04 - Simple Algebra

Lean: First Steps

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Algebra

• Let's ask Lean to do algebra - this is new.



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Task

• Show the following is true

$$(a+b)(a-b) = a^2 - b^2$$

• a and b are integers.



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Maths

• Let's try to write out a proof, aiming to show and justify each step.

$$(a+b)(a-b) = a^2 - ab + ba - b^2$$

= $a^2 - b^2$

by algebra

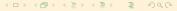
by algebra

 Simple proof - no need for any hypotheses, no substitutions, and no arithmetic.



Maths

- Step by step:
 - We start with the expression we want to prove something about, (a + b)(a b).
 - We expand out the brackets, multiplying every combination of terms inside the brackets, to give $a^2 ab + ba b^2$.
 - We collect like terms, and find that ba and -ba cancel out, to give the desired result $a^2 b^2$.
- All we needed for this proof was basic algebra, specifically, multiplying out brackets, and collecting like terms.



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Code

```
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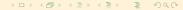
import Mathlib.Tactic

example {a b : Z} : (a - b) * (a + b) = a^2 - b^2 := by

calc

(a - b) * (a + b) = a^2 - a*b + a*b - b^2 := by ring

_ = a^2 - b^2 := by ring
```



Code

- Notice there are no hypotheses to declare.
- Proof body starts with the expression we want to prove something about, (a - b) * (a + b)...
- ... then states this is equal to a^2 a*b + a*b b^2.
- This is justified by the ring tactic, which can do basic algebra.
- The final line states this is a^2 b^2, again justied by the ring tactic.

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Code

- First use of the ring tactic was to justify multiplying out the brackets.
- Second use of the ring tactic was to justify collecting like terms, and cancelling the -a*b and +a*b terms.
- We could combine both steps into one, using the ring tactic just once - try it!

calc

$$(a - b) * (a + b) = a^2 - b^2 := by ring$$

Intentional Error

- What would happen if we changed $\{a \ b : \mathbb{Z}\}\ to \{a \ b : \mathbb{N}\}$?
- Try it!
- The Infoview shows errors the ring tactic failed. Why?

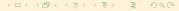
Intentional Error

- If a and b are natural numbers, then (a b) doesn't make sense if b > a.
- ... because natural numbers can't be negative integers.
- More formal Lean fails to apply ring tactic because $\mathbb N$ is not a ring under subtraction.
- This illustrates one of the benefits of Lean. It can catch apparently small assumptions that end up invalidating our results.

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Easy Exercise

- Write a Lean program to prove $(a + b)^2 = a^2 + b^2$ if we know ab = 0, where $a, b \in \mathbb{Z}$.
- This will require both the ring and rewriting tactics.



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