Proof Theory

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Basic Logic

Satisfiability of Sets of Formulas

Definition 1 If v is a valuation, this is, a mapping from the atoms to the set $\{t, f\}$.

Definition 2 [4] Let Σ denote a set of well-formed formulas and t a valuation. Define

$$\Sigma^t = \begin{cases} T & \text{if for each } \beta \in \Sigma, \beta^t = T \\ F & \text{otherwise} \end{cases}$$

When $\Sigma^t = T$, we say that t satisfies Σ. A set Σ is satisfiable iff there is some valuation t such that $\Sigma^t = T$.

Definition 3 Let Σ be a set of formulas, and let α be a formula, we say that

1. α is a logical consequence of Σ, or
2. Σ (semantically) entails α, or
3. $\Sigma \models \alpha$,

if and only if for all truth valuations t, if $\Sigma^t = T$ then also $\alpha^t = T$. We write $\Sigma \not\models \alpha$ for there exists a truth valuation t such that $\Sigma^t = T$ and $\alpha^t = F$.

Annotation 4 For example, $\Sigma = \{p_1, p_2, \cdots, p_n\}$ could be a set of premises and let α could be the conclusion that we want to derive.

Natural Deduction

Remark 5 Natural deduction is a kind of proof calculus in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning.

Judgments and Propositions

Definition 6 A *judgment* is somthing we may know, this is, an object of knowledge. A judgment is *evident* if we in fact know it.

Annotation 7 "A is false" (see classical logic), "A is true at time t" (see temporal logic), "A is necessarily true" or "A is possibly true" (see modal logic), "the program M has type " (see programming languages and type theory), "A is achievable from the available resources" (see linear logic).

Introduction and Elimination

Definition 8 Inference rules that introduce a logical connective is the conclusion are known as *introduction rules*. i.e., to conclude "A and B *true*" for propositions A and B, one requires evidence for "A *true*" and B *true*. As an inference rule:

$$\frac{A\ true \quad B\ true}{A \wedge B\ true} \wedge I$$

Here $\wedge I$ stands for "conjunction introduction".

Definition 9 Inference rules that describe how to deconstruct information about a compound proposition into information about its consitiuents are elimination rules. i.e., from $A \wedge B$ *true*, we can conclude A *true* and B *true*:

$$\frac{A \wedge B\ true}{A\ true} \wedge E_L \qquad \frac{A \wedge B\ true}{B\ true} \wedge E_R$$

Annotation 10 The meaning of conjunction is determinded by its *verifications*.

Hypothetical Derivations

Definition 11 A *hypothetical judgment* is $J_1, \cdots, J_n \vdash J$, where judgments $J_1, \cdots, J_n$ are unproved assumptions, and the judgment J is the conclusion. A *hypothetical deduction*(derivation) for $J_1, \cdots, J_n \vdash J$ has the form

$$\begin{array}{c} J_1 \quad \cdots \quad J_n \\ \vdots \\ J \end{array}$$

which means J is derivable from $J_1, \cdots, J_n$.

Annotation 12 上面的 $J_1, \cdots, J_2$ 都可以替换成关于 J_i 的一个 hypothetical derivation.

Definition 13 In the natural deduction calculus, an assumption is discharged when the conclusion of an inference does not depend on it, although one of the premises of the inference does[1].

Annotation 14 Once the appropriate rules have been completed, these are known as discharged assumptions, and are not included in the pool of assumptions on which the conclusion of the rule depends[3].

Annotation 15 hypothetical derivation 要求最后的 conclusion 依赖的 poof of assumptions 不是空的.

Theorem 16 Deduction theorem

$$T, P \vdash Q \iff T \vdash P \to Q$$

.

Annotation 17 在 deduction theorem 中我们注意到第一个 hypothetical judgment 里面的 antecedent Q 被去掉了，在第二个 hypothetical judgment 的 succedent 里面作为一个 implication 的 antecedent 出现了，这里我们就可以说 assumption Q is discharged，即现在的 conclusion 已经不依赖它了. 那么我们是如何构造 deduction theorem 里面的 implication 的呢？下面接着看

Definition 18 (implication) If B is true under the assumption that A is true, formly written $A \supset B$. The corresponed introduction and elimination rule as follow

$$\dfrac{\dfrac{}{A\ true}\,u \\ \vdots \\ B\ true}{A \supset B\ true} \supset I^u \qquad \dfrac{A \supset B \quad A\ true}{B\ true} \supset E$$

Annotation 19 Why indexed u In the introduction rule, the antecedent named u is discharged in the conclusion. This is a mechanism for delimiting the scope of the hypothesis: its sole reason for existence is to establish "$B\ true$"; it cannot be used for any other purpose, and in particular, it cannot be used below the introduction.

上面这段话出自 natural deduction 的 wiki，这个 $u$scope 了 assumption $A\ true$ 的开端，因为 $A \supset B$ 并不依赖 $A\ true$, 它描述只是 if $A\ true$ then $B\ true$. 同时最后的 introduction rule 会将这个 assumption $A\ true$ discharged 掉，表示 scope 在这里已经结束了. 而 implication rule 会将上述 derivation 直接总结得到一个结论，即

$$A \vdash B \Rightarrow \cdot \vdash A \to B.$$

Example 20 Considering the following proof of $A \supset (B \supset (A \wedge B))$

$$\dfrac{\dfrac{\dfrac{\overline{A\ true}\ u \quad \overline{B\ true}\ w}{A \wedge B\ true}\ \wedge I}{B \supset (A \wedge B)\ true}\ I^w}{A \supset (B \supset (A \wedge B))\ true}\ I^u.$$

这整个 derivation 不是 hypothetical 的，因为两个 assumptions A *true* 和 B *true* 都已经被 discharged，因此它实际上一个 complete proof!

Definition 21 (disjunction) The elimination rule for disjunction:

$$\dfrac{A \vee B\ true \quad \begin{array}{c}\overline{A\ true}\ u \\ \vdots \\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}\ w \\ \vdots \\ C\ true\end{array}}{C\ true}\ \vee E^{u,w}$$

both assumption u, w are discharged at the disjunction elimination rule.

Definition 22 The falsehood elimination rule:

$$\dfrac{\perp\ \ true}{C\ true}\ \perp E$$

Annotation 23 falsehood 可以看做一个 zero-ary disjunction, 啥都不用考虑直接可以得到任意的 conclusion??? There is no proof for $\perp$ *true*, so its sound to conclude arbitrary propositions.

Harmony

Definition 24 Local soundness shows that the elimination rules are not strong: no matter how we apply eliminations rules to the result of an introduction we cannot gain any new information.

Definition 25 Local completeness shows that the elimination rules are not weak: there is alwayws a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the the results by apply intruduction ruls.

Annotation 26 local soundness 告诉你通过 elimination 压缩得到的东西不会比你已经知道的东西强 (not strong), 而 local completeness 告诉你合并通过 elimination 压缩得到的东西会得到全部你知道的信息.

Definition 27 Given two deduction of same judgment, we use the notion

$$\begin{array}{c}\mathcal{D}\\ A\ true\end{array} \Longrightarrow_R \begin{array}{c}\mathcal{D}'\\ A\ true\end{array}$$

for the local reduction of a deduction $\mathcal{D}$ to another deduction D' of same judgement $A\ true$. Simliarly, we have local expansion

$$\begin{array}{c}\mathcal{D}'\\ A\ true\end{array} \Longrightarrow_E \begin{array}{c}\mathcal{D}\\ A\ true\end{array}$$

Definition 28 (substitution Principle) If

$$\begin{array}{c}\overline{A\ true}\ u\\ \mathcal{E}\\ C\ true\end{array}$$

is a hypothetical proof of $\mathcal{C}\ true$ under the undischarged hypothesis $A\ true$ labelled u, and

$$\begin{array}{c}\mathcal{D}\\ A\ true\end{array}$$

is a proof of $A\ true$ then

$$\begin{array}{c}\dfrac{\mathcal{D}}{A\ true}\ u\\ \mathcal{E}\\ C\ true\end{array}$$

is our notation for substituting $\mathcal{D}$ for all uses of the hypothesis labelled u in $\mathcal{E}$. This deduction, also sometime written as $[\mathcal{D}/u]\mathcal{E}$ no longer depends on u.

Example 29 If given a elimination rule of disjunction as follow

$$\frac{A \vee B\ true}{A\ true}\ \vee E_L$$

The rule a little bit stronger, since we would not be able to reduce

$$\dfrac{\dfrac{B\ true}{A \vee B\ true}\ \vee I_R}{A\ true}\ \vee E_L$$

As u can see it's not local soundness.

Verifications and Uses

Definition 30 a verification should be a proof that only analyzes the consitituents of a proposition.

Annotation 31 natural deduction 实际上像 constructive logic 或者 intuitive logic, 不像 classic logic, 例如 Proposition $A \vee (A \supset B)$ 在 classic logic 就是正确的，因为我们 A 和 B 都需要给定是 true/false tag, 但是在 natural deduction 里面我们好像没有办法来处理. 更甚，如果我们要证明一个 B 是 accpeted in natural deduction, 你可能首先需要证明 $A \supset B$ 和 B 都是 accepted，就是根据结构来做 derivation.

Definition 32 Writing $A\uparrow$ for the judgment "A has a verification". Naturally, this should mean that A is true, and that the evidence for that has a special form.

Definition 33 Writing $A\downarrow$ for the judgment "A may be used". $A\downarrow$ should be the case when either A *true* is a hypothesis, or A is deduced from a hypothesis via elimination rules.

Definition 34 For conjunction.

$$\frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow} \wedge I \qquad \frac{A \wedge B\downarrow}{A\downarrow} \wedge E_L \qquad \frac{A \wedge B\downarrow}{B\downarrow} \wedge E_R$$

Definition 35 For implication

$$\frac{\begin{array}{c}\overline{A\downarrow}\,^{u} \\ \vdots \\ B\uparrow\end{array}}{A \supset B\uparrow} \supset^u \qquad \frac{A \supset B\downarrow \quad A\uparrow}{B\downarrow} \supset E$$

implication introduction rule 里面的 $B\uparrow$ 表示没看懂，因为这里的 B 显然是来自 elimination 的结果. 为什么 implication elimination 里面需要 $A\uparrow$ 呢?

Example 36

$$\frac{\dfrac{\overline{A \wedge B\ true}\,^{u}}{A\ true} \wedge E_L}{(A \wedge B) \supset A\ true} \supset I^u$$

Definition 37 For disjunction

$$\frac{A\uparrow}{A \vee B\uparrow} \vee I_L \qquad \frac{B\uparrow}{A \vee B\uparrow} \vee I_R \qquad \frac{A \vee B\downarrow \quad \begin{array}{c}\overline{A\uparrow}\,^{u} \\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}\,^{w} \\ \vdots \\ C\uparrow\end{array}}{C\uparrow} \vee E^{u,w}$$

Definition 38 For truth and falsehood.

$$\frac{}{\top\uparrow}\,\top I \qquad \frac{\bot\downarrow}{C\uparrow}\,\bot E$$

Annotation 39 $\bot\downarrow$ signifies a contradiction from our hypotheses.

Definition 40 For atomic propositions.

$$\frac{P\downarrow}{P\uparrow}\downarrow\uparrow.$$

Annotation 41 对于 atomic props，我们只能对它赋予一个 property，没有关于它的 verification. 因此上述的规则是在进行一个转换，只要我们 assumption 了关于它的一个 property，就默认它已经被 verified.

Soundness and Completeness of Natural Deduction

Definition 42 [5]Soundness of natural deduction means that the conclusion of proof is always a logical consequence of the premises. That is

$$\text{If } \Sigma \vdash \alpha, \text{then } \sigma \vDash \alpha.$$

Definition 43 Completeness of natural deduction means that all logical consequences in propositional logic are provable in natural deduction. That is,

$$\text{If } \Sigma \vDash \alpha, \text{then } \sigma \vdash \alpha.$$

Notational Definition

Definition 44 A notational definition gives the meaning of the general form of a proposition in terms of another proposition whose meaning has already been defined.

Example 45 We can define logical equivalence, written $A \equiv B$ as

$$(A \supset B) \wedge (B \supset A).$$

Example 46 We can define negation $\neg A$ as

$$\neg A = (A \supset \perp).$$

Derived Rules of Inference

Example 47

$$\frac{A \supset B\ \mathit{true} \quad B \supset C\ \mathit{true}}{A \supset C\ \mathit{true}}$$

is a derived rule of inference. Its derivation is the following:

$$\dfrac{\dfrac{B \supset C\ \mathit{true} \qquad \dfrac{A \supset B\ \mathit{true} \quad \dfrac{}{A\ \mathit{true}}\,u}{B\ \mathit{true}}\supset E}{C\ \mathit{true}}\supset E}{A \supset C\ \mathit{true}}\supset I^u$$

Annotation 48 关于 derivation 的推导这里有一些 strategies 在里面

- 使用 introduction rule 从下至上，即我们想要什么;
- 使用 elimination rule 从上至下，即我们知道什么.

Curry-Howard Conrrespondence

Definition 49 Curry-Howard correspondence is between the natural deduction and simply-typed λ-calculus at three levels

- propositions are types;
- proofs are programs; and
- simplification of proofs is evaluation of programs.

That is

Types	Propositions
Unit types (1)	Truth ($\top$)
Product type ($\times$)	Conjunction ($\wedge$)
Union type ($+$)	Disjunction ($\vee$)
Function type ($\to$)	Implication ($\supset$)
Void types (0)	False ($\perp$)

Every typing rule has a correspondence with a deduction rule.

Example 50 The typing derivation of the term $\lambda a.\ \lambda b.\ \langle a, b\rangle$ can be seen as a deduction tree proving $A \supset B \supset A \wedge B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{a : A \in \Gamma}{\Gamma \vdash a : A}\,var \quad \dfrac{b : B \in \Gamma}{\Gamma \vdash b : B}\,var}{\Gamma \vdash \langle a, b\rangle : A \times B}\,pair}{\Gamma \vdash \lambda y : B.\ \ \langle a, y\rangle : B \to A \times B}\,abs}{\Gamma \vdash \lambda x : A.\ \lambda y : B.\ \ \langle x, y\rangle : A \to B \to A \times B}\,abs \quad \Longleftrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A\ true}\,u \quad \dfrac{}{B\ true}\,w}{A \wedge B\ true}\,\wedge I}{B \supset A \wedge B\ true}\,\supset I^w}{A \supset B \supset A \wedge B\ true}\,\supset I^u$$

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