07 - Proof By Cases

Lean: First Steps

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October 27, 2024

Proof By Cases

- A **proof by cases** divides a task into separate cases,
- .. and proves each one leads to the desired conclusion.

Task

- Given

$$(x = 3) \vee (x = -3)$$

- where $x \in \mathbb{R}$, show

$$x^2 = 9$$

Disjunction

- The symbol $\vee$ means “logical or”.
- The statement $P \vee Q$ means either P is true, or Q is true, or possibly even both are true.
- Statements of the form $P \vee Q$ are called **disjunctions**.

- Aim is to show $x^2 = 9$ follows from the hypothesis

$$(x = 3) \vee (x = -3)$$

- Hypothesis tells us that either $x = 3$ is true, or $x = -3$ is true.
- We don't know which, so we have to consider **both** cases, and show the conclusion follows from **each case**.

Maths

$$\begin{tikzcd} (x = 3) \lor (x = -3) \arrow[d] \\ x^2 = 9 \end{tikzcd}$$

$$\begin{tikzcd} x = 3 \arrow[dr] & & x = -3 \arrow[dl] \\ & x^2 = 9 & \end{tikzcd}$$

Maths

- Structured proof

$$(x = 3) \lor (x = -3) \qquad \text{given fact} \qquad (1)$$

$$\begin{aligned} \text{case } x &= 3 && \text{using fact (1)} && (2) \\ x^2 &= (3)^2 && \text{using case (2)} \\ &= 9 \end{aligned}$$

$$\begin{aligned} \text{case } x &= -3 && \text{using fact (1)} && (3) \\ x^2 &= (-3)^2 && \text{using case (3)} \\ &= 9 \end{aligned}$$

$$(x = 3) \lor (x = -3) \implies x^2 = 9 \qquad \square$$

- The given fact (1) splits into two cases, $x = 3$ and $x = -3$.
  - The first case (2) is $x = 3$. This case gives us $x^2 = (3)^2$, leading to the conclusion $x^2 = 9$.
  - The second case (3) is $x = -3$. This case gives us $x^2 = (-3)^2$, also leading to the conclusion $x^2 = 9$.
- The two cases $x = 3$ and $x = -3$ are sufficient to fully cover the hypothesis $(x = 3) \vee (x = -3)$.
  - There is no third case.
- So we have shown that, given the hypothesis, x^2 is indeed always 9.

Code

```
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import Mathlib.Tactic

example {x : ℤ} (h : x = 3 ∨ x = -3) : x^2 = 9 := by
  obtain ha | hb := h
  · calc
    x^2 = (3)^2 := by rw [ha]
    _ = 9 := by norm_num
  · calc
    x^2 = (-3)^2 := by rw [hb]
    _ = 9 := by norm_num
```

Code

- The hypothesis is a disjunction. The letter `v` denotes “logical or”.
- `obtain ha | hb := h` splits the disjunctive hypothesis `h` into `ha` and `hb`.
- The Infoview will show `ha : x = 3` and `hb : x = -3`.
- The two **calc** sections, one after the other, prove the goal for each **case**.

Code

- The **calc** sections have a dot preceding them, `· calc`.
- This **focussing dot** is good style when writing sub-proofs within a larger proof.
  - Visually see, at a glance, the structure of the proof.
  - Infoview restricts information to the current goal.

Infoview

- Placing the cursor before `obtain` shows only one goal, the **overall proof goal**.

```
1 goal
x : ℤ
h : x = 3 ∨ x = -3
⊢ x ^ 2 = 9
```

Infoview

- Moving the cursor after the `obtain` instruction, just before the first `· calc`, shows the **two sub-goals** and their hypotheses `ha` and `hb`.

```
2 goals
case inl
x : ℤ
ha : x = 3
⊢ x ^ 2 = 9

case inr
x : ℤ
hb : x = -3
⊢ x ^ 2 = 9
```

Infoview

- Moving the cursor past the focussing dot, placing it just before `calc`, shows **only the sub-goal** and hypothesis relevant to the first case.

```
1 goal
case inl
x : ℤ
ha : x = 3
⊢ x ^ 2 = 9
```

Easy Exercise

- Write a Lean program to prove $x^2 - 3x + 2 = 0$, where $x \in \mathbb{R}$, given

$$(x = 1) \lor (x = 2)$$

- In your proof create two cases $x = 1$ and $x = 2$ from the given hypothesis $(x = 1) \lor (x = 2)$.