12 - Odd Numbers

Lean: First Steps

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Using Definitions: Odd Numbers

- There is a huge body of commonly agreed knowledge that mathematicians refer to in their own proofs.
- Lots of these lemmas, theorems, and definitions are in Mathlib .
- We'll start by using the definition of an odd number.

Task

• Show the integer 13 is odd.

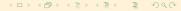


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Task

- To show 13 is odd, we need to show it meets the **definition** of odd.
 - An **odd** integer is of the form 2k + 1, where k is an integer.
 - If there exists an integer k such that n = 2k + 1, then n is odd.
- The task has become an existence proof.
 - If we can find an integer k such that 13 = 2k + 1, then we have shown 13 is odd.



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Maths

13 is odd

proof objective

$$\exists k \in \mathbb{Z}[n=2k+1] \implies n \text{ is odd}$$

definition of odd

$$\exists k \in \mathbb{Z}[13 = 2k + 1]$$

sufficient goal, using (1)

use
$$k = 6$$

 $13 = 2(6) + 1$

chosen example using (2)

(2)

(1)

$$13 = 2(6) + 1 \implies 13$$
 is odd

by definition (1)

Maths

- We start with the proof objective, to show 13 is odd.
- We then state the **definition**, that n is odd if it can be written in the form 2k + 1, where k is an integer.
- To show 13 is odd, it is **sufficient** to show it can be written in the form 2k + 1.
 - New goal, to show there exists an integer k such that 13 = 2k + 1.
 - We choose k = 6, and confirm that 13 = 2(6) + 1.
- We have shown 13 can be written in the form 2k + 1.
- So, by the definition of odd, we have shown 13 is odd.



Code

```
-- 12 - Definition: Odd Number
import Mathlib.Tactic
example : Odd (13: Z) := by
  dsimp [Odd]
  use 6
  norm_num
```

Code

- The proof objective states that 13 is Odd.
 - 13 interpreted as a natural number.
 - (13: \mathbb{Z}) specifies 13 as an integer.
- Odd is defined in Mathlib.
- dsimp [Odd] expands that definition in the Infoview.
 - From being displayed as Odd 13, to \exists k, 13 = 2 * k + 1.
- dsimp has no effect on the proof itself.



Code

- After this point, the proof proceeds as a simple existence proof.
 - The instruction use 6 tells Lean we want to try 6 for k.
 - This changes the goal to 13 = 2 * 6 + 1.
 - We resolve this goal by arithmetic, using norm_num.
- For this simple goal, there is no need for a multi-line calc section.

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What is Odd?

- Odd is a definition in Mathlib, and happens to be a function.
- That function Odd takes a number, and outputs a proposition involving that number.
 - Odd applied to 13, output is a (true) proposition; ∃k such that
 13 = 2*k + 1.
 - Odd applied to 14, output is a (false) proposition; $\exists k$ such that 14 = 2*k + 1.
- Interesting to see the definition of Odd inside Mathlib:

def Odd (a : α) : Prop := \exists k, a = 2 * k + 1



Infoview

- Infoview is very useful when working with definitions and existence.
- Placing the cursor before dsimp [Odd] shows the original proof goal.

```
⊢ Odd 13
```

 Moving the cursor to the end of the line after dsimp [Odd] shows the goal is now displayed using the definition of Odd.

$$\vdash \exists k, 13 = 2 * k + 1$$

Placing the cursor after use 6 shows the goal is now specific to
 k = 6.

$$\vdash \exists k, 13 = 2 * 6 + 1$$



Easy Exercise

- Write a Lean program to prove the integer 14 is even.
- Your proof should use Mathlib's definition Even for even numbers.
- Use **dsimp** to see how the definition is applied to 14.