

Lattice

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目录

1	Ordered Sets	2
2	Semilattices, Lattices and Complete Lattices	6
2.1	Semilattice	6
2.2	Lattice	8
2.3	Complete Lattice	9
2.4	Closure System	11

Ordered Sets

Definition 1.1. **Partially ordered set** is a system $\mathcal{P} = (P, \leq)$ where P is a nonempty set and \leq is a binary relation on P satisfying, for all $x, y, z \in P$,

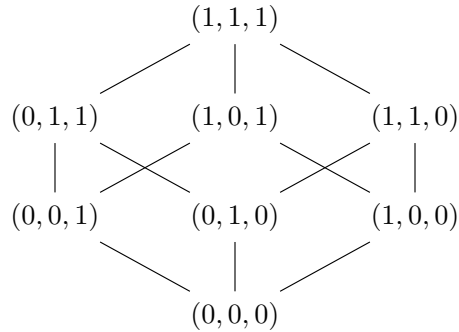
1. $x \leq x$, (reflexivity)
2. if $x \leq y$ and $y \leq x$, then $x = y$, (antisymmetry)
3. if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

Definition 1.2. \mathcal{C} is a **chain** if for every $x, y \in \mathcal{C}$, either $x \leq y$ or $y \leq x$.

chain 上的元素都可以相互比较, 所以它是 totally ordered.

Definition 1.3. We say that x is **covered** by y in \mathcal{P} , written $x \prec y$, if $x \leq y$ and there is no $z \in P$ with $x \leq z \leq y$.

Definition 1.4. **Hasse diagram** for a finite partially order set \mathcal{P} : the elements of P are represented by points in the plane, and a line is drawn from a up to b precisely when $a \prec b$.



Definition 1.5. Given a partially order set, f is a **order preserving map** satisfying the condition $x \leq y$ implies $f(x) \leq f(y)$.

Definition 1.6. Given two posets (P, \leq_P) and (Q, \leq_Q) , an **order isomorphism** from (P, \leq_P) to (Q, \leq_Q) is a bijective order preserving map.

Definition 1.7. Given two posets (P, \leq_P) and (Q, \leq_Q) , an **order embedding** from (P, \leq_P) to (Q, \leq_Q) is a both order-preserving and order-reflecting map that $x \leq y \iff f(x) \leq f(y)$.

相比 order isomorphism 而言稍微弱一点, 不需要是一个 surjective.

Definition 1.8. An **ideal** I of a partially ordered set \mathcal{P} is a subset of the elements of P which satisfy the property that if $x \in \mathcal{P}$ and exists $y \in I$ with $x \leq y$, then $x \in I$.

衍生自 the ideal of ring, 后面我们将会看见 the ideal of lattice.

Definition 1.9. Given an ordered set $\mathcal{P} = (P, \leq)$. The **dual of P** is another poset $\mathcal{P}^d = (P, \leq^d)$ with the order relation defined by $x \leq^d y \iff y \leq x$.

Definition 1.10. The dual notion of an ideal is called a **filter** that F is a subset of P such $x \geq y \in F$ implies $x \in F$

类似的还有 principle ideal 和 principle filter. 就是通过一个元素生成的.

Definition 1.11. The poset \mathcal{P} has a **maximum**(element) if there exists $x \in P$ such that $y \leq x$ for all $x \in P$.

An element $x \in P$ is **maximal** if there is no element $y \in P$ with $x \leq y$ and $x \neq y$.

maximum 是一个名词表示最大值 (greatest), maximal 是一个形容词表示极大的意思. 在 poset 中可能不只有一个 maximal element.

Lemma 1.12. The following are equivalent for an poset \mathcal{P} .

1. Every nonempty subset $S \subseteq P$ contains an element minimal in S .
2. \mathcal{P} contains no infinite descending chain

$$a_0 > a_1 > a_2 > \cdots$$

这里去掉等号是指 $a_0 \neq a_1 \neq a_2 \neq \cdots$

3. If

$$a_0 \geq a_1 \geq a_2 \geq \cdots$$

in \mathcal{P} , then there exists k such that $a_n = a_k$ for all $n \geq k$.

这个 lemma 被称为 descending chain condition(DCC). 对偶地也有 ascending chain condition(ACC). original 'a partially ordered set \mathcal{P} requires that all decreasing sequences in \mathcal{P} become eventually constant'.

证明. (2) \Rightarrow (3) 前提只存在 finite descending chain. 假设 (3) 不成立, 且 $a_0 \geq a_1 \geq a_2 \geq \cdots$ 是 infinite chain. 则对于任意的 k , 都能找到 $n \geq k$ 使得 $a_n \neq a_k$ 且 $a_k \geq a_n$, 那么 $a_k > a_n$. 这样从 $k = 0, 1, 2, \cdots$ 开始我们每次都可以找到 $a_{n_0} > a_{n_1} > \cdots$. 这样我们实际构造了一个 infinite descending chain, 这是和前提矛盾的. 若 $a_0 \geq a_1 \geq a_2 \geq \cdots$ 是一个 finite chain, 它的最后一个元素显然是满足 (3), 这和假设是矛盾的.

(3) \Rightarrow (2) 也是分 infinite chain 和 finite chain 来讨论, finite 是显然的, infinite 的时候可以把它变成 finite.

(1) \Rightarrow (2) (1) 前提满足下, 假设 (2) 不成立, 即 \mathcal{P} 存在 infinite descending chain. 把这个 chain 上的元素取出来组成一个 subset S , 那么任取 a_k 都有 $a_{k+1} \leq a_k$. 即找不到 minimal.

(2) \Rightarrow (1) (2) 前提满足下, 假设 (1) 不成立. 这里需要用一下[选择公理](#)了, 定义 S 上一个选择函数 $f: S \rightarrow T$, 其中 $T \subseteq S$. 让 $a_0 = f(S)$, 递归地定义对任意的 $i \in \omega$ 有 $a_{i+1} = f(\{s \in S \mid s < a_i\})$. 接下来让这个 definition make sense, (2) 前提下 S 是没有 minimal, 所以 $\{s \in S \mid s \leq a_i\}$ 不是 empty set. 这样就找到了一个 infinite descending chain, 与假设矛盾.

(1) \Rightarrow (2) \Rightarrow (3)

(3) \Rightarrow (2) \Rightarrow (1)

done well!

□

Lemma 1.13. Let \mathcal{P} be an poset satisfyint the DCC. If $\varphi(x)$ is statement such that

1. $\varphi(x)$ holds for all minimal elements of P , and
2. whenever $\varphi(y)$ holds for all $y < x$, then $\varphi(x)$ holds,

then $\varphi(x)$ is true for every element of P .

这个 lemma 有点意思, 如果对 P 上所有的 minimal element m 都有命题 $\varphi(m)$ 成立, 且 \mathcal{P} 满足 DCC. 那么再加上一个条件: 只要对任意元素 $x \in P$, 满足 $y < x$ 都有 $\varphi(y)$ 成立. 则对任意元素 $x \in P$ 都有 $\varphi(x)$ 成立.

证明. 其实 (1) 是 (2) 的一个 special case. 在 (1)(2)hold 的情况下, 我们试想一下 $\varphi(x)$ 没有被 hold 住的是哪些元素呢? 即对于某个 x , 存在 $y < x$ 使得 $\varphi(y)$ 没有被 hold. 递归地, 我们再去考虑这个 y . 那么这里就存在一条 descending chain 在这里, 由于 \mathcal{P} 是满足 DCC, 所以这个 descending chain 是 infinite 的. 这条 chain 的结尾显然是一个 minimal element, 但是它是满足 $\varphi(x)$. 所以实际上是不存在这里的 x 不满足 $\varphi(x)$. □

Definition 1.14. Let \mathcal{P} be poset. Two elements a and b of \mathcal{P} are called **comparable** if $a \leq b$ or $a \geq b$. Otherwise, they are called **incomparable**.

[可比性](#).

Definition 1.15. An **antichain** in \mathcal{P} is a subset A of \mathcal{P} in which each pair of different element are incomparable.

Definition 1.16. Define the **width** of an poset \mathcal{P} by

$$w(\mathcal{P}) = \sup\{|A| \mid A \text{ is an antichain in } \mathcal{P}\}$$

where $|A|$ denotes the cardinality(集合的势) of A .

Definition 1.17. We define the **chain-covering-number** CCN $c(\mathcal{P})$ to be the least cardinal number k , such that P is a union of k chains(finite) of P , means $P = \bigcup C_i$

另一种 covering number, 有趣.

Lemma 1.18. Suppose $P = \bigcup C_i$ where $i \in I$, then $w(\mathcal{P}) \leq |I|$.

证明. 因为 $|A \cap C_i| \leq 1$ for $i \in I$. 也就是说你把 A 里面的元素分开塞到 C_i 上, 每次都只能塞一个. 那么最多你可以每个 C_i 上都塞一个. \square

Theorem 1.19. (Dilworth, 1950) Let \mathcal{P} be a finite poset. $w(\mathcal{P})$ is width. Then \mathcal{P} is a union of $w(\mathcal{P})$ -chains.

证明. TODO. \square

Semilattices, Lattices and Complete Lattices

Semilattice

Definition 2.1. A **semilattice** is an algebra $\mathcal{S} = (S, *)$ satisfying, for all $x, y, z \in S$,

1. $x * x = x$,
2. $x * y = y * x$,
3. $x * (y * z) = x * (y * z)$.

where $*$ is binary operator. 换句话说 **semilattice** 就是一个 **idempotent commutative semigroup**(幂等交换半群).

Theorem 2.2. In a semilattice \mathcal{S} , define $x \leq y$ if $x * y = x$. Then (S, \leq) is a poset in which every pair of elements has a greater lower bound.

Conversely, given an poset P with that property, define $x * y = g.l.b(x, y)$. Then $(P, *)$ is a semilattice.

semilattice 上弄了一个特殊的 poset 出来, 它最好的性质就是任意两个元素都有一个下确界. 把 $*$ 换成 \cap , 然后把 \leq 换成 \subseteq , 可能就很熟悉了. A semilattice with the above ordering is usually called **meet semilattice**.

证明. 先证明这个是一个 poset.

1. $x * x = x$ implies $x \leq x$,
2. if $x \leq y$ and $x \geq y$, then $x = x * y = y * x = y$,
3. if $x \leq y$ and $y \leq z$. then $x * z = (x * y) * z = x * (y * z) = x * y = x$, so $x \leq z$.

这个 greater lower bound 就是 $x * y$. 首先证明它是一个 lower bound, $x * (x * y) = x * y$ and $y * (x * y) = x * y$, 所以 $x * y$ 是一个 lower bound. 再来证明所有的 lower bound 都比它小, 假设 $z \leq x$ 和 $z \leq y$, 即 z 是 $\{x, y\}$ 的一个 lower bound. 那么 $z * (x * y) = z * y = z$, 所以 $z \leq (x * y)$. 最后 $x * y$ 的一个 greater lower bound.

□

对偶地, 使得 $x \geq y \iff x * y = x$, 则称 \mathcal{S} 为是一个 **join semilattice**. 自然地在 (S, \leq) 下任意的 pair 都有一个 least upper bound $x \vee y$.

Definition 2.3. A **homomorphism** between two semilattice is a map $f: \mathcal{S} \rightarrow \mathcal{T}$ with the property that $f(x * y) = f(x) * f(y)$. An **isomorphism** is a homomorphism that injective and surjective.

nothing new...

Theorem 2.4. Let \mathcal{S} be a meet semilattice. Define $\phi: \mathcal{S} \rightarrow \mathcal{O}(\mathcal{S})$ by

$$\phi(x) = \{ y \in \mathcal{S} \mid y \leq x \}$$

where $\mathcal{O}(\mathcal{S})$ is collection of all order ideals of \mathcal{S} . Then \mathcal{S} is isomorphic $(\mathcal{O}(\mathcal{S}), \cap)$ (注意这里是 \mathcal{S} 的 image).

怎么感觉这些 ideal 都是 principle ideal.

证明. \cap 表示 set inclusion, ϕ 是 order-preserving 和 order-reflecting 还是比较 obvious. 所以 ϕ 是一个 order embedding of \mathcal{S} into $\mathcal{O}(\mathcal{S})$. Moreover $\phi(x \wedge y) = \phi(x) \cap \phi(y)$ because $x \wedge y$ is the greatest lower bound of $\{x, y\}$, so that $z \leq x \wedge y$ if and only if $z \leq x$ and $z \leq y$. \square

Lattice

Definition 2.5. A **lattice** is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying, for all $x, y, z \in L$,

1. $x \wedge x = x$ and $x \vee x = x$,
2. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$,
4. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.

就第四个在我们眼里似乎没有那么自然, 它叫 absorption laws(吸收律), 它在这里可以保证后面 \wedge 和 \vee 定义了相同的 partial order(虽然是 dual). 前三个我们知道 lattice 同时在两种 binary operator 都是 semilattice, 所以我们只要在 lattice 上定义前面合适的 partial order, 它就是 both meet and join semilattice.

Theorem 2.6. In a lattice \mathcal{L} , define $x \leq y$ if and only if $x \wedge y = x$. Then (L, \leq) is a poset in which every pair of elements has a greatest lower bound and a least upper bound.

证明. 给定一个 pair (x, y) . 前面已经证明了 $x \wedge y$ 是它的一个 greater lower bound. 再根据 lattice definition 的第四条的第一个式子, $x \vee y$ 是它的一个 upper bound, 第二式子说明当 $x \geq y$ 时, 有 $x \vee y = x$, 对偶地 $x \vee y$ 是 least upper bound.

这里若 $x \wedge y = x$, 则 $x \vee y = (x \wedge y) \vee y = y$. 类似地 $x \vee y = y$, 则 $x \wedge y = x \wedge (x \vee y) = x$. 所以有一个很重要的结论就是 $x \wedge y = x \iff x \vee y = y$. \square

类似的我们可以通过一个 poset 构造 lattice.

Theorem 2.7. Given an poset \mathcal{P} with that above property, define $x \wedge y = \sup\{x, y\}$ and $x \vee y = \inf\{x, y\}$. Then (P, \wedge, \vee) is a lattice.

所以实际上 lattice 可以有两种定义第一种是前面的代数定义, 第二种就是在 poset 上定义 join 和 meet 操作, 这一点要清楚.

the definitions of sublattice, homomorphism and isomorphism).

Complete Lattice

Definition 2.8. For a subset A of a poset P , let A^u denote the set of all upper bounds of A ,

$$\begin{aligned} A^u &= \{x \in P \mid x \geq a \text{ for all } a \in A\} \\ &= \bigcap_{a \in A} \uparrow a \end{aligned}$$

where $\uparrow a = \{x \in P \mid x \geq a\}$. Dually, A^l is the set of all lower bounds of A ,

$$\begin{aligned} A^l &= \{x \in P \mid x \leq a \text{ for all } a \in A\} \\ &= \bigcap_{a \in A} \downarrow a \end{aligned}$$

where $\downarrow a = \{x \in P \mid x \leq a\}$.

思考一个问题 poset P 的一个 subset A 什么时候 least upper bound? 很显然 A^u 一定不是空的, 更确切地说 A^u 有一个 greatest lower upper z , 而且 $z \in A^u$, 根据 z 的 definition 它是 A 的 least upper bound. 这种情况下我们就说 the join of A exists, and write $z = \bigvee A$. 对偶地, 考虑 A 的 greatest lower bound, 则 A^l 一定不为空, 那么 A^l 里面是有一个 lower upper bound 的 w , 根据 w 的 definition 它是 A 的 greatest lower bound. 这种情况下我们就说 the meet of A exists, and write $w = \bigwedge A$.

这样我们 define 两个特殊的 meet 和 join 作用在一个集合上.

Theorem 2.9. Let \mathcal{S} be a finite meet semilattice with greatest element 1. Then \mathcal{S} is a lattice with join operation defined by

$$x \vee y = \bigwedge \{x, y\}^u = \bigwedge (\uparrow x \cap \uparrow y).$$

证明. \mathcal{S} 有 greatest element, 则 A^u 肯定不是空了, 至少这个 greatest element 里面. $\bigwedge A^u$ 就是要找 A^u 的 lower upper bound. 由于 \mathcal{S} 是一个 finite lattice, 所以 A^u 也是 finite. A^u 里面的元素做有限次 meet 操作得到就是一个 lower upper bound, 但是你还得说明它在 A^u 里面. 这是很显然的, $x \wedge z_1 \wedge \cdots \wedge z_k = x$ 其中 $z_i \in A^u$, 所以 $\bigwedge A^u$ 是它的一个 upper bound.

还得 proof 一下它是一个 lattice, 上面只是证明了这个东西是 well behaved. Lattice definition 中前三条还是比较明显的.

$$x \wedge (x \vee y) = x$$

这也很显然, 因为 $x \vee y \in \{x, y\}^u$.

$$x \vee (x \wedge y) = x$$

因为 $x \wedge y$ 是 $\{x, y\}$ 的一个 greatest lower bound, 有 $x \geq x \wedge y$, 那么 $\inf(x, x \wedge y) = x$. □

这个 theorem 告诉我们: if a finite poset P has a greatest element and every pair of elements has a meet, then P is a lattice.

Theorem 2.10. Every finite subset of a lattice has a greatest lower bound and a least upper bound.

证明. \mathcal{L} 是 finite, 则它的 subset 也是 finite. 前面我们知道 lattice 中任意一个 pair 都有 greatest lower bound 和 least upper bound, 这是 meet 和 join 定义下的 partial order 所带来的性质. 在 finite subset 里面先挑两个出来做 meet 或者 join 可以得到 inf 和 sup 它们也是属于 L 的, 再从剩下的 subset 里面再挑一个出来做同样的操作, 这个操作只会做有限多次, 所以最终我可以得到这个 subset 的 greatest lower bound 和 least upper bound. \square

这个性质在 infinite lattice 下可能就无法成立. 例如 infinite subset 上述操作可能根本就停不下. 由此我们定义另外一个概念.

Definition 2.11. Given poset \mathcal{L} . If every subset A of \mathcal{L} has a greatest lower bound $\bigwedge A$ and a least upper bound $\bigvee A$, then \mathcal{L} is called complete lattice.

subset 其中就包含了 pair, 所以它是一个 lattice 还是比较明显的. 此外, finite lattice 是 complete 的, 并且所有 complete lattice 都包含 greatest element 和 least element.

Definition 2.12. a complete meet semilattice is an poset \mathcal{S} with greatest element and the property that every nonempty subset A of S has a greatest lower bound $\bigwedge A$.

下面这个 theorem 让我们抛弃了更强的 finite, 只需要 complete 就可以在 meet semilattice 上构造一个 lattice 出来.

Theorem 2.13. If \mathcal{L} is a complete meet semilattice, then \mathcal{L} is a complete lattice with the join operation defined by

$$\bigvee A = \bigwedge A^u = \bigwedge (\bigcap_{a \in A} \uparrow a).$$

证明. 和前面在 finite meet semilattice 上构造 lattice 类似, 这里 finite 换成了 complete. 这里我们直接就可以知道 $\bigwedge A^u$ 是有意义的, 前面也证明了它还是一个 A 的 upper bound. 那么 $\bigwedge A$ 的 definition 是满足 A 的 least upper bound. \square

Closure System

Definition 2.14. A **closure system** on a set X is a collection \mathcal{C} of subsets of X that is closed under arbitrary intersections (任意的交). The sets in \mathcal{C} are called closed set.

Example 2.15. 有一些 *closure system* 的例子

1. *closed subsets of topological space,*
2. *subgroups of group,*
3. *subspace of vector space.*
4. *convex subsets of euclidean space \mathbb{R}^n ,*
5. *order ideals of an poset.*

By convention, 把 $\bigcap \emptyset = X$ 的话, closure system 就是一个 complete lattice.

Definition 2.16. A **closure** operator on a set X is a map $\Gamma: \mathfrak{P}(X) \rightarrow \mathfrak{P}(X)$ satisfying, for all $A, B \subseteq X$,

1. $A \subseteq \Gamma(A)$,
2. $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$,
3. $\Gamma(\Gamma(A)) = \Gamma(A)$.

closure 的 general definition 有点意思.