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Mass Properties Measurement Handbook-

by

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1.0 Abstract There has been some discussion at recent SAWE International Conferences regarding the creation of a SAWE sponsored mass properties seminar, with the ultimate goal being the certification of Mass Properties Engineers by the SAWE. This paper presents a review of the methods used to measure Center of Gravity Location, Moment of Inertia, Product of Inertia, and Weight. The authors have attempted to discuss all the elements of mass properties measurement, so that this paper can be used as a textbook. This will be condensed and edited at a later date for incorporation into the SAWE Weight Engineering Handbook.

Much of the material in this paper has been gleaned from previous papers written by the senior staff engineers at Space Electronics (Boynton, Wiener, and Bell). We have provided a bibliography at the end of this paper which references some of these papers, so that readers wishing to delve further into these subjects can obtain information on mathematical derivations of error sources, etc.

2.0 Steps in Making a Mass Properties measurement

There are 9 steps required to measure the mass properties of an object:

2.1 Define the particular mass properties you need to measure and the required measurement accuracy Sometimes this task has already been done by someone else, but other times you may be asked to measure mass properties without being given much guidance regarding what is to be measured or the accuracy required.

2.2 Choose the correct type of measuring instrument This choice will be driven by the availability of existing equipment, accuracy required, cost, and suitability for the measurement environment (i.e. production vs. research).

2.3 Define the coordinate system on the object to be used as the mass properties reference axes Any object has an infinite number of values for CG location, moment of inertia, and product of inertia, depending on where the reference axes are assigned. The axes may be related to the geometric centerline of the vehicle, a line of thrust, or may depend on the attachment interface to another stage of the vehicle.

2.4 Define the position of the object on the mass properties measuring machine There are an infinite number of ways a payload can be mounted on a mass properties machine. While the mass properties of the payload are fixed, the measured data will be dependent on the orientation of the payload relative to the measurement coordinate system. We are talking about basic position, not how accurate this position is. For example, a rocket can be mounted on the machine with its nose up or its nose down. The fins can be parallel to the X axis of the machine or the Y axis (or for that matter can be oriented at any angle). To avoid confusion, you need to make a drawing or sketch of the position of the payload on the machine, so you can interpret the measured data correctly. The X axis of the mass properties instrument will not necessarily correspond to the X axis of the payload.

2.5 Determine the dimensional accuracy of the object being measured This can be the limiting factor on accuracy. For example, you can=t measure CG of a cylindrical object with an accuracy of 0.005 inch if the outer surface of the object has a runout of 0.020 inch.

2.6 Design the fixture required to mount the object at a precise location relative to the measuring instrument This will require a means of determining the location of the measurement axis of the instrument as well as a means of accurately supporting the object on the instrument.

2.7 Verify the position of the object on the instrument. There are clever techniques which can make this relatively easy.

2.8 Make the mass properties measurement. This can be the quickest part of the job.

2.9 Report the mass properties data After the measurement is made, you will have to report the data to someone else. You need to define which axis is X, which axis is Y, etc. Your X may be someone else's Y. Even within one company, one department may call the roll axis X and another department may call it Y. If you submit the data without defining the axes, each group will use its own set of coordinates in interpreting the data. These problems can be minimized by using the Recommended Practice for Mass Properties Reporting which is summarized in the figures below. No amount of discussion will replace the value of a single sketch showing the orientation of the payload and definition of the measurement coordinate system.

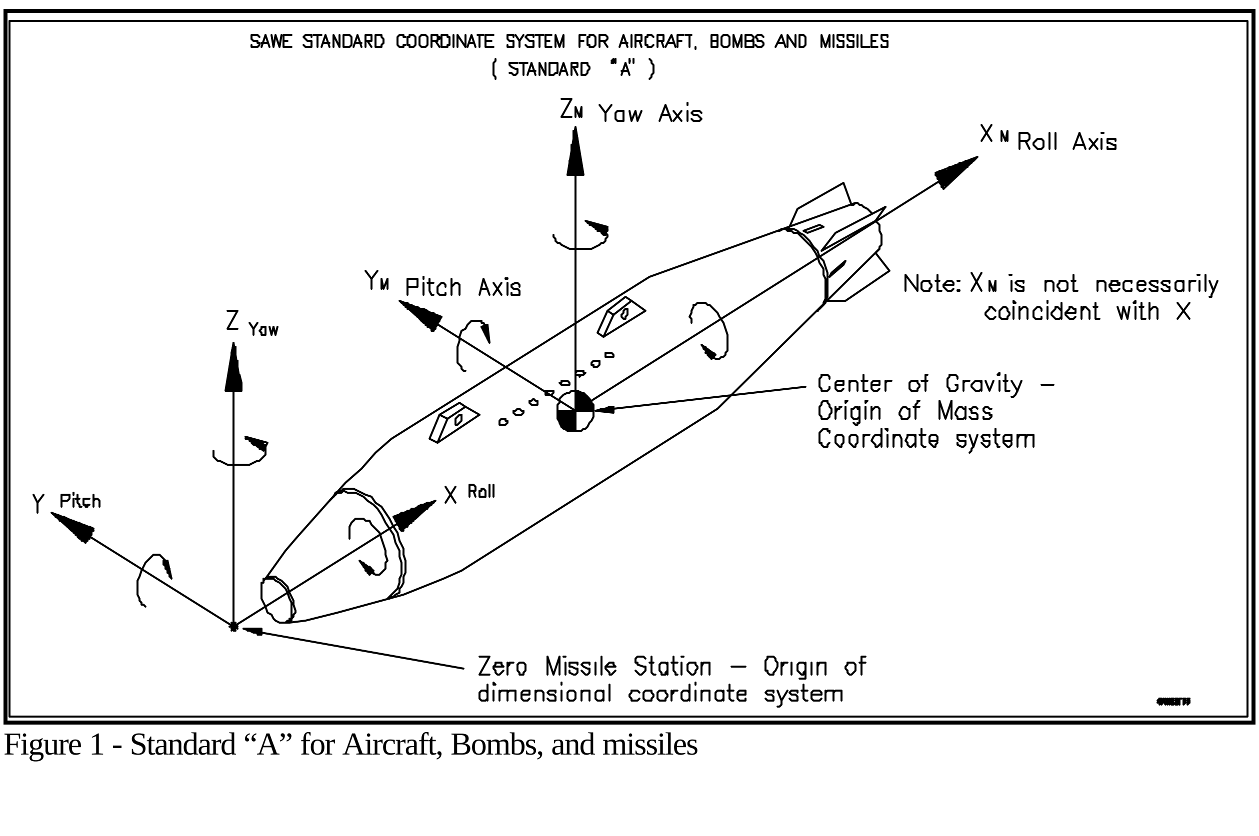


Figure 1 - Standard “A” for Aircraft, Bombs, and missiles

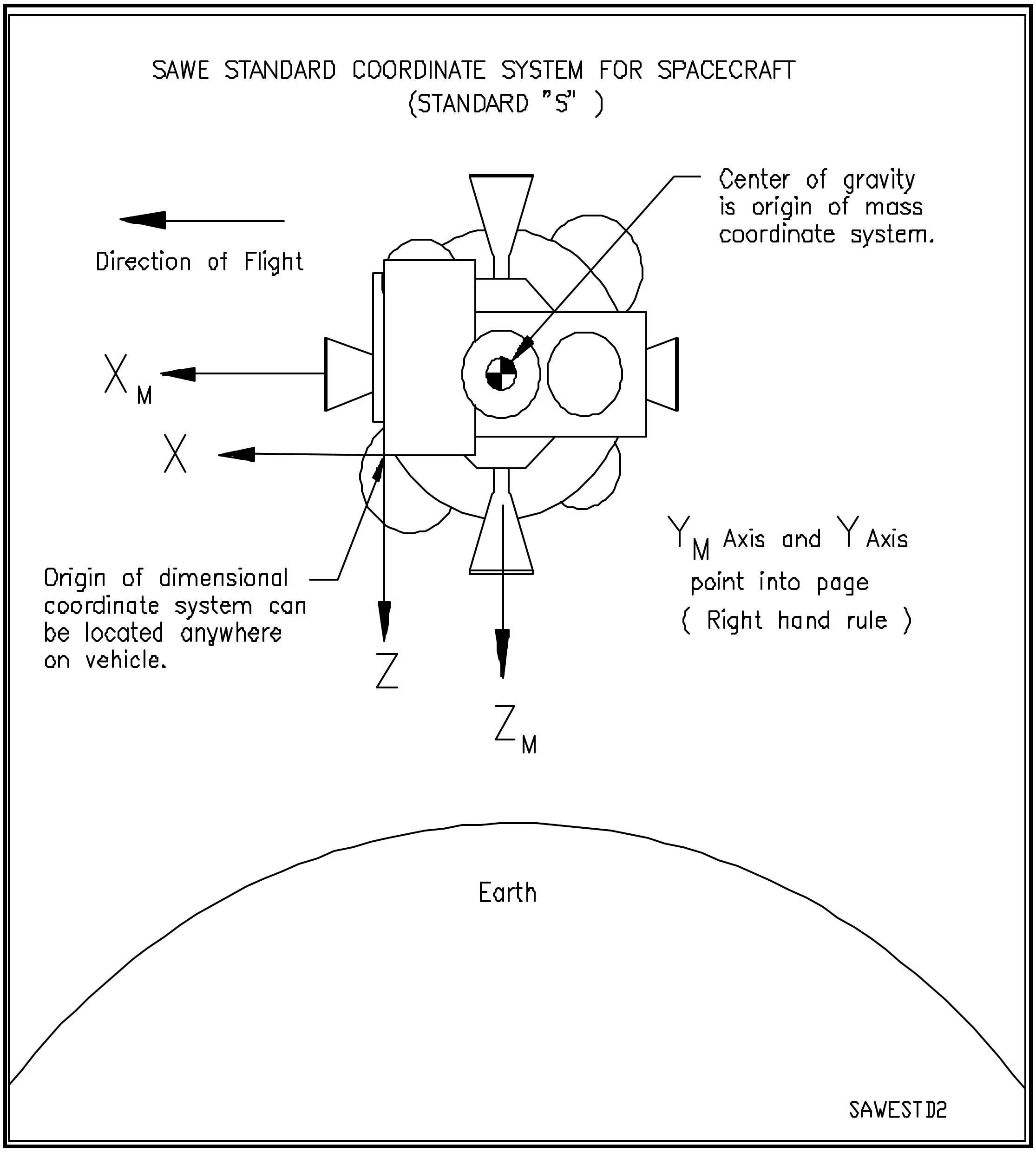


Figure 2 - Standard “S” for vehicles which orbit the earth

3.0 Establishing a frame of reference (relating payload and instrument coordinate systems)

3.1 Choosing the frame of reference The flight dynamics engineer works with two different vehicle frames of reference: the body frame, defined by the structure of the missile, and the inertial frame, defined by the mass properties of the vehicle. Many people involved with mass properties do not fully understand the difference between these two frames of reference, or how you translate vector quantities from one frame to the other (a process involving Euler angles and an ordered sequence of three matrix rotations about the axes of the source frame).

The body frame is a reference system which is related to the physical structure of the vehicle. This frame is easy to define for a perfect ideal vehicle shape, but may be hard to locate on a real vehicle, because of loose manufacturing tolerances and other practical problems.

The inertial frame is a reference system defined by the principal axes of the vehicle. This can be crudely calculated, but it is necessary to make measurements of the real vehicle to accurately determine the location of this inertial frame relative to the body frame. Measurements are made on a mass properties instrument which determines CG location, moments of inertia, and (if necessary) products of inertia. These measurements define the inertial frame relative to the body frame *within the tolerance limitations of both the structure and the measuring instruments.*

Often the goal is to align the two frames of reference so one principal axis coincides with the roll axis, etc. This is not always the goal, however. Certain reentry vehicles are deliberately designed with a misalignment to produce a coning action during reentry, and smart weapons scan the target in a circular pattern by using a similar misalignment.

3.2 Interpreting the data The following is a review of some general characteristics of mass properties data which must be observed when deciding on the orientation and coordinate system to be used for mass properties measurements.

3.2.1 Moment of inertia can only be positive, so there is never any uncertainty regarding sign. However, you should determine whether this magnitude should be expressed about the geometric centerline of the vehicle or about its CG, about an axis parallel to the geometric centerline or rotated so the data is about the principal axes. In most cases, there will not be a big difference in these three magnitudes. This can lead to confusion, since it will not be immediately obvious that the wrong data is being presented. Space Electronics mass properties instruments always report MOI and CG relative to the instrument centerline. If user data is entered, then these properties will also be reported relative to the payload datum and coordinate system.

3.2.2 Center of gravity coordinates can be positive or negative. You should determine whether your positive axis agrees with the definition of axes used by the recipient of your data. Furthermore, CG distance can be expressed along a coordinate system defined by the geometry of the vehicle or along the principal axes. We recommend you provide a sketch which clearly shows the axes and their algebraic signs.

3.2.3 Product of inertia can also be positive or negative. Since this quantity is derived by multiplying the incremental masses by two different distances, the POI sign is even more prone to error than the sign of the CG data. We frequently hear the comment: "I can calculate POI, but I never get the sign right". What usually happens is not that the sign is wrong, but that the mass properties engineer and the recipient of his data are using different coordinate systems.

3.2.4 About vs. along Moment of inertia is expressed <u>about</u> an axis. CG coordinates can be expressed as a distance <u>along</u> an axis or as an unbalance moment <u>about</u> an axis (CG along X corresponds to the CG unbalance moment about Y or Z). POI is <u>relative to</u> two axes. (or it can be a tilt angle in a plane defined by two axes).

Six types of information are required to establish a mass properties reference system:

1. The location of the reference axes origin.
2. The mathematical symbols used to define the reference axes.
3. The zero point along each axis
4. The direction of positive values along each axis.
5. The positive direction for rotation about each axis
6. A zero rotation angle reference about each axis

3.3 Dimensional Errors - In the previous section we discussed the <u>definition</u> of axes and polarities. In this section we are concerned with the <u>difference</u> between the ideal and the real object. In order to make accurate measurements, the payload must have precisely defined measurement axes. If the object is a smooth ground cylinder, then it is obvious where the axes are located. However, on real parts,

- flat surfaces are not perfectly flat;
- round surfaces are not perfectly round;
- concentric surfaces are not exactly on the same center;
- perpendicular surfaces are not exactly perpendicular;
- some surfaces are soft or poorly shaped (cork, thick paint).

The effect of all these non ideal conditions is that the datum for the payload coordinate system can be no better than the accumulated uncertainties of the datum surfaces.

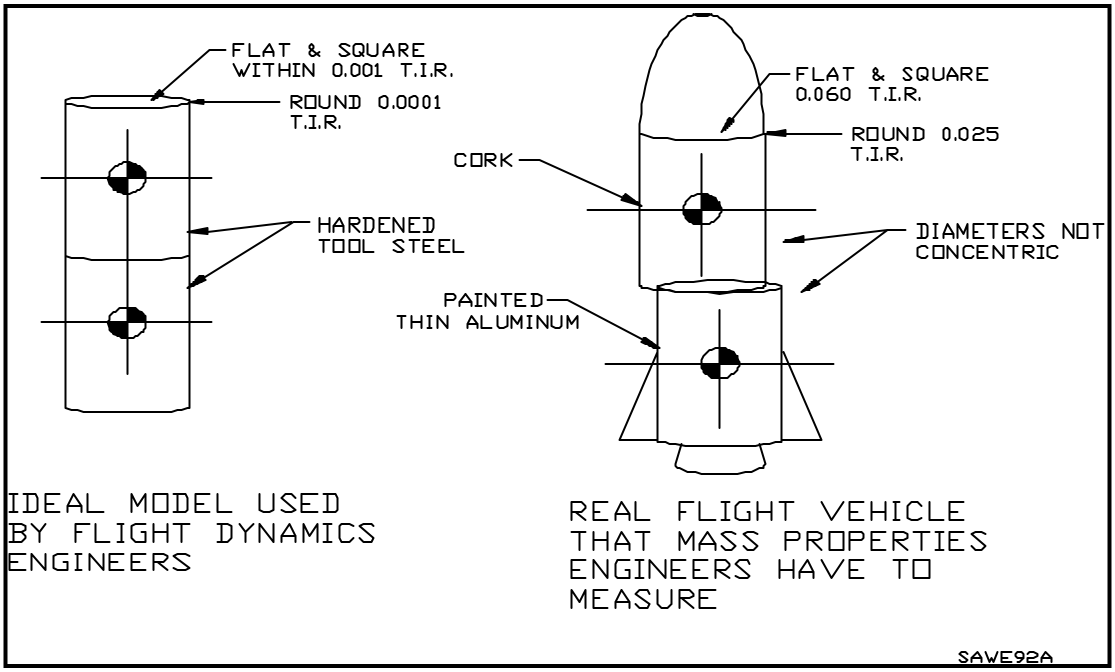


Figure 3

Often mass properties are defined relative to a coordinate system that includes the centerline of a vehicle, and two orthogonal axes located at a specific missile "station" (distance along the centerline). What do you do if the missile is made of three different sections, each with its own centerline? Until the measurement axes have been defined to the satisfaction of all parties involved with mass properties, there is no sense proceeding.

Recommendation: When you first receive an object to measure, do a dimensional inspection of the object. If the object outer surface is less accurate

3.3.1 Unrealistic CG tolerances Your CG measurement accuracy can only be as good as the machining accuracy of the reference datum. For example, if you must measure and correct the radial CG of a rocket within 0.003 inch of the centerline of the rocket motor flange, and you dial indicate this flange and find that it is out of round by 0.015" TIR, then the specification for CG accuracy is impossible to achieve. However, there may be a common sense compromise which will allow the measurement to proceed. For example, you might discover that the flange is close to an ellipse, so you can establish a center of the large and small diameter and relate the CG to that point. A@ best fit circle@ program is often used to locate a centerline for out of round cylindrical parts. You may find that there is a flat or other anomaly area on the flange, so you can ignore that part and base your center on the part that appears more round.

A better solution might be to find out what is really the point of the specification. For example, it might turn out that what is really required is that the CG be centered on the exit cone of the rocket motor when this motor is rotated to a straight ahead position. If you know this, then you can negotiate a much more meaningful test which accomplishes the flight objective.

3.3.2 Determining realistic mass properties tolerances To determine if the stated tolerances are realistic, the payload uncertainties, fixture uncertainties, and instrument uncertainties must be determined and compared to the specified measurement tolerances. The sum of these three uncertainty sources must be less than the measurement tolerance by a factor of at least 3 with a factor of 5 or more being desirable. To determine the object uncertainties a two step analysis is recommended. Keep in mind, this analysis is just for the object. Another, similar, analysis must be performed as part of the fixture design process to be sure that the fixture does not use up more than 10 or 20 percent of the allowable mass properties measurement tolerance.

STEP 1 = Calculate the required mechanical dimensional tolerances necessary in order to be 10 times better than the accuracy specification for mass properties. For example, if CG accuracy required is 0.005 inch, then you must know the location of the reference axes to an accuracy better than 0.0005 inch. .

STEP 2 = Do a dimensional inspection of the object. If the object outer surface or other datum surfaces have accumulated tolerances less accurate than the tolerances calculated in step 1, then you have a problem.

In cases where there is no precise surface on the object and the critical axis is the aerodynamic centerline, then this can be determined using multiple measurements at different heights along the object and then entering this data in a computer program which determines the best fit solution. Space Electronics manufactures a system consisting of a very rigid dial indicator stand with two electronic dial indicators. The outputs of the dial indicators are connected to the computer, automating the process of determining the single line which represents the geometric centerline of the vehicle.

3.3.3 Dimensional tolerance for product of inertia measurement Objects which are spin balanced must generally be dimensionally defined even more accurately than those requiring CG measurement.

Tolerances for product of inertia are trickier to calculate. The best approach is to first calculate the axis tilt corresponding to the object POI tolerance, and then relate this to TIR runout of two reference diameters or between a reference diameter and a perpendicular axis.

3.3.4 Establishing hard points on the object If you have an influence in the early stages of a design, maybe you can convince the project engineer to add two precision datum rings to the object. This will give you a reliable interface for your fixture and will also give you something to measure to determine if the object is located correctly in the fixture. The example below is for a rocket. But could apply equally well to a satellite. Engineers who align the guidance system will find these rings invaluable. Motor nozzles can be located relative to these rings.

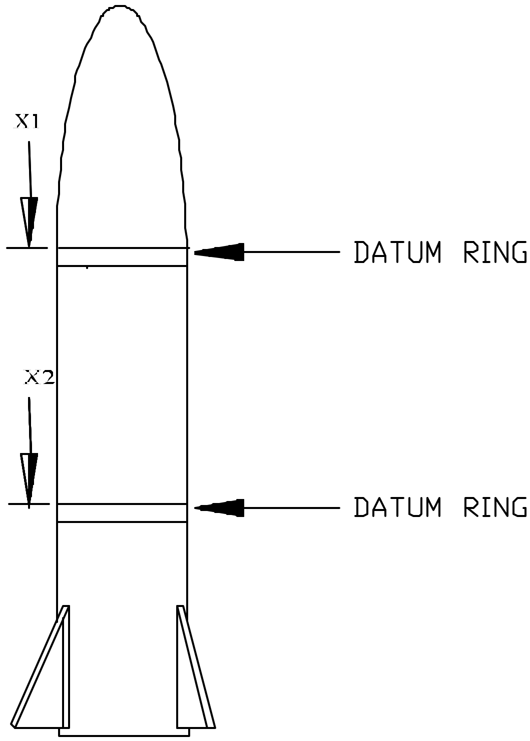


Figure 4 - Precision rings eliminate uncertainty regarding measurement axes

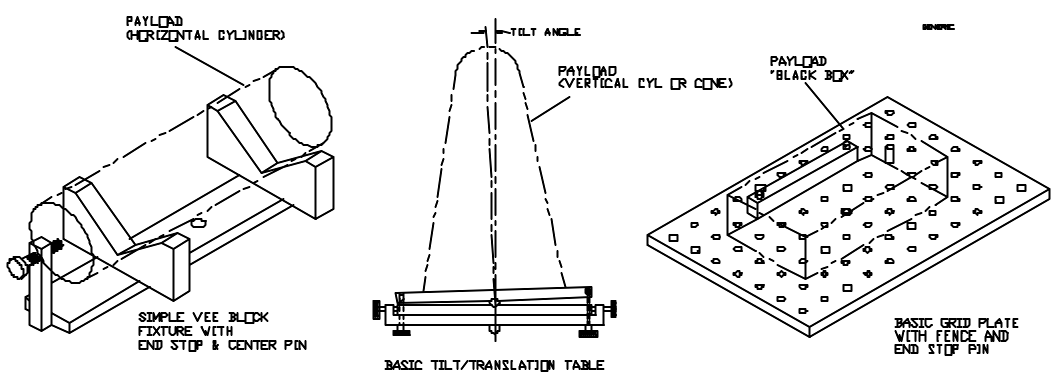


Figure 5 - Standard Generic Fixtures Available from Space Electronics

4.0 Choosing a fixture

The number one source of measurement error in most mass properties measurements is the inability to accurately position the object being measured relative to the measurement axis of the instrument. Traditionally this has been accomplished using a precision fixture which supports and locates the object. For horizontal measurements of cylinders, vee blocks are commonly used. For vertical cylinders or cones, an adjustable fixture is often used and the object is centered using dial indicators. Rectangular objects are usually fixtured using a grid plate fixture that has "fence" type end stops which interface with a precise pattern of holes on the fixture.

4.1 Importance of fixturing accuracy The accuracy of mass properties measurement is only as good as the accuracy of the fixture used to support the payload. If you are measuring a smooth ground cylinder, then fixturing accuracy of 0.001 inch is not hard to achieve. This is a typical accuracy when fixturing a shaft for a turbine rotor. Objects such as this have precision surfaces where the bearings are supported. Not only are these surfaces almost perfectly round and true, they also generally lie along a longitudinal axis within 0.001 inch TIR. When you're in the jet engine business, fixturing is relatively easy because everything is made so precisely.

Many satellites represents the mass properties engineer's worst nightmare. There is no outer skin or controlled "hard point". It is just a clutter of irregular objects attached to a thin walled structure, and is so fragile that it can only be held at certain places, none of which are dimensionally controlled within the tolerances required for the measurement accuracy needed.

4.2 Mass properties fixtures perform three basic functions:

4.2.1. The fixture must locate the object in a repeatable and rigid manner relative to the mass properties instrument. For maximum measurement accuracy, the nominal CG of the object should be as close as possible to the measurement axis of the instrument. For spin balancing, the axis of rotation of the object must be coincident with the axis of rotation of the balancing machine.

4.2.2.. The fixture must provide a means to precisely relate the object coordinate system to the mass properties instrument coordinate system, so that measurements made relative to the machine axes can be expressed relative to the object axes.

4.2.3. The fixture should be balanced relative to the measuring instrument so that the full range of the instrument is available to measure the object. Static balance is adequate for CG and MOI measurements, but fixtures must be dynamically balanced for POI measurements. With the advent of computer controlled balancing machines, it is no longer necessary to have the fixture balanced perfectly. However, we recommend that fixture unbalance be no greater than 5 times the balance specification for the object.

4.3 No Detachable Parts Any detachable or movable parts or hardware on the fixture complicate the process of obtaining accurate tare measurements. Avoid using mounting hardware if possible to simplify measurement of fixture tare. For moment of inertia, all mounting hardware must be included in the tare readings. If hardware is necessary, then it must be made rigidly captive so that it remains with the fixture when the tare reading is made. For CG measurements, the hardware should be symmetrical about the centerline so it does not introduce unbalance. If possible, the hardware should be oriented vertically, so that a change in the amount of thread engagement will not alter the tare CG or MOI.

4.4 Low Windage Design of the fixture should minimize aerodynamic drag. This is important for both POI and MOI measurements. Excess windage will result in increased damping during MOI measurements. The error due to drag can be reduced by measuring in a helium atmosphere (see SAWE Paper 2024 entitled "Using Helium to Predict the Mass Properties of a Object in the Vacuum of Space", by Boynton, Bell, and Wiener). If the fixture is used for spin balancing, the outer surface should be as smooth as possible to reduce the forces due to turbulence, which will obscure the forces due to unbalance, limiting the sensitivity of the machine. It may be desirable to make a shroud around the fixture. The shroud shape and clearance will influence the POI measurement.

4.5 Provisions for loading the object in the fixture If the object weighs more than about 75 pounds, then some thought must be given on how to mount it in the fixture. If these concerns are not addressed up front, considerable delay and cost may be incurred later.

4.6 Verifying object position If possible you should design the fixture so the position of the object can be verified after it is installed in the fixture. This may require that you provide access openings in the fixture so the object can be probed with a dial indicator. Never assume that the precision of the fixture will insure that the object is located correctly. Variations in object diameter and runout can result in unacceptable fixturing errors.

4.7 Defining the instrument axes High accuracy mass properties machines such as the Space Electronics KSR series have a mounting table which rotates, making it easy to determine the measurement axes with great precision. The measurement axis is simply the center of rotation of the object mounting table. Often the object can be dial indicated to align the object with this axis. The 0 degree mark on the mounting plate usually corresponds to the +X axis and the 90 degree mark corresponds to the +Y axis. One of the best ways to assure repeatable fixture positioning is to use a round pin to engage the instrument center bushing and a diamond pin (relieved locating pin) at a relatively large distance from the center to provide angular alignment.

CG instruments which use the 2, 3, or 4 point weighing method do not have a rotating mounting plate and therefore have no well defined measurement axis. This constitutes a major source of error and is one of the reasons why the three point reaction force method of CG measurement is less accurate than the rotating table method.

4.8 The four basic types of object/fixture interfaces There are four basic types of object/fixture interfaces:

4.8.1 attachment point interface, where the fixture emulates the actual interface between the object and a mating part. This is suitable for rocket/missile stages.

4.8.2 hard point interface, where a system of hard points or rings is used for inspection, alignment and assembly reference.

4.8.3 adjustable interface, where the object has no well-defined hard points and a sophisticated (and time-consuming) method must be used to determine the position of the axes. The fixture is then adjusted to move the object so its axes are coincident with the machine axes. A novel method of dealing with the problem of fixturing irregular objects involves the use of video imaging equipment. A TV camera is mounted above the object, and views the object as it is slowly rotated. A digital computer acquires the video images and calculates the mean center of the vehicle. If the object is a cone, then the camera can view two different heights and direct the operator to adjust both concentricity and tilt.

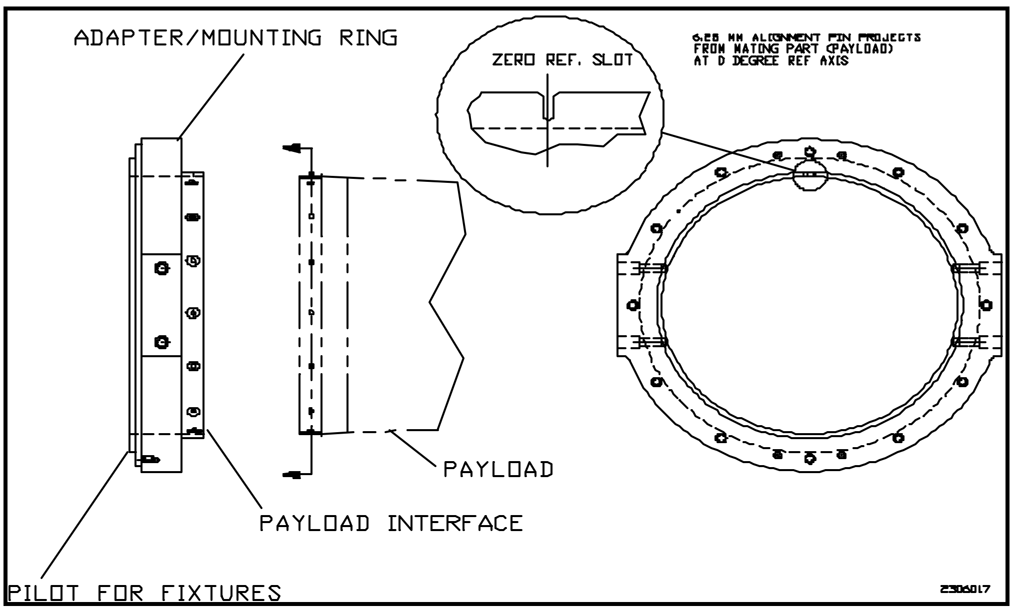


Figure 6 – Adapter/Mounting Ring Fixture Simulates the Actual Attachment Surface

4.8.4 calculated interface, where the object is placed in approximately the correct position, a measurement is made with the object in this position, electronic probes sense the position of the object relative to the machine, and the data is then corrected mathematically so it is expressed relative to the object axes.

4.9 Vee block fixture Cylindrical objects are often supported in a vee block fixture. The vee block does not depend on exact fit between diameters, since the cylinder sits tightly in the vee, no matter what its diameter is. One end of the object can be slid against a stop to establish the location along the length (X axis). The object can be rolled to an angle of 0^o, 90^o, 180^o, and 270^o to allow you to measure both radial axes..

4.10 Measurement tricks - turning uncertainties into correctable errors It's easy to measure the mass properties of a perfect cylinder. Not only can the object be located precisely relative to the instrument, but if the object is fixtured in a vee block, several valuable tricks can be used to maximize measurement accuracy.

4.10.1 Trick number 1. The cylinder can be turned end for end and re-measured. This establishes the location of the end stop. If the cylinder were a uniform solid and it were fixtured so one half its length were exactly on the centerline of the instrument, then turning the cylinder end for end would give the same CG location. In practice, the CG is not usually centered along the length, and the end stop is not located at exactly half the distance from the center of rotation of the instrument. However, if we know the length exactly (which is easy to measure), and we make a CG measurement from either end, then we can eliminate the fixture end stop uncertainty and correct the end stop position error. This method works for real test objects as well as cylinders as long as the test object can be turned end for end in the fixture.

First measure the CG from one end as shown in position 1, the left view of figure 7. This gives the distance from the machine center CG1. The turn the part end for end and measure CG2 as shown in position 2. The two CG locations (for any object) will be equidistant from the mid line of the part at L/2. This attribute permits calculating the CG offset (d) from the end stop (for position 1).

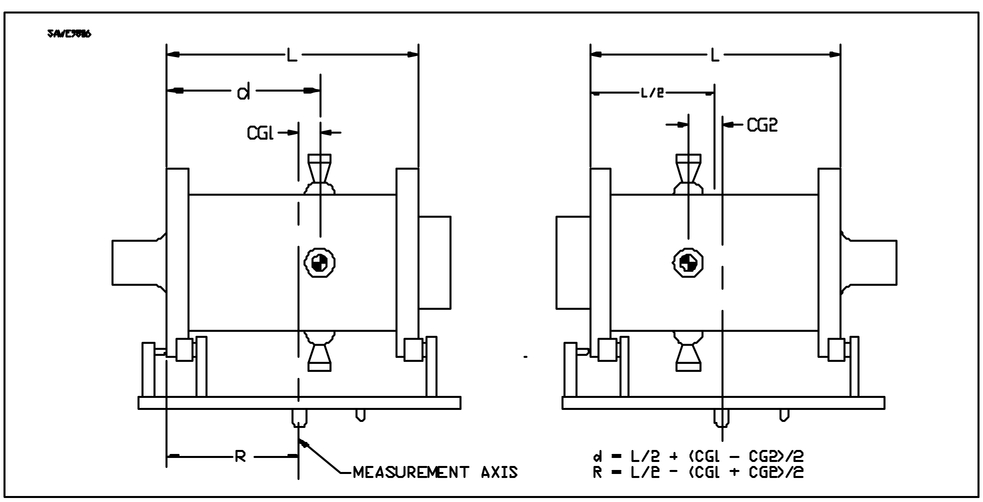
$$d = L/2 + (CG1 - CG2)/2$$

If several test objects of the same type are to be measured, the true end stop location may also be determined and used as a reference datum distance (R) as measured from the measurement axis. This distance will be:

$$R = L/2 - (CG1-CG2)/2$$

For successive part measurements the CG offset from the datum (d) then becomes R + CG1.

In practice, you should locate the end stop so the nominal payload CG location is as close as possible to the measurement axis of the instrument to eliminate second order uncertainties.



4.10.2 Trick number 2. The cylindrical test object can be rolled 180° in the fixture (vee block) and remeasured. This establishes the centerline of the vee block.

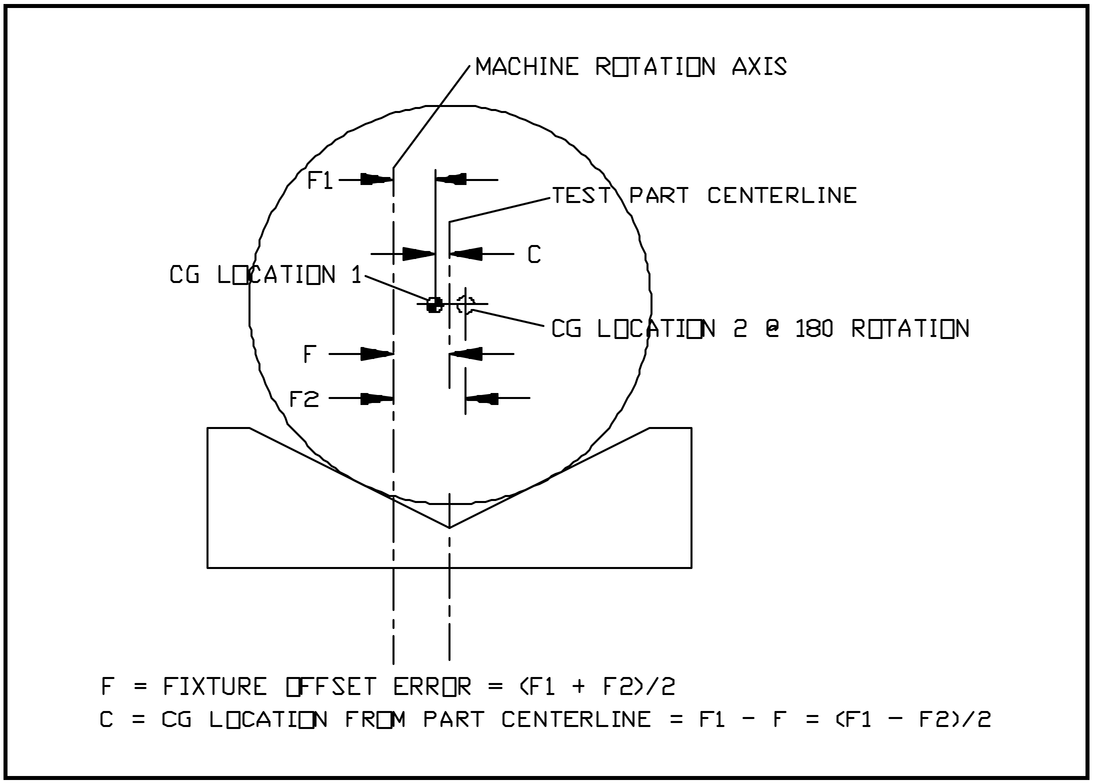


Figure 8-Work Reversal Method Eliminates Fixture Error

The process here is similar to the end for end case. If the vee block were centered on the measurement axis of the instrument, rolling the object 180° will change the sign of the CG offset, but the magnitude will remain the same. If the magnitudes differ, then this indicates that the vee block is not exactly centered. You then have two choices: if there is a large difference, then you can reposition the vee; if the difference is small, then computing half the difference between the two readings gives the true CG offset. As in the previous case, the exact position of the adaptor no longer has any effect on the answer. If several test objects must be measured, the fixture offset may be calculated and used as a correction for the measurements.

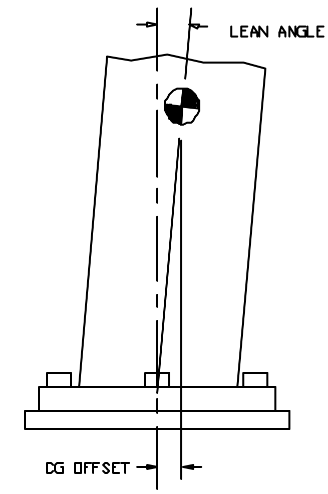
Note: Tricks 1 and 2 are called the "work reversal method".

Note: This method eliminates fixture position error, but does not eliminate errors shown below due to poor fit in the fixture.

NOTE:. It is very important to note that these methods only work if the cylinder is round and square, and the vee block contacts the cylinder on precision surfaces. Otherwise, you have no way of separating fixture misalignment from machining errors in the fixture.

4.11 Effect of fixturing error

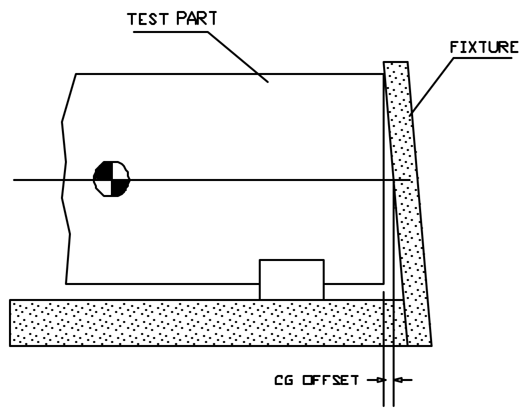
4.11.1 Effect of fixturing error on mass properties measurements Dimensional inaccuracy in fixtures adds directly to the inaccuracy of mass properties measurements. For example, if fixture error causes the test part to lean slightly to one side, then the instrument will indicate an apparent CG offset. Rotating the test table 360 degrees will not detect this lean error, since it causes the test object CG to go through a maximum and minimum reading in a manner similar to that which occurs for true CG offset. This same lean for a tall slender object will cause a horizontal shift in mass at the upper end of the object causing an inaccurately large MOI to be measured. The POI measurement will also be adversely affected. Fixturing is generally more critical for CG and POI measurements than MOI, as discussed below.



OUT OF SQUARE TEST PART INTERFACE
MAY CAUSE TRUE CG OFFSET

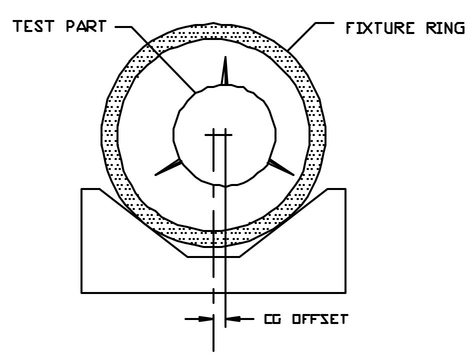
OUT OF SQUARE FIXTURE MUST BE
SHIMMED TO ELIMINATE LEAN ERROR

4.11.2 Effect of fixture error on CG Fixturing errors in the horizontal plane have a direct one to one relationship to CG error. A 0.01 inch fixturing error translates to a 0.01 inch measurement error. The goal for fixturing accuracy should be about one fifth to one eighth of the allowed CG tolerance. For example, if object CG must be within +/-0.008 inch of the centerline of the object, then to meet the one eighth goal, the interface surface of the object must be round and concentric with the centerline within +/- 0.001 inch.



OUT OF SQUARE REFERENCE SURFACE
CAUSES CG OFFSET ERROR

The interface must also be perpendicular to the centerline to a tolerance that insures the part does not lean more than 0.001 inch at
the CG height, causing an erroneous CG offset. The required tolerance for lean can be calculated from the formula:



FIXTURE RING CONCENTRICITY ERROR
CAUSES CG OFFSET

$$TIR = DX/8H$$

where TIR = total indicator vertical runout at diameter D of the object/table interface
X = CG offset tolerance of object
H = CG height of the object

This formula results in a perpendicularity tolerance which is one eighth the allowed CG offset, X. The tolerance on perpendicularity is surprisingly tight for most tests. For a rocket with a 100 inch CG height, an interface diameter of 10 inches, and a CG offset tolerance, X, of +/- 0.004 inch, the required perpendicularity (to meet the 1/8 ratio) would be 0.000,050 inch TIR!

Clearly, this is impossible. One solution is to mount the object on a tilt/translation fixture and center the object *at the CG height*. A more practical and cost effective solution is to fixture the object horizontally to avoid this difficult perpendicularity requirement.

4.11.3 POI: The magnitude of POI error is proportional to the difference between the axial and transverse moments of inertia of the test item (if these are equal, then alignment is not critical; if they are very different, such as would occur with a long thin shaft, then alignment is the limiting factor in measurement accuracy). Never measure POI without first dial indicating the object at two different heights to make certain the object is centered and does not lean. Do not rely on the fit in a fixture to establish position. For a slender object, a runout at the top of the object of 0.005 inch TIR may be enough tilt to cause the object to fail the POI specification. If you then add correction weights to compensate for this, you will be creating an unbalance!

MOI: Fixturing accuracy is not critical except if the payload is tall and thin. The reason for this is that the error is proportional to the square of the ratio of radius of gyration ("k") and fixture offset error ("d"). Generally the fixturing error is less than 1% of the radius of gyration, so the resulting error will be less than 0.01%. This relationship is derived below, using the well-known formula for translations of axes.

$$I = Mk^2 + Md^2 = Mk^2\left(1 + \frac{d^2}{k^2}\right)$$

$$\text{If } \frac{d}{k} < \frac{1}{100}, \text{ then } \frac{d^2}{k^2} < \frac{1}{10{,}000}$$

4.11.4 Using a Precision Dummy Payload One very convincing method to verify fixture accuracy is to construct a precision test weight with known mass properties which interfaces with the fixture in the same way as the real payload. For example, this weight might be a simple cylinder of constant diameter. If the mass of a solid cylinder would be too large, but you need a large diameter to interface with the fixture, you can use a small diameter solid cylinder with a larger diameter disc attached to each end.

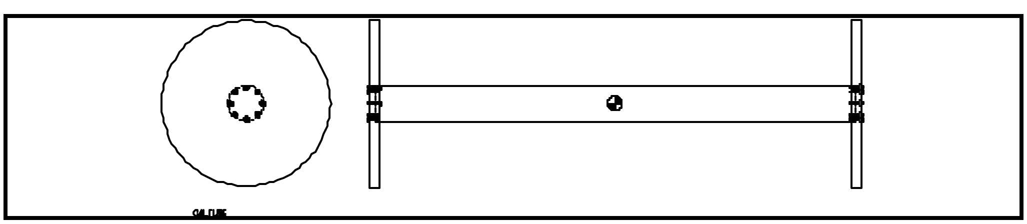


Figure 10 - Solid metal calibration weight simulates mass and diameter of low density rocket.

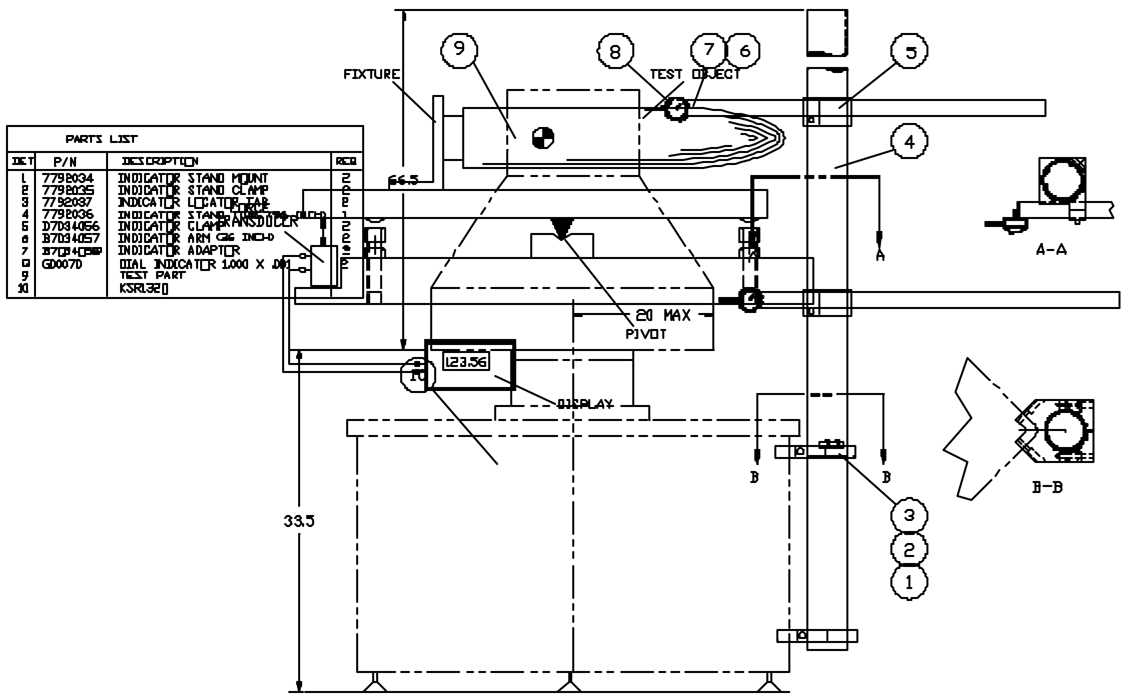


Figure 4 - A good way of fixturing an object for POI measurement is to dial indicate the object and adjust its position.

5.0 Methods used to measure CG Location

There are three basic <u>static</u> methods used to measure the CG of an object and 2 dynamic. Static methods depend only on the force of gravity acting through the test object CG and are preferred over dynamic methods.

In contrast to static methods, the dynamic methods require spinning the object or oscillating it to measure MOI in several positions. Consequently, dynamic methods are generally less accurate and more difficult to accomplish than static methods.

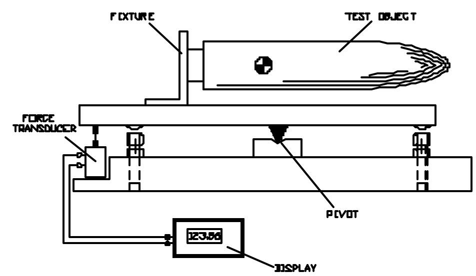
5.1 Static Methods

Unbalance Moment Method This method uses a pivot axis which supports most of the weight of the test object. The table CG machine senses the overturning moment produced by a displacement of the test object CG from the pivot center of the table.

Multiple Point Weighing Method --(also called "3-Point Weight and CG Instrument") or "Reaction Method") The CG of an aircraft is traditionally determined by placing scales or load cell platforms under the three wheels of the aircraft and calculating the CG location from the difference in force measurement at these three points. An instrument can be constructed on this same principle, wherein a test platform is supported by three or more load cells.

Mechanical Repositioning Method This method uses a pivot axis which supports all of the weight of the test object. To measure CG, the object is moved so that a balance is achieved about the pivot point.

5.2 Description of Unbalance Moment Method This method uses a pivot axis which supports most of the weight of the test object. The pivot is shown as a knife edge in the following illustration. However, modern instruments use a spherical air bearing as the pivot. A force transducer senses the overturning moment produced by a displacement of the test object CG from the pivot center of the table. Measuring this moment and dividing by the test object weight will yield the CG displacement from the center.



5.2.1 Instrument pivot type - Pivot friction affects the sensitivity of the instrument. The best instruments use a gas bearing. Crossed-web flexures are equally low in friction, but they have a bending moment which can result in a non-linearity for soft transducer machines. Knife edges have moderate sensitivity when new, but they rapidly deteriorate as the edge is worn down and are easily damaged. Roller or ball bearings have relatively high friction which seriously limits accuracy. These bearings can also be damaged by impact.

Air Bearing Pivots Air bearings consist of a precision rotor and a precision stator separated by an air gap that is less than 0.0005 inches thick. Air is introduced to the gap through jewel orifices that meter the air and provide dynamic centering of the bearing. Machining accuracy on these bearings is better than 30 millionths of an inch. This is what makes air bearings so expensive and difficult to make.

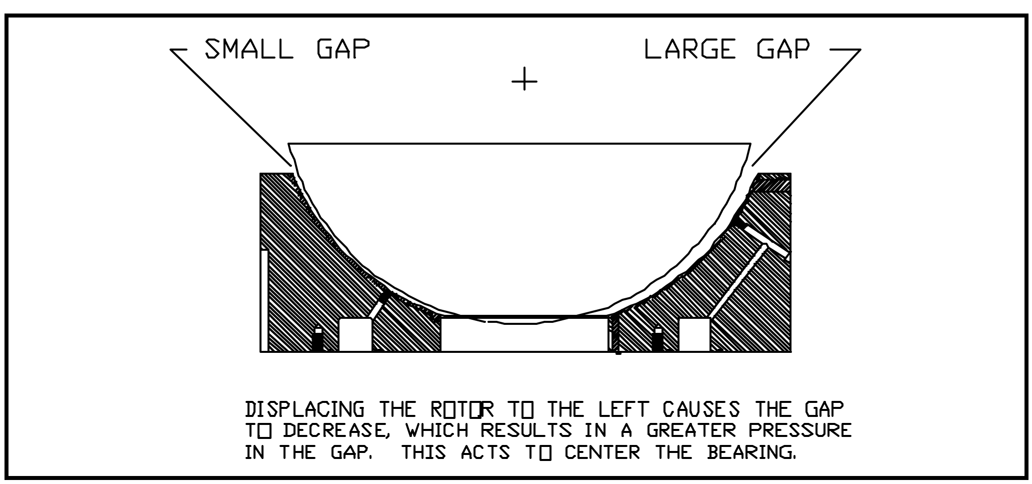


Figure 7 - Partially choked flow through small orifices produces dynamic centering of air bearing.

Dynamic centering action Contrary to intuition, air bearings have greater stiffness and precision than any other type of bearing.

The reason is that an air bearing is a dynamic device. If air is supplied through a single opening to the gap between the ball and the cup of a spherical air bearing, then the bearing would only operate successfully if the external forces were exactly in the center of the upper plate. A side load would cause the rotor of the bearing to move sideways so that one edge rubbed against the stator. Increasing the amount of air pressure in the plenum of the bearing would not improve the situation, since the additional available air would flow out the side which had the larger gap.

Air bearings made by Space Electronics minimize this effect by using independently supplied segments and small diameter jewel orifices which operate in a partially choked condition.

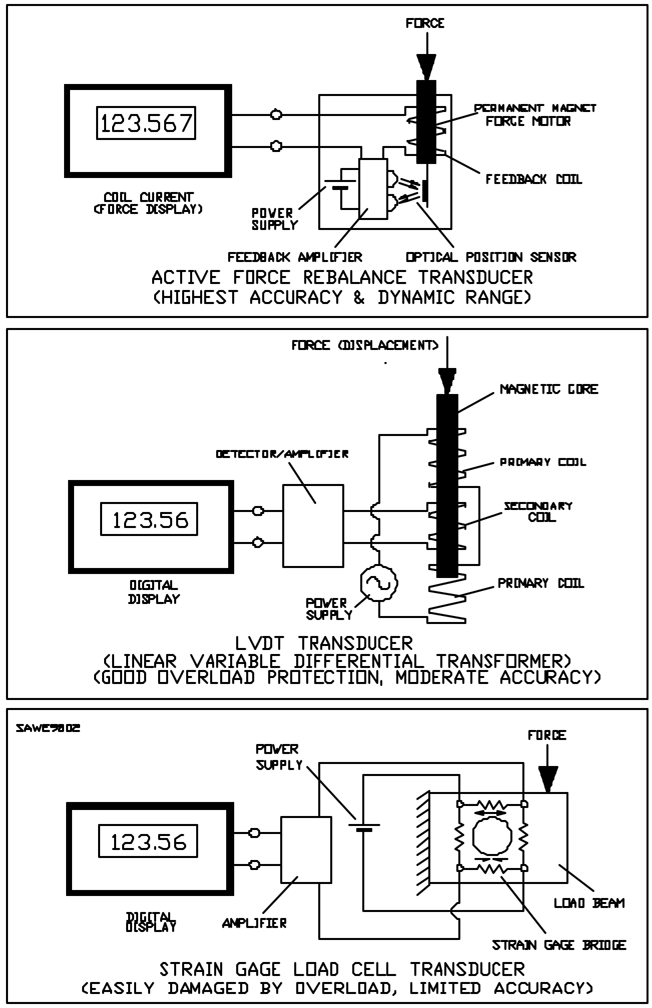
Under conventional operation with the payload centered on the interface table, the amount of air flow through each orifice is such that a pressure drop of approximately one half the pressure in the plenum occurs.

A minute movement of the rotor of the bearing results in a restriction of the flow on the side of the bearing which has the smaller gap and an increase in flow on the opposite side of the bearing. This produces a self-compensating or centering action of the bearing, since a reduction of the air flow on the low side increases the pressure in the gap on that side and an increase in flow on the high side reduces the pressure on that side. Proper selection of orifice sizes and cavity configurations permits the bearing to remain centered within about .0001 when subject to side loading.

5.2.2 Instrument Transducer type - The best instruments use an active force restoration transducer (introduced to the CG measurement industry in 1988 by Space Electronics) to measure moment.. This type of transducer can be built with a dynamic range of 300,000 to 1, a linearity of 0.001%, is very stiff (so that lean error is minimized), and has excellent overload protection.

Load cells can be used in place of the active rebalance transducer. These have high stiffness, but overload protection is very poor, so that the instrument has to be protected by air cylinder lockouts or other means while the test object is being loaded. Dynamic range of the load cell is limited to about 2000 to 1.

A third transducer type is a torsion rod/LVDT system which measures the lean angle that results from the unbalance moment. These transducers have good overload protection, but they are very soft, so that lean error is large for tall test objects and considerable time is required for the system to stabilize after the test object is installed or rotated. Linearity and dynamic range are slightly better than load cells.



Instruments can also be made which contain no electrical transducer. These instruments use sliding weights or weights which are driven by a lead screw. These instruments are sensitive and linear; their disadvantage is that there is no electrical output to a computer or printer, and skill and time are required to make a measurement.

5.2.3 Rotary table CG instrument Adding a rotary table to the CG instrument greatly improves measurement accuracy. Most systematic errors are automatically eliminated or can be eliminated in machine setup. The table rotates the test article to four locations (0, 90, 180, & 270 degrees) where static CG measurements are made.

> **!** The axis of measurement becomes the center of rotation of the table. This eliminates the need to accurately determine the relationship between the instrument pivot axis and the mounting surface of the instrument.

! For cylindrical test parts, or parts that can be accurately located in a fixture with a cylindrical reference surface at the nominal CG location, a dial indicator may be used to bring the part or fixture centerline concentric with the center of rotation of the instrument to within extremely close tolerances. This eliminates zero reference offset errors.

! For tall cylindrical parts, two dial indicator readings may be made: one close to the table and another at a location well above the table. A tilt table, shimming, or other means of adjustment, will allow the operator eliminate errors due to the part axis leaning away from the machine rotational axis.

! Another common system error occurs from improper leveling of the machine. This causes lean, which may be interpreted as CG offset. With the rotary table machine, this error is eliminated by taking readings for each axis which are 180 degrees apart. The lean error is equal for both measurements and is therefore subtracted from the result.

! Taking two readings for each axis also eliminates other systematic errors such as transducer zero offsets.

5.2.4 Spherical gas bearing rotary table instrument When the rotary table concept is implemented with a spherical gas bearing, the resulting instrument is capable of higher accuracy than any other method. The bearing acts as both a pivot and a rotary table. The use of a gas bearing makes it easy to design an instrument which can measure both CG and Moment of Inertia. For maximum accuracy, the overturning moment produced by a displacement of the test object CG from the center of rotation of the table is sensed using a force restoration transducer. Measuring this moment and dividing by the test object weight will yield the CG displacement from the center. This type of instrument does not measure weight so a separate scale must be used. The weight data can be automatically acquired by the mass properties machine software and used to calculate CG location.

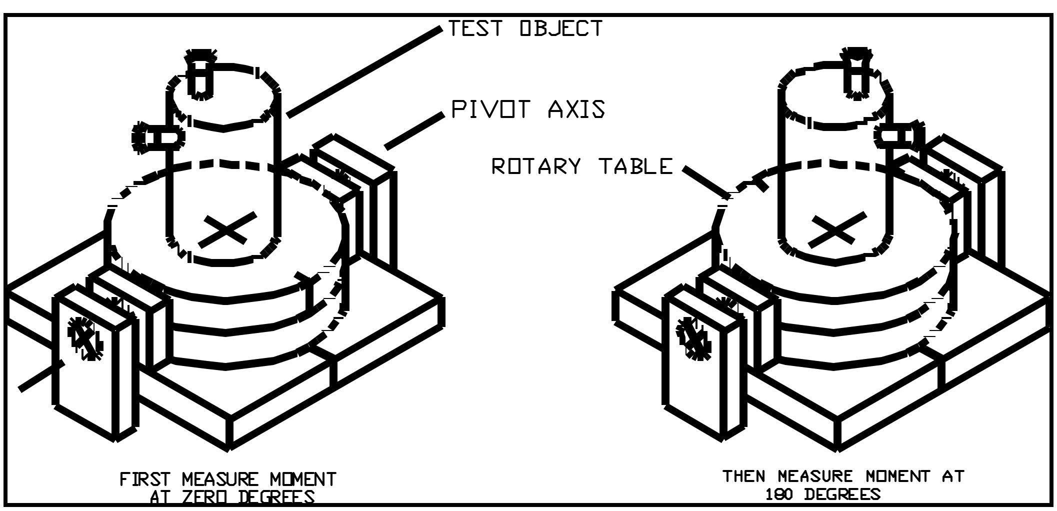
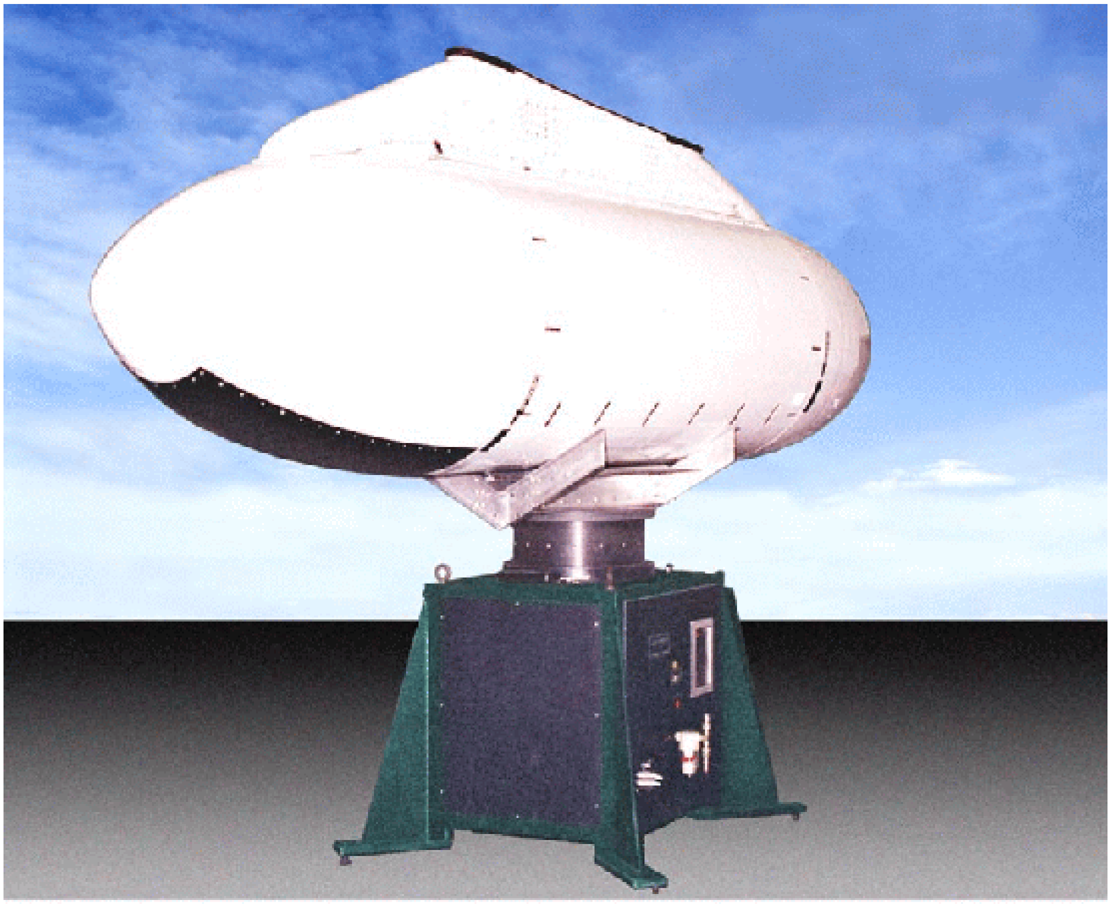


Figure 15



The Object in the photo is supported on spherical air bearing. Inverted torsion pendulum measures moment of inertia and force restoration transducer measures two-axis center of gravity location.

Photo shows Space Electronics Model KSR Mass Properties Instrument with object mounted in vee fixture. Both center of gravity and moment of inertia are measured in a single setup. Instrument CG measurement uncertainty is +/- 0.001 inch and moment of inertia measurement uncertainty is 0.1%.

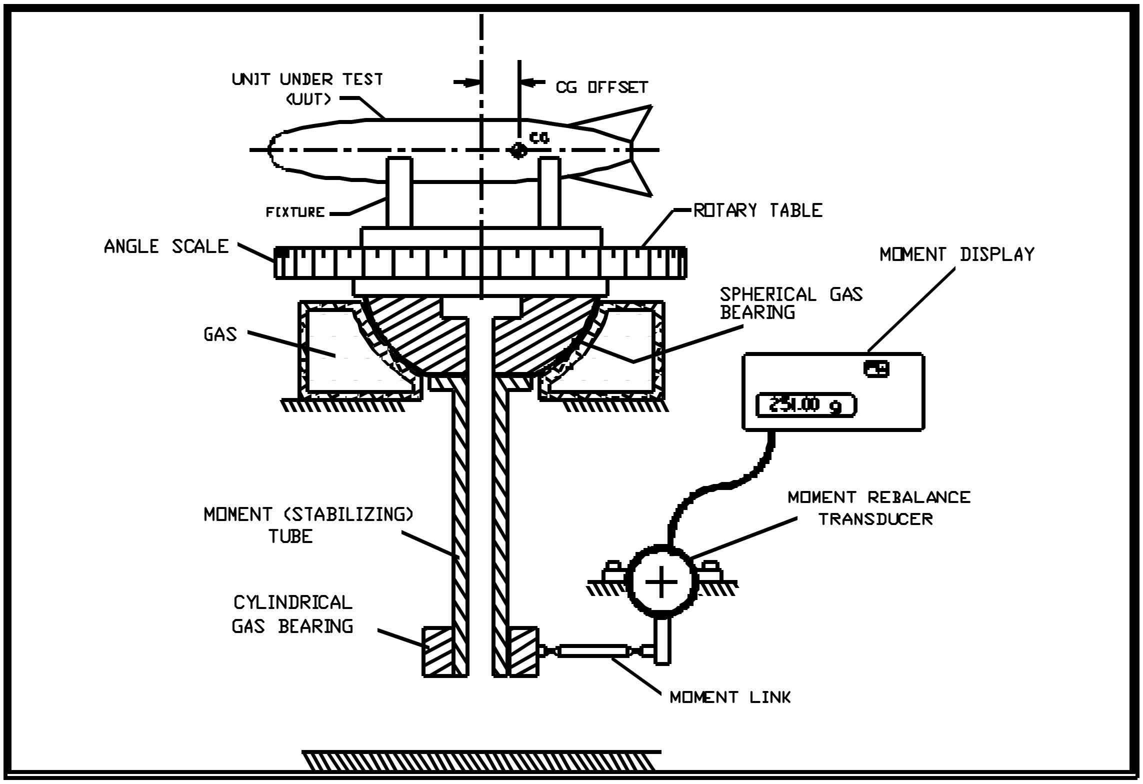


Figure 16 Basic elements of the Space Electronics Model KSR instrument. The spherical gas bearing creates both a precision rotary table and a frictionless pivot. Active force rebalance transducer measures overturning moment due to CG offset from center of rotation.

SUMMARY OF BENEFITS & SHORTCOMINGS

Benefits:

1. Accuracy is higher for this type of instrument than any other method.
2. This instrument is easily configured to measure MOI as well as CG moment.
3. Fixturing error is minimized since the rotary table allows cylindrical parts to be dial indicated. 4. Levelling error is eliminated by using the rotary table to take data readings at test part locations which are separated by 180 degrees.

Shortcomings:

1. A separate weight platform must be used to determine test part weight.
2. It is a more expensive system for CG measurement than the 3-point method. (Cost is less than the spin balance method).
3. It is slower than the 3-point method which may make it less suitable for high volume use.

5.3 Multiple Point Weighing Method -- A test platform is supported by three or more load cells, and the CG location is calculated from the difference in force measurement at these three points. In the past, the accuracy of this method has been limited by the dynamic range of load cells, so that these instruments were not suitable for projectile and missile measurements. The introduction of force rebalance technology to CG measurement by Space Electronics in 1988 has reduced force measurement errors by a factor of 30. When this technology is applied to the Multiple Point Weighing Method, accuracy improvement is great enough so that this method now becomes acceptable for many applications. This instrument measures weight as well as CG.

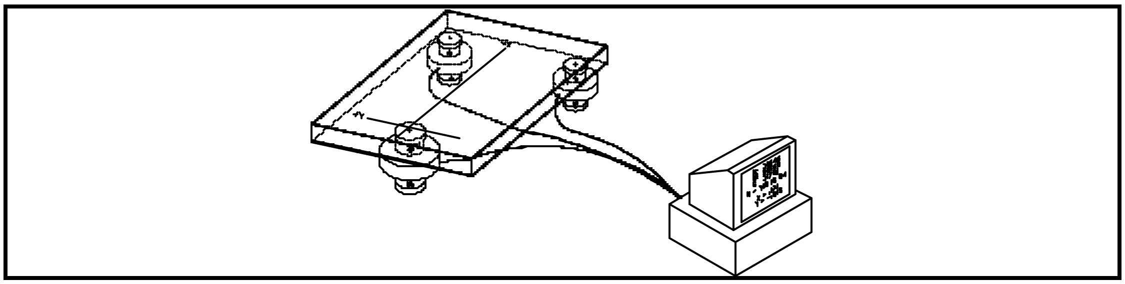


Figure 17 Multiple Point Weighing CG Machine - Fast and easy to use - moderate accuracy.

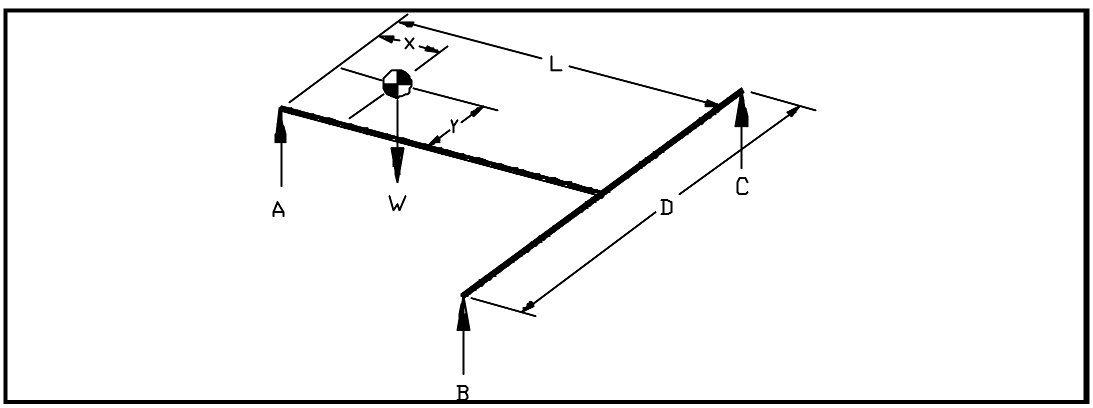


Figure 18

5.3.1 Calculating weight and CG location

To determine part weight (W) and CG coordinates X and Y, three force transducers are typically used to support a frame which in turn supports the object.

$$\boxed{W = A + B + C}$$

where A, B, and C are force readings on the three force transducers.

To determine CG, take moments about A, where X and Y are the CG measurement coordinates.
If all the transducers outputs are set to zero when fixturing is in place, the equations above are used to determine the CG location of the test part. In practice, tare readings are subtracted from the part

$$\Sigma M_x = (B+C)L - WX = 0$$

$$\Sigma M_y = \frac{CD}{2} - \frac{BD}{2} - WY = 0 = \frac{D}{2}(C-B) - WY$$

$$X = \frac{(B+C)L}{W}$$

$$Y = \frac{(C-B)D}{2W}$$

measurements and the values above represent the <u>net</u> A, B, and C forces required to support the part weight and CG offset moment.

SUMMARY OF MAJOR BENEFITS AND SHORTCOMINGS

Benefits

1. Measures both CG and weight.
2. By using the latest force rebalance transducers and optimum geometry, sensitivity is adequate for most applications
3. For a given CG offset moment capacity and part weight, it is the lowest cost automatic system.
4. It is most suitable for very heavy parts with tight CG location tolerances
5.This is the fastest CG measurement method. Total time to make a measurement of 2 axis CG is less than 30 seconds.

Shortcomings

1. A separate instrument must be used to measure MOI if this quantity is required.
2. It is highly sensitive to and not readily correctable for lean error caused by leveling.
3. The machine axis zero point is difficult to define. Unlike rotary table machines, the object cannot be dial indicated. Fixturing errors may be relatively large.

5.4 Repositioning Method of Cg Measurement A third method of CG measurement is the free-pivot method where the test object is balanced on a pivot and allowed to tilt. The test object is moved relative to the pivot of the instrument until a balance is obtained. Some means is then required to measure the final position of the object This method of CG measurement is:

Inexpensive
Very time-consuming
Generally the least accurate of all methods

The first two techniques discussed do not require an “instrument”. CG is measured using common objects that are laying around the lab. Both of these methods have a fundamental flaw: there is no accurate method to transfer the answer to the coordinate system of the test object.

5.4.1 Hanging pivot If an object is suspended from a knife-edge, the CG of the object will lie directly below the center of pivot of the knife-edge. The position of the test object is moved relative to this pivot point until a balance condition is achieved. Torpedos and projectiles can sometime be measured using this method.

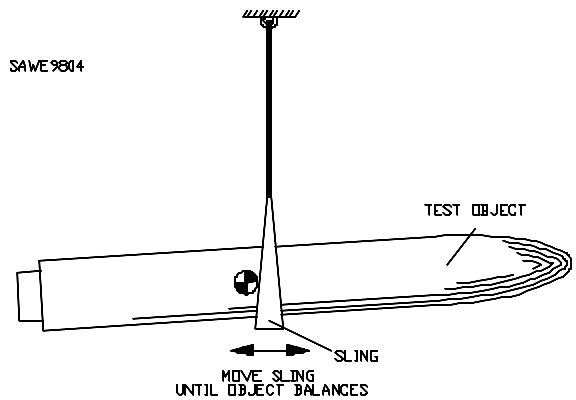


Figure 19

Hanging systems such as this one where the test object CG is always below the pivot point have the disadvantage that their sensitivity is low. Shifting the lateral CG of the test object results in only a small change in the level condition of the instrument. The amount of tilt which results from a CG offset is a function of the CG <u>height</u> difference between the object and the knife edge. Reducing this distance increases sensitivity. However, it also increases measurement time, since the object rocks back and forth at a slower rate when sensitivity is increased.   Each time the item is moved, the operator must wait for the system to settle again. In this respect, this method is like using the old beam balance scales. For very sensitive systems, it can take hours to make a single measurement. (In contrast, force rebalance techniques are up to 100 times more accurate but only require seconds for a single measurement).

The sensitivity of the instrument is directly dependent on the accuracy of the means used to detect the level of the instrument. Bubble levels are available with the sensitivities as great as one division for an angle change of 0.00005" per ft. A more practical figure to use in determining level accuracy, however, is approximately 0.002" per ft., since the position of the bubble level relative to the measurement axis of the instrument is difficult to adjust to a closer tolerance than this.   Even the smallest error in the position of the bubble level relative to the structure of the instrument results in a large error in measured CG location.

This method is extraordinarily tedious to use. The object must be lifted and repositioned 10 or 20 or 30 times, depending on the accuracy required. Each time the swinging must be allowed to damp out and the level condition read. Since there is no readout to indicate how far to move the object, the process is strictly trial and error. The initial cost of this method is the lowest of any type of CG measuring technique, but often the labor required to make a measurement more than offsets the cost, causing this method to be in fact, the most expensive of all techniques.

Once you have obtained an accurate balance, how do you relate the final position of the object to the pivot axis of the knife edge? Unless you can do this accurately, you have accomplished nothing by tediously balancing the object. One method is to use a transit. You center the crosshairs on the knife edge pivot and then swing down to view the object. The object is marked, and then removed from the CG fixture and placed on a coordinate measuring machine, where the location of the mark is measured. Generally the accuracy of this method is about ±0.040 inch.

5.4.2 Measuring CG using a "broom handle" The simplest method of measuring CG is to balance the object on a round rod. This method only works if :

a) the object has a low profile
b) the object has a rigid surface that will not be indented if its entire weight is supported on a narrow rod
c) the surface of the object is flat and smooth.

Since the total CG of the test object lies above the pivot axis, the object can tip in either direction when the pivot point is near its CG. This deadband results in an error in CG location. The magnitude of this error is proportional to the amount of angular tilt of the test object which is permitted. Reducing this tilt will decrease the magnitude of this error. However, the small amount of travel makes the instrument extremely tedious to operate - there is no advance warning that the balance point is being reached and extremely fine adjustments of the test part position must be added to prevent overshooting this very narrow point.

To get a better feeling for the problems in using this method, lay a ruler at a right angle on top of a round pen. You will notice that the ruler can never be made to balance on top of the pen. The reason is that the CG of the ruler is above the pivot point (contact point between objects), so that the ruler has two states of equilibrium. The trick to using this method is to roll the pen in one direction until the ruler flops over, and mark the contact point with the pivot; then roll the pen in the other direction and mark where it flops back. The CG of the ruler is near the center of these two points. (It is not exactly at the center, because of pivot friction, and the error introduced by inequalities of tilt angle when the object is at rest at its two equilibrium positions). This method works pretty well for a ruler, because it is long and thin. The length allows you to limit the amount of tilt, and the low CG height limits the deadband.

Now try using this method to measure the CG of a coffee cup. You will notice that the distance between the two CG marks increases, since the CG is higher and moves a greater lateral distance when the cup moves from one equilibrium state to another. Furthermore, it is hard to limit the tilt so that it is equal in both directions.

One catch to this method is that there is no way to accurately transfer the answer to the coordinate system of the object. You can see when the object tilts, but how do you relate this location to the CG along the length of the object?

Note: Cylindrical objects cannot be tested using this method, since this would result in the intersection of two rounded surfaces, causing the entire weight of the object to be supported on a single point. In most instances, the object would be damaged. Furthermore, the object will tend to rotate about a vertical axis, so that its orientation relative to the pivot axis will be altered.

5.4.3 CG Measuring structures which make use of repositioning The concepts discussed above can be incorporated into an instrument with a mounting surface and an accurate relationship between pivot and center pilot on the mounting surface. These instruments require the object to be moved. Therefore, they represent obsolete technology and are mentioned only to make this summary historically correct.

Figure 20 illustrates the same type of instrument in an unstable condition. There is no vertical counterweight so that the CG of the moving system is above the pivot point for any test object. Rotation stops are provided to prevent excessive motion of the test object. This unstable condition results in a hysteresis or deadband that limits the accuracy of the instrument. Decreasing the gap between the stops reduces the deadband, increasing sensitivity. It is impractical to reduce this gap more than a certain amount. The ultimate accuracy with this type of instrument can be obtained by leveling the instrument with one stop in contact, and then moving the object until the platform tips. This technique has, in effect, reduced the deadband to zero.

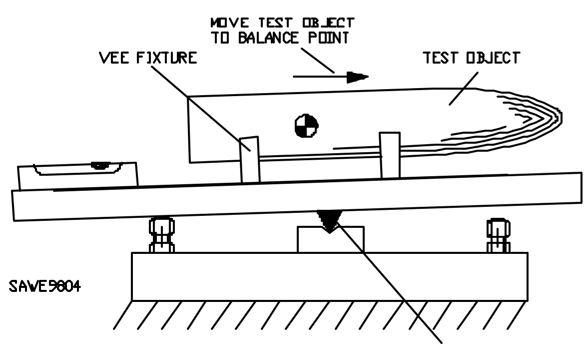


Figure 20

Figure 21 illustrates a stable type of free pivot instrument. In this instrument, the lower counterweight is adjusted vertically until the total system CG is slightly below the pivot point.

The instrument will then exhibit maximum sensitivity but will be stable. This instrument will only work for an object which has a straight surface and can be slid sideways in the fixture.

One type of re-positioning CG instrument which we used to manufacture in the 1970's consisted of a test platform which is mechanically coupled to a counterweight. A motion of this

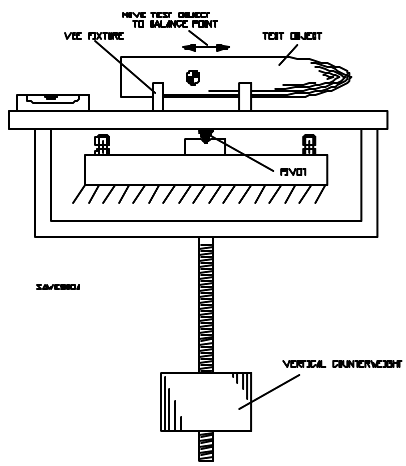


Figure 21

test platform results in an equal and opposite motion of the counterweight maintaining the balance of the structure of the instrument when the test part is re-positioned.

All free balance CG measuring systems have the same disadvantages. The measurement is extemely tedious to perform, since re-positioning of the object often requires that it be lifted temporarily. After the object is moved, the new position of the object must be determined in order to identify the location of the object CG. This is the major accuracy limitation of this method. In contract, the object is fixture at a precisely known location when using moment measuring instruments. Repositioning instruments often require a relatively elaborate fixture which is usable for only one particular type of test part.. The sensitivity of the techniques which pivot the test part above or below its CG is very poor, but the sensitivity of the methods which approximately align the pivot axis with the CG is high. However, these high sensitivity systems require the use of vertical counterweights; errors in the alignment of these counterweights with the pivot axis of the instrument can cause large measurement errors unless the structure is re-balanced after each height adjustment is made. Unstable free balance systems have the additional disadvantage that the test operator has no way of knowing when he is approaching a balance condition.

SUMMARY OF BENEFITS AND SHORTCOMINGS

Benefits

1. Lowest cost method.
2. Can achieve high sensitivity (but not necessarily high accuracy).
3. Inherently safe in explosive environments.

Shortcomings

1. Accuracy is generally limited because of difficulty of determining location of object
2. Tedious and time consuming to operate.
3. Can’t be used for irregularly shaped objects.
4. Accuracy depends on skill of operator.

5.5 Dynamic methods of measuring CG

In contrast to the methods described above, these methods require that the test object must be moved during measurement. The data obtained during these measurements is related to product of inertia as well as CG; some method must be used to eliminate the effect of product of inertia in order to derive CG location.

Spin Balance Method -- The test object is rotated and force transducers sense the reactions on the bearings which support the part during rotation. These forces are due to both gravity and centrifugal force (the higher the spin speed, the less significant the gravity force is). The CG location of the part may then be separated from the dynamic unbalance of the part using calculations that involve the magnitude of the bearing forces and their phase relationship. This method was used extensively before the 1970's, when force measuring technology was rather crude. However, since the development of single-point load cells with solid state amplification there is no longer any justification for using this method.

Moment of Inertia Method -- The test object is mounted on an inverted torsion pendulum (moment of inertia instrument) and successive moment of inertia measurements made for at least three positions of the test object. The CG location can then be calculated from the small change in MOI which results from moving the principal axis of the object.

These methods are described in more detail in the following sections..

5.5.1 Spin Balance Method of Measuring CG -- The test object is rotated and force transducers sense the reactions on the bearings which support the part during rotation. These forces are due to both gravity and centrifugal force (the higher the spin speed, the less significant the gravity force is). The CG location of the part may then be separated from the dynamic unbalance of the part using calculations that involve the magnitude of the bearing forces and their phase relationship. Spin balance machines rotate the test item at speeds ranging from 50 RPM to 10,000 RPM and measure the reaction forces acting against the bearings in the machine due to dynamic unbalance (a combination of CG offset and product of inertia).

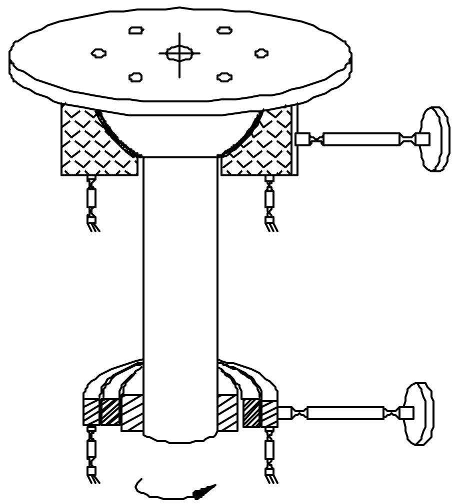


Figure 22 - The spin balance method of CG measurement is expensive and accuracy is often very limited..

At first analysis it might appear that it doesn't make any difference whether CG is measured in a static or a dynamic mode. However, there are a number of considerations which make the spin method of CG measurement unsatisfactory. If you need to measure the longitudinal CG of a long test object, (i.e. 10 meter long rocket), then static measurement is the only way. Spinning such a test object would require tremendous power and generate high winds in the test laboratory. If you are measuring the CG of a partially filled fuel tank, then the fuel will ride up the sides of the tank if you spin it, because of the centrifugal force. This results in an erroneous CG measurement. If the test object has extended solar panels, then centrifugal forces may damage or deflect them. For objects with a large CG offset, the force limits of the transducers will be exceeded if you spin the part. If a part has a very large product of inertia unbalance, CG measurement will be more accurate in the static mode, since the small CG forces do not have to be separated from the larger POI forces. Finally, and perhaps the most important consideration, when an irregularly shaped object is spun, aerodynamic forces cause large variations in the measured CG offset, severely limiting measurement accuracy.

<u>Using a balancing machine to reduce unbalance</u> -- A spin balance machine is the ideal device to balance an object which rotates. For this application, it makes little difference whether the unbalance is caused by CG offset or product of inertia; the goal is to reduce the unbalance to acceptable limits. The balancing machine will instruct the operator as to the ballast weight which must be added to achieve balance. Since the ultimate goal is to reduce unbalance, a 5% error in measurement has little consequence. For large unbalances, this 5% error means that a 20 to 1 reduction is balance is possible for each iteration. It would be hard to do better than that even if the machine were more accurate, since balance weights are not always the correct mass and it is hard to place them in exactly the right location. When balance has been achieved, a 5% error also has little consequence: 5% of a very small unbalance is insignificant.

Using a balancing machine to measure CG --The situation is totally different when a balancing machine is used to measure CG rather than correct unbalance. In this case, the effect of product of inertia unbalance must be subtracted from total unbalance to obtain CG. Often the product of inertia term is much larger than the CG term, so that the CG is proportional to the difference between two similar large numbers. In effect this is similar to weighing a man by having him drive a dump truck onto a truck scale, obtain the gross weight, and then have the man step out of the truck, and get the net weight of the truck. The difference is the weight of the man. A static CG machine does not respond to product of inertia at all, since there are no centrifugal forces. On a balancing machine, however, CG measurement accuracy will be severely limited if the product of inertia of the object is large. This situation is likely to occur if a tall rocket is measured. The rocket must be mounted vertically in the machine since the windage in the horizontal mode would be so great that the test lab would become a wind tunnel. It is not uncommon is these instances for product of inertia forces to be as much as 100 times greater than the forces due to CG offset. Since these product forces must be subtracted from the CG offset forces, a 5% error in the measurement of the product forces will result in a 500% error in CG measurement!

SUMMARY OF BENEFITS AND SHORTCOMINGS

Benefits

1. By selecting spin speed, machine can be either low sensitivity/high offset range or high sensitivity/low offset range.
2. Sensitivity can be very high at high spin speeds. However, in most cases maximum spin speed is limited by structural limits or the windage of the test object.
3. Measures product of inertia as well as CG.

Shortcomings

1. High product of inertia of test item often obscures CG, limiting accuracy.
2. Air turbulence during spin can produce a large uncertainty in CG measurement Some objects cannot be spun because solar panels or other protrusions would break off due to windage forces.
3. On some test items, spinning alters their CG. Other items cannot tolerate the centrifugal or vibratory force that would occur at the spin speed.
4. Cost of this type of instrument is higher than any other.
5. Fixturing must rigidly support the test item when being subject to relatively high vibratory forces during spin. This makes fixture design much more expensive and difficult than fixtures for static machines.

5.5.2 Moment of Inertia Method of Measuring CG -- The test object is mounted on an inverted torsion pendulum (moment of inertia instrument) and successive moment of inertia measurements made for at least three positions of the test object. The CG location can then be calculated from the small change in MOI which results from moving the principal axis of the object. This method was developed by Space Electronics in the early 1970's and described in SAWE papers #1169 and #1440. As these papers show, this is the least accurate method of CG determination. Its only advantage is the low cost if you already were in possession of a MOI instrument.

Center of gravity is determined on a torsion pendulum by making use of the parallel axis theorem.

If the moment of inertia of the object about axis A-A through its CG is I_A, then the moment of inertia through axis B-B is

$$I_B = I_A + d^2M$$

Where M is the mass of the object and "d" is the distance between axis A-A and axis B-B. Note that the minimum measured moment of inertia of an object occurs when the axis of measurement coincides with the CG of the object.

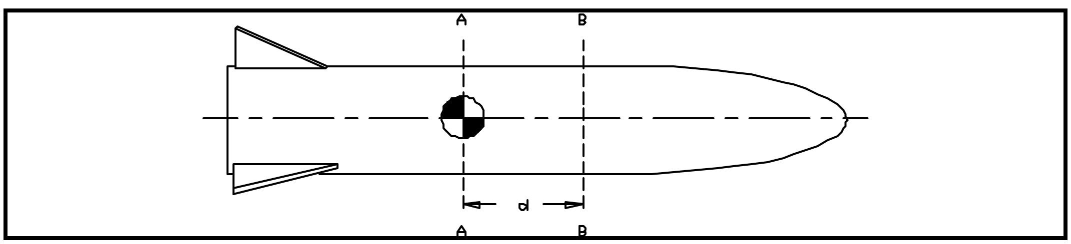


Figure 23 - Center of gravity is determined using the parallel axis theorem.

For the object shown in Figure 23 above the longitudinal CG can be determined by mounting the object in a vee block fixture on a moment of inertia instrument; several moment of inertia measurements are then made at different object positions to determine the object location resulting in the smallest measured moment of inertia.. When the measured moment of inertia is a minimum the CG of the test object is coincident with the axis of measurement.

This is an extremely tedious procedure, and requires some means of determining part position at minimum moment of inertia. Furthermore, this method only works for single axis CG. A better method is to measure moment of inertia at three known positions and calculate CG from this data using the methods described in the SAWE papers referenced above. Referring to Figure 24, the distance the test part was moved is known, the two values of moment of inertia are measured, the test part is weighed to determine the value of M, and the distance to the CG, d_b , is calculated.

If the displacements between the three measurement positions are made small, then the sensitivity of this method is abysmal. Accuracy of better than 0.1 inch is difficult to obtain. If the displacements are made large relative to the radius of gyration of the test object, then the accuracy improves from a theoretical standpoint. However, torsion pendulums do not operate successfully with large CG offsets, due to the gravity pendulum error, so that the increased measurement error partially offsets the gain in sensitivity, and the accuracy of measurement is still worse than other methods.

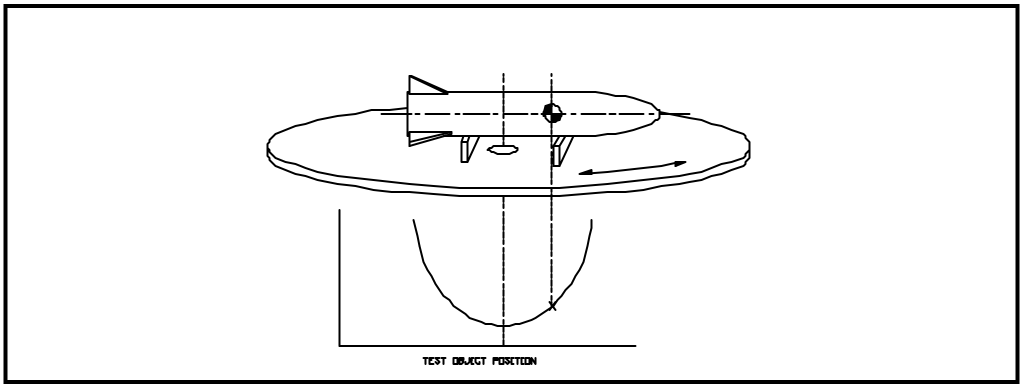


Figure 24 - Test object is repositioned in vee blocks until point of minimum moment of inertia is determined

SUMMARY OF BENEFITS AND SHORTCOMINGS

Benefits

1. If you already have a moment of inertia instrument, it can be configured to measure CG with relatively little extra cost.
2. Machine performs with creditable accuracy when measuring small diameter solid steel test weights (unfortunately very few real test objects are of this type).

<u>Shortcomings</u>

1. Accuracy is poor for realistic test parts (ones whose radius of gyration is considerable larger than the CG offset from the center of the instrument).
2. Each measurement requires the test object to be positioned in three different locations, so that an elaborate and expensive fixture is required.
3. The instrument does not give a direct readout of CG, so that corrections cannot be made to the object and the resulting CG shift observed.
4. Error increases dramatically if the machine is even slightly out of level.
5. Test items containing fluids cannot be measured accurately
6. The test is tedious to run.

6.0 MOI measurement

Every engineer knows he can measure moment of inertia by hanging an object from a wire, twisting it to start it oscillating, and then timing the period of oscillation. However, anyone who has actually tried this finds that the object swings from side to side and rocks up and down rather than rotating smoothly about an axis, making it difficult to accurately time the period of oscillation. Furthermore, there are a number of practical problems involved in hanging most test articles from a wire. How do you attach the wire to the object? Where do you attach the upper end of the wire (particularly if the object weighs more than 1000 kg)? How do you calibrate the device, and what do you do to correct for the change in calibration when the weight of the test object stretches the wire?

6.1 Inverted Torsion Pendulum Modern moment of inertia instruments consist of an inverted torsion pendulum which oscillates in a rotational sense and a means of measuring the exact period of oscillation of the torsion pendulum. Instead of hanging from a torsion rod or wire, the test object rests on a precision rotary table attached to the top of the instrument. Low friction bearings support the table and payload while constraining the motion of this torsion member to pure rotation. Air bearings provide the best performance. Unlike custom-made hanging wire, trifilar, or compound pendulum systems, measurements are made about a well defined axis, a minimum amount of fixturing is required, and elaborate computational techniques are not necessary.

.

The measurement of the moment of inertia of the test part is based on the change in the natural frequency of oscillation of the torsion pendulum resulting from the addition of the test part mass. This change in natural frequency is compared with the change in natural frequency which occurs when a calibration mass of known moment of inertia is placed on the instrument.

<u>Step 1</u> The object is secured to the table with its CG aligned with the axis of the bearing. The part is rotated and released. It will then oscillate about the fixed axis of the instrument and the total time for one complete oscillation can be displayed on a digital period counter. The total combined moment of inertia of the test object, its fixture, and the instrument itself can be calculated from the formula:

$I_x = CT_x^2$ TOTAL MOMENT OF INERTIA

where I_x is equal to the total moment of inertia, C is the calibration constant of the instrument, (a function of its torsional stiffness), and T_x is the period of oscillation in seconds.

<u>Step 2</u> The test object is then removed from the instrument and the "tare" moment of inertia of the instrument and the fixture determined by measuring the oscillation time period without the test object.

$I_o = CT_o^2$ TARE MOMENT OF INERTIA

Step 3 The moment of inertia of the test object is then the difference between the total inertia and tare inertia.

$$I = I_x - I_o \qquad \text{NET MOMENT OF INERTIA OF OBJECT}$$

in order to establish the value of the calibration constant, C, of the instrument, MOI calibration standards are measured. MOI calibration standards are precision weights of simple geometry, known mass and known physical dimensions.

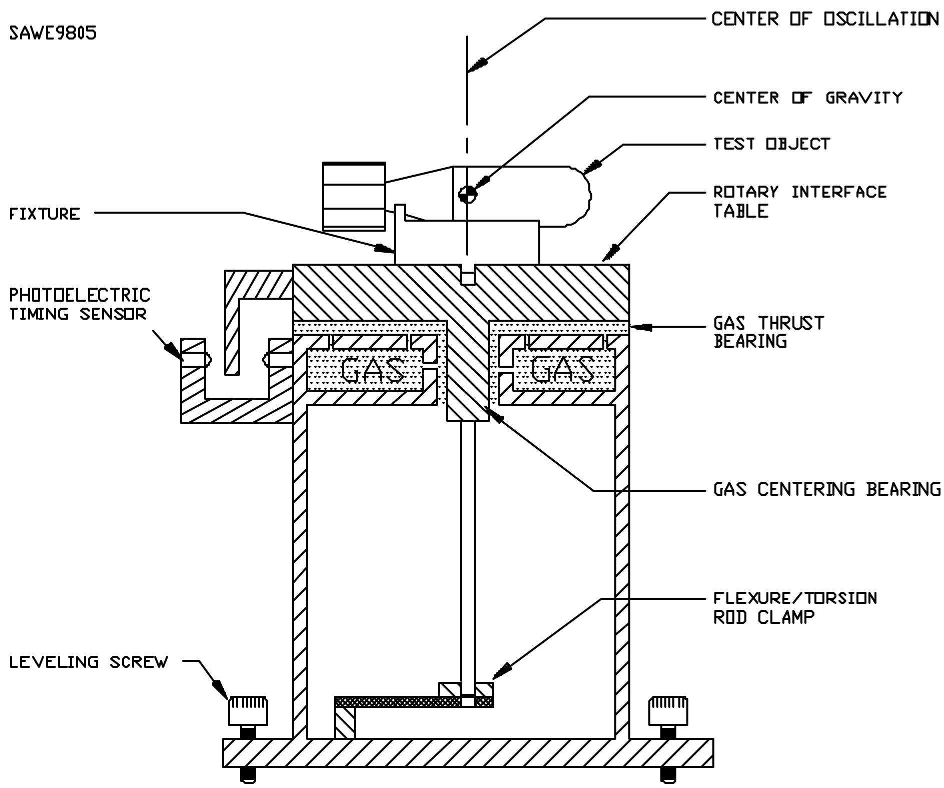


Figure 25

The calibration procedure is identical with the procedure for measuring the moment of inertia of a object of unknown MOI, except that in the computation, the inertia is a known quantity and the value of the calibration constant is the unknown which must be solved for.

Because the weight of the object is supported by the air bearing, these instruments are linear over a wide range of test part weight and moment of inertia. Only a single calibration measurement is required to establish the value of the calibration constant used for all measurements.

6.2 Description of MOI Instrument

A greatly simplified cross section view of a MOI instrument is shown in figure 25. The instrument consists of three basic parts: an air bearing, a torsion rod, and a photocell system for timing the period of oscillation. The test part is first mounted in a test fixture (designed and fabricated by the user) which locates the test part in proper orientation to the measurement axis of the instrument and couples the test part rigidly to the interface table of the instrument during the testing. The test fixture itself is aligned with the interface table of the instrument through the use of a precision pilot in the interface table.

To measure the moment of inertia of the test part, gas pressure is applied to the storage cavities in the gas bearing. The gas then flows through small diameter orifices to the gap between the gas bearing and the interface table, floating the interface table on a film of gas. The damping in the torsion pendulum therefore consists only of the internal damping in the torsion rod. The interface table is temporarily twisted in a counterclockwise direction until it contacts a stop; then it is sharply released, resulting in an oscillatory motion of the interface table due to the stiffness of the torsion rod (which is rigidly attached to the interface table and clamped to a flexure at its lower end). The time period of oscillation is then measured by connecting a digital period counter to the photoelectric timing sensor. As the interface table oscillates, a timing pulse is emitted as the table reaches its mid-point of oscillation when traveling in a clockwise direction. The first timing pulse starts the counter, a second timing pulse stops the counter exactly one period later.

6.3 Keeping the Test Part CG Coincident with the Rotational Axis It is important that the center of gravity of the test part and fixture be positioned so that it is as close as possible to the rotational axis of the torsion pendulum. Otherwise measurement error will increase for a number of reasons.

Most significant of these is the so-called "gravity pendulum" error which occurs because the axis of rotation of the torsion pendulum can never be made exactly vertical. If a part is rotated about an axis which does not fall on its center of gravity, then the force of gravity acting through the center of gravity will tend to bring the center of gravity to its low point (i.e. the direction in which the axis of rotation of the torsion pendulum is tilted), resulting in a change in the effective calibration constant of the instrument. This effect can be minimized by leveling the table and re-positioning the test part so that its center of gravity lies close to the rotational axis of the instrument.

A second source of error with offset center of gravity is the classical axis translation error. Mathematically, this increase is equal to the test part mass times the square of the offset distance. Since this increase can be exactly calculated, it can be subtracted from the measured moment of inertia; therefore it does not constitute an uncertainty. For small offsets, (less than .01 times the radius of gyration) this effect is negligible. For larger offsets, instruments are available which measure both CG and MOI. The software then automatically reports the MOI about the center of oscillation and about the CG. To utilize this feature, the test part must be weighed.

If it is desired to measure a test part through a point other than its center of gravity, the most accurate method of accomplishing this is to first measure the moment of inertia of the part about its center of gravity and then mathematically translate this measured value to the new axis by adding the translation factor R^2 M where R is the distance between the CG axis and the desired axis, and M is the mass of the test part. This procedure also requires the weighing of the test part.

6.4 Effect of air mass - For large lightweight objects, the measured mass properties are often different from the calculated values. In particular, measured moment of inertia can be 10% to 20% larger than calculated. The reason for this is that air has significant mass and alters the mass properties in two ways:

1. Air trapped **inside** the payload will increase its mass by an amount equal to the unoccupied volume in the payload times the density of air (0.0754 pounds per cubic foot). For example, the air trapped in a 4 foot diameter, 2 foot long satellite weighs approximately 4 lbs. We call this the <u>entrapped</u> air effect.

2. Air dragged or pushed along by any protrusions on the outer surface of the payload can dramatically increase moment of inertia. For example, the roll moment of inertia of a missile flying in air is much larger than the roll MOI of the missile in a vacuum. We call this the <u>entrained</u> air effect.

How you handle this difference depends on whether the payload operates in the vacuum of space or in air. If the payload flies in a vacuum, then measured values must be decreased to eliminate the effect of air mass. The best way of doing this is to make a second measurement in helium and then extrapolate the value in vacuum (see SAWE paper No. 2024 by Boynton and Wiener). Calculated values remain unchanged.

If the payload flies in air, then measured values remain unchanged and represent the true mass properties.

6.5 Minimum MOI which can be measured

Moment of inertia instruments have an amazing dynamic range. A MOI instrument which is designed to measure objects weighing up to 3000 pounds can often detect the change in MOI due to the addition of an object weighing 0.1 pound. However, accuracy is reduced when the MOI of an object is smaller than the tare MOI of the instrument. The error is primarily due to thermal expansion and contraction of the instrument and fixture during the time between tare and object MOI measurement. For example, if a payload has a MOI of 10 lb-in^2, and the instrument has a tare MOI of 10,000 lb-in^2, then a 0.1% change in tare due to an ambient temperature change will result in a 100% error in the measured MOI of the payload. Reducing short-term temperature change can increase the usable range of an instrument. We have found that improved accuracy can be achieved by simply shutting off the heating or air conditioning system during the interval of time between tare and object measurement. Temperature control systems that frequently cycle are not desirable.

6.6 Damping

Air bearing MOI instruments themselves have very small losses, and the effect of this damping can generally be ignored. For payloads which introduce significant damping though air turbulence while oscillating, the actual period of oscillation is greater than the undamped natural period by an amount determined by the damping ratio, z. If the torsion pendulum is being used as an instrument to measure moment of inertia, then the measured moment of inertia will be greater than the true value. This error can be eliminated if the following equation is used in place of equation XX. The quantity z^2 is the error.

$$I = C\,T^2\,(1-z^2)$$

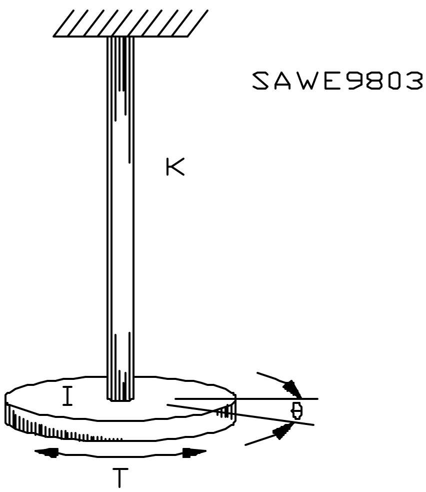


Figure 26

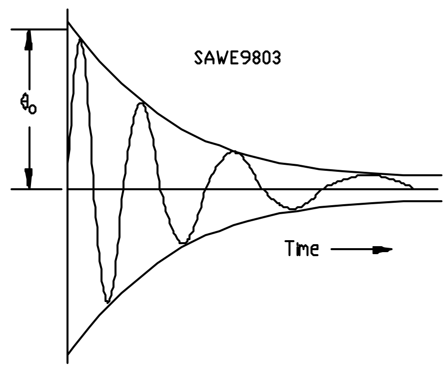


Figure 27

In order to make use of this equation, the value of the damping ratio, z, must be determined. This is accomplished by noting the rate at which the amplitude of oscillation decays. If we define the logarithmic decrement as the natural logarithm of the ratio of any two successive amplitudes, then the log decrement, d, of the starting amplitude, a_o, as compared to the peak amplitude, a_n, after n cycles have elapsed is given by the equation:

$$d = 1/n\,(\ln\ a_o/a_n)$$

For small values of z, the logarithmic decrement, d, can be related to z by the following relationship.

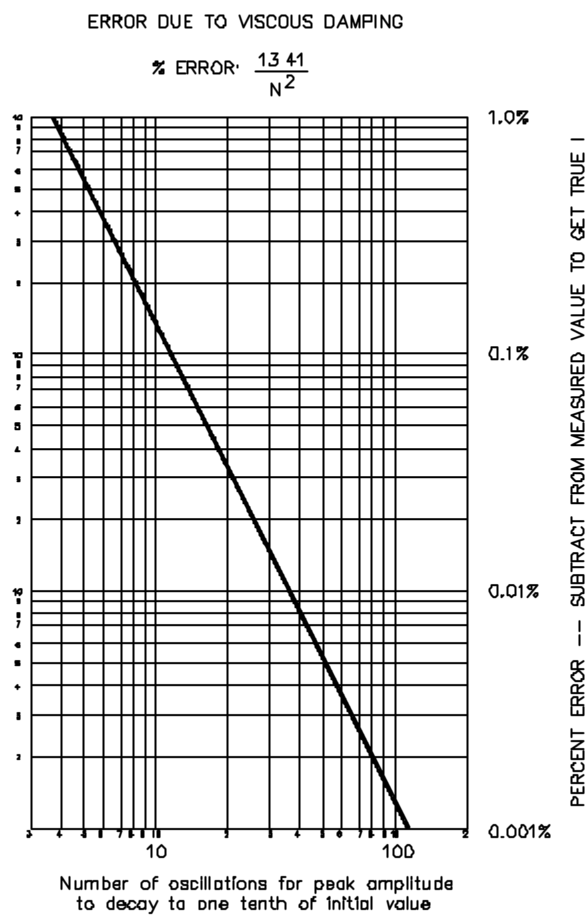
$$d = 2\,B\,z$$

If we now count the number of oscillations of our torsion pendulum, n, for a decay in peak amplitude of 10/1, we may combine the above equations and solve for the error resulting from damping.

% error due to damping : $100\, z^2$

$$\%\, e = \frac{(\ln 10)^2\, 100}{(2\pi n)^2} = \frac{13.41}{n^2}$$

A graphical solution to this equation is given in figure 28. To correct the measured value of moment of inertia, the amount shown on the graph should be subtracted from the measured value to yield the true value. Note that the error is insignificant if more than 50 oscillations are required for the amplitude to decay to one tenth of its original value.



6.7 Hanging wire torsion pendulum Although hanging wire pendulums are not accurate enough for satellite and missile measurements, they are useful for measuring the MOI of an aircraft. In fact, no torsion pendulum instrument currently exists which is large enough for aircraft measurement, so there is no choice. Fortunately, the MOI tolerance of an aircraft is not critical and the resulting accuracy is acceptable.

This method consists of hanging an object from a wire, twisting it to start it oscillating, and then timing the period of oscillation. Although it sound like a simple device, the structure required to support the upper end of the wire can be very expensive, and some accurate means is required to time the period of oscillation. One problem with the hanging wire method is that the object swings from side to side and rocks up and down rather than rotating smoothly about an axis, making it difficult to acquire accurate time period data.

It is essential that the center of gravity of the object be aligned horizontally with the center of the torsion rod. Otherwise, the moment the object is released, there will be a couple generated and the motion of the pendulum will be sideways as well as torsional. There is a serious practical problem when measuring heavy objects using this method: how do you attach the object and adjust its position so the CG is in the center of the rod?

A single hanging (steel) wire has a torsional stiffness (k) which is proportional to the fourth power of the diameter:

$$k = \frac{1,178,000 d^4}{L} \ inch\text{-}lb/radian$$ where L = length and d = diameter (inches)

The equation of motion for this pendulum is:

$$I = Ct^2 = \frac{Kt^2}{4\pi^2}$$

However, the actual period of oscillation will not follow this formula because the wire stretches under load, and there will be both a swinging pendulum effect and a rocking pendulum. Typical measurement uncertainty with this method is about 3%.

It is possible to hold the object level and eliminate the rocking mode of oscillation by using three wires rather than one. This is called a trifilar pendulum. It is the method most commonly used for large heavy payloads such as aircraft. Unfortunately, using three wires introduces a nonlinearity. As the pendulum twists, the wires restrain each other, causing the object to lift slightly. So the torsion constant becomes a function of test object weight as well as torsional stiffness. The oscillation period will increase as oscillation amplitude decreases. Therefore, it is necessary to use a timing system that triggers at a specific amplitude.

When using a single wire system, it is obvious when the object CG is misaligned, because the object will hang at an angle. However, if you use a trifilar pendulum, the object will not tilt appreciably, so some independent means is necessary to align the object CG such as incorporating load cells into the wires to adjust for equal loading of all three wires.

Shown here is a Space Electronics POI series Spin Balance Machine with a 12,000 pound capacity. An "L" Fixture permits 3-axis measurement by rotating the test object about its horizontal axis. The unique gas bearing design of the Space Electronics POI series Spin Balance Machines allows measurement of dynamic unbalance, product of inertia (POI), moment of inertia (MOI), and center of gravity (CG) with a single setup. Its slow spin speed (10 - 300 RPM) minimizes centrifugal forces on the payload during dynamic unbalance measurements.

7.0 Measuring Product of Inertia

There are two methods which can be used to measure POI:

1. The object can be rotated in a spin balancing machine, and the reaction forces measured against the bearings. POI can then be determined by performing calculations that involve the magnitude of the bearing forces and their phase relationship. For the measurement of rockets and satellites, the spin speed is usually about 100 RPM. This minimizes the effect of air turbulence. A special high sensitivity spin balance machine is required, which differs greatly in construction from the type of high speed balancing machine that is used to measure automobile crankshafts and jet engine rotors.

2. Objects such as control fins and satellites with extended solar panels cannot be measured using the spin method, because of the large, non-repeatable errors which are introduced by the entrained and entrapped air and turbulence. In these instances, product of inertia can be determined by making a series of moment of inertia measurements with the object oriented in 6 different positions. Product of inertia can then be calculated using formulas which involve the rotation angles of the different fixture positions. Moment of inertia is measured by oscillating the object on a torsion pendulum. Since the object moves very slowly during this measurement, there are negligible centrifugal and windage forces exerted on the object. Furthermore, the mass of the entrapped and entrained air can be compensated for by making a second set of measurements in helium, and extrapolating the data to predict the mass properties in a vacuum

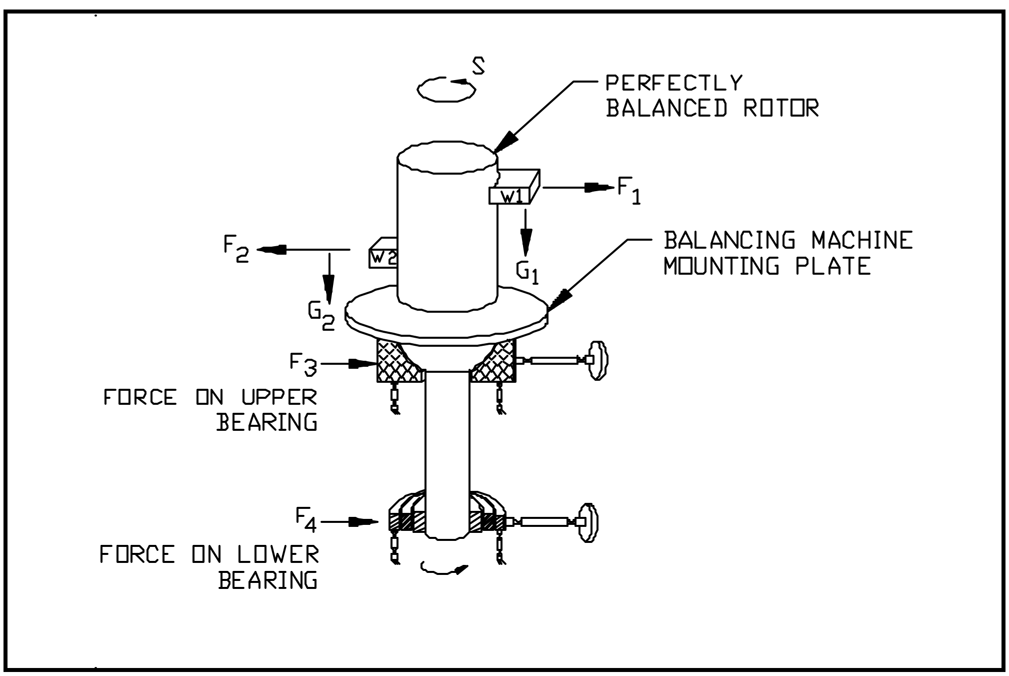
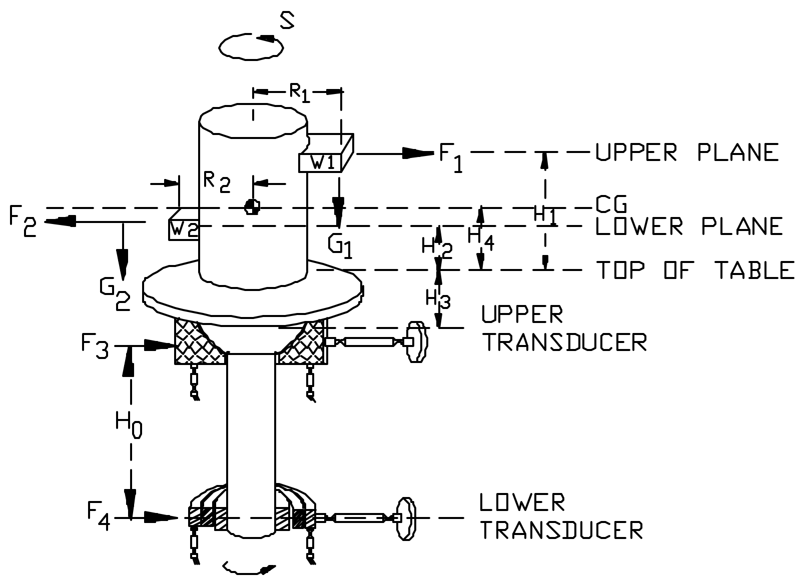


Figure 29

.SPIN BALANCE METHOD

Product of inertia is generally measured using a spin balance machine. In this type of machine, the object is rotated at a fixed speed, and the reaction forces against the upper and lower spindle bearings are measured. Product of inertia is then calculated automatically by the machine's on line computer, using formulas that involve the vertical spacing between the upper and lower bearings, and the height of the CG object above the mounting surface of the machine. The CG location of the part may be separated mathematically from the dynamic unbalance of the part.



7.1 Balancing Machine Theory -- When the test object spins, there are two forces acting through the CG of the object: gravity forces acting downward and centrifugal forces acting horizontally (the higher the spin speed, the less significant the gravity force is). The magnitude of the downward gravity force is equal to the weight of the object in pounds (M1). The magnitude of the horizontal centrifugal force is:

Centrifugal force (lbs) $F_1 = M_1 \times R_1 \times S^2$

where M_1 = mass of unbalance in slugs
R_1 = radius of CG in feet
S = speed in radians per second

Converting the mass into weight and the speed into RPM:

$$(\text{lbs})\ F_1 = \frac{W_1 \times R_1 \times (\text{RPM})^2}{35207}$$

where W_1 = weight of unbalance mass in lbs
R_1 = radius of CG of unbalance in inches
RPM = speed of rotation in RPM
35207 = constant to transform units

The forces applied to the bearings of the balancing machine will depend on the geometry of the balancing machine and the type of unbalance. If neither a CG offset or a product of inertia unbalance is present in the rotating test object, then the forces on the bearings will be zero. A product of inertia (with no CG offset) results in equal forces being applied 180 degrees out of phase on the two bearings (the couple due to the product of inertia is offset by an equal couple on the bearings of the balancing machine). A CG offset will cause a different force to be applied to each bearing. In order to evaluate the accuracy of a balancing machine in measuring CG, it is necessary to know the relative values of POI and CG offset. For a flywheel, POI is usually small, and the balancing machine will be able to accurately measure CG offset. For a tall rocket, the reverse is true.

If the goal of the measurement is to ballast the test object for minimum POI and CG offset about a particular axis, then the measurement becomes much more accurate. As the unbalance is reduced in successive iterations, the residual unbalance can be measured on a more sensitive scale and the magnitude of POI can be reduced so that some CG sensitivity results.

To calculate the CG offset from the measured bearing forces, we can make use of the fact that the sum of the moments around each transducer must be zero (since the system is stable). Independent calculations can be made at the X and Y axes, and the resultant of these two calculations determined to yield the magnitude and angle of the CG offset.

7.1.1 Two transducers are required to separate POI from CG offset In order to separate the unbalance due to POI from the unbalance due to CG offset, it is necessary to use a force transducer on each of the spindle bearings in the balancing machine. Then both CG offset and product of inertia are obtained in a single measurement. If two measurements are made, then it is theoretically possible to use a single transducer to measure both CG and product of inertia. Static measurements are first made to determine CG offset. Since the gravity component of CG offset does not change with speed, it could be concluded that this component can be separated from POI. In the real world, however, it doesn't work. The reason is that the large forces resulting from the CG offset must be subtracted from the smaller forces due to POI. Since the CG offset is measured under static conditions where forces are small, there is an error. This small static force error becomes a very large error when spinning at 100 RPM.

7.1.2 Errors due to air turbulence The force on the transducers of the balancing machine increases as the square of spin speed. In some POI machine designs, the transducers chosen are velocity pickups whose sensitivity also increases with speed. The net effect in this case is that the sensitivity of these balancing machines increases as the cube of the speed. At first it might seem that there would no limit to the sensitivity which could be obtained with this method. However, there are a number of factors which limit the speed of the machine.

The machine must be rigid enough not to go into resonance at high spin speeds. The test item must be strong enough to withstand the centrifugal forces. The outer surface of the test item must be smooth and round, so that the test item will not fan the air and create turbulence. Solid metal shafts, such as turbine rotors, are smooth and round and may be spun at a high speed when balancing them. Aerospace hardware, however, usually cannot be spun above 200 RPM without damage occurring, and windage forces on the irregular outer surface of the vehicle increase as the square of the spin speed, so that the random vibration produced by air turbulence provides another limitation.   This vibration limits the real sensitivity of the balancing machine.

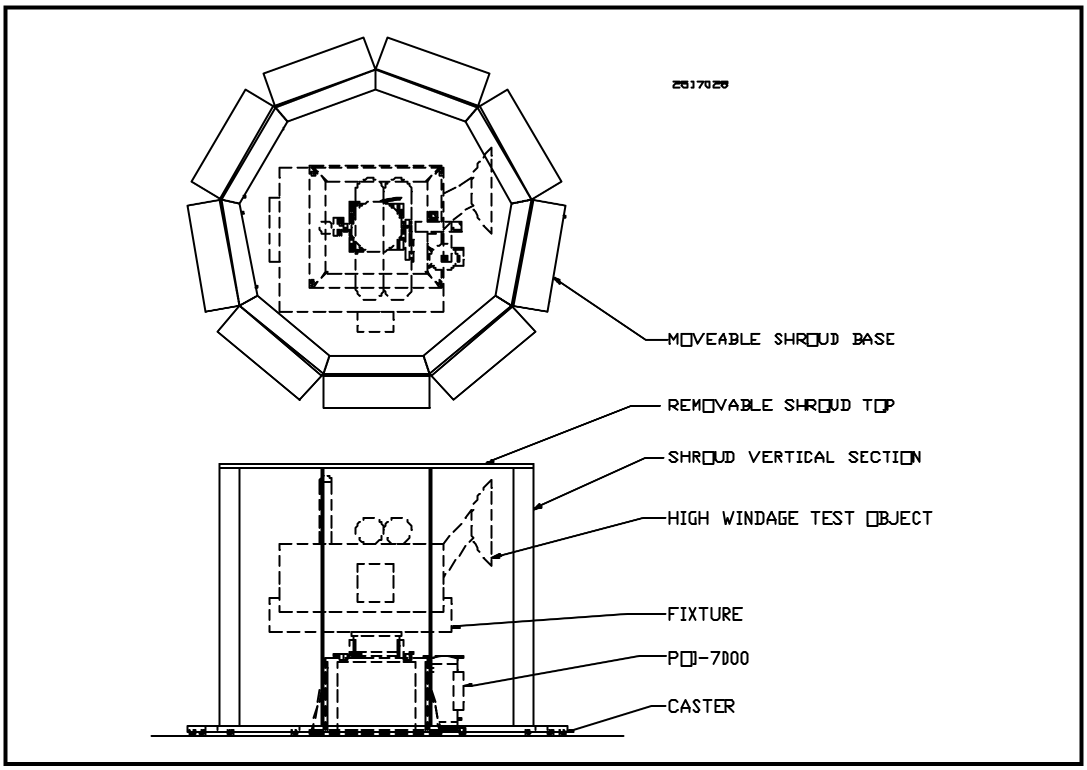


Figure 31 Typical Shroud Installation

The error due to air mass can be minimized by surrounding the spin balance machine with a shroud. Typical shrouds are cylindrical in overall shape and adjustable in height and radius so that they can be made to enclose the spinning object as closely as possible. Additional improvement can be obtained by placing a circular cap on the shroud to close the overhead opening.

The objective is to enclose the smallest possible volume of air, while allowing the test object to spin with adequate clearance. The walls of the shroud must approximate a smooth cylinder as much as possible. This will allow all of the air within the shroud to turn at the same rate as the test object and minimize the resulting forces on the transducer. Note: the need for a shroud is not a special requirement of a particular balancing machine. Any balancing machine manufactured by <u>any</u> company will have the same limitation.

SUMMARY OF BENEFITS AND SHORTCOMINGS

Benefits

1. By selecting spin speed, machine can be either low sensitivity/high offset range or high sensitivity/low offset range.
2. Sensitivity can be very high at high spin speeds. However, in many cases maximum spin speed is limited by structural limits or the windage of the test object.
3. Measures CG and well as product of inertia.

Shortcomings

1. Air turbulence during spin produces uncertainty in measurement;
2. Some objects cannot be spun because solar panels or other protrusions would break off due to windage forces, or they cannot tolerate centrifugal or vibratory force that occurs at the spin speed.
3. Spin balance machines are expensive.

7.2 Moment of Inertia Method of Measuring POI

This method uses a torsion pendulum to determine POI by making use of the relationship between POI and MOI of an object. Special fixtures must be constructed to move the object to a number of positions while keeping both the object and the fixture CG near the center of oscillation. The moment of inertia of the object is measured in each orientation. The tare MOI of the fixtures must then be measured and subtracted from the measurement with the object. The net MOI of the object in the different orientations is then used to determine the POI of the object The calculations are quite complex, so an on-line computer is used.

To better understand the concept, refer to Mohr's Circle on the following page. The axes of minimum and maximum moment of inertia correspond to the axes where the product of inertia is zero. These are called the principal axes. The product of inertia is a maximum at an angle of 45^{o} from these principal axes.

If the test part were fixtured so that it could be rotated through an angle C about a horizontal axis (i.e. the Z axis) and MOI measured about numerous axes in the X-Y plane, including the X and Y axes, the MOI would be found to vary sinusoidally. If the angle C ranges over 180 degrees, the maximum and minimum values of MOI can be seen in a plot of MOI vs C.

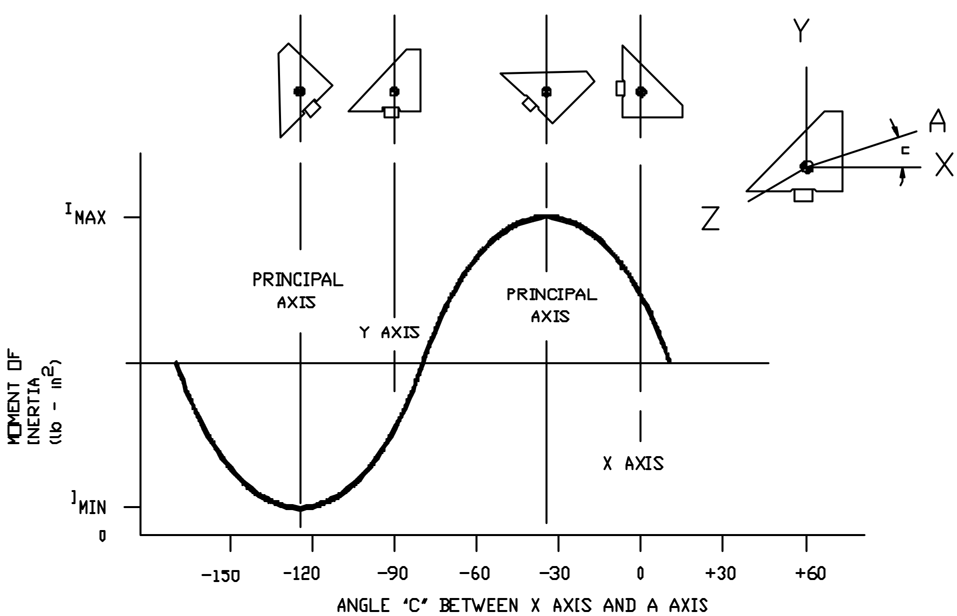
The axes about which the maximum and minimum MOIs are measured are the Principal Axes. For all other axes the moment of inertia I_{Axy}, about an axis (A) in the X-Y plane at an angle C from the +X axis, and the product of inertia P_{xy}, are related through the equation:

$$I_{Axy} = I_{yy}\sin^2 C + I_{xx}\cos^2 C - P_{xy}\sin 2C$$

Solving this equation for P_{xy} forms the basis for the MOI method of POI determination.

$$P_{xy} = \frac{(I_{yy}\sin^2 C + I_{xx}\cos^2 C - I_{Axy})}{\sin 2C}$$

7.2.1 NUMBER OF MOI MEASUREMENTS For the general case, the total number of MOI measurements needed for POI calculations is nine: three in each of three mutually perpendicular planes. If the intersections of these planes are selected to be the coordinate axes, then the MOI about each of these axes will be common to two planes, thus reducing the total number of measurements to six: three about the X, Y, and Z axes, and three about axes at 45 degrees between the X-Y, Y-Z, and Z-X axes. If vacuum data is required, the same six MOI measurements must also be repeated in a helium atmosphere.



7.2.2 Mohr's Circle for Moments of Inertia

Given:

(1) The moment of inertia values I_X, I_Y for an object about its center of gravity, where the center of gravity lies at the origin of a set of mutually perpendicular axes X-Y.

(2) The corresponding value for the product of inertia, P_{XY}.

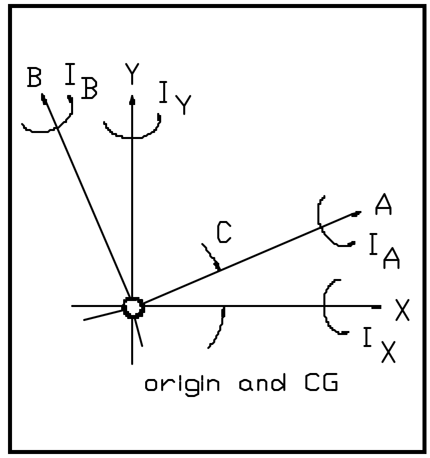


Figure 33

Mohr's circle is then constructed using the layout geometry shown below. The following information may then be obtained.

(1) The location of the principal axes about which the moments of inertia are maximum and minimum and the products of inertia are zero.

(2) The corresponding maximum and minimum values of moments of inertia.

(3) The moments and products of inertia for any other set of mutually perpendicular axes A-B whose origin lies at the center of gravity of the given object and rotated C degrees from the original axes X-Y. Reference, the figure to the right.

(4) The maximum values for the products of inertia about axes located 45° from the principal axes.

Layout Geometry

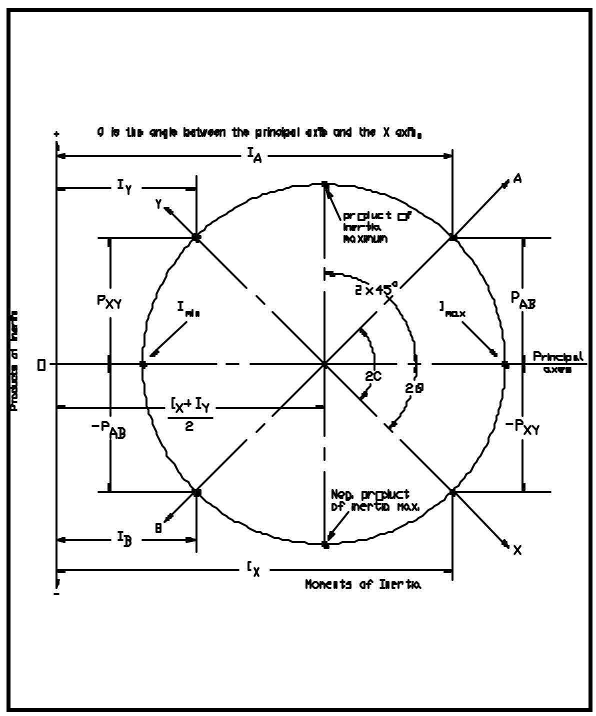


Figure 34

The radius of the Circle is:

$$R = \sqrt{\left(\frac{I_X - I_Y}{2}\right)^2 + (P_{XY})^2}$$

SUMMARY OF BENEFITS AND SHORTCOMINGS

Benefits

1. The cost of a moment of inertia instrument is much less than the cost of a spin balance machine. If you already have a moment of inertia instrument, it can be configured to measure POI at a relatively minor cost. (However, the cost of fixturing and labor is considerably greater than for spin balancing machine measurements).
2. This method puts a minimum stress on the object being measured and is ideal for fragile satellites which cannot be spun on a balancing machine.
3. The effect of air mass and turbulence is minimized.

Shortcomings

1. Accuracy is poorer than with a spin balance machine.
2. Each measurement requires the test object to be positioned in up to nine different positions, so that elaborate and expensive fixturing is required.
3. The test is <u>very</u> tedious to run and labor cost is high. Measuring the POI of a single object can require as much as 10 hours including set up.
4. The instrument does not give a direct readout of POI, so that corrections cannot be made to the object and the resulting POI shift observed.
5. Error increases dramatically if the machine is even slightly out of level.

8.0 Considerations in Choosing a Mass Properties Instrument

There are a wide variety of mass properties measuring instruments available today. The choice of which one to use in part depends on what properties you want to measure, the accuracy required, the degree of automation required, and budgetary restrictions. If you are measuring CG, you need to determine whether you need to measure along a single axis, or along more than one axis (some CG instruments are only capable a single axis measurement). In addition, you need to choose the size of the instrument. This is usually governed by the weight of the largest object you need to measure.

Since CG sensitivity decreases as the size of the instrument gets larger, the selection of an instrument may involve a tradeoff. Frequently, management attempts to purchase an instrument which will serve present and anticipated needs. This may result in the selection of too large an instrument, resulting in limited accuracy for the present requirements.

8.1 What properties you want to measure -- There are instruments available which measure only one mass property; Center of Gravity Location (CG), Moment of Inertia (MOI), Product of Inertia (POI), Weight or Mass. Many instruments can measure 2 or more of these properties due to inherent characteristics or by adding relatively low cost options. Using a combined function instrument often eliminates the effort, cost, and risk involved in moving the test item to another instrument and/or fixture. Some of the most common dedicated and combined instruments measure:

Dedicated

CG (measured statically) only*
MOI only
POI and CG (measured dynamically)
Weight only

Combined

CG and Weight
MOI and CG (measured statically)
POI, and CG (measured both statically and dynamically)
MOI, POI, and CG (measured dynamically)
MOI, POI, and CG (measured both statically and dynamically)

* Static measurement of CG depends on measuring a moment balance condition where forces are generated only by gravity. Dynamic measurement of CG involves moments generated by gravity *and* by centrifugal forces generated as the test article rotates.

Need to measure both CG and MOI -- In order to measure MOI, you will need an instrument with a gas bearing rotary table. Instruments such as the Space Electronics KSR series have this combined capability.

Need to measure both CG and weight -- Multiple point weighing type instruments have this capability. These instruments are available with three load cell technology or with the more accurate force cell technology (Space Electronics WCG series).

Need to measure both CG and POI -- If you have a requirement to measure dynamic balance as well as CG, then a spin balance machine would be a good solution, particularly if your goal was to ballast the test object for minimum POI and CG offset about a particular axis. Space Electronics POI series instrument are available with a separate static CG feature. This improves CG measurement accuracy over what would normally be available with a spin balance machine.

Need maximum CG accuracy --Instruments with a gas bearing rotary table and force restoration moment measuring technology are the most accurate. Space Electronics KSR series instruments are of this type.

Need an explosion proof design -- All types are available with this option. Since repositioning CG instruments are purely mechanical, they are intrinsically safe in explosive environments.

Need to measure CG of objects weighing more than 25,000 pounds -- Three-point reaction force machines have been made to measure objects as heavy as the space shuttle. Gas bearing rotary table machines are limited to about 25,000 pounds because of practical problems in building gas bearings larger than this.

9.0 Weight Measurement

9.1 Types of scales Purely mechanical scales have largely gone the way of the dinosaur. Nearly all scales currently in use have electronic digital displays and computer interface capability. The most common scales still use strain gage load cells with typical accuracy of one part in 2000. These scales lend themselves to computer interfacing at relatively low cost. Pulsed DC power supplies and linearization circuits have allowed accuracies to one part in 5000 at slightly higher cost.

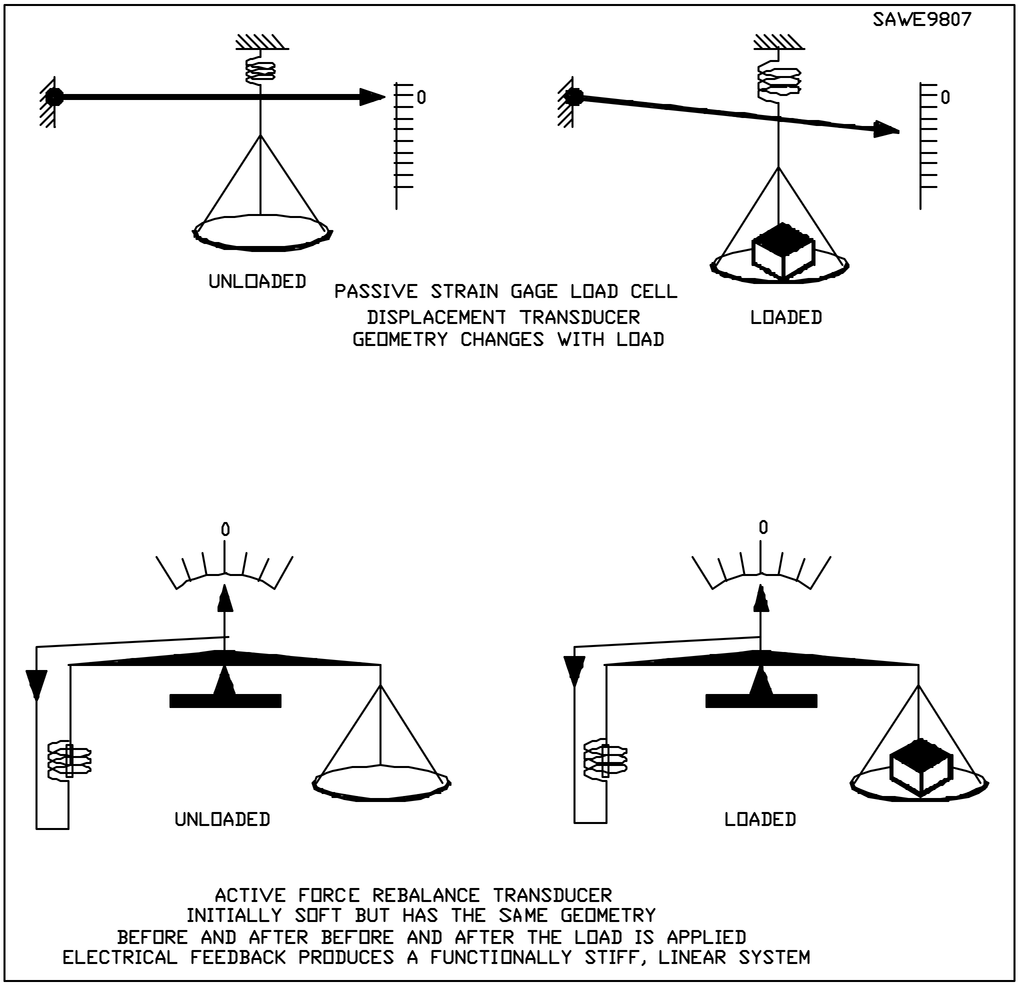
The next level of price with improved rangeability comes with new ceramic capacitive strain gages (for purposes of this paper, we will define rangeability as the ratio between load capacity and accuracy). These scales can be made with accuracies up to one part in 50,000. Transducer stiffness is comparable with strain gage beam cells. These transducers have one drawback: they can be damaged if a hard object is dropped on the scale, since the ceramic spring is brittle and cannot withstand shock. Some newer scales using this technology incorporate spring shock dampers to eliminate this problem.

The biggest innovation has been the application of force restoration technology to weight measurement. This technology has been in use since the late 1950's in both electronic and pneumatic process control transducers. The newest generation of electronic force rebalance transducers can achieve accuracies on the order of one part in 10 million in laboratory balances. In the more common bench scale ranges up to 25 lb, the accuracy can approach one part in 1 million. In the larger sizes, 75 to 13,000 lb, accuracies are typically one part in 25,000 to one part in 50,000. At this time, the maximum load ratings available are on the order of 13,000 lb.

These scales are highly programmable to accommodate many weighing conditions: stability (i.e. animal weighing), parts counting on weight basis, selectable units of measurement, etc. They are fully compatible with computer interfacing. The major disadvantages are price, and slow response time. The slow response time is due to the fact that a closed loop rebalance circuit is used.

9.2 Force Restoration Principle When a load is applied, the transducer deflects. A current driven restoring force is applied through a closed loop control system until the unloaded geometry is restored. The applied current is then related to the applied force. Since the loaded geometry after the restoring force is applied is the same as the unloaded geometry, the transducer is inherently linear like the time honored balance beam scale. This is unlike the strain gage load cell which relies on the deformation of the sensitive spring element to generate an output. High accuracy mass properties measuring instruments for static CG and moment measurement use this force restoration technology.

9.3 Comparison of scale types The table below compares scales with the three transducer types described above.. Relative cost is a comparison of the cost of a given scale to a no frills strain gage scale of the same load capacity.


SAWE9807
0
UNLOADED
PASSIVE STRAIN GAGE LOAD CELL
DISPLACEMENT TRANSDUCER
GEOMETRY CHANGES WITH LOAD
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UNLOADED
0
LOADED
ACTIVE FORCE REBALANCE TRANSDUCER
INITIALLY SOFT BUT HAS THE SAME GEOMETRY
BEFORE AND AFTER BEFORE AND AFTER THE LOAD IS APPLIED
ELECTRICAL FEEDBACK PRODUCES A FUNCTIONALLY STIFF, LINEAR SYSTEM

Figure 35

Summary Table of Weight Scale Characteristics

Transducer Type	Load Range	Typical Rangeablility	Relative Cost	Comments
Strain Gage Load Cell	up to millions of lbs	1/2,000 to 1/5,000	1 to 2	- Least rugged - Insufficient sensitivity for laboratory scales - Nearly unlimited capacities and configurations available.
Capacitive Load Cell	Fractional to 50 lb	1/10,000 to 1/50,000	1.3 to 2.5	- Sensitive to shock - limited capacities available - high rangeability
Force Rebalance Load Cell	Micro gram to 13,000 lb	1/20,000,000 to 1/20,000	1.5 to 5.0	- Wide range of capacities available - highest rangeability - not suitable for dynamic measurements - Most optional features

9.4 Corner Loading Error When a scale manufacturer quotes the accuracy of his scale, usually he is referring to the accuracy when an object is placed on the platform so that its CG is in the center of the scale. If you place the object off center, then a moment will be created which tends to tip the platform of the scale. On many scales this will introduce an error. Depending on the internal mechanism in the scale, this error can be as large as 0.5%. High quality scales use parallel beam flexures and other compensating mechanisms, so that this effect can be as small as 0.001%. You can test the scale you are using by first placing a test weight in the center of the scale and measuring its weight. Then you move the weight to each of the corners of the scale and remeasure the weight. You may be surprised at how large the change is.

9.5 Weight vs Mass The mass of an object is fixed and is the same whether the object is on the earth or in outer space. Weight, on the other hand, is a force which depends on several factors which are related to the location of the scale. With the advent of force restoration technology, scales such as the Space Electronics YST Series have uncertainties in the order of 0.003% of full scale. With these scales, it is possible to measure an object at different locations of the earth and observe significantly different values for weight.

The force a mass exerts on a scale is affected by four factors:

1. The gravitational mass attraction to the earth at the particular location, which is in part related to the altitude

2. The gravitational mass attraction to the sun and moon at the particular location, which may reach 0.003% of the acceleration of earth gravity at certain dates during the year when the sun and moon align

3. The centrifugal force due to the rotation of the earth, which varies from zero at the north pole to a maximum value at the equator

4. The buoyancy of the object as it floats in a sea of air. This can be compensated for by determining the enclosed volume, and calculating the weight of the displaced air (whose density can vary due to the weather).

These factors combine to result in a change in the weight of an object of almost 1% over the surface of the earth, and about 0.2% over the contiguous USA! We put the exclamation point at the end of the sentence, because we frequently see specifications for weight accuracy of 0.02%, and none of these specifications mention the location on earth where this measurement is to take place. If a mass weighs 100 pounds at one location is the USA, it could weigh 99.8 lbs somewhere else in the USA.

To get around this problem, and contrary to popular beliefs, the world, including the USA, uses *Mass* rather than *Weight* to standardize and calibrate scales. To calibrate a scale, a standard calibration mass is placed on the scale. The scale is then adjusted until it reads the appropriate *standard weight*. The standard weight is the weight the mass would have at standard gravity (32.174 ft/sec^2).

Note: the traditional "Scales of Justice" balance beam compares one mass against another mass, and therefore the measurement does not vary with changes in gravitational field strength.

A problem occurs when a scale is calibrated at one location and then moved to another location to weigh an object. For large scale capacities, it is often not possible to bring a calibration weight to the new site, either because this weight is not available, or because of the problems of shipping a calibration mass weighing many thousands of pounds. Therefore, it is necessary to correct for the change in the acceleration of gravity between the site where the scale was calibrated, and the site where the object is being measured. The National Geodetic Information Center in Rockville, Md has data on the weight correction required for many locations on earth. If this data is not available, then another method is to determine the correction is to calibrate a small scale at the first site, and ship this scale to the new site, together with its calibration weight. The small calibration weight is then remeasured. If the measured value of this small weight is 0.05% high, then the acceleration of gravity is 0.05% higher at this new location, and the measurement of the large object must be divided by 1.0005 to correct for this change in the acceleration of gravity.

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Trifilar Pendulum: Measurement and Error Analysis

Myles Cooper and Paul Titchener

December 17, 2013

1 ABSTRACT

Trifilar pendulums are easy-to-make devices that can be used to experimentally derive the moment of inertia of an object about its center of mass. However, the equations of motion used to obtain this datum make use of several simplifying assumptions. In this project, the full frequency response of the system is recorded and the amplitude decay over time is fit to a simple viscous damping model using a custom quadrature encoder and MATLAB, respectively. This model is then used to calculate damping ratio equation 1.2, which is used to adjust oscillatory period in an attempt to make moment of inertia calculation more accurate. The equation that determines the moment of inertia of the object being measured is:

$$I_{object} = \frac{R^2 g \tau^2}{4\pi^2 L}(M_{platform} + M_{object}) - I_{platform} \tag{1.1}$$

The equation that determines the damping factor of that measurement is:

$$\zeta = \frac{-c * \tau_d}{\sqrt{c^2 * \tau_2^2 + 4\pi^2}} \tag{1.2}$$

The results of this project implied that, for the system tested, forcing the damping ratio to fit a viscous damping model based on amplitude decay does not change the error percentage of moment of inertia calculation by a statistically significant amount. Subsequently, error calculations were performed which concluded that measurement error is worse when the ratio of mass to moment of inertia is high.

2 Background

This project involved building and testing a trifilar pendulum. At its simplest, a trifilar pendulum is a platform rotating about a point via three evenly-spaced vertical strings attached to the outsides of the platform. When the disk is rotated, the strings are pulled to an angle and the disk is lifted very slightly. Then when it is released, the angled strings put a torque on the platform that turns it in the other direction, twisting it up a small distance on the other side of its equilibrium. Then it oscillates for a period of time as determined by damping. The period of this oscillation is proportional to the moment of inertia, such that the moment of inertia can be determined through measuring the period of oscillation.

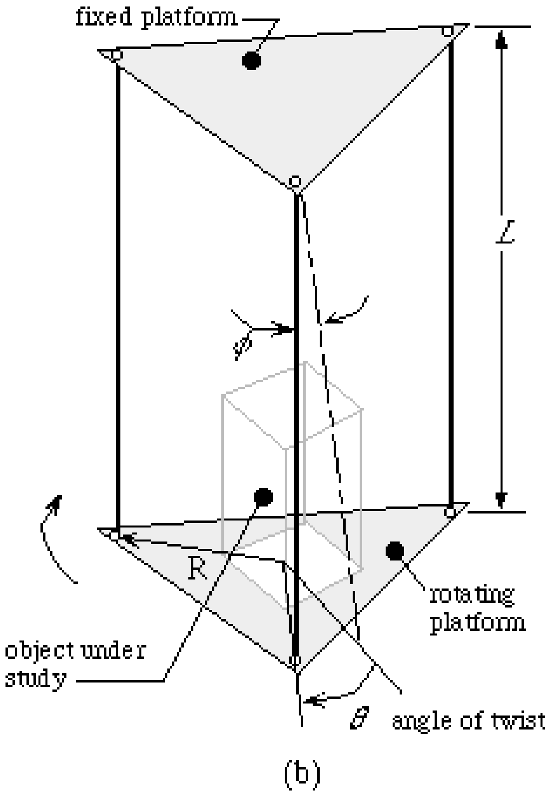

Figure 2.1: A basic trifilar pendulum diagram.
http://www.me.utexas.edu/ me244L/labs/filar/filaroverview.html

These measurement systems are commonly used for determining moments of inertia in many fields such as the automotive or aerospace industries. In these situations, an object is put into a special frame such that it can be suspended about many different axes and then entire moment of inertia tensor can be measured. [4] From the equations of motion, a relationship between the period of oscillation, mass of the system, and moment of inertia about the central axis can be made, shown below. This relationship can be used to measure the moment of inertia about the center of mass of any item placed on the disk.

$$I_{object} = \frac{R^2 g \tau^2}{4\pi^2 L}(M_{platform} + M_{object}) - I_{platform} \tag{2.1}$$

Where:
I refers to moment of inertia through center of mass about the z-axis of the system
R is the distance from the center of the disk to each string
g is acceleration due to gravity (approximated to by 9.8 m/s2)
τ is the period of oscillation
L is the length of the strings
m refers to mass
Reaching the general equation of motion of the disk would be quite complicated as the motion is three dimensional and non-linear. However, the motion of the disk can be simplified to two dimensions by using small angle approximation and a very large ratio between the length of the strings and the radius of the disk. These approximations are commonly accepted as reasonable and are used in this project. [2] [3] The derivation of the equations of motion for this project was based on the assignment detailed in a class design project from Brown University[1].

3 Equation of Motion Derivation

This derivation determines the moment of inertia of an object from the period of its oscillation on a trifilar pendulum. There are several important assumptions to note. First, the small angle approximation is used for both $sin(\theta) = \theta$ and $cos(\theta) = 1$. In addition, the vertical motion of the platform is assumed to be zero, due to the small angle approximation.

3.1 Diagram

This diagram is taken from a Brown University Assignment on finding the equations of motion of a trifiar pendulum.

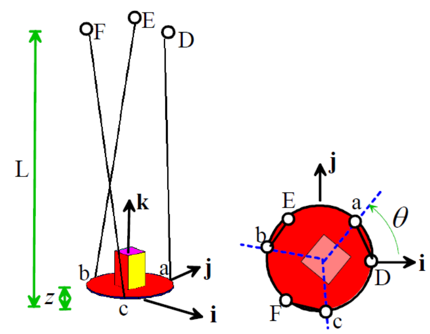

Figure 3.1: A basic trifilar pendulum diagram [1]

3.2 Location of Fixture Points

The derivation begins by finding the locations of the fixture points of the ends of strings. The reference frame used throughout this derivation is centered at the center of the platform in its static equilibrium with the z axis pointed upwards and the x axis pointed towards connection point A.

$$Rd = \begin{bmatrix} R \\ 0 \\ L \end{bmatrix} \tag{3.1}$$

$$Re = \begin{bmatrix} -.5R \\ .866R \\ L \end{bmatrix} \tag{3.2}$$

$$Rf = \begin{bmatrix} -.5R \\ -.866R \\ L \end{bmatrix} \tag{3.3}$$

3.3 Location of Fixture Points on the Disk

$$Rd = \begin{bmatrix} R * cos(\theta) \\ R * sin(\theta) \\ z \end{bmatrix} \tag{3.4}$$

$$Re = \begin{bmatrix} R * cos(\theta + \frac{2\pi}{3}) \\ R * sin(\theta + \frac{2\pi}{3}) \\ z \end{bmatrix} \tag{3.5}$$

$$Rf = \begin{bmatrix} R * cos(\theta - \frac{2\pi}{3}) \\ R * sin(\theta - \frac{2\pi}{3}) \\ z \end{bmatrix} \tag{3.6}$$

3.4 Unit Vectors along Strings

The unit vector along each string is equal to the position of the upper fixture point minus the disk fixture point divided by the length of the string.

$$S1 = \frac{Rd - Ra}{L} = \begin{bmatrix} R - R * cos(\theta) \\ -R * sin(\theta) \\ L - z \end{bmatrix} * \frac{1}{L} \tag{3.7}$$

$$S2 = \frac{Re - Rb}{L} = \begin{bmatrix} -.5R - Rcos(\theta + \frac{2\pi}{3}) \\ .866R - Rsin(\theta + \frac{2\pi}{3}) \\ L - z \end{bmatrix} * \frac{1}{L} \tag{3.8}$$

$$S3 = \frac{Rf - Rc}{L} = \begin{bmatrix} -.5R - Rcos(\theta - \frac{2\pi}{3}) \\ -.866R - Rsin(\theta - \frac{2\pi}{3}) \\ L - z \end{bmatrix} * \frac{1}{L} \tag{3.9}$$

3.5 Small Angle Assumptions

At this point, certain assumptions must be made to linearize the model. These all stem from the small angle approximation of $sin(\theta)$ and $cos(\theta)$. The final result of this is that the vertical travel of the pendulum can be assumed to be zero.

$$z = 0 \tag{3.10}$$

From this we know that:

$$\ddot{z} = 0 \tag{3.11}$$

Therefore, the pendulum is not accelerating vertically and the equation F = ma can be used to determine the tension in the strings.

$$3T - mg = 0 \tag{3.12}$$

$$T = mg/3 \tag{3.13}$$

At this point we can use an Euler equation, as we know the direction of the moment arms, the direction of the tension force and the magnitude of the tension force.

$$I\ddot{\theta} = T\frac{R_a \times (R_d - R_a)}{L} + T\frac{R_b \times (R_e - R_b)}{L} + T\frac{R_c \times (R_f - R_c)}{L} \tag{3.14}$$

When these cross products are evaluated, the result is that:

$$I\ddot{\theta} + \frac{mgR^2}{L}\theta = 0 \tag{3.15}$$

When this equation is solved the final result for moment of inertia is that

$$I_{object} = \frac{R^2 g\tau^2}{4\pi^2 L}(M_{platform} + M_{object}) - I_{platform} \tag{3.16}$$

4 Learning Objectives

The goal of this project was to build a trifilar pendulum and determine how much the accuracy of the resultant moment of inertia calculation could be improved by taking damping into account. Another goal was to automate our calculation as much as possible by making a custom quadrature encoder that recorded data from each run, sent it to an Arduino Uno, and then had a computer process the data in MATLAB to yield damping ratio. This automation makes it very easy to run multiple tests on varying objects and decreases the labor needed to analyze each run. A final goal was to run error analysis on the data and draw conclusions about the accuracy of the experimental setup overall.

5 System Model

This model takes into account length of the strings, radius of the platform, moment of inertia of the platform, and the moment of inertia of the test object. It specifically neglects displacement on the z-axis, makes small-angle approximations for trigonometric functions of θ, and ignores all sources of damping (friction in strings, air resistance, and, in our case, friction from our stabilization peg). Damping was calculated but was determined to be statistically insignificant.

5.1 Mechanical

The pendulum used was built from relatively low-cost materials. A top and bottom base support were measured and cut from MDF as truncated equilateral triangles (with an extension on the bottom base to support an encoder). The support beams are pre-cut wood banister beams, chosen for their straightness, consistent length and square ends. The supporting wire is Kevlar spear fishing wire. This cable was selected because of its low stretch. Our model does not account for stretch in the strings and this string reduced error from that source. This wire may have added more frictional error than expected due to its large diameter and stiffness, which will be discussed later in results. The plate is .118" laser cut acrylic. A steel pin was placed through the center of the plate and into a delrin block to horizontally constrain the plate's motion to rotation about a fixed point. We determined the moment of inertia of or plate very precisely through first creating the system in SolidWorks, and then creating a custom material to ensure that the density of our acrylic was being accurately accounted for.

Figure 5.1: Mechanical System

6 Electrical System

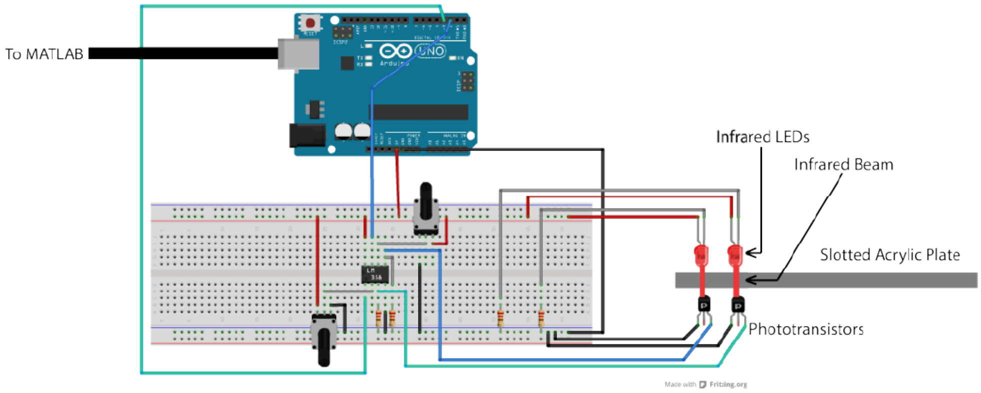

Figure 6.1: The circuit for the measurement system of the pendulum. Two IR leds are positioned such that they will return four different counts over a rotation of 3 degrees. The information from the diodes is send to an op-amp that sets a cutoff voltage above which the signal to the arduino is 5v and below which it is zero. The arduino then logs the counts and determines what direction the system is moving in and how many counts have passed. This data is then sent to matlab which plots the position of the system and determines the period.

Measurements were taken from the spinning disk by a quadrature encoder made from two well-placed LED-photo-transistor pairs mounted on a custom-milled aluminum block that allowed the disk to spin in between the photo-transistors and LEDs. The disk was laser cut from acrylic with precisely-designed slots that allowed the encoder to detect direction and amplitude of oscillation as the disk rotated over time. The LED-phototransistor pairs transmitted signal to an Arduino Uno corresponding to whether or not the LED-to-photo-transistor beam was blocked by the acrylic plate at a given instant. A simple amplification circuit was placed between these pairs and the Arduino so that the signal from the pairs would arrive as a well-defined square wave signaling that a pair was 'blocked' or 'unblocked'. The Arduino then processed the relative signals of the pairs to determine amplitude over time. These data were then transmitted to MATLAB, which plotted the data for verification and processed the numbers to calculate moment of inertia and error. This system was very effective in that data could be captured quickly and easily. It was also accurate to less than a degree. The only major issue with it was that when the disk spun too rapidly, the encoder would miss counts and give unreliable data. However, this only occurred when the disk was displaced by significantly more than an acceptable angle for small-angle assumptions. In this report, it is assumed that the encoder did not miss counts and that the data are accurate.

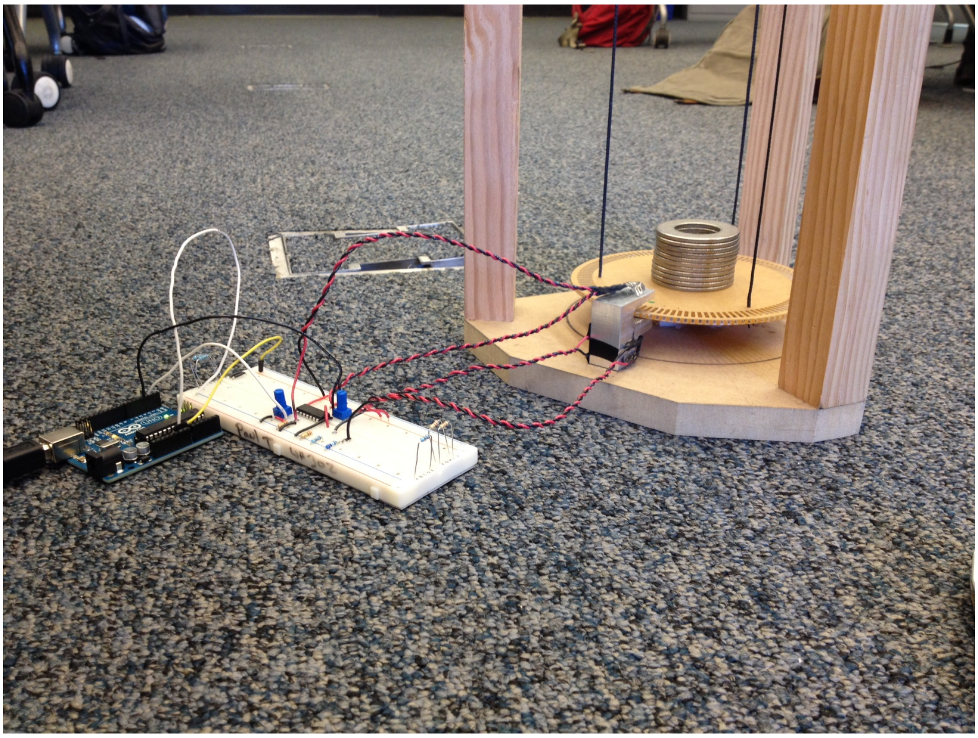

Figure 6.2: A close up image of the encoder and plate working together.

7 Results

Using the quadrature encoder and MATLAB processing, full frequency responses were extracted for analysis. MATLAB calculated time between peaks and averaged them to come up with period of oscillation. It also used the "peaks" function to trace the degradation of amplitude over time. By assuming that this decay fit the form $Xe^{-\zeta\omega_n t}$ (viscous damping model), the natural logarithm of that curve could be fit to a first order polynomial with the "polyfit" MATLAB function and a damping ratio could then by calculated by:

$$\zeta = \frac{-c * \tau_d}{\sqrt{c^2 * \tau_2^2 + 4\pi^2}} \tag{7.1}$$

Where:
c is the coefficient of the linear term produced by fitting the natural logarithm of the amplitude degradation to a line.
T_d is the measured (damped) period of oscillation.
ζ was then used to calculate the natural period of oscillation using:

$$\tau_n = \tau_d \sqrt{1 - \zeta^2} \tag{7.2}$$

To calculate results, we first plotted the data in MATLAB and then determined peaks. Finally, we determined the time between the peaks to find the final period of oscillation. An example of this raw data is shown below. Damping values were also calculated but were statistically insignificant so have not been heavily analyzed.

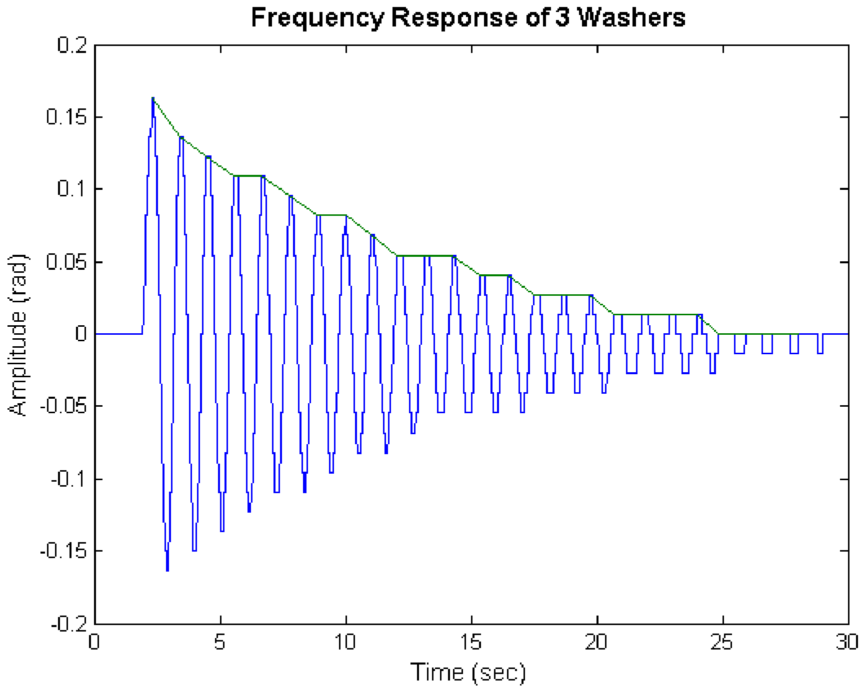

Figure 7.1: Frequency response with its corresponding amplitude decay curve. All runs looked similar to this, varying in frequency and specific decay curve.

Theoretical moments of inertia were calculated for several samples by massing each with scales and measuring geometry with digital calipers. Then, moments of inertia (MOI) were experimentally derived using the trifilar pendulum and error percentages were calculated using both the measured period of oscillation and the corrected period using ζ. Finally, a student's t-test was used to determine if the moment of inertia derived with the corrected period was significantly different from the moment of inertia derived with the original, damped period. In all cases, $p>.5$, so they are not significantly different. Therefore, in all following analysis, the damped frequency will be used for measurements. The tables produced are shown below:

Wood Block

Trial #	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.039839
1	6.0900E-05	6.1108E-05	6.0946E-05	0.3415%	0.0755%	
2		6.4850E-05	6.3940E-05	6.4860%	4.9918%	**mass/MOI**
3		6.0050E-05	5.8900E-05	-1.3957%	-3.2841%	654.1707718
4		6.1706E-05	6.1520E-05	1.3235%	1.0181%	
5		6.0616E-05	6.0414E-05	-0.4663%	-0.7980%	***t* -test (p)**
Averages		6.1666E-05	6.1144E-05	1.2578%	0.4007%	0.669296217

ABS block

Trial #	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.07113
1	7.6923E-05	7.6847E-05	7.6738E-05	-0.0983%	-0.2400%	
2		7.4789E-05	7.4669E-05	-2.7737%	-2.9297%	**mass/MOI**
3		7.5230E-05	7.5130E-05	-2.2004%	-2.3304%	924.6957331
4		7.8018E-05	7.7927E-05	1.4240%	1.3057%	
5		7.7799E-05	7.7700E-05	1.1393%	1.0104%	***t* -test (p)**
Averages		7.6537E-05	7.6433E-05	-0.5018%	-0.6368%	0.914062296

2 Wood Blocks

Trial #	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.079678
1	1.2180E-04	1.1497E-04	1.1200E-04	-5.6076%	-8.0497%	
2		1.2500E-04	1.2297E-04	2.6273%	0.9606%	**mass/MOI**
3		1.1648E-04	1.1618E-04	-4.3678%	-4.6141%	654.1707718
4		1.2349E-04	1.2329E-04	1.3875%	1.2233%	
5		1.2120E-04	1.2093E-04	-0.4926%	-0.7143%	***t* -test (p)**
Averages		1.2023E-04	1.1907E-04	-1.2906%	-2.2388%	0.703253945

Washers

# Washers	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass/unit (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.031
0	0	2.5267E-06	2.3766E-06	N/A	N/A	
1	1.2880E-05	2.2062E-05	2.1965E-05	71.29%	70.54%	**Mass/MOI**
2	2.5760E-05	3.9323E-05	3.9292E-05	52.65%	52.53%	2406.832298
3	3.8640E-05	5.4671E-05	5.4595E-05	41.49%	41.29%	
4	5.1520E-05	6.4311E-05	6.4216E-05	24.83%	24.64%	***t* -test (p)**
5	6.4400E-05	8.0973E-05	8.0866E-05	25.73%	25.57%	0.983052405
6	7.7280E-05	9.4432E-05	9.4362E-05	22.19%	22.10%	
7	9.0160E-05	1.0265E-04	1.0256E-04	13.85%	13.76%	
8	1.0304E-04	1.1881E-04	1.1872E-04	15.30%	15.21%	
9	1.1592E-04	1.3308E-04	1.3294E-04	14.80%	14.69%	
10	1.2880E-04	1.4646E-04	1.4637E-04	13.71%	13.64%	

Figure 7.2: Data collected from trial runs. Theoretical moments of inertia, moments of inertia calculated from both τ_n and τ_d, errors and error factors are reported. The statistical significance of the difference between the data using τ_n and τ_d is reported. In no trial is the difference significant.

For analysis, the samples can be split into two categories: blocks and washers. The three test blocks pictured below are wood, ABS plastic, and wood from left to right. The rightmost sample consists of two identical wood blocks of the same mass and dimensions as the leftmost sample stacked vertically in the same orientation as the leftmost sample. These samples each ran through 5 trials on the trifilar pendulum and the calculated moments of inertia were recorded. These data showed that, based on a student t-test, using the corrected period Tn as calculated using a viscous damping model did not change the calculated moment of inertia by a statistically significant amount.

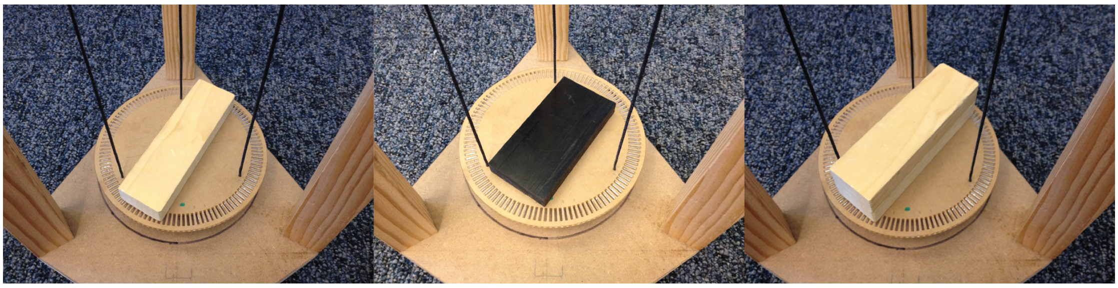

Figure 7.3: Block samples. From left to right: single block, ABS block, and two wooden blocks.

Large steel washers were also tested for several reasons. They can be modeled as thick hollow cylinders for theoretical MOI calculation, which is simple and well-understood formula. Washers are also very consistent in their geometries and are plentiful, so many of them could be gathered and treated as having the same moment of inertia (this was verified using a scale and digital calipers). Additionally, they can be stacked in a column which, in the theoretical MOI calculation, affects the mass term, but does not affect the mass to moment of inertia ratio.

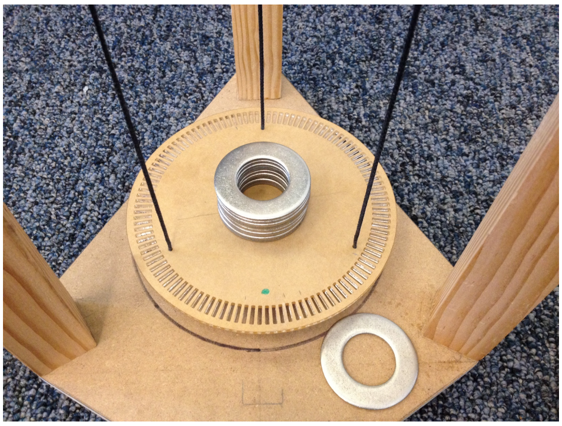

Figure 7.4: Washer arrangement for testing.

Washer MOI were measured by incrementing the number of washers stacked in the center of the platform up to ten washers and measuring MOI using the trifilar pendulum for each number of washers. As with the blocks, error percentage was calculated using both Td and Tn. As with the blocks, a student t-test showed no statistically significant change in MOI calculation between the two periods. It was observed, however, that error decreased dramatically as the number of washers in the stack increased until about seven washers were stacked, at which point error percentage leveled out. Since stacking the washers only increases the mass term of the theoretical MOI calculation, this relationship implies a correlation between accuracy of MOI measurement and mass of sample object. This idea is explored in more detail in the "Diagnosis and Discussion" section.

8 Error Analysis

Our system measured moments of inertia with consistently low errors in certain situations. When the ratio of mass to moment of inertia was low, the system was far more accurate. In addition, on very small masses, friction increased error drastically. However, this error was reduced through increasing the amount of mass used.

8.1 Measurement Error of Encoder

Our trifilar pendulum largely performed very well. It measured the moments of inertia of the blocks of wood and ABS with error margins of less than 3%. However, the error was significantly higher when we measured items such as steel washers. Rather than under 3%, it was always over 12% and as high as 71.29%. This error is potentially due to a combination of several factors.

As with many mechanical systems, precision in a trifilar pendulum is difficult to achieve. When objects have a low moment of inertia, period must be measured accurately to more than 1/100th of a second. In addition, the platform's moment of inertia must be determined very precisely such as not to offset the measure. Finally, an object must be perfectly centered on the platform such that its moment of inertia does not increase unreasonably and such that there is not excessive loading of the pin keeping the system rotating with 1 degree of freedom. If an object is off-center, then friction damping increases drastically due to the additional forces applied to the pin and the resulting friction. However, there are ways to avoid these errors. Objects must be chosen such that their ratio of mass of moment to inertia is as high as possible and they must be centered as well as possible on the platform. These specific error factors will be discussed in the following sections.

8.1.1 Derivation of Encoder Error

This derivation begins with determining the average tick lengths of a data set. In this case, it is done with data from 11 washers, however it should be the same for any sample, as it is dependent on the sensor, not the sample.

$$\tau_{actual} = \frac{d+c-a-b}{2} \tag{8.1}$$

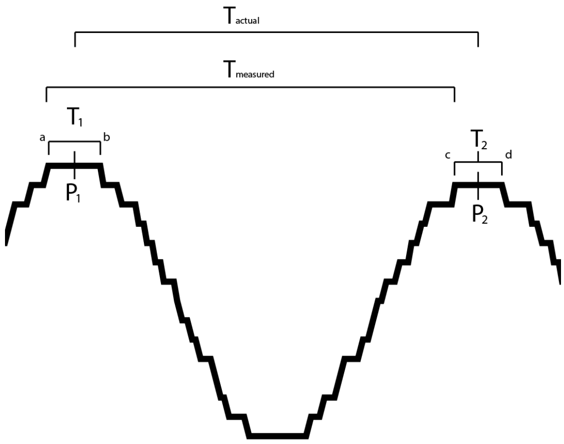

Figure 8.1: Variables for Encoder Error Calculation

$$\tau_{measured} = c - a \tag{8.2}$$

$$error = \pm(\tau_{actual} - \tau_{measured}) \tag{8.3}$$

$$error = \frac{(d-c)-(b-a)}{2} = \frac{T_1 - T_2}{2} \tag{8.4}$$

When this equation is expanded to get the average error, the worst period (Average plus two times standard deviation) is subtracted from the average period.

$$error_{avg} = \frac{T_{avg} - (T_{avg} - 2StDev)}{2} = StDev \tag{8.5}$$

Therefore, our measurement error from the encoder is twice the Standard Deviation of the tick length at the maximums. Using the data above error in period calculation is determined to be $\pm.07997s$, which is slightly larger than, but roughly equal to the deviations in measurements observed during testing.

8.1.2 Error Multiplication by Mass to Moment of inertia ratio

During our testing, it was observed that the error on measurements of objects that had a high $\frac{Mass}{MomentofInertia}$ ratio had significantly (p<.05) higher errors than objects that had a comparatively low one. This was especially observed in the differences between the errors recorded when measure the moments of inertia of the washers ($\frac{Mass}{MomentofInertia}$ = 2406.8) and the moments of inertia of wooden and ABS blocks (($\frac{Mass}{MomentofInertia}$ = 654.1 and 924.7, respectively). We determined that this change is due to the equation that is used. The derivation of the error scaled by this ratio is shown here.

$$error = \frac{I_{experimental} - I_{theoretical}}{I_{theoretical}} \tag{8.6}$$

From this point onward, the constants in the moment of inertia equation shall be referred to as C.

$$C = \frac{r^2 g}{4\pi^2 L} \tag{8.7}$$

$$error = \frac{(C\tau^2(M_o + M_p) - I_p) - I_t}{I_t} \tag{8.8}$$

The M_p term and the I_p term will cancel, resulting in:

$$error = \frac{C\tau^2 M_o}{I_t} - 1 \tag{8.9}$$

In this equation, as the ratio of $\frac{Mass}{MomentofInertia}$ increases, error increases proportionally.

8.1.3 Error due to friction

Error due to friction was not a focus of this study, and has therefore not been calculated. However, it is clear that there is a proportion of the error that is due to friction. In the washer data, as mass increases, error decreases, in a function that can be linked to $\frac{a}{x}$ where a is the force due to friction. This is due to the fact that as mass increases, the force due to the washers increases. However, the force due to friction always stays the same. Therefore, as force due to the washers increases, it eventually overwhelms the force due to friction, decreasing the error. However, friction is always present and therefore the error will never decline to zero.

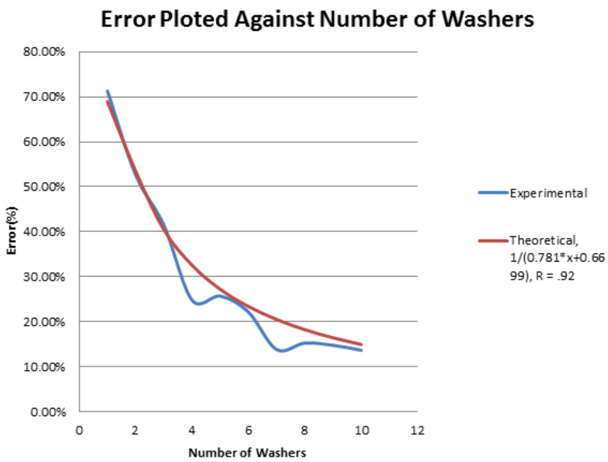

Figure 8.2: Error as number of washer increase. The data fits very well to a $\frac{a}{x}$ fitline, showing that friction is likely the cause of the excessive errors for low numbers of washers

8.1.4 Errors Combined

When all of the error calculations are combined the end result is shown below:

$$error = C(\tau_t \pm .08)^2 * \frac{M_o}{I_t} - 1 \tag{8.10}$$

Where $\tau_t \pm .08$ is the theoretical value that τ should be plus or minus the possible measurement error.

$$error = C(\tau_t^2 \pm .16\tau_t + .0064) * \frac{M_o}{I_t} - 1 \tag{8.11}$$

$C\tau_t^2 M_o$ is the theoretical moment of inertia for the object. Therefore it divided by I_t and the fraction is equal to 1.
The error is now:

$$error = C(\pm.16\tau_t + .0064) * \frac{M_o}{I_t} \tag{8.12}$$

However, this can be reduced further, because τ_t is mass and moment of inertia ratio by the following equation:

$$I_t = C * T_t^2 * M_o \tag{8.13}$$

$$T_i = \sqrt{\frac{1}{\frac{Mo}{I_t}C}} \tag{8.14}$$

Therefore, the final error calculation, with friction neglected, is:

$$error = C(\pm.16\sqrt{\frac{1}{\frac{Mo}{I_t}C}} + .0064) * \frac{M_o}{I_t} \tag{8.15}$$

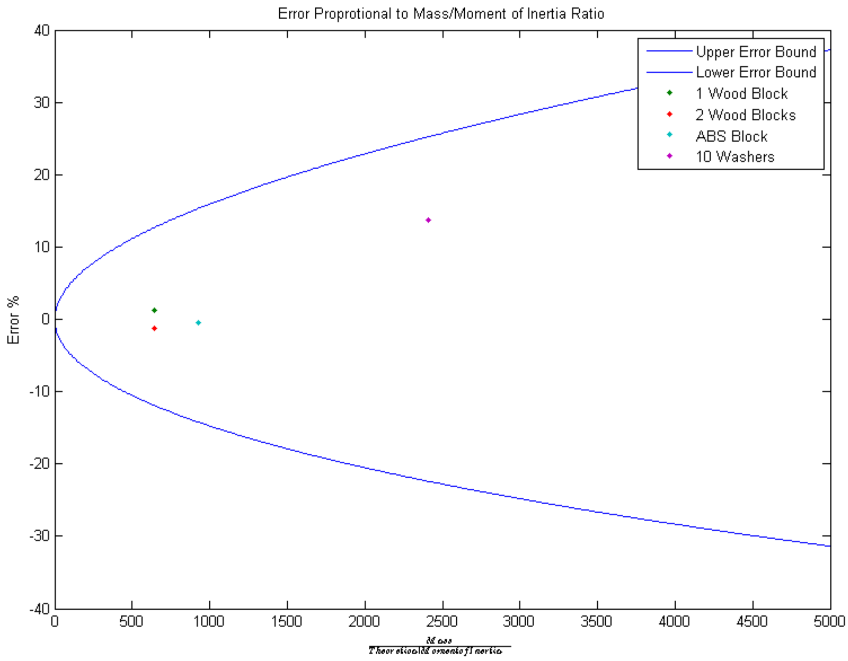

Figure 8.3: Error bounds plotted with sample errors from our measurements. The error used for the washer is the washer measurement with the largest number of washers. This should diminish the effects of friction upon this measurement. All of the measurements are reasonably within the error bounds. Most of our measurements are much better than the expected error bounds, but this is simply due to random chance. The fact that they are within the bounds is indicative that our error analysis is correct.

9 Damping Discussion

The damping on this system is a combination of linear damping due to friction and exponential damping due to air resistance. The original plan of this project was to determine the undamped, natural frequency of oscillation and then use that information to calculate a more accurate moment of inertia. However, when this calculation was performed, the difference between the two data sets was not statistically significant ($p>.5$), and in certain situations it increased the error. When friction damping was modeled and simulated, the differences in period were also not significant. Therefore, all of the final numbers used in this project are the damped frequency measure directly from our test setup.

In addition, the error on the calculation of damping was much greater than the error of the unprocessed measurement. Therefore, the inclusion of the natural frequency calculation would further increase error and therefore is undesirable.

10 Improvement

This model added sophistication to the commonly used basic trifilar pendulum setup by recording the full oscillatory response of the pendulum instead of just the period. This allowed analysis of the degradation in amplitude over time to account for damping in the system. One improvement that could have been made to increase accuracy is to increase the size of the entire system. While this would potentially increase error for smaller objects as we discovered with our washer measurements, it would allow the encoder higher resolution and would reduce the effects of friction of the stabilizing peg in the middle and the thickness of the support strings relative to the moment of the platform.
Another improvement to the system would be to make it more permanent. This would allow the pendulum to be on a consistently level surface and would allow the encoder to be fixed to that surface, allowing for better calibration of the system.

11 Reflection

The first lesson from this project was the importance of both accuracy and precision. Seemingly trivial factors like string width, levelness of the test surface, and how long the pendulum had sat without use all affected experimental results significantly. Especially when working with abstracted, heavily simplified math, it is important to make the experimental setup match the system's assumptions as best as possible.
Another lesson learned from this project is the importance peer review conducted as frequently as possible. Several delays in this project were caused by overlooking simple but fundamental errors in math, code, circuitry, etc. Minimizing these errors is important to reduce wasted time; catching an error quickly can save hours of debugging later.

12 Conclusion

In this project, a trifilar pendulum was built and outfitted with a custom quadrature encoder to detect the full frequency response of the system's oscillations. Sample blocks and larger washers were tested on this pendulum and error percentages between measured and theoretical moments of inertia were calculated to arrive at the following results:

1. Fitting the system to a viscous damping model by adjusting the oscillatory period based on decay of amplitude of oscillation over time does not yield a statistically significant change in the error percentage of the moment of inertia calculation of the test object.

2. Samples of lower mass-moment of inertia ratios resulted in lower errors.

13 Future Use

This project was a good application of the skills taught in Dynamics. It incorporated a design aspect and requires real-world problem assessment that would otherwise not be taught in

the course. In the time provided, however, the project was potentially too open-ended. More might have been gained from a more specific assignment as proposed below with a pre-built tifilar pendulum. It would involve less construction time and thus could focus more on the dynamics behind the system, while still including some debugging and real-world issues.
Proposed Trifilar Pendulum Assignment:

1. Derive the equation of motion to describe the rotation of the trifilar pendulum disk over time. Assume that the initial angular offset of the pendulum is small and that the displacement of the disk along the z-axis is negligible. What must be true about the relative dimensions of the disk and the strings for the latter assumption to be true?

2. Use this equation to find a relationship between moment of inertia of the plate and the parameter of the system (mass, disk radius, string length, etc.). How would you adapt this equation to find the moment of inertia of an object placed on top of the disk with its center of mass located directly above the disk's center?

3. Use the trifilar pendulum in the classroom and a stopwatch to measure the period of an object that you have the theoretical moment of inertia for (either via CAD or calculation). Using the relationship found above, how close is your experimental measurement to your theoretical moment of inertia? What techniques can you try to improve accuracy?

4. For each of these techniques, calculate percent error of experimental moment of inertia to theoretical. From these, what factors seem most important in reducing experimental error? Can you connect this error to the ratio of mass to moment of inertia of your test object(s)?

14 References

References

[1] A Trifilar Pendulum to Measure Mass Moments of Inertia. Brown.edu. Brown University, n.d. Web. 14 Dec. 2013.

[2] Du Bois, J. L., N. A. J. Lieven, and S. Adhikari. "Error Analysis in Trifilar Inertia Measurements." Experimental Mechanics 49.4 (2009): 533-40. Print.

[3] Moment of Inertia Test Lab. N.p., 24 Sept. 2003. Web. 14 Dec. 2013

[4] Williams, Huw. "Measuring the Inertia Tensor." Lecture. IMA Mathematics 2007 Conference. 26 Apr. 2007. Web. 14 Dec. 2013.

15 APENDIX A: ARDUINO DATALOGING CODE

```
/*
  Button

 Turns on and off a light emitting diode(LED) connected to digital
 pin 13, when pressing a pushbutton attached to pin 2.


 The circuit:
 * LED attached from pin 13 to ground
 * pushbutton attached to pin 2 from +5V
 * 10K resistor attached to pin 2 from ground

 * Note: on most Arduinos there is already an LED on the board
 attached to pin 13.


 created 2005
 by DojoDave <http://www.0j0.org>
 modified 30 Aug 2011
 by Tom Igoe

 This example code is in the public domain.

 http://www.arduino.cc/en/Tutorial/Button
 */

// constants won't change. They're used here to
// set pin numbers:
const int LED1pin = 2;
const int LED2pin = 3;// the number of the pushbutton pin
const int ledPin =  13;      // the number of the LED pin

// variables will change:
int LED1 = 0;
int LED2 = 0;
int time = 0;
int STATE;
int STATEold = 0;
int dir;
int DIREC;
int count;
```

```
void setup() {
  // initialize the LED pin as an output:
  pinMode(ledPin, OUTPUT);
  // initialize the pushbutton pin as an input:
  pinMode(LED1, INPUT);
  pinMode(LED2, INPUT);
  Serial.begin(9600);
}

void loop(){
  // read the state of the pushbutton value:
  LED1 = digitalRead(LED1pin);
  LED2 = digitalRead(LED2pin);

  if(LED1==HIGH){
      if(LED2==HIGH){
        STATE = 1;
      }
      else{
        STATE = 2;
      }
  }
  else{
      if(LED2==HIGH){
        STATE = 4;
      }
      else{
        STATE = 3;
      }
  }

DIREC =  STATE-STATEold;

if(DIREC==-1||DIREC==3){
  dir = 1;
}

if(DIREC==1||DIREC==-3){
  dir = -1;
}
else{
```

```
  dir = dir;
}

if(STATE!=STATEold){
  count = count + dir;
}

Serial.print(millis());
Serial.print(" ");
Serial.println(count);
//Serial.print(" ");
//Serial.print(STATE);
//Serial.print(" ");
//Serial.println(dir);


STATEold = STATE;


}
```

16 Appendix B: MATLAB Datalogging Code

```
% Communications MatLab <--> Arduino
% Matlab file 1 for use with Arduino file 1
clc;
clf
clear all;

areaDensity = 0.0035928916
Density = areaDensity/2.9972

format long g;
numcyc=20;
t=[];
v=[];

s1 = serial('COM3');    % define serial port
s1.BaudRate=9600;               % define baud rate
set(s1, 'terminator', 'LF');    % define the terminator


resolution = 3

for i = 1:600;
    x(i) = i;
end;


%---------------------------------------------------------------
%Starting to pull data


fopen(s1);
try                             % use try catch to ensure fclose
                                % signal the arduino to start collection
w=fscanf(s1,'%s');              % must define the input % d or %s, etc.
if (w=='A');
    display(['Collecting data']);
    fprintf(s1,'%s/n','A');     % establishContact just wants
                                % something in the buffer
end

i=0;
```

```
t0=tic;
intermediate(1) = 0;    %Initialize intermediate as 0
intermediate(2) = 0;

%%
while (i<(numcyc*100)); %While sensor is less
    %than 180 degrees yaw and 150 degrees pitch, letting it run a full
    %cycle
    clear v
    i=i+1; %increase index
    t(i)=toc(t0);
    t(i)=t(i)-t(1);

    v=fscanf(s1,'%c');        % must define the input % d or %s, etc.

    intermediate = sscanf(v,'%g') %intermediate values pulled from arduino
    a(i,1) = intermediate(1);     %Yaw
    a(i,2) = intermediate(2);     %Pitch

end


catch me;
    fclose(s1);                    %Close serial port s1


end
fclose(s1);
a(:,2) =  a(:,2)*.782*pi/180;
a(:,2) = a(:,2)-a(end,2);

plot(a(:,1)/1000,a(:,2))

[pks,locs] = findpeaks(a(:,2));
dsize = length(pks);

time = a(locs,1)/1000;

hold all
plot(time,pks,'.')


logpks = log(pks);
plot(time,logpks)
```

```
for i = 2:dsize-6
    pks2(i-1) = pks(i);
    time2(i-1) = time(i);
end
fit = polyfit(time2,log(pks2),1);




%%

pertot = 0;
for i=2:dsize
    per(i) = time(i)-time(i-1);
    pertot = pertot + per(i);
end

avgper = pertot/(dsize-1)

%%
% %finding period using fzero
%
% zeros = find(a(:,2)==0);
% timez = a(zeros,1);
% dsizez = length(timez);
% pertotz = 0;
% for i=2:dsizez
%     perz(i) = timez(i)-timez(i-1);
%     pertotz = pertotz + perz(i);
% end
%
% avgperz = pertotz/(dsizez-1)
%
% Tz = avgperz

%%
%damping
c = fit(1);
T = avgper;
zeta = (-c.*T)./sqrt(c.^2.*T.^2+4*pi^2);

Wd = 2*pi/avgper
```

```
Wn  = Wd/(sqrt(1-zeta^2));

Tn = 2*pi/Wn;

r = .05715;
g = 9.8;
l = .88185;
Mp = .0583;
Ip = 1.71655*10^-4; %as found by SOLIDWORKS
Ip2 =  .000158%1.0765e-04; %Version 2        6.4088*10^-5 %As derived experimentally

%Washer Masses (kg)
W1 = .02842;
W2 = .02853;
W3 = .02849;
W4 = .02848;
W5 = .02671;

TMOM =    (3.832 +   3.842 +    3.834 +   3.834 +   3.597 ) * 10^-5;

num = 10;
Mo = .0311*num%.039839*2%.0311*num;%W4+W3+W2+W1+W5 %.0311*num;% W1+W2%

MOM = (r^2*9.8*T^2/(4*pi^2*.8818)*(Mo+Mp))-.000114143 %(((r^2*g*T^2)/(4*pi^2*l))*(Mp+Mo)

TMOMBig = 0.00001288*num%6.09e-5*2%5.174*10^-5*num; %(3.834+3.834+3.842+3.597+3.832)*10^-


%Small Washer Moments%  1.0e-05 *of Inertia
%
%      3.832 (W1)
%      3.842 (W2)
%      3.834 (W3)
%      3.834 (W4)
%      3.597 (W5)

EPC = ((TMOMBig-MOM)/TMOMBig)*100;

nTME = [num;TMOMBig;MOM;EPC];

%Td -Tn comparison
TdTn = [T;Tn]

MOMn = r^2*9.8*Tn^2/(4*pi^2*.8818)*(Mo+Mp)-.000114143; %
```

```
EPCn = ((TMOMBig - MOMn)/TMOMBig)*100;

errors = [EPC;EPCn]
MOM
```

17 APPENDIX C: MATLAB ERROR ANALYSIS CODE

```
%Making PColor of error
clear all
r = .05715;
g = 9.8;
l = .88185;
Mp = .0583;
Ip = 1.71655*10^-4; %as found by SOLIDWORKS
Ip2 = .000158%1.0765e-04; %Version 2


C = r^2*g/(4*pi^2*l);
% for i = 1:50;
%     for j = 1:500;
%         t(i,j) = i/25;
%
%         Rat(i,j) = j*10;
%         error1(i,j) = (C.*(.16.*t(i,j)+.0064).*Rat(i,j)-1).*100;
%         error2(i,j) = (C.*(-.16.*t(i,j)+.0064).*Rat(i,j)-1).*100;
%
%
%     end
% end
Ratio = linspace(1,5000,500)
t = sqrt(1./(Ratio*C))

error1 = ((C.*(.16.*t+.0064).*Ratio)).*100;
error2 = ((C.*(-.16.*t+.0064).*Ratio)).*100;

%figure(3)
%clf(3)
%pcolor(t,Rat,error1);
%figure(4)
%clf(4)
%pcolor(t,Rat,error2);
figure(5)
clf(5)
plot(Ratio,error1)
hold on
plot(Ratio,error2);
```

```
xlabel('$\frac{Mass}{Theoretical Moment of Inertia}$','interpreter','latex')
ylabel('Error %')
title('Error Proprotional to Mass/Moment of Inertia Ratio')
hold all
plot(645.17,1.25,'.')
plot(645.17,-1.29,'.')
plot(924.69,-.5,'.')
plot(2406,13.71,'.')

legend('Upper Error Bound','Lower Error Bound','1 Wood Block','2 Wood Blocks','ABS Block
```

18 APPENDIX D: MATLAB FRICTION ANALYSIS CODE

This code simulates the effect of friction. It was used used to determine that friction has a minimal effect on the period for most situations and that it could be safely ignored.

```
%Paul Titchener
%Dynamics Fall 2013 Problem Set 2
%Problem 4: Roller Coaster


function res = roller()
%parameters
g = 9.81

%Initial ConVxditions

theta = .26
thetad = .00001
 %options = odeset('RelTol',1e-1);


[t,Y] = ode45(@rollerstates, [0,10],[theta, thetad]);


figure(2)
hold all
plot(t,Y(:,1))
[pks,locs] = findpeaks(Y(:,1))

time = t(locs)
clear period
for i = 2:4
    period(i-1) = time(i)-time(i-1)
end
avgper = mean(period)

end

function res = rollerstates(t,Z)
r = .05715;
g = 9.8;
l = .88185;
Mp = .0583;
Ip = 1.71655*10^-4; %as found by SOLIDWORKS
```

```
Ip2 =  .00012;%1.0765e-04; %Version 2        6.4088*10^-5 %As derived experimentally
A = .0001;

thetad = Z(2);
%if Z(2) == 0
   % thetadd = (-r^2*Mp*g/l*Z(1))/Ip;  %-A*Mp*Z(1)/abs(Z(1))
%else
thetadd = (-A*Mp*Z(2)/abs(Z(2))-r^2*Mp*g/l*Z(1))/Ip2;  %-A*Mp*Z(1)/abs(Z(1))
%end

res = [thetad;thetadd];
t;

end
```