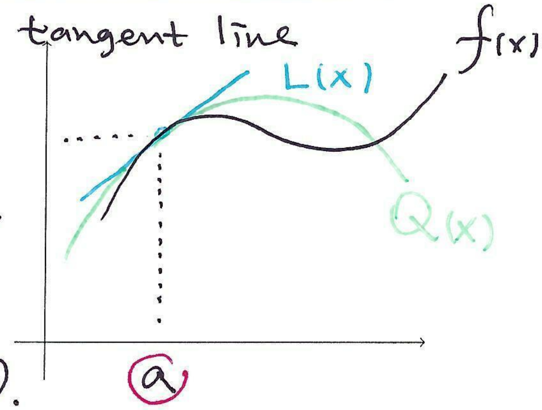
MATH 152 – Section 11.10, Taylor and Maclaurin Series

Recall that the Linear approximation of $y = f(x)$ at $x = a$ is the tangent line



$$L(x) = f(a) + f'(a)(x-a)$$

which satisfies $L(a) = f(a)$, $L'(a) = f'(a)$.

This $L(x)$ is the unique linear function with the property.

If a linear function $y = g(x)$ passes $(a, f(a))$

$$g(x) = f(a) + m(x-a) = f(a) + f'(a)(x-a).$$

If $g(a) = f(a)$ and $g'(a) = f'(a)$, $f(a) = \text{constant term}$, $g'(a) = m = f'(a)$

We can consider a quadratic function $Q(x)$ such that

$$Q(a) = f(a) \quad Q'(a) = f'(a), \quad Q''(a) = f''(a)$$

Any quadratic function $Q(x)$ passing $(a, f(a))$ can be written as

$$\rightarrow Q(x) = f(a) + A(x-a) + B(x-a)^2$$

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

If $Q(a) = f(a)$, Constant term $= f(a)$

With $Q'(x) = A + 2B(x-a)$, and $Q''(x) = 2B$

if $Q'(a) = f'(a)$ and $Q''(a) = f''(a)$ ↘

$$Q'(a) = A = f'(a) \qquad Q''(a) = 2B = f''(a) \Rightarrow B = \frac{1}{2}f''(a)$$

This $Q(x)$ is nothing but the quadratic approximation.

Similarly, the cubic approximation $C(x)$ is the only a cubic function satisfying ↙

$$C(a) = f(a) \quad C'(a) = f'(a) \quad C''(a) = f''(a) \quad C'''(a) = f'''(a)$$

It turns out that ↙ c_3

$$C(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3\cdot 2}(x-a)^3$$

We determine c_3 by $C'''(x) = 3\cdot 2\, c_3 \Rightarrow c_3 = \dfrac{C'''(a)}{3\cdot 2} = \dfrac{f'''(a)}{3\cdot 2}$

We continue adding higher-order terms in this way to build the Taylor series.

n^{th} derivative

For a series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ we determine coefficients c_n by

$$c_0 = f(a) \quad c_1 = f'(a) \quad c_2 = \frac{f''(a)}{2!} \quad c_3 = \frac{f'''(a)}{3!} \quad \cdots \quad c_n = \frac{f^{(n)}(a)}{n!}$$

Theorem (Taylor series of f at $x =$ a

$0! = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin series is a special case of Taylor series when $a =$ 0

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Maclaurin series for elementary functions

$x = 0$

(1) For $f(x) = e^x$, we have $f'(x) = f''(x) = \cdots =$ e^x

$$c_0 = f(0) = 1, \quad c_1 = f'(0) = 1, \quad c_2 = \frac{f''(0)}{2!} = \frac{1}{2!}, \quad c_3 = \frac{f'''(0)}{3!} = \frac{1}{3!}$$

$$e^x = 1 + 1 \cdot x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \frac{1}{n!}x^n$$

e^{2025}

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$R = \infty$

The radius of convergence is

$$a_{n+1} \cdot \frac{1}{a_n} = \frac{x \cdot \cancel{x^n}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} = \frac{x}{n+1} \to 0 < 1, \text{ as } n \to \infty$$

This means that $f(x) = e^x$ can be represented by the above series for all real number x.

By replacing x with x^2 we have

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots$$

$x = 0$

(2) $f(x) = \cos x$

Since $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, and $f^{(4)}(x) = \cos x$

we have $c_0 = f(0) = \cos 0 = 1$, $c_1 = f'(0) = \cancelto{0}{-\sin 0}$

$\cos \to -\sin \to -\cos \to \sin \to \cos$

$c_2 = f''(0) = -1$, $c_3 = \cancelto{0}{0}$

$$\cos x = f(0) + \cancelto{0}{f'(0)x} + \frac{f''(0)}{2!}x^2 + \cancelto{0}{\frac{f'''(0)x^3}{3!}} + \frac{f^{(4)}}{4!}x^4 + \dots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

Note that $\cos x$ is an even function while $\sin x$ is an odd function

(3) $f(x) = \sin x$

$c_0 = f(0) = \cancelto{0}{0}$ $c_1 = f'(0) = 1$ $c_2 = \cancelto{0}{\frac{f''(0) = 0}{2!}}$ $c_3 = \frac{-f'''(0)}{3!} = \frac{-1}{3!}$

$$\sin x = \cancelto{0}{f(0)} + f'(0)x + \cancelto{0}{\frac{f''(0)}{2}x^2} + \frac{f'''(0)x^3}{3!} + \cancelto{0}{\frac{f^{(4)}(0)x^4}{4!}} + \dots$$

$2(n+1)$
$= 2n+2$

$$= x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

The convergence radius is $R = \infty$

Cos: $$\frac{a_{n+1}}{a_n} = \frac{x^{2(n+1)}}{2(n+1)!} \cdot \frac{(2n)!}{x^{2n}} = \frac{\cancel{x^{2n}} \cdot x^2}{(2n+2)(2n+1) \cdot \cancel{2n!}} \cdot \frac{\cancel{(2n)!}}{\cancel{x^{2n}}} \to 0$$

(4) $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ $x = 0$

With $f'(x) = +(1-x)^{-2}$ $f''(x) = 2(1-x)^{-3}$ $f'''(x) = +3 \cdot 2(1-x)^{-4}$

we have $c_0 = f(0) = \frac{1}{1}$, $C_1 = f'(0) = 1$, $C_2 = \frac{2}{2!} = 1$, $C_3 = \frac{3!}{3!} = 1$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n.$$

Important Maclaurin Series and Their Radii of Convergence

11.9 $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$ ← odd

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$ ← even

11.9 $$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$ ← odd

11.9 $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

Example 1 Use a known Maclaurin series to obtain a Maclaurin series for the given function.

(1) $f(x) = x\,e^{4x}$

Recall that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = \sum_{0}^{\infty} \frac{4^n x^n}{n!}$$

$$x\,e^{4x} = x \sum \frac{4^n x^n}{n!} = \sum \frac{4^n x^{n+1}}{n!}$$

We compute the radius of convergence $R = \infty$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1} x^{n+2}}{(n+1)!} \frac{n!}{4^n x^{n+1}} = \frac{4 \cdot x}{n+1} \to 0 < 1 \text{ as } n \to \infty$$

(2) $f(x) = \sin\left(\frac{\pi x}{5}\right) = \sum \frac{(-1)^n}{2n+1!}\left(\frac{\pi x}{5}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{\pi}{5}\right)^{2n+1}}{(2n+1)!} x^{2n+1}$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1!}$

$R=\infty$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{\pi}{5}\right)^{2n+3} x^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{\left(\frac{\pi}{5}\right)^{2n+1} x^{2n+1}}\right| = \left|\frac{\left(\frac{\pi}{5}\right)^2 \cdot x^2}{(2n+3)(2n+2)}\right| \to 0$

(3) $f(x) = 8x^2 \tan^{-1}(9x^3) = 8x^2 \sum \frac{(-1)^n (9x^3)^{2n+1}}{2n+1} = \sum \frac{(-1)^n 8 \cdot 3^{4n+2} x^{6n+5}}{2n+1}$

$\tan^{-1}(x) = \sum \frac{(-1)^n x^{2n+1}}{2n+1}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{8 \cdot 3^{4n+6} \cdot x^{6n+11}}{2n+3} \frac{2n+1}{8 \cdot 3^{4n+2} x^{6n+5}}\right| = \left|3^4 \cdot x^6 \cdot \frac{2n+1}{2n+3}\right| \to |3^4 \cdot x^6|$

$\Rightarrow |3^4 x^6| < 1 \Rightarrow |x^6| < \frac{1}{3^4} \Rightarrow -\frac{1}{x^{2/3}} < x < \left(\frac{1}{3^4}\right)^{\frac{1}{6}} = \frac{1}{3^{2/3}}$

Example 2 Use MacLaurin series to compute the indefinite integral.

(1) $\int x\cos(x^4)\,dx = \int \sum dx = \sum \int \frac{(-1)^n x^{8n+1}}{2n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+2}}{2n!\,(8n+2)}$

$+C$

$x\cos(x^4) = x \sum \frac{(-1)^n (x^4)^{2n}}{2n!} = \sum_{0} \frac{(-1)^n x^{8n+1}}{2n!} = x^1 + \cdots$

(2) $\int \frac{\cos(x)-1}{x}\,dx = \int \frac{\sum_{n=1} \frac{(-1)^n x^{2n}}{(2n)!}}{x}$ W.A $= \sum_{n=1} \left(\frac{(-1)^n x^{2n-1+1}}{(2n)!\,2n}\right) + C$

or

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} = \frac{1 \cdot 1}{1} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos x - 1 = \sum_{n=1}$

$\int \frac{\cos x}{x} - \frac{1}{x}\,dx = \int \frac{\sum (-1) x^{2n-1}}{2n!} dx - \int \frac{1}{x} dx = \sum_{n=0} \frac{(-1) x^{2n}}{(2n)!\,2n} - \ln x$

$+C$

Example 3 Find the sum of the series. Identify a relevant MacLaurin series

(1) $$\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 = \underbrace{\frac{1}{1}}_{n=0} + \underbrace{\frac{2}{1}}_{n=1} + \frac{(2)^2}{2} + \cdots \quad \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$$

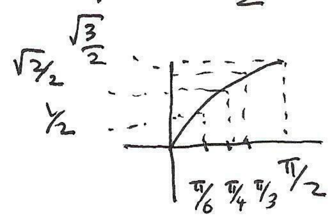
(2) $$\sum_{n=1}^{\infty} \frac{2^n}{n!} = \sum_{n=0} \frac{2^n}{n!} - 1 = e^2 - 1.$$

(3) $$\sum_{n=0}^{\infty} (-1)^n \frac{7^{n+1} x^{2n}}{n!} = \sum 7 \frac{\left((-1)\, 7\, x^2\right)^n}{n!} = 7 \sum_{n=0}^{\infty} \frac{\left(-7x^2\right)^n}{n!}$$

$$e^{\square} = \sum \frac{\square^n}{n!} \qquad = 7 \cdot e^{-7x^2}$$

(4) $$\sum_{n=0}^{\infty} \frac{(-1)^n\, \pi^{2n}}{4^{2n}\,(2n)!} = \sum \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{4} \right)^{2n} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

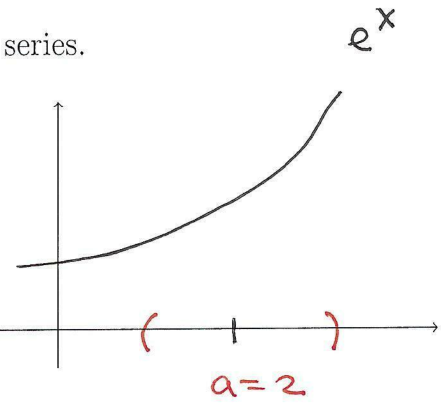


(5) $$\sum_{n=0}^{\infty} \frac{7(-1)^n\, \pi^{2n+1}}{6^{2n+1}\,(2n+1)!} = 7 \sum \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{6} \right)^{2n+1} = 7 \cdot \sin \frac{\pi}{6} = \frac{1}{2} \cdot 7$$

$$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Remark. Depending on the center a, one gets a different Taylor series.

For example, the Taylor series for $f(x) = e^x$ at 2 is

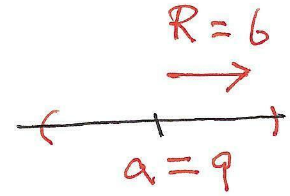


$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

$$\text{vs} \quad \sum \frac{x^n}{n!} \text{ at } x=0$$

$$f' = e^x = \cdots = f^{(n)}, \quad f^{(n)}(2) = e^2$$

Example 4 Find the Taylor series for f centered at 9 if $f^{(n)}(9) = \dfrac{(-1)^n n!}{6^n(n+4)}$.



$|x-9|<6$

The coefficient c_n is determined by

$$c_n = \frac{f^{(n)}(9)}{n!} = \frac{(-1)^n \cancel{n!}}{6^n(n+4)} \cdot \frac{1}{\cancel{n!}} =$$

$$f(x) = \sum_{n=0}^{\infty} c_n(x-9)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{6^n(n+4)}(x-9)^n$$

The radius of convergence R is 6.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-9)^{\cancel{n}+1}}{6^{\cancel{n}+1}(n+5)} \cdot \frac{\cancel{6^n}(n+4)}{\cancel{(x-9)^n}}\right| = \left|\frac{x-9}{6} \cdot \frac{n+4}{n+5}\right| \to \left|\frac{x-9}{6}\right| < 1$$

$$\frac{n+4}{n+5} \to 1$$

Find $f^{(23)}(9)$ for the Taylor series above

$$f^{(23)}(9) = \frac{(-1)^{23}(23)!}{6^{23}(27)}, \quad c_{23} = \frac{-1}{6^{23}(27)}$$

☆

Example 5 Find the Taylor series for the given center $x = a$

(1) $f(x) = \dfrac{1}{x} = x^{-1}$ centered at $a = 5$.

$f'(x) = -x^{-2}$ $\qquad$ $f''(x) = 2 \cdot x^{-3}$ $\qquad$ $f'''(x) = -3 \cdot 2\, x^{-4}$

$c_0 = f(5) = \dfrac{1}{5}$, $\quad c_1 = f'(5) = -\dfrac{1}{5^2}$, $\quad c_2 = \dfrac{\cancel{2}}{5^3 \cancel{2!}} = \dfrac{f''(5)}{2!}$ $\quad c_3 = f'''(5) \cdot \dfrac{1}{3!} = \dfrac{-\cancel{3!}}{\cancel{3!}} \dfrac{1}{5^4}$

With $c_n = (-1)^n \dfrac{1}{5^{n+1}}$

$$\frac{1}{x} = \sum_{n=0} \frac{f^{n}(5)}{n!}(x-5)^n = \sum_0 \frac{(-1)^n}{5^{n+1}} \cdot \frac{(x-5)^n \cancel{n!}}{\cancel{n!}} = \sum_0 \frac{(-1)^n (x-5)^n}{5^{n+1}}$$

(2) $f(x) = e^{2x}$ centered at $a = 4$

$f'(x) = 2 \cdot e^{2x}$ $\quad f''(x) = 2^2 \cdot e^{2x}$ $\quad f^{(n)}(x) = 2^n \cdot e^{2x}$

$$c_n = \frac{f^{(n)}(4)}{n!} = \frac{2^n \cdot e^{2\cdot 4}}{n!}$$

$$f(x) = e^{2x} = \sum c_n (x-4)^n = \sum_{n=0}^{\infty} \frac{e^8 \cdot 2^n}{n!}(x-4)^n$$

We can find the radius of convergence $R = \infty$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{e^8} 2^{n+1}(x-4)^{n+1}}{(n+1)!} \frac{n!}{\cancel{e^8} \cdot \cancel{2^n}(x-4)^n} = \frac{2}{n+1}(x-4) \to 0$$

Extra topic - Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The MacLaurin series $e^x = \sum \frac{x^n}{n!}$ implies

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \cdots\right)$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

(See Math 308 Chapt. 3 ...)

since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and

$i = \sqrt{-1}$

$$\cos\theta = \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots$$

$$\sin\theta = \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$