05 - Inequalities

Lean: First Steps

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August 25, 2024

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Inequalities

- Previously we used an inequality as the final proof step, to establish 4 > 1.
 - Simple numerical comparison.
- Here we have an inequality in the hypothesis of the theorem we want to prove.
 - Allows more general theorems.

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Task

Given

$$b = a^2$$
$$a \ge 2$$

our task is to show

$$b \ge 4$$

• where $a, b \in \mathbb{N}$



- Conclusion holds for any $a \ge 2$.
 - In the first chapter a was very specifically constrained to a=4.
 - So this theorem is much broader in scope.
- First chapter's proof concluded 4 > 1, justified by a simple numerical comparison.
 - Can't do that here because b in $b \ge 4$ could be any one of an infinity of natural numbers.
- So there is something fundamentally different about this task.

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• Structured proof.

$$b = a^2$$
 given fact (1)

$$a \ge 2$$
 given fact (2)

$$b = a^2$$
 using fact (1)

$$\geq (2)^2$$
 using fact (2)

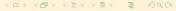
- First list the given facts, $b = a^2$ and $a \ge 2$.
- Start with the expression we want to prove something about, b. This is a^2 , justified by the first given fact.
- Our aim is to express b as a comparison to 4, so we want to resolve a.
- The second fact $a \ge 2$ gives us $a^2 \ge (2)^2$.
- Finally, we say $(2)^2 = 4$, giving us the desired conclusion.



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- Two things worth pointing out about this proof.
- How do we justify $a \ge 2 \implies a^2 \ge (2)^2$?
 - Probably didn't think about it much skipping over justification.
 - Commonly accepted result $m \le n \implies m^i \le n^i$, where $m, n, i \in \mathbb{N}$.
- Easy to misread the proof as stating b = 4.
 - Correct reading is $b = a^2$, and $a^2 \ge (2)^2$, and finally $(2)^2 = 4$. Summarising this chain of inequalities gives $b \ge 4$.



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Code

```
-- 05 - Simple Inequality

import Mathlib.Tactic

example {a b : N} (h1 : b = a^2) (h2: a \ge 2) : b \ge 4 := by

calc

b = a^2 := by rw [h1]

_ \ge (2)^2 := by rel [h2]
_ = 4 := by norm_num
```

Code

- Nothing structually different about this proof.
- Second hypothesis, proof objective, and proof body now use an inequality

 rather than an equality =.
- The rel tactic is new. Used to justify statements about relations, such as "greater than or equal to" ≥.
- Here the rel tactic is combining a² and the hypothesis a ≥ 2 to give a² ≥ (2)².



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Looking Inside The rel Tactic

- The rel tactic is using a lemma from the Lean maths library, Nat.pow_le_pow_left, which encodes $n \leq m \implies n^i \leq m^i$, where $m, n, i \in \mathbb{N}$.
- Recall that a large body of fundamental results have been encoded into the Lean Mathlib library as theorems and lemmas.
- Interesting to see how this lemma is declared in the Mathlib library

```
abbrev Nat.pow_le_pow_left {n : Nat} {m : Nat} (h : n < m)</pre>
    (i : Nat) :
    n \hat{i} \leq m \hat{i}
```

Easy Exercise

• Write a Lean program to prove a < c if we know a < b and $b \le c$, where $a, b, c \in \mathbb{N}$.

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