# 07 - Proof By Cases

Lean: First Steps

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# **Proof By Cases**

- A proof by cases divides a task into separate cases,
- .. and proves each one leads to the desired conclusion.

## Task

Given

$$(x = 3) \lor (x = -3)$$

• where  $x \in \mathbb{R}$ , show

$$x^2 = 9$$



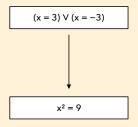
# Disjunction

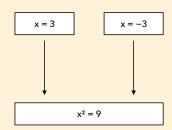
- The symbol ∨ means "logical or".
- The statement  $P \lor Q$  means either P is true, or Q is true, or possibly even both are true.
- Statements of the form  $P \lor Q$  are called **disjunctions**.

• Aim is to show  $x^2 = 9$  follows from the hypothesis

$$(x = 3) \lor (x = -3)$$

- Hypothesis tells us that either x = 3 is true, or x = -3 is true.
- We don't know which, so we have to consider both cases, and show the conclusion follows from each case.





Structured proof

$$(x=3) \lor (x=-3) \qquad \text{given fact} \qquad (1)$$

case 
$$x = 3$$
 using fact (1) (2)  
 $x^2 = (3)^2$  using case (2)  
 $= 9$ 

case 
$$x = -3$$
 using fact (1) (3)  

$$x^{2} = (-3)^{2}$$
 using case (3)  

$$= 9$$

$$(x=3) \lor (x=-3) \implies x^2=9$$

- The given fact (1) splits into two cases, x = 3 and x = -3.
  - The first case (2) is x = 3. This case gives us  $x^2 = (3)^2$ , leading to the conclusion  $x^2 = 9$ .
  - The second case (3) is x = -3. This case gives us  $x^2 = (-3)^2$ , also leading to the conclusion  $x^2 = 9$ .
- The two cases x = 3 and x = -3 are sufficient to fully cover the hypothesis  $(x = 3) \lor (x = -3)$ .
  - There is no third case.
- So we have shown that, given the hypothesis,  $x^2$  is indeed always 9.

#### Code

```
-- 07 - Proof by Cases
import Mathlib.Tactic
example \{x : \mathbb{Z}\}\ (h : x = 3 \ v \ x = -3) : x^2 = 9 := by
  obtain ha | hb := h
  calc
    x^2 = (3)^2 := by rw [ha]
    _ = 9 := by norm_num
  calc
    x^2 = (-3)^2 := by rw [hb]
    _ = 9 := by norm_num
```

#### Code

- The hypothesis is a disjunction. The letter v denotes "logical or".
- obtain ha | hb := h splits the disjunctive hypothesis h into ha and hb.
- The Infoview will show ha : x = 3 and hb : x = -3.
- The two calc sections, one after the other, prove the goal for each case.

#### Code

- The calc sections have a dot preceding them, · calc.
- This focussing dot is good style when writing sub-proofs within a larger proof.
  - Visually see, at a glance, the structure of the proof.
  - Infoview restricts information to the current goal.

#### Infoview

 Placing the cursor before obtain shows only one goal, the overall proof goal.

```
1 goal x : \mathbb{Z} h : x = 3 \lor x = -3 \vdash x ^2 = 9
```

#### Infoview

Moving the cursor after the obtain instruction, just before the first calc, shows the two sub-goals and their hypotheses ha and hb.

```
2 goals
case inl
x : \mathbb{Z}
ha : x = 3
\vdash x ^2 = 9

case inr
x : \mathbb{Z}
hb : x = -3
\vdash x ^2 = 9
```

### Infoview

 Moving the cursor past the focussing dot, placing it just before calc, shows only the sub-goal and hypothesis relevant to the first case.

```
1 goal

case inl

x : \mathbb{Z}

ha : x = 3

\vdash x ^2 = 9
```

# Easy Exercise

• Write a Lean program to prove  $x^2 - 3x + 2 = 0$ , where  $x \in \mathbb{R}$ , given

$$(x=1) \lor (x=2)$$

• In your proof create two cases x = 1 and x = 2 from the given hypothesis  $(x = 1) \lor (x = 2)$ .