# An efficient isogeometric Galerkin method for the estimation of the Karhunen-Loève series expansion of a random field

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**Abstract**

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## 1. Introduction

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## 2. Notation

| | |
|---|---|
| $\mathcal{D}$ | physical space |
| $\hat{\mathcal{D}}$ | parametric space |
| $d$ | dimension of the parametric space |
| $D$ | dimension of the physical space |
| $k$ | parametric coordinate direction $k = 1, 2, \ldots, d$ |
| $K$ | physical coordinate direction $k = 1, 2, \ldots, D$ |
| $\xi_k$ | $k$th parametric coordinate |
| $x_K$ | $K$th physical coordinate |
| $n_k$ | number of basis functions in $k$th direction, $n_k \in \mathbb{N}^+$ |
| $p_k$ | polynomial degree of the basis in the $k$th direction, $p_k \in \mathbb{N}_0^+$ |
| $\Xi_k$ | knot vector corresponding to the $k$th parametric direction, $\Xi_k := [0 = \xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,n_k+p_k+1} = 1] \in \mathbb{R}^{n_k+p_k+1}$, $\xi_{k,1} \leq \xi_{k,2} \leq \cdots \leq \xi_{k,n_k+p_k+1}$ |
| $B^k_{i_k,p_k}$ | $i_k$th B-spline basis function of $p_k$th order in the $k$th parametric direction, $1 \leq i_k \leq n_k$ |
| $P^K_{i_1 \ldots i_d}$ | $K$th coordinate of a control point corresponding to the $(i_1, \ldots, i_d)$ set of basis functions, where a particular index $i_k = 1, 2, \ldots, n_k$ |
| $\mathcal{B}^k$ | univariate B-spline space in the $k$th parametric coordinate direction |
| $\mathcal{B}^k$ | multivariate B-spline space |
| $\mathcal{N}$ | NURBS space |

Assuming a convention for summation (reduction) with respect to repeated lower indices, which are not present on both sides of an equation, a point evaluation expressed in a long, general form would read

$$x_K\left(\xi_1, \ldots, \xi_d\right) = P^K_{i_1 \ldots i_d} B^1_{i_1,p_1}(\xi_1) \cdots B^d_{i_d,p_d}(\xi_d) \tag{1}$$

or, for a special case in a short form,

$$p_X = P^X_{mn} B^u_{m,p} B^v_{n,q} \tag{2}$$

$$p_Y = P^Y_{mn} B^u_{m,p} B^v_{n,q}\,, \tag{3}$$

which is kind of aesthetically pleasing, if we use the tensor notation everywhere. The space of univariate rational B-splines in the $k$th parametric coordinate direction could be defined as

$$\mathcal{B}^k := \mathcal{B}^k(\Xi_k; p_k) := \operatorname{span}\{B^k_{i_k,p_k}(\xi_k)\}_{i_k=1,\ldots,n_k} \tag{4}$$

and the tensor product space as

$$\mathcal{B} := \bigotimes_{k=1}^{d} \mathcal{B}^k(\Xi_k; p_k) = \bigotimes_{k=1}^{d} \operatorname{span}\{B^k_{i_k,p_k}(\xi_k)\}_{i_k=1,\ldots,n_k}. \tag{5}$$

A NURBS (non-uniform rational B-spline) can be seen as a projection of a spline from a $D+1$ dimensional non-rational B-spline space on to a $D$ dimensional rational B-spline space with respect to some rational weights $W_{i_1 \ldots i_d}$. Assuming a scalar projection operator

$$R(\xi_1, \ldots, \xi_d) = W_{i_1 \ldots i_d} B^1_{i_1,p_1}(\xi_1) \cdots B^d_{i_d,p_d}(\xi_d) \tag{6}$$

the $x_K$ coordinates of the NURBS projection may be computed as

$$x_K\left(\xi_1,\ldots,\xi_d\right)=\tilde{P}^K_{i_1\ldots i_d}B^1_{i_1,p_1}(\xi_1)\cdots B^d_{i_d,p_d}(\xi_d)R^{-1}(\xi_1,\ldots,\xi_d),\quad K=1,\ldots,D \tag{7}$$

where $\tilde{P}^K_{i_1\ldots i_d}$ denotes the weighted control points of the corresponding non-rational B-spline,

$$\tilde{P}^K_{i_1\ldots i_d}=P^K_{i_1\ldots i_d}W_{i_1\ldots i_d}. \tag{8}$$

The NURBS space is defined as

$$\mathcal{N}:=\mathcal{N}(\Xi_1,\ldots,\Xi_d;p_1,\ldots,p_d;W_{i_1\ldots i_d}) \tag{9}$$

$$:=\operatorname{span}\{W_{i_1\ldots i_d}B^1_{i_1,p_1}(\xi_1)\cdots B^d_{i_d,p_d}(\xi_d)R^{-1}(\xi_1,\ldots\xi_d)\}_{i_k=1,\ldots,n_k}. \tag{10}$$

The expression for the $A$ tensor from the classical Galerkin method on the non-rational B-spline solution subspace would read

$$A_{i_1\ldots i_d j_1\ldots j_d}=\int_{\hat{\mathcal{D}}}\int_{\hat{\mathcal{D}}}\Gamma(\xi_1,\ldots,\xi_d,\xi'_1,\ldots,\xi'_d)B^1_{i_k,p_k}(\xi_k)\cdots B^d_{i_d,p_d}(\xi_d)B^1_{j_k,p_k}(\xi'_k)\cdots B^d_{j_d,p_d}(\xi'_d)$$
$$\times\det(J_{\xi_k})\det(J_{\xi'_k})\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_d\mathrm{d}\xi'_1\cdots\mathrm{d}\xi'_d. \tag{11}$$

In the proposed efficient Galerkin method the solution subspace will be set to

$$\mathcal{W}:=\operatorname{span}\{B^1_{i_1,p_1}(\xi_1)\cdots B^d_{i_d,p_d}(\xi_d)\mathrm{det}(J_\xi)^{-1}\}_{i_k=1,\ldots,n_k}. \tag{12}$$

by that we will abandon the isoparametric concept, since the map $x_K:\hat{\mathcal{D}}\to\mathcal{D}$ will remain a NURBS map. Furthermore, the kernel $\Gamma:\hat{\mathcal{D}}\to\mathbb{R}$ will be approximated by

$$\Gamma(\xi_1,\ldots,\xi_d,\xi'_1,\ldots,\xi'_d)\approx g_{k_1\ldots k_d l_1\ldots l_d}\hat{B}^1_{k_1,\hat{p}_1}(\xi_1)\cdots\hat{B}^d_{k_d,\hat{p}_d}(\xi_d)\hat{B}^1_{k_1,\hat{p}_1}(\xi'_1)\cdots\hat{B}^d_{k_d,\hat{p}_d}(\xi'_d). \tag{13}$$

From now on a hat over the symbols will be used to refer to the space approximating the kernel. The tensor $A$ utilizing the approximated kernel will have the form

$$A_{i_1\ldots i_d j_1\ldots j_d}=g_{k_1\ldots k_d l_1\ldots l_d}M_{k_1\ldots k_d i_1\ldots i_d}M_{l_1\ldots l_d j_1\ldots j_d} \tag{14}$$

where the tensor $M$ is defined as

$$M_{k_1\ldots k_d i_1\ldots i_d}:=\int_{\hat{\mathcal{D}}}\hat{B}^1_{k_1,\hat{p}_1}(\xi_1)\cdots\hat{B}^d_{k_d,\hat{p}_d}(\xi_d)B^1_{i_1,p_1}(\xi_1)\cdots B^d_{i_d,p_d}(\xi_d)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_d. \tag{15}$$

with $1\leq k_k\leq\hat{n}_k$ and $1\leq i_k\leq n_k$. Observing the structure of $M$ it is easy to see that it can be rewritten as a tensor product of

$$M^k_{k_k i_k}:=\int_{\xi_{k,1}}^{\xi_{k,i_k+p_k+1}}\hat{B}^k_{k_k,\hat{p}_k}(\xi_k)B^k_{i_k,i_k}(\xi_k)\,\mathrm{d}\xi_k \tag{16}$$

namely,

$$M_{k_1\ldots k_d i_1\ldots i_d}=M^1_{k_1 i_1}\cdots M^d_{k_d i_d} \tag{17}$$

The control points $g$ of the kernel approximation might by computed by means of some quasi-interpolant or as the Greville ordinates for a given abscissae, i.e. integration points. It is also mandatory, that the parametrization of the kernel approximation is equivalent to the parametrization of the solution space and the geometrical mapping. To solve the generalized matrix eigenvalue problem it will be necessary to reorder the computed tensors such that $A_{i_1\ldots i_d j_1\ldots j_d}\to[A_{i_1\ldots i_d j_1\ldots j_d}]_{ij}$. There are methods for generalized tensor eigenvalue problems, but the way of least resistance would be reordering. Since $A$ is dense, it would be straight forward to redefine the access methods instead.

## 3. Conclusion

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## References