

# 11 - Existence

Lean: First Steps

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# Existence Proofs

- The aim of an existence proof is simply to show something exists.

- Show **there exists** a natural number  $n$ , such that

$$n^2 + 1 = 10$$

- In mathematical notation, where the symbol  $\exists$  means “there exists”

$$\exists n \in \mathbb{N} [n^2 + 1 = 10]$$

- The essence of an existence proof is to demonstrate an object exists that satisfies any given conditions.
- Calculating, deriving, or even guessing, what that object is, is not the central point.

- After some mental trial and error we find that  $n = 3$  works.
- We don't need to justify how we arrived at our example. We only need to show it satisfies the condition  $n^2 + 1 = 10$ .

$$\exists n \in \mathbb{N}[n^2 + 1 = 10]$$

proof objective

use  $n = 3$

$$\begin{aligned}n^2 + 1 &= (3)^2 + 1 \\ &= 10\end{aligned}$$

chosen example (1)

using (1)

by arithmetic



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```
-- 11 - Existence proof

import Mathlib.Tactic

example :  $\exists n: \mathbb{N}, n^2 + 1 = 10$  := by
  use 3
  calc
     $3^2 + 1 = 10$  := by norm_num
```

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- The proof header has no separate definition of variable types, and no hypothesis. The entire theorem is in the proof objective.
- Proof objective  $\exists n: \mathbb{N}, n^2 + 1 = 10$  says there exists a natural number  $n$ , **such that**  $n^2 + 1 = 10$ .
- `use 3` tells Lean that we propose 3.
- `Infview` confirms `use 3` changes the goal from  $\exists n, n^2 + 1 = 10$  to  $3^2 + 1 = 10$ .
- We can use a simple **calc** section to show that  $3^2 + 1$  is indeed 10.



# Bad Example

- Let's see what happens if we choose a bad example, say  $n = 4$ .
- This means changing the code to use 4.
- Placing the cursor after use 4 shows the proof goal.

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$\vdash 4 \wedge 2 + 1 = 10$

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- This goal is **impossible** to prove, because 17 is not 10.

# Simpler Code

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```
-- 11 - Existence proof

import Mathlib.Tactic

example :  $\exists n: \mathbb{N}, n^2 + 1 = 10$  := by
  use 3
  norm_num
```

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- The goal after `use 3` is  $\vdash 3^2 + 1 = 10$ .
- This is simple enough to apply the `norm_num` tactic.
- Doesn't really need a `calc` section, which is better suited to derivations requiring several steps.

# Easy Exercise

- Write a Lean program to prove there exists a natural number greater than 5.
- In mathematical notation, the proof objective is as follows, and should translate into Lean code fairly directly.

$$\exists n \in \mathbb{N} [n > 5]$$

- As an optional extra challenge, try writing the proof in the concise form shown above.