03 - Symbols, No Numbers

Lean: First Steps

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Symbolic Statements

- Let's see Lean prove purely symbolic statements.
- Because these don't mention specific numbers, they can be stronger, more general, statements.

Task

- Given the following facts about $x, y, z, c \in \mathbb{R}$.

$$z = y^2$$

$$y = x + c$$

- Our task is to prove

$$z = (x + c)^2$$

- If true, this statement holds for **any** values of $x, c \in \mathbb{R}$.

Maths

- The task is not difficult. Even so, let's write out a structured proof.

$$z = y^2 \qquad \text{given fact} \tag{1}$$

$$y = x + c \qquad \text{given fact} \tag{2}$$

$$\begin{aligned} z &= y^2 && \text{using fact (1)} \\ &= (x+c)^2 && \text{substitution using fact (2)} \qquad \square \end{aligned}$$

Maths

- We start by listing the two given facts, $z = y^2$ and $y = x + c$.
- We want to prove something about z. The first fact gives us $z = y^2$.
- We'd like to express y in terms of x and c, if possible.
- The second fact tells us $y = x + c$. We can use it to substitute for the y in y^2. This gives us $z = (x + c)^2$, our objective.

- This proof has no numbers, it is entirely symbolic.

Code

```
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import Mathlib.Tactic

example {x y z c: ℝ} (h1 : z = y^2) (h2: y = x + c) : z = (x +
    c)^2 := by
  calc
    z = y^2 := by rw [h1]
    _ = (x + c)^2 := by rw [h2]
```

Code

- Notice how similar this code is to that from the previous chapter.
- We can see in the proof header how we
  - define the variables `x`, `y`, `z` and `c` as real numbers
  - list the two hyoptheses
  - specify the overall proof objective `z = (x + c)^2`.

Code

- The main body of the proof is also similar.
- We first establish that `z = y^2`, justifying it by the first hypothesis `h1`.
- Then we say this is the same as `(x + c)^2`, justifying it with second hypothesis `h2`.
- That completes the proof, which is one line shorter than the previous one because we don't need to do a numerical calculation.

Easy Exercise

- Write a Lean program to prove that $z = x$ given $z = y$ and $y = x$, where $x, y, z \in \mathbb{R}$.