Proof Theory

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Natural Deduction

Remark 1 Natural deduction is a kind of proof calculus in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning.

Judgments and Propositions

Definition 2 A *judgment* is somthing we may know, this is, an object of knowledge. A judgment is *evident* if we in fact know it.

Annotation 3 "A is false" (see classical logic), "A is true at time t" (see temporal logic), "A is necessarily true" or "A is possibly true" (see modal logic), "the program M has type " (see programming languages and type theory), "A is achievable from the available resources" (see linear logic).

Introduction and Elimination

Definition 4 Inference rules that introduce a logical connective is the conclusion are known as *introduction rules.* i.e., to conclude "A and B *true*" for propositions A and B, one requires evidence for "A *true*" and B *true*. As an inference rule:

$$\frac{A\ true \quad B\ true}{A \wedge B\ true}\ \wedge I$$

Here $\wedge I$ stands for "conjunction introduction".

Definition 5 Inference rules that describe how to deconstruct information about a compound proposition into information about its consitiuents are elimination rules. i.e., from $A \wedge B$ *true*, we can conclude A *true* and B *true*:

$$\frac{A \wedge B\ true}{A\ true}\ \wedge E_L \qquad \frac{A \wedge B\ true}{B\ true}\ \wedge E_R$$

Annotation 6 The meaning of conjunction is determinded by its *verifications.*

Hypothetical Derivations

Definition 7 A *hypothetical judgment* is $J_1, \cdots, J_n \vdash J$, where judgments $J_1, \cdots, J_n$ are unproved assumptions, and the judgment J is the conclusion. A *hypothetical deduction*(derivation) for $J_1, \cdots, J_n \vdash J$ has the form

$$\begin{matrix} J_1 & \cdots & J_n \\ & \vdots & \\ & J & \end{matrix}$$

which means J is derivable from $J_1, \cdots, J_n$.

Annotation 8 上面的 $J_1, \cdots, J_2$ 都可以替换成关于 J_i 的一个 hypothetical derivation.

Definition 9 In the natural deduction calculus, an assumption is discharged when the conclusion of an inference does not depend on it, although one of the premises of the inference does[1].

Annotation 10 Once the appropriate rules have been completed, these are known as discharged assumptions, and are not included in the pool of assumptions on which the conclusion of the rule depends[3].

Annotation 11 hypothetical derivation 要求最后的 conclusion 依赖的 poof of assumptions 不是空的.

Theorem 12 Deduction theorem

$$T, P \vdash Q \iff T \vdash P \to Q$$

.

Annotation 13 在 deduction theorem 中我们注意到第一个 hypothetical judgment 里面的 antecedent Q 被去掉了，在第二个 hypothetical judgment 的 succedent 里面作为一个 implication 的 antecedent 出现了，这里我们就可以说 assumption Q is discharged，即现在的 conclusion 已经不依赖它了. 那么我们是如何构造 deduction theorem 里面的 implication 的呢？下面接着看

Definition 14 (implication) If B is true under the assumption that A is true, formly written $A \supset B$. The corresponed introduction and elimination rule as follow

$$\dfrac{\dfrac{}{A\ true}u \atop {\vdots \atop B\ true}}{A \supset B\ true} \supset I^u \qquad \dfrac{A \supset B \quad A\ true}{B\ true} \supset E$$

Annotation 15 Why indexed u In the introduction rule, the antecedent named u is discharged in the conclusion. This is a mechanism for delimiting the scope of the hypothesis: its sole reason for existence is to establish "$B\ true$"; it cannot be used for any other purpose, and in particular, it cannot be used below the introduction.

上面这段话出自 natural deduction 的 wiki，这个 u 实际上就是代指了从 A 推出 B 这中间可能的 derivation，现在我们通过 introduction rule 将它总结成了 $A \supset B$，因此 premise 实际上" 已经没有用了"，对照 discharge. 即

$$A \vdash B \Rightarrow \cdot \vdash A \to B.$$

Example 16 Considering the following proof of $A \supset (B \supset (A \wedge B))$

$$\dfrac{\dfrac{\dfrac{\overline{A\ true}^{\ u} \quad \overline{B\ true}^{\ w}}{A \wedge B\ true} \wedge I}{B \supset (A \wedge B)\ true} I^w}{A \supset (B \supset (A \wedge B))\ true} I^u.$$

这整个 derivation 不是 hypothetical 的，因为两个 assumptions $A\ true$ 和 $B\ true$ 都已经被 discharged，因此它实际上一个 complete proof!

Definition 17 (disjunction) The elimination rule for disjunction:

$$\dfrac{A \vee B\ true \quad \begin{array}{c}\overline{A\ true}^{\ u} \\ \vdots \\ C\ true\end{array} \quad \begin{array}{c}\overline{B\ true}^{\ w} \\ \vdots \\ C\ true\end{array}}{C\ true} \vee E^{u,w}$$

both assumption u, w are discharged at the disjunction elimination rule.

Definition 18 The falsehood elimination rule:

$$\dfrac{\perp\ true}{C\ true} \perp E$$

Annotation 19 falsehood 可以看做一个 zero-ary disjunction, 啥都不用考虑直接可以得到任意的 conclusion??? There is no proof for $\perp\ true$, so its sound to conclude arbitrary propositions.

Harmony

Definition 20 Local soundness shows that the elimination rules are not strong: no matter how we apply eliminations rules to the result of an introduction we cannot gain any new information.

Definition 21 Local completeness shows that the elimination rules are not weak: there is alwayws a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the the results by apply intruduction ruls.

Definition 22 (substitution Principle) If

$$\begin{array}{c} \overline{A\ true}\ u \\ \mathcal{E} \\ C\ true \end{array}$$

is a hypothetical proof of $\mathcal{C}$ *true* under the undischarged hypothesis A *true* labelled u, and

$$\begin{array}{c} \mathcal{D} \\ A\ true \end{array}$$

is a proof of A *true* then

$$\begin{array}{c} \dfrac{\mathcal{D}}{A\ true}\ u \\ \mathcal{E} \\ C\ true \end{array}$$

is our notation for substituting $\mathcal{D}$ for all uses of the hypothesis labelled u in $\mathcal{E}$. This deduction, also sometime written as $[\mathcal{D}/u]\mathcal{E}$ no longer depends on u.

Example 23 If given a elimination rule of disjunction as follow

$$\frac{A \vee B\ true}{A\ true}\ \vee E_L$$

The rule a little bit stronger, since we would not be able to reduce

$$\cfrac{\cfrac{B\ true}{A \vee B\ true}\ \vee I_R}{A\ true}\ \vee E_L$$

As u can see it's not local soundness.

Verifications and Uses

Definition 24 a verification should be a proof that only analyzes the consitituents of a proposition.

Annotation 25 natural deduction 实际上像 constructive logic 或者 intuitive logic, 不像 classic logic, 例如 Proposition $A \vee (A \supset B)$ 在 classic logic 就是正确的，因为我们 A 和 B 都需要给定是 true/false tag, 但是在 natural deduction 里面我们好像没有办法来处理. 更甚，如果我们要证明一个 B 是 accpeted in natural deduction, 你可能首先需要证明 $A \supset B$ 和 B 都是 accepted，就是根据结构来做 derivation.

Definition 26 Writing $A\uparrow$ for the judgment "A has a verification".

Definition 27 Writing $A\downarrow$ for the judgment "A may be used". $A\downarrow$ should be the case when either A *true* is a hypothesis, or A is deduced from a hypothesis via elimination rules.

Definition 28 For conjunction.

$$\frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow} \wedge I \qquad \frac{A \wedge B\downarrow}{A\downarrow} \wedge E_L \qquad \frac{A \wedge B\downarrow}{B\downarrow} \wedge E_R$$

Definition 29 For implication

$$\frac{\begin{array}{c}\overline{A\downarrow}^{\,u} \\ \vdots \\ B\uparrow\end{array}}{A \supset B\uparrow} \supset^u \qquad \frac{A \supset B\downarrow \quad A\uparrow}{B\downarrow} \supset E$$

implication introduction rule 里面的 $B\uparrow$ 表示没看懂，因为这里的 B 显然是来自 elimination 的结果. 为什么 implication elimination 里面需要 $A\uparrow$ 呢?

Example 30

$$\frac{\dfrac{\overline{A \wedge B\ true}^{\,u}}{A\ true} \wedge E_L}{(A \wedge B) \supset A\ true} \supset I^u$$

参考文献

[1] John Slaney. The Logic Notes. http://users.cecs.anu.edu.au/~jks/LogicNotes/

[2] The relation between deduction theorem and discharged. https://math.stackexchange.com/questions/3527285/what-does-discharging-an-assumption-mean-in-natural-deduction

[3] Definition:Discharged Assumption. https://proofwiki.org/wiki/Definition:Discharged_Assumption