关于 Maple Algebra 的这一路

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Definition 1.1. A labelled transition system (LTS) is a tuple $(S, \Lambda, \rightarrow)$ where S is set of states, Λ is set of labels, and $\rightarrow$ is relation of labelled transitions (i.e., a subset of $S \times \Lambda \times S$). A $(p, \alpha, q) \in \rightarrow$ is written as $p \xrightarrow{\alpha} q$.

Annotation 1.2. TODO: categorical semantics: F-coalgebra

Definition 1.3. [1]Let $T = (S, \Lambda, \rightarrow)$ be a labelled transition system. The set of traces $Tr(s)$, for $s \in S$ is the minimal set satisfying

- $\varepsilon \in Tr(s)$.
- $\alpha\ \sigma \in Tr(s)$ if $\{\, s' \in S \mid s \xrightarrow{\alpha} s' \text{ and } \sigma \in Tr(s') \,\}$.

Definition 1.4. Two states p, q are trace equivalent iff $Tr(p) = Tr(q)$.

Definition 1.5. (Simultation) Given two labelled transition system $(S_1, \Lambda, \rightarrow_1)$ and $(S_2, \Lambda, \rightarrow_2)$, relation $R \subseteq S_1 \times S_2$ is a simulation iff, for all $(p, q) \in R$ and $\alpha \in \lambda$ satisfies

$$\text{for any } p \xrightarrow{\alpha}_1 p',\ \text{then there exists } q' \text{ such that } q \xrightarrow{\alpha}_2 q' \text{ and } (p', q') \in R$$

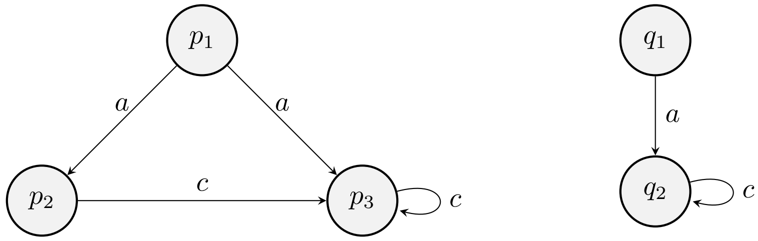
$$\begin{tikzcd}
p \arrow[r, "\alpha"] \arrow[d, dashed, "R"'] & p' \arrow[d, dashed, "R"] \\
q \arrow[r, "\alpha"'] & q'
\end{tikzcd}$$

Definition 1.6. We say q simulates p if there exists a simulation R includes (p, q) (i.e., $(p, q) \in R$), written $p < q$.

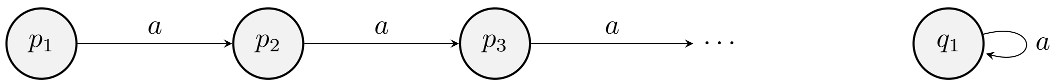
Definition 1.7. (Bisimultation) Given two labelled transition system $(S_1, \Lambda, \rightarrow_1)$ and $(S_2, \Lambda, \rightarrow_2)$, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse $\overline{R}$ are simulations, for all $(p, q) \in R$ and $\alpha \in \Lambda$ satisfies

$$\text{for any } p \xrightarrow{\alpha}_1 p',\ \text{then there exists } q' \text{ such that } q \xrightarrow{\alpha}_2 q' \text{ and } (p', q') \in R$$
$$\text{for any } q \xrightarrow{\alpha}_2 q',\ \text{then there exists } p' \text{ such that } p \xrightarrow{\alpha}_1 p' \text{ and } (p', q') \in R$$

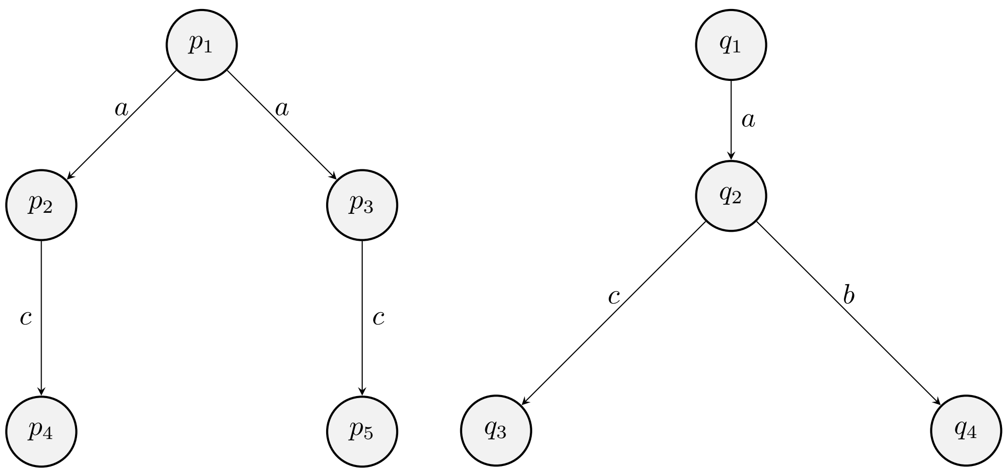
Example 1.8. 一些 bisimulation 的例子



关于上面两个 transition system 的 bisimultaion 为 $R = \{(p_1, q_1), (p_2, q_2), (p_3, q_2)\}$. 还有一个比较有点特别的例子



如果关于上图这样 bisimulation R 存在，那么 $(p_i, q_1) \in R$ for every i. 再看一个不是 bisimulation 的例子



这里不满足 $(p_3, q_2) \notin R$.

Definition 1.9. (Bisimilarity) Given two states p and q in S, p is bisimilar to q, written $p \sim q$, if and only if there is a bisimulation R such that $(p, q) \in R$.

Definition 1.10. The bisimilarity relation $\sim$ is the union of all bisimulations.

Lemma 1.11. The bisimulation has some properties:

- The identity relation id is a bisimulation(with two same LTS).
- The empty relation $\perp$ is a bisimulation.

- (closed under union) The $\bigcup_{i\in I} R_i$ of a family of bisimulations $(R_i)_{i\in I}$ is a bisimulation.

Lemma 1.12. [2] The bisimilarity relation $\sim$ is equivalence relation (i.e., reflexivity, symmetry, transitivity).

证明. 其中 reflexivity, symmetry 是比较显然的. Transitivity 稍微麻烦一点, 我们用 relation composition 定义新的 relation$R_3 = R_1; R_2$, 此时有 $(p,q) \in R_3$, 因此只要证明 R_3 is bisimulation 足够了. 取任意一个 $(p_1, q_1) \in R_3$, 那么按照 R_3 的定义, 存在 $(p_1, r_1) \in R_1$ 和 (r_1, q_1). 由 $p_1 \sim r_1$ 那么对于任意的 $p_1 \xrightarrow{\alpha} p_1'$, 存在 $r_1 \xrightarrow{\alpha} r_1'$ 满足 $(p_1', r_1') \in R_1$. 再由 $r_1 \sim q_1$, 存在 $q_1 \xrightarrow{\alpha} q_1'$ 满足 $(r_1', q_1') \in R_2$. 于是按照 R_3 的定义也有 $(p_1', q_1') \in R_3$. 再由 R_2 is bisimulation, 从 $(r_1, q_1) \in R_2$ 按照上述的思路往回证明即可, 最终 R_3 is bisimulation. □

Definition 1.13. [3] An LTS is called deterministic if for every state p and action α, there is at most one state q such that $p \xrightarrow{\alpha} q$.

Lemma 1.14. In a deterministic LTS, two states are bisimilar if and only if they are trace equivalent,

$$s_1 \sim s_2 \iff Tr(s_1) = Tr(s_2)$$

证明. 先证 $\Rightarrow$, 设满足 $s_1 \sim s_2((s_1, s_2) \in R$ and R is bisimultaion), 设 $\sigma_{s_1} \in Tr(s_1)$, 其中 σ_{s_1} 为 sequence $(\alpha_i)_{i\in I}$ where I is a indexed famliy. 由于 $s_1 \sim s_2$, 那么对于 $s_1 \xrightarrow{\alpha_1} s_1'$, 存在 $s_2 \xrightarrow{\alpha_1} s_2'$, 于是 $(s_1', s_2') \in R$, 根据 σ 长度做 induction 可以证明 $\sigma_{s_1} \in Tr(q)$. 再反过来证明 $\sigma_{s_2} \in Tr(s_2)$ 也同样有 $\sigma_{s_2} \in Tr(s_1)$. 最终 $Tr(s_1) = Tr(s_2)$.

对于 $\Leftarrow$, 我们可以用 $Tr(s_1) = Tr(s_2)$ 构造一个 bisimulation, 定义 relation R 为

$$Tr(s_1) = Tr(s_2) \iff (s_1, s_2) \in R.$$

只要能证明 R bisimulation 即可. 首先我们来说明在 deterministic 限制下一个比较好性质: 若 $Tr(s_1) = Tr(s_2)$ 且当 $s_1 \xrightarrow{\alpha} s_1', s_2 \xrightarrow{\alpha} s_2'$, 那么 $Tr(s_1') = Tr(s_2')$. 这样对于任意地 $(s_1, s_2) \in R$, 它们 accept 相同 action 对应的 transition $(s_1', s_2') \in R$. 因此 $s_1 \sim s_2$. □

Definition 1.15. (Weak Bisimultation) Given two labelled transition system $(S_1, \Lambda, \rightarrow_1)$ and $(S_2, \Lambda, \rightarrow_2)$, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse $\overline{R}$ are simulations, for all $(p,q) \in R$ and $\alpha \in \Lambda \cup \{\tau\}$ satisfies

$$\text{for any } p \xrightarrow{\alpha}_1 p', \text{ then there exists } q' \text{ such that } q \xrightarrow{\tau*\ \alpha\ \tau*}{}_2^* q' \text{ and } (p', q') \in R$$
$$\text{for any } q \xrightarrow{\alpha}_2 q', \text{ then there exists } p' \text{ such that } p \xrightarrow{\tau*\ \alpha\ \tau*}{}_1^* p' \text{ and } (p', q') \in R$$

where $\rightarrow^*$ is multi-transition.

Annotation 1.16. 对于 LTS 的一些想法:

- 如果你想用 transition system 来做 reasoning 可以考虑把它和 Kripke frame 联系起来, 同时要构造一些 modality 来设计方便做 reasoning 的 calculus.

- (*bisimulation proof method*) 对于两个特别的 states 来说，我们应该如何找到这样 bisimulation 来满足 $(p, q) \in R$?
- 对于两个特别的 LTS 来说，我们怎样以 bisimulation 思考它们是否 equivalent? bisimulation 的最初定义应该叫做 strong bisimulation, 它建立的是一种 strong equivalence, 而 weak bisimulation 建立是一种 observation equivalence.

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