ME104: Engineering Mechanics II
Discussion Week 6 of 15

Ekaterina Antimirova, Prashant Pujari

Topics:

- Conservation of angular momentum
- Orbital motion

***Midterm I* is *next week* on Thursday Oct, 10 (in class)**

- Closed-book
- One page (two sides) of self-prepared handwritten notes is allowed

Angular momentum

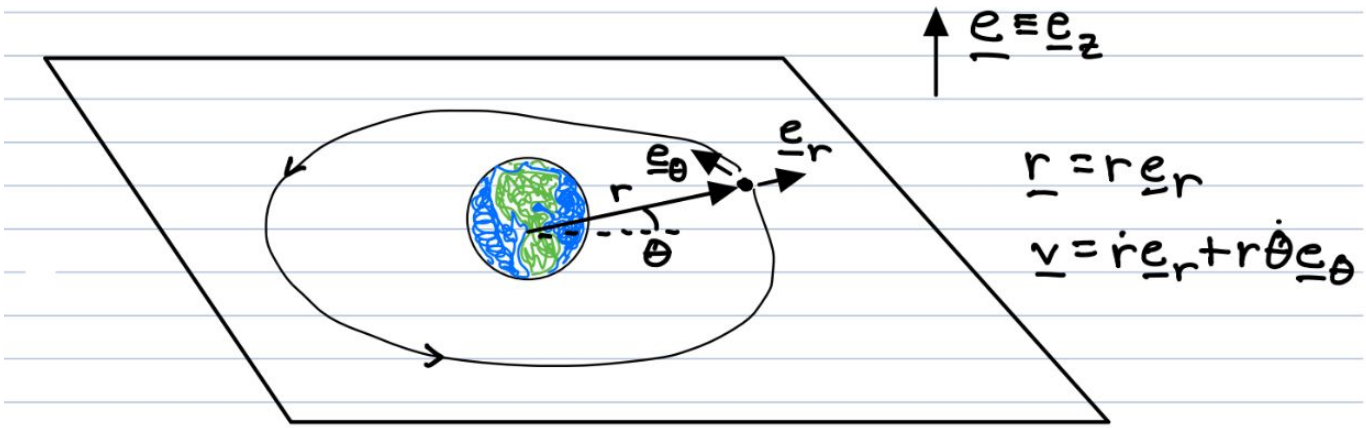
Linear momentum: the product of mass and velocity ⇒ **G** = m**v**

Angular momentum: moment of the linear momentum about the origin ⇒ $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$

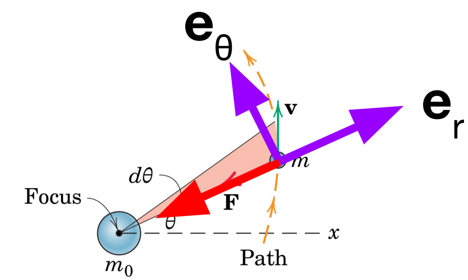
- (If) Fixed axis: $\mathbf{H}_O$ perpendicular to **r** and **v** which must share a single plane (x-y-z analogy)

- Plane polar coordinates: $\mathbf{H}_O = mr\mathbf{e}_r \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = mr^2\dot{\theta}\mathbf{e}_z = H_O\mathbf{e}_z$

- Time derivative of angular momentum: $\dot{\mathbf{H}}_O = m\dot{\mathbf{r}} \times \mathbf{v} + m\mathbf{r} \times \dot{\mathbf{v}} = m\mathbf{v} \times \mathbf{v} + \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m\mathbf{a}$
- Conservation of angular momentum: if $\mathbf{r} \times m\mathbf{a}$ is zero in $\mathbf{E}_z$ direction



Derivation of orbital motion trajectory



$$\mathbf{F}_g = -G\frac{m_0 m}{|\mathbf{r}|^2}\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right)$$

Step 1: Kinematics $\mathbf{a}_{polar} = (\ddot{r} - r\omega^2)\mathbf{e}_r + (r\alpha + 2\dot{r}\omega)\mathbf{e}_\theta$

Step 2: FBD and energies

Kinetic energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

Potential energy: $U_g = -\frac{mM_eG}{r} = -\frac{\beta m}{r}$

Step 3: Equilibrium equations and energy balance

$$\sum F_r = ma_r = m(\ddot{r} - r\omega^2) = -G\frac{mm_0}{r^2} \qquad \sum F_\theta = ma_\theta = m(r\alpha + 2\dot{r}\omega) = 0$$

Energy is conserved: $const = \frac{1}{2}mv^2 - \frac{\beta m}{r} \Rightarrow const = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\beta m}{r}$

Step 4: Solve by eliminating dependence on time, use cons. of ang. moment. bcz m**a** (= **F**) is parallel to **r**

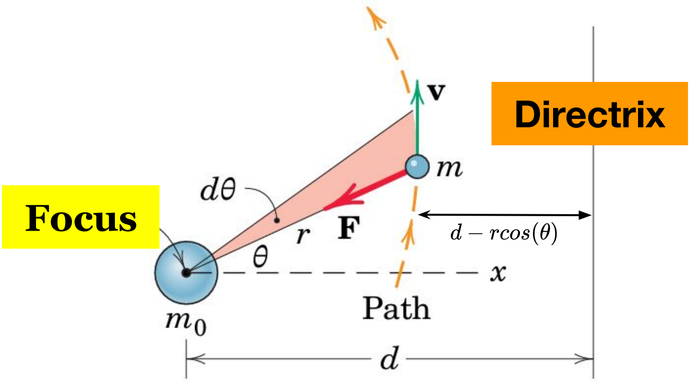
Via conservation of angular momentum: $\dot{\theta} = \frac{H_O}{mr^2}$ and $\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{H_O}{mr^2}$

$$\Rightarrow const = \frac{1}{2}m\left(\left(\frac{dr}{d\theta}\right)^2\left(\frac{H_0}{mr^2}\right)^2 + r^2\frac{H_0^2}{m^2r^4}\right) - \beta m/r \Leftrightarrow \frac{dr}{d\theta} = \left(\frac{2}{m}\left(const + \frac{\beta m}{r}\right) - \frac{H_0^2}{m^2r^2}\right)\frac{m^2r^4}{H_0^2}$$

$$\Rightarrow r(\theta) = \frac{p}{1+e\cdot\cos(\theta-\phi)}$$

$p =$ "trajectory parameter" $= \frac{H_0^2}{m^2\beta}$ $\quad e =$ "eccentricity" $= \left(1 + \frac{2E_0H_0^2}{m^3\beta^2}\right)^{1/2}$ $\quad \phi =$ "offset angle" $= \theta_i - \cos^{-1}\left(\frac{p - r_i}{er_i}\right)$

Conic sections



Definition: a conic section is formed by the locus of a point which moves so that the ratio e of its distance from a point (focus) to a line (directrix) is constant

$$e = \frac{r}{d - r\cos(\theta)} \Leftrightarrow \frac{1}{r} = \frac{1}{d}\cos(\theta) + \frac{1}{ed}$$

The three possible trajectories:

1. **Ellipse (e < 1)**: r is a *min* when θ = 0 and is a *max* when $\theta = \pi$

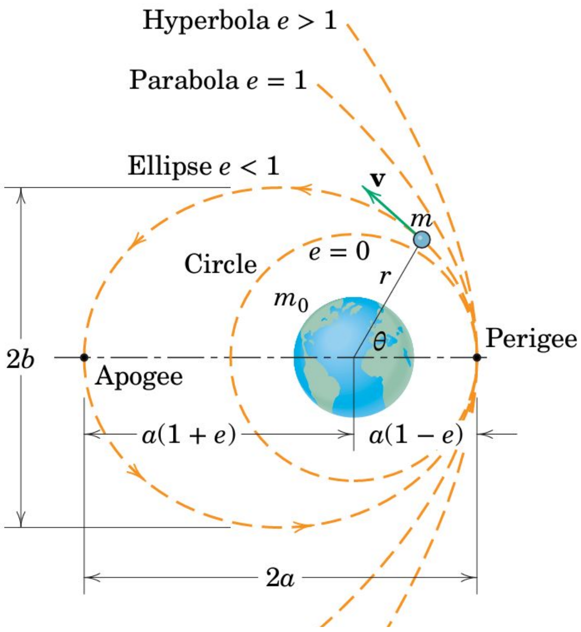
$$\frac{1}{r} = \frac{1 + e\cos\theta}{a(1 - e^2)} \qquad r_{\min} = a(1 - e) \qquad r_{\max} = a(1 + e)$$

2. **Parabola (e = 1)**: radius vector becomes infinite as θ approaches π

$$\frac{1}{r} = \frac{1}{d}(1 + \cos\theta)$$

3. **Hyperbola (e > 1)**

$$\frac{1}{-r} = \frac{1}{d}\cos(\theta - \pi) + \frac{1}{ed}$$



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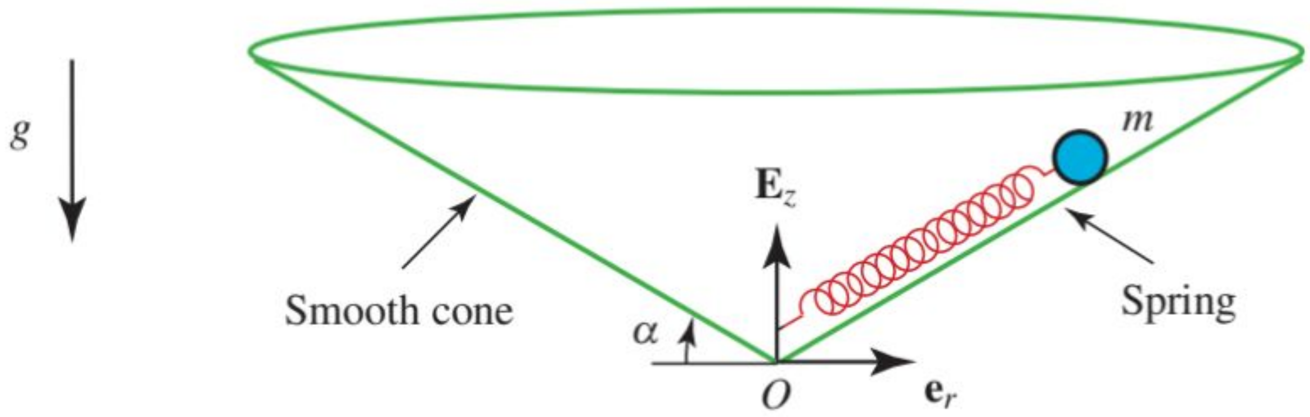
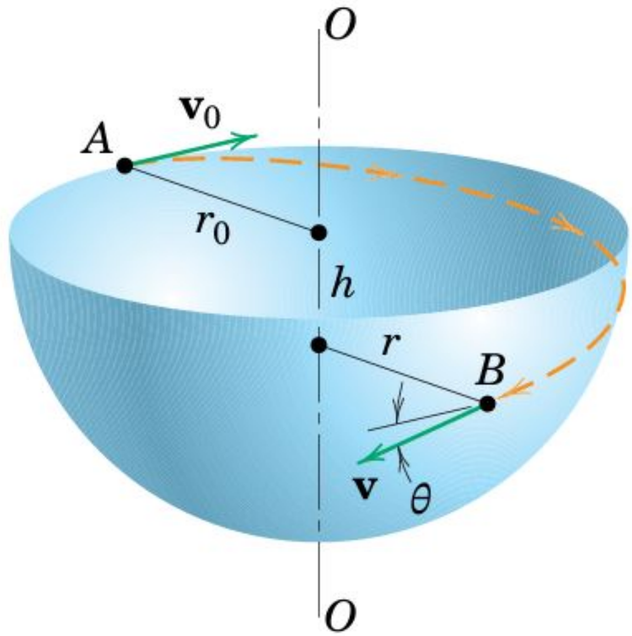
Problem 1: Conservation of angular momentum



Show that $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved (O'Reilly notes)

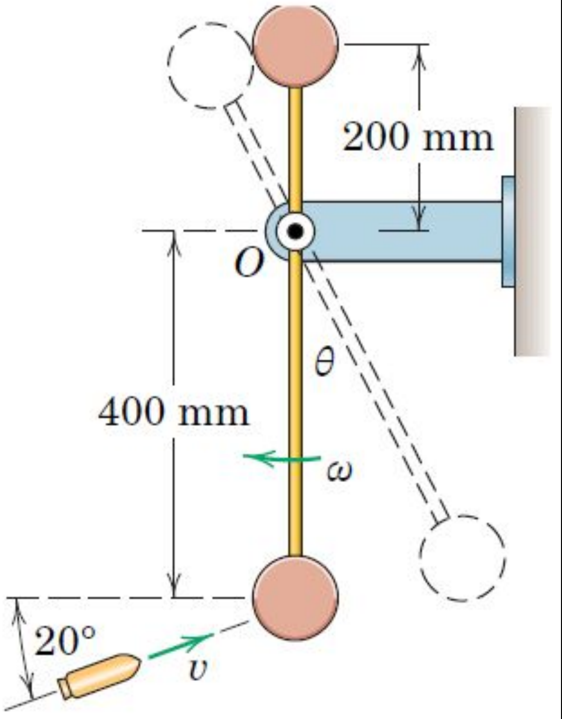
Problem 2: Conservation of angular momentum

A small mass particle is given an initial velocity $\mathbf{v}_0$ tangent to the horizontal rim of a smooth hemispherical bowl at a radius r_0 from the vertical centerline, as shown at point A. As the particle slides past point B, a distance h below A and a distance r from the vertical centerline, its velocity $\mathbf{v}$ makes an angle θ with the horizontal tangent to the bowl through B. Determine θ.



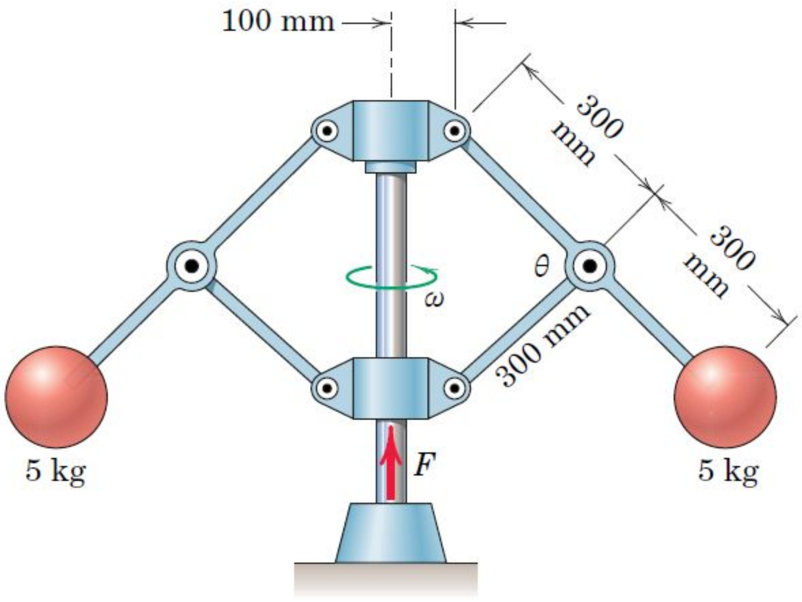
Problem 3

A pendulum consists of two 3.2-kg concentrated masses positioned as shown on a light but rigid bar. The pendulum is swinging through the vertical position with a clockwise angular velocity $\omega = 6$ rad/s when a 50-g bullet traveling with velocity $v = 300$ m/s in the direction shown strikes the lower mass and becomes embedded in it. Calculate the angular velocity ω' which the pendulum has immediately after impact and find the maximum angular deflection θ of the pendulum.



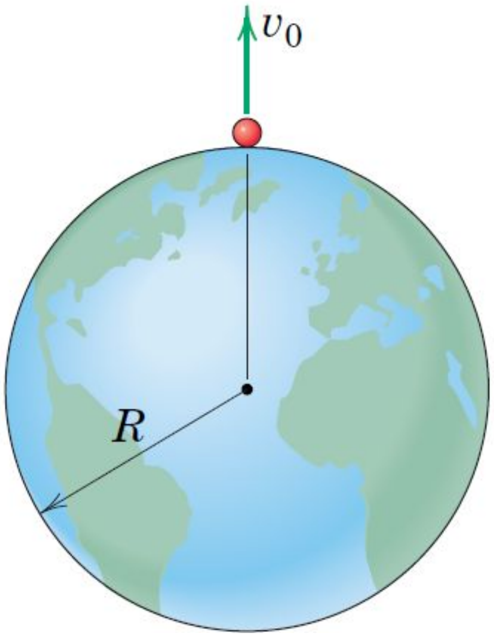
Problem 4

The assembly of two 5-kg spheres is rotating freely about the vertical axis at 40 rev/min with $\theta = 90°$. If the force F which maintains the given position is increased to raise the base collar and reduce θ to 60°, determine the new angular velocity ω. Also determine the work U done by F in changing the configuration of the system. Assume that the mass of the arms and collars is negligible.



Problem 5

A projectile is launched from the north pole with an initial vertical velocity v_0. What value of v_0 will result in a maximum altitude of $R/2$? Neglect aerodynamic drag and use $g = 9.825\ \mathrm{m/s^2}$ as the surface-level acceleration due to gravity.



Problem 6

A projectile is launched from B with a speed of 2000 m/s at an angle α of 30° with the horizontal as shown. Determine the maximum altitude $h_{\max}$.