MISC

Log

$\ln(mn) = \ln(m) + \ln(n)$

$\ln\left(\frac{m}{n}\right) = \ln(m) - \ln(n)$

$\ln(m^r) = r\ln(m)$

Exponent

$(ab)^x = a^x b^x$

$(a^x)^y = a^{xy}$

$a^x a^y = a^{x+y}$

Summation

$\sum_{i=1}^{n} ar^{i-1} = a\frac{1-r^n}{1-r}$

Integrals

$\int \frac{1}{x}dx = \ln|x|$

$\int e^{ax}dx = \frac{1}{a}e^{ax}$

$\int a^x dx = \frac{a^x}{\ln(a)}$

$\int \ln(x)dx = x\ln(x) - x$

$\int \cos(x)dx = \sin(x)$

$\int \sin(x)dx = -\cos(x)$

Derivatives

$(e^x)' = e^x$

$(\ln(x))' = \frac{1}{x}$

$\sin(x) = \cos(x)$

$\cos(x) = -\sin(x)$

$(fg)' = fg' + f'g$

$\frac{1}{f} = -\frac{f'}{f^2}$

$(f(g(x)))' = f'(g(x))g'(x)$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

COUNTING

n choose k

nb of combinations (any order)

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

permutations

nb of ways of ordering n elements (order matters)

$n!$

subsets of n elements

2^n

partitions

n objects into r groups

$\frac{n!}{n_1!n_2!...n_r!}$

PROB BASICS

Properties

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$P(A\cup B) \le P(A) + P(B)$

Conditional

$P(A \mid B) = \frac{P(A\cap B)}{P(B)}$

$P(A\cap B) = P(B)P(A \mid B)$

Total Prob Theorem

$P(B) = P(A_1\cap B) + ... + P(A_n \cap B)$

$= P(A_1)P(B \mid A_1) + ...$

Bayes

$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$

Independence

$P(A \mid B) = P(A)$

$P(A\cap B) = P(A)P(B)$

DISCRETE DISTROS

Bernouilli

$P(X=1) = p$

$E[X] = p$

$Var(X) = p(1-p)$

Uniform

$p_X(x) = \frac{1}{b-a+1}$

$E[X] = \frac{a+b}{2}$

$Var(X) = \frac{(b-a+1)^2 - 1}{12}$

Binomial

k successes in n trials

$p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$

$E[X] = np$

$Var(X) = np(1-p)$

Geometric

number of trials until success

$p_X(k) = (1-p)^{k-1}p$

$E[X] = \frac{1}{p}$

$Var(X) = \frac{1-p}{p^2}$

Poisson

how many occurrences k in τ given rate λ

$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$

$E[N_\tau] = \lambda\tau$

$Var(N_\tau) = \lambda\tau$

CONT. DISTROS

$P(a \le x \le b) = \int_a^b f_X(x)dx$

Disjoint

$P(1 \le x \le 3 \text{ or } 4 \le x \le 5)$

$= P(1 \le x \le 3) + P(4 \le x \le 5)$

Properties

$f(x) \ge 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$

CDF Properties

· $\to_{x\to\infty} 1$ and $\to_{x\to-\infty} 0$

·increasing/monotonic

·right-continuous ___*---

Uniform

$f_X(x) = \frac{1}{b-a}$

$F_X(x) = \frac{x-a}{b-a}$

$E[X] = \frac{a+b}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

Exponential

time to wait for something

$f_X(x) = \lambda e^{-\lambda x}$

$P(X \ge a) = \int_a^\infty \lambda e^{-\lambda x}dx = e^{-\lambda a}$

$F_X(x) = 1 - e^{-\lambda x}$

$E[X] = \frac{1}{\lambda}$

$E[X^2] = \frac{2}{\lambda^2}$

$Var(X) = \frac{1}{\lambda^2}$

NORMALS

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$E[X] = \mu$

$Var(X) = \sigma^2$

Linear Functions

$Y = aX + b$ with $X \sim N(\mu, \sigma^2)$

$Y = N(a\mu + b, a^2\sigma^2)$

Indie Sum

$Z \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Tables

$\Phi(-2) = P(Y \le -2)$

$= 1 - P(Y \le 2) = 1 - \Phi(2)$

Standardising

$X \sim N(\mu, \sigma^2)$ and $Y \sim N(0,1)$

$Y = \frac{X-\mu}{\sigma}$

$X = \mu + \sigma Y$

CONDITIONAL VARS

same for PDF

$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

Multiplication Rule

$p_{X,Y}(x,y) = p_Y(y)p_{X|Y}(x \mid y)$

$p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y \mid x)p_{Z|X,Y}(z \mid x,y)$

$p_{X,Y|Z}(x,y \mid z) = \frac{p_{X,Y,Z}(x,y,z)}{p_Z(z)}$

MULTIPLE VARS

$\sum_x\sum_y p_{X,Y}(x,y) = 1$

$P((X,Y)\in B) = \int\int_{(x,y)\in B} f_{X,Y}(x,y)dxdy$

Marginals

$p_X(x) = \sum_y p_{X,Y}(x,y)$

$f_X(x) = \int f_Y(y)f_{X|Y}(x \mid y)dy$

⚠ ranges: what values can Y take when X = x?

$= \int f_{X,Y}(x,y)dx$

Expected Value Rule

$E[g(X,Y)] = \sum_x\sum_y g(x,y)p_{X,Y}(x,y)$

$E[g(X,Y)] = \int E[g(x,y) \mid Y=y]f_Y(y)dy$

$E[g(X,Y) \mid Y=y] = \int g(x,y)f_{X|Y}(x \mid y)dy$

CDF

$F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(s,t)dsdt$

EXPECTATIONS

Expected Value

$E[g(x)] = \sum_x g(x)p_X(x)$

$E[g(x)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$

Linearity of Expectations

$E[aX+b] = aE[X] + b$

Total Expectation Th.

$E[X] = \sum_y p_Y(y)E[X \mid Y=y]$

$E[X] = \int_{-\infty}^{\infty} f_Y(y)E[X \mid Y=y]dy$

$E[X] = \sum_i P(A_i)E[X \mid A_i]$

Cond. Expectation

$E[g(x) \mid Y=y] = \sum_x g(x)p_{X|Y}(x \mid y)$

Iterated Expectation

$E[E[X \mid Y]] = E[X]$

INDEPENDENCE

If Indie

$E[XY] = E[X]E[Y]$

$Var(X+Y) = Var(X) + Var(Y)$

$p_{X,Y|Z}(x,y \mid z) = p_{X|Z}(x \mid z)p_{Y|Z}(y \mid z)$

$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$f_{X|Y}(x \mid y) = f_X(x)$

$Cov(X,Y) = 0$

MIXED RV

$X = Y$ (discrete) w.p p

Z (continuous) w.p (1-p)

$F_X(x) = pF_Y(x) + (1-p)F_Z(x)$

$E[X] = pE[Y] + (1-p)E[Z]$

VARIANCE

$Var(x) = E\left[(x-\mu)^2\right]$ and $\sigma = \sqrt{Var(X)}$

Properties

$Var(aX+b) = a^2Var(X)$

$Var(X) = E[X^2] - (E[X])^2$

Dependent Sum

$Var(X_1 + ... + X_n) = \sum_{i=1}^{n} Var(X_i) + \sum_{\{i,j\}:i\ne j} Cov(X_i, X_j)$

Independent Sum

$Var(X+Y) = Var(X) + Var(Y)$

Law of Total Var

$Var(X) = E[Var(X \mid Y)] + Var(E[X \mid Y])$

DERIVED DISTROS

PMF function of discrete RV

$p_Y(y) = P(g(x) = y) = \sum_{x:g(x)=y} p_X(x)$

Linear Functions

$Y = aX + b$

$p_Y(y) = p_X\left(\frac{y-b}{a}\right)$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

g is monotonic

$f_Y(y) = f_X(h(y))\left|\frac{dh}{fy}(y)\right|$

general case

1) find CDF: $F_Y(y) = P(g(x) \le y)$
2) derive CDF for PDF

CONVOLUTIONS

$Z = X + Y$

$p_Z(z) = \sum_x p_X(x) p_Y(z-x)$

$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$

COVARIANCE

$Cov(X,Y) = E[(X - E[X])(Y - E[Y])$

Direction

$Cov(X,Y) > 0$ same sign

If Indie

$Cov(X,Y) = 0$

⚠ inverse not true

Properties

$Cov(X,X) = Var(X)$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$Cov(aX+b, Y) = aCov(X,Y)$

$Cov(X, Y+Z) = Cov(X,Y) + Cov(Y,Z)$

CORRELATION COEF.

$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

FRESH START/MEMORYLESSNESS

Exponential

$f_{X|X>t}(x \mid x > t) = t + f_X(x)$

Bernouilli/Poisson

$P(A \mid B) = P(A)$

i.e. prob of two arrivals (A) after 1 arrival (B) = prob of 2 arrivals (A)

BERNOUILLI PROCESS

requires indie, time homogen.

Properties

$S = X_1 + ... + X_n$

$P(S = K) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[S] = np$

$Var(S) = np(1-p)$

Time until 1st success

$T_1 = \min\{i: X_i = 1\}$

$P(T_1 = k) = (1-p)^{k-1}p$

Time of kth arrival

$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$

$E[Y_k] = \frac{k}{p}$

$Var(Y_k) = \frac{k(1-p)}{p^2}$

Merging

$Z_t = g(X_t, Y_t) \sim Ber(p + q - pq)$

$\Rightarrow$ prob either or both have arrival at time t

Splitting

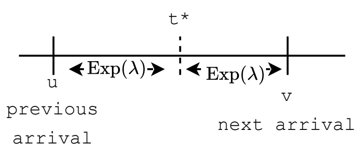
flip a coin with prob q

$A \sim Ber(qp)$

$B \sim Ber((1-q)p)$

⚠ these streams are not indie

INTER-ARRIVAL TIMES / R.INCIDENCE



we arrive at t*
u,v are each $Exp(\lambda)$ away from t*

$\Rightarrow$ $E[V-U]$ is twice the expectation of $\text{Exp}(\lambda)$

POISSON PROCESS

indie, time homogen. seq of exp

λ: arrival rate

$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$

$E[N_\tau] = \lambda\tau$

$Var(N_\tau) = \lambda\tau$

$\lambda = E\frac{N_\tau}{\tau}$

Time of kth arrival / Erlang

$$\begin{aligned} f_{Y_k} &= \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} \\ &= Erlang(k) \\ &= Erlang\left(\frac{k}{2}\right) + Erlang\left(\frac{k}{2}\right) \end{aligned}$$

Sum

⚠ must be indie

M: $Poisson(\mu)$ N: $Poisson(v)$

M+N: $Poisson(\mu + v)$

Merging

A: λ_A B: λ_B

$\lambda = \lambda_A + \lambda_B$

$P(k^{th} \text{arrival is A}) = \frac{\lambda_A}{\lambda_A + \lambda_B}$

$P(\text{k arrivals are A})$ is $\text{Binomial}\left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)$

Splitting

flip a coin with prob q

⚠ these streams <u>are</u> indie

A: $\lambda_A = \lambda q$

B: $\lambda_B = \lambda(1-q)$

Multiple Engine Example

3 engines with death rate λ_e
rate until 1st dies is $\lambda = 3\lambda_e$
then rate until 2nd dies $\lambda = 2\lambda_e$

Min

$$\begin{aligned} P(&\min\{X,Y,Z\} \ge t) \\ &= P(X \ge t, Y \ge t, Z \ge t) \\ &= e^{-3\lambda t} \end{aligned}$$

$\Rightarrow$ have 3 merged Poissons and want to know first arrival

$\Rightarrow$ $\min\{X,Y,Z\}$ is $\text{Exp}(3\lambda)$

$E[\min\{X,Y,Z\}] = \frac{1}{3\lambda}$

Max

$$\begin{aligned} P(&\max(T_1, T_2, T_3) \le t) \\ &= P(T_1 \le t)P(T_2 \le t)P(T_3 \le t) \\ &= \left(1 - e^{-\lambda t}\right)^3 \end{aligned}$$

then derive this to get PDF