

A quantum mechanical model of alpha particle emission

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Research question: What is the relation between the half life of a radioactive nucleus and the released energy of its alpha decay?

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Chapter 1

Introduction

“If you think you understand Quantum Mechanics, you don’t understand Quantum Mechanics” was the phrase used by the Nobel Prize winning physicist Richard Feynman to describe this field. Quantum Mechanics rapidly caught my attention after finding out all the strange and bizarre phenomena it accurately predicted. Among these phenomena is Quantum Tunneling, in which particles can cross a region where the potential energy is greater than its kinetic energy, which is completely prohibited in classical mechanics. In the process of investigating the way in which we mathematically describe this phenomenon, I found out that α -decay was the result of an alpha particle tunneling away from the nucleus. I was amazed by this fact, and, as a future theoretical physicist, decided to inquire about this topic in order to mathematically describe it and compare my results with the experimental data. In this sense, the question this investigation aims to answer is: What is the relation between the half life of a radioactive nucleus and the released energy of its alpha decay?

Chapter 2

Theoretical Framework

2.1 Quantum Mechanics

During the last century, it was gradually discovered (by Planck, Heisenberg, Born, and more) that subatomic particles can only be described statistically, in contrast with the deterministic way in which classical systems behave. This means that, knowing the initial conditions, one could deduce the state of a classical system at a certain point in time. This does not happen in Quantum Mechanics: one can only know the probability distribution of the different physical quantities that describe the system.¹

2.1.1 The Shrödinger Equation and the Wave Function

The Shrödinger Equation, developed by Erwin Shrödinger in 1926, can be understood as the quantum-mechanical analog to $F = ma$: by knowing the forces acting on our classical system, we can deduce its equation of motion through Newton's Second Law; so, by knowing the potential energy present in our quantum system, one can find out its wave function through the complex second order partial differential equation known as Shrödinger's Equation²:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V\Psi(x, t) \quad (2.1)$$

¹For the sake of a more comprehensive analysis, the mathematical background necessary to develop the physical theory will be skipped and left to the appendix (check Appendix A).

²See A.1.1 and A.1.3 for more on complex numbers and differential equations.

where Ψ is the wave function (a function that can output complex values) and V is the potential previously mentioned. The way in which the wave function describes the state of a quantum system (e.g. a single particle) is through its square modulus, as it gives us the probability density function of the particle's position (Shankar, 1994 [15]). In other words, the probability of finding our particle between a and b is given by:

$$P_{ab} = \int_a^b |\Psi|^2 dx = \int_a^b \Psi \Psi^* dx \quad (2.2)$$

where Ψ^* is the complex conjugate of Ψ . Necessarily, the particle that we are studying has to be somewhere in space, so the probability of finding the particle throughout it has to be 1, hence:

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1 \quad (2.3)$$

This property is called unitarity.

Furthermore, each wave function comes with its correspondent probability current (sometimes called particle flux), given by:

$$J(x, t) = \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) \quad (2.4)$$

The probability current can be understood by comparing making a simile with the electric current. The latter tells us how much the electric charge is changing in a place per unit time, meanwhile the other one tells us how much the probability for the particle to be in a specific region in space is changing per unit time. Its formula is derived by differentiating $|\Psi|^2$. The other way of understanding this quantity, according to Zwiebach (2016) [17], is as a particle flux. Sometimes, we are not able to describe quantum systems referring to single particles, but to a number of them. Hence, in this cases, the probability current will be proportional to the number of particles entering a region per second.

2.1.2 Stationary Solutions

It is usually very hard to find solutions to the Shrödinger Equation analytically, but one way in which it is usually solved is by proposing a separable solution, which has a time

dependant factor multiplied by a time-independent factor: $\Psi(x, t) = f(t)\psi(x)$. Replacing our proposed solution in the equation, and assuming V is time-independent (which happens to be common), we get:

$$i\hbar \frac{\partial}{\partial t}(f(t)\psi(x)) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}(f(t)\psi(x)) + V(f(t)\psi(x)) \quad (2.5)$$

Which is equivalent to:

$$i\hbar \psi(x) \frac{\partial}{\partial t} f(t) = -\frac{\hbar^2}{2m} f(t) \frac{\partial^2}{\partial x^2} \psi(x) + f(t)V\psi(x) \quad (2.6)$$

Dividing both sides by $f(t)\psi(x)$:

$$i\hbar \frac{1}{f} \frac{df}{dt} = -\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V \quad (2.7)$$

If we take a closer look, we can see that the LHS is a function of t , meanwhile the RHS, of x . The only way in which two functions with different variables can be equal is if they are constant. Therefore, we could set each side equal to a constant E ³:

$$i\hbar \frac{1}{f} \frac{df}{dt} = E \quad (2.8)$$

$$-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2\psi}{dx^2} + V = E \quad (2.9)$$

Rearranging both equations we get:

$$\frac{df}{dt} = -\frac{iE}{\hbar} f \quad (2.10)$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi \quad (2.11)$$

³The reason why it is called E is because it can be shown that if we take the expected value of the energy of the particle, we get this constant as a result. See Griffiths [9] for more.

We can actually solve the time dependant equation (by rearranging and integrating both sides⁴) to get that $f(t) = e^{-\frac{iE}{\hbar}t}$. Therefore, if we want to solve for the wave function, one must solve for ψ in the Time-Independent Shrödinger Equation (Eq. 2.11) and then multiply this solution by $f(t)$.

An important aspect about this kind of solutions is that they are stationary, as the characteristics of the system it describes does not depend on time.⁵ Finally, another crucial condition every wave function satisfies is that it is continuous and differentiable everywhere (Griffiths, 1995 [9]).

2.1.3 Step Potential

We will now studied the behaviour of the wave function subject to what is called a step potential (0 when x is negative and V_0 when its positive):

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$

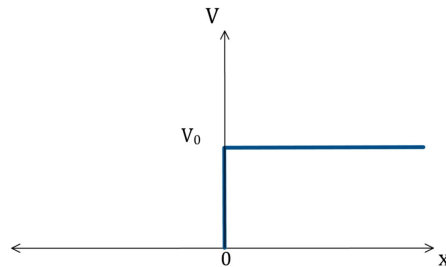


Figure 2.1: Step Potential. Own Image. Made using OneNote.

We will take $V_0 < E$ for reasons that will then become apparent. In the region $x < 0$ the Time-Independent Shrödinger equation reads:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \quad (2.12)$$

⁴See A.1.3 for the general way to solve these type of equations.

⁵This is true because, by computing the square modulus of Ψ , we get: $\Psi\Psi^* = (\psi e^{-\frac{iE}{\hbar}t})(\psi^* e^{\frac{iE}{\hbar}t}) = \psi^2$, which is time-independent.

Which can be turned into:

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad (2.13)$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$. Solving the second order linear differential equation gives⁶:

$$\psi_1(x) = Ae^{-ikx} + Be^{ikx} \quad (2.14)$$

This is the wave function of a free particle⁷. Then, in the region $x > 0$, we have:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0\psi = E\psi \quad (2.15)$$

Which can be turned into:

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E - V_0)}{\hbar^2}\psi = -k'^2\psi \quad (2.16)$$

with $k' = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$. As it has the same form of the previous equation, the wave function in this region is:

$$\psi_2(x) = Ce^{-ik'x} + De^{ik'x} \quad (2.17)$$

As mentioned before, the wave function must be continuous and differentiable everywhere. Therefore, $\psi_1(0) = \psi_2(0)$ and $\psi_1'(0) = \psi_2'(0)$. These conditions yield the following equations:

$$A + B = C + D \quad (2.18)$$

$$k(A - B) = k'(C - D) \quad (2.19)$$

We have two equations and four variables, so we cannot solve the system. We know that by the property of unitarity of the wave function, the following condition applies:

$$\int_{-\infty}^{\infty} \psi^* \psi dx = 1 \quad (2.20)$$

⁶See A.1.3. for more in this.

⁷A particle is called to be free when it is in a region of 0 potential.

Breaking up the integral into the two regions for the different wave functions we get:

$$\int_{-\infty}^0 \psi_1^* \psi_1 dx + \int_0^{\infty} \psi_2^* \psi_2 dx = 1 \quad (2.21)$$

Now, computing the integral of the square modulus for each wave function:

$$\int_{-\infty}^0 \psi_1^* \psi_1 dx = \int_{-\infty}^0 A^2 + B^2 + 2AB \cos(kx) dx \quad (2.22)$$

$$\int_0^{\infty} \psi_2^* \psi_2 dx = \int_0^{\infty} C^2 + D^2 + 2CD \cos(k'x) dx \quad (2.23)$$

Which actually diverge⁸! It is then said that these wavefunctions are not normalizable, in the sense that there is no way to adjust the coefficients such that unitarity is conserved. According to Griffiths (1995) [9], the physical interpretation of this is that there cannot be such thing as a free stationary particle, which makes sense because, if $V = 0$, the only energy the particle could have is kinetic. Therefore, what does these wave functions describe? To answer this question, we can multiply each ψ by their time-dependent factor $e^{-\frac{iE}{\hbar}t}$ to get the actual wavefunction:

$$\Psi_1(x) = Ae^{-i(kx + \frac{E}{\hbar}t)} + Be^{i(kx - \frac{E}{\hbar}t)} \quad (2.24)$$

$$\Psi_2(x) = Ce^{-i(k'x + \frac{E}{\hbar}t)} + De^{i(k'x - \frac{E}{\hbar}t)} \quad (2.25)$$

It is easy to see that the wave functions for the particle in the first and second region are in the form:

$$y(x, t) = ae^{i(kx - \omega t)} \quad (2.26)$$

which is the equation of a traveling wave⁹. This fact is very interesting, as it is telling us that matter has wave-like behavior! Therefore, according to Griffiths (1995) [9], one possible interpretation for the not normalizable solutions is that, actually, single particles behave

⁸This means that the integrals blow to infinity.

⁹See A.1.2 for a deeper explanation on this.

like “wave packets”¹⁰ (see figure 2.2), and, thus, what the wavefunctions are describing are the addition of a lot of wave packets that make up larger waves. So, these wave functions describe a group of particles moving in the same direction, making the square modulus now mean the summed up probabilities of finding each particle in space, making possible an overall probability greater than one.

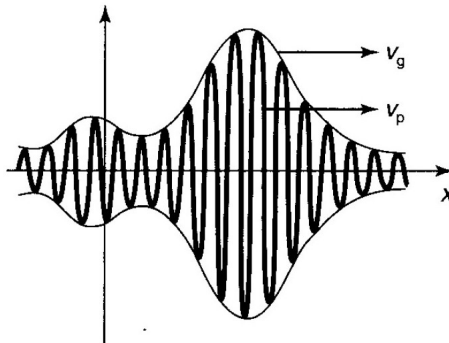


Figure 2.2: Wave Packet. Extracted from Griffiths D. (1995) [9].

Let us assume that the particles in this system are being emitted from $-\infty$. Therefore, the first term of the first wave function (Ψ_1) describes a wave that is “hitting” the region of high potential (called incident wave), as it travels from $-\infty$ to 0. The second term describes another wave that goes from one from 0 to $-\infty$ which represents a wave departing from the potential towards the left, or “bouncing off” from it (called reflected wave). Finally, the first term of the second wave function (Ψ_2) describes one going from 0 to ∞ , which represents a wave departing from the potential towards the right (transmitted wave). One final wave is left (the one coming from ∞ towards 0), but this one does not make physical sense (as there is no way in which particles emitted from negative infinity travel from infinity towards 0). Therefore, we can set $D = 0$. Now, we can solve for B and C as a function of the amplitude of the incident wave (A) to get:

$$B = \frac{1 - k_2/k_1}{1 + k_2/k_1} A \quad (2.27)$$

¹⁰Wave packets can be understood as waves that are concentrated in a small region of space. For example, a “clap” sound will make up a wave packet of air molecules. These were famously introduced by Einstein in 1905 in order to explain the behaviour of the photon, an example of a quantum particle.

$$C = \frac{2}{1 + k_2/k_1} A \quad (2.28)$$

We then define two quantities known as transmission and reflection coefficients, T and R respectively, as follows:

$$T = \frac{J_{transmitted}}{J_{incident}} \quad (2.29)$$

$$R = \frac{J_{reflected}}{J_{incident}} \quad (2.30)$$

(where the J 's are the probability currents). According to Zwiebach (2016) [17], these two quantities gives us the probability of a particle to be transmitted (in the case of T) or reflected (on that of R). The reason why this is true is that T divides the number of particles (per second) that are being transmitted between the ones that are arriving to the potential rise, meanwhile R divides the number of reflected particles between the ones arriving. Computing the coefficients we get:

$$T = \frac{4k'/k}{(1 + k'/k)^2} \quad (2.31)$$

$$R = \left(\frac{1 - k'/k}{1 + k'/k} \right)^2 \quad (2.32)$$

It is important to notice that they both have to add up to 1, as a particle is either transmitted or reflected.

If we would study the same system classically (particles being emitted towards a region with constant potential), when the particles arrive to the region of $x > 0$, due to conservation of energy, they will only slow down and continue their path. However, due to the wave-like behavior of matter, there is actually a chance that the particle will return to where it was coming.

2.1.4 Quantum Tunneling

Let us now analyze one type of potential (often called potential barrier) which is zero everywhere except a region between 0 and a , in which it equals V_0 :

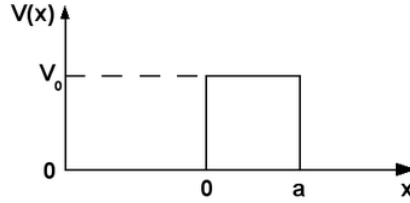


Figure 2.3: Potential Barrier. Public Domain.

We therefore have three regions: $x < 0$, $0 \leq x \leq a$ and $a \leq x$. As seen in the last subsection, solving the time-independent Schrödinger equation in the first region gives:

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad (2.33)$$

where $k^2 = \frac{2mE}{\hbar^2}$. Consequently:

$$\psi_1 = Ae^{-ik_1x} + Be^{ik_1x} \quad (2.34)$$

Now, the equation in the second region looks like this:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0\psi = E\psi \quad (2.35)$$

And, imposing the condition that $E < V_0$ gives:

$$\psi_2 = Ce^{-k_2x} + De^{k_2x} \quad (2.36)$$

The equation in the third region has the same form as that of the first one (they have identical potentials), so:

$$\psi_3 = Ee^{-ik_1x} + Fe^{ik_1x} \quad (2.37)$$

We know that ψ and its derivative must be continuous, so we get the following equations:

$$A + B = C + D \quad (2.38)$$

$$Ce^{-ak_2} + De^{ak_2} = Ee^{-ik_1a} + Fe^{ik_1a} \quad (2.39)$$

$$ik_1A - ik_1B = k_2D - k_2C \quad (2.40)$$

$$k_2De^{k_2a} - k_2Ce^{-k_2a} = ik_1Ee^{ik_1a} - ik_1Fe^{-ik_1a} \quad (2.41)$$

by computing the values of the wave function and its derivative at the boundaries.

With the same argument as in Section 2.1.3, we can set $F = 0$. We can now get every coefficient with respect to that of the incident wave. Computing the transmission coefficient we get:

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 k_2a} \quad (2.42)$$

Classically, the particle would not be able to cross this potential barrier: kinetic energy can not be negative. However, due to the wave-like behavior of quantum particles, these are able to suddenly trespass these barriers. In Figure 2.4, we can see how the wave function decreases exponentially inside the barrier, but still has a non-zero amplitude in $x > a$. This phenomenon is called Quantum Tunneling.

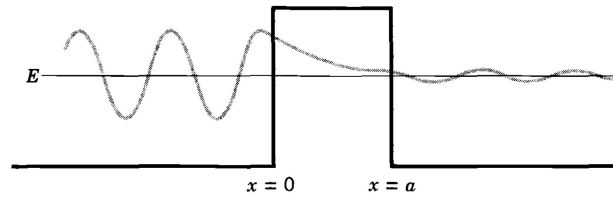


Figure 2.4: Wave function through potential barrier. Extracted from Krane, K. (1988) [11].

2.2 Nuclear Physics

2.2.1 Basic Concepts

We know that there are certain isotopes that are radioactive. This radioactivity, according to Krane (1988)[11], is caused due to an instability in the nucleus: one force (Coulomb repulsion or the Strong Nuclear Force) may be much stronger than the other, causing an excess of energy. In order to free some energy, atomic nuclei rely on what are called decay mechanisms. Some examples of these mechanisms include: Fission, Alpha Decay, Beta Decay, etc. Additionally, if we have a sample of N_0 radioactive nuclei, we know that the number of unstable nuclei left after some time t is given by:

$$N(t) = N_0 e^{-\lambda t} \quad (2.43)$$

Where λ is the decaying constant, that corresponds to the probability of an atom to decay per unit time. There is a relevant quantity called half life ($t_{1/2}$) defined as the time in which half of the particles in a given sample decay, given by $\ln(2)/\lambda$.

2.2.2 Alpha decay

Alpha decay is a decay mechanism by which an atomic nucleus emits an alpha particle, which is composed of two protons and two neutrons (${}^4_2\text{He}$). This type of decay usually happens on nuclei with high atomic numbers due to an unbalance in the forces that maintain the nucleons together: the electrostatic force increases approximately with Z^2 (as $F = kq^2/r^2$) meanwhile the Strong Nuclear Force, approximately with A (Krane, 1988 [11]). Furthermore, the reason why certain nuclei "choose" to decay through alpha emission can be seen with an analysis of the energy before and after the process. Assuming the primordial nucleus¹¹ is at rest, we have that the initial energy is:

$$E = m_Z c^2 \quad (2.44)$$

¹¹The primordial nucleus is the nucleus that will decay.

The final energy of the system will thus have the rest energy of the daughter nucleus, of the emitted particle (call it X) and the kinetic energy of both of them:

$$E = m_{Z'}c^2 + m_Xc^2 + K_{Z'} + K_X \quad (2.45)$$

Hence:

$$m_Zc^2 = m_{Z'}c^2 + m_Xc^2 + K_{Z'} + K_X \quad (2.46)$$

$$(m_Z - m_{Z'} - m_X)c^2 = K_{Z'} + K_X = Q \quad (2.47)$$

where Q equals the liberated energy (the mass defect), which equals the kinetic energy of both particles. With this equation, it is clear that, for the primordial nucleus (Z) to free as much energy as possible, X , the particle emitted, necessarily has to have a very low mass. With atomic nuclei, a very low mass also means high stability¹². According to Krane (1988) [11], the alpha particle is a good candidate for the X particle due to its high stability and low mass.

Furthermore, as mentioned before, quantum tunneling is the mechanism by which an alpha particle is emitted from an unstable nucleus. The short range of the attractive Strong Nuclear Force (approximately the nucleus's radius) produces a strong negative potential "inside" it. For distances greater than the nucleus's radius, only the electric potential produced by the protons prevails, which has a strong and positive character. Therefore, as a simplified model, in this investigation, the potential energy in the system containing only an atomic nucleus will contain two regions (considering r as the distance from the center of the nucleus): one constant negative potential $-V_0$ inside the nucleus ($r < a$, where a is the atomic nucleus) and the Coulomb potential between the daughter nucleus (of charge $Z'e$, where e is the elemental charge) and the alpha particle (of charge $2e$) which equals $\frac{1}{4\pi\epsilon_0} \frac{(Z'e)(2e)}{r}$. In summary:

¹²This happens due to the fact that the more stable the particle is, the less energy it contains, and, as $E = mc^2$, little mass is added to the mass of the nucleus.

$$V(r) = \begin{cases} -V_0 & r < a \\ \frac{1}{4\pi\epsilon_0} \frac{2Z'e^2}{r} & r > a \end{cases}$$

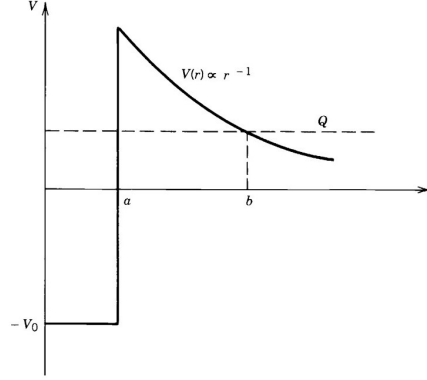


Figure 2.5: Nuclear Potential. Extracted from Krane, K. (1988). [11]

Additionally, we may assume that the alpha particle forms inside the nucleus and behaves like a single particle¹³. Furthermore, the energy of the alpha particle outside the nucleus will be Q , as it is the energy liberated by the decay process. Hence, the region outside the nucleus in which the alpha particle will be classically "allowed" to exist will be where its distance from the center of the nucleus is greater than b , such that $b = \frac{1}{4\pi\epsilon_0} \frac{2Z'e^2}{Q}$ (e.g. when the potential energy equals its kinetic energy). Therefore, there will exist a potential barrier in the region $a < r < b$ (see Figure 2.5).

Furthermore, according to Chi (2012) [13], there exists a law that relates the energy liberated (Q) during an alpha decay process and the half-life of the radioactive nucleus called the Geiger-Nutall Law:

$$\log t_{1/2} = aQ^{-\frac{1}{2}} + b \quad (2.48)$$

Where a and b are constants that depend on the element decaying. This law predicts that Q and $t_{1/2}$ are negatively correlated. This makes sense because, if Q is relatively big, it means that the nucleus has a lot of energy to free and thus it is very unstable, meaning that it will rapidly decay.

¹³This model is called the one-body model, firstly introduced by Gurny and Condon as well as Gamow [8].

Chapter 3

Theoretical Development

We will now try to estimate the probability per second of an atomic nucleus to decay through the emission of an alpha particle. We will do so by a quantum mechanical description of the alpha particle. In that sense, we will first solve the Schrödinger equation for each region of the potential (taking in consideration a 1-D model of the atomic nucleus), then establish the equations for the coefficients of the solutions, proceed with the calculation of the probability currents and the transmission coefficients, following with the computation of the decaying constants and half-lives and finally the comparison between our values and the experimental ones.

3.1 Solving the Schrödinger Equation

Starting with the Schrödinger equation, the potential that will be used, as it was already established, is the following one:

$$V(x) = \begin{cases} -V_0 & 0 < x < a \\ \frac{1}{4\pi\epsilon_0} \frac{2Z'e^2}{x} & x > a \end{cases}$$

We then get two equations to solve (one for each region of the potential), that are¹:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi_1}{dx^2} - V_0 = Q\psi_1 \quad (3.1)$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi_2}{dx^2} + \frac{1}{4\pi\epsilon_0} \frac{2Z'e^2}{x} \psi_2 = Q\psi_2 \quad (3.2)$$

From previous work, we know that:

$$\psi_1 = Ae^{i\kappa x} + Be^{-i\kappa x} \quad (3.3)$$

where $\kappa^2 = \frac{2m(Q+V_0)}{\hbar^2}$. The second wave function is significantly more difficult to solve analytically, so we will leave it to a computer software (Wolfram Mathematica²). Rearranging the second equation we are left with:

$$\frac{d^2\psi_2}{dx^2} - \frac{mZ'e^2}{\pi\epsilon_0\hbar^2} \frac{1}{x} \psi_2 = -\frac{2mE}{\hbar^2} \psi_2 \quad (3.4)$$

$$\frac{d^2\psi_2}{dx^2} = \left(\frac{k'}{x} + (ik)^2\right) \psi_2 \quad (3.5)$$

where $k' = mZ'e^2/\pi\epsilon_0\hbar^2$ and $k^2 = 2mQ/\hbar^2$. Solving this differential equation yields³:

$$\psi_2 = Cxe^{-ikx}U\left(1 + \frac{k'}{2ik}, 2, 2ikx\right) \quad (3.6)$$

Where U is the hypergeometric function of second kind, defined by:

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt \quad (3.7)$$

¹Notice that we are only taking in consideration "the right side" of the system e.g. ignoring the region for $x < 0$. This, in theory, should not be a problem because the probability of the alpha particle to decay must be the same no matter what "side" it decays from. So, it turns out to be more convenient as we are left with only two differential equations (instead of three) and we get the same results.

²The Wolfram code used to solve the differential equation plus further calculations is in [5]

³The actual solution involved one more term: $Dxe^{-ikx}F_1^1\left(1 + \frac{k'}{2ik}, 2, 2ikx\right)$, where F_1^1 is the Kummer confluent hypergeometric function. The reason why this term was discarded is in the Appendix B.

Where $\Gamma(a)$ is the well-known Gamma function, defined as:

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt \quad (3.8)$$

So, the wave function in the $x > a$ region, after some simplification, reads:

$$\psi_2 = \frac{Cxe^{-ikx}}{\Gamma(1 - ik'/2k)} \int_0^\infty e^{-2ikxt} \left(1 + \frac{1}{t}\right)^{ik'/2k} dt \quad (3.9)$$

For the sake of simplicity, we will define $U(x) = xe^{-ikx}U(1 - \frac{ik'}{2k}, 2, 2ikx)$ (a U function with only one entry). Therefore, the solutions to the Shrödinger equations are:

$$\begin{aligned} \psi_1(x) &= Ae^{i\kappa x} + Be^{-i\kappa x} \\ \psi_2(x) &= CU(x) \end{aligned} \quad (3.10)$$

Until know, the solution makes sense as we have three waves: one incident to the potential barrier ($Ae^{i\kappa x}$), one reflected from that potential ($Be^{-i\kappa x}$) and so $CU(x)$ should be the transmitted one. The next step, as its usually done for computing transmission and reflection coefficients (see Griffiths (1995)[9] and Krane (1988)[11]), is to solve for the coefficients. The equations we have for doing so are the properties of continuity and differetiability of the wave function:

$$\psi_1(a) = \psi_2(a) \quad (3.11)$$

$$\psi'_1(a) = \psi'_2(a) \quad (3.12)$$

Solving the system of equations we get:

$$A = A \quad (3.13)$$

$$B = \frac{2i\kappa}{U'(a)e^{-i\kappa a} + i\kappa U(a)e^{-i\kappa a}} A \quad (3.14)$$

$$C = \frac{i\kappa e^{-i\kappa a} e^{i\kappa a} + i\kappa e^{i\kappa a} e^{-i\kappa a}}{U'(a)e^{-i\kappa a} + i\kappa U(a)e^{-i\kappa a}} A \quad (3.15)$$

Before continuing with the computation, we wish to know how the wave function and the probability density distribution looks like for some radioactive nucleus. Choosing the decay process of ^{224}Th , we have that $Z = 90$, $A = 224$, $a = 7.22 \times 10^{-15}$ m, $b = 2.34 \times 10^{-14}$, $Q = 8.89$ MeV and $V_0 = 35$ MeV.⁴⁵ Here is a plot of $\text{Re}(\psi)$:

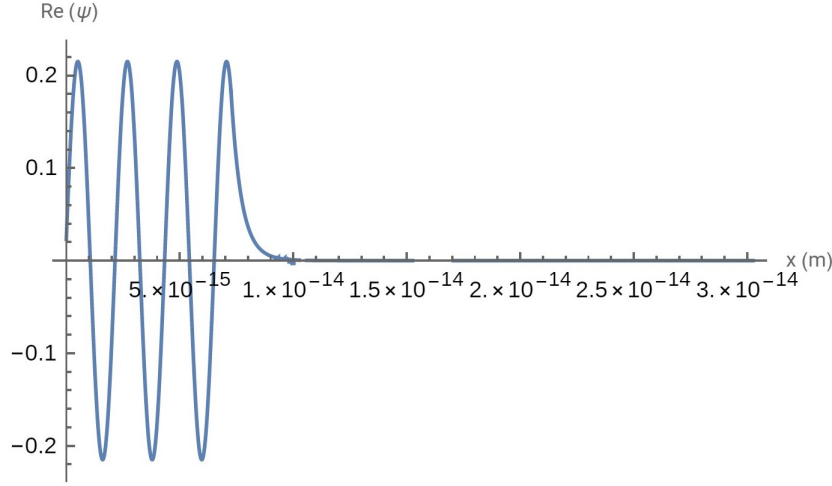


Figure 3.1: Real part of the wave function through the nuclear potential. Own Image. Made using Wolfram Mathematica [4].

What we see in Figure 3.1 is that in the region $x < a = 7.22 \times 10^{-15}$ m, the wave function behaves just like a regular sinusoidal function, but, once it gets to a , it starts rapidly decaying, looking as if it approached 0. Figure 3.2 shows the wave function rapidly decaying inside the region in which the potential is greater than the particle's energy ($a < x < b$). Once the wave function leaves this region, it starts to oscillate again (Figure 3.3). Notice that the amplitude of this oscillations is 10^9 times smaller than the ones inside the nucleus, which means that there is a very small probability that the alpha particles suddenly appears outside of the nucleus. This result makes a lot of sense, because, if we estimate that the velocity v of the particle equals $\sqrt{2Q/m}$ (this is, if the kinetic energy Q equals $\frac{1}{2}mv^2$), we will get that the particle encounters itself near $|x| = a$, $v/2a = 1.58 \times 10^{21}$

⁴For this investigation, we will assume that the potential inside the nucleus is the same for every nucleus and equals $V_0 = 35$ MeV. This, according to Krane (1988), is a “typical potential for heavy nuclei”.

⁵We are using Fermi's approximation of the nuclear radius $R = R_0 A^{1/3}$, where $R_0 = 1.2$ fm.

times per second (we will call this quantity f for frequency).⁶ This means that the particle will "bounce off" the boundaries of the nucleus a lot of times per second, each try having a really low probability of tunnelling out, so the magnitudes compensate each other.

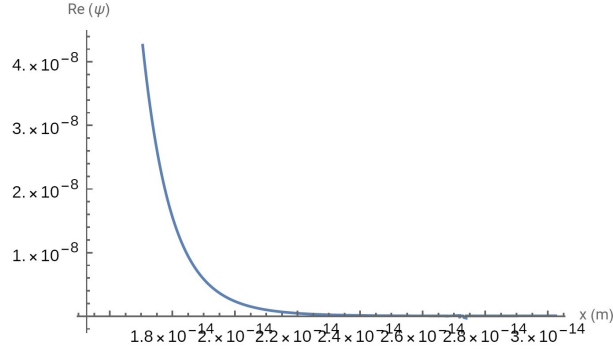


Figure 3.2: Real part of the wave function inside region where $V > Q$. Own Image. Made using Wolfram Mathematica [5].

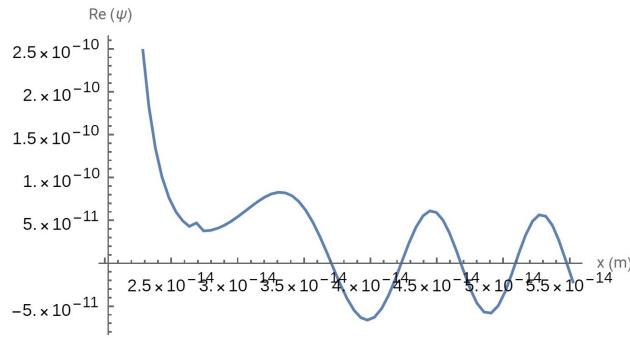


Figure 3.3: Real part of the wave function through the classically allowed region. Own Image. Made using Wolfram Mathematica [5].

3.2 Calculation of Transmission Coefficient

Returning to the computation, now that we have the coefficients, we have to remember the definition of probability current given in the first chapter:

$$J(x, t) = \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) \quad (3.16)$$

⁶This estimation is assuming that the particle behaves like a classical particle. For a more precise estimation, one could compute the expected value of the velocity inside the nucleus.

If we want to compute the transmission, we have to divide the probability current of the transmitted wave with the incident one, correspondingly. Therefore:

$$T = \frac{\Psi_{tran} \frac{\partial \Psi_{tran}}{\partial x} - \Psi_{tran}^* \frac{\partial \Psi_{tran}}{\partial x}}{\Psi_{inc} \frac{\partial \Psi_{inc}}{\partial x} - \Psi_{inc}^* \frac{\partial \Psi_{inc}}{\partial x}} \quad (3.17)$$

And here we have the probability of the alpha particle to tunnel out of the nucleus! We cannot write an exact formula because the functions (specially $U(x)$) are too complex for it to be practical, thus, when calculating the coefficients, we will rely on Wolfram Mathematica ⁷. In relation to the ^{224}Th nucleus, this probability turns out to be 2.74×10^{-18} , which is very small as expected. According to Krane (1988)[11], with this quantity, we can now estimate the probability by unit time the alpha particle will escape the nucleus (fT), which, by definition, is equal to λ , the decaying constant. Knowing λ , we can easily compute the half life of ^{224}Th , which turns out to be about 0.551 s, very close to the real half life of about 1 s.

⁷For all the calculations, see [5]

Chapter 4

Data Analysis and Comparison

We are now going to compute the theoretical values we get for the half-life of different nuclei, plot the values for each element, perform the linear regression (for the Geiger-Nutall Law, $\log t_{1/2} = aQ^{-\frac{1}{2}} + b$) and compare the experimental and theoretical data ¹.

For the Thorium isotopes ($Z = 90$) we get²:

A	Q (10^{-12} J)	$t_{1/2}$ Theo (s)	$t_{1/2}$ Exp (s)
208	1.464	3.44×10^{-3}	1.7×10^{-3}
210	1.434	77.4×10^{-3}	17×10^{-3}
212	1.424	13×10^{-3}	36×10^{-3}
214	1.404	26.1×10^{-3}	100×10^{-3}
216	1.443	5.12×10^{-3}	26.8×10^{-3}
218	1.728	407×10^{-9}	109×10^{-9}
220	1.584	29.4×10^{-6}	9.7×10^{-6}
222	1.452	2.77×10^{-3}	2.2×10^{-3}
224	1.319	0.551	1
226	1.183	329.8	1830
228	1.034	1.64×10^6	1×10^6
230	0.914	7.32×10^9	2.37×10^{12}
232	0.804	8.44×10^{13}	4.4×10^{17}

Table 4.1: Experimental and Theoretical Values for the Half Lives of Thorium Isotopes.

¹In Table 4.2 (and others) we find to statistical parameters: R^2 and r . The first one is called the determination coefficient. The more it approaches 1, the better the data fits the best fit line. The other one, r , is the Pearson correlation coefficient. If it approaches 1, the data is positively correlated; if it approaches 0, it has no correlation; and if it approaches -1, it is negatively correlated.

²Data extracted from Audi, G. (2003) [1]; National Data Nucleon Center [2]; and Holden, Norman E. (2003) [10].

Plotting $\log(t_{1/2})$ vs. $Q^{-1/2}$ (remembering Geiger-Nutall Law) we get³:

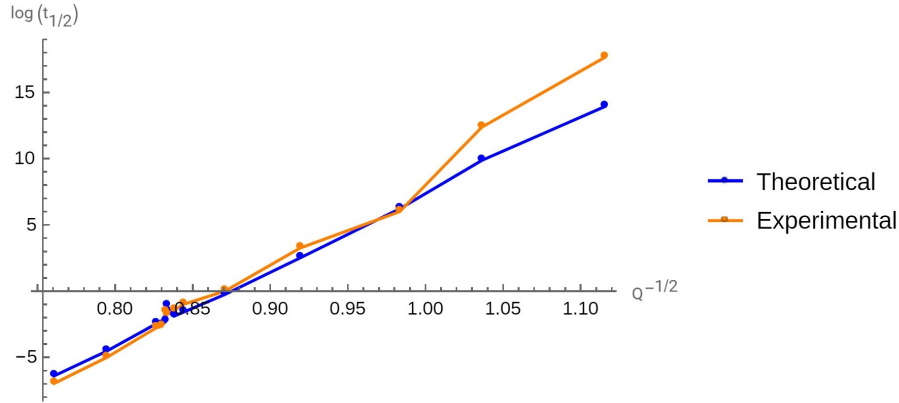


Figure 4.1: Comparison between the plot of $\log(t_{1/2})$ vs. $Q^{-1/2}$ for the theoretical and experimental values from Thorium isotopes. Note: the values for $t_{1/2}$ are measured in seconds, and the ones from $Q^{-1/2}$, in $10^6 \text{ } 1/\sqrt{J}$. Plotted using Wolfram Mathematica [4].

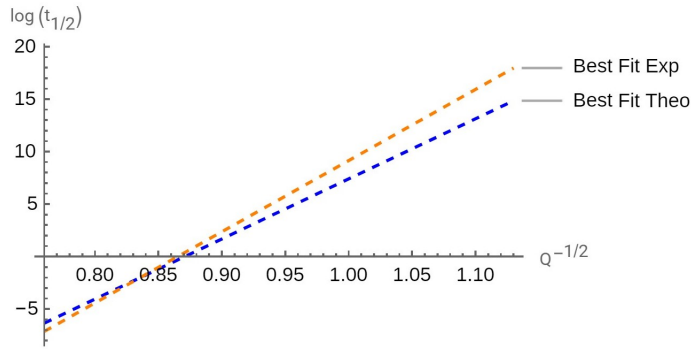


Figure 4.2: Comparison between the best fit line of the plot of $\log(t_{1/2})$ vs. $Q^{-1/2}$ for the theoretical and experimental values from Thorium isotopes. Note: the values for $t_{1/2}$ are measured in seconds, and the ones from $Q^{-1/2}$, in $10^6 \text{ } 1/\sqrt{J}$. Plotted using Wolfram Mathematica [4].

Parameter	Theoretical	Experimental
a ($10^{-6}\sqrt{J}$)	56.588	67.066
b	-49.309	-58.063
R^2	0.9969	0.9902
r	0.9984	0.9951

Table 4.2: Statistical parameters of the lines of best fit for the graph of $\log(t_{1/2})$ vs. $Q^{-1/2}$. Calculated using Excel

³Check B.1 for the tables.

Now, doing the same for Uranium isotopes ($Z = 92$)⁴:

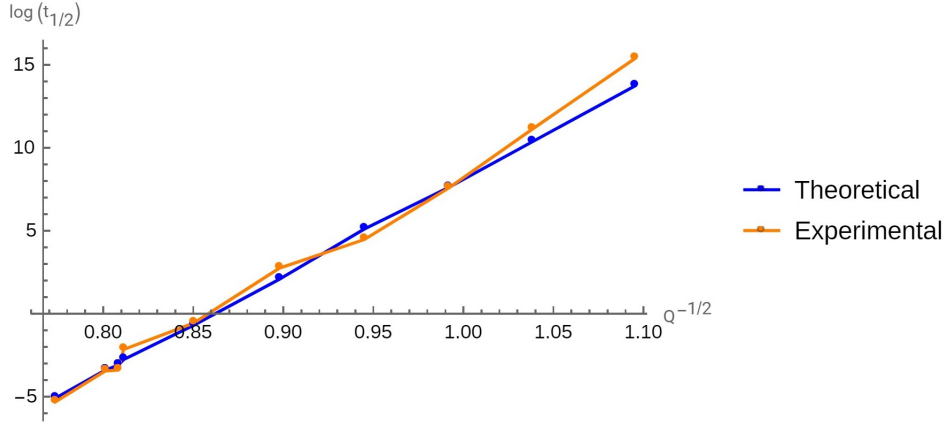


Figure 4.3: Comparison between the plot of $\log(t_{1/2})$ vs. $Q^{-1/2}$ for the theoretical and experimental values from Uranium isotopes. Note: the values for $t_{1/2}$ are measured in seconds, and the ones from $Q^{-1/2}$, in $10^6 \text{ 1}/\sqrt{J}$. Plotted using Wolfram Mathematica [4].

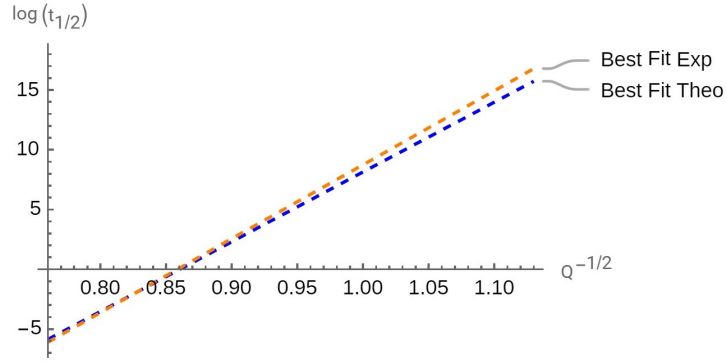


Figure 4.4: Comparison between the best fit line of the plot of $\log(t_{1/2})$ vs. $Q^{-1/2}$ for the theoretical and experimental values from Uranium isotopes. Note: the values for $t_{1/2}$ are measured in seconds, and the ones from $Q^{-1/2}$, in $10^6 \text{ 1}/\sqrt{J}$. Plotted using Wolfram Mathematica [4].

Parameter	Theoretical	Experimental
a ($10^{-6}\sqrt{J}$)	58.273	61.709
b	-50.166	-53.006
R^2	0.9999	0.9929
r	0.9999	0.9964

Table 4.3: Statistical parameters of the lines of best fit for the graph of $\log(t_{1/2})$ vs. $Q^{-1/2}$. Calculated using Excel.

⁴See Section 2.2 from the Appendix for tables. Data extracted from the NUBASE2020 (2021) [6]; and Meija, Juris (2016) [7].

Lastly, repeating the process for Plutonium isotopes ($Z = 94$)⁵:

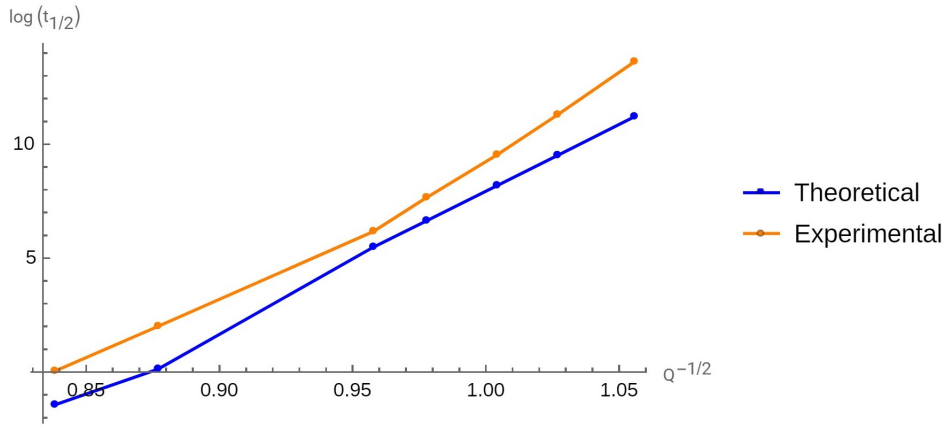


Figure 4.5: Comparison between the plot of $\log(t_{1/2})$ vs. $Q^{-1/2}$ for the theoretical and experimental values from Plutonium isotopes. Note: the values for $t_{1/2}$ are measured in seconds, and the ones from $Q^{-1/2}$, in $10^6 \text{ } 1/\sqrt{J}$. Plotted using Wolfram Mathematica [4].

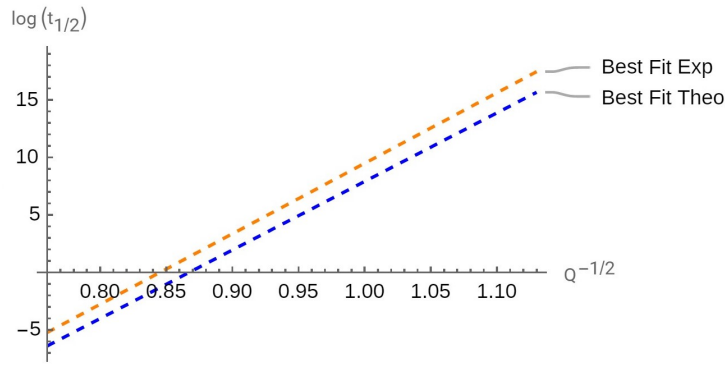


Figure 4.6: Comparison between the best fit line of the plot of $\log(t_{1/2})$ vs. $Q^{-1/2}$ for the theoretical and experimental values from Plutonium isotopes. Note: the values for $t_{1/2}$ are measured in seconds, and the ones from $Q^{-1/2}$, in $10^6 \text{ } 1/\sqrt{J}$. Plotted using Wolfram Mathematica [4].

Parameter	Theoretical	Experimental
a ($10^{-6}\sqrt{J}$)	59.636	61.324
b	-51.728	-51.838
R^2	0.9999	0.9929
r	0.9978	0.9861

Table 4.4: Statistical parameters of the lines of best fit for the graph of $\log(t_{1/2})$ vs. $Q^{-1/2}$. Calculated using Excel.

⁵Data extracted from Audi, G. (2003) [1]; National Data Nucleon Center [2]; and Holden, Norman E. (2003) [10].

Element	$a_{theo} (10^{-6}\sqrt{J})$	$a_{exp} (10^{-6}\sqrt{J})$	% of error	b_{theo}	b_{exp}	% of error
Th ⁹⁰	56.588	67.066	15.6	-49.309	-58.063	15.1
U ⁹²	58.273	61.709	5.57	-50.166	-53.006	5.36
Pu ⁹⁴	59.636	61.324	2.75	-51.728	-51.838	0.212

Table 4.5: Summary of the Geiger-Nutall coefficients for each element.

Will all this data, we will now proceed to the analysis of the results.

Chapter 5

Conclusions and Discussion of Results

5.1 General Conclusions

As we can see in Figure 4.1, 4.3 and 4.5, the theoretical values agree to a great extent with the experimental data. In the case of Thorium, the percentage of error between the a 's and b 's of the theoretical and experimental value were 15.6 and 15.1 respectively; for Uranium, 5.57 and 5.36; and, for Plutonium, 2.75 and 0.212. We can thus conclude that the element that was better described by the model was Plutonium due to its relatively low percentage of error, followed by Uranium and then Thorium. Nevertheless, by examining the values of the theoretical and experimental half lives for themselves (see Table B.1, B.3 and B.5 in Appendix), we realize that Uranium and Thorium are in general more accurate than Plutonium (with some exceptions that will be discussed later). By examining the graph of the lines of best fit for the Plutonium experimental and theoretical values (Figure 4.6) as well as its table of data (Table B.5), we can see that there is a kind of "systematic error" in the sense that it looks as if the theoretical values were being shifted by a constant value. So, the half live values are in general more accurate for Uranium and Thorium, but the percentages of error of the Geiger-Nutall coefficients are greater than the ones of Plutonium. This discrepancy occurs because in the cases of Uranium and Thorium, the model tends to be much less accurate with high values of $t_{1/2}$ (by underestimating them)

which makes the slope and the y -intercept deviate away from the real value. Therefore, we can conclude that the model is exact up to a certain extent, as for values of $t_{1/2} \gg 1$ s, the ratio between the experimental and theoretical values tends to be very significant.

There are two reasons why this probably happens. The first one is related with the fact that for the calculations for every nuclei, the same potential well depth was used (35 MeV), which is probably not the best option for very heavy nuclei (like ^{230}Th , ^{232}Th , ^{238}U , etc.). According to Liang, B. (2016) [12] the potential well depth inside the nucleus is positively correlated with its mass number. Taking α as a value proportional to A , we can see this relation in Figure 5.1. This asserts that for more accuracy with nuclei with big A (approximately ≥ 230), the average potential value should be adjusted. The other possible reason why the discrepancy mentioned above happens is that, according to Krane (1988) [11], the nucleons of mass number $A \geq 230$ are greatly deformed, making the approximation of $R = R_0 A^{1/3}$ not very accurate. This fits perfectly with our results, as it is right with the isotopes ^{230}Th , ^{232}Th , ^{238}U , etc. that the differences between the calculated and the experimental values get bigger. Additionally, these two facts could also explain the constant shift in the values of Plutonium, as the mass numbers used for the calculations ranged from 228 to 244, making them considerably out of the range of accuracy for the model. Furthermore, we can see that the graphs of $\log(t_{1/2})$ vs. $Q^{-1/2}$

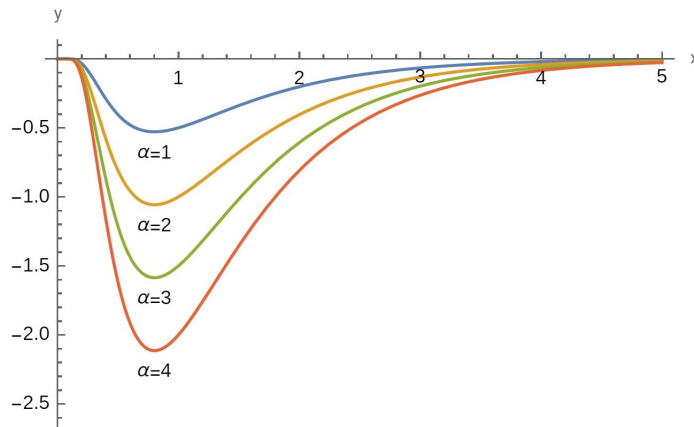


Figure 5.1: Function representing the potential well in the nucleus. Note: the y and x axis correspond to the potential and position respectively. Their units are meaningless, as only the shape of the function is wished to be analyzed. Taking α as an quantity directly proportional to A , it can be seen that the bigger A is, the deeper the potential well becomes. Own Image. Made using Wolfram Mathematica [5].

for Th have a coefficient of determination for the theoretical and experimental values of 0.9969 and 0.9902 respectively, for U, of 0.9999 and 0.9929, and, for Pu, of 0.9978 and 0.9891 (see Table 4.2, 4.3 and 4.4). These values are, in general, very close to 1, which means that the values adjust very precisely to a linear equation. Additionally, the values of the Pearson correlation coefficient for these graphs are 0.9984 and 0.9951 for Th, 0.9999 and 0.9964 for U, and 0.9998 and 0.9946 for Pu (see Table 4.2, 4.3 and 4.4). These values are also very near 1, meaning that they have a very strong positive correlation. With the analysis of these two values, we can conclude that the Geiger-Nutall law holds true, as the data adjusts in a very good manner to the equation $\log t_{1/2} = aQ^{-\frac{1}{2}} + b$.

5.2 Discussion

One of first physicists to ever attempt to explain quantum mechanically the alpha decay process was Gamow, G. in 1928 [8]. In his paper, he estimated that the probability of transmission by considering the potential outside the nucleus as potential barriers of infinitesimal length dr . He found out that this probability was given by:

$$P = e^{-2G} \quad (5.1)$$

where G is known as the Gamow factor, which equals:

$$G = \sqrt{\frac{2m}{\hbar^2 Q}} \int_a^b \sqrt{V(r) - Q} dr \quad (5.2)$$

In his book, Krane, K. (1988)[11] calculated some half lives using this formula. It seemed to know have worked very well, as every value he got was different from the experimental value for around two orders of magnitude. Still, the Gamow factor is useful when explaining the extreme sensitivity the half life has for the liberated energy. Furthermore, Sivasankaran B.R. published a paper in 2019 [16] in which he used Gamow's theory (but not precisely the same formula as Krane, K.) to compute the theoretical values of a considerable number of isotopes. His results were very accurate. Nevertheless, they were limited in the sense that he only calculated half lives that were in the order of

10^{-1} to 10^1 , meanwhile the values worked in this investigation ranged from 10^{-6} to 10^6 s, maintaining a high level of accuracy.

Additionally, Samanta, C. (2007) [14] published a paper in which the half lives of super-heavy nuclei ($Z = 102 - 120$) were calculated theoretically using the WKB approximation, which does not get to solve for the wave functions but takes in consideration a more accurate potential for the nucleus. The results were, in general, very accurate, comparable with the ones achieved in this work. The value of this paper is that the accuracy maintains for bigger A , which, in this investigation, tend to be the most inaccurate.

5.3 Final Considerations

We can conclude that the strict quantum mechanical calculations done in this investigation achieve a considerable accuracy for the experimental results for nuclei of mass number between $218 \leq A \leq 232$. It will be interesting to further this investigation with the following considerations:

- Estimating the average potential energy inside the nucleus for each isotope.
- Calculating the wave function inside the nucleus using a more realistic potential (as the one presented by Liang, B. (2016).
- Considering a 3-D model of the nucleus.
- Test the model for isotopes that decay via nuclear fission (adjusting the constants and the potential).

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Appendix A

Mathematical Preliminaries

A.1 Mathematical Framework

In order to understand the physics displayed in this investigation, it is necessary to define some mathematical concepts.

A.1.1 Complex Numbers

Complex numbers are numbers with a real and imaginary part, written in the form $z = a + ib$, where i is the imaginary unit defined as $\sqrt{-1}$ (a being the real part and ib the imaginary one). They can be represented in what is known as an Argand diagram, which functions as a coordinate system in which the "x-axis" is the real number line and the "y-axis", the imaginary number line:

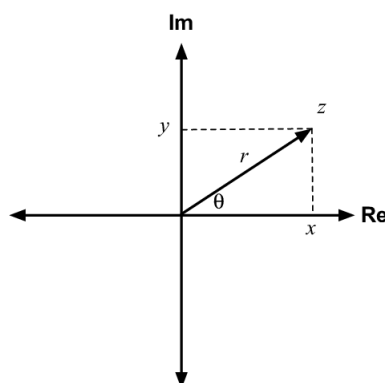


Figure A.1: Argand Diagram

Each complex number has a complex conjugate, defined as $\bar{z} = a - ib$, which corresponds to a reflection along the Real axis. Furthermore, complex numbers can be represented by their distance from the origin (or modulus, $|z| = \sqrt{a^2 + b^2}$) and the angle they form with the horizontal axis (or argument, $\theta = \arctan(b/a)$ see Figure 2.1) through the following equation: $z = |z|e^{i\theta}$ (this is known as the "polar form").¹

A.1.2 Travelling Waves

We know that the equation that describes the oscillation of single objects in time is:

$$y = A \cos(\omega t + \phi) \quad (\text{A.1})$$

By definition, a traveling wave, which, by definition is a wave that travels at constant speed (e.g. sound), can be seen as a collection of points in different position oscillating in time (e.g. air particles oscillating with respect of their positions). So, according to Chong (2020), the equation that describes a travelling wave is the following:

$$y = A \cos(kx - \omega t + \phi) \quad (\text{A.2})$$

that makes sense because, if we set $x = x_0$ (e.g. one point in the wave), the equation becomes only $A \cos(\omega t + \phi')$ (where $\phi' = \phi + kx_0$), which is an oscillation in time, just as was mentioned before. Furthermore, fixing the phase of the wave (which will mean a point in the wave itself, like a maximum or a minimum) to be ϕ_0 , we get that:

$$\phi_0 = kx - \omega t + \phi \quad (\text{A.3})$$

Rearranging:

$$x = \frac{\phi_0 - \phi}{k} + \frac{\omega}{k}t \quad (\text{A.4})$$

So, the position of the fixed point in the wave travels just like an object with no acceleration moving to the right (notice that Equation A.4 looks just like $x = x_0 + vt$). Hence, the

¹this equivalence is true due to Euler's formula: $e^{i\alpha} = \cos \alpha + i \sin \alpha$

wave is moving to the right, as all points in it have the same velocity. If we were to change ω to $-\omega$, the point will be moving to the left. Additionally, by taking in consideration that $e^{i\theta} = \cos \theta + i \sin \theta$, we can state a more general formula for a traveling wave (Chong, 2020 [3]), namely:

$$y = Ae^{i(kx - \omega t)} \quad (\text{A.5})$$

The real part of this formula will be:

$$\text{Re}(y) = \text{Re}(Ae^{i(kx - \omega t)}) \quad (\text{A.6})$$

Taking in consideration that $A = |A|e^{i\arg(A)}$ (where $\arg(A)$ is the argument of A), we can say that:

$$\text{Re}(y) = |A|\text{Re}(e^{i(kx - \omega t + \arg(A))}) \quad (\text{A.7})$$

So, it makes sense to represent a wave in this manner as the coefficient A englobes the amplitude and the phase.

A.1.3 Differential Equations

Differential equations (D.E.'s) are equations that relate a function, its variables and derivatives. In this investigation, we are only going to focus on three types of differential equations: ordinary first order separable, ordinary linear second order homogeneous, and (briefly) second order partial. The first type of D.E.'s (first order separable) are in the form $\frac{dy}{dx} = f(y) \cdot g(x)$, y is a function of x , f of y and g of x . These are solved simply by placing all the x 's to one side and y 's to the other:

$$\frac{1}{f(y)} dy = g(x) dx \quad (\text{A.8})$$

And integrating both sides.

$$\int \frac{1}{f(y)} dy = \int g(x) dx \quad (\text{A.9})$$

So that we are left with:

$$F(y) = G(x) + C \quad (\text{A.10})$$

(where F and G are the anti-derivatives of f and g respectively and C the constant of integration) and then we could, if we wanted, isolate y as a function of x .

The second type of D.E.'s to be studied are ordinary linear second order homogeneous, which have the general form:

$$ay'' + by' + cy = 0 \quad (\text{A.11})$$

(again, y is a function of x). This equation is "ordinary" because it only involves a single-variable function; linear due to a reason explained afterwards; second order because its higher derivative is the second one; and homogeneous because it can be set equal to 0 (e.g. $ay'' + by' + cy = p(x)$ will be inhomogeneous). So, as each derivative is linearly dependent on the other one, it is logical to guess an answer with the form $y = e^{\lambda x}$ (usually called "ansatz") as $y' = \lambda y$ and $y'' = \lambda y' = \lambda^2 y$. If we insert our guessed solution to the equation, we get:

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0 \quad (\text{A.12})$$

As $e^{\lambda x} \neq 0$ for any real value of the exponent, we could divide the equation by $e^{\lambda x}$ to get:

$$a\lambda^2 + b\lambda + c = 0 \quad (\text{A.13})$$

Solving for λ :

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (\text{A.14})$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (\text{A.15})$$

We therefore get two solutions:

$$y_1 = e^{\lambda_1 x} \quad (\text{A.16})$$

$$y_2 = e^{\lambda_2 x} \quad (\text{A.17})$$

To get a more general solution, we may recall the "linear" in the name for this types of equations. If we insert any linear combination² of solutions to this differential equations,

²Briefly, a linear combination of the solutions will be the sum of the solutions, each of them multiplied

namely $y = Ay_1 + By_2$, we get:

$$a(Ay_1'' + By_2'') + b(Ay_1' + By_2') + c(Ay_1 + By_2) = 0 \quad (\text{A.18})$$

$$A(ay_1'' + by_1' + cy_1) + B(ay_2'' + by_2' + cy_2) = 0 \quad (\text{A.19})$$

By definition, each of the two terms is equal to 0, so we conclude that if y_1 and y_2 are solutions to this type of differential equations, then $y = Ay_1 + By_2$ is also a solution. Another point of discussion is the nature of λ_1 and λ_2 , as they can be real and equal, real and different or complex conjugates. For this investigation, we are only interested in the complex conjugate solutions, and more specifically the ones with $b = 0$, which implies: $\lambda_1 = -\lambda_2 = i\sqrt{\frac{c}{a}} = i\omega$. The general solution therefore looks like this:

$$y = Ae^{i\omega x} + Be^{-i\omega x} \quad (\text{A.20})$$

Before discussing the next type of D.E's, it is necessary to briefly present partial derivatives. Partial derivatives are one type of derivatives that operate on multi-variable functions and give as a result the function differentiated only with respect to one variable (the other ones are treated like constants). For example, if we take the partial derivative with respect to x of $f(x, y) = x^2y$ (denoted $\frac{\partial f}{\partial x}$), we get $\frac{\partial f}{\partial x} = 2yx$.

Therefore, a second order partial differential equation is a differential equation that relates a multi-variable function to its variables and partial derivatives.

by a constant.

Appendix B

Tables

B.1 Thorium

A	Q (10^{-12} J)	$t_{1/2}$ Theo (s)	$t_{1/2}$ Exp (s)
208	1.464	3.44×10^{-3}	1.7×10^{-3}
210	1.434	77.4×10^{-3}	17×10^{-3}
212	1.424	13×10^{-3}	36×10^{-3}
214	1.404	26.1×10^{-3}	100×10^{-3}
216	1.443	5.12×10^{-3}	26.8×10^{-3}
218	1.728	407×10^{-9}	109×10^{-9}
220	1.584	29.4×10^{-6}	9.7×10^{-6}
222	1.452	2.77×10^{-3}	2.2×10^{-3}
224	1.319	0.551	1
226	1.183	329.8	1830
228	1.034	1.64×10^6	1×10^6
230	0.914	7.32×10^9	2.37×10^{12}
232	0.804	8.44×10^{13}	4.4×10^{17}

Table B.1: Experimental and Theoretical Values for the Half Lives of Thorium Isotopes.

A	$Q^{-1/2}$ (10^6 J)	$\log t_{1/2}$ Theo (s)	$\log t_{1/2}$ Exp (s)
208	0.7608	-6.3904	-6.9626
210	0.7945	-4.5317	-5.0132
212	0.8265	-2.4639	-2.7696
214	0.8210	-2.5575	-2.6576
216	0.8325	-2.2907	-1.5719
218	0.8334	-1.1113	-1.7696
220	0.8381	-1.8861	-1.4437
222	0.8441	-1.5834	-1.0000
224	0.8707	-0.2588	0.0000
226	0.9194	2.5183	3.2625
228	0.9834	6.2148	6.0021
230	1.0460	9.8645	12.3739
232	1.1155	13.9263	17.6435

Table B.2: Experimental and Theoretical Values for the Logarithm of Half Lives of Thorium Isotopes and the liberated energy to the $-1/2$.

B.2 Uranium

A	Q (10^{-12} J)	$t_{1/2}$ Theo (s)	$t_{1/2}$ Exp (s)
216	1.5192	1.697×10^{-3}	6.900×10^{-3}
218	1.5580	0.394×10^{-3}	0.350×10^{-3}
222	1.6723	7.95×10^{-6}	4.7×10^{-6}
224	1.5307	7.720×10^{-4}	3.960×10^{-4}
226	1.3834	0.199	0.269
228	1.2407	117.570	546
230	1.1099	1.241×10^5	0.291×10^5
232	1.0170	3.898×10^7	3.571×10^7
234	0.9280	2.224×10^{10}	1.273×10^{11}
238	0.8338	5.181×10^{13}	2.316×10^{15}

Table B.3: Experimental and Theoretical Values for the Half Lives of Uranium Isotopes.

A	$Q^{-1/2}$ (10^6 J)	$\log t_{1/2}$ Theo (s)	$\log t_{1/2}$ Exp (s)
216	0.8113	-2.7703	-2.1612
218	0.8012	-3.4045	-3.4559
222	0.7733	-5.0997	-5.3279
224	0.8083	-3.1124	-3.4023
226	0.8502	-0.7008	-0.5702
228	0.8978	2.0703	2.7372
230	0.9492	5.0939	4.4644
232	0.9916	7.5908	7.5529
234	1.0381	10.3472	11.1047
238	1.0951	13.7144	15.3648

Table B.4: Experimental and Theoretical Values for the Logarithm of Half Lives of Uranium Isotopes and the liberated energy to the $-1/2$.

B.3 Plutonium

A	Q (10^{-12} J)	$t_{1/2}$ Theo (s)	$t_{1/2}$ Exp (s)
228	1.4230	0.0364	1.100
230	1.300	1.3569	102.0
236	1.090	3.0526×10^5	1.4816×10^6
238	1.0460	4.3420×10^6	4.5464×10^7
240	0.9917	1.5217×10^8	3.4012×10^9
242	0.9483	3.2300×10^9	1.9449×10^{11}
244	0.8972	1.6148×10^{11}	4.1472×10^{13}

Table B.5: Experimental and Theoretical Values for the Half Lives of Plutonium Isotopes.

A	$Q^{-1/2}$ (10^6 J)	$\log t_{1/2}$ Theo (s)	$\log t_{1/2}$ Exp (s)
228	0.8383	-1.4384	0.0414
230	0.87706	0.1326	2.0086
236	0.9578	5.4847	6.1707
238	0.9778	6.6377	7.6577
240	1.0042	8.1823	9.5316
242	1.0269	9.5092	11.2887
244	1.0557	11.2081	13.6178

Table B.6: Experimental and Theoretical Values for the Logarithm of Half Lives of Plutonium Isotopes and the liberated energy to the $-1/2$.

Appendix C

Specifications

The original solution to the Shrödinger equation in Section 3.1 was:

$$\psi_2 = Cxe^{-ikx}U\left(1 + \frac{k'}{2ik}, 2, 2ikx\right) + Dxe^{-ikx}F_1^1\left(1 + \frac{k'}{2ik}, 2, 2ikx\right) \quad (\text{C.1})$$

Where F_1^1 , the Kummer confluent hypergeometric function and U , the hypergeometric function of second kind. The reason why the second term was discarded was that, in theory, we should only have three waves: one wave inside the nucleus traveling to the right, which will describe the incident wave to the potential barrier; one wave inside the nucleus travelling to the left, which will describe the reflected wave from the potential barrier; and a wave travelling from a towards ∞ , which will represent the transmitted wave. We get the first two waves from the first solution (ψ_1). If we examine the second solution (ψ_2) we can see that it has two terms, each of them representing one wave, meaning that probably one of them should be eliminated. As we can not determine analytically which wave travels where due to their high complexity¹, we solved this problem numerically by performing all the calculations that follow considering first $C = 0$ and then $D = 0$, and saw which one made more physical sense. It turned out that, when calculating the probability currents considering $C = 0$, the particle flux of the transmitted wave was 0, letting us know that that one (the term with F_1^1 , the Kummer confluent hypergeometric

¹Even though each one of the terms of the second wave function contains e^{-ikx} (meaning that they in theory travel to the same direction), we cannot discard the fact that, while simplifying or modifying the special functions, one of them suddenly contains a e^{2ikx} (or something similar) that will make it travel in the opposite direction.

function) was the wave travelling in the wrong direction.