Proof Theory

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目录

Basic Logic

Logical Framework

Definition 1 Meta logic is the logic that used to formalize anthor logic

Definition 2 Object logic is the logic that formalized from meta logic.

Definition 3 Implementations of meta logics to represent and reason about object logics are called logical frameworks.

Annotation 4 上面提到的 implementations 实际上就是 languages，例如 Prolog 是这样的一个 language，它以 Horn clause 作为 meta logic. 你要实现的 object logic 可以通过这个 language 来进行 encoding，也就是所谓的 representating，同时还可以在 encoding 基础上进行 reasoning.

First Order Logic

Example 5 给定一个 statement:

"Every natural number is even or odd, but not both."

如果我们想要构造一个 system 去 accept 这个 statement，我们需要说明 natural number 是什么？even 和 odd 又是什么？如何描述 even 和 odd 是冲突的？这里引出这个 system 需要描述的三个重要部分: objects, their properties, and relations between them. 同时如何描述 every natural number？这里又涉及到了 quantifiter.

Annotation 6 我遵循描述一个 logical system, 首先定义里面的 terms, 再描述由这些 terms 构成的 formulas.

Definition 7 A signature σ consist of *constant symbols*, *function symbols* and *predicate symbols*.

Definition 8 Let σ-*terms* be a set of terms in first order logic, it defined by the follow inductive process:

1. Each variable $x \in \sigma$-$terms$.
2. Each constant symbol $a \in \sigma$, then $a \in \sigma$-$terms$.
3. If $t_1, \cdots, t_2 \in \sigma$-$terms$ and f is k-ary function symbol in σ, then $f(t_1, \cdots, t_k) \in \sigma$-$terms$.

Let $\mathcal{F}_{\sigma\text{-}terms}$ be a set of formulas in first order logic defined by follow inductive process:

1. Given terms $t_1, \cdots, t_k \in \sigma$-$terms$ and k-ary predicate symbol P then $P(t_1, \cdots, t_k)$ is a term.
2. All formulas relate to logical connective in propositional logic.
3. If $F \in \mathcal{F}_{\sigma\text{-}terms}$ and x is a variable, then $\forall x.F$ and $\exists x.F$ are both in $\mathcal{F}_{\sigma\text{-}terms}$.

Annotation 9 在理解 propositional logic 的基础上，只要记住几个关键词 *function*, *predicate*, and *quantifier* 就足以刻画 first order logic.

Satisfiability of Sets of Formulas

Definition 10 If v is a valuation, this is, a mapping from the atoms to the set $\{t, f\}$.

Definition 11 [4] Let Σ denote a set of well-formed formulas and t a valuation. Define

$$\Sigma^t = \begin{cases} T & \text{if for each } \beta \in \Sigma, \beta^t = T \\ F & \text{otherwise} \end{cases}$$

When $\Sigma^t = T$, we say that t satisfies Σ. A set Σ is satisfiable iff there is some valuation t such that $\Sigma^t = T$.

Definition 12 Let Σ be a set of formulas, and let α be a formula, we say that

1. α is a logical consequence of Σ, or
2. Σ (semantically) entails α, or
3. $\Sigma \models \alpha$,

if and only if for all truth valuations t, if $\Sigma^t = T$ then also $\alpha^t = T$. We write $\Sigma \not\models \alpha$ for there exists a truth valuation t such that $\Sigma^t = T$ and $\alpha^t = F$.

Annotation 13 For example, $\Sigma = \{p_1, p_2, \cdots, p_n\}$ could be a set of premises and let α could be the conclusion that we want to derive.

Classic Propositional Modal Logic

Definition 14 [8]Let Σ be a set of propositional letters or atomic propositions. The set $F_P(\Sigma)$ of formulas of classical propositional modal logic is the smallest set with:

1. If $A \in \Sigma$ is a propositional letter, then $A \in F_P(\Sigma)$;
2. If $\phi, \psi \in F_P(\Sigma)$, then $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi) \in F_P(\Sigma)$;
3. If $\phi \in F_P(\Sigma)$, then $(\Box\phi), (\Diamond\phi) \in F_P(\Sigma)$.

Definition 15 Let $\mathcal{S}$ be a system of modal logic, this is $F_P(\Sigma)$ with a set of axioms and rules. If axioms and rules as follow

$$\text{all propostional tautologies} \qquad \text{(P)}$$

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) \qquad \text{(Kripke axiom)}$$

$$\Box\phi \to \phi \qquad \text{(T)}$$

$$\Box\phi \to \Box\Box\phi \qquad \text{(4)}$$

$$\frac{\phi \quad \phi \to \psi}{\psi} \qquad \text{(modus ponens)}$$

$$\frac{\phi}{\Box\phi} \qquad \text{(Gödel)}$$

We call it modal logic $\mathcal{S}4$.

Annotation 16 Kriple axiom 原本的形式应为

$$\Box\phi \wedge \Box(\phi \to \psi) \to \Box\psi$$

上面是它经常用的等价形式. Axiom T 是指若 ϕ is necessary, 那么 ϕ is true. Axiom 4 是指 ϕ is necessary, 那么命题"ϕ is necessary" is necessary, 有点别扭，举个形象的例子如果 box 是指某个人知道某件事，假设我知道 A *true*, 那么我肯定知道我知道 A *true*. 最后一个叫 Gödel translation, 它将 intuitionistic logic 里面的 formulas 转换到 modal logic 里面.

Definition 17 Let $\mathcal{S}$ be a system of modal logic. For a formula ψ and a set of formulas Φ, we write $\Phi \vdash_{\mathcal{S}} \psi$ and say that ψ can be derived from Φ(or is provable from Φ), iff there is a proof of ψ that uses only the formulas of Φ and the axioms and proof rules of $\mathcal{S}$. That is, we define $\Phi \vdash_{\mathcal{S}} \psi$ inductively as:

$$\Phi \vdash_{\mathcal{S}} \psi$$

iff $\psi \in \Phi$ or there is an instance

$$\frac{\phi_1 \quad \cdots \quad \phi_n}{\psi}$$

of a proof rule of $\mathcal{S}$ with conclusion ψ and some number $n \geq 0$ of premises such that for all $i = 1, 2, \cdots, n$, the premises ϕ_i is derivable, i.e:

$$\Phi \vdash_{\mathcal{S}} \phi_i$$

When the case $n = 0$ corresponds to axioms.

Annotation 18 现在以 $\Box$ 表示 provable 的视角来看待前面提到的 axioms. 首先是

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) \quad \text{(Kripke axiom)}$$

若 $\phi \to \psi$ is provable 且 ϕ is provable, 那么则 ψ is provable.

$$\Box\phi \to \phi \quad \text{(T)}$$

若 ϕ is provable, 那么 ϕ should be true.

$$\Box\phi \to \Box\Box\phi \quad (4)$$

若 ϕ is provable, 那么 ϕ should be provably provable, 也就是我们肯定知道存在一个 proof.

$$\frac{\phi}{\Box\phi} \quad \text{(Gödel)}$$

若 ϕ is proven, 那么 ϕ should be provable.

Definition 19 A Kripke frame (W, ρ) consists of a non-empty set W and a relation $\rho \subseteq W \times W$ on worlds. The element of W are called possible worlds and ρ is called accessibility relation.

Definition 20 A Kripke structure $K = (W, \rho, v)$ consists of Kripke frame (W, ρ) and a mapping $v : W \to \Sigma \to \{true, false\}$ that assigns truth-values to all the propositional letters in all worlds.

Definition 21 Given a Kriple structure $K = (W, \rho, v)$, the interpretation $\vDash$ of modal formulas in worlds s is defined as

- $K, s \vDash A$ iff $v(s)(A) = true$;
- $K, s \vDash \phi \wedge \psi$ iff $K, s \vDash \phi$ and $K, s \vDash \psi$;
- $K, s \vDash \phi \vee \psi$ iff $K, s \vDash \phi$ or $K, s \vDash \psi$;
- $K, s \vDash \neg\phi$ iff it is not the case that $K, s \vDash \phi$;

- $K, s \vDash \Box\phi$ iff $K, t \vDash \phi$ for all worlds t with $s\rho t$;
- $K, s \vDash \Diamond\phi$ iff $K, t \vDash \phi$ for some worlds t with $s\rho t$.

Annotation 22 最后两个关于 modality $\Box$ 和 $\Diamond$ 定义是最重要的，它们借助 accessible possible world 来 make sense. 可以通过它们的 nesting 形式来描述更长的路径即 $\Box\Box, \Diamond\Diamond, \Box\Diamond$.

Definition 23 Given a Kripke structure $K = (W, \rho, v)$, formula ϕ is vaild in K, written $K \vDash \phi$, iff $K, s \vDash \phi$ for all worlds $s \in W$.

Definition 24 (local consequence) Let ψ be a formula and Φ a set of formulas. Then we write $\Phi \vDash_l \psi$ if and only if, for each Kripke structure $K = (W, \rho, v)$ and each world $s \in W$, we have $K, s \vDash \Phi$ implies $K, s \vDash \psi$.

Definition 25 (global consequence) Let ψ be a formula and Φ a set of formulas. Then we write $\Phi \vDash_g \psi$ if and only if, for each Kripke structure $K = (W, \rho, v)$, if for all world $s \in W : K, s \vDash \Phi$, then for all world $s \in W : K, s \vDash \psi$.

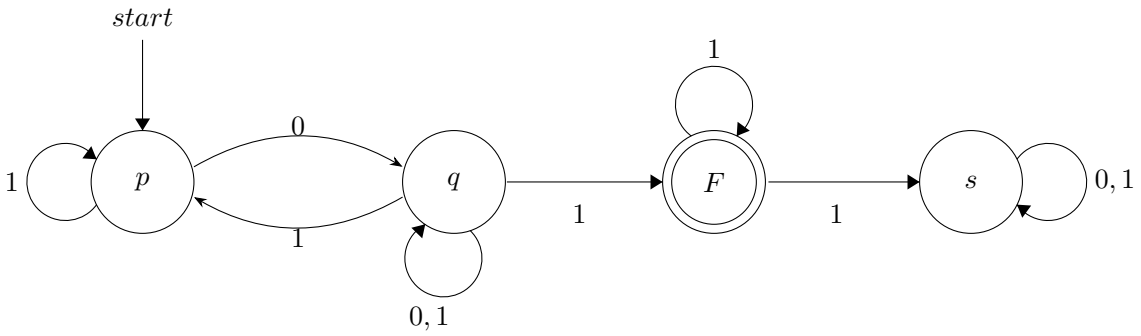
Annotation 26 local consequence 和 global consequence 的区别就是 assumption 是在某个 world 里面还是在所有的 worlds 里面.

Definition 27 A formula ϕ is vaild or a tautology, iff $\emptyset \vDash_l \phi$, which we write $\vDash \phi$. A set of formulas Φ is called satisifable, iff there is a Kripke structure K and a world s with $K, s \vDash \Phi$.

Lemma 28 (local deduction theorem) For formulas ϕ, ψ we have

$$\phi \vDash_l \psi \iff \vDash_l \phi \to \psi.$$

Annotation 29 (view of finite automata) 对于 Kripke frame 的第一反应应该是 finite automata, 但是对于一个给定的 finite automata 我们还需要一些额外的说明. 例如



每一个 state 里面存在一个 proposition，它表示这个 proposition is hold at this state，自然地 states 就变成了 possible worlds. state 现在可以接受多个输入 $\{0,1\}$，那么这里就表示我们有两个 relations ρ_0 和 ρ_1，对应我们需要两个 pair 来构建不同的 modality $(\Box_0, \Diamond_0)$ 和 $(\Box_1, \Diamond_1)$，它们都是用于描述某个 state 的 successor. 因此这里可以对应上一个 Kripke structure, 对上图我们可以列举几个 vaild formula.

$$\begin{aligned} &K \vDash \neg\Diamond_0 F && \text{does not end with 0} \\ &K \vDash p \to \Diamond_0 p && p \text{ has a 1-loop} \\ &K \vDash \Diamond_0\ true && \text{never stuck with input 0} \\ &K \vDash \Diamond_1\ true && \text{never stuck with input 1} \end{aligned}$$

再看一个稍微复杂一点

$$K \vDash F \to \Diamond_0(\neg\Diamond_0 F \wedge \neg\Diamond_1 F)$$

它意思如果某个状态 σ 下 F is hold, 那么 σ accept 0 的 successors $\{s_i\}$ 中每个 s_i 的 successors 都无法 hold F，显然这是成立的.

Definition 30 A system $\mathcal{S}$ of proof rules and axioms of modal logic is sound iff, for all formulas ψ and all sets of formulas Φ:

$$\Phi \vdash_S \psi \text{ implies } \Phi \vDash_g \psi$$

Annotation 31 上述 soundness 实际在建立关于 axiomatic modal logic 和 semantic modal logic 之间的一座桥，这座桥需要每一个 axiom make sense.

Lemma 32 Kripke axiom $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ is sound.

Proof 首先给定任意一个 Kripke structure K. 我们需要证明

$$K, s \vDash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi).$$

因此假设其前提

$$\begin{aligned} &K, s \vDash \Box(\phi \to \psi) \\ &K, s \vDash \Box\phi \end{aligned}$$

那么对应所有满足 $s\rho t$ 的 successor t, 都有

$$\begin{aligned} &K, t \vDash \phi \to \psi \\ &K, t \vDash \phi \end{aligned}$$

自然地这里有 $K, t \vDash \psi$，于是 $K, s \vDash \Diamond\psi$. Q. E. D.

Lemma 33 Gödel Rule $\dfrac{\phi}{\Box\phi}$ is sound.

PROOF 注意这里的结论是建立在 global assumption 上的，即 $K, s \vDash \phi$ for any $s \in W$，证明过程是显然的. Q. E. D.

Lemma 34 A Kripke frame (W, ρ) is reflexive, that ρ is reflexive, if and only if $K, s \vDash \Box q \to q$ for all Kripke structures $K = (W, \rho, v)$.

PROOF ($\Rightarrow$) 若 (W, ρ) is reflexive, 这是显然的.

($\Leftarrow$) 若 $K, s \vDash \Box q \to q$ for all Kripke structures $K = (W, \rho, v)$. 假设存在一个 r such that $(r, r) \notin \rho$, 构造一个比较巧妙地 valuation v

$$v(s)(q) = \begin{cases} true & \text{if } r \rho s \\ false & \text{otherwise} \end{cases}$$

那么显然有 $K, r \vDash \Box q$，根据前提这里有 $K, r \vDash q$，而根据 valuation 这里就 r 存在一个 successor 是它自己，即 (r, r) 与假设矛盾. Q. E. D.

Lemma 35 A Kriple frame (W, ρ) is transitive, that ρ is transitive, if and only if $K, s \vDash \Box q \to \Box\Box q$ for all Kripke structures $K = (W, \rho, v)$.

PROOF ($\Rightarrow$) 若 (W, ρ) is transitive, 给定 $K, s \vDash \Box q$，对于 s 的任意一个 successor $t (s \rho t)$ 则有 $K, t \vDash p$，进一步对 t 的任意一个 successor $r (t \rho r)$，考虑 transitive $s \rho r$，那么有 $K, r \vDash p$. 由于 t 和 r 的任意性，因此 $K, s \vDash \Box\Box p$.

($\Leftarrow$) 若 Kriple frame 满足对任意的 valuation v 都有 $K, s \vDash \Box q \to \Box\Box q$. 假设 (W, ρ) 不是 transitive，那么存在 $r_1, r_2, r_3 \in W$ such that $r_1 \rho r_2, r_2 \rho r_3$ and $(r_1, r_3) \notin \rho$. 构造一个 valuation v

$$v(s)(q) = \begin{cases} true & \text{if } r_0 \rho s \\ false & \text{otherwise} \end{cases}$$

那么 $K, r_0 \vDash \Box q$，但是因为 $(r_0, r_3) \notin \rho$，因此 $K, r_0 \nvDash \Box\Box q$，和假设前提矛盾了. Q. E. D.

Annotation 36 这座需要两边的支撑一样高，给定特定 axiomatic modal logic, 我们得到找到与之对应的 semantic modal logic，我们的手法就是 sketch it from basic Kripke frame. 当我们尝试构造了一部分之后，我们需要让其 make sense，上述 lemma 利用 formula 来 characterize 是一个不错的选择.

Definition 37 (characterization)Let C be a class of Kripke frames and ϕ a formula in modal logic. Formula ϕ characterizes C, if for every Kripke frame (W, ρ):

$$(W, \rho) \in C \text{ iff for each } v : K, s \vDash \phi \text{ holds for } K = (W, \rho, v).$$

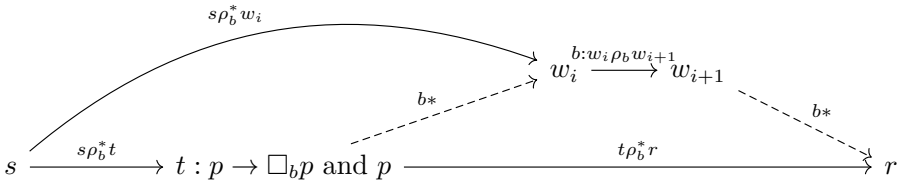
Theorem 38 (soundness of $\mathcal{S}4$)The Kriple proof rules for $\mathcal{S}4$ are sound for the class of reflexive and transitive frames.

Theorem 39 The conjunction of the following two multimodal formulas

$$\Box_a p \to (p \wedge \Box_a \Box_b p)$$
$$\Box_a(p \to \Box_b p) \to (p \to \Box_a p)$$

characterizest the class of all multimodal kripke frames (W, ρ_a, ρ_b) such that ρ_a is the reflxive, transitive closure of ρ_b.

Proof ($\Leftarrow$) 如果 $(W, \rho_a, \rho_b)$is Kripke frame where ρ_a is the reflexive, transitive closure of ρ_b. 对于一个 formula 只要注意到 $\Box_a \Box_a p \to \Box_a \Box_b p$ 即可，可以从需要考虑的 successors 数量来证明. 对于第二个 formula, 先给一个思考图



这里 $\rho_a = \rho_b^*$. 这里证明手法是

$$\Box_a(p \to \Box_b p) \to \Box_a(p \to \Box_a p) \text{ and } \Box_a(p \to \Box_a p) \to (p \to \Box_a p)$$

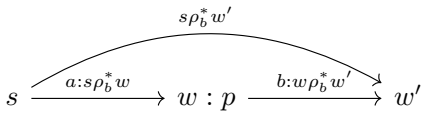
最重要是证明第一个 implication，第二个 implication 是前面已经证明过的 reflexive. 对于第一个 implication 它描述的是首先给出前提 (1)$\Box_a(p \to \Box_b p)$ 即 $s\rho_b^* t$. 然后我们想要将 t 中 $p \to \Box p$ 扩展至 $p \to \Box_b^* p$，因此再给一个假设前提 t holds p，我们来考察 $\Box_b^* p$ 是否成立即 $t\rho_b^* r$. 这里我们需要分解 $t\rho_b^* r$ 使其为 $w_i \rho_b w_{i+1}$ for all $i < n$，其中 $w_0 = t$ 和 $w_n = r$. 利用数学归纳法证明 $K, w_i \vdash p$，这里就不详细描述了，和后面一个证明过程类似，但是说明几点: (1)$s\rho_b^* w_i$ 送来了 $p \to \Box_b p$ (2) 假设前提保证了 $K, w_i \vDash p$. 因此 $K, w_{i+1} \vDash p$.

($\Rightarrow$) 如果 $(W, \rho_a, \rho_b)$is Kripke frame such that above formulas are vaild in it for any valution v. 我们得证明 $\rho_a = \rho_b^*$. 这种证明两个集合相等的手法，还是用两边证.

先证$\rho_a \subseteq \rho_b^*$. 取任意的 $(s,t) \in \rho_a$，我们得证明 $(s,t) \in \rho_b^*$. 还是构造一个特殊的 valuation

$$v(w)(q) = \begin{cases} true & \text{if } (s,w) \in \rho_b^* \\ false & \text{otherwise} \end{cases}$$

我们的思路是首先证明第二个 formula 的前提 (1) $\Box_a(p \to \Box_b p)$，从而得到对应的 conclusion (2) $(p \to \Box_a p)$，由给定的 v 结合 ρ_b^* 的 reflexive 性质，自然地有 $K, s \vDash p$，在使用一下 (2) 得到 $K, t \vDash p$, 这样就有 $(s,t) \in \rho_b^*$. 证明 (1) 思路是依然是假设前提: 给定 $s\rho_a w$ 且 $K, w \vDash p$，实际上 $(s,w) \in \rho_b^*$. 考虑下面的思考过程

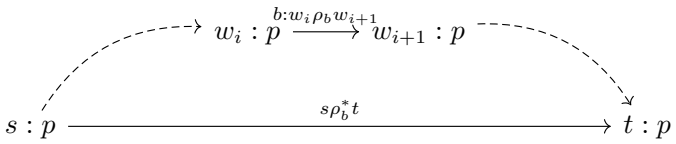


另外又给了一个 w' 满足 $w\rho_b w'$，再根据 ρ_b^* 的 transitive 得到 $s\rho_b^* w'$，从而 $K, w' \vDash p$.

再证$\rho_a \supseteq \rho_b^*$. 取任意的 $(s,t) \in \rho_b^*$，我们要证明 $(s,t) \in \rho_a$. 依然构造一个类似的 valuation

$$v(w)(q) = \begin{cases} true & \text{if } (s,w) \in \rho_a \\ false & \text{otherwise} \end{cases}$$

我们的思路: 由于我们构造地特别的 v 有 $K, s \vDash \Box_a p$, 借助命题中的第一个 formula 得到对应的 conclusion (1) $K, s \vDash p \wedge \Box_a \Box_b p$. 考虑下面的思考过程



我们考虑将 $s\rho_b^* t$ 拆开，设 $w_i \rho_b w_{i+1}$ for all $i < n$，其中 $w_0 = s$ 和 $w_n = t$，这是可以做到的，考虑 closure 的构造过程. 再用一下数学归纳法证明 $K, w_i \vDash p$，在 $i = 0$ 显然是成立的，假设 w_i 成立，那么根据 v 即有 $s\rho_a w_i$，再利用一下 (1) 可以得到 $K, w_i \vDash \Box_b p$，因此 $K, w_{i+1} \vDash p$. 最终 $K, t \vDash p$，那么 $(s,t) \in \rho_a$. Q. E. D.

Annotation 40 回顾上面的证明手法，我们如果想要刻画两个 possible worlds 是否存在某种关系，例如 $(r,t) \overset{?}{\in} \rho$，我们可以额外借助一个 formula p 和 valuation v，仅使得所有 w 满足 $r\rho w$ 都 hold p. 这样如果我们能利用额外和 p 相关的条件间接证明 t holds p, 那么就可以证明 $s \to t$. 我们应该意识到 relations 是 Kripke frame 固有的性质，与 valuation 无关因此这里我们可以任意的定义它.

Natural Deduction

Remark 41 Natural deduction is a kind of proof calculus in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning.

Judgments and Propositions

Definition 42 A *judgment* is somthing we may know, this is, an object of knowledge. A judgment is *evident* if we in fact know it.

Annotation 43 "A is false" (see classical logic), "A is true at time t" (see temporal logic), "A is necessarily true" or "A is possibly true" (see modal logic), "the program M has type " (see programming languages and type theory), "A is achievable from the available resources" (see linear logic).

Introduction and Elimination

Definition 44 Inference rules that introduce a logical connective is the conclusion are known as *introduction rules*. i.e., to conclude "A and B *true*" for propositions A and B, one requires evidence for "A *true*" and B *true*. As an inference rule:

$$\frac{A\ true \quad B\ true}{A \wedge B\ true}\ \wedge I$$

Here $\wedge I$ stands for "conjunction introduction".

Annotation 45 实际上面的 inference rule 的 general form 应该是

$$\frac{A\ prog \quad B\ prog \quad A\ true \quad B\ true}{A \wedge B\ true}\ \wedge I$$

这里才能帮助后面的 $\vDash$ make sense.

Definition 46 Inference rules that describe how to deconstruct information about a compound proposition into information about its consitiuents are elimination rules. i.e., from $A \wedge B$ *true*, we can conclude A *true* and B *true*:

$$\frac{A \wedge B\ true}{A\ true}\ \wedge E_L \qquad \frac{A \wedge B\ true}{B\ true}\ \wedge E_R$$

Annotation 47 The meaning of conjunction is determinded by its *verifications*.

Hypothetical Derivations

Definition 48 A *hypothetical judgment* is $J_1, \cdots, J_n \vdash J$, where judgments $J_1, \cdots, J_n$ are unproved assumptions, and the judgment J is the conclusion. A *hypothetical deduction*(derivation) for $J_1, \cdots, J_n \vdash J$ has the form

$$\begin{array}{c} J_1 \quad \cdots \quad J_n \\ \vdots \\ J \end{array}$$

which means J is derivable from $J_1, \cdots, J_n$.

Annotation 49 上面的 $J_1, \cdots, J_2$ 都可以替换成关于 J_i 的一个 hypothetical derivation.

Definition 50 In the natural deduction calculus, an assumption is discharged when the conclusion of an inference does not depend on it, although one of the premises of the inference does[1].

Annotation 51 Once the appropriate rules have been completed, these are known as discharged assumptions, and are not included in the pool of assumptions on which the conclusion of the rule depends[3].

Annotation 52 hypothetical derivation 要求最后的 conclusion 依赖的 poof of assumptions 不是空的.

Theorem 53 Deduction theorem

$$T, P \vdash Q \iff T \vdash P \to Q$$

.

Annotation 54 在 deduction theorem 中我们注意到第一个 hypothetical judgment 里面的 antecedent Q 被去掉了，在第二个 hypothetical judgment 的 succedent 里面作为一个 implication 的 antecedent 出现了，这里我们就可以说 assumption Q is discharged，即现在的 conclusion 已经不依赖它了. 那么我们是如何构造 deduction theorem 里面的 implication 的呢？下面接着看

Definition 55 (implication) If B is true under the assumption that A is true, formly written $A \supset B$. The corresponed introduction and elimination rule as follow

$$\frac{\begin{array}{c}\overline{A\ true}^{\,u} \\ \vdots \\ B\ true\end{array}}{A \supset B\ true}\ \supset I^u \qquad \frac{A \supset B\ true \quad A\ true}{B\ true}\ \supset E$$

Annotation 56 Why indexed u In the introduction rule, the antecedent named u is discharged in the conclusion. This is a mechanism for delimiting the scope of the hypothesis: its sole reason for existence is to establish "$B\ true$"; it cannot be used for any other purpose, and in particular, it cannot be used below the introduction.

上面这段话出自 natural deduction 的 wiki，这个 $u$scope 了 assumption $A\ true$ 的开端，因为 $A \supset B$ 并不依赖 $A\ true$, 它描述只是 if $A\ true$ then $B\ true$. 同时最后的 introduction rule 会将这个 assumption $A\ true$ discharged 掉，表示 scope 在这里已经结束了. 而 implication rule 会将上述 derivation 直接总结得到一个结论，即

$$A \vdash B \Rightarrow \cdot \vdash A \to B.$$

Example 57 Considering the following proof of $A \supset (B \supset (A \wedge B))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A\ true}^{\,u} \quad \overline{B\ true}^{\,w}}{A \wedge B\ true}\wedge I}{B \supset (A \wedge B)\ true} I^{w}}{A \supset (B \supset (A \wedge B))\ true} I^{u}.$$

这整个 derivation 不是 hypothetical 的，因为两个 assumptions $A\ true$ 和 $B\ true$ 都已经被 discharged，因此它实际上一个 complete proof!

Definition 58 (disjunction) The elimination rule for disjunction:

$$\dfrac{A \vee B\ true \quad \begin{array}{c}\overline{A\ true}^{\,u}\\ \vdots \\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}^{\,w}\\ \vdots \\ C\ true\end{array}}{C\ true} \vee E^{u,w}$$

both assumption u, w are discharged at the disjunction elimination rule.

Definition 59 The falsehood elimination rule:

$$\dfrac{\bot\ true}{C\ true}\ \bot E$$

Annotation 60 falsehood elimination 的意义在哪？首先你应该主要到一个特殊等价命题 $A \vee \bot = A$，从 $\vee$ 的 introduction rule 来看这意味 $\bot\ true \vdash A\ true$，由于 A 是任意的，因此我们得到了 $\bot\ true \vdash C\ true$.

Harmony

Definition 61 Local soundness shows that the elimination rules are not strong: no matter how we apply eliminations rules to the result of an introduction we cannot gain any new information.

Definition 62 Local completeness shows that the elimination rules are not weak: there is always a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the the results by apply intruduction ruls.

Annotation 63 local soundness 告诉你通过 elimination 压缩得到的东西不会比你已经知道的东西强 (not strong), 而 local completeness 告诉你合并通过 elimination 压缩得到的东西会得到全部你知道的信息.

Definition 64 Given two deduction of same judgment, we use the notion

$$\begin{matrix}\mathcal{D}\\ A\ true\end{matrix} \Longrightarrow_R \begin{matrix}\mathcal{D}'\\ A\ true\end{matrix}$$

for the local reduction of a deduction $\mathcal{D}$ to another deduction D' of same judgement $A\ true$. Simliarly, we have local expansion

$$\begin{matrix}\mathcal{D}'\\ A\ true\end{matrix} \Longrightarrow_E \begin{matrix}\mathcal{D}\\ A\ true\end{matrix}$$

Definition 65 (substitution Principle) If

$$\begin{matrix}\overline{A\ true}\ u\\ \mathcal{E}\\ C\ true\end{matrix}$$

is a hypothetical proof of $\mathcal{C}\ true$ under the undischarged hypothesis $A\ true$ labelled u, and

$$\begin{matrix}\mathcal{D}\\ A\ true\end{matrix}$$

is a proof of $A\ true$ then

$$\begin{matrix}\dfrac{\mathcal{D}}{A\ true}\ u\\ \mathcal{E}\\ C\ true\end{matrix}$$

is our notation for substituting $\mathcal{D}$ for all uses of the hypothesis labelled u in $\mathcal{E}$. This deduction, also sometime written as $[\mathcal{D}/u]\mathcal{E}$ no longer depends on u.

Example 66 If given a elimination rule of disjunction as follow

$$\frac{A \vee B\ true}{A\ true}\ \vee E_L$$

The rule a little bit stronger, since we would not be able to reduce

$$\dfrac{\dfrac{B\ true}{A \vee B\ true}\ \vee I_R}{A\ true}\ \vee E_L$$

As u can see it's not local soundness.

Verifications and Uses

Definition 67 a verification should be a proof that only analyzes the consitituents of a proposition.

Annotation 68 [9] 在 natural deduction 中由于 local reduction 的存在，可能会让一个证明过程变得非常的冗余，例如在证明 conjunction commutativity

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \wedge B\ true}^{\,u}}{A\ true}\wedge E_1 \quad \dfrac{\overline{A \wedge B\ true}^{\,u}}{B\ true}\wedge E_2}{A \wedge B\ true}\wedge I}{B\ true}\wedge E_2 \quad \dfrac{\dfrac{\overline{A \wedge B\ true}^{\,u}}{A\ true}\wedge E_1}{B \wedge A\ true}\wedge I}{(A \wedge B) \supset (B \wedge A)\ true}\supset I^u$$

其中左上角的 local reduction 显然是冗余的. 这样对于谈论某个具体 proposition 的 proof 时就会出现问题, 因为 the shape of proof is not decidable. 同时我们也希望未来能够设计出一个 tool 用于 deviates proofs automatically, 也就是 search proof automatically. 因此从 natural deduction 上诞生了一个新的 calculus，它会在 syntax level 上来施加一些限制，借此限制 the shape of proof. 最后我们将证明这个 calculus 引入的 restrictions 不会产生 side-effect.

Definition 69 Writing $A\uparrow$ for the judgment "A has a verification". Naturally, this should mean that A is true, and that the evidence for that has a special form.

Definition 70 Writing $A\downarrow$ for the judgment "A may be used". $A\downarrow$ should be the case when either $A\ true$ is a hypothesis, or A is deduced from a hypothesis via elimination rules.

Annotation 71 我觉得下述两种理解方式更为明确易懂

- $A\uparrow$ denotes that we are searching for a verfication of A;
- $A\downarrow$ denotes that we are allowed to use A.

Annotation 72 上述两个 definitions 里面隐藏着非常重要但有点不正式的结论:If A has a vertification then A true, 反之依然. 后面我们将形式化地证明它们.

Definition 73 For conjunction.

$$\dfrac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\wedge I \qquad \dfrac{A \wedge B\downarrow}{A\downarrow}\wedge E_L \qquad \dfrac{A \wedge B\downarrow}{B\downarrow}\wedge E_R$$

Definition 74 For implication

$$\dfrac{\begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots\\ B\uparrow\end{array}}{A \supset B\uparrow}\supset^u \qquad \dfrac{A \supset B\downarrow \quad A\uparrow}{B\downarrow}\supset E$$

Annotation 75 (why implication) In order to have a verification of $A \supset B$, we need a proof of B and we are given an assumption A to work with. Therefore, we will need a verification of B and we are allowed to use A.

When using an implication statement in a proof, we need to show that the antecedent holds, so we need a verification of it. Only then we are allowed to use the consequent.

Example 76

$$\dfrac{(A \supset A) \supset B\ ? \quad \dfrac{\dfrac{A\ ?}{A\ ?}}{A \supset A\ ?}}{\dfrac{B\ ?}{((A \supset A) \supset B) \supset B \uparrow}}$$

Example 77

$$\dfrac{\dfrac{\dfrac{\overline{A \wedge B\ true}\ u}{A\ true}\ \wedge E_L}{(A \wedge B) \supset A\ true}\ \supset I^u$$

那么它对应上 verification 和 use

$$\dfrac{\dfrac{A \wedge B \downarrow}{A \downarrow}\ \wedge E_L}{(A \wedge B) \supset A\ ?}\ \supset I^u$$

一切都非常奇怪，这个 verification 和 use 到底是怎样对应 truth? 从前面两个例子都可以清晰地感觉到一个阻力，即

$$\begin{array}{c} A \downarrow \\ ??? \\ A \uparrow \end{array}$$

就是当我们在 use A 的时候，实际上存在一个 A has a verification.

Definition 78 For disjunction

$$\dfrac{A \uparrow}{A \vee B \uparrow}\ \vee I_L \qquad \dfrac{B \uparrow}{A \vee B \uparrow}\ \vee I_R \qquad \dfrac{A \vee B \downarrow \quad \begin{array}{cc} \overline{A \uparrow}\ u & \overline{B \downarrow}\ w \\ \vdots & \vdots \\ C \uparrow & C \uparrow \end{array}}{C \uparrow}\ \vee E^{u,w}$$

Definition 79 For truth and falsehood.

$$\dfrac{}{\top \uparrow}\ \top I \qquad \dfrac{\bot \downarrow}{C \uparrow}\ \bot E$$

Annotation 80 $\bot \downarrow$ signifies a contradiction from our hypotheses.

Annotation 81 the elimination rule of disjunction and falsehood 里面出现 conclusion $C \uparrow$ 也很奇怪，为什么不是 $C \downarrow$?

Definition 82 For atomic propositions.

$$\frac{P\downarrow}{P\uparrow}\downarrow\uparrow.$$

Annotation 83 当引入上述的 arrow switch 之后我们可以回答前面的种种问题了. 首先是 example 76, 假设其中的 A, B 都是 atomic proposition, 则

$$\dfrac{\dfrac{\overline{(A\supset A)\supset B\downarrow}\,w \quad \dfrac{\dfrac{\overline{A\downarrow}\,u}{A\uparrow}\downarrow\uparrow}{A\supset A\uparrow}\supset I^u}{\dfrac{B\downarrow}{B\uparrow}\downarrow\uparrow}}{((A\supset A)\supset B)\supset B\uparrow}\supset I^w$$

同时如果将 implication emilination 的 premise 换成 $A\downarrow$, 在找 $A\supset A\downarrow$ 的 proof 时就被卡住了. example 77类似. 那么有一个很自然的问题这个 arrow switch 能不能推广到任意的 propositions 上呢? 本质上是没有问题的, 例如

$$\dfrac{\dfrac{A\supset A\downarrow \quad \dfrac{\overline{A\downarrow}\,u}{A\uparrow}\downarrow\uparrow}{\dfrac{A\downarrow}{A\uparrow}\downarrow\uparrow}}{A\supset A\uparrow}\supset I^u$$

但是这样的语法又会使得 proof search space 变大, 并不符合我们的初衷, 因此我们只将 arrow switch 放在的了 atomic proposition 上, 这样做的后果你也可以看到, 需要将 connectives 都展开.

再来思考另外一种 arrow switch

$$\frac{F\downarrow}{F\uparrow}\uparrow\downarrow$$

这个人在本质上也是没有问题的, 当我们有一个关于 F 的 verification, 我们当然可以 use it. 但是引入它同样会造成我们的 proof search space, 就像 classical logic 中的 tautologies, 我们可以在任何 proof 中使用它, 但是有时候是没有意义的. 同在 emilinotation rule of disjunction and falsehood 中的 conclusion 中我们都是使用的 verification, 而不是 use, 也是为了防止后续使得我们的 proof 变得复杂.

$$\begin{array}{c}A\downarrow\\ \vdots\\ C\uparrow\end{array}$$

Theorem 84 (Global Soundness) If $A\uparrow$ and $C\uparrow$ then $C\uparrow$

Annotation 85 Global Soundness 意味着如果 if the verification formula of C under the verification formula A, 那么在 C 中使用 A, 并不会得到任何其他 new informations.

Theorem 86 (Global Completeness) If $A\downarrow$, then $A\uparrow$.

Annotation 87 Global completeness 意味着如果我们确定 formula A 在某种情况下可以使用，那么在相同的 assumptions 下我们可以推导出一个关于它的 verification. 这在前面关于 $\downarrow\uparrow$ 转换规则的 annotation 中已经见识过一个特殊例子了.

Notational Definition

Definition 88 A notational definition gives the meaning of the general form of a proposition in terms of another proposition whose meaning has already been defined.

Example 89 We can define logical equivalence, written $A \equiv B$ as

$$(A \supset B) \wedge (B \supset A).$$

Example 90 We can define negation $\neg A$ as

$$\neg A = (A \supset \bot) \Longrightarrow \dfrac{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}{\neg A}\ \neg I$$

We also can give the introduction rule of falsehood.

$$\dfrac{\neg A \quad A}{\bot}\ \bot I$$

so $\bot$ actually means any contradictions. moreover double negation is coming.

Annotation 91 notaional definition 可以看做用已有的东西构造出一些东西. 与之对应的是我们可以直接符号化的给出某个新的定义，称之为 symbolic definition.

Soundness and Completeness of Native Natural Deduction

Definition 92 [5]Soundness of natural deduction means that the conclusion of proof is always a logical consequence of the premises. That is

$$\text{If } \Sigma \vdash \alpha, \text{then } \Sigma \vDash \alpha.$$

Definition 93 Completeness of natural deduction means that all logical consequences in propositional logic are provable in natural deduction. That is,

$$\text{If } \Sigma \vDash \alpha, \text{then } \Sigma \vdash \alpha.$$

Annotation 94 其中 $\Sigma \vdash \alpha$，表示存在一个以 Σ 作为 premise 得到 conclusion 为 α 的 proof. 而 $\Sigma \vDash \alpha$，就考虑两端的 proposition 加上 truth-falsehood 了，即如 $\Sigma^t = True$ 则有 $\alpha^t = True$.

对于 soundness 的证明，我们需要根据 α 的结构来做归纳，而后再考虑赋予其 true/false 来考虑. 这里记录一下对于结构归纳它是怎样对应一般归纳法命题 $P(n)$ 结构上，这里的 n 应该对应 α 的 bottom-up derivation 里面的 maximum depth of line.

而对于 completeness 的证明，相对来说会复杂一点. 我们需要下面 3 个 lemma. 有一个疑问不引入 negation 是不是还说明不了 completeness?

Lemma 95 If $\Sigma = \{\alpha_0, \alpha_1, \cdots, \cdots, \alpha_n\}$ and $\Sigma \vDash \beta$, then

$$\emptyset \vDash (\alpha_0 \to (\alpha_1 \to (\cdots \to (\alpha_n \to \beta) \cdots))).$$

Annotation 96 Deduction theorem 体现的淋漓尽致，将 β 完美转换成了一个 tautology.

Lemma 97 For any well-form formula γ containing atoms $p_1, p_2, \cdots, p_n$ and any valuation t, we have

1. If $\gamma^t = True$ then $\widehat{p}_1, \widehat{p}_2, \cdots, \widehat{p}_n \vdash \gamma$;
2. If $\gamma^t = False$ then $\widehat{p}_1, \widehat{p}_2, \cdots, \widehat{p}_n \vdash \neg\gamma$;

where defines $\widehat{p}_i$ as follow

$$\widehat{p}_i = \begin{cases} p_i & \text{if } p_i^t = True \\ \neg p_i & \text{if } p_i^t = False \end{cases}$$

Example 98 若 $\gamma = p \to q$，我们可以构造一个真值表

p	q	$p \to q$	Claim
T	T	T	$p, q \vdash p \to q$
T	F	F	$p, \neg q \vdash \neg(p \to q)$
F	T	T	$\neg p, q \vdash p \to q$
F	F	T	$\neg p, \neg q \vdash p \to q$

那么上面的 claims 是怎么来的呢？我们可以来分别证明, 对于第一行

$$\dfrac{\dfrac{\overline{p\ true}^{\ u} \quad q\ true}{q\ true}}{p \to q\ true}\ u$$

感觉有点奇怪，这里需要用到 vars inference rule, 这里相对于对 $q \vdash p \to q$ 的 weaken premise. 对于第二行

$$\dfrac{\dfrac{\overline{p \to q\ true}^{\ u} \quad p\ true}{q} \quad \neg q\ true}{\dfrac{\bot}{\neg(p \to q)\ true}\ u}$$

对于第三行

$$\dfrac{\dfrac{\overline{p\ true}^{\ u} \quad \neg p\ true}{\bot}}{p \to q\ true}\ u$$

对于第四行，和第三行类似. 可以看的出来这个 lemma 非常深刻，只要将 atoms 调整为在当前 valuation 下都是 true 的命题，结论再对应调整，就可以构造一个对应的 proof.

Lemma 99 For any well-formed formula γ, if $\emptyset \vDash \gamma$, then $\emptyset \vdash \gamma$.

Annotation 100 Lemma 99一句话概况就是 tautologies are provable. 其证明过程可以用 Lemma 97来说明. 现在 γ 是一个 tautology，那么对于所有的 valuation 都有 $\gamma^t = true$，这有什么用呢？这里还需要引入另外一种 tautology $p \vee \neg p$，配合 emilination rule of vee，即

$$\dfrac{(p_1 \vee \neg p_1) \quad (p_2 \vee \neg p_2) \quad \cdots \quad (p_n \vee \neg p_n) \qquad \begin{matrix}\overline{p_1} \ \cdots \ \overline{p_n} \\ \vdots \\ \gamma\end{matrix} \qquad \cdots \qquad \begin{matrix}\overline{\neg p_1} \ \cdots \ \overline{\neg p_n} \\ \vdots \\ \gamma\end{matrix}}{\gamma}$$

这里需要考虑有 2^n 个 cases，每一个对应一种 valuation，又因为 γ 是 tautology，因此最后的 conclusion 也都是 γ.

Lemma 101 If $\emptyset \vdash (\alpha_0 \to (\alpha_1 \to (\cdots \to (\alpha_n \to \beta)\cdots))$, then $\{\alpha_0, \alpha_1, \cdots, \alpha_n\} \vdash \beta$, that is, $\Sigma \vdash \beta$.

Derived Rules of Inference

Example 102

$$\frac{A \supset B\ true \quad B \supset C\ true}{A \supset C\ true}$$

is a derived rule of inference. Its derivation is the following:

$$\dfrac{\dfrac{B \supset C\ true \qquad \dfrac{A \supset B\ true \quad \overline{A\ true}^{\,u}}{B\ true}\supset E}{C\ true}}{A \supset C\ true}\supset I^{u}$$

Annotation 103 关于 derivation 的推导这里有一些 strategies 在里面

- 使用 introduction rule 从下至上，即我们想要什么;
- 使用 elimination rule 从上至下，即我们知道什么.

Example 104 Modus tollens(这玩意不就是逆否命题)

$$\frac{A \to B \quad \neg B}{\neg A}\ MT.$$

Curry-Howard Conrrespondence

Definition 105 Curry-Howard correspondence is between the natural deduction and simply-typed λ-calculus at three levels

- propositions are types;
- proofs are programs; and
- simplification of proofs is evaluation of programs.

That is

Types	Propositions
Unit types (1)	Truth ($\top$)
Product type ($\times$)	Conjunction ($\wedge$)
Union type ($+$)	Disjunction ($\vee$)
Function type ($\to$)	Implication ($\supset$)
Void types (0)	False ($\bot$)

Every typing rule has a correspondence with a deduction rule.

Example 106 The typing derivation of the term $\lambda a.\ \lambda b.\ \langle a, b\rangle$ can be seen as a deduction tree proving $A \supset B \supset A \wedge B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{a : A \in \Gamma}{\Gamma \vdash a : A}\ var \quad \dfrac{b : B \in \Gamma}{\Gamma \vdash b : B}\ var}{\Gamma \vdash \langle a, b\rangle : A \times B}\ pair}{\Gamma \vdash \lambda y : B.\ \langle a, y\rangle : B \to A \times B}\ abs}{\Gamma \vdash \lambda x : A.\ \lambda y : B.\ \langle x, y\rangle : A \to B \to A \times B}\ abs \iff \dfrac{\dfrac{\dfrac{\overline{A\ true}\ u \quad \overline{B\ true}\ w}{A \wedge B\ true}\ \wedge\wedge I}{B \supset A \wedge B\ true}\ \supset I^w}{A \supset B \supset A \wedge B\ true}\ \supset I^u$$

Annotation 107 从上面例子中看的出来，the inference rule of natural deduction 缺点什么，我也可以给原本每个 inference rule 都加上 the annotation for proof terms. [6] 那么这里 $M : A$ 有两种解释:

1. M is proof term for proposition A;
2. M is a program of type A.

这样解释 Curry-Howard ismorphism 或许方便一点. 让 proof terms make sense: 我们有"if $M : A$ then $A\ true$", 反过来"if $A\ true$ then $M : A$". 例如我们可以将 the proof term of $A \wedge B\ true$ 看做一个 pair 包含两个 subterm，一个关于 $A\ true$ 和另一个关于 $B\ true$.

$$\dfrac{M : A \quad N : B}{\langle M, N\rangle : A \wedge B}\ \wedge I$$

那么 the elimination rule of conjunction 对应一个 natural projection.

$$\frac{M : A \wedge B}{\pi_1 M : A}\ \wedge E_L \quad \frac{M : A \wedge B}{\pi_2 M : B}\ \wedge E_R$$

Example 108 通过 Curry-Howard isomorphism 我们可以将我们想要证明的 judgment 转换到 type system 中，你会看到非常的便利! 例如

$$(A \supset (B \wedge C)) \supset (A \supset B) \wedge (A \supset C)\ true$$

等价于

$$\lambda x.\ \langle \lambda y.\ \pi_1(x\ y), \lambda y.\ \pi_2(x\ y)\rangle : (A \to B \times C) \to (A \to B) \times (A \to C)$$

一个 implication 被转换成了对应的 abstraction，此时我们肯定会想如果给一个 false proposition 是不是就转不了？例如

$$(A \supset B) \supset (B \supset A)$$

显然我们无法在现有 type system 构造出一个合理的 abstraction 使得 $(A \to B) \to (B \to A)$.

迎面走来的问题是: 给定一个 proposition true, 是否有其他的 term with type 和它对应呢？显然是有的，

$$\lambda z.\ \lambda x.\ \langle \lambda y.\ \pi_1(x\ y), \lambda y.\ \pi_2(x\ y)\rangle\ z'$$

那这是不是违反 Curry-Howard isomorphsim 了呢？其实并不是，这里的对应是指 proof terms 和 deduction of proposition true, 显然 deduction 变了，对应的 proof terms 也要变.

More Delicate

Natural Deduction in Sequent Nation

Definition 109 A sequent is a pariticular form of hypothetical judgement

$$A_1, \cdots, A_n \vdash C.$$

where $A_1, \cdots, A_n$ and C are well-defined formulas.

Definition 110 The correspondence between natural deduction and natural deduction in sequent nation.

$\dfrac{A\ true \quad B\ true}{A \wedge B\ true}\ \wedge I$	$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge I$
$\dfrac{A \wedge B\ true}{A\ true}\ \wedge E_1 \quad \dfrac{A \wedge B\ true}{B\ true}\ \wedge E_2$	$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge E_1 \quad \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge E_2$
$\dfrac{A\ true}{A \vee B\ true}\ \vee I_1 \quad \dfrac{B\ true}{A \vee B\ true}\ \vee I_2$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee I_1 \quad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee I_2$
$\dfrac{A \vee B\ true \quad \begin{array}{c}\overline{A\ true}^{\,u} \\ \vdots \\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}^{\,w} \\ \vdots \\ C\ true\end{array}}{C\ true}\ \vee E^{u,w}$	$\dfrac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \vee E$
$\dfrac{\begin{array}{c}\overline{A\ true}^{\,u} \\ \vdots \\ B\ true\end{array}}{A \supset B\ true}\ \supset I^u$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset I$
$\dfrac{A \supset B\ true \quad A\ true}{B\ true}\ \supset E$	$\dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}\ \supset E$
$\dfrac{}{\top\ true}\ \top I$	$\dfrac{}{\Gamma \vdash \top}\ \top I$
$\dfrac{\bot\ true}{C\ true}\ \bot E$	$\dfrac{\Gamma \vdash \bot}{\Gamma \vdash C}\ \bot E$
Hypothesis discharging	$\dfrac{}{\Gamma, A \vdash A}\ hyp$
Substitution	$\dfrac{\Gamma, A \vdash C \quad \Gamma \vdash A}{\Gamma \vdash C}\ subst$

Annotation 111 (detail of correspondence) 其中 Γ 是一个 set of formulas，它可以是 empty set. 思考上述 sequent 形式下的 natural deduction，我们应该用 bottom-up 的视角来观察. 试想我们在没有 additional assumptions 证明一个 formulas，在最开始 Γ 应该是 empty 的，随着我们不断 apply 上述规则过程中将不断的填充 Γ. 那么什么时候证明算接结束了呢? 在 natural deduction 中我们从下往上使用 introduction rules，并添加相应的 assumptions，再从上往下使用 emilination rules，直到它们在中途相遇，这时候我们的证明就结束了，当证明结束的时候，此时所有的 assumptions 都应该被 discharge 了，这个操作对应到 sequent 形式下就是上述 hyp rule，在利用 sequent 构造 proof 的时候，总是以 hyp rule 结束的.

显然 sequent 提供了一种收集 assumptions 的方式，使得 assumptions 和需要证明的 formula 总是在一个 level，一旦某个 assumption 建立之后，以此往后的证明过程中这个 assumption 都是 visible 的.

Example 112 A proof in sequent form.

$$\dfrac{\dfrac{\dfrac{}{A\supset B,(A\wedge C)\vdash A\supset B}\,hyp \quad \dfrac{\dfrac{}{A\supset B,(A\wedge C)\vdash A\wedge C}\,hyp}{A\supset B,(A\wedge C)\vdash A}\,\wedge E_1}{A\supset B,(A\wedge C)\vdash B}\,\supset E \quad \dfrac{\dfrac{}{A\supset B,(A\wedge C)\vdash A\wedge C}\,hyp}{A\supset B,(A\wedge C)\vdash C}\,\wedge E_2}{\dfrac{\dfrac{A\supset B,(A\wedge C)\vdash (B\wedge C)}{A\supset B\vdash (A\wedge C)\supset (B\wedge C)}\,\supset I}{\vdash (A\supset B)\supset ((A\wedge C)\supset (B\wedge C))}\,\supset I}\,\wedge I$$

Definition 113 We say a rule is admissible if all proofs using the rule can be transformed into proofs that do not use the rule.

Lemma 114 The *subst* rule is admissible.

Proof 我们可以将其替换为等价的形式

$$\dfrac{\Gamma,A\vdash C \quad \Gamma\vdash A}{\Gamma\vdash C}\,subst \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma,A\vdash C}{\Gamma\vdash A\supset C}\,\supset I \quad \Gamma\vdash A}{\Gamma\vdash C}\,\supset E$$

Sequent Calculus

Definition 115 A sequent is a pariticular form of hypothetical judgement

$$A_1\ left, \cdots, A_n\ left \vdash C\ right.$$

where $A\ left$ corresponds to a proposition that can be used ($A \downarrow$) and $C\ right$ corresponds to a proposition we have to verify ($C \uparrow$). The right rules decompose C in analogy with introduction rules from the perspective of "bottom-up", while the left rule decompose one of the hypotheses, in analogy with elimination rules, but "upside-down".

Definition 116 引入上述 sequent 及其 inferences 是为了正式地说明 proof search, 即从 conclusion 到 premises 的 derivations.

Definition 117 The initial rule

$$\frac{}{\Gamma, P\ left \vdash P\ right}\ init$$

where P is atomic proposition.

Definition 118 The left rules and right rules

$\dfrac{\Gamma, A \wedge B\ left, A\ left \vdash C\ right}{\Gamma, A \wedge B\ left \vdash C\ right}\ \wedge L_1$ $\dfrac{\Gamma, A \wedge B\ left, B\ left \vdash C\ right}{\Gamma, A \wedge B\ left \vdash C\ right}\ \wedge L_2$	$\dfrac{\Gamma \vdash A\ right \quad \Gamma \vdash B\ right}{\Gamma \vdash A \wedge B\ right}\ \wedge R$
$\dfrac{\Gamma, A \supset B\ left \vdash A\ right \quad \Gamma, A \supset B\ left, B\ left \vdash C\ right}{\Gamma, A \supset B\ left \vdash C\ right}\ \supset L$	$\dfrac{\Gamma, A\ left \vdash B\ right}{\Gamma \vdash A\ left \supset B\ right}\ \supset R$
$\dfrac{\Gamma, A \vee B\ left, A\ left \vdash C\ right \quad \Gamma, A \vee B\ left, B\ left \vdash C\ right}{\Gamma, A \vee B\ left \vdash C\ right}\ \vee L$	$\dfrac{A\ right}{\Gamma \vdash A \vee B\ right}\ \vee R_1$ $\dfrac{B\ right}{\Gamma \vdash A \vee B\ right}\ \vee R_2$
	$\dfrac{}{\Gamma \vdash \top\ right}\ \top R$
$\dfrac{\Gamma, \bot\ left}{C\ right}\ \bot L$	

Annotation 119 The above rules we can use $\Gamma \Rightarrow A$ instead of them.

Annotation 120 这里 frank 给出的 left rules 怪怪的，因为 conclusion 里面的 assumptions 依然出现在了 primises 里面，这让人很奇怪，虽然不影响其正确性. frank 对此的意见是这只是一种 weakening 操作，同时他想表达一个"monotonicity of hypotheses" 的概念: 在 bottom-up 形式下的 proof 中一旦建立某个 assumption，那么它在后续的构造过程中同样 available.

我的感觉是 left rules 应该和 right rules 一样，right rules 在 simplify conclusion，而 left rules 也应该去 simplify hypotheses. 这里 simplify 是指去掉 formula 里面存在的 connectives.

Example 121 The proof in sequent calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \supset B, (A \wedge C), A \Rightarrow A}\, init}{A \supset B, (A \wedge C) \Rightarrow A}\, \wedge L_1 \quad \dfrac{}{A \supset B, (A \wedge C), B \Rightarrow B}\, init}{A \supset B, (A \wedge C) \Rightarrow B}\, \supset L \quad \dfrac{\dfrac{}{A \supset B, (A \wedge C), C \Rightarrow C}\, init}{A \supset B, (A \wedge C) \Rightarrow C}\, \wedge L_2}{A \supset B, (A \wedge C) \Rightarrow (B \wedge C)}\, \wedge R$$
$$\dfrac{A \supset B, (A \wedge C) \Rightarrow (B \wedge C)}{A \supset B \Rightarrow (A \wedge C) \supset (B \wedge C)}\, \supset R$$
$$\dfrac{A \supset B \Rightarrow (A \wedge C) \supset (B \wedge C)}{\Rightarrow (A \supset B) \supset ((A \wedge C) \supset (B \wedge C))}\, \supset R$$

Theorem 122 (from verifications to sequent calculus) Given hypotheses $\Gamma = (A_1 \uparrow, \cdots, A_n \uparrow)$, it correpsonds to $\widehat{\Gamma} = (A_1\ left, \cdots, A_n\ left)$. Then we have

1. If $\Gamma \vdash C \uparrow$ then $\widehat{\Gamma} \vdash C\ right$;
2. If $\Gamma \vdash A \downarrow$ and $\widehat{\Gamma}, A\ left \vdash C\ right$ then $\widehat{\Gamma} \vdash C\ right$.

Proof 这里需要对 $\Gamma \vdash C \uparrow$ 和 $\Gamma \vdash A \downarrow$ 做 mutual induction. 记录几个 representative cases.

Case 1 若

$$\dfrac{\Gamma, C_1 \downarrow \vdash C_2 \uparrow}{\Gamma \vdash C_1 \supset C_2 \uparrow}\, \supset I$$

则

(1) $\widehat{\Gamma}, C_1\ left \vdash C_2\ right$ hyp.1 from premise1

(2) $\widehat{\Gamma} \vdash C_1 \supset C_2\ right$ $\supset R$. (1)

Case 2 若

$$\dfrac{\Gamma \vdash P \downarrow}{\Gamma \vdash P \uparrow}\, \downarrow\uparrow$$

则

(1) $\widehat{\Gamma}, P\ left \vdash P\ right$ *init*

(2) $\widehat{\Gamma} \vdash P\ right$ hyp.2 from premise1

Case 3 若

$$\frac{\Gamma \vdash A_1 \supset A_2 \downarrow \quad \Gamma \vdash A_1 \uparrow}{\Gamma \vdash A_2 \downarrow} \supset E$$

则

(1)	$\widehat{\Gamma}, A_2\ left \vdash C\ right$	assumption
(2)	$\widehat{\Gamma}, A_1 \supset A_2\ left, A_2\ left \vdash C\ right$	weakening(1)
(3)	$\widehat{\Gamma} \vdash A_2\ right$	hyp.1 from premise1
(4)	$\widehat{\Gamma}, A_1 \supset A_2\ left \vdash A_2\ right$	weakening(3)
(5)	$\widehat{\Gamma}, A_1 \supset A_2\ left \vdash C\ right$	$\supset L$(2)(4)
(6)	$\widehat{\Gamma} \vdash C\ right$	hyp.1 from premise2

Case 4 若

$$\frac{}{\Gamma', A \downarrow \vdash A \downarrow}\ hyp$$

则

(1)	$\widehat{\Gamma}, A\ left, A\ left \vdash C\ right$	assumption
(2)	$\widehat{\Gamma}, A\ left \vdash C\ right$	contraction(1)

Annotation 123 注意这里的 hypotheses 是两个部分，因为 verification calculus 的 elimintation rule 存在，会导致 use 同样出现在左端. 实际这里缺一个过程，应该像 ND in Seq 那样，我们也应该对 verificatio calculus 也做一个 sequent 形式的变换.

Theorem 124 (substitution of uses) If $\Gamma \vdash A \downarrow$ then

1. if $\Gamma, A \downarrow \vdash B \downarrow$ then $\Gamma \vdash B \downarrow$, and
2. if $\Gamma, A \downarrow \vdash C \uparrow$ then $\Gamma \vdash C \uparrow$,

Proof 这里需要对 $\Gamma, A \downarrow \vdash B \downarrow$ 和 $\Gamma, A \downarrow \vdash C \uparrow$ 做 mutual induction. 还是列举几个代表性的 cases.

Case 1 Base case

$$\frac{}{\Gamma \vdash \top \uparrow}\ \top I$$

根据假设 $\Gamma, A \downarrow \vdash \top \uparrow$，这里显然有 $\Gamma, \vdash \top \uparrow$.

Case 2 若

$$\frac{\Gamma' \vdash C \supset B \downarrow \quad \Gamma' \vdash C \uparrow}{\Gamma' \vdash B \downarrow} \supset E$$

其中 $\Gamma' = (\Gamma, A\downarrow)$. 那么

$$\begin{array}{lll} (1) & \Gamma \vdash A\downarrow & \text{assumption} \\ (2) & \Gamma \vdash C \supset B\downarrow & \text{hyp.1} \\ (3) & \Gamma \vdash C\uparrow & \text{hyp.2} \\ (4) & \Gamma \vdash B\downarrow & \supset E(3)(4) \end{array}$$

Theorem 125 (from sequent calculus to verifications) If $\widehat{\Gamma} \vdash C\ right$ then $\Gamma \vdash C\uparrow$.

Proof 注意这里有个 abuse symbol 了，结合前面的 theorem，可能会想成我们构造了一个 isomorphism，其实不是这样的，我仅仅讨论从一边到另一边，并不是 composition! 这里依然对 $\widehat{\Gamma} \vdash C\ right$ 做 structure induction. 列举几个代表性 cases.

Case 1 Base case

$$\frac{}{\widehat{\Gamma} \vdash \top\ right}\ \top R$$

显然有 $\Gamma \vdash \top\uparrow$.

Case 2 若

$$\frac{\widehat{\Gamma}, A\ left \vdash B\ right}{\widehat{\Gamma} \vdash A \supset B\ right}\ \supset R$$

则

$$\begin{array}{lll} (1) & \Gamma, A\uparrow \vdash B\uparrow & \text{hyp} \\ (2) & \Gamma \vdash A \supset B\uparrow & \supset I(1) \end{array}$$

Case 3 若

$$\frac{\widehat{\Gamma}, A \supset B\ left \vdash A\ right \quad \widehat{\Gamma}, A \supset B\ left, B\ left \vdash C\ right}{\widehat{\Gamma}, A \supset B\ left \vdash C\ right}\ \supset L$$

则

$$\begin{array}{lll} (1) & \Gamma, A \supset B\downarrow \vdash A\uparrow & \text{hyp} \\ (2) & \Gamma, A \supset B\downarrow \vdash A \supset B\downarrow & \textit{hyp}\ \text{rule} \\ (3) & \Gamma, A \supset B\downarrow \vdash B\downarrow & \supset E(1)(2) \\ (4) & \Gamma, A \supset B\downarrow, B\downarrow \vdash C\uparrow & \text{hyp} \\ (n) & \Gamma, A \supset B \vdash C\uparrow & \textit{subst}(124) \end{array}$$

Definition 126 (another of substitution) The rule of cut

$$\frac{\Gamma \vdash A\ right \quad \Gamma, A\ left \vdash C\ right}{\Gamma \vdash C\ right}\ cut$$

Annotation 127 注意 *cut* rule 是在用 the vertification of A 去替换 the use of A，这和前面 substitution of uses 是不太一样的.

Theorem 128 (admissibility of cut) If $\Gamma \vdash A\ right$ and $\Gamma, A\ left \vdash C\ right$ then $\Gamma \vdash C\ right$.

Proof 证明 *cut* rule 是个技术活. 我们要做一个 nested structure induction. 首先给定 *cut* rule 的 shape

$$\frac{\overset{\mathcal{D}}{\Gamma \Rightarrow A} \quad \overset{\mathcal{E}}{\Gamma, A \Rightarrow C}}{\Gamma \Rightarrow C}\ cut$$

这里我们要对 triple $(C, \mathcal{D}, \mathcal{E})$ 做归纳. 有三种 base cases:

1. 若 $\mathcal{E}$ 是 *init* rule. 这里两个地方可以 apply *init* rule. 此时 A 为 atomic.

   (a) 若 A 不同于 C. 那么这里可以马上知道 $A \in \Gamma$，因此可以 emilinate 掉 cut.

   $$\frac{\overset{\mathcal{D}}{\Gamma \Rightarrow C} \quad \dfrac{}{\Gamma, C \Rightarrow A}\ init}{\Gamma \Rightarrow A}\ cut \rightsquigarrow \frac{}{\Gamma \Rightarrow A}\ init$$

   (b) 若 $C = A$. 那么这里显然可以直接用 $\mathcal{D}$ 得到 $\Gamma \Rightarrow A$.

   $$\frac{\overset{\mathcal{D}}{\Gamma \Rightarrow A} \quad \dfrac{}{\Gamma, A \Rightarrow A}\ init}{\Gamma \Rightarrow A}\ cut \rightsquigarrow \overset{\mathcal{D}}{\Gamma \Rightarrow A}$$

2. 若 $\mathcal{D}$ 是 *init* rule. 此时 C 是 atomic，那么可以知道 $C \in \Gamma$，因此 $\Gamma', C, C \Rightarrow A$，再用一下 contraction 就有 $\Gamma \Rightarrow A$，因此这里可以用 $\mathcal{E}$ 作为 proof derivation.

   $$\frac{\dfrac{}{\Gamma \Rightarrow C}\ init \quad \overset{\mathcal{E}}{\Gamma, C \Rightarrow A}}{\Gamma \Rightarrow A}\ cut \rightsquigarrow \overset{\mathcal{E}}{\Gamma \Rightarrow A}.$$

3. 若 C 是 atomic. 此时有可能 $\mathcal{D}$ 和 $\mathcal{E}$ 都不是 *init* rule, 此时我们需要一个 lemma 131 来使得它们变成前面两种情况.

那么我们怎么来用这三个 base case，这里就展示一下 nested induction 是咋 worked. 首先我们给定一个命题 $cut(F, l, r)$: 如果 l 是一个关于 $\Gamma \Rightarrow F$ 的 cut-free proof，r 是一个关于 $\Gamma, F \Rightarrow C$ 的 cut-free proof，那么我们可以构造一个关于 $\Gamma \Rightarrow C$ 的 cut-free proof. 来正式开始我的 induction.

- [11]Base case(1): $\forall l.\forall r.cut(atom, l, r)$

  对应前面的 base case(3).

- Induction step(1): to show $\forall l.\forall r.cut(F+1, l, r)$

  IH(1): $\forall l.\forall r.cut(F, l, r)$

  The proof procceeds by induction on l:

  - Base case(2): $\forall l.\forall r.cut(F+1, init, r)$ 对应前面的 base case(2).
  - Induction step(2): to show $\forall l.\forall r.cut(F+1, l+1, r)$

    IH(2): $\forall l.\forall r.cut(F+1, l, r)$

    The proof procceeds by induction on r:

    1. Base case(3): $\forall l.\forall r.cut(F+1, l+1, init)$
    2. Induction step(3): to show $\forall l.\forall r.cut(F+1, l+1, r+1)$

       IH(3): $\forall l.\forall r.cut(F+1, l+1, r)$

       Now we show this by cases. As a bleow example:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow A} \quad \overset{\mathcal{E}}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \wedge B} \wedge L_1 \quad \dfrac{\overset{\mathcal{F}}{\Gamma, A \wedge B, A \Rightarrow C}}{\Gamma, A \wedge B \Rightarrow C}}{\Gamma \Rightarrow C}\, cut$$

$$\Downarrow$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow A} \quad \dfrac{\dfrac{\mathcal{D} + \text{ weakening}}{\Gamma, A \Rightarrow A} \quad \dfrac{\mathcal{E} + \text{ weakening}}{\Gamma, A \Rightarrow B}}{\Gamma, A \Rightarrow A \wedge B} \wedge R \quad \dfrac{\overset{\mathcal{F}}{\Gamma, A \wedge B, A \Rightarrow C}}{\Gamma, A \Rightarrow C}\, cut}{\Gamma \Rightarrow C}\, cut$$

       on the upper-most cut, we apply IH(3), since $F+1$ and $l+1$ are unchanged, but right branch is now r, which is smaller. and on the lower cut, we apply IH(1), since we have F as the cut-formula. Q. E. D.

Definition 129 We call rank deduction the rewriting operation that premutes the cut rule over other rules in a proof.

Annotation 130 "premutes it up" 是啥意思呢? 就是指某个 cut rule" 往上移", 例如

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A \wedge B, A \Rightarrow A}\ init}{\Gamma, A \wedge B \Rightarrow A}\ \wedge L_1 \qquad \overset{\mathcal{E}}{\Gamma, A \wedge B, A \Rightarrow C}}{\Gamma, A \wedge B \Rightarrow C}\ cut$$

$$\Downarrow$$

$$\dfrac{\dfrac{}{\Gamma, A \wedge B, A \Rightarrow A}\ init \qquad \begin{array}{c}\mathcal{E} + \text{ weakening} \\ \Gamma, A \wedge B, A, A \Rightarrow C\end{array}}{\dfrac{F, A \wedge B, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1}\ cut$$

此时 cup 拥有了 smaller left branch, 这其中为我们后续的 nested induction hypothesis 提供了途径. 再来看一个例子

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow A} \qquad \overset{\mathcal{E}}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \wedge B} \qquad \dfrac{\overset{\mathcal{F}}{\Gamma, A \wedge B, A \Rightarrow C}}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1}{\Gamma \Rightarrow C}\ cut$$

$$\Downarrow$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow A} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D} + \text{ weakening} \\ \Gamma, A \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{E} + \text{ weakening} \\ \Gamma, A \Rightarrow B\end{array}}{\Gamma, A \Rightarrow A \wedge B}\ \wedge R \qquad \overset{\mathcal{F}}{\Gamma, A \wedge B, A \Rightarrow C}}{\Gamma, A \Rightarrow C}\ cut}{\Gamma \Rightarrow C}\ cut$$

此时 *cut rule* 中的两个 premises 中的 cut formula 都是被 apply 了相应的 rules, 这种 shape 的 cut 我称为principal cases, 它和第一个例子不太相同. 此时下面这个 cup 拥有了 smaller left branch, 而上面这个 cup 拥有了 smaller right branch.

Lemma 131 The *cut* rule permutes up all other rules that do not operate on the cut formula.

Proof 换句话说就是对于任意的 *cut* rule, 我们都可以重排它的 primises, 把 cut 放到 right place. 那么这里分别要对它的 primises 应用 sequent calculus 的规则, 两个 primises 就是 20cases. td Q. E. D.

Annotation 132 引入 *cut* rule 不利于做 proof search.

Definition 133 (generalization of *init* rule) The rule of identity

$$\dfrac{}{\Gamma, A\ left \vdash A\ right}\ id$$

Theorem 134 (admissibility of identity) $\Gamma, A\ left \vdash A\ right$ fir arbitrary propositions A and contexts Γ.

PROOF 对 A 做 structure induction. 列举几个代表性的 cases.

Case 1 若 $A = P$, 根据 *init* rule 显然有 $\Gamma, A\ left \vdash A\ right$.

Case 2 若 $A = B \supset C$. 则

(1)	$\Gamma, B\ left \vdash B\ right$	hyp
(2)	$\Gamma, B \supset C\ left, B\ left \vdash B\ right$	weakening
(3)	$\Gamma, C\ left \vdash C\ right$	hyp
(4)	$\Gamma, B \supset C\ left, B\ left, C\ left \vdash C\ right$	weakning
(5)	$\Gamma, B \supset C\ left, B\ left \vdash C\ right$	$\supset L(2)(4)$
(6)	$\Gamma, B \supset C\ left \vdash B \supset C\ right$	$\supset R(5)$

Case 3 若 $A = B \wedge C$. 则

(1)	$\Gamma, B \wedge B, B \Rightarrow B$	hyp + weakening
(2)	$\Gamma, B \wedge C \Rightarrow C$	$\wedge L_1(1)$
(3)	$\Gamma, B \wedge C, C \Rightarrow C$	hyp + weakening
(4)	$\Gamma, B \wedge C \Rightarrow B$	$\wedge L_2(3)$
(5)	$\Gamma, B \wedge C \Rightarrow B \wedge C$	$\wedge R(2)(4)$

Theorem 135 (from natural deduction to sequent calculus) If $\Gamma \vdash A\ true$ then $\widehat{\Gamma} \vdash A\ right$.

PROOF 依然对 $\Gamma \vdash A\ true$ 做 structure induction. 列举几种代表性 cases.

Case 1 若

$$\frac{A\ true}{\Gamma', A\ true}\ hyp$$

此时 $\Gamma = (\Gamma', A\ true)$, 根据 *identity* rule(133) 有 $\widehat{\Gamma}, A\ left \vdash A\ right$.

Case 2 若

$$\frac{\Gamma, B\ true \vdash C\ true}{\Gamma \vdash B \supset C\ true} \supset I$$

则

(1)	$\widehat{\Gamma}, A\ left \vdash B\ right$	hyp
(2)	$\widehat{\Gamma} \vdash A \supset B\ right$	$\supset R(1)$

Case 3 若

$$\frac{\Gamma \vdash C \supset B\ true \quad \Gamma \vdash C\ true}{\Gamma \vdash B\ true} \supset E$$

则

(1)	$\widehat{\Gamma} \vdash C \supset B\ right$	hyp
(2)	$\widehat{\Gamma} \vdash C\ right$	hyp
(3)	$\widehat{\Gamma}, C \supset B\ left, C\ left \vdash C\ right$	*identity*
(4)	$\widehat{\Gamma}, C \supset B\ left, C\ left, B\ left \vdash B\ right$	*identity*
(5)	$\widehat{\Gamma}, C \supset B\ left, C\ left \vdash B\ right$	$\supset L(3)(4)$
(6)	$\widehat{\Gamma}, C\ left \vdash B\ right$	$cut(1)(5)$
(7)	$\widehat{\Gamma} \vdash B\ right$	$cut(2)(6)$

Theorem 136 (truth and verification) $A\ true$ iff $A\uparrow$.

Proof (⇒) 从 $A\updownarrow$ 到 $A\ true$ 是比较显然的，直接将所有的 arrow 都换成 true 即可，对于 arrow switch $\downarrow\uparrow$，此时 premise 和 conclusion 都是相同，因此这里可以在转换中去掉.

(⇐) 这里就需要多出中间一步. 利用 Theorem 135 将 $\cdot \vdash A\ true$ 转换到 $\cdot \vdash A\ right$; 再利用 Theorem 125 将 $\cdot \vdash A\ right$ 转换到 $\cdot \vdash A\ \uparrow$. Q. E. D.

Annotation 137 How everything is related

Natural Deduction

$\equiv$

ND in sequent nation

$\Updownarrow$

Sequent Calculus(with cut, identity and weakening)

$\updownarrow$

Sequent Calculus

$\equiv$

Verification Calculus

where $\equiv$ means the systems are equivalent, $\Updownarrow$ means the systems are sound and complete each other, and $\updownarrow$ means the system are shown to be equivalent by showing the rules cut, identity and weakening are admissible.

Validity

Definition 138 *A vaild* if $\bullet \vdash A\ true$ where $\bullet$ is emphasizing that there are no truth hypotheses(different from $\cdot$ that represents empty collection of hypotheses), and we call $\bullet \vdash A\ true$ is categorical judgement. Written ΔA for reflecting the notion of validity as a proposition.

Annotation 139 其中 $\Box A$ 表示一个 proposition claimed A is vaild, 因此 $\Box A\ true$ 表示这个 proposition 成立. 那么关于它的 introduction rule 是什么? 很自然地由 $A\ valid$ 的 definition 有

$$\frac{\bullet \vdash A\ true}{\Gamma \vdash \Box A\ true}\ \Box I$$

那么它的 elimination rule 又是什么呢? 第一次尝试

$$\frac{\Gamma \vdash \Box A\ true}{\bullet \vdash A\ true}\ \Box E$$

看起来是 local soundness，通过它得到的 infos 还行. 但是实际上有问题

$$\frac{\overline{\Box A\ true \vdash \Box A\ true}}{\bullet \vdash A\ true}\ \Box E$$

这等于我们可以 no assumption 推出所有 proposition 都是 valid，因此这个 elimination rule 有点太强了. 那么我们考虑让它弱一点，第二次尝试

$$\frac{\Gamma \vdash \Box A\ true}{\Gamma \vdash A\ true}\ \Box E$$

这里确实是 local soundness，但却不是 local completeness

$$\frac{\dfrac{\Gamma \vdash \Box A\ true}{\Gamma \vdash A\ true}\ \Box E}{\Gamma \vdash \Box A\ true}\ ?$$

我们得改变一下思路，如果 $A\ vaild$，那么其他 premise 包含 $A\ vaild$ 的 judgement 那么实际上都是可以去掉 $A\ vaild$，但也仅仅局限以此，这才是 emilination 故事的主线.

Definition 140 Then general judgement form

$$\underbrace{u_1 :: B_1\ vaild, \cdots, u_k :: B_k\ vaild}_{\Delta}; \underbrace{x_1 : A_1\ true, \cdots, x_n : A_n\ true}_{\Gamma} \vdash C\ true$$

Definition 141 The introduction rule and elimination rule of $A\ vaild$ as follow

$$\frac{\Delta; \bullet \vdash A\ true}{\Delta; \Gamma \vdash \Box A\ true}\ \Box I \qquad \frac{\Delta; \Gamma \vdash \Box A\ true \quad \Delta, u :: A\ vaild; \Gamma \vdash C\ true}{\Delta; \Gamma \vdash C\ true}\ \Box E$$

Theorem 142 Local soundness and local completeness of above introduction and elimination rule are held

Annotation 143 可以看到 emilination rule 变成了 substitution，而不是从单纯从本身要得到什么，后面会看见更多这样的东西.

Example 144 Proof of $\cdot;\cdot \vdash \Box A \supset A$.

$$\dfrac{\dfrac{\dfrac{}{\cdot; x : \Box A\ true \vdash \Box A\ true}\ x \quad \dfrac{}{u :: A\ vaild; x : \Box A\ true \vdash A\ true}\ u}{\cdot; x : \Box A\ true}\ \Box E^u}{\cdot;\cdot \vdash (\Box A \supset A)\ true}\ \supset I^x$$

Box is Powerful

Definition 145 $\Box$ is $\Box$.

Definition 146 A term box$M$ means M is a quated source expression such that there are not any free variables x.

Definition 147 And $\Box A$ is necessity modality.

Possibility

Definition 148 We use $\Diamond A$ for possibility modality.

Annotation 149 $\Diamond A$ 就是一个 claim A is possible 的命题. 通常在 classic modal logic 里面我们定义 A is possible if its negation is not necessary, that is $\Diamond A = \neg\Box\neg A$. 但是这种手法在现在我们讨论的 intuitionistic logic 无法奏效，我们希望的是有一个直观的 introduction rule 来得到它，也就是我们需要一些 explicit evidences，一开始就它的 negation 那显然是做不到的.

Definition 150 [7]The definition of possiblity.

$$\frac{\Delta, \Gamma \vdash A\ true}{\Delta, \Gamma \vdash A\ poss}\ poss$$

Definition 151 The introduction and emilination rule of possiblity.

$$\frac{\Delta; \Gamma \vdash A\ poss}{\Delta; \Gamma \vdash \Diamond A\ true}\ \Diamond I \qquad \frac{\Delta; \Gamma \vdash \Diamond A\ true \quad \Delta; x : A\ true \vdash C\ poss}{\Delta; \Gamma \vdash C\ poss}\ \Diamond E$$

Annotation 152 注意这里的 emilination rule 里面的第二个 premise 中的 hypothesis 只有 $A\ true$，即我们 under assumption $A\ true$, we conclude $C\ poss$.

Theorem 153 Local soundness and completeness are held.

Annotation 154 td; 对上述 inference rule 的理解.

Example 155 Proof of $\Box(A \supset B) \supset \Diamond A \supset \Diamond B$.

Proof Searching

Simplification

Annotation 156 目的是去掉 sequent calculus 里面 the duplication of main formula. 为了区分 simplified sequent，我们用 $\rightarrow$ 代替 $\Rightarrow$

Definition 157 [12]Simplified sequent calculus defined as follow

$\dfrac{\Gamma, A, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge L$	$\dfrac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B} \wedge R$
$\dfrac{\Gamma, A \rightarrow C \quad \Gamma, B \rightarrow C}{\Gamma, A \vee B \rightarrow C} \vee L$	$\dfrac{A}{\Gamma \rightarrow A \vee B} \vee R_1$ $\dfrac{B}{\Gamma \rightarrow A \vee B} \vee R_2$
$\dfrac{\Gamma, A \supset B \rightarrow A \quad \Gamma, B \rightarrow C}{\Gamma, A \supset B \rightarrow C} \supset L$	$\dfrac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B} \supset R$
	$\dfrac{}{\Gamma \rightarrow \top} \top R$
$\dfrac{}{\Gamma, \bot \rightarrow C} \bot L$	
$\dfrac{}{\Gamma, P \rightarrow P}$	

Annotation 158 为什么 implication emilination 的 left premise 中的 main formual 没有去掉? 这是值得探讨的问题，right premise 能去掉因为

$$\dfrac{\dfrac{}{\Gamma, A \supset B \Rightarrow A \supset B}\, id \quad \Gamma, A \supset B, B \vdash C}{\Gamma, B \Rightarrow C}\, cut$$

同时更加底层的原因是 $\vdash B \supset (A \supset B)$. 而你此时思考 $\Gamma, A \supset B \supset A$ 时，应当思考是否存在一个关于 A 的 proof 里面需要用到 $A \supset B$? 注意此时的 *cut* rule 在这里是无法奏效的，我好想想不到这样 proof td.

Theorem 159 Simplified sequent calculus is soundness and completeness, that is

- If $\Gamma \rightarrow C$ then $\Gamma \Rightarrow C$;
- If $\Gamma \Rightarrow C$ then $\Gamma \rightarrow C$.

Proof 依旧是 straightforward structure induction. Q. E. D.

Invertibility

Annotation 160 Why Invertibility? [13] 我们想构造一个 automation for proof searching，可能会面临一些问题. 假设我们想要找 $\Gamma \vdash C$ 的 verification，其中 Γ 包含 $n-1$ 个 formuals，这意味我们要从 n 个 formuals(加上 C)，挑一个出来作为 main formual，对其 apply 对应的 rules. 如果我们挑到正确的那个 rule，将会导致都后续一定会出错，因此我们要考虑 backtrack. 例如证明 $B \to A \vee B$ 时，我们对 right side 挑 $\vee R_1$ rule

$$\frac{\Gamma \to A}{\Gamma \to A \vee B} \vee R_1,$$

这显然就出问题了. 这样看起来我们得找到一个可以接受的 search algorithm，这里有策略是我们可以尽可能选 right choice，直到选不出时候，我们再 make a choice. 这样做的好处是我们可以尽可能减少 backtrack，同时一旦选择了 bad choice，那么无论你后续怎么选都还是错的，因此我还可以直接 remove 掉 bad choice. 那么如何选 right choice，就引出了 invertible rule 的概念.

Definition 161 A rule p is called invertible in a sequent calculus system if a proof of its conclusion implies the existence of proofs of each of its premises.

Annotation 162 简而言之就是可以从 conclusion is vaild 推出 premise is vaild.

Lemma 163 The left rule for disjunction is invertible.

$$\frac{\Gamma, A \to C \quad \Gamma, B \to C}{\Gamma, A \vee B \to C} \vee L$$

Proof 很直接.

$$\frac{\dfrac{\dfrac{}{\Gamma, A \to A}\, id}{\Gamma, A \to A \vee B} \vee R_1 \quad \Gamma, A, A \vee B \to C}{\Gamma, A \to C}\, cut \qquad \frac{\dfrac{\dfrac{}{\Gamma, B \to B}\, id}{\Gamma, B \to A \vee B} \vee R_2 \quad \Gamma, B, A \vee B \to C}{\Gamma, B \to C}\, cut$$

Lemma 164 Both rules for conjunction are invertible.

Lemma 165 The right rule for implication is invertible.

Proof that is

$$\frac{\Gamma, A \to A \supset B \qquad \dfrac{\dfrac{}{\Gamma, A, A \supset B \to A}\, init \quad \dfrac{}{\Gamma, A, B \to B}\, init}{\Gamma, A, A \supset B \to B}}{\Gamma, A \to B}$$

Annotation 166 为什么 the left rule of implication is not invertible? 因为从 conclusion 推不出 left premise.

Definition 167 Proof searching algorithm. Assuming we want prove $\Gamma \to C$

1. Starting with formula on right C, apply invertible rules as long as we can
2. When the only rule left to be applied on the right is non-invertible, we say C^+, stop working there and move to left.
3. Process the formulas in the left context Ω in order:
   (a) If the rule to be applied is invertible, go ahead and apply it, keeping the possibly generated subformuals in the front of the list.
   (b) If the rule to be applied is non-invertible or the formula is atomic, move it to a side context Γ^- to delay working with it.
4. When we have absolutely no other choice, we can either apply a noninvertible rule on the right or on the left. Then we move the focus to the newly generated formulas in the hope they require inveritible.

that Ω is context of left side, which is ordered and holds any formuals; and Γ^- which holds only atoms or formuals that require non-invertible left rules.

Annotation 168 接下来就是详细解释 Ω 和 Γ^- 是如何构造的.

Definition 169 We will label the sequent arrows with R or L, depending on whether we are on a right inversion phase or left inversion phase.

$$\Gamma^-; \Omega \xrightarrow{R} C$$
$$\Gamma^-; \Omega \xrightarrow{L} C$$

Definition 170 Formalization of proof searching algorithm for $\Gamma \to C$.

1. Start: $\cdot; \Omega \xrightarrow{R} C$, where Γ^- is empty and $\Omega = \Gamma$.
2. Process right side:
   (a) The right inversion phase consists of applying right invertibale rule:

$$\frac{\Gamma^-; \Omega \xrightarrow{R} A \quad \Gamma^-; \Omega \xrightarrow{R} B}{\Gamma^-; \Omega \xrightarrow{R} A \wedge B} \wedge R$$
$$\frac{\Gamma^-; \Omega, A \xrightarrow{R} B}{\Gamma^-; \Omega \xrightarrow{R} A \supset B} \supset R$$
$$\frac{}{\Gamma^-; \Omega \xrightarrow{R} \top} \top R$$

(b) If we reach an atom on the right side, we either check if it is in Γ^- and close the branch with *init*, or we move to apply left inversion rules.

$$\frac{P \in \Gamma^-}{\Gamma^-; \Omega \xrightarrow{R} P}\ \mathit{init}$$

$$\frac{P \notin \Gamma^- \quad \Gamma^-; \Omega \xrightarrow{L} P}{\Gamma^-; \Omega \xrightarrow{R} P}\ \mathrm{LR}_P$$

(c) The only other cases left are when the right formula is disjunction or $\bot$. At this pointwe stop working on the right and move to the left.

$$\frac{\Gamma^-; \Omega \xrightarrow{L} A \vee B}{\Gamma^-; \Omega \xrightarrow{R} A \vee B}\ \mathrm{LR}_\vee$$

$$\frac{\Gamma^-; \Omega \xrightarrow{L} \bot}{\Gamma^-; \Omega \xrightarrow{R} \bot}\ \mathrm{LR}_\bot$$

3. Process left side:

(a) The left inversion phase processes the formulas in Ω in order, always taking the rightmost one.

$$\frac{\Gamma^-; \Omega, A, B \xrightarrow{L} C^+}{\Gamma^-; \Omega, A \wedge B \xrightarrow{L} C^+}\ \wedge L$$

$$\frac{\Gamma^-; \Omega, A \xrightarrow{L} C^+ \quad \Gamma^-; \Omega, B \xrightarrow{L} C^+}{\Gamma^-; \Omega, A \vee B \xrightarrow{L} C^+}\ \vee L$$

(b) If the first formual in Ω is $\bot$, we can close the branch; If it is $\top$ we can remove it form our list.

$$\frac{}{\Gamma^-; \Omega, \bot \xrightarrow{L} C^+}\ \bot\, L$$

$$\frac{\Gamma^-; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega, \top \xrightarrow{L} C^+}\ \top L$$

(c) If we encounter an atom, we can close the branch if it is equal to the right side or we can move it. and only case left is an implication, we also move it to Γ^-.

$$\frac{}{\Gamma^-; \Omega, P \xrightarrow{L} P}\ \mathit{init}$$

$$\frac{\Gamma^-, P; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega, P \xrightarrow{L} C^+}\ \mathrm{shift}_P$$

$$\frac{\Gamma^-, A \supset B; \Omega \xrightarrow{L} C^+}{\Gamma^-; \Omega, A \supset B \xrightarrow{L} C^+}\ \mathrm{shift}_\supset$$

4. End:

   (a) We will end-up with a sequent where Ω is empty. Its time to make a choice by applying non-invertible rules.

$$\frac{\Gamma^-;\cdot \xrightarrow{R} A}{\Gamma^-;\cdot \xrightarrow{R} A \vee B}\vee R_1 \qquad \frac{\Gamma^-;\cdot \xrightarrow{R} B}{\Gamma^-;\cdot \xrightarrow{R} A \vee B}\vee R_2$$

$$\frac{\Gamma^-, A \supset B;\cdot \xrightarrow{L} A \quad \Gamma^-; B \xrightarrow{L} C^+}{\Gamma^-, A \supset B;\cdot \xrightarrow{L} C^+}\supset L$$

   (b) After a choice is made, we try to go back to an inversion phase by using the sequent arrow corresponding to where auxiliary formuals went.

   (c) If it is failed at one choice, then we can backtrack to choice points.

Annotation 171 上述算法的核心我们称其为focuing，它是 proof search 中非常重要的一个环节.

Contraction-free

Annotation 172 在 simplified sequent calculus 中 left implication rule 里面的 left premise 还是留着 main formula, 即

$$\frac{\Gamma, A \supset B \to A \quad \Gamma, B \to C}{\Gamma, A \supset B \to C} \supset L$$

这时你如果考虑对上面的 left primise 再使用一下 $\supset L$，你会发现又得到了 $\Gamma, A \supset B \to A$，这意味你可能陷入在 proof searching 中陷入 loop. 为了尝试解决这个问题，同时让我们的 proof searching 变成更加的 decidable，我需要分解 $\supset L$，以归纳 A 的结构为切入点.

Example 173 如果 A 是 atom，那么此时的 $\subset L$ 对应的 instance 为

$$\frac{\Gamma, a \supset B \to a \quad \Gamma, B \to C}{\Gamma, a \supset B \to C} \supset L$$

这里分三种情况讨论 left premise 的 subproof:

1. 如果 $a \in \Gamma$, 那么显然我们是可以 close 掉 left subproof，直接以 right premise 作为前提.
2. 如果继续对 left premise 继续使用 $\subset L$，前面说了没有意义，更何况这里 A 还只是一个 atom. 那么什么时候重复 apply $\subset L$ 有意义呢? 除非我们还想从 A 里面提取点其他的信息，例如 $A = A_1 \vee A_2$，可以两次 apply $\subset L$，再利用一下 $\vee R_1$ 和 $\vee R_2$ 就能得到不一样的信息.
3. 如果从 Γ 里面挑一个 formula 出来 apply rules，那么这里可能有两种选择，要么 apply invertible rule 或者 non-invertible rule. 对于 non-invertible rule 而言，这里只能选择 $\supset L$，因为此时 right side 是一个 atom. 当我们考虑 delay applying $\supset L$ for $a \supset B$，apply invertible rule and other left implicaion as possible we can, 我们最终会回到第一种情况.

因此我们分析告诉我们在这种情况下，我可以将 $\supset L$ 化简为

$$\frac{\Gamma, a, B \to C}{\Gamma, a, a \supset B \to C} \, a \supset L$$

注意我们在 conclusion 中显式的标注 a 的存在，才能 closed 掉 left branch.

Example 174 如果 A 是 $A_1 \wedge A_2$，考虑以下 logical equivalent:

$$(A_1 \wedge A_2) \supset B \equiv A_1 \supset (A_2 \supset B)$$

为什么要这样换呢？可以对比一下下面的两个 derivations:

$$\dfrac{\dfrac{\Gamma, (A_1 \wedge A_2) \supset B \to A_1 \quad \Gamma, (A_1 \wedge A_2) \supset B \to A_2}{\Gamma, (A_1 \wedge A_2) \supset B \to A_1 \wedge A_2} \wedge R \quad \Gamma, B \to C}{\Gamma, (A_1 \wedge A_2) \supset B \to C} \supset L$$

$$\dfrac{\Gamma, A_1 \supset (A_2 \supset B) \to A_1 \quad \dfrac{\Gamma, A_2 \supset B \to A_2 \quad \Gamma, B \to C}{\Gamma, A_2 \supset B \to C} \supset L}{\Gamma, A_1 \supset (A_2 \supset B) \to C} \supset L$$

很显然后者我们将所有 antecedent 里面 implication 都简化了，每个 implication 都只依赖一个 assumption，这样是有利于我们做归纳的. 因此在这里情况下可以简化 $\supset L$ 为

$$\dfrac{\Gamma, A_1 \supset (A_2 \subset B) \to C}{\Gamma, (A_1 \vee A_2) \subset B \to C} \wedge \subset L$$

Example 175 若 A 为 $A_1 \vee A_2$，这里依然使用一个 logical equivalent:

$$(A_1 \vee A_2) \supset B \equiv (A_1 \supset B) \wedge (A_2 \supset B)$$

这样我们就不需要再使用 $\vee R$ 来从 $A_1 \vee A_2$ 获得信息，因此这里简化的 $\subset L$ 为

$$\dfrac{\Gamma, A_1 \supset B, A_2 \supset B \to C}{\Gamma, (A_1 \vee A_2) \supset B \to C} \vee \supset L$$

Example 176 若 A 为 $A_1 \supset A_2$，我们可以尝试先 derivation 一下

$$\dfrac{\dfrac{\Gamma, (A_1 \supset A_2) \supset B, A_1 \to A_2}{\Gamma, (A_1 \supset A_2) \supset B \to A_1 \supset A_2} \supset R \quad \Gamma, B \to C}{\Gamma, (A_1 \supset A_2) \supset B \to C} \supset L$$

其中这里有一个 logical equivalent:

$$(A_1 \supset A_2) \supset B \wedge A_1 \equiv (A_2 \supset B) \wedge A_1$$

因此这里简化的 $\supset L$ 为

$$\dfrac{\Gamma, (A_2 \supset B), A_1 \to A_2 \quad \Gamma, B \to C}{\Gamma, (A_1 \supset A_2) \supset B \to C} \supset\supset L$$

Example 177 若 A 为 $\top$，显然有

$$\dfrac{\Gamma, B \to C}{\Gamma, \top \subset B \to C} \top \supset L$$

因为 left branch $\Gamma, \top \subset B \to \top$ 显然是可以直接 close 掉的.

Example 178 若 A 为 $\bot$，这里有一个显然的 logical equivalent:

$$\bot \subset B \equiv A$$

因此这里简化的 $\supset L$ 为

$$\frac{\Gamma \to C}{\Gamma, \bot \subset B \to C} \bot \supset L$$

Definition 179 We call simplified sequent calculus with above compound left rules except original left implication G4ip: Contraction-free calculus for intuitionistic logic.

Theorem 180 (Soundness) If the sequent $\Gamma \to C$ is derivable in G4ip, then it is derivable in sequent calculus.

Proof 首先将 G4ip 中的 derivation 写成二维的形式 $\Gamma \overset{\mathcal{D}}{\to} C$，对 $\mathcal{D}$ 做 structure induction. 首先是 bases cases，即 $\mathcal{D}$ 为 empty 的 cases，显然这些 cases 对应的 rules 在 Gi4p 和 sequent calculus 中是保持一致. 再假设 $\mathcal{D}, \mathcal{E}$ is a proof，其对应在 sequent calculus 中的 proof 为 $\mathcal{D}', \mathcal{E}'$. 下面就是证明以它们为 subproof 的 cases，其中只需要证明关于 implication rules，整个过程是非常 straightforward. Q. E. D.

Annotation 181 要证明 completeness of derivability in G4ip，我们得先证明 terminitation of proof searching in G4ip，因为现在没有 soundness 中的前提条件了. 那是什么原因导致我们会担心会不会 terminated 呢? 因为 $\wedge \supset L$

$$\frac{\Gamma, A_1 \supset (A_2 \subset B) \to C}{\Gamma, (A_1 \vee A_2) \subset B \to C} \wedge \subset L$$

看起来 premise 和 conclusion 似乎有相同的规模，这里能否保证 termination? 这里引入 weight 来证明它，很奇妙的规则，至少我还没有洞悉它.

Definition 182 (Weight) For each propositional formula A, we assign it a weight as follow:

- $w(A) = w(\top) = w(\bot) = 2$ for atomic A;
- $w(A \wedge B) = w(A)(1 + w(B))$;
- $w(A \vee B) = 1 + w(A) + w(B)$;
- $w(A \supset B) = 1 + w(A)w(B)$.

Lemma 183 For each rule in G4ip, its premises have a strictly lower weight than its conclusion.

Proof 需要每个例子都验证一遍. 这里我只验证一下上面提到的例子.

$$\begin{aligned} w(A_1 \supset (A_2 \supset B)) &= 1 + w(A_1)(1 + w(A_2)(B)) \\ &= 1 + w(A_1) + w(A_1)w(A_2)w(B) \end{aligned}$$

$$\begin{aligned} w((A_1 \wedge A_2) \supset B) &= 1 + w(A_1)(1 + w(A_2))w(B) \\ &= 1 + w(A_1)w(B) + w(A_1)w(A_2)w(B) \end{aligned}$$

其中 $w(A_1) \geq 2$，因此 $w(A_1 \supset (A_2 \supset B)) \leq w((A_1 \wedge A_2) \supset B)$. Q. E. D.

Theorem 184 (Termination) Proof search in G4ip is terminating

Proof It is straightforward by Lemma 183. Q. E. D.

Theorem 185 (Completeness) If the sequent $\Gamma \to C$ is provable in sequent calculus, then it is provable in G4ip.

Proof 我们需要更细粒度的 structure induction 来处理 $\supset L$，因此我们这里采用induction on the weight of the sequent，这是非常重要的一个 trick. 对于给定一个 sequent，我们得思考 sequent bigger than it，同时我们最好以 proof searching 的视角去看待，这样会让我们讨论的 proof 更加的具体. 这里记录几个 representative cases.

Case 1 若 $\Gamma = \Gamma', A \wedge B$

根据 proof searching 中 invertibility 的策略，可以得到一个 provable premise $\Gamma', A, B \to C$，而它的 weight 小于原 sequent，因此这里可以使用 induction hypothesis 得到它在 G4ip 也是 provable, 接着再使用 $\wedge L$ in G4ip 即可.

Case 2 若 $\Gamma = \Gamma', a, a \supset B$

这里你会发现 sequent calculus 里面没有 rules 可以用，但是我们可以站在更高的一点地方来看它. 如果 $\Gamma', a, a \supset B \to C$ is provable, 那么存在一个 derivation $\mathcal{D}$ end with it，若我们可以从这个 derivation 推出 $a \supset L$ 的 premise，我们就可以正常使用 $a \supset L$ 得到我们想要的结果，这实际就是要说明 $a \supset L$ is invertible.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, a, B, a \to B}\, id}{\Gamma, a, B \to a \supset B}\, \supset R \qquad \begin{matrix}\mathcal{D} \\ \Gamma, a, a \supset B \vdash C\end{matrix}}{\Gamma', a, B \to C}\, cut$$

Logical Programming

Backward Chaining

Annotation 186 Proof searching as computation

Definition 187 We define Horn clauses includes two classes: G denotes goal clauses and P denotes program clauses. They are defined by the following grammer (where A denotes an atom):

$$G \coloneqq A \mid G \wedge G \mid \exists x.G$$
$$P \coloneqq A \mid B \supset A \mid \forall x.P$$
$$B \coloneqq A \mid B \wedge B$$

In words: goals are existentially quantified conjunction of atoms. Programs are either atoms or implications where the antecedent is conjunction of atoms and succedent is an atom, both universally quantified.

Definition 188 A logic program is a sequent $\mathcal{P} \to G$ where the $\mathcal{P}$ contains only program clauses and G is a goal clause.

Example 189 给定下述 logic program:

$$\begin{array}{c} \forall x.plus(0,x,x), \\ \forall x.\forall y.\forall z.plus(x,y,z) \supset plus(s(x),y,s(z)) \end{array} \to \exists x.plus(s(0),s(s(0)),x)$$

其中 $plus$ 是一个 predicate symbol, 而 0 和 s 是 function symbols. 如果我们希望上述 logica program processing(computing) like proof searching in sequent calculus，即以 proof searching 的方式去构造这个 sum. 那么首先我们得思考要构建一个怎样的 proof system?

Definition 190 (Representative proof system) Each clause will be represented as a rule: atoms are rules without premises, implications are rules where the premises are the atoms in B and the conclusion is the succdent. The program computing sums we had before will thus become:

$$\frac{}{plus(0,X,X)}\ plus_0 \qquad \frac{plus(X,Y,Z)}{plus(s(X),Y,s(Z))}\ plus_s$$

Annotation 191 回到 Example 189，我们可以构造一个 proof 说明 x 可以为 $s(s(s(x)))$.

$$\dfrac{\dfrac{}{plus(0,s(s(0)),s(s(0)))}\ plus_0}{plus(s(0),s(s(0)),s(s(s(x))))}\ plus_s$$

Definition 192 Bottom-up proof search is called backward chaining; top-down proof search is called forward chaining.

Annotation 193 Backward chaining 指的就是从 conclusion 推 hypotheses，这个过程包含的几个重要环节为:

1. 从 conclusion 找对应的 inference rules，这个过程我们称为pattern match.
2. 如果某个地方 proof searching 进行不下去了导致 failed，需要回到到某个点重新选择 inference rules，或者没有这样点导致整体完全 failed. 这个过程我们称为backtracking.
3. 以 Exmaple 189为例，它的 conclusion 含有一个未知的 variable x，那这个时候该如何进行第一步呢? 我们首先用一个 symbol X 来占位，那么此时 conclusion 为 $plus(s(0), s(s(0)), X)$，现在它其实可以叫做我们当前的 goal. 此时我们只有第二个 program clause 可以用，因为第一个 program clause require 第一个数得是 0. 那么问题又来了

$$\frac{plus(0, s(s(0)), ?)}{plus(s(0), s(s(0)), X)}\ plus_s$$

   这里我们也可以用另外一个 symbol Y 来占位，即 $plus(0, s(s(0)), Y)$，此时的 goal 只有第一个 program clause 可以对应，同时我必须将 Y 替换为 $s(s(0))$，因为此时没有 far premises 了. 将占位 symbols 替换为指定的 terms 过程我们就称为unification. 当 Y 替换之后，X 配合 $plus_s$ rule 就自然地推出来了. 在最开始 X 这里，我们还不能进行 unify 操作，因为还不是那么明显，因此我们可以 delay it. 后续我们将详细介绍 unification 操作.
4. 如果 apply 某个 inference rule 得到了多个 premises，这样我们可能有多个 goal 需要去 resolve，这里采用depth-first的手法.

Example 194 为什么 disjunction 没有出现在上面的 Horn clauses 里面呢?我们可以来看一个 derivable sequent:

$$p(a) \vee p(b) \to \exists x.p(x)$$

这里我们显然是无法找到 x 满足 $p(x)$，那么我们的 unify 操作在这里是行不通的.

Prolog

Definition 195 Prolog is a logic programming language implementing backward chaining on Horn clauses.

Example 196 一段 Prolog 程序实际上就是一堆 clauses, 这些 clauses 可以分为两类:

1. A clause may be a fact we know about the world. For example:

   mother(li, maple).
   father(hu, maple).

   Each of those is an atom(predicate) and they are interpreted as a logical formula: e.g., the first fact is $mother(li, maple) \supset \top$, but why?

2. A clause may be a rule in the shape:

   head :- body

   meaning that head is true if body is true. The head is atom, but the body can be conjunction of atoms. We can have following ruls for family relations, for example:

   parent(X,Y) :- mother(X, Y).
   parent(X,Y) :- father(X, Y).
   sibling(X,Y) :- parent(Z,X), parent(Z,X).

   It also has a representation as logical formulas, for example, the last clause is
   $\forall x.\forall y.\forall z.(parent(z, x) \wedge parent(z, y) \supset sibling(x, y))$.

当我们有了上面这些 clauses 之后, 我们就可以来做一些 querys, 例"maggie 的 parent 是谁?", parent(X, maggie). Prolog 以它作为 current goal, 开始搜索 head 是 parent 的那些 clauses, 找到对应的 body 里面的 subgoals, 接着周而复始, 直到最后 facts, 这过程充斥着我们提到的 backtracking 和 unification.

Example 197 一个非常神奇的例子: quicksort.

```
quicksort([], []).
quicksort([X|Xs], Ys) :- partition(Xs, X, Ls, Gs),
                         quicksort(Ls, Sl),
                         quicksort(Gs, Sr),
                         append(Sl, [X|Sr], Ys).

partition([], _, [], []).
partition([X|L], P, [X|Ls], Gs) :- X < P,
        partition(L, P, Ls, Gs).
partition([X|L], P, Ls, [X|Gs]) :- X >= P,
        partition(L, P, Ls, Gs).
```

配合 trace 理解它:

```
 1    1  Call: quicksort([2,3,1],_29) ?
 2    2  Call: partition([3,1],2,_100,_101) ?
 3    3  Call: 3<2 ?
 3    3  Fail: 3<2 ?
 3    3  Call: 3>=2 ?
 3    3  Exit: 3>=2 ?
 4    3  Call: partition([1],2,_153,_87) ?
 5    4  Call: 1<2 ?
 5    4  Exit: 1<2 ?
 6    4  Call: partition([],2,_140,_87) ?
 6    4  Exit: partition([],2,[],[]) ?
 4    3  Exit: partition([1],2,[1],[]) ?
 2    2  Exit: partition([3,1],2,[1],[3]) ?
 7    2  Call: quicksort([1],_233) ?
 8    3  Call: partition([],1,_259,_260) ?
 8    3  Exit: partition([],1,[],[]) ?
 9    3  Call: quicksort([],_284) ?
 9    3  Exit: quicksort([],[]) ?
10    3  Call: quicksort([],_309) ?
10    3  Exit: quicksort([],[]) ?
11    3  Call: append([],[1],_337) ?
11    3  Exit: append([],[1],[1]) ?
 7    2  Exit: quicksort([1],[1]) ?
```

```
12    2  Call: quicksort([3],_363) ?
13    3  Call: partition([],3,_389,_390) ?
13    3  Exit: partition([],3,[],[]) ?
14    3  Call: quicksort([],_414) ?
14    3  Exit: quicksort([],[]) ?
15    3  Call: quicksort([],_439) ?
15    3  Exit: quicksort([],[]) ?
16    3  Call: append([],[3],_467) ?
16    3  Exit: append([],[3],[3]) ?
12    2  Exit: quicksort([3],[3]) ?
17    2  Call: append([1],[2,3],_29) ?
17    2  Exit: append([1],[2,3],[1,2,3]) ?
 1    1  Exit: quicksort([2,3,1],[1,2,3]) ?
```

其中第一列数字表示具体哪个 call, 第二列数字表示 the depth of goal, 其中带下划线的数字表示占位的 variable, 等待被 unified. 可以看到第 8 次 call 的时候第一次 unify.

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[6] Lecture Notes on Proofs as Programs.
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[7] Computational Interpretations of Modalities.
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[8] Classic Modal Logic.
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[9] Verifications and uses.
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[10] Sequent Calculus.
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[11] Cut and Identity eliminations.
https://web2.qatar.cmu.edu/cs/15317/lectures/08-cutelimination.pdf

[12] Restricted sequent calculus.
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[13] Invertibility.
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