Lattice

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目录

The Elements of Universal Algebra

Definition and Examples of Algebras

One of the aims of universal algebra is to extract, whenever possible, the common elements of several seemingly different types of algebraic structures.

Definition 1.1. For A a nonempty set and n a nonnegative integer we define $A^0 = \{\emptyset\}$, and, for $n > 0$, A^n is the set of n-tuples of elements from A. An n-ary operation (or function) on A is any function f from A^n to A; n is the arity (or rank) of f. A finitary operation is an n-ary operation, for some n. The image of $\langle a_1, \cdots, a_n \rangle$ under an n-ary operation f is denoted by $f(a_1, \cdots, a_n)$. An operation f on A is called a nullary operation (or constant) if its arity is zero; it is completely determinded by the image $f(\emptyset)$ in A of the only element $\emptyset$ in A^0, and as such it is convenient to identify it with the element $f(\emptyset)$. Thus a nullary operation is thought of as an element of A. An operation f on A is unary, binary, or ternay if its arity is 1,2, or 3, respectively.

Definition 1.2. A language (or type) of algebras is a set $\mathcal{F}$ of function symbols such a nonnegative integer n is assgined to each member f of $\mathcal{F}$. This integer is called the arity (or rank) of f, and f is said to be an n-ary function symbol. The subset of n-ary function symbols in $\mathcal{F}$ is denoted by $\mathcal{F}_n$.

Definition 1.3. If $\mathcal{F}$ is a language of algebras then an algebra $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair $\langle A, F \rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the language $\mathcal{F}$ such that corresponding to each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the universe (全域) (or underlying set) of $\mathbf{A} = \langle A, F \rangle$, and the $f^{\mathbf{A}}$ ' s are called the fundamental operations (基本运算) of $\mathbf{A}$. (In practice we prefer to write just f for $f^{\mathbf{A}}$ —this convention creates an ambiguity which seldom causes a problem. However, in this chapter we will be unusually careful.) If $\mathcal{F}$ is finite, say $\mathcal{F} = \{f_1, \cdots, f_k\}$, we often write $\langle A, f_1, \cdots, f_k \rangle$ for $\langle A, F \rangle$, usually adopting the convention:

$$arity f_1 \geq \cdots \geq arity f_k.$$

An algebra $\mathbf{A}$ is unary if all of its operations are unary, and it is mono-unary if it has just on unary operation.

抽象来说一个 algebra 就是一个集合和一堆 operations 构成的, 在目前已经学到的代数中 operation 的 arity 大多数不会超过 2(够 modern).

Example 1.4. $\mathbf{A}$ is a groupoid if it has just one binary operation; this operation is usually denoted by $+$ or $\cdot$, and we write $a + b$ or $a \cdot b$ for the image of $\langle a, b \rangle$ under this operation, and call it the sum or product of a and b, respectively.

Definition 1.5. An alegebra $\mathbf{A}$ is finite if $|\mathbf{A}|$ is finite, and trivial if $|\mathbf{A}| = 1$.

Example 1.6. Some well-known algebras

- Group
- Semigroup (半群) and Monoid (幺半群)
- ...

Isomorphic Algebras and Subalgebras

Definition 1.7. Let **A** and **B** be two algebras of the same type $\mathcal{F}$. Then a function $\alpha\colon A\to B$ is an isomorphism from **A** to **B** if α is one-to-one and onto, and for every n-ary $f \in \mathbf{F}$, and for $a_1, \cdots, a_n \in A$, we have

$$\alpha f^{\mathbf{A}}(a_1, \cdots, a_n) = f^{\mathbf{B}}(\alpha a_1, \cdots, \alpha a_n).$$

We say **A** is isomorphic to B, if there is a isomorphism from **A** to **B**.

老样子元素上保持 bijective，对应的运算结果也保持一致.

Definition 1.8. Let **A** and **B** be two algebras of the same type. Then **B** is a subalgebra of A if $B \subseteq A$ and every fundamental operation of **B** is the restriction of the corresponding operation of **A**, i.e., for each function symbol f, $f^{\mathbf{B}}$ is $f^{\mathbf{A}}$ restricted to B; we write simply $\mathbf{B} \leq \mathbf{A}$.

A subuniverse of **A** is a subset B of A which is closed under the fundamental operations of **A**, i.e., if f is a fundamental n-ary operation of **A** and $a_1, \cdots, a_n \in B$ we wound required $f(a_1, \cdots, a_n) \in B$.

这个 restriction 就是只对定义域做限制的意思，subalgebra 是一种构造新代数的方法，后面会学到另外几种.

Definition 1.9. Let **A** and **B** be of the same type. A function $\alpha\colon A\to B$ is an embedding of **A** into **B** if α is one-to-one and satisfies

$$\alpha f^{\mathbf{A}}(a_1, \cdots, a_n) = f^{\mathbf{B}}(\alpha a_1, \cdots, \alpha a_n).$$

Such an α is also called a monomorphism. We say **A** can be embedded in **B** if there is an embedding of **A** into **B**.

单纯地去掉 surjective.

Theorem 1.10. *If $\alpha\colon A\to B$ is an embedding, then $\alpha(A)$ is a subuniverse of $\mathbb{B}$.*

证明. 我们需要说 $\alpha(A)$ 在 n-ary operation 下保持封闭. 因为 α 是应 embedding，所以对应一个 n-ary operation f 和 $a_1, \cdots, a_n \in A$ 有

$$f^{\mathbf{B}}(\alpha a_1, \cdots, \alpha a_n) = \alpha f^{\mathbf{A}}(a_1, \cdots, c_n) \in \alpha(A).$$

已经证闭. □

Algebraic Lattices and Subuniverses

这一章阐述 algebraic lattice 出现在 universe algebra 的原因.

Definition 1.11. Given an algebra $\mathbf{A}$ define, for every $X \subseteq A$,

$$\mathrm{Sg}(X) = \bigcap \{\, B \mid X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathbf{A} \,\}.$$

We read $\mathrm{Sg}(X)$ as "the subuniverse generated by X".

Definition 1.12. A closure operator C on the set A is an algebraic closure operator if for $X \subseteq A$

$$C(X) = \bigcup \{\, C(Y) \mid Y \subseteq X \text{ and } Y \text{ is finite} \,\}.$$

Theorem 1.13. If we are given an algebra $\mathbf{A}$, then Sg is an algebraic closure operator on A.

证明. 很明显任意地 subuniverses 交还是一个 subuniverse，所有的 subuiverses 构成了一个 closure system，所以 Sg 是一个 closure operator. 对于任意的 $X \subseteq A$ 我们定义

$$E(X) = X \cup \{\, f(a_1, \cdots, a_n) \mid f \text{ is a fundamental } n\text{-ary operation on } A \text{ and } a_1, \cdots, a_n \in X \,\}.$$

然后定义它的 n 次复合 $E^n(X)$ 为

$$E^0(X) = X$$
$$E^{n+1}(X) = E(E^n(X)).$$

由于 A 上所有 fundamental operation 都是 finitary, 且有

$$X \subseteq E(X) \subseteq \cdots \subseteq E^n(X).$$

接下来我们来证明下面的式子

$$\mathrm{Sg}(X) = X \cup E(X) \cup E^2(X) \cup \cdots .$$

思路是 (1)$\mathrm{Sg}(X) \subseteq X \cup E(X) \cup E^2(X) \cup \cdots$ 和 (2)$X \cup E(X) \cup E^2(X) \cup \cdots \subseteq \mathrm{Sg}(X)$.

(1) 我们从 $X \cup E(X) \cup E^2(X) \cup \cdots$ 这种构造出发，实际上这种构造得到的就是包含 X 的最小 subuniverse. 考虑它的 bound，可能存在某个最小的 k 使得 $E^{k+1}(E^k(X)) = E^k(X)$，那么当 $i > k$ 时都有 $E^i(X) = E^k(X)$，无疑问现在 $E^k(X)$ 是一个 subuniverse 并且包含 X，这其实就说明了 $\mathrm{Sg}(X) \subseteq E^k(X)$. 如果你不能找到这样的 k，那么会有若 $i > k$，那么 $E^k(X) \subset E^i(X)$，但是它是 bound by A，且 $\mathrm{Sg}(A) = A$，这是显然地.

(2) 任取 $x \in \mathrm{Sg}(X)$, 我们用 $\{Z_i\}_{i \in I}$ 表示所有 such that $X \subseteq Z$ 且 Z 是 $\mathbf{A}$ 的一个 subuniverse. 那么对任意的 i，有 $x \in Z_i$. 那么我们更有

$$\mathrm{Sg}(X) = E^n(Z_1) \cap E^n(Z_2) \cap \cdots .$$

对任意的 n 成立. 并且我们还有

$$E^n(X) \subseteq E^n(Z_1) \cap E^n(Z_2) \cap \cdots .$$

这是因为每一项都有 $E^n(X) \subseteq E^n(Z_i)$. 那么我们把 n 取遍，就可以得到

$$X \cup E(X) \cup E^2(X) \cup \cdots \subseteq \mathrm{Sg}(X).$$

综上 $\mathrm{Sg}(X) = X \cup E(X) \cup E^2(X) \cup \cdots$.

那么对于任意的 $a \in \mathrm{Sg}(X)$，都有 $a \in E^n(X)$ 其中 $n < \omega$ (omega 表示无穷远它比任何的自然数都大). 那么存在 finite $Y \subseteq X$，使得 $a \in E^n(X)$. 这个过程就是把生成 a 的哪些元素在 $E^{n-1}(X)$ 里面挑出来，然后循环往复，n 是 finite 的，所以 Y 也是有限的. 这就说明了 $\mathrm{Sg}(X) = \bigcap \mathrm{Sg}(Y_i)$，因为每一个 a 都可以找到一个 finite Y. $\square$

Corollary 1.14. If $\mathbf{A}$ is an algebra then $\mathcal{L}_{Sg}$, the lattice of subuniverses of $\mathbf{A}$, is an algebraic lattice.