Proof Theory

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Natural Deduction

Remark 1 Natural deduction is a kind of proof calculus in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning.

Judgments and Propositions

Definition 2 A *judgment* is somthing we may know, this is, an object of knowledge. A judgment is *evident* if we in fact know it.

Annotation 3 "A is false" (see classical logic), "A is true at time t" (see temporal logic), "A is necessarily true" or "A is possibly true" (see modal logic), "the program M has type " (see programming languages and type theory), "A is achievable from the available resources" (see linear logic).

Introduction and Elimination

Definition 4 Inference rules that introduce a logical connective is the conclusion are known as *introduction rules*. i.e., to conclude "A and B *true*" for propositions A and B, one requires evidence for "A *true*" and B *true*. As an inference rule:

$$\frac{A\ true \quad B\ true}{A \wedge B\ true}\ \wedge I$$

Here $\wedge I$ stands for "conjunction introduction".

Definition 5 Inference rules that describe how to deconstruct information about a compound proposition into information about its consitiuents are elimination rules. i.e., from $A \wedge B$ *true*, we can conclude A *true* and B *true*:

$$\frac{A \wedge B\ true}{A\ true}\ \wedge E_L \qquad \frac{A \wedge B\ true}{B\ true}\ \wedge E_R$$

Hypothetical Derivations

Definition 6 A *hypothetical judgment* is $J_1, \cdots, J_n \vdash J$, where judgments $J_1, \cdots, J_n$ are unproved assumptions, and the judgment J is the conclusion. A *hypothetical deduction*(derivation) for $J_1, \cdots, J_n \vdash J$ has the form

$$\begin{array}{c} J_1 \quad \cdots \quad J_n \\ \vdots \\ J \end{array}$$

which means J is derivable from $J_1, \cdots, J_n$.

Annotation 7 上面的 $J_1, \cdots, J_2$ 都可以替换成关于 J_i 的一个 hypothetical derivation.

Definition 8 In the natural deduction calculus, an assumption is discharged when the conclusion of an inference does not depend on it, although one of the premises of the inference does[1].

Annotation 9 Once the appropriate rules have been completed, these are known as discharged assumptions, and are not included in the pool of assumptions on which the conclusion of the rule depends[3].

Annotation 10 hypothetical derivation 要求最后的 conclusion 依赖的 poof of assumptions 不是空的.

Theorem 11 Deduction theorem

$$T, P \vdash Q \iff T \vdash P \to Q$$

.

Annotation 12 在 deduction theorem 中我们注意到第一个 hypothetical judgment 里面的 antecedent Q 被去掉了，在第二个 hypothetical judgment 的 succedent 里面作为一个 implication 的 antecedent 出现了，这里我们就可以说 assumption Q is discharged，即现在的 conclusion 已经不依赖它了. 那么我们是如何构造 deduction theorem 里面的 implication 的呢？下面接着看

Definition 13 If B is true under the assumption that A is true, formly written $A \supset B$. The corresponed introduction and elimination rule as follow

$$\dfrac{\dfrac{\overline{A\ true}\ u}{\vdots}\atop B\ true}{A \supset B\ true} \supset I^u \qquad \dfrac{A \supset B \quad A\ true}{B\ true} \supset E$$

Annotation 14 Why indexed u In the introduction rule, the antecedent named u is discharged in the conclusion. This is a mechanism for delimiting the scope of the hypothesis: its sole reason for existence is to establish "$B\ true$"; it cannot be used for any other purpose, and in particular, it cannot be used below the introduction.

上面这段话出自 natural deduction 的 wiki，这个 u 实际上就是代指了从 A 到 B 这中间可能的 derivation，现在我们通过 introduction rule 将它总结成了 $A \supset B$，因此 premise 实际上"已经没有用了"，对照 discharge. 美妙！

Example 15 Considering the following proof of $A \supset (B \supset (A \wedge B))$

$$\dfrac{\dfrac{\dfrac{\overline{A\ true}\ u \quad \overline{B\ true}\ w}{A \wedge B\ true} \wedge I}{B \supset (A \wedge B)\ true} I^w}{A \supset (B \supset (A \wedge B))\ true} I^u.$$

这整个 derivation 不是 hypothetical 的，因为两个 assumptions $A\ true$ 和 $B\ true$ 都已经被 discharged，因此它实际上一个 complete proof!

参考文献

[1] John Slaney. The Logic Notes. http://users.cecs.anu.edu.au/~jks/LogicNotes/

[2] The relation between deduction theorem and discharged. https://math.stackexchange.com/questions/3527285/what-does-discharging-an-assumption-mean-in-natural-deduction

[3] Definition:Discharged Assumption. https://proofwiki.org/wiki/Definition:Discharged_Assumption