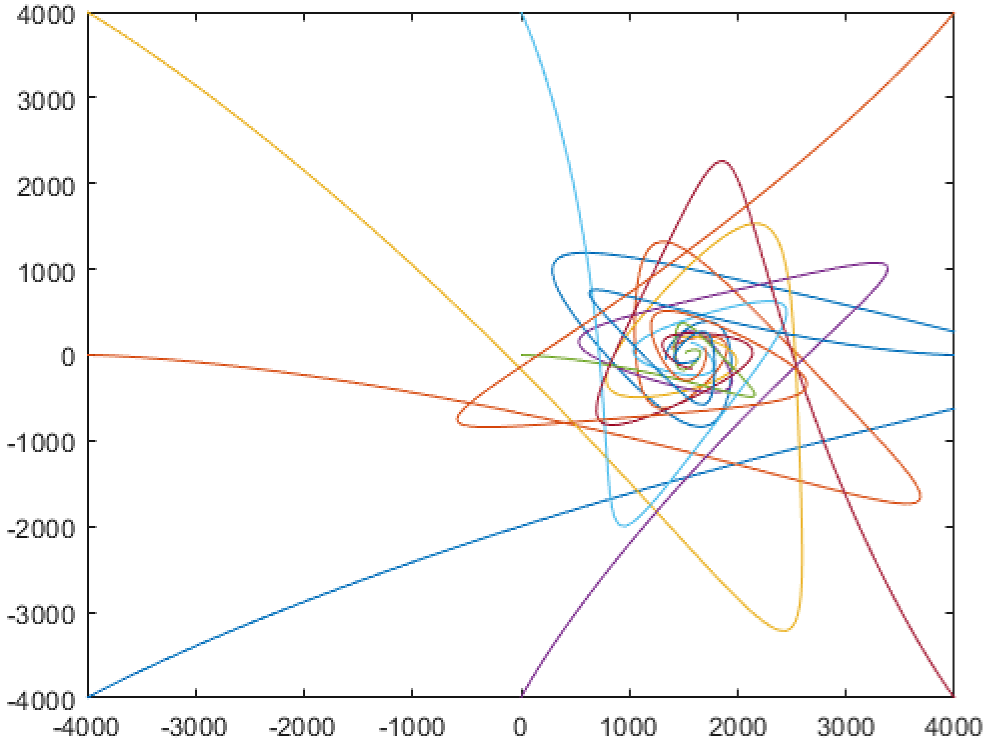
$$\Rightarrow \quad a^x_{rel} = \omega^2 r^x_{rel} + 2\omega V^y_{rel} - \frac{C_s V^x_{rel}}{m} - \frac{F_0}{m}(r^x_{rel} - r_0)$$

$$\Rightarrow \quad a^y_{rel} = \omega^2 r^y_{rel} - 2\omega V^x_{rel} - \frac{C_s V^y_{rel}}{m} - \frac{F_0}{m} r^y_{rel}$$

$$\frac{d}{dt}\begin{bmatrix} r^x_{rel} \\ r^y_{rel} \\ V^x_{rel} \\ V^y_{rel} \end{bmatrix} = \begin{bmatrix} V^x_{rel} \\ V^y_{rel} \\ \omega^2 r^x_{rel} + 2\omega V^y_{rel} - \frac{C_s V^x_{rel}}{m} - \frac{F_0}{m}(r^x_{rel} - r_0) \\ \omega^2 r^y_{rel} - 2\omega V^x_{rel} - \frac{C_s V^y_{rel}}{m} - \frac{F_0}{m} r^y_{rel} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} y(3) \\ y(4) \\ \omega^2 y(1) + 2\omega y(4) - \frac{C_s y(3)}{m} - \frac{F_0}{m}(y(1) - r_0) \\ \omega^2 y(2) - 2\omega y(3) - \frac{C_s y(4)}{m} - \frac{F_0}{m} y(2) \end{bmatrix}$$

```
clear all; close all

global m cs F0 d Omega x0;

R=4000;
Omega=2*pi;
x0=1500;
F0=1000;
m=1;
cs=15;
d=100;

for xini= [-4000:4000:4000]
    for yini= [-4000:4000:4000]
[t,U]=ode45(@hurricane,[0:.001:.5],[xini,0,yini,0]);
plot(U(:,1),U(:,3))
axis([-4000,4000,-4000,4000])
hold on
end
end

%%%%%%%%%%%%%%%%%%%%

function Udot = hurricane(t,U)

global m cs F0 d Omega x0;

x=U(1);
xdot=U(2);
y=U(3);
ydot=U(4);

xddot=(1/m)*(-F0*(x-x0)-cs*xdot+m*x*Omega^2+2*m*Omega*ydot);
yddot=(1/m)*(-F0*(y)-cs*ydot+m*y*Omega^2-2*m*Omega*xdot);

Udot=[xdot;xddot;ydot;yddot];

end
```

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