05 - Inequalities

Lean: First Steps

Tariq Rashid

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- Previously we used an inequality as the final proof step, to establish $4 > 1$.
  - Simple numerical comparison.
- Here we have an inequality in the hypothesis of the theorem we want to prove.
  - Allows more general theorems.

Task

- Given

$$b = a^2$$
$$a \geq 2$$

- our task is to show

$$b \geq 4$$

- where $a, b \in \mathbb{N}$

Maths

- Conclusion holds for **any** $a \geq 2$.
  - In the first chapter a was very specifically constrained to $a = 4$.
  - So this theorem is much broader in scope.
- First chapter's proof concluded $4 > 1$, justified by a simple numerical comparison.
  - Can't do that here because b in $b \geq 4$ could be any one of an infinity of natural numbers.
- So there is something fundamentally different about this task.

Maths

- Structured proof.

$$b = a^2 \qquad \text{given fact} \qquad (1)$$

$$a \geq 2 \qquad \text{given fact} \qquad (2)$$

$$\begin{aligned} b &= a^2 && \text{using fact (1)} \\ &\geq (2)^2 && \text{using fact (2)} \\ &= 4 && \text{using arithmetic} \quad \square \end{aligned}$$

- First list the given facts, $b = a^2$ and $a \geq 2$.
- Start with the expression we want to prove something about, b. This is a^2, justified by the first given fact.
- Our aim is to express b as a comparison to 4, so we want to resolve a.
- The second fact $a \geq 2$ gives us $a^2 \geq (2)^2$.
- Finally, we say $(2)^2 = 4$, giving us the desired conclusion.

Maths

- Two things worth pointing out about this proof.
- How do we justify $a \geq 2 \implies a^2 \geq (2)^2$?
  - Probably didn't think about it much - skipping over justification.
  - Commonly accepted result $m \leq n \implies m^i \leq n^i$, where $m, n, i \in \mathbb{N}$.
- Easy to misread the proof as stating $b = 4$.
  - Correct reading is $b = a^2$, and $a^2 \geq (2)^2$, and finally $(2)^2 = 4$. Summarising this chain of inequalities gives $b \geq 4$.

Code

```
-- 05 - Simple Inequality

import Mathlib.Tactic

example {a b : ℕ} (h1 : b = a^2) (h2: a ≥ 2) : b ≥ 4 := by
  calc
    b = a^2 := by rw [h1]
    _ ≥ (2)^2 := by rel [h2]
    _ = 4 := by norm_num
```

Code

- Nothing structually different about this proof.
- Second hypothesis, proof objective, and proof body now use an inequality $\geq$ rather than an equality =.
- The **rel** tactic is new. Used to justify statements about **relations**, such as "greater than or equal to" $\geq$.
- Here the **rel** tactic is combining `a^2` and the hypothesis `a` $\geq$ `2` to give `a^2` $\geq$ `(2)^2`.

Looking Inside The rel Tactic

- The **rel** tactic is using a **lemma** from the Lean maths library, `Nat.pow_le_pow_left`, which encodes $n \leq m \implies n^i \leq m^i$, where $m, n, i \in \mathbb{N}$.

- Recall that a large body of fundamental results have been encoded into the Lean **Mathlib** library as theorems and lemmas.

- Interesting to see how this lemma is declared in the Mathlib library

```
abbrev Nat.pow_le_pow_left {n : Nat} {m : Nat} (h : n ≤ m)
    (i : Nat) :
    n ^ i ≤ m ^ i
```

Easy Exercise

- Write a Lean program to prove $a < c$ if we know $a < b$ and $b \leq c$, where $a, b, c \in \mathbb{N}$.