09 - “Or” Goal

Lean: First Steps

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"Or" Goal

- We've seen disjunctive and conjunctive **hypotheses**.
- Here we look at a disjunctive proof **goal**.

Task

- For integer $x = -1$, show that

$$(x^2 = 1) \vee (x^3 = 1)$$

- $x^3 = 1$??

Maths

- Clearly $(-1)^3 = 1$ is not true.
- $P \vee Q$ being true means at least one of P and Q is true.
- If Q is false, but P is true, then the disjunction $P \vee Q$ is still true.
- If we can prove $x^2 = 1$, then we have proven the disjunction $(x^2 = 1) \vee (x^3 = 1)$.

Maths

- Step-by-step proof

$$
\begin{aligned}
x &= -1 && \text{given fact} && (1)\\
(x^2 = 1) \lor (x^3 = 1) & && \text{proof objective} && (2)
\end{aligned}
$$

$$
x^2 = 1 \qquad \text{sufficient goal (2)} \qquad (3)
$$

$$
\begin{aligned}
x^2 &= (-1)^2 && \text{using fact (1)}\\
&= 1 && \text{using arithmetic}
\end{aligned}
$$

$$
(x = -1) \implies (x^2 = 1) \lor (x^3 = 1) \qquad \square
$$

Choice

- Consider proof objective

$$(x^2 = 1) \lor (x^3 = -1)$$

- Both statements $x^2 = 1$ and $x^3 = -1$ can be proven to be true.
- We can choose which **one** of the two statements we want to prove. One is sufficient.

Code

```
-- 09 - Disjunctive "or" Goal

import Mathlib.Tactic

example {x : ℤ} (h : x = -1) : x^2 = 1 ∨ x^3 = 1 := by
  left
  calc
    x^2 = (-1)^2 := by rw [h]
    _ = 1 := by norm_num
```

Code

- The objective is a disjunction, and uses the symbol $\vee$ to denote “logical or”.
- We state our intention to prove only the “left” part of the disjunction using `left`.
- Infoview will confirm `left` replaces the goal `x^2 = 1 ∨ x^3 = 1` with `x^2 = 1`.
- Rest of the proof uses the familiar **calc** to show `x^2 = 1`.

Infoview

- Placing the cursor before `left` shows the original proof goal.

```
x : ℤ
h : x = -1
⊢ x ^ 2 = 1 ∨ x ^ 3 = 1
```

- Placing the cursor on the next line after `left` confirms the proof goal has been replaced by a smaller, but sufficient, statement.

```
x : ℤ
h : x = -1
⊢ x ^ 2 = 1
```

- Manipulating the proof goal is quite normal.

Easy Exercise

- A longer disjunction could be $P \lor Q \lor R \lor S$.
  - `left` selects the left-most statement P as the new goal.
  - `right` selects the remainder $Q \lor R \lor S$ as the new goal.
- For integer x, given that $x = -1$, write a Lean program to show

$$(x = 1) \lor (x^2 = 1) \lor (x^3 = 1)$$

- You'll need both **right** and **left** to write the proof.