11 - Existence

Lean: First Steps

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Tariq Rashid 11 - Existence October 27, 2024 1/11

Existence Proofs

• The aim of an existence proof is simply to show something exists.

Tariq Rashid 11 - Existence October 27, 2024 2 / 11

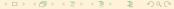
Task

• Show there exists a natural number n, such that

$$n^2 + 1 = 10$$

• In mathematical notation, where the symbol \exists means "there exists"

$$\exists n \in \mathbb{N} \ [n^2 + 1 = 10]$$



Tariq Rashid 11 - Existence October 27, 2024 3 / 11

Task

- The essence of an existence proof is to demonstrate an object exists that satisfies any given conditions.
- Calculating, deriving, or even guessing, what that object is, is not the central point.

Tariq Rashid 11 - Existence October 27, 2024 4 / 11

Maths

- After some mental trial and error we find that n = 3 works.
- We don't need to justify how we arrived at our example. We only need to show it satisfies the condition $n^2 + 1 = 10$.



Tariq Rashid 11 - Existence October 27, 2024 5 / 11

Maths

$$\exists n \in \mathbb{N}[n^2 + 1 = 10]$$

proof objective

use
$$n = 3$$

 $n^2 + 1 = (3)^2 + 1$
 $= 10$

chosen example (1)

using (1)

by arithmetic



Code

```
-- 11 - Existence proof

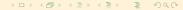
import Mathlib.Tactic

example : ∃ n: N, n^2 + 1 = 10 := by

use 3

calc

3^2 + 1 = 10 := by norm_num
```



Tariq Rashid 11 - Existence

Code

- The proof header has no separate definition of variable types, and no hypothesis. The entire theorem is in the proof objective.
- Proof objective \exists n: \mathbb{N} , $n^2 + 1 = 10$ says there exists a natural number n, such that $n^2 + 1 = 10$.
- use 3 tells Lean that we propose 3.
- Infoview confirms use 3 changes the goal from \exists n, n^2 + 1 = 10 to 3^2 + 1 = 10.
- We can use a simple calc section to show that 3^2 + 1 is indeed 10.



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Bad Example

- Let's see what happens if we choose a bad example, say n=4.
- This means changing the code to use 4.
- Placing the cursor after use 4 shows the proof goal.

$$+4^2 + 1 = 10$$

This goal is impossible to prove, because 17 is not 10.

Simpler Code

```
-- 11 - Existence proof

import Mathlib.Tactic

example : ∃ n: N, n^2 + 1 = 10 := by

use 3

norm_num
```

- The goal after use 3 is \vdash 3^2 + 1 = 10.
- This is simple enough to apply the **norm_num** tactic.
- Doesn't really need a calc section, which is better suited to derivations requiring several steps.

Easy Exercise

- Write a Lean program to prove there exists a natural number greater than 5.
- In mathematical notation, the proof objective is as follows, and should translate into Lean code fairly directly.

$$\exists n \in \mathbb{N} \ [n > 5]$$

 As an optional extra challenge, try writing the proof in the concise form shown above.

Tariq Rashid 11 - Existence October 27, 2024 11 / 11