08 - "And" Hypothesis

Lean: First Steps

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"And" Hypotheses

- Previously looked at a "**logical or**" hypothesis.
- Here we'll look at a "**logical and**" hypothesis.

Task

- Given

$$(x = 5) \wedge (y = x + 3)$$

- where $x, y \in \mathbb{Z}$, show

$$y = 8$$

Conjunction

- The symbol $\wedge$ means "logical and".
- The statement $P \wedge Q$ means both P and Q are true.
- Statements of the form $P \wedge Q$ are called **conjunctions**.

Maths

- A hypothesis $P \wedge Q$ is the same as hypotheses P and Q both being true.
- .. proof proceeds just like earlier examples with two hypotheses.
- Conjunctive hypotheses are **not particularly interesting**,
  - ... but we do need to know how to handle them.

Maths

- Structured proof

$$
\begin{array}{lll}
(x = 5) \wedge (y = x + 3) & \text{given fact} & (1) \\
 & & \\
x = 5 & \text{derived fact (1)} & (2) \\
y = x + 3 & \text{derived fact (1)} & (3) \\
 & & \\
y = x + 3 & \text{using fact (3)} & \\
\quad = (5) + 3 & \text{using fact (2)} & \\
\quad = 8 & \text{using arithmetic} & \square
\end{array}
$$

Maths

- From the single conjunctive hypothesis $(x = 5) \wedge (y = x + 3)$ we derive two smaller hypotheses, both of which are true:
  - $x = 5$
  - $y = x + 3$
- We start with $y = x + 3$ from derived fact (3), then use $x = 5$ from derived fact (2) to finally conclude $y = 8$.

Code

```
-- 08 - Conjunctive "and" Hypothesis

import Mathlib.Tactic

example {x y : ℤ} (h : x = 5 ∧ y = x + 3) : y = 8 := by
  obtain ⟨ ha , hb ⟩ := h
  calc
    y = x + 3 := by rw [hb]
    _ = (5) + 3 := by rw [ha]
    _ = 8 := by norm_num
```

Code

- The hypothesis is a conjunction, uses the symbol $\wedge$ for "logical and".
- The instruction `obtain ⟨ ha, hb ⟩ := h` splits the conjunctive hypothesis `h` into `ha` and `hb`, and then removes `h`.
- Notice the comma and angle brackets `⟨⟩` used to split a conjunction, unlike the vertical bar `|` and no brackets used to split a disjunction.
- The rest of the proof uses the familiar **calc** structure to show `y = 8`.

Infoview

- Placing the cursor before `obtain` shows only one hypothesis `h`.

```
x y : ℤ
h : x = 5 ∧ y = x + 3
⊢ y = 8
```

Infoview

- Placing the cursor on the next line after `obtain` shows `h` has been replaced by `ha` and `hb`.

```
x y : ℤ
ha : x = 5
hb : y = x + 3
⊢ y = 8
```

- Notice the hypothesis `h` has been removed.

Easy Exercise

- Write a Lean program to show $y \geq 8$, where $x, y \in \mathbb{Z}$, given the conjunctive hypothesis

$$(x \geq 5) \wedge (y = x + 3)$$