Lattice

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Ordered Sets

Definition 1.1. Partially ordered set is a system $\mathcal{P} = (P, \leq)$ where P is a nonempty set and $\leq$ is a binary relation on P satisfying，for all $x, y, z \in P$,

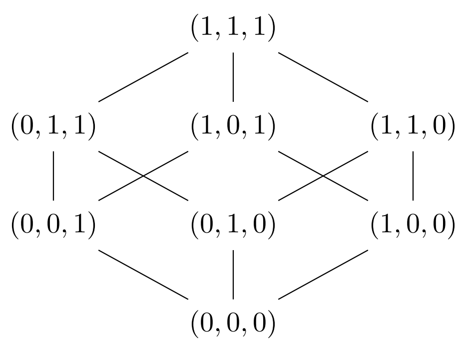
1. $x \leq x$, (reflexivity)
2. if $x \leq y$ and $y \leq x$,then $x = y$, (antisymmetry)
3. if $x \leq y$ and $y \leq z$,then $x \leq z$. (transitivity)

Definition 1.2. $\mathcal{C}$ is a chain if for every $x, y \in \mathcal{C}$， either $x \leq y$ or $y \leq x$.

chain 上的元素都可以相互比较，所以它是 totally ordered.

Definition 1.3. We say that x is covered by y in $\mathcal{P}$， written $x \prec y$， if $x \leq y$ and there is no $z \in P$ with $x \leq z \leq y$.

Definition 1.4. Hasse diagram for a finite partially order set $\mathcal{P}$: the elements of P are represented by points in the plane, and a line is drawn from a up to b precisely when $a \prec b$.



Definition 1.5. Given a partially order set，f is a order preserving map satisfying the condition $x \leq y$ implies $f(x) \leq f(y)$.

Definition 1.6. Given two posets $(P, \leq_S)$ and $(Q, \leq_Q)$， an order isomorphism from $(P, \leq_S)$ to $(Q, \leq_Q)$ is a bijective order preserving map.

Definition 1.7. Given two posets $(P, \leq_S)$ and $(Q, \leq_Q)$， an order embedding from $(P, \leq_S)$ to $(Q, \leq_Q)$ is a both order-preserving and order-reflecting map that $x \leq y \iff f(x) \leq f(y)$.

相比 order isomorphism 而言稍微弱一点，不需要是一个 surjective.

Definition 1.8. An ideal I of a partially ordered set $\mathcal{P}$ is a subset of the elements of P which satisfy the property that if $x \in \mathcal{P}$ and exists $y \in I$ with $x \leq y$，then $x \in I$.

衍生自 the ideal of ring，后面我们将会看见 the ideal of lattice.

Definition 1.9. Given an ordered set $\mathcal{P} = (P, \leq)$. The dual of P is another poset $\mathcal{P}^d = (P, \leq^d)$ with the order relation defined by $x \leq^d y \iff y \leq x$.

Definition 1.10. The dual notion of an ideal is called a filter that F is a subset of P such $x \geq y \in F$ implies $x \in F$

类似的还有 principle ideal 和 principle filter. 就是通过一个元素生成的.

Definition 1.11. The poset $\mathcal{P}$ has a maximum(element) if there exists $x \in P$ such that $y \leq x$ for all $x \in P$.

An element $x \in P$ is maximal if there is no element $y \in P$ with $x \leq y$ and $x \neq y$.

maximum 是一个名词表示最大值 (greatest)，maximal 是一个形容词表示极大的意思. 在 poset 中可能不只有一个 maximal element.

Lemma 1.12. The following are equivalent for an poset $\mathcal{P}$.

1. Every nonempty subset $S \subseteq P$ contains an element minimal in S.
2. $\mathcal{P}$ contains no infinite descending chain

$$a_0 > a_1 > a_2 > \cdots .$$

   这里去掉等号是指 $a_0 \neq a_1 \neq a_2 \neq \cdots$

3. If

$$a_0 \geq a_1 \geq a_2 \geq \cdots$$

   in $\mathcal{P}$，then there exists k such that $a_n = a_k$ for all $n \geq k$.

这个 lemma 被称为 descending chain condition(DCC). 对偶地也有 ascending chain condition(ACC). original 'a partially ordered set $\mathcal{P}$ requires that all decreasing sequences in $\mathcal{P}$ become eventually constant'.

证明. (2) $\Rightarrow$ (3) 前提只存在 finite descending chain. 假设 (3) 不成立，且 $a_0 \geq a_1 \geq a_2 \geq \cdots$ 是 infinite chain. 则对于任意的 k，都能找到 $n \geq k$ 使得 $a_n \neq a_k$ 且 $a_k \geq a_n$，那么 $a_k > a_n$. 这样从 $k = 0, 1, 2, \cdots$ 开始我们每次都可以找到 $a_{n_0} > a_{n_1} > \cdots$. 这样我们实际构造了一个 infinite descending chain，这是和前提矛盾的. 若 $a_0 \geq a_1 \geq a_2 \geq \cdots$ 是一个 finite chain，它的最后一个元素显然是满足 (3)，这和假设是矛盾的.

(3) $\Rightarrow$ (2) 也是分 infinite chain 和 finite chain 来讨论，finite 是显然的，infinite 的时候可以把它变成 finite.

(1) $\Rightarrow$ (2) (1) 前提满足下，假设 (2) 不成立，即 $\mathcal{P}$ 存在 infinite descending chain. 把这个 chain 上的元素取出来组成一个 subset S，那么任取 a_k 都有 $a_{k+1} \leq a_k$. 即找不到 minimal.

(2) $\Rightarrow$ (1) (2) 前提满足下，假设 (1) 不成立. 这里需要用一下选择公理了，定义 S 上一个选择函数 $f\colon S \to T$，其中 $T \subseteq S$. 让 $a_0 = f(S)$，递归地定义对任意的 $i \in \omega$ 有 $a_{i+1} = f(\{\, s \in S \mid s < a_i \,\})$. 接下来让这个 definition make sense，(2) 前提下 S 是没有 minimal, 所以 $\{\, s \in S \mid s \leq a_i \,\}$ 不是 empty set. 这样就找到了一个 infinite descending chain，与假设矛盾.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3)

(3) $\Rightarrow$ (2) $\Rightarrow$ (1)

done well! □

Lemma 1.13. Let $\mathcal{P}$ be an poset satisfyint the DCC. If $\varphi(x)$ is statement such that

1. $\varphi(x)$ holds for all minimal elements of P， and
2. whenever $\varphi(y)$ holds for all $y < x$， then $\varphi(x)$ holds，

then $\varphi(x)$ is true for every element of P.

这个 lemma 有点意思，如果对 P 上所有的 minimal element m 都有命题 $\varphi(m)$ 成立，且 $\mathcal{P}$ 满足 DCC. 那么再加上一个条件: 只要对任意元素 $x \in P$，满足 $y < x$ 都有 $\varphi(y)$ 成立. 则对任意元素 $x \in P$ 都有 $\varphi(x)$ 成立.

证明. 其实 (1) 是 (2) 的一个 special case. 在 (1)(2)hold 的情况下，我们试想一下 $\varphi(x)$ 没有被 hold 住的是哪些元素呢? 即对于某个 x，存在 $y < x$ 使得 $\varphi(y)$ 没有被 hold. 递归地，我们再去考虑这个 y. 那么这里就存在一条 descending chain 在这里，由于 $\mathcal{P}$ 是满足 DCC，所以这个 descending chain 是 infinite 的. 这条 chain 的结尾显然是一个 minimal element，但是它是满足 $\varphi(x)$. 所以实际上是不存在这里的 x 不满足 $\varphi(x)$. □

Definition 1.14. Let $\mathcal{P}$ be poset. Two elements a and b of $\mathcal{P}$ are called comparable if $a \leq b$ or $a \geq b$. Otherwise, they are called incomparable.

可比性.

Definition 1.15. An antichain in $\mathcal{P}$ is a subset A of $\mathcal{P}$ in which each pair of different element are incomparable.

Definition 1.16. Define the width of an poset $\mathcal{P}$ by

$$w(\mathcal{P}) = \sup\{\, |A| \mid \text{A is an antichain in } \mathcal{P} \,\}$$

where $|A|$ denotes the cardinality(集合的势) of A.

Definition 1.17. We define the chain-covering-number CCN $c(\mathcal{P})$ to be the least cardinal number k, such that P is a union of k chains(finite) of P, means $P = \bigcup C_i$

另一种 covering number，有趣.

Lemma 1.18. *Suppose* $P = \bigcup C_i$ *where* $i \in I$，*then* $w(\mathcal{P}) \le |I|$.

证明. 因为 $|A \cap C_i| \le 1$ for $i \in I$. 也就是说你把 A 里面的元素分开塞到 C_i 上，每次都只能塞一个. 那么最多你可以每个 C_i 上都塞一个. □

Theorem 1.19. (Dilworth ,1950) Let $\mathcal{P}$ be a finite poset. $w(\mathcal{P})$ is width. Then $\mathcal{P}$ is a union of $w(\mathcal{P})$-chains.

证明. TODO. □

Semilattices, Lattices and Complete Lattices

Definition 2.1. A semilattice is an algebra $\mathcal{S} = (S, *)$ satisfying，for all $x, y, z \in \mathcal{S}$,

1. $x * x = x$,
2. $x * y = y * x$,
3. $x * (y * z) = x * (y * z)$.

where $*$ is binary operator. 换句话说 semilattice 就是一个 idempotent commutative semigroup(幂等交换半群).

Theorem 2.2. In a semilattice $\mathcal{S}$，define $x \leq y$ if $x * y = x$. Then $(S, \leq)$ is a poset in which every pair of elements has a greater lower bound.

Conversely，given an poset P with that property，define $x * y = g.l.b(x, y)$. Then $(P, *)$ is a semilattice.

semilattice 上弄了一个特殊的 poset 出来，它最好的性质就是任意两个元素都有一个下确界. 把 $*$ 换成 $\cap$, 然后把 $\leq$ 换成 $\subseteq$，可能就很熟悉了. A semilattice with the above ordering is usually called meet semilattice.

证明. 先证明这个是一个 poset.

1. $x * x = x$ implies $x\ leq x$,
2. if $x \leq y$ and $x \geq y$, then $x = x * y = y * x = y$,
3. if $x \leq y$ and $y \leq z$. then $x * z = (x * y) * z = x * (y * z) = x * y = x$, so $x \leq z$.

   这个 greater lower bound 就是 $x*y$. 首先证明它是一个 lower bound, $x*(x*y) = x*y$ and $y*(x*y) = x*y$, 所以 $x * y$ 是一个 lower bound. 再来证明所有的 lower bound 都比它小，假设 $z \leq x$ 和 $z \leq x$，即 z 是 $\{x, y\}$ 的一个 bound. 那么 $z * (x * y) = z * y = z$，所以 $z \leq (x * y)$. 最后 $x * y$ 的一个 greater lower bound.

$\square$

对偶地，使得 $x \geq y \iff x * y = x$，则称 $\mathcal{S}$ 为是一个join semilattice.

Definition 2.3. A homomorphism between two semilattice is a map $f \colon \mathcal{S} \to \mathcal{T}$ with the property that $f(x*y) = f(x) * f(y)$. An isomorphism is a homomorphism that injective and surjective.

nothing new...

Theorem 2.4. Let $\mathcal{S}$ be a meet semilattice. Define $\phi\colon S \to \mathcal{O}(S)$ by

$$\phi(x) = \{\, y \in S \mid y \leq x \,\}$$

where $\mathcal{O}(S)$ is collection of all order ideals of $\mathcal{S}$. Then $\mathcal{S}$ is isomophic $(\mathcal{O}(S), \cap)$(注意这里是 $\mathcal{S}$ 的 image).

怎么感觉这些 ideal 都是 principle ideal.

证明. $\cap$ 表示 set inclusion，ϕ 是 order-preserving 和 order-reflecting 还是比 obvious. 所以 ϕ 是一个 order embedding of $\mathcal{S}$ into $\mathcal{O}(\mathcal{S})$. Moreover $\phi(x \wedge y) = \phi(x) \cap \phi(y)$ because $x \wedge y$ is the greate lower bound of $\{x, y\}$, so that $z \leq x \wedge v$ if and only if $z \leq x$ and $z \leq y$. $\square$

Definition 2.5. A lattice is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying，for all $x, y, z \in L$,

1. $x \wedge x = x$ and $x \vee x = x$,
2. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$,
4. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.

就第四个在我们眼里似乎没有那么自然，它叫 absorption laws(吸收律)，它在这里可以保证后面 $\wedge$ 和 $\vee$ 定义了相同的 partial order(虽然是 dual). 前三个我们知道 lattice 同时在两种 binary operator 都是 semilattice，所以我们只要在 lattoce 上定义前面合适的 partial order，它就是 both meet and join semilattice.

Theorem 2.6. *In a lattice $\mathcal{L}$，define $x \leq y$ if and only if $x \wedge y = x$. Then $(L, \leq)$ is a poset in which every pair of elements has a greatest lower bound and a least upper bound.*

证明. 给定一个 pair (x, y). 前面已经证明了 $x \wedge y$ 是它的一个 greater lower bound. 再根据 lattice definition 的第四条的第一个式子，$x \vee y$ 是它的一个 upper bound，第二式子说明当 $x \geq y$ 时，有 $x \vee y = x$，对偶地 $x \vee y$ 是 least upper bound.

这里若 $x \wedge y = x$，则 $x \vee y = (x \wedge y) \vee y = y$. 类似地 $x \vee y = y$，则 $x \wedge y = x \wedge (x \vee y) = x$. 所以有一个很重要的结论就是$x \wedge y = x \iff x \vee y = y$. $\square$

类似的我们可以通过一个 poset 构造 lattice.

Theorem 2.7. *Given an poset $\mathcal{P}$ with that above property，define $x \wedge y = \sup\{x, y\}$ and $x \vee y = \inf\{x, y\}$. Then $(P, \wedge, \vee)$ is a lattice.*