

# Proof Theory

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## Basic Logic

### Satisfiability of Sets of Formulas

**Definition 1** If  $v$  is a **valuation**, this is, a mapping from the atoms to the set  $\{t, f\}$ .

**Definition 2** [4] Let  $\Sigma$  denote a set of well-formed formulas and  $t$  a valuation. Define

$$\Sigma^t = \begin{cases} T & \text{if for each } \beta \in \Sigma, \beta^t = T \\ F & \text{otherwise} \end{cases}$$

When  $\Sigma^t = T$ , we say that  $t$  **satisfies**  $\Sigma$ . A set  $\Sigma$  is **satisfiable** iff there is some valuation  $t$  such that  $\Sigma^t = T$ .

**Definition 3** Let  $\Sigma$  be a set of formulas, and let  $\alpha$  be a formula, we say that

1.  $\alpha$  is a **logical consequence** of  $\Sigma$ , or
2.  $\Sigma$  **(semantically) entails**  $\alpha$ , or
3.  $\Sigma \models \alpha$ ,

if and only if for all truth valuations  $t$ , if  $\Sigma^t = T$  then also  $\alpha^t = T$ . We write  $\Sigma \not\models \alpha$  for there exists a truth valuation  $t$  such that  $\Sigma^t = T$  and  $\alpha^t = F$ .

**Annotation 4** For example,  $\Sigma = \{p_1, p_2, \dots, p_n\}$  could be a set of premises and let  $\alpha$  could be the conclusion that we want to derive.

## Classic Propositional Modal Logic

**Definition 5** Let  $\Sigma$  be a set of propositional letters or atomic propositions. The set  $F_P(\Sigma)$  of formulas of classical propositional modal logic is the smallest set with:

1. If  $A \in \Sigma$  is a propositional letter, then  $A \in F_P(\Sigma)$ ;
2. If  $\phi, \psi \in F_P(\Sigma)$ , then  $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi) \in F_P(\Sigma)$ ;
3. If  $\phi \in F_P(\Sigma)$ , then  $(\Box\phi), (\Diamond\phi) \in F_P(\Sigma)$ .

**Definition 6** Let  $\mathcal{S}$  be a system of modal logic, this is  $F_P(\Sigma)$  with a set of axioms and rules. If axioms and rules as follow

$$\begin{array}{ll}
 \text{all propostional tautologies} & \text{(P)} \\
 \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) & \text{(Kripke axiom)} \\
 \Box\phi \rightarrow \phi & \text{(T)} \\
 \Box\phi \rightarrow \Box\Box\phi & \text{(4)} \\
 \frac{\phi \quad \phi \rightarrow \psi}{\psi} & \text{(modus ponens)} \\
 \frac{\phi}{\Box\phi} & \text{(Gödel)}
 \end{array}$$

We call it modal logic  $\mathcal{S}4$ .

**Annotation 7** Kripke axiom 原本的形式应为

$$\Box\phi \wedge \Box(\phi \rightarrow \psi) \rightarrow \Box\psi$$

上面是它经常用的等价形式. Axiom T 是指若  $\phi$  is necessary, 那么  $\phi$  is true. Axiom 4 是指  $\phi$  is necessary, 那么命题“ $\phi$  is necessary” is necessary, 有点别扭, 举个形象的例子如果 box 是指某个人知道某件事, 假设我知道  $A$  true, 那么我肯定知道我知道  $A$  true. 最后一个叫 Gödel translation, 它将 intuitionistic logic 里面的 formulas 转换到 modal logic 里面.

**Definition 8** Let  $\mathcal{S}$  be a system of modal logic. For a formula  $\psi$  and a set of formulas  $\Phi$ , we write  $\Phi \vdash_{\mathcal{S}} \psi$  and say that  $\psi$  can be derived from  $\Phi$ (or is provable from  $\Phi$ ), iff there is a proof of  $\psi$  that uses only the formulas of  $\Phi$  and the axioms and proof rules of  $\mathcal{S}$ . That is, we define  $\Phi \vdash_{\mathcal{S}} \psi$  inductively as:

$$\Phi \vdash_{\mathcal{S}} \psi$$

iff  $\psi \in \Phi$  or there is an instance

$$\frac{\phi_1 \quad \cdots \quad \phi_n}{\psi}$$

of a proof rule of  $\mathcal{S}$  with conclusion  $\psi$  and some number  $n \geq 0$  of premises such that for all  $i = 1, 2, \dots, n$ , the premises  $\phi_i$  is derivable, i.e:

$$\Phi \vdash_{\mathcal{S}} \phi_i$$

When the case  $n = 0$  corresponds to axioms.

**Annotation 9** 现在以  $\Box$  表示 provable 的视角来看待前面提到的 axioms. 首先是

$$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) \text{ (Kripke axiom)}$$

若  $\phi \rightarrow \psi$  is provable 且  $\phi$  is provable, 那么则  $\psi$  is provable.

$$\Box\phi \rightarrow \phi \text{ (T)}$$

若  $\phi$  is provable, 那么  $\phi$  should be true.

$$\Box\phi \rightarrow \Box\Box\phi \text{ (4)}$$

若  $\phi$  is provable, 那么  $\phi$  should be provably provable, 也就是我们肯定知道存在一个 proof.

$$\frac{\phi}{\Box\phi} \text{ (Gödel)}$$

若  $\phi$  is proven, 那么  $\phi$  should be provable.

**Definition 10** A Kripke frame  $(W, \rho)$  consists of a non-empty set  $W$  and a relation  $\rho \subseteq W \times W$  on worlds. The element of  $W$  are called possible worlds and  $\rho$  is called accessibility relation.

**Definition 11** A Kripke structure  $K = (W, \rho, v)$  consists of Kripke frame  $(W, \rho)$  and a mapping  $v : W \rightarrow \Sigma \rightarrow \{true, false\}$  that assigns truth-values to all the propositional letters in all worlds.

**Definition 12** Given a Kripke structure  $K = (W, \rho, v)$ , the interpretation  $\models$  of modal formulas in worlds  $s$  is defined as

- $K, s \models A$  iff  $v(s)(A) = true$ ;
- $K, s \models \phi \wedge \psi$  iff  $K, s \models \phi$  and  $K, s \models \psi$ ;
- $K, s \models \phi \vee \psi$  iff  $K, s \models \phi$  or  $K, s \models \psi$ ;
- $K, s \models \neg\phi$  iff it is not the case that  $K, s \models \phi$ ;

- $K, s \models \Box\phi$  iff  $K, t \models \phi$  for all worlds  $t$  with  $spt$ ;
- $K, s \models \Diamond\phi$  iff  $K, t \models \phi$  for some worlds  $t$  with  $spt$ .

**Definition 13** Given a Kripke structure  $K = (W, \rho, v)$ , formula  $\phi$  is **vaild** in  $K$ , written  $K \models \phi$ , iff  $K, s \models \phi$  for all worlds  $s \in W$ .

**Definition 14** (**local consequence**) Let  $\psi$  be a formula and  $\Phi$  a set of formulas. Then we write  $\Phi \models_l \psi$  if and only if, for each Kripke structure  $K = (W, \rho, v)$  and each world  $s \in W$ , we have  $K, s \models \Phi$  implies  $K, s \models \psi$ .

**Definition 15** (**global consequence**) Let  $\psi$  be a formula and  $\Phi$  a set of formulas. Then we write  $\Phi \models_g \psi$  if and only if, for each Kripke structure  $K = (W, \rho, v)$ , if for world  $s \in W : K, s \models \Phi$ , then for world  $s \in W : K, s \models \psi$ .

**Annotation 16** local consequence 和 global consequence 的区别就是 assumption 是在某个 world 里面还是在所有的 worlds 里面.

**Definition 17** A formula  $\phi$  is **vaild** or a tautology, iff  $\emptyset \models_l \phi$ , which we write  $\models \phi$ . A set of formulas  $\Phi$  is called **satisfiable**, iff there is a Kripke structure  $K$  and a world  $s$  with  $K, s \models \Phi$ .

**Lemma 18** (**local deduction theorem**) For formulas  $\phi, \psi$  we have

$$\phi \models_l \psi \iff \models_l \phi \rightarrow \psi.$$

## Natural Deduction

**Remark 19** Natural deduction is a kind of proof calculus in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning.

### Judgments and Propositions

**Definition 20** A *judgment* is something we may know, this is, an object of knowledge. A judgment is *evident* if we in fact know it.

**Annotation 21** "A is false" (see classical logic), "A is true at time t" (see temporal logic), "A is necessarily true" or "A is possibly true" (see modal logic), "the program M has type " (see programming languages and type theory), "A is achievable from the available resources" (see linear logic).

### Introduction and Elimination

**Definition 22** Inference rules that introduce a logical connective in the conclusion are known as *introduction rules*. i.e., to conclude " $A$  and  $B$  true" for propositions  $A$  and  $B$ , one requires evidence for " $A$  true" and  $B$  true. As an inference rule:

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

Here  $\wedge I$  stands for "conjunction introduction".

**Annotation 23** 实际上面的 inference rule 的 general form 应该是

$$\frac{A \text{ prog} \quad B \text{ prog} \quad A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

这里才能帮助后面的  $\models$  make sense.

**Definition 24** Inference rules that describe how to deconstruct information about a compound proposition into information about its constituents are elimination rules. i.e., from  $A \wedge B$  true, we can conclude  $A$  true and  $B$  true:

$$\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_L \quad \frac{A \wedge B \text{ true}}{B \text{ true}} \wedge E_R$$

**Annotation 25** The meaning of conjunction is determined by its *verifications*.

## Hypothetical Derivations

**Definition 26** A *hypothetical judgment* is  $J_1, \dots, J_n \vdash J$ , where judgments  $J_1, \dots, J_n$  are unproved assumptions, and the judgment  $J$  is the conclusion. A *hypothetical deduction*(derivation) for  $J_1, \dots, J_n \vdash J$  has the form

$$\begin{array}{c} J_1 \quad \cdots \quad J_n \\ \vdots \\ J \end{array}$$

which means  $J$  is derivable from  $J_1, \dots, J_n$ .

**Annotation 27** 上面的  $J_1, \dots, J_n$  都可以替换成关于  $J_i$  的一个 hypothetical derivation.

**Definition 28** In the natural deduction calculus, an assumption is discharged when the conclusion of an inference does not depend on it, although one of the premises of the inference does[1].

**Annotation 29** Once the appropriate rules have been completed, these are known as discharged assumptions, and are not included in the pool of assumptions on which the conclusion of the rule depends[3].

**Annotation 30** hypothetical derivation 要求最后的 conclusion 依赖的 pool of assumptions 不是空的.

**Theorem 31** Deduction theorem

$$T, P \vdash Q \iff T \vdash P \rightarrow Q$$

.

**Annotation 32** 在 deduction theorem 中我们注意到第一个 hypothetical judgment 里面的 antecedent  $Q$  被去掉了, 在第二个 hypothetical judgment 的 succedent 里面作为一个 implication 的 antecedent 出现了, 这里我们就可以说 assumption  $Q$  is discharged, 即现在的 conclusion 已经不依赖它了. 那么我们是如何构造 deduction theorem 里面的 implication 的呢? 下面接着看

**Definition 33** (implication) If  $B$  is true under the assumption that  $A$  is true, formally written  $A \supset B$ . The corresponded introduction and elimination rule as follow

$$\frac{\frac{\overline{A \text{ true}}^u \quad \vdots \quad B \text{ true}}{A \supset B \text{ true}} \supset I^u \quad \frac{A \supset B \quad A \text{ true}}{B \text{ true}} \supset E$$

**Annotation 34** Why indexed  $u$  In the introduction rule, the antecedent named  $u$  is discharged in the conclusion. This is a mechanism for delimiting the scope of the hypothesis: its sole reason for existence is to establish " $B \text{ true}$ "; it cannot be used for any other purpose, and in particular, it cannot be used below the introduction.

上面这段话出自 natural deduction 的 wiki, 这个 *uscope* 了 assumption  $A \text{ true}$  的开端, 因为  $A \supset B$  并不依赖  $A \text{ true}$ , 它描述只是 if  $A \text{ true}$  then  $B \text{ true}$ . 同时最后的 introduction rule 会将这个 assumption  $A \text{ true}$  discharged 掉, 表示 scope 在这里已经结束了. 而 implication rule 会将上述 derivation 直接总结得到一个结论, 即

$$A \vdash B \Rightarrow \cdot \vdash A \rightarrow B.$$

**Example 35** Considering the following proof of  $A \supset (B \supset (A \wedge B))$

$$\frac{\frac{\frac{\overline{A \text{ true}}^u \quad \overline{B \text{ true}}^w}{A \wedge B \text{ true}} \wedge I}{B \supset (A \wedge B) \text{ true}} I^w}{A \supset (B \supset (A \wedge B)) \text{ true}} I^u.$$

这个整个 derivation 不是 hypothetical 的, 因为两个 assumptions  $A \text{ true}$  和  $B \text{ true}$  都已经被 discharged, 因此它实际上是一个 complete proof!

**Definition 36** (**disjunction**) The elimination rule for disjunction:

$$\frac{\frac{\overline{A \text{ true}}^u \quad \overline{B \text{ true}}^w}{A \vee B \text{ true}} \quad \frac{\begin{array}{c} \vdots \\ C \text{ true} \end{array} \quad \begin{array}{c} \vdots \\ C \text{ true} \end{array}}{C \text{ true}} \vee E^{u,w}}$$

both assumption  $u, w$  are discharged at the disjunction elimination rule.

**Definition 37** The falsehood elimination rule:

$$\frac{\perp \text{ true}}{C \text{ true}} \perp E$$

**Annotation 38** falsehood elimination 的意义在哪? 首先你应该主要到一个特殊等价命题  $A \vee \perp = A$ , 从  $\vee$  的 introduction rule 来看这意味  $\perp \text{ true} \vdash A \text{ true}$ , 由于  $A$  是任意的, 因此我们得到了  $\perp \text{ true} \vdash C \text{ true}$ .



## Harmony

**Definition 39** **Local soundness** shows that the elimination rules are not strong: no matter how we apply eliminations rules to the result of an introduction we cannot gain any new information.

**Definition 40** **Local completeness** shows that the elimination rules are not weak: there is always a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the the results by apply intruduction rules.

**Annotation 41** local soundness 告诉你通过 elimination 压缩得到的东西不会比你已经知道的东西强 (not strong), 而 local completeness 告诉你合并通过 elimination 压缩得到的东西会得到全部你知道的信息.

**Definition 42** Given two deduction of same judgment, we use the notion

$$\frac{\mathcal{D}}{A \text{ true}} \Longrightarrow_R \frac{\mathcal{D}'}{A \text{ true}}$$

for the **local reduction** of a deduction  $\mathcal{D}$  to another deduction  $\mathcal{D}'$  of same judgement  $A \text{ true}$ . Similiarly, we have **local expansion**

$$\frac{\mathcal{D}'}{A \text{ true}} \Longrightarrow_E \frac{\mathcal{D}}{A \text{ true}}$$

**Definition 43** (**substitution Principle**) If

$$\frac{\frac{\mathcal{D}}{A \text{ true}}}{\mathcal{E}}^u \frac{}{C \text{ true}}$$

is a hypothetical proof of  $C \text{ true}$  under the undischarged hypothesis  $A \text{ true}$  labelled  $u$ , and

$$\frac{\mathcal{D}}{A \text{ true}}$$

is a proof of  $A \text{ true}$  then

$$\frac{\frac{\mathcal{D}}{A \text{ true}}}{\mathcal{E}}^u \frac{}{C \text{ true}}$$

is our notation for substituting  $\mathcal{D}$  for all uses of the hypothesis labelled  $u$  in  $\mathcal{E}$ . This deduction, also sometime written as  $[\mathcal{D}/u]\mathcal{E}$  no longer depends on  $u$ .

**Example 44** If given a elimination rule of disjunction as follow

$$\frac{A \vee B \text{ true}}{A \text{ true}} \vee E_L$$

The rule a little bit stronger, since we would not be able to reduce

$$\frac{\frac{B \text{ true}}{A \vee B \text{ true}} \vee I_R}{A \text{ true}} \vee E_L$$

As u can see it's not local soundness.

## Verifications and Uses

**Definition 45** a verification should be a proof that only analyzes the constituents of a proposition.

**Annotation 46** natural deduction 实际上像 constructive logic 或者 intuitive logic, 不像 classic logic, 例如 Proposition  $A \vee (A \supset B)$  在 classic logic 就是正确的, 因为我们  $A$  和  $B$  都需要给定是 true/false tag, 但是在 natural deduction 里面我们好像没有办法来处理. 更甚, 如果我们要证明一个  $B$  是 accepted in natural deduction, 你可能首先需要证明  $A \supset B$  和  $B$  都是 accepted, 就需要根据其结构 bottom-up 来做 derivation.

**Definition 47** Writing  $A \uparrow$  for the judgment "A has a verification". Naturally, this should mean that  $A$  is true, and that the evidence for that has a special form.

**Definition 48** Writing  $A \downarrow$  for the judgment "A may be used".  $A \downarrow$  should be the case when either  $A$  true is a hypothesis, or  $A$  is deduced from a hypothesis via elimination rules.

**Annotation 49** 上述两个 definitions 里面隐藏着非常重要但有点不正式的结论: If  $A$  has a verification then  $A$  true, 反之亦然. 后面我们将形式化地证明它们.

**Definition 50** For conjunction.

$$\frac{A \uparrow \quad B \uparrow}{A \wedge B \uparrow} \wedge I \quad \frac{A \wedge B \downarrow}{A \downarrow} \wedge E_L \quad \frac{A \wedge B \downarrow}{B \downarrow} \wedge E_R$$

**Definition 51** For implication

$$\frac{\overline{A \downarrow}^u \quad \vdots \quad B \uparrow}{A \supset B \uparrow} \supset^u \quad \frac{A \supset B \downarrow \quad A \uparrow}{B \downarrow} \supset E$$

implication introduction rule 里面的  $B \uparrow$  表示没看懂, 因为这里的  $B$  显然是来自 elimination 的结果. 为什么 implication elimination 里面需要  $A \uparrow$  呢?

**Example 52**

$$\frac{\overline{A \wedge B \text{ true}}^u \quad \frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_L}{(A \wedge B) \supset A \text{ true}} \supset I^u$$

**Definition 53** For disjunction

$$\frac{A \uparrow}{A \vee B \uparrow} \vee I_L \quad \frac{B \uparrow}{A \vee B \uparrow} \vee I_R \quad \frac{\overline{A \uparrow}^u \quad \overline{B \downarrow}^w \quad \vdots \quad C \uparrow \quad \vdots \quad C \uparrow}{C \uparrow} \vee E^{u,w}$$

**Definition 54** For truth and falsehood.

$$\frac{}{\top \uparrow} \top I \quad \frac{\perp \downarrow}{C \uparrow} \perp E$$

**Annotation 55**  $\perp \downarrow$  signifies a contradiction from our hypotheses.

**Definition 56** For atomic propositions.

$$\frac{P \downarrow}{P \uparrow} \downarrow \uparrow.$$

**Annotation 57** 对于 atomic props, 我们只能对它赋予一个 property, 没有关于它的 verification. 因此上述的规则是在进行一个转换, 只要我们 assume 了关于它的一个 property, 就默认它已经被 verified.

## Soundness and Completeness of Natural Deduction

**Definition 58** [5]Soundness of natural deduction means that the conclusion of proof is always a logical consequence of the premises. That is

$$\text{If } \Sigma \vdash \alpha, \text{ then } \Sigma \models \alpha.$$

**Definition 59** Completeness of natural deduction means that all logical consequences in propositional logic are provable in natural deduction. That is,

$$\text{If } \Sigma \models \alpha, \text{ then } \Sigma \vdash \alpha.$$

**Annotation 60** 其中  $\Sigma \vdash \alpha$ , 表示存在一个以  $\Sigma$  作为 premise 得到 conclusion 为  $\alpha$  的 proof. 而  $\Sigma \models \alpha$ , 就考虑两端的 proposition 加上 truth-falsehood 了, 即如  $\Sigma^t = \text{True}$  则有  $\alpha^t = \text{True}$ .

对于 soundness 的证明, 我们需要根据  $\alpha$  的结构来做归纳, 而后再考虑赋予其 true/false 来考虑. 这里记录一下对于结构归纳它是怎样对应一般归纳法命题  $P(n)$  结构上, 这里的  $n$  应该对应  $\alpha$  的 bottom-up derivation 里面的 maximum depth of line.

而对于 completeness 的证明, 相对来说会复杂一点. 我们需要下面 3 个 lemma. 有一个疑问不引入 negation 是不是还说明不了 completeness?

**Lemma 61** If  $\Sigma = \{\alpha_0, \alpha_1, \dots, \dots, \alpha_n\}$  and  $\Sigma \models \beta$ , then

$$\emptyset \models (\alpha_0 \rightarrow (\alpha_1 \rightarrow (\dots \rightarrow (\alpha_n \rightarrow \beta) \dots))).$$

**Annotation 62** Deduction theorem 体现的淋漓尽致, 将  $\beta$  完美转换成了一个 tautology.

**Lemma 63** For any well-form formula  $\gamma$  containing atoms  $p_1, p_2, \dots, p_n$  and any valuation  $t$ , we have

1. If  $\gamma^t = \text{True}$  then  $\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_n \vdash \gamma$ ;
2. If  $\gamma^t = \text{False}$  then  $\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_n \vdash \neg \gamma$ ;

where defines  $\widehat{p}_i$  as follow

$$\widehat{p}_i = \begin{cases} p_i & \text{if } p_i^t = \text{True} \\ \neg p_i & \text{if } p_i^t = \text{False} \end{cases}$$

**Example 64** 若  $\gamma = p \rightarrow q$ , 我们可以构造一个真值表

$p$	$q$	$p \rightarrow q$	Claim
$T$	$T$	$T$	$p, q \vdash p \rightarrow q$
$T$	$F$	$F$	$p, \neg q \vdash \neg(p \rightarrow q)$
$F$	$T$	$T$	$\neg p, q \vdash p \rightarrow q$
$F$	$F$	$T$	$\neg p, \neg q \vdash p \rightarrow q$

那么上面的 claims 是怎么来的呢? 我们可以来分别证明, 对于第一行

$$\frac{\overline{p \text{ true}}^u \quad q \text{ true}}{\frac{q \text{ true}}{p \rightarrow q \text{ true}}^u}$$

感觉有点奇怪, 这里需要用到 vars inference rule, 这里相对于对  $q \vdash p \rightarrow q$  的 weaken premise. 对于第二行

$$\frac{\frac{\overline{p \rightarrow q \text{ true}}^u \quad p \text{ true}}{q} \quad \neg q \text{ true}}{\frac{\perp}{\neg(p \rightarrow q) \text{ true}}^u}$$

对于第三行

$$\frac{\overline{p \text{ true}}^u \quad \neg p \text{ true}}{\frac{\perp}{q \text{ true}}^u \quad p \rightarrow q \text{ true}}^u$$

对于第四行, 和第三行类似. 可以看的出来这个 lemma 非常深刻, 只要将 atoms 调整为在当前 valuation 下都是 true 的命题, 结论再对应调整, 就可以构造一个对应的 proof.

**Lemma 65** For any well-formed formula  $\gamma$ , if  $\emptyset \models \gamma$ , then  $\emptyset \vdash \gamma$ .

**Annotation 66** Lemma 64 一句话概况就是 tautologies are provable. 其证明过程可以用 Lemma 62 来说明. 现在  $\gamma$  是一个 tautology, 那么对于所有的 valuation 都有  $\gamma^t = \text{true}$ , 这有什么用呢? 这里还需要引入另外一种 tautology  $p \vee \neg p$ , 配合 emilination rule of *vee*, 即

$$\frac{\begin{array}{ccccccc} \overline{p_1} & \cdots & \overline{p_n} & & \overline{\neg p_1} & \cdots & \overline{\neg p_n} \\ (p_1 \vee \neg p_1) & (p_2 \vee \neg p_2) & \cdots & (p_n \vee \neg p_n) & \vdots & \cdots & \vdots \\ & & & & \gamma & & \gamma \end{array}}{\gamma}$$

这里需要考虑有  $2^n$  个 cases, 每一个对应一种 valuation, 又因为  $\gamma$  是 tautology, 因此最后的 conclusion 也都是  $\gamma$ .

**Lemma 67** If  $\emptyset \vdash (\alpha_0 \rightarrow (\alpha_1 \rightarrow (\cdots \rightarrow (\alpha_n \rightarrow \beta) \cdots)))$ , then  $\{\alpha_0, \alpha_1, \cdots, \alpha_n\} \vdash \beta$ , that is,  $\Sigma \vdash \beta$ .

## Notational Definition

**Definition 68** A **notational definition** gives the meaning of the general form of a proposition in terms of another proposition whose meaning has already been defined.

**Example 69** We can define logical equivalence, written  $A \equiv B$  as

$$(A \supset B) \wedge (B \supset A).$$

**Example 70** We can define negation  $\neg A$  as

$$\neg A = (A \supset \perp) \implies \frac{A \quad \vdots \quad \perp}{\neg A} \neg I$$

We also can give the introduction rule of falsehood.

$$\frac{\neg A \quad A}{\perp} \perp I$$

so  $\perp$  actually means any contradictions. moreover double negation is coming.

**Annotation 71** notational definition 可以看做用已有的东西构造出一些东西. 与之对应的是我们可以直接符号化的给出某个新的定义, 称之为 symbolic definition.

## Derived Rules of Inference

**Example 72**

$$\frac{A \supset B \text{ true} \quad B \supset C \text{ true}}{A \supset C \text{ true}}$$

is a derived rule of inference. Its derivation is the following:

$$\frac{B \supset C \text{ true} \quad \frac{A \supset B \text{ true} \quad \overline{A \text{ true}}}{B \text{ true}} \supset E}{\frac{C \text{ true}}{A \supset C \text{ true}} \supset I^u} \supset E^u$$

**Annotation 73** 关于 derivation 的推导这里有一些 strategies 在里面

- 使用 introduction rule 从下至上，即我们想要什么；
- 使用 elimination rule 从上至下，即我们知道什么。

**Example 74** Modus tollens(这玩意不就是逆否命题)

$$\frac{A \rightarrow B \quad \neg B}{\neg A} MT.$$



## Curry-Howard Correspondence

**Definition 75** Curry-Howard correspondence is between the natural deduction and simply-typed  $\lambda$ -calculus at three levels

- propositions are types;
- proofs are programs; and
- simplification of proofs is evaluation of programs.

That is

Types	Propositions
Unit types (1)	Truth ( $\top$ )
Product type ( $\times$ )	Conjunction ( $\wedge$ )
Union type ( $+$ )	Disjunction ( $\vee$ )
Function type ( $\rightarrow$ )	Implication ( $\supset$ )
Void types (0)	False ( $\perp$ )

Every typing rule has a correspondence with a deduction rule.

**Example 76** The typing derivation of the term  $\lambda a. \lambda b. \langle a, b \rangle$  can be seen as a deduction tree proving  $A \supset B \supset A \wedge B$ .

$$\begin{array}{c}
 \frac{\frac{a : A \in \Gamma}{\Gamma \vdash a : A} \text{ var} \quad \frac{b : B \in \Gamma}{\Gamma \vdash b : B} \text{ var}}{\Gamma \vdash \langle a, b \rangle : A \times B} \text{ pair} \\
 \frac{\Gamma \vdash \lambda y : B. \langle a, y \rangle : B \rightarrow A \times B}{\Gamma \vdash \lambda x : A. \lambda y : B. \langle x, y \rangle : A \rightarrow B \rightarrow A \times B} \text{ abs}
 \end{array}
 \iff
 \begin{array}{c}
 \frac{\frac{\overline{A \text{ true}}^u \quad \overline{B \text{ true}}^w}{A \wedge B \text{ true}} \wedge \wedge I}{\frac{B \supset A \wedge B \text{ true}}{A \supset B \supset A \wedge B \text{ true}} \supset I^w} \supset I^u
 \end{array}$$

**Annotation 77** 从上面例子中看的出来, the inference rule of natural deduction 缺点什么, 我也可以给原本每个 inference rule 都加上 the annotation for proof terms. [6] 那么这里  $M : A$  有两种解释:

1.  $M$  is proof term for proposition  $A$ ;
2.  $M$  is a program of type  $A$ .

这样解释 Curry-Howard ismorphism 或许方便一点. 让 proof terms make sense: 我们有”if  $M : A$  then  $A \text{ true}$ ”, 反过来”if  $A \text{ true}$  then  $M : A$ ”. 例如我们可以将 the proof term of  $A \wedge B \text{ true}$  看做一个 pair 包含两个 subterm, 一个关于  $A \text{ true}$  和另一个关于  $B \text{ true}$ .

$$\frac{M : A \quad N : B}{\langle M, N \rangle : A \wedge B} \wedge I$$

那么 the elimination rule of conjunction 对应一个 natural projection.

$$\frac{M : A \wedge B}{\pi_1 M : A} \wedge E_L \quad \frac{M : A \wedge B}{\pi_2 M : B} \wedge E_R$$

**Example 78** 通过 Curry-Howard isomorphism 我们可以将我们想要证明的 judgment 转换到 type system 中, 你会看到非常的便利! 例如

$$(A \supset (B \wedge C)) \supset (A \supset B) \wedge (A \supset C) \text{ true}$$

等价于

$$\lambda x. \langle \lambda y. \pi_1(x y), \lambda y. \pi_2(x y) \rangle : (A \rightarrow B \times C) \rightarrow (A \rightarrow B) \times (A \rightarrow C)$$

一个 implication 被转换成了对应的 abstraction, 此时我们肯定会想如果给一个 false proposition 是不是就转不了? 例如

$$(A \supset B) \supset (B \supset A)$$

显然我们无法在现有 type system 构造出一个合理的 abstraction 使得  $(A \rightarrow B) \rightarrow (B \rightarrow A)$ .

迎面走来的问题是: 给定一个 proposition true, 是否有其他的 term with type 和它对应呢? 显然是有的,

$$\lambda z. \lambda x. \langle \lambda y. \pi_1(x y), \lambda y. \pi_2(x y) \rangle z'$$

那这是不是违反 Curry-Howard isomorphism 了呢? 其实并不是, 这里的对应是指 proof terms 和 deduction of proposition true, 显然 deduction 变了, 对应的 proof terms 也要变.

## More Delicate

### Validity

**Definition 79**  $A$  *valid* if  $\bullet \vdash A \text{ true}$  where  $\bullet$  is emphasizing that there are no truth hypotheses (different from  $\cdot$  that represents empty collection of hypotheses), and we call  $\bullet \vdash A \text{ true}$  is **categorical judgement**. Written  $\Delta A$  for reflecting the notion of validity as a proposition.

**Annotation 80** 其中  $\Box A$  表示一个 proposition claimed  $A$  is valid, 因此  $\Box A \text{ true}$  表示这个 proposition 成立. 那么关于它的 introduction rule 是什么? 很自然地由  $A$  *valid* 的 definition 有

$$\frac{\bullet \vdash A \text{ true}}{\Gamma \vdash \Box A \text{ true}} \Box I$$

那么它的 elimination rule 又是什么呢? 第一次尝试

$$\frac{\Gamma \vdash \Box A \text{ true}}{\bullet \vdash A \text{ true}} \Box E$$

看起来是 local soundness, 通过它得到的 infos 还行. 但是实际上有问题

$$\frac{\Box A \text{ true} \vdash \Box A \text{ true}}{\bullet \vdash A \text{ true}} \Box E$$

这等于我们可以 no assumption 推出所有 proposition 都是 valid, 因此这个 elimination rule 有点太强了. 那么我们考虑让它弱一点, 第二次尝试

$$\frac{\Gamma \vdash \Box A \text{ true}}{\Gamma \vdash A \text{ true}} \Box E$$

这里确实是 local soundness, 但却不是 local completeness

$$\frac{\frac{\Gamma \vdash \Box A \text{ true}}{\Gamma \vdash A \text{ true}}}{\Gamma \vdash \Box A \text{ true}} \Box E \quad ?$$

我们得改变一下思路, 如果  $A$  *valid*, 那么其他 premise 包含  $A$  *valid* 的 judgement 那么实际上都是可以去掉  $A$  *valid*, 但也仅仅局限以此, 这才是 emilination 故事的主线.

**Definition 81** Then general judgement form

$$\underbrace{u_1 :: B_1 \text{ valid}, \dots, u_k :: B_k \text{ valid}}_{\Delta}; \underbrace{x_1 : A_1 \text{ true}, \dots, x_n : A_n \text{ true}}_{\Gamma} \vdash C \text{ true}$$

**Definition 82** The introduction rule and elimination rule of  $A$  *vaild* as follow

$$\frac{\Delta; \bullet \vdash A \text{ true}}{\Delta; \Gamma \vdash \Box A \text{ true}} \Box I \quad \frac{\Delta; \Gamma \vdash \Box A \text{ true} \quad \Delta, u :: A \text{ vaild}; \Gamma \vdash C \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} \Box E$$

**Theorem 83** Local soundness and local completeness of above introduction and elimination rule are held

**Annotation 84** 可以看到 emilination rule 变成了 substitution，而不是从单纯从本身要得到什么，后面会看见更多这样的东西.

**Example 85** Proof of  $\cdot; \cdot \vdash \Box A \supset A$ .

$$\frac{\frac{\cdot; x : \Box A \text{ true} \vdash \Box A \text{ true} \quad x \quad \frac{u :: A \text{ vaild}; x : \Box A \text{ true} \vdash A \text{ true}}{\Box E^u}}{\cdot; x : \Box A \text{ true}} \supset I^x$$

## Box is Powerful

**Definition 86**  $\Box$  is  $\Box$ .

**Definition 87** A term  $\text{box}M$  means  $M$  is a quoted source expression such that there are not any free variables  $x$ .

**Definition 88** And  $\Box A$  is necessity modality.

## Possibility

**Definition 89** We use  $\Diamond A$  for possibility modality.

**Annotation 90**  $\Diamond A$  就是一个 claim  $A$  is possible 的命题. 通常在 classic modal logic 里面我们定义  $A$  is possible if its negation is not necessary, that is  $\Diamond A = \neg \Box \neg A$ . 但是这种手法在现在我们讨论的 intuitionistic logic 无法奏效, 我们希望的是有一个直观的 introduction rule 来得到它, 也就是我们需要一些 explicit evidences, 一开始就它的 negation 那显然是做不到的.

**Definition 91** [7] The definition of possibility.

$$\frac{\Delta, \Gamma \vdash A \text{ true}}{\Delta, \Gamma \vdash A \text{ poss}} \text{ poss}$$

**Definition 92** The introduction and emilation rule of possibility.

$$\frac{\Delta; \Gamma \vdash A \text{ poss}}{\Delta; \Gamma \vdash \Diamond A \text{ true}} \Diamond I \quad \frac{\Delta; \Gamma \vdash \Diamond A \text{ true} \quad \Delta; x : A \text{ true} \vdash C \text{ poss}}{\Delta; \Gamma \vdash C \text{ poss}} \Diamond E$$

**Annotation 93** 注意这里的 emilation rule 里面的第二个 premise 中的 hypothesis 只有  $A \text{ true}$ , 即我们 under assumption  $A \text{ true}$ , we conclude  $C \text{ poss}$ .

**Theorem 94** Local soundness and completeness are held.

**Annotation 95** td; 对上述 inference rule 的理解.

**Example 96** Proof of  $\Box(A \supset B) \supset \Diamond A \supset \Diamond B$ .

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