04 - Simple Algebra

Lean: First Steps

Tariq Rashid

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Algebra

- Let's ask Lean to do algebra - this is new.

Task

- Show the following is true

$$(a+b)(a-b) = a^2 - b^2$$

- a and b are integers.

Maths

- Let's try to write out a proof, aiming to show and justify each step.

$$
\begin{aligned}
(a+b)(a-b) &= a^2 - ab + ba - b^2 && \text{by algebra} \\
&= a^2 - b^2 && \text{by algebra} \qquad \square
\end{aligned}
$$

- Simple proof - no need for any hypotheses, no substitutions, and no arithmetic.

- Step by step:
  - We start with the expression we want to prove something about, $(a+b)(a-b)$.
  - We expand out the brackets, multiplying every combination of terms inside the brackets, to give $a^2 - ab + ba - b^2$.
  - We collect like terms, and find that ba and $-ba$ cancel out, to give the desired result $a^2 - b^2$.
- All we needed for this proof was basic algebra, specifically, multiplying out brackets, and collecting like terms.

Code

```
-- 04 - Simple Algebra

import Mathlib.Tactic

example {a b : ℤ} : (a - b) * (a + b) = a^2 - b^2 := by
  calc
    (a - b) * (a + b) = a^2 - a*b + a*b - b^2 := by ring
    _ = a^2 - b^2 := by ring
```

Code

- Notice there are no hypotheses to declare.
- Proof body starts with the expression we want to prove something about, `(a - b) * (a + b)`...
- ... then states this is equal to `a^2 - a*b + a*b - b^2`.
- This is justified by the **ring** tactic, which can do basic algebra.
- The final line states this is `a^2 - b^2`, again justied by the **ring** tactic.

Code

- First use of the ring tactic was to justify **multiplying out the brackets**.
- Second use of the ring tactic was to justify **collecting like terms**, and **cancelling** the `-a*b` and `+a*b` terms.
- We could combine both steps into one, using the ring tactic just once - try it!

```
calc
  (a - b) * (a + b) = a^2 - b^2 := by ring
```

Intentional Error

- What would happen if we changed `{a b : ℤ}` to `{a b : ℕ}`?
- Try it!
- The Infoview shows errors - the ring tactic failed. Why?

Intentional Error

- If a and b are natural numbers, then $(a - b)$ doesn't make sense if $b > a$.
- ... because natural numbers can't be negative integers.
- More formal - Lean fails to apply ring tactic because $\mathbb{N}$ is not a ring under subtraction.
- This illustrates one of the benefits of Lean. It can catch apparently small assumptions that end up invalidating our results.

Easy Exercise

- Write a Lean program to prove $(a+b)^2 = a^2 + b^2$ if we know $ab = 0$, where $a, b \in \mathbb{Z}$.
- This will require both the ring and rewriting tactics.