13 - Disequality

Lean: First Steps

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Using Lemmas

- In addition to **definitions**, like Odd and Even, Mathlib also contains **lemmas** and **theorems** we can use.
- Here we'll use a lemma to support a simple disequality proof.

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Task

• Given a natural number n < 5, show that

$$n \neq 5$$

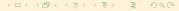
- $n \neq 5$ is a **disequality**
- n < 5 is an inequality



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Task

- Common knowledge : given $a, b \in \mathbb{N}$, if a < b is true, then $a \neq b$.
- That common knowledge might seem trivial, but we'll think of it as a small lemma.
- If we can show n < 5, then we can conclude $n \neq 5$, using that lemma.



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Maths

$$n < 5$$
 hypothesis (1)
 $n \neq 5$ proof objective (2)

$$a < b \implies a \neq b$$
 existing lemma (3)

$$n < 5$$
 sufficient goal, by lemma (3) (4)

$$n < 5$$
 using (1) (5)

$$n < 5 \implies n \neq 5$$
 by lemma (3)

Maths

- We start with the hypothesis n < 5, and our proof objective $n \neq 5$.
- We know about a lemma (3) applicable to natural numbers, that if a < b then $a \neq b$.
 - If we can prove n < 5, then we can conclude $n \neq 5$.
 - This changes our proof goal from $n \neq 5$ to n < 5.
- n < 5 is given by hypothesis (1).
- So n < 5, and by lemma (3) we finally conclude $n \neq 5$.



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Code

```
-- 13 - Lemma: Not Equal from Less Than
import Mathlib.Tactic
example {n : N} (h: n < 5): n ≠ 5 := by
apply ne_of_lt
exact h</pre>
```



Code

- Proof header declares n as a natural number, establishes hypothesis h: n < 5, specifies proof objective $n \neq 5$.
- apply applies a lemma or theorem to the current goal, usually resulting in a change in goal.
- Here, it applies a lemma named ne_of_lt, which means "not equal from less than".
 - we can prove the "not equal" goal by proving a "less than" goal.
- The Infoview will show that apply ne_of_lt does indeed change the current proof goal from $n \neq 5$ to n < 5.

Code

- The current goal is now n < 5.
- We could use apply h to resolve the goal.
- Since the goal matches exactly hypothesis h, we can use exact h.
- It may be helpful to correlate this new code back to the maths proof:
 - apply ne_of_lt corresponds to line (4) of the maths proof
 - exact h corresponds to line (5).



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- We can use apply wherever we use exact.
- The benefit of exact is that it is stricter than apply.
 - The hypothesis or lemma must exactly match the current goal, and if a misunderstanding has led to that not being true, it will be exposed immediately.

Infoview

 Placing the cursor before apply ne_of_lt shows the original proof goal.

```
\begin{array}{l} n : \mathbb{N} \\ h : n < 5 \\ \vdash n \neq 5 \end{array}
```

 Moving the cursor to the beginning of the next line after apply ne_of_lt shows the goal has changed.

```
n : \mathbb{N}
h : n < 5
\vdash n < 5
```

Lemmas & Theorems

- The distinction between what is called a lemma or a theorem in Mathlib is not precise.
- Ultimately it doesn't matter as both are used in the same way.
- Searching for suitable lemmas and theorems in Mathlib is currently not ideal. Many do conform to a naming convention, which helps.

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Easy Exercise

- Write a Lean program to prove $n \neq 5$, given n > 5, where n is a natural number.
- The proof will be almost exactly the same as this chapter's example, except the lemma will be "not equal from greater than".
- Work out the required lemma's Mathlib name fom the naming convention, or search the online Lean documentation to find it.

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