01 - First Proof

Lean: First Steps

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Task

- Let's start with a very simple task. Imagine we're given the following fact.

$$a = 4$$

- Our task is to prove that

$$a > 1$$

- Here, a, 1 and 4 are all natural numbers.

Maths

- Seems obvious that if $a = 4$ then $a > 1$.
- Challenge is thinking about easy tasks in a structured way.
- So how can we justify $a > 1$, given $a = 4$?
  - Question - how can we justify $4 > 1$?
  - Answer - the natural numbers $\mathbb{N}$ are ordered by "greater than" $>$.

Maths

- Let's write down these thoughts in a structured way.

$$
\begin{aligned}
a &= 4 && \text{given fact} \\
&> 1 && \text{by the ordering of natural numbers} \qquad \square
\end{aligned}
$$

- We'll need this kind of structured step-by-step thinking to write proofs in Lean.

Code

```
-- 01 - First Proof

import Mathlib.Tactic

example {a : ℕ} (h1: a = 4) : a > 1 := by
  calc
    a = 4 := by rw [h1]
    _ > 1 := by norm_num
```

Code

- The proof starts on the line beginning with `example`. This line states the **theorem** we want to prove.
- `{a : ℕ}` tells Lean the variable `a` is a natural number.
- `(h1 : a = 4)` is the given fact, or **hypothesis**.
  - It's given a label `h1` so we can refer to it later.
- `a > 1` is the statement we want to prove.
- Finally the `:= by` signals we're about to prove that statement.

Code

The last three lines of code prove the theorem.

- We state `a = 4`, and **justify** it by refering to the given fact, previously labelled `h1`.
- We complete the proof by saying this is `> 1`, and **justify** it by the ordering of natural numbers.

Tactics

- We justified $4 > 1$ with `norm_num`. How did this happen?
- The **Peano axioms** describe the natural numbers. On top of these axioms, we can define addition, then the "greater than" $>$ relation.
- We did not go that deep in our maths proof!
- For a large body of fundamental results, we don't need to prove them every time we use them. The same idea applies to Lean proofs.
- Many of those fundamental results have been written in Lean and packaged up as **tactics**, ready for us to use.
- `norm_num` is a tactic that includes knowledge about the order of natural numbers.

Easy Exercise

- The Lean program above proves $a > 1$ given $a = 4$.
- Change it to prove $a < 10$ given $a = 4$.
- Tip: the **norm_num** tactic understands "less than" $<$, as well as "greater than" $>$.