07 - Proof By Cases

Lean: First Steps

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Proof By Cases

- A proof by cases divides a task into separate cases,
- .. and proves each one leads to the desired conclusion.

Task

Given

$$(x = 3) \lor (x = -3)$$

• where $x \in \mathbb{R}$, show

$$x^2 = 9$$



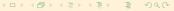
Disjunction

- The symbol ∨ means "logical or".
- The statement $P \lor Q$ means either P is true, or Q is true, or possibly even both are true.
- Statements of the form $P \lor Q$ are called **disjunctions**.

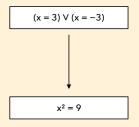
• Aim is to show $x^2 = 9$ follows from the hypothesis

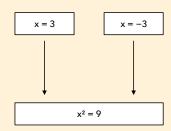
$$(x = 3) \lor (x = -3)$$

- Hypothesis tells us that either x = 3 is true, or x = -3 is true.
- We don't know which, so we have to consider both cases, and show the conclusion follows from each case.



5 / 15





Structured proof

$$(x=3) \lor (x=-3) \qquad \text{given fact} \qquad (1)$$

case
$$x = 3$$
 using fact (1) (2)
 $x^2 = (3)^2$ using case (2)
 $= 9$

case
$$x = -3$$
 using fact (1) (3)

$$x^{2} = (-3)^{2}$$
 using case (3)

$$= 9$$

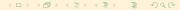
$$(x=3) \lor (x=-3) \implies x^2=9$$

- The given fact (1) splits into two cases, x = 3 and x = -3.
 - The first case (2) is x = 3. This case gives us $x^2 = (3)^2$, leading to the conclusion $x^2 = 9$.
 - The second case (3) is x = -3. This case gives us $x^2 = (-3)^2$, also leading to the conclusion $x^2 = 9$.
- The two cases x = 3 and x = -3 are sufficient to fully cover the hypothesis $(x = 3) \lor (x = -3)$.
 - There is no third case.
- So we have shown that, given the hypothesis, x^2 is indeed always 9.



Code

```
-- 07 - Proof by Cases
import Mathlib.Tactic
example \{x : \mathbb{Z}\}\ (h : x = 3 \ v \ x = -3) : x^2 = 9 := by
  obtain ha | hb := h
  calc
    x^2 = (3)^2 := by rw [ha]
    _ = 9 := by norm_num
  calc
    x^2 = (-3)^2 := by rw [hb]
    _ = 9 := by norm_num
```



Code

- The hypothesis is a disjunction. The letter v denotes "logical or".
- obtain ha | hb := h splits the disjunctive hypothesis h into ha and hb.
- The Infoview will show ha : x = 3 and hb : x = -3.
- The two calc sections, one after the other, prove the goal for each case.

Code

- The calc sections have a dot preceding them, calc.
- This focusing dot is good style when writing sub-proofs within a larger proof.
 - Visually see, at a glance, the structure of the proof.
 - Infoview restricts information to the current goal.

Infoview

 Placing the cursor before obtain shows only one goal, the overall proof goal.

```
1 goal x : \mathbb{Z} h : x = 3 \lor x = -3 \vdash x ^ 2 = 9
```

Infoview

Moving the cursor after the obtain instruction, just before the first

 calc
 shows the two sub-goals and their hypotheses ha and
 hb

```
2 goals
case inl
x : \mathbb{Z}
ha : x = 3
\vdash x ^2 = 9

case inr
x : \mathbb{Z}
hb : x = -3
\vdash x ^2 = 9
```

Infoview

 Moving the cursor past the focussing dot, placing it just before calc, shows only the sub-goal and hypothesis relevant to the first case.

```
1 goal

case inl

x : \mathbb{Z}

ha : x = 3

\vdash x ^2 = 9
```

Easy Exercise

• Write a Lean program to prove $x^2 - 3x + 2 = 0$, where $x \in \mathbb{R}$, given

$$(x=1) \lor (x=2)$$

• In your proof create two cases x = 1 and x = 2 from the given hypothesis $(x = 1) \lor (x = 2)$.

