Lattice

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目录

The Elements of Universal Algebra

Definition and Examples of Algebras

One of the aims of universal algebra is to extract, whenever possible, the common elements of several seemingly different types of algebraic structures.

Definition 1.1. For A a nonempty set and n a nonnegative integer we define $A^0 = \{\emptyset\}$, and, for $n > 0$, A^n is the set of n-tuples of elements from A. An n-ary operation (or function) on A is any function f from A^n to A; n is the arity (or rank) of f. A finitary operation is an n-ary operation, for some n. The image of $\langle a_1, \cdots, a_n \rangle$ under an n-ary operation f is denoted by $f(a_1, \cdots, a_n)$. An operation f on A is called a nullary operation (or constant) if its arity is zero; it is completely determinded by the image $f(\emptyset)$ in A of the only element $\emptyset$ in A^0, and as such it is convenient to identify it with the element $f(\emptyset)$. Thus a nullary operation is thought of as an element of A. An operation f on A is unary, binary, or ternay if its arity is 1,2, or 3, respectively.

Definition 1.2. A language (or type) of algebras is a set $\mathcal{F}$ of function symbols such a nonnegative integer n is assgined to each member f of $\mathcal{F}$. This integer is called the arity (or rank) of f, and f is said to be an n-ary function symbol. The subset of n-ary function symbols in $\mathcal{F}$ is denoted by $\mathcal{F}_n$.

Definition 1.3. If $\mathcal{F}$ is a language of algebras then an algebra $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair $\langle A, F \rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the language $\mathcal{F}$ such that corresponding to each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the universe (全域) (or underlying set) of $\mathbf{A} = \langle A, F \rangle$, and the $f^{\mathbf{A}}$ ' s are called the fundamental operations (基本运算) of $\mathbf{A}$. (In practice we prefer to write just f for $f^{\mathbf{A}}$ —this convention creates an ambiguity which seldom causes a problem. However, in this chapter we will be unusually careful.) If $\mathcal{F}$ is finite, say $\mathcal{F} = \{f_1, \cdots, f_k\}$, we often write $\langle A, f_1, \cdots, f_k \rangle$ for $\langle A, F \rangle$, usually adopting the convention:

$$arity f_1 \geq \cdots \geq arity f_k.$$

An algebra $\mathbf{A}$ is unary if all of its operations are unary, and it is mono-unary if it has just on unary operation.

抽象来说一个 algebra 就是一个集合和一堆 finitary operations 构成的, 在目前已经学到的代数中 operation 的 arity 大多数不会超过 2(够 modern).

Example 1.4. $\mathbf{A}$ is a groupoid if it has just one binary operation; this operation is usually denoted by $+$ or $\cdot$, and we write $a + b$ or $a \cdot b$ for the image of $\langle a, b \rangle$ under this operation, and call it the sum or product of a and b, respectively.

Definition 1.5. An alegebra $\mathbf{A}$ is finite if $|\mathbf{A}|$ is finite, and trivial if $|\mathbf{A}| = 1$.

Example 1.6. Some well-known algebras

- Group
- Semigroup (半群) and Monoid (幺半群)
- ...

Isomorphic Algebras and Subalgebras

Definition 1.7. Let **A** and **B** be two algebras of the same type $\mathcal{F}$. Then a function $\alpha\colon A{\to}B$ is an isomorphism from **A** to **B** if α is one-to-one and onto, and for every n-ary $f \in \mathbf{F}$, and for $a_1, \cdots, a_n \in A$, we have

$$\alpha f^{\mathbf{A}}(a_1, \cdots, a_n) = f^{\mathbf{B}}(\alpha a_1, \cdots, \alpha a_n).$$

We say **A** is isomorphic to B, if there is a isomorphism from **A** to **B**.

老样子元素上保持 bijective，对应的运算结果也保持一致.

Definition 1.8. Let **A** and **B** be two algebras of the same type. Then **B** is a subalgebra of A if $B \subseteq A$ and every fundamental operation of **B** is the restriction of the corresponding operation of **A**, i.e., for each function symbol f, $f^{\mathbf{B}}$ is $f^{\mathbf{A}}$ restricted to B; we write simply $\mathbf{B} \leq \mathbf{A}$.

A subuniverse of **A** is a subset B of A which is closed under the fundamental operations of **A**, i.e., if f is a fundamental n-ary operation of **A** and $a_1, \cdots, a_n \in B$ we wound required $f(a_1, \cdots, a_n) \in B$.

这个 restriction 就是只对定义域做限制的意思，subalgebra 是一种构造新代数的方法，后面会学到另外几种.

Definition 1.9. Let **A** and **B** be of the same type. A function $\alpha\colon A{\to}B$ is an embedding of **A** into **B** if α is one-to-one and satisfies

$$\alpha f^{\mathbf{A}}(a_1, \cdots, a_n) = f^{\mathbf{B}}(\alpha a_1, \cdots, \alpha a_n).$$

Such an α is also called a monomorphism. We say **A** can be embedded in **B** if there is an embedding of **A** into **B**.

单纯地去掉 surjective.

Theorem 1.10. *If $\alpha\colon A{\to}B$ is an embedding, then $\alpha(A)$ is a subuniverse of $\mathbb{B}$.*

证明. 我们需要说 $\alpha(A)$ 在 n-ary operation 下保持封闭. 因为 α 是应 embedding，所以对应一个 n-ary operation f 和 $a_1, \cdots, a_n \in A$ 有

$$f^{\mathbf{B}}(\alpha a_1, \cdots, \alpha a_n) = \alpha f^{\mathbf{A}}(a_1, \cdots, c_n) \in \alpha(A).$$

已经证闭. □

Algebraic Lattices and Subuniverses

这一章阐述 algebraic lattice 是如何自然地出现在了 universe algebra 里面.

Definition 1.11. Given an algebra $\mathbf{A}$ define, for every $X \subseteq A$,

$$\mathrm{Sg}(X) = \bigcap\{\, B \mid X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathbf{A} \,\}.$$

We read $\mathrm{Sg}(X)$ as "the subuniverse generated by X".

Definition 1.12. A closure operator C on the set A is an algebraic closure operator if for $X \subseteq A$

$$C(X) = \bigcup\{\, C(Y) \mid Y \subseteq X \text{ and } Y \text{ is finite} \,\}.$$

Theorem 1.13. If we are given an algebra $\mathbf{A}$, then Sg is an algebraic closure operator on A.

证明. 很明显任意地 subuniverses 交还是一个 subuniverse，所有的 subuiverses 构成了一个 closure system，所以 Sg 是一个 closure operator. 对于任意的 $X \subseteq A$ 我们定义

$$E(X) = X \cup \{\, f(a_1, \cdots, a_n) \mid f \text{ is a fundamental } n\text{-ary operation on } A \text{ and } a_1, \cdots, a_n \in X \,\}.$$

然后定义它的 n 次复合 $E^n(X)$ 为

$$E^0(X) = X$$
$$E^{n+1}(X) = E(E^n(X)).$$

由于 A 上所有 fundamental operation 都是 finitary, 且有

$$X \subseteq E(X) \subseteq \cdots \subseteq E^n(X).$$

接下来我们来证明下面的式子

$$\mathrm{Sg}(X) = X \cup E(X) \cup E^2(X) \cup \cdots.$$

思路是 (1)$\mathrm{Sg}(X) \subseteq X \cup E(X) \cup E^2(X) \cup \cdots$ 和 (2)$X \cup E(X) \cup E^2(X) \cup \cdots \subseteq \mathrm{Sg}(X)$.

(1) 我们从 $X \cup E(X) \cup E^2(X) \cup \cdots$ 这种构造出发，实际上这种构造得到的就是包含 X 的最小 subuniverse. 考虑它的 bound，可能存在某个最小的 k 使得 $E^{k+1}(E^k(X)) = E^k(X)$，那么当 $i > k$ 时都有 $E^i(X) = E^k(X)$，无疑问现在 $E^k(X)$ 是一个 subuniverse 并且包含 X，这其实就说明了 $\mathrm{Sg}(X) \subseteq E^k(X)$. 如果你不能找到这样的 k，那么会有若 $i > k$，那么 $E^k(X) \subset E^i(X)$，但是它是 bound by A，且 $\mathrm{Sg}(A) = A$，这是显然地.

(2) 任取 $x \in \mathrm{Sg}(X)$, 我们用 $\{Z_i\}_{i \in I}$ 表示所有 such that $X \subseteq Z$ 且 Z 是 $\mathbf{A}$ 的一个 subuniverse. 那么对任意的 i，有 $x \in Z_i$. 那么我们更有

$$\mathrm{Sg}(X) = E^n(Z_1) \cap E^n(Z_2) \cap \cdots.$$

对任意的 n 成立. 并且我们还有

$$E^n(X) \subseteq E^n(Z_1) \cap E^n(Z_2) \cap \cdots .$$

这是因为每一项都有 $E^n(X) \subseteq E^n(Z_i)$. 那么我们把 n 取遍，就可以得到

$$X \cup E(X) \cup E^2(X) \cup \cdots \subseteq \mathrm{Sg}(X).$$

综上 $\mathrm{Sg}(X) = X \cup E(X) \cup E^2(X) \cup \cdots$.

那么对于任意的 $a \in \mathrm{Sg}(X)$，都有 $a \in E^n(X)$ 其中 $n < \omega$ (omega 表示无穷远它比任何的自然数都大). 那么存在 finite $Y \subseteq X$，使得 $a \in E^n(Y)$. 这个过程就是把生成 a 的哪些元素在 $E^{n-1}(X)$ 里面挑出来，然后循环往复，n 是 finite 的，所以 Y 也是有限的. 这就说明了 $\mathrm{Sg}(X) = \bigcap \mathrm{Sg}(Y_i)$，因为每一个 a 都可以找到一个 finite Y.

还是写一下严格的归纳假设吧，首先我们点出一个关系. 注意到 $E^{\cdot}$ 是单调的，那么给定两个集合 X, Y，因为 $X \subseteq X \cup Y$ 和 $Y \subseteq X \cup Y$，所以 $E^k(X) \cup E^k(Y) \subseteq E^k(X \cup Y)$.

我们要证明 (1) 任意的 $x \in E^k(X)$ 都有 finite set $Y \subseteq X$ 使得 $x \in E^k(Y)$.

那么当 $k = 0$ 时，

$$a \in E^0(X) = X \Rightarrow a \in E^(\{a\}),$$

所以这里可以取 $Y = \{a\}$.

假设对任意 $k < n$，都有命题 (1) 成立，那么当 $k = n$ 时，

$$\begin{aligned} a \in E^n(X) \Leftrightarrow & a \in E(E^{n-1}(X)) \\ &= E^{n-1}(X) \cup \{\, f(a_1, \cdots, a_n) \mid f \text{ is a fundamental } t\text{-ary operation on } A \text{ and } a_1, \cdots, a_n \in X \,\} \\ &= E^{n-1}(X) \cup B_n, \end{aligned}$$

要么 $a \in E^{n-1}(X)$ 或者 $a \in B_n$. 如果 $a \in E^{n-1}(X)$，那么归纳假设就完成了. 如果 $a \in B_n$，则 $a \in E(\{a_1, \cdots, a_t\})$，其中 $a_1, \cdots, a_t \in E^{n-1}(X)$ (这里用 t 是不想和这个上标 n 弄混)，那么根据假设有 $a_i \in E^{n-1}(Y_i)$，其中 finite set $Y_i \subseteq X$，再替换一下

$$a \in E(\{a_1, \cdots, a_t\}) \subseteq E(\bigcup_1^t E^{n-1}(Y_i)) \subseteq E(E^{n-1}(\bigcup_1^t Y_i)) = E^n(\bigcup_1^t Y_i).$$

所以这里取 $Y = \bigcup_{i=1}^t Y_i$，它确实是一个 finite set.

□

Corollary 1.14. If $\mathbf{A}$ is an algebra then $\mathcal{L}_{\mathrm{Sg}}$, the lattice of subuniverses of $\mathbf{A}$, is an algebraic lattice.

Definition 1.15. Given an algebra $\mathbf{A}$, Sub($\mathbf{A}$) denotes the set of subuniverses of A, and Sub($\mathbf{A}$) is the corresponding algebraic lattice, the lattice of subuniverses of $\mathbf{A}$. For $X \subseteq A$ we say X generates $\mathbf{A}$ if $\mathrm{Sg}(X) = A$. The algebra $\mathbf{A}$ is finitely generated if it has a finite set of generators.

如果 A 有有限多个生成元，那么就说 algebra $\mathbf{A}$ 是有限生成的.

Theorem 1.16. (closed sets 对应某个 algebra 的 universes) Let Γ be an algebraic closure operator on a set X. Then there is an $\mathbf{A}$ on the set X such that the subuniverse of $\mathbf{A}$ are precisely the closed sets of Γ.

证明. Basic: 我们要构造出来一个 algebra $\mathbf{A}$，首先我们得确定在哪个集合上，再给定它的 operations.

我们的目标: 当前的前提是给定 X 上的 algebraic closure operator Γ，以 X 为 underlying set，构造其上的 operations 使得对应的 algebra $\mathbf{A}$ 的 universes 都是 Γ 对应的 closed set.

对任意的 finite set $B \subseteq X$ 和任意的 $b \in \Gamma(B)$，我们定义 n-ary operation 如下，其中 $n = |B|$.

$$f_{B,b}(a_1, \cdots, a_n) = \begin{cases} b & \text{if } B = \{a_1, \cdots, a_n\} \\ a_1 & \text{otherwise.} \end{cases}$$

接下来我们证明 Γ 对应的 closed set A 是一个在上面定义的 operations 下是封闭的. 这里明显地有

$$f_{B,b}(a_1, \cdots, a_n) \subseteq \Gamma(\{a_1, \cdots, a_n\}).$$

这里我们再利用一个结论（证明在 lattice note 里面），(1)Γ 是 algebraic closure operation，那么若 $A \subseteq X$ 是一个 closed set 当且仅当 $\Gamma(F) \subseteq A$ 对所有的 finite set $F \subseteq A$ 成立. 那么任取对任意上述的 operation，我们取 $a_1, \cdots, a_n \in A$，那么

$$f_{B,b}(a_1, \cdots, a_n) \subseteq \Gamma(\{a_1, \cdots, a_n\}) \subseteq A$$

所以上述所有的 operations 在 A 下是封闭.

反过来我们还要证明一个 universe S 是 closed set. 还是用 (1) 这个结论，我们证明所有的 finite set $B \subseteq S$，$\Gamma(B) \subseteq S$，其中

$$\Gamma(B) = \{\, f_{B,b}(a_1, \cdots, a_n) \mid B = a_1, \cdots, a_n, b \in \Gamma(B) \,\}.$$

自然地，因为 $f_{B,b}(a_1, \cdots, a_n)$ 在 S 下是封闭的，这个 operation 的构造就巧到这里，所以 $\Gamma(B) \subseteq S$. □

Theorem 1.17. (algebraic lattice 同构于某个 algebra 的 universes 构成的 lattice) If $\mathcal{L}$ is an algebraic lattice, then $\mathcal{L} \cong \mathrm{Sub}(\mathbf{A})$, for some algebra A.

证明. 这个命题是上面命题的一个推论，这里的 $\mathbf{A}$ 的 underlying set 可以设为 $\mathcal{L}^c$ 即 $\mathcal{L}$ 上所有的 compact elements. 关于这一点去看 lattice note 里面 $\mathcal{L} \cong \mathcal{C}_\Delta$ 部分. 那么接下来就是证明 $\mathrm{Sub}(A) = \mathcal{C}_\Delta$，这里有更比上面更简单的 operations 的构造方式. 对任意的 $a \in \mathcal{C}_\Delta$, 定义一个 unary operation

$$f_a(x) = \begin{cases} a & \text{if } a \leq x \\ x & \text{otherwise.} \end{cases}$$

把这样的 unary operation 作用到某个 $x \subseteq \mathcal{L}^c$，它生成就是 principle ideal $\downarrow x$，自然地到某个 $A \subseteq lattice^c$，它就是 $\Delta(A)$. $\square$

The Irredundant Basis Theorem

Definition 1.18. Let C be a closure operator on A. For $n < \omega$, let C_n be the function definied on $\mathrm{Su}(A)$ (power set) by

$$C_n(X) = \bigcup\{\, C(Y) \mid Y \subseteq X,\ |Y| \leq n \,\}.$$

We say that C is n-ary if

$$C(X) = C_n(X) \cup C_n^2(X) \cup \cdots ,$$

where

$$C_n^1(X) = C_n(X),$$
$$C_n^{k+1}(X) = C_n(C_n^k(X)).$$

也可以用 $C(X) = C_n^{\omega}(X)$ 来表示，其中 ω 表示基数最小 infinite set 的基数. Tarski 原本 C_n 的 definition 里面去掉了 $|Y| \leq n$ 中的等号，这个 n 也叫做 the rank of (A, C).

Lemma 1.19. Let $\mathbf{A}$ be an algebra all of whose fundamental operations have arity at most n. Then Sg is an n-ary closure operator on A.

证明. 从 Theorem 1.13 我们得到

$$E(X) \subseteq (\mathrm{Sg})_n(X) \subseteq \mathrm{Sg}(X),$$

第一个 $\subseteq$ 由 $E(X)$ 的 definition，第二个 $\subseteq$ 由 Sg 的 definition. 因此

$$\begin{aligned} \mathrm{Sg}(X) &= X \cup E(X) \cup E^2(X) \cup \cdots \\ &\subseteq (\mathrm{Sg})_n(X) \cup (\mathrm{Sg})_n^2(X) \cup \cdots \\ &\subseteq \mathrm{Sg}(X). \end{aligned}$$

所以 $\mathrm{Sg}(x) = (\mathrm{Sg})_n(X) \cup (\mathrm{Sg})_n^2(X) \cup \cdots$. □

Theorem 1.20. Let C be a closure operator on A. A' is closed set such that $C(A') = A'$ and C' is the restriction of C to the family of subsets of A'. Then C' is also a closure operator; if, moreover, C is n-ary,then so is C'.

证明. C' 是 closure operator 这是显然的，对于任意的 $B' \subseteq A'$, $C'(B') = C(B')$，所以它们两个 rank 是相同的. □

Definition 1.21. The set X is a minimal generating set for Y if X generates Y and no proper subset of X generates Y.

Definition 1.22. Let C be a closure operator on S. A set $X \subseteq S$ is called irredundant if $y \notin C(X - \{y\})$ for every $y \in X$.

Definition 1.23. Suppose C is a n-ary closure operator on S. A minimal generating set of S is called an irredundant basis. Let $\mathrm{IrB}(C) = \{\, n \le \omega \mid S$ has an irredundant basis of n elements $\}$.

Proposition 1.24. If S has a finite basis B, then it also has a finite irredundant basis B', so $\mathrm{IrB}(S)$ is not empty.

Proposition 1.25. If C is finite ary, then every basis for S includes a finite irredundant basis. So that no infinite basis of S is irredundant. Thus $\mathrm{IrB}(S)$ is the set of cardinalities of all irredundant basis for S.

Theorem 1.26. If C is an n-ary closure operator on S with $n \ge 2$, and if $i < j$ with $i, j \in \mathrm{IrB}(C)$ such that

$$\{i+1, \cdots, j-1\} \cap \mathrm{IrB}(C) = \emptyset,$$

then $j - i \le n - 1$. In particular, if $n = 2$ then $\mathrm{IrB}(C)$ is a convex subset of ω, i.e., a sequence of consecutive numbers.

证明. 让 B 是一个 irreducible basis 且 $|B| = j$. 让 K 表示那些 irreducible basis A 且 $|A| \le i$ 构成的集合. 我们的策略是从 K 里面挑一个 A_0 出来, 让这个 $A_0 \subseteq C_n^{t+1}(B)$ 但是 $A_0 \nsubseteq C_n^t(B)$, 那么再取一点 $a_0 \in C_n^{t+1}(B) - C_n^t(B)$, 把这个 a_0 替换成 $c_1, \cdots, c_m \in C_n^t(B)$, 这样构成了一个新的 base A_1, 它里面是有一个 irredundant basis A_2 的, 且这个 $|A_2| < i + n$ 的. 这是基本的想法，下面来阐述一下这个想法的细节。

首先 B 是一个 irreducible basis 且 C 是一个 n-ary closure operator，那么

$$A \subseteq S = C(B) = C_n^\omega(B).$$

我们由前面的命题可以知道 A 是一个 finite set 且是一个 base，那么 $A \nsubseteq B$. 且 $C_n^{k+1} \supseteq C_n^k$，那么是可以找到某个 t 使得 $A \subseteq C_n^{t+1}(B)$ 且 $A \nsubseteq C_n^t(B)$ 的，这就是为什么可以找到上面 A_0 的原因. 并且如果我们找到这样的 $A_0 \in K$，那么

$$A_0 \nsubseteq C_n^t(B) \text{ implies } A \nsubseteq C_n^t(B),$$

对任意的 $A \in K$ 成立. 假设不成立，会得到 A 不是 irreducible 的，这是矛盾的. 我们再考虑对于 $A \in K$ 且 $A \subseteq C_n^{t+1}(B)$，我们假设

$$|A_0 \cap (C_n^{t+1}(B) - C_n^t(B))| \le |A \cap (C_n^{t+1}(B) - C_n^t(B))|,$$

我们让 A_0 更靠近 B 一点，就是指 $C_n^{t+1}(B)$ 和 $C_n^t(B)$ 中间这个环，包含更少一点 A_0 中的点. 这样做的目的就是让后面得到的新 base 的基数尽可能的小. 然后我们选择一点 $a_0 A_0 \cap (C_n^{t+1}(B) - C_n^t(B))$. 那么肯定是存在 $b_1, \cdots, b_m \in C_n^t(B)$，其中 $m \le n$，使得

$$a_0 \in C_n(b_1, \cdots, b_m).$$

然后把 a_0 替换掉得到的新 base $A_1 = (A - \{a_0\}) \cup \{b_1, \cdots, b_m\}$，那么 $A_0 \subseteq C(A_1)$，所以

$$C(A_0) \subseteq C(A_1).$$

那么在 A_1 上我们是可以找到一个 irredundant basis $A_2 \subseteq A_1$. 现在我们有 $|A_2| < |A_0| + n$. 如果 $|A_0| + n \leq j$, 那么这个 A_2 会和我们选择的 A_0 造成矛盾，即

$$|A_2 \cap (C_n^{t+1}(B) - C_n^t(B))| < |A_0 \cap (C_n^{t+1}(B) - C_n^t(B))|.$$

所以这里 $|A_0| + n \geq j$，其中 $|A_0| \leq i$，那么就可以得到 $j - i \leq n$.

我们来理一下关系，这个命题如果两个 irredundant basis 的基数之间距离达到一定程度，你找不到一个 irreducible basis 的基数在它们之间. 这个距离如何描述呢? $j - i < n$. 上面证明的方法是尝试在 irreducible basis 基数小于 i 的 basis 的基础上构造一个新的 irreducible basis 出来，看它落在哪里. 现在我们构造了出来了这个新的 irreducible basis A_2，我们只知道 $|A_2| < i + n$，并不知道它和 j 有什么关系，所以我们考虑这个 upper bound 和 j 的关系. 然后我们知道了这个 upper bound 是大于的 j 的， $\square$