CS 278 HW8 assignment Solutions

1) Solution is in the textbook.
2) Solution is in the textbook.
3) Find the mistake(s) in the following proof fragment:

Theorem: For any integer $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

"**Proof (by mathematical induction):**
Certainly the theorem is true for $n = 1$ because $1^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = 1$. So the base case is true.
For inductive step, suppose that for some integer $k \geq 1$, $k^2 = \frac{k(k+1)(2k+1)}{6}$.
We must show that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}."$$

(Note that it is a proof fragment, not the whole proof.)

Answer: In the inductive step, both the inductive hypothesis and what is to be shown are wrong. The inductive hypothesis should be

Suppose that for some integer $k \geq 1$, $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

And what is to be shown should be

$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

4) Find the mistake(s) in the following "proof" by mathematical induction:

Theorem: For all integers $n \geq 1$, $3^n - 2$ is even.

"**Proof (by mathematical induction):**
Suppose the theorem is true for an integer k, where $k \geq 1$. That is, suppose that $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is
even. But
$3^{k+1} - 2 = 3^k \cdot 3 - 2 = 3^k\,(1 + 2) - 2 = (3^k - 2) + 3^k \cdot 2$.
Now $3^k - 2$ is even by inductive hypothesis. Therefore, $3^k - 2 = 2m$ for some integer m. Hence,
$(3^k - 2) + 3^k \cdot 2 = 2m + 3^k \cdot 2 = 2(m+3^k)$ which is even (because $m+3^k$ is an integer). It follows that $3^{k+1} - 2$ is even, which is what we needed to show."

Answer: Base case is missing.

5) Let P(n) be the following property: $\sum_{j=1}^{n} \frac{j(j+1)}{2} = \frac{n(n+1)(n+2)}{6}$. Use mathematical induction to prove that P(n) is true for all integers $n \geq 1$.

Proof by induction.

Base case: For n=2 we have 1+3 = 4 and $\frac{2(2+1)(2+2)}{6}$= 4.

Inductive step: Assume that for some k≥1, P(k) is true. That is, $\sum_{j=1}^{k} \frac{j(j+1)}{2} = \frac{k(k+1)(k+2)}{6}$.

We need to show that P(k+1) is true. That is, $\sum_{j=1}^{k+1} \frac{j(j+1)}{2} = \frac{(k+1)(k+1+1)(k+1+2)}{6}$.

LHS = $\sum_{j=1}^{k+1} \frac{j(j+1)}{2} = \sum_{j=1}^{k} \frac{j(j+1)}{2} + \frac{(k+1)(k+1+1)}{2}$ by inductive hypothesis

$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+1+1)}{2} = \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+1+1)}{6} = \frac{k(k+1)(k+2)+3(k+1)(k+2)}{6}$

$=\frac{(k+1)(k+2)(k+3)}{6}$=RHS, as desired.

6) Prove by mathematical induction that $2^n < (n+1)!$, for all integers n ≥ 2.

Proof by induction on n.

Base case: Show that for n = 2, $2^n < (n+1)!$:
$2^n = 4 < 6 = (2+1)!$, so the inequality holds for n = 2.

Inductive step: Assume that k is a positive integer such that k ≥ 2 and $2^k < (k+1)!$.
We need to show that $2^{k+1} < ((k+1)+1)!$.
$2^{k+1} = 2 \cdot 2^k < 2 \cdot (k+1)!$ by the inductive hypothesis
Since k ≥ 2, we have that k+2 > 2 or 2 < k+2. Therefore,
$2 \cdot (k+1)! < (k+2) \cdot (k+1)! = (k+2)!$
Combining the above inequalities together we get
$2^{k+1} < (k+2)!$
This is what we needed to show.