11 - Existence

Lean: First Steps

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Existence Proofs

- The aim of an existence proof is simply to show something exists.

Task

- Show **there exists** a natural number n, such that

$$n^2 + 1 = 10$$

- In mathematical notation, where the symbol $\exists$ means "there exists"

$$\exists n \in \mathbb{N} \; [n^2 + 1 = 10]$$

Task

- The essence of an existence proof is to demonstrate an object exists that satisfies any given conditions.

- Calculating, deriving, or even guessing, what that object is, is not the central point.

Maths

- After some mental trial and error we find that $n = 3$ works.
- We don't need to justify how we arrived at our example. We only need to show it satisfies the condition $n^2 + 1 = 10$.

Maths

$\exists n \in \mathbb{N}[n^2 + 1 = 10]$	proof objective	
use $n = 3$	chosen example	(1)
$n^2 + 1 = (3)^2 + 1$	using (1)	
$= 10$	by arithmetic	□

Code

```
-- 11 - Existence proof

import Mathlib.Tactic

example : ∃ n: ℕ, n^2 + 1 = 10 := by
  use 3
  calc
    3^2 + 1 = 10 := by norm_num
```

Code

- The proof header has no separate definition of variable types, and no hypothesis. The entire theorem is in the proof objective.

- Proof objective `∃ n: ℕ, n^2 + 1 = 10` says there exists a natural number `n`, **such that** `n^2 + 1 = 10`.

- `use 3` tells Lean that we propose `3`.

- Infoview confirms `use 3` changes the goal from `∃ n, n^2 + 1 = 10` to `3^2 + 1 = 10`.

- We can use a simple **calc** section to show that `3^2 + 1` is indeed `10`.

Bad Example

- Let's see what happens if we choose a bad example, say $n = 4$.
- This means changing the code to `use 4`.
- Placing the cursor after `use 4` shows the proof goal.

```
⊢ 4 ^ 2 + 1 = 10
```

- This goal is **impossible** to prove, because 17 is not 10.

Simpler Code

```
-- 11 - Existence proof

import Mathlib.Tactic

example : ∃ n: ℕ, n^2 + 1 = 10 := by
  use 3
  norm_num
```

- The goal after `use 3` is `⊢ 3^2 + 1 = 10`.

- This is simple enough to apply the **norm_num** tactic.

- Doesn't really need a **calc** section, which is better suited to derivations requiring several steps.

Easy Exercise

- Write a Lean program to prove there exists a natural number greater than 5.

- In mathematical notation, the proof objective is as follows, and should translate into Lean code fairly directly.

$$\exists n \in \mathbb{N}\ [n > 5]$$

- As an optional extra challenge, try writing the proof in the concise form shown above.