Devoir Maison Analyse de Données

1 VQ diagonalizable

1.1 Q diagonal

This the case we use in our "cours".
We have $Q = \text{diag}(\dots q_j \dots)$ with $q_j > 0$ (as Q is a scalar product). We can write

$$Q^{1/2} = \begin{pmatrix} \sqrt{q_1} & & & \\ & \sqrt{q_2} & & \\ & & \ddots & \\ & & & \sqrt{q_p} \end{pmatrix} \quad Q^{-1/2} = \begin{pmatrix} 1/\sqrt{q_1} & & & \\ & 1/\sqrt{q_2} & & \\ & & \ddots & \\ & & & 1/\sqrt{q_p} \end{pmatrix}$$

and we have

$$\begin{aligned} Q^{1/2}Q^{1/2} &= Q \\ Q^{1/2}Q^{-1/2} &= I_p \end{aligned}$$

• VQ diagonalizable
$Q^{1/2}VQ^{1/2}$ is symmetric hence diagonalisable (and can be written as $B'B$ thus is semi definite positive).
Let us call $(b^{(k)}, \lambda_k \geq 0)$ its eigenvector / eigenvalue. Let take it with norm 1 $(b^{(k)\prime}b^{(k)} = 1)$

$$\begin{aligned} Q^{1/2}VQ^{1/2}b^{(k)} &= \lambda_k b^{(k)} \\ Q^{-1/2}Q^{1/2}VQ^{1/2}b^{(k)} &= \lambda_k Q^{-1/2}b^{(k)} \\ VQ^{1/2}Q^{1/2}Q^{-1/2}b^{(k)} &= \lambda_k Q^{-1/2}b^{(k)} \\ VQa^{(k)} &= \lambda_k a^{(k)} \end{aligned}$$

using in the last equation the following definition $a^{(k)} = Q^{-1/2}b^{(k)}$
As $Q^{-1/2}$ is a matrix of an one to one endomorphism $a^{(k)}$ is not 0 and is thus an eigenvector of VQ with $\lambda_k \geq 0$ its eigenvalue.
• Norm

$$\begin{aligned} b^{(k)\prime}b^{(k)} &= 1 \\ b^{(k)\prime}Q^{-1/2}Q^{1/2}Q^{1/2}Q^{-1/2}b^{(k)} &= 1 \\ a^{(k)\prime}Qa^{(k)} &= 1 \end{aligned}$$

Thus we get a Q-norm 1 for $a^{(k)}$ as required.

1.2 Q not diagonal

As Q is a symmetric definite positive matrix (scalar product matrix) it is diogonalizable with an orthonormal basis of eigenvectors. We thus have

$$Q = U\Delta U'$$

where U is a matrix of orthonormal eigenvectors in each columns and $\Delta = \text{diag}(\delta_i)$ the diagonal matrix of eigenvalues ($\delta_i > 0$)
We can write

$$Q^{1/2} = U\Delta^{1/2}U'$$
$$Q^{-1/2} = U\Delta^{-1/2}U'$$

and we have

$$Q = Q^{1/2}Q^{1/2}$$
$$I_p = Q^{1/2}Q^{-1/2} = Q^{-1/2}Q^{1/2}$$

We can follow along the same lines as in the diagonal case to find that VQ is diagonalizable.