# 05 - Inequalities

Lean: First Steps

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## Inequalities

- Previously we used an inequality as the final proof step, to establish 4 > 1.
  - Simple numerical comparison.
- Here we have an inequality in the hypothesis of the theorem we want to prove.
  - Allows more general theorems.

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# Task

Given

$$b = a^2$$
$$a \ge 2$$

our task is to show

$$b \ge 4$$

• where  $a, b \in \mathbb{N}$ 



- Conclusion holds for any  $a \ge 2$ .
  - In the first chapter a was very specifically constrained to a=4.
  - So this theorem is much broader in scope.
- First chapter's proof concluded 4 > 1, justified by a simple numerical comparison.
  - Can't do that here because b in  $b \ge 4$  could be any one of an infinity of natural numbers.
- So there is something fundamentally different about this task.

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• Structured proof.

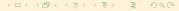
$$b = a^2$$
 given fact (1)

$$a \ge 2$$
 given fact (2)

$$b = a^2$$
 using fact (1)

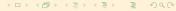
$$\geq (2)^2$$
 using fact (2)

- First list the given facts,  $b = a^2$  and  $a \ge 2$ .
- Start with the expression we want to prove something about, b. This is  $a^2$ , justified by the first given fact.
- Our aim is to express b as a comparison to 4, so we want to resolve a.
- The second fact  $a \ge 2$  gives us  $a^2 \ge (2)^2$ .
- Finally, we say  $(2)^2 = 4$ , giving us the desired conclusion.



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- Two things worth pointing out about this proof.
- How do we justify  $a \ge 2 \implies a^2 \ge (2)^2$ ?
  - Probably didn't think about it much skipping over justification.
  - Commonly accepted result  $m \le n \implies m^i \le n^i$ , where  $m, n, i \in \mathbb{N}$ .
- Easy to misread the proof as stating b = 4.
  - Correct reading is  $b = a^2$ , and  $a^2 \ge (2)^2$ , and finally  $(2)^2 = 4$ . Summarising this chain of inequalities gives  $b \ge 4$ .



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#### Code

```
-- 05 - Simple Inequality

import Mathlib.Tactic

example {a b : N} (h1 : b = a^2) (h2: a \ge 2) : b \ge 4 := by

calc

b = a^2 := by rw [h1]

_ \ge (2)^2 := by rel [h2]
_ = 4 := by norm_num
```

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#### Code

- Nothing structually different about this proof.
- Second hypothesis, proof objective, and proof body now use an inequality 

   rather than an equality = .
- The rel tactic is new. Used to justify statements about relations, such as "greater than or equal to"  $\geq$ .
- Here the **rel** tactic is combining a^2 and the hypothesis a  $\geq$  2 to give a^2  $\geq$  (2)^2.

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# Looking Inside The rel Tactic

- The **rel** tactic is using a **lemma** from the Lean maths library, Nat.pow\_le\_pow\_left, which encodes  $n \le m \implies n^i \le m^i$ , where  $m, n, i \in \mathbb{N}$ .
- Recall that a large body of fundamental results have been encoded into the Lean Mathlib library as theorems and lemmas.
- Interesting to see how this lemma is declared in the Mathlib library

```
abbrev Nat.pow_le_pow_left \{n: Nat\} = \{m: Nat\} = (h: n \le m) (i: Nat): n \cap i \le m \cap i
```



# Easy Exercise

• Write a Lean program to prove a < c if we know a < b and  $b \le c$ , where  $a, b, c \in \mathbb{N}$ .

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