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A new view of relationship between atomic posets and complete (algebraic) lattices

Abstract: In the context of the atomic poset, we propose several new methods of constructing the complete lattice and the algebraic lattice, and the mutual decision of relationship between atomic posets and complete lattices (algebraic lattices) is studied.

1 Introduction

Order theory can formally be seen as a subject between lattice theory [23–25, 34, 48] and graph theory [6, 22, 36]. Indeed, one can say with good reason that lattices are special types of ordered sets, which are in turn special types of directed graphs. Yet this would be much too simplistic an approach. In each theory the distinct strengths and weaknesses of the given structure can be explored. This leads to general as well as discipline specific questions and results. Of the three research areas mentioned, order theory undoubtedly is the youngest. In recent years, as order and partial ordered set theory were widely applied in the combinatorics [1, 9, 13, 37, 43], fuzzy mathematics [7, 32, 40, 42, 44], computer science [2, 39], and even in the social science [14, 15] etc.

A poset consists of a set together with a binary relation that indicates that, for certain pairs of elements in the set, one of the elements precedes the other. Such a relation is called a partial order to reflect the fact that not every pair of elements needs to be related: for some pairs, it may be that neither element precedes the other in the poset. Thus, partial orders generalize the more familiar total orders, in which every pair is related. A finite poset can be visualized through its Hasse diagram (discrete graphs), which depicts the ordering relation [35]. This area of order theory was investigated in a series of papers by Erné [16, 18] and independently by Chajda, Halaš, Larmerová, Rachůnek, Niederle [8, 26–29, 31], and later by Joshi, Kharat, Mokbel, Mundlik, Waphare [33, 45, 46] and many others. In [19], they are mainly interested in ideal-theoretic properties and various degrees of (finite or infinite) distributivity in atomic posets. However, we are more interested in atoms of atomic posets. And it is conceivable that the role of the atomic elements is very important (each element in the boolean lattice can be expressed by atomic elements i.e. $a = \bigvee\{x \in \mathcal{A}(B) | x \leq a\}$) in the Boolean lattice [11]. Similarly, atoms in atomic posets also deserve a keen attention.

In this paper, we stress the importance of the two kinds of operators (C-operator and D-operator) in the study of the theoretical aspect of atomic posets. Specifically, we first define two relation operators (C-operator and D-operator) between the non-atomic element and the atomic element, and get series of related properties. Almost

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immediately, two kinds of operators above are combined to construct complete (algebraic) lattices, and used to study the relation between atomic posets and complete (algebraic) lattices.

The work of this paper is organized as follows. We shall first briefly introduce poset and related concepts. In Section 3, two kinds of operators above are combined to construct complete lattices, and used to study the relation between atomic posets and complete lattices. In Section 4, two kinds of operators above are combined to construct algebraic lattices, and used to study the relation between atomic posets and algebraic lattices.

2 Preliminaries

By a partial order on set P we mean a binary relation $\leq$ on P which is reflexive, antisymmetric and transitive, and by a partially ordered set we mean a non-empty set P together with a partial $\leq$ on P. Less familiar is the symbol $\|$ used to denote non-comparability: we write $x \| y$ if $x \not\leq y$ and $y \not\geq x$. We say P has a bottom element if there exist $0 \in P$ (called bottom) with the property that $0 \leq x$ for all $x \in P$. An element $x \in P$ is an upper bound of S if $s \leq x$ for all $s \in L$. A lower bound is defined dually. The set of all upper bounds of L is denoted by S^u (read as "L upper") and the set of all lower bounds by L^l (read as "L lower").

Throughout this article, 0 denotes the least element in a poset.

Definition 2.1 ([11]). *Let P be an ordered set and $x, y \in P$. We say x is covered by y (y covers x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there is no element z of P with $x < z < y$.*

Observe that if P is finite, $x < y$ if and only if there exist a finite sequence of covering relations $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$. Thus, in the finite case, the order relation determines, and is determined by the covering relation.

Definition 2.2 ([25]). *A subset D of a poset P is directed provided it is nonempty and every finite subset of D has an upper bound in D.*

Definition 2.3 ([4, 11]). *Let P be a non-empty ordered set.*
 (i) *If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a lattice;*
 (ii) *If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a complete lattice.*

Definition 2.4 ([24]). *Let P and Q be ordered sets. A map $\varphi : P \to Q$ is said to be*
 (i) *order-preserving if if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q;*
 (ii) *order-embedding (and we write $\varphi : P \hookrightarrow Q$) if $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q;*
 (iii) *order-isomorphism if φ is onto and $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q.*

Definition 2.5 ([24]). *Let L and K be lattices. A map $f : L \to K$ is said to be a lattice homomorphism if f is join-preserving and meet-preserving, that is, for all $a, b \in L$,*

$$f(a \vee b) = f(a) \vee f(b) \ and \ f(a \wedge b) = f(a) \wedge f(b).$$

A bijective lattice homomorphism is a lattice isomorphism.

Proposition 2.6 ([11]). *Let L and K be lattices and $f : L \to K$ is a map. f is a lattice isomorphism if and only if it is an order-isomorphism.*

Lemma 2.7 ([11]). *Let X be a set and $\mathcal{L}$ be a family of subsets of X, ordered by inclusion, such that*
 (i) *$\bigcap_{i \in I} \in \mathcal{L}$ for every non-empty family $\{A_i\}_{i \in I} \subseteq \mathcal{L}$, and*
 (ii) *$X \in \mathcal{L}$.*
That is to say that $\mathcal{L}$ is a topped intersection structure on X. Then $\mathcal{L}$ is a complete lattice in which

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i,$$

$$\bigvee_{i \in I} A_i = \bigcap \{B \in \mathcal{L} | \bigcup_{i \in I} A_i \subseteq B\}.$$

Lemma 2.8 ([11]). *Let P and Q be ordered sets and $\varphi \colon P \to Q$ be an order-isomorphism map. Then φ preserves all existing joins and meets.*

Definition 2.9 ([11]). *Let L be a complete lattice and let $k \in L$.*
 (i) *k is called finite (in L), for every directed set D in L,*

$$k \leq \bigsqcup D \Rightarrow k \leq d \ for \ some \ d \in D.$$

The set of finite elements of L is denoted $F(L)$
(ii) *k is said to be compact if, for every subset S of L,*

$$k \leq \bigvee S \Rightarrow k \leq \bigvee T \ for \ some \ finite \ subset \ T \ of \ S.$$

The set of compact elements of L is denoted $K(L)$.

Lemma 2.10 ([11]). *Let L be a complete lattice. Then $F(L) = K(L)$*

Definition 2.11 ([11]). *A complete lattice L is said to be algebraic if, for each $a \in L$,*

$$a = \bigvee \{k \in K(L) | k \leq a\}.$$

Definition 2.12 ([20]). *A poset is said to be directed complete if every directed subset has a sup. A directed complete algebraic poset L is called an algebraic domain.*

Definition 2.13 ([19]). *Let P be a poset. If P has a least element 0, then $x \in P$ is called an atom if $0 \prec x$. If P has no least element, then $x \in P$ is called an atom when x is a minimal element in P.*

All finite posets are atomic. The set of atoms of P is denoted by $\mathcal{A}(P)$ and let $\mathcal{A}_0(P) = \mathcal{A}(P) \cup \{0\}$ and $P_0 = P \setminus \mathcal{A}_0(P)$. The poset P is called atomic if, given $a(\neq 0)$ in P, there exists $x \in \mathcal{A}(P)$ such that $x \leq a$.

In atomic posets, according to the relationship between the non-atomic element and the atomic element, We define the relation operator naturally, called C-operator and D-operator.

Definition 2.14. *Let P be an atomic poset.*
(1) *(C-operator) $\forall a \in P_0$, denote $C_a = \{x \in \mathcal{A}(P) | x \leq a\}$;*
(2) *(D-operator) $\forall b \in \mathcal{A}(P)$, denote $D_b = \{x \in P_0 | b \leq x\}$.*

C-operator and D-operator can be viewed as operators between $\mathcal{A}(P)$ and P_0. Naturally, we can define $C_A = \bigcup_{a \in A} C_a$ for any $A \subseteq P_0$, and $D_B = \bigcup_{b \in B} D_b$ for any $B \subseteq P_0$. Then we can define two kinds of operators between $\mathcal{A}(P)$ and P_0 via C_a and D_b.

Based on C-operators and D-operators, we generate several new relational operators as follows:

Definition 2.15. *Let P be an atomic poset.*
(1) *(C^o-operator) $\forall A \subseteq P_0$, denote $C_A^o = \{x \in \mathcal{A}(P) | D_x \subseteq A\}$;*
(2) *(D^o-operator) $\forall B \subseteq \mathcal{A}(P)$, denote $D_B^o = \{x \in P_0 | C_x \subseteq B\}$;*
(3) *($\overline{C}$-operator) $\forall A \subseteq P_0$, denote $\overline{C}_A = \{x \in \mathcal{A}(P) | A \subseteq D_x\}$;*
(4) *($\overline{D}$-operator) $\forall B \subseteq \mathcal{A}(P)$, denote $\overline{D}_B = \{x \in P_0 | B \subseteq C_x\}$.*

It is obvious that $\overline{C}_A = \bigcap\limits_{a \in A} C_a$, $\overline{D}_B = \bigcap\limits_{b \in B} D_b$. In Definition 2.14 and 2.15, these operators are very reasonable. Then, we study some properties of several operators.

Proposition 2.16. *Let P be an atomic poset. $A, A_i \subseteq P_0$, $B, B_i \subseteq \mathcal{A}(P)$, for $i \in I$, $a, b \in P_0$, $c \in \mathcal{A}(P)$. Then*

(1) $a \le b \Rightarrow C_a \subseteq C_b$;

(2) $A_1 \subseteq A_2 \Rightarrow C_{A_1} \subseteq C_{A_2}$, $B_1 \subseteq B_2 \Rightarrow D_{B_1} \subseteq D_{B_2}$;

(3) $A_1 \subseteq A_2 \Rightarrow C^o_{A_1} \subseteq C^o_{A_2}$, $B_1 \subseteq B_2 \Rightarrow D^o_{B_1} \subseteq D^o_{B_2}$;

(4) $A_1 \subseteq A_2 \Rightarrow \overline{C}_{A_1} \supseteq \overline{C}_{A_2}$, $B_1 \subseteq B_2 \Rightarrow \overline{D}_{B_1} \supseteq \overline{D}_{B_2}$;

(5) $a \in \overline{D}_{C_a}, c \in \overline{C}_{D_c}, A \subseteq \overline{D}_{\overline{C}_A}, B \subseteq \overline{C}_{\overline{D}_B}$;

(6) $A \subseteq D^o_{C_A}, C_{D^o_B} \subseteq B$;

(7) $\overline{C}_A = \overline{C}_{\overline{D}_{\overline{C}_A}}$, $\overline{D}_B = \overline{D}_{\overline{C}_{\overline{D}_B}}$;

(8) $D^o_{C_{D^o_B}} = D^o_B$, $C_{D^o_{C_A}} = C_A$;

(9) $\overline{C}_{\bigcup\limits_{i \in I} A_i} = \bigcap\limits_{i \in I} \overline{C}_{A_i}$, $\overline{D}_{\bigcup\limits_{i \in I} B_i} = \bigcap\limits_{i \in I} \overline{D}_{B_i}$;

(10) $A \subseteq \overline{D}_B \Leftrightarrow \overline{C}_A \supseteq B$.

Proof. We consider cases of (6), (7), (8), (10), and the other proofs are similar.

(6) (i) $D^o_{C_A} = \{x \in P_0 | C_x \subseteq C_A\}$. If $a \in A$, then there must be $C_x \subseteq C_A$. So $a \in D^o_{C_A}$ and therefore $A \subseteq D^o_{C_A}$.

(ii) $D^o_B = \{x \in P_0 | C_x \subseteq B\}$, $C_{D^o_B} = \bigcup\limits_{x \in D^o_B} C_x$. If $b \in C_{D^o_B}$, then there exists $x_0 \in D^o_B$ such that $b \in C_{x_0} \subseteq B$. So $b \in B$ and therefore $C_{D^o_B} \subseteq B$.

(7) Since $A \subseteq \overline{D}_{\overline{C}_A}$, then $\overline{C}_A \supseteq \overline{C}_{\overline{D}_{\overline{C}_A}}$. Since $B \subseteq \overline{C}_{\overline{D}_B}$, then $\overline{C}_A \subseteq \overline{C}_{\overline{D}_{\overline{C}_A}}$. Therefore $\overline{C}_A = \overline{C}_{\overline{D}_{\overline{C}_A}}$.

(8) Since $C_{D^o_B} \subseteq B$, then $D^o_{C_{D^o_B}} \subseteq D^o_B$. Since $D^o_{C_A} \supseteq A$, then $D^o_{C_{D^o_B}} \supseteq D^o_B$. Therefore $D^o_{C_{D^o_B}} = D^o_B$.

(10) ($\Rightarrow$) Since $A \subseteq \overline{D}_B = \{x \in P_0 | B \subseteq C_x\}$, then $\forall a \in A$, $B \subseteq C_a$ and so $B \subseteq \bigcap\limits_{a \in A} C_a = \overline{C}_A$.

($\Leftarrow$) Since $B \subseteq \overline{C}_A = \{x \in \mathcal{A}(P) | A \subseteq D_x\}$, then $\forall b \in B$, $A \subseteq D_b$ and so $A \subseteq \bigcap\limits_{b \in B} D_b = \overline{D}_B$. $\square$

3 Constructing complete lattices and mutual decision

The construction of complete lattice [11, 12, 17, 30, 38, 49] is very essential branch in the research of various order structures. In 3.1 and 3.2, several operators ($C_A, \overline{C}_A, \overline{D}_B$ and D^o_B) are worked on atomic posets, and then a complete lattice is generated. Subsequently, we find that complete lattices and posets are mutually corresponding. Thus, in the theoretical study of posets, we can see the crucial role of the content of this section.

3.1 Complete lattices via $(\overline{C}_A, \overline{D}_B, =)$

Let P be an atomic poset. A pair (A, B) satisfies $A \subseteq P_0$, $B \subseteq \mathcal{A}(P)$, $\overline{C}_A = \{x \in \mathcal{A}(P) | A \subseteq D_x\} = B$ and $\overline{D}_B = \{a \in P_0 | B \subseteq C_x\} = A$. This implies $A = \overline{D}_{\overline{C}_A}$, $B = \overline{C}_{\overline{D}_B}$. The set of all those pairs of P is denoted by $\mathcal{B}(P)$.

For pairs (A_1, B_1) and (A_2, B_2) in $\mathcal{B}(P)$ we write $(A_1, B_1) \le (A_2, B_2)$ if $A_1 \subseteq A_2$. Also $A_1 \subseteq A_2$ implies $\overline{C}_{A_1} \supseteq \overline{C}_{A_2}$, and the reverse implication is also valid, so $A_1 = \overline{D}_{\overline{C}_{A_1}}$ and $A_2 = \overline{D}_{\overline{C}_{A_2}}$. We therefore have

$$(A_1, B_1) \le (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2.$$

We can then see easily that the relation $\le$ is an order on $\mathcal{B}(P)$. As we can see in Theorem 3.1, $< \mathcal{B}(P); \le >$ is a complete lattice.

Theorem 3.1. *Let P be an atomic poset. Then $< \mathcal{B}(P); \leq>$ is a complete lattice in which join and meet are given by*

$$\bigvee_{i \in I} (A_i, B_i) = (\overline{D}_{\bigcap_{i \in I} \overline{C}_{A_i}}, \bigcap_{i \in I} B_i)$$

$$\bigwedge_{i \in I} (A_i, B_i) = (\bigcap_{i \in I} A_i, \overline{C}_{\bigcap_{i \in I} \overline{D}_{B_i}}).$$

Proof. Define $\mathcal{B}(P_0) := \{A \subseteq P_0 | A = \overline{D}_{\overline{C}_A}\}$. The map $\mu : (A, B) \to A$ gives an order-isomorphism between $\mathcal{B}(P)$ and $\mathcal{B}(P_0)$.

We shall prove that $\mathcal{B}(P_0)$ is a topped intersection structure. Let $A_i \in \mathcal{B}(P_0)$ for $i \in I$. Then $\overline{D}_{\overline{C}_{A_i}} = A_i$ for each i. By (5) in Proposition 2.16

$$\bigcap_{i \in I} A_i \subseteq \overline{D}_{\overline{C}_{\bigcap_{i \in I} A_i}}$$

Also $\bigcap_{i \in I} A_i \subseteq A_i$ for all $i \in I$, which, by (4) in Proposition 2.16, implies that

$$\overline{D}_{\overline{C}_{\bigcap_{i \in I} A_i}} \subseteq \overline{D}_{\overline{C}_{A_i}} = A_i \ \text{for all } i \in I,$$

whence

$$\overline{D}_{\overline{C}_{\bigcap_{i \in I} A_i}} \subseteq \bigcap_{i \in I} A_i.$$

Therefore $\overline{D}_{\overline{C}_{\bigcap_{i \in I} A_i}} = \bigcap_{i \in I} A_i$ and hence $\bigcap_{i \in I} A_i \in \mathcal{B}(P_0)$. Also, $P_0 \subseteq \overline{D}_{\overline{C}_{P_0}}$. So that $P_0 = \overline{D}_{\overline{C}_{P_0}}$, which shows that $\mathcal{B}(P_0)$ is topped.

By Theorem 2.7, $\mathcal{B}(P_0)$ is a complete lattice in which meet is given by intersection. A formula for the join is given in Theorem 2.7 but we shall proceed more directly. We claim that

$$\bigvee_{i \in I} A_i = \overline{D}_{\overline{C}_{\bigcup_{i \in I} A_i}} = \overline{D}_{\bigcap_{i \in I} \overline{C}_{A_i}}.$$

Let $A = \overline{D}_{\bigcap_{i \in I} \overline{C}_{A_i}}$. Certainly $A = \overline{D}_{\overline{C}_A}$, by (7) in Proposition 2.16, and $\bigcup_{i \in I} A_i \subseteq A$, by (2) in Proposition 2.16. Hence A is an upper bound for $\{A_i\}_{i \in I}$ in $\mathcal{B}(P_0)$. Also, if X is an upper bound in $\mathcal{B}(P_0)$ for $\{A_i\}_{i \in I}$, then

$$\bigcup_{i \in I} A_i \subseteq X \Rightarrow A \subseteq \overline{D}_{\overline{C}_X} = X.$$

Therefore A is indeed the required join. We may now appeal to Theorem 2.8 to deduce that $\mathcal{B}(P)$ is a complete lattice in which joins and meets are given by

$$\bigvee_{i \in I} (A_i, B_i) = (\overline{D}_{\bigcap_{i \in I} \overline{C}_{A_i}}, \bigcap_{i \in I} B_i)$$

$$\bigwedge_{i \in I} (A_i, B_i) = (\bigcap_{i \in I} A_i, \overline{C}_{\bigcap_{i \in I} \overline{D}_{B_i}}). \qquad \square$$

Theorem 3.2. *Let P be an atomic poset and $\mathcal{B}(P)$ be the complete lattice by Theorem 3.1. Then $\varphi(P_0)$ is join-dense in $\mathcal{B}(P)$ and $\varphi(\mathcal{A}(P))$ is meet-dense in $\mathcal{B}(P)$.*

Proof. Let $(A, B) \in \mathcal{B}(P)$. Then

$$\bigvee \varphi(A) = \bigvee_{g \in A} \varphi(g)$$

$$= \bigvee_{g \in A} (\overline{D}_{C_g}, C_g)$$

$$= (\overline{D}_{\bigcap_{g \in A} \overline{C}_{\overline{D}_{C_g}}}, \bigcap_{g \in A} C_g)$$

$$=(\overline{D}_{\bigcap_{g\in A} C_g}, \bigcap_{g\in A} C_g)$$

However, according to Definition 2.15

$$\bigcap_{g\in A} C_g = \overline{C}_{\bigcup_{g\in A}\{g\}} = \overline{C}_A = B.$$

Since (A, B) and $\bigvee \varphi(A)$ are elements of $\mathcal{B}(P)$ with the same second coordinate, $\bigvee \varphi(A) = (A, B)$. Consequently $\varphi(P_0)$ is join-dense in $\mathcal{B}(P)$ and $\varphi(\mathcal{A}(P))$ is meet-dense in $\mathcal{B}(P)$. □

Theorem 3.3. *For every complete lattice L, there is an atomic poset P such that L is order-isomorphic to $\mathcal{B}(P)$.*

Proof. Suppose $(L, \sqsubseteq)$ is a complete lattice. Define the atomic poset $P = \{0\}\bigcup \mathcal{A}(P)\bigcup P_0$, where $\mathcal{A}(P) = L$ and $P_0 = L$. Further, in P, let $a\|b$ for $\forall a, b \in \mathcal{A}(P)$; let $a \leq b$ iff $a \sqsubseteq b$ for $\forall a \in \mathcal{A}(P), b \in P_0$; let $a \leq b$ iff $a\|b$ for $\forall a, b \in P_0$. As L is a lattice, it is easy to see that P is an atomic poset. We want to show that $(L, \sqsubseteq)$ is order-isomorphic to $\mathcal{B}(P)$.

First note that for any $X \subseteq P_0$, we have

$$\begin{aligned}
\overline{C}_X &= \{b \in \mathcal{A}(P)|X \subseteq D_b\}\\
&= \{b \in \mathcal{A}(P)|\forall x \in X, b \leq x\}\\
&= \{b \in L|\forall x \in X, b \sqsubseteq x\}\\
&= \bigcap_{x\in X} \{\downarrow_L(x)\}\\
&= \downarrow_L(\bigwedge X)
\end{aligned}$$

Among them, $\downarrow_D(x)$ means the upper set of x in L. On the other hand, for any $Y \subseteq \mathcal{A}_P$,

$$\begin{aligned}
\overline{D}_Y &= \{a \in P_0|Y \subseteq C_a\}\\
&= \{a \in P_0|\forall y \in Y, y \leq a\}\\
&= \{a \in L|\forall y \in Y, y \sqsubseteq a\}\\
&= \bigcap_{y\in Y} \{\uparrow_L(y)\}\\
&= \uparrow_L(\bigvee Y)
\end{aligned}$$

Therefore, $X \in \mathcal{B}(P_0)$ iff $\overline{D}_{\overline{C}_P} = X$, or

$$\uparrow_L(\bigvee(\downarrow_L(\bigwedge X))) = X$$

Since $\uparrow_L \bigvee(\downarrow_L \bigwedge X) = \uparrow_L(\bigwedge X)$, hence, $X \subseteq \mathcal{B}(P_0)$ iff $X = \uparrow_L(\bigwedge X)$. In other words, $\mathcal{B}(P_0)$ are precisely the up-closed subsets of L generated by a single element. Hence, a subset of P_0 belongs to $\mathcal{B}(P_0)$ if and only if it is a principal filter.

The mapping $x \mapsto \uparrow x$ provides an order-isomorphism between L and $\mathcal{B}(P_0)$. Since $\mathcal{B}(P_0)$ is isomorphic to $\mathcal{B}(P)$, therefore $(L, \sqsubseteq)$ is order-isomorphic to $\mathcal{B}(P)$. □

Example 3.4. *Let $L = \{a, b, c, d\}$ be a complete lattice, the Hasse diagram of L is illustrated by Figure 1. We can get an atomic poset P by Theorem 3.3, whose Hasse diagram is illustrated by Figure 1, and can also get a complete lattice $\mathcal{B}(P)$ which is isomorphic to L by Theorem 3.1. In Figure 1, a_1 and a_2 in P is a in L, b_1 and b_2 in P is b in L, c_1 and c_2 in P is c in L, d_1 and d_2 in P is d in L. In $\mathcal{B}(P)$, $A = (\{d_2\}, \{a_1, b_1, c_1, d_1\})$, $B = (\{b_2, d_2\}, \{a_1, b_1\})$, $C = (\{c_2, d_2\}, \{a_1, c_1\})$, $D = (\{a_2, b_2, c_2, d_2\}, \{a_1, b_1, c_1, d_1\})$.*
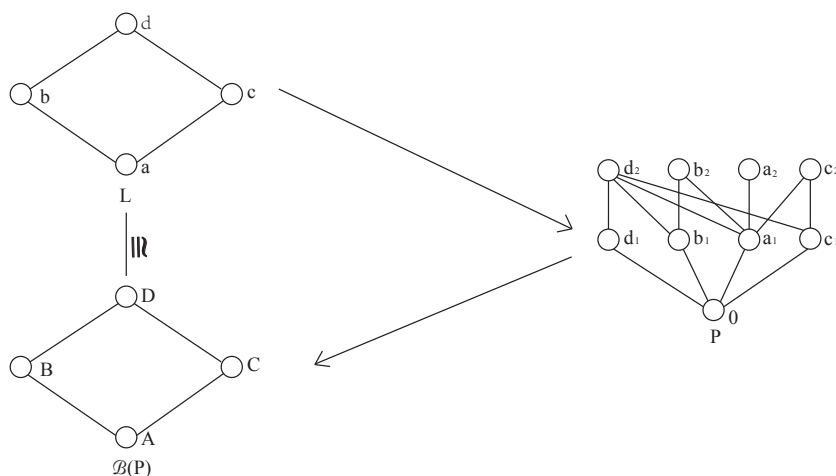
3.2 Complete lattices via $(C_A, D_B^o, =)$

Let P be an atomic poset. A set A satisfies $A \subseteq P_0$, $D_{C_A}^o = A$. The set of all those sets of P is denoted by $\mathcal{B}^o(P)$.

For set A_1 and A_2 in $\mathcal{B}^o(P)$ we write $A_1 \leq A_2$ if $A_1 \subseteq A_2$. We can then see easily that the relation $\leq$ is an order on $\mathcal{B}^o(P)$. As we see in Theorem 3.5, $< \mathcal{B}^o(P); \leq>$ is a complete lattice.

Theorem 3.5. *Let P be an atomic poset. Then $< \mathcal{B}^o(P); \leq>$ is a complete lattice in which join and meet are given by*

$$\bigvee_{i \in I} A_i = D_{\bigcup\limits_{i \in I} C_{A_i}}^o, \quad \bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i.$$

Proof. We shall prove that $\mathcal{B}^o(P)$ is a topped intersection structure. Let $A_i \in \mathcal{B}^o(P)$ for $i \in I$. Then $D_{C_{A_i}}^o = A_i$ for each i. By (6) in Proposition 2.16

$$\bigcap_{i \in I} A_i \subseteq D_{C_{\bigcap\limits_{i \in I} A_i}}^o$$

Also $\bigcap\limits_{i \in I} A_i \subseteq A_i$ for all $i \in I$, which, by (3) in Proposition 2.16, implies that

$$D_{C_{\bigcap\limits_{i \in I} A_i}}^o \subseteq D_{C_{A_i}}^o = A_i \ for \ all \ i \in I,$$

whence

$$D_{C_{\bigcap\limits_{i \in I} A_i}}^o \subseteq \bigcap_{i \in I} A_i.$$

Therefore $D_{C_{\bigcap\limits_{i \in I} A_i}}^o = \bigcap\limits_{i \in I} A_i$ and hence $\bigcap\limits_{i \in I} A_i \in \mathcal{B}^o(P)$. Also, $P_0 \subseteq D_{C_{P_0}}^o$. So that $P_0 = D_{C_{P_0}}^o$, which shows that $\mathcal{B}^o(P)$ is topped.

By Theorem 2.7, $\mathcal{B}^o(P)$ is a complete lattice in which meet is given by intersection. A formula for the join is given in Theorem 2.7 but we shall proceed more directly. We claim that

$$\bigvee_{i \in I} A_i = D_{C_{\bigcup\limits_{i \in I} A_i}}^o = D_{\bigcup\limits_{i \in I} C_{A_i}}^o.$$

Let $A = D_{\bigcup\limits_{i \in I} C_{A_i}}^o$. Certainly $A = D_{C_A}^o$, by (8) in Proposition 2.16, and $\bigcup\limits_{i \in I} A_i \subseteq A$, by (2) in Proposition 2.16. Hence A is an upper bound for $\{A_i\}_{i \in I}$ in $\mathcal{B}^o(P)$. Also, if X is an upper bound in $\mathcal{B}^o(P)$ for $\{A_i\}_{i \in I}$, then

$$\bigcup_{i \in I} A_i \subseteq X \Rightarrow A \subseteq D_{C_X}^o = X.$$

Therefore A is indeed the required join. We may now appeal to Theorem 2.8 to deduce that $\mathcal{B}^o(P)$ is a complete lattice in which joins and meets are given by

$$\bigvee_{i \in I} A_i = D^o_{\bigcup_{i \in I} C_{A_i}}, \quad \bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i. \qquad \square$$

Theorem 3.6. *For every complete lattice L, there is an atomic poset P such that L is order-isomorphic to $\mathcal{B}^o(P)$.*

Proof. Suppose $(L, \le)$ is a complete lattice. Define the atomic poset $P = \mathcal{A}(P) \bigcup P_0$, where $\mathcal{A}(P) = L$ and $P_0 = L$. Further, in P,

(1) let $a \| b$ for $\forall a, b \in \mathcal{A}(P)$;

(2) let $a \le b$ for $\forall a \in \mathcal{A}(P), b \in P_0$ iff $a = b$ in case that a is the least element in L, $a \le b$ in case that b is the largest element in L, in other cases $a \not\ge b$ in L;

(3) let $a \le b$ for $\forall a, b \in P_0$ iff $a \le b$ in L.

Under the order relation defined in P, it is easy to see that P is an atomic poset. We want to show that $(L, \le)$ is order-isomorphic to $\mathcal{B}(P)$.

First note that for any $X \subseteq P_0$ which does not contain the least and largest elements, we have

$$
\begin{aligned}
C_X &= \{b \in \mathcal{A}(P) | \exists x \in X, b \le x\} \\
&= \{b \in L | \exists x \in X, b \not\ge x\} \\
&= L \setminus \{b \in L | \forall x \in X, b \ge x\} \\
&= L \setminus \bigcap_{x \in X} \uparrow_L(x) \\
&= L \setminus \uparrow_L(\bigvee X)
\end{aligned}
$$

Among them, $\uparrow_L(x)$ means the upper set of x in L. On the other hand,

$$
\begin{aligned}
D^o_{C_X} &= \{a \in P_0 | C_a \subseteq C_X\} \\
&= \{a \in L | L \setminus \uparrow a \subseteq L \setminus \uparrow_L(\bigvee X)\} \\
&= \{a \in L | \uparrow a \supseteq \uparrow(\bigvee X)\} \\
&= \{a \in L | a \le \bigvee X\} \\
&= \downarrow_L(\bigvee X)
\end{aligned}
$$

If $X (\subseteq P_0)$ which contains the least or largest elements, we can easily check that $D^o_{C_X} = \downarrow(\bigvee X)$. Therefore, we have $\{D^o_{C_X} | X \subseteq P_0\} = \{\downarrow x | x \in L\}$. Hence, $X \subseteq \mathcal{B}^o(P)$ iff $X = \downarrow_L (\bigvee X)$. In other words, $\mathcal{B}^o(P)$ are precisely the lower sets of L generated by a single element. It is obvious that $\mathcal{B}^o(P)$ is isomorphic to L. $\qquad \square$

Example 3.7. *Let $L = \{a, b, c, d\}$ be a complete lattice, the Hasse diagram of L is illustrated by Figure 2. We can get an atomic poset P by Theorem 3.6, whose Hasse diagram is illustrated by Figure 2, and can also get a complete lattice $\mathcal{B}^o(P)$ which is isomorphic to L by Theorem 3.5. In Figure 2, a_1 and a_2 in P is a in L, b_1 and b_2 in P is b in L, c_1 and c_2 in P is c in L, d_1 and d_2 in P is d in L. In $\mathcal{B}^o(P)$, $A = \{a_2\}$, $B = \{a_2, b_2\}$, $C = \{a_2, c_2\}$, $D = \{a_2, b_2, c_2, d_2\}$.*

Fig. 2. The figure in Example 3.7
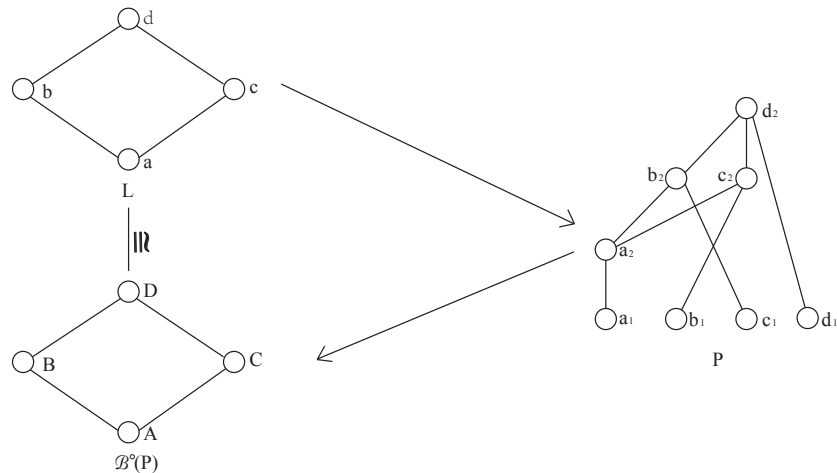
4 Constructing algebraic lattices and mutual decision

The construction of algebraic lattice [21, 41, 47] is very essential branch in the research of various order structures. In 4.1 and 4.2, Several operators(C_A, $\overline{C}_A$, $\overline{D}_B$ and D_B^o) are worked on atomic posets, and then a algebraic lattice is generated. Subsequently, we find that algebraic lattices and posets are mutually corresponding. Thus, in the theoretical study of posets, we can see the crucial role of the content of this section.

4.1 Algebraic lattices via $(\overline{C}_A, \overline{D}_B, \subseteq)$

Let P be an atomic poset. A set A satisfies $A \subseteq P_0$ and for every finite subset $X \subseteq A$, $\overline{D}_{\overline{C}_X} \subseteq A$. The set of all those sets of P is denoted by $\mathcal{F}(P)$.

For A_1, A_2 in $\mathcal{F}(P)$ we write $A_1 \leq A_2$ iff $A_1 \subseteq A_2$. We can see easily that the relation $\leq$ is an order on $\mathcal{F}(P)$. As we can see in Theorem 4.1, $< \mathcal{F}(P); \leq>$ is an algebraic lattice.

Theorem 4.1. *Let P be an atomic poset. Then $< \mathcal{F}(P); \leq>$ forms an algebraic lattice.*

Proof. We first show that $< \mathcal{F}(P); \leq>$ is a complete lattice. To show that $< \mathcal{F}(P); \leq>$ is a complete lattice it suffices to show that $< \mathcal{F}(P); \leq>$ is a topped intersection structure by Lemma 2.7. Given any subset $T \subseteq \mathcal{F}(P)$, it suffices to show that $\bigcap T \in \mathcal{F}(P)$. Suppose X is a finite subset of $\bigcap T$. Then $X \subseteq t$ for each $t \in T$. Since each $t \in \mathcal{F}(P)$, we have $\overline{D}_{\overline{C}_X} \subseteq t$ for each $t \subseteq T$. This implies $\overline{D}_{\overline{C}_X} \subseteq \bigcap T$ and so $\bigcap T \in \mathcal{F}(P)$. It is easy to see $L_0 \in \mathcal{F}(P)$ and so $< \mathcal{F}(P); \leq>$ is a topped intersection structure.

To show that $< \mathcal{F}(P); \leq>$ is algebraic, note that $\overline{D}_{\overline{C}_X}$ is a compact element for each finite X in L_0. To see this, we first show $\overline{D}_{\overline{C}_X} \in \mathcal{F}(P)$. Let X_1 be a finite subset of $\overline{D}_{\overline{C}_X}$, then $\overline{D}_{\overline{C}_{X_1}} \subseteq \overline{D}_{\overline{C}_{(\overline{D}_{\overline{C}_X})}} = \overline{D}_{\overline{C}_X}$ by Proposition 2.16, which implies $\overline{D}_{\overline{C}_X} \in \mathcal{F}(P)$. Then let $\{A_i | i \in I\}$ be a directed subset of $\mathcal{F}(P)$ such that

$$\overline{D}_{\overline{C}_X} \subseteq \bigcup_{i \in I} A_i.$$

By Proposition 2.16, $X \subseteq \overline{D}_{\overline{C}_X}$. Therefore $X \subseteq \bigcup_{i \in I} A_i$. Since X is finite and $\{A_i | i \in I\}$ is directed, $X \subseteq A_k$ for some $k \in I$. But $A_k \in \mathcal{F}(P)$, therefore $\overline{D}_{\overline{C}_X} \subseteq A_k$. By Definition 2.9 and Lemma 2.10, $\overline{D}_{\overline{C}_X}$ is a compact element for each finite X.

Next we will show that for any $T \in \mathcal{F}(P)$, $T = \bigcup\{\overline{D}_{\overline{C}_X} | X \subseteq^{fin} T\}$. For any $X \subseteq^{fin} T$, as $T \in \mathcal{F}(P)$, we have $\overline{D}_{\overline{C}_X} \subseteq T$. Then $\bigcup\{\overline{D}_{\overline{C}_X} | X \subseteq^{fin} T\} \subseteq T$. As $X \subseteq \overline{D}_{\overline{C}_X}$, So $T = \bigcup X \subseteq \bigcup\{\overline{D}_{\overline{C}_X} | X \subseteq^{fin} T\}$. Therefore $T = \bigcup\{\overline{D}_{\overline{C}_X} | X \subseteq^{fin} T\}$. Therefore $< \mathcal{F}(P); \leq >$ forms an algebraic lattice by Definition 2.11.   □

Corollary 4.2. *Let P be a finite atomic poset. Then $\mathcal{F}(P) = \{\overline{D}_{\overline{C}_X} | X \subseteq L\}$.*

Proof. First we will show $\mathcal{F}(P) \subseteq \{\overline{D}_{\overline{C}_X} | X \subseteq P\}$. $\forall A \in \mathcal{F}(P)$, we have $A \subseteq P_0$ and for every finite subset $X \subseteq A$, $\overline{D}_{\overline{C}_X} \subseteq A$. As P is finite, we have that A is finite and $\overline{D}_{\overline{C}_A} \subseteq A$. Since $A \subseteq \overline{D}_{\overline{C}_A}$ by Proposition 2.16, therefore $A = \overline{D}_{\overline{C}_A}$. So $\mathcal{F}(P) \subseteq \{\overline{D}_{\overline{C}_X} | X \subseteq P\}$. Then we will show $\{\overline{D}_{\overline{C}_X} | X \subseteq P\} \subseteq \mathcal{F}(P)$. Since we show in Theorem 4.1, $\overline{D}_{\overline{C}_X}$ is a compact element in $< \mathcal{F}(P); \leq >$ for each finite X in P_0. Since P is finite, so $\{\overline{D}_{\overline{C}_X} | X \subseteq L\} \subseteq \mathcal{F}(P)$. Therefore $\mathcal{F}(P) = \{\overline{D}_{\overline{C}_X} | X \subseteq P\}$.   □

Theorem 4.3. *For every algebraic lattice D, there is an atomic poset P such that D is order-isomorphic to $\mathcal{F}(P)$.*

Proof. Suppose $(D, \sqsubseteq)$ is an algebraic lattice. Define a poset $(P, \leq) = \{0\} \bigcup \mathcal{A}(P) \bigcup P_0$ with $\mathcal{A}(P) = D$, $P_0 = K(D)$, where $K(D)$ stands for the set of compact elements of D. Further, in P, let $a \| b$ for $\forall a, b \in \mathcal{A}(P)$; let $a \leq b$ iff $b \sqsubseteq a$ for $\forall a \in \mathcal{A}(P), b \in P_0$; let $a \leq b$ iff $b \sqsubseteq a$ for $\forall a, b \in P_0$. As D is an algebraic lattice, it is easy to see that P is an atomic poset. We want to show that $(D, \sqsubseteq)$ is order-isomorphic to $\mathcal{F}(P)$.

First note that for any $X \subseteq P_0$, we have

$$
\begin{aligned}
\overline{C}_X &= \{b \in \mathcal{A}(P) | X \subseteq D_b\} \\
&= \{b \in \mathcal{A}(P) | \forall x \in X, x \leq b\} \\
&= \{b \in D | \forall x \in X, x \sqsubseteq b\} \\
&= \bigcup_{x \in X} \{\uparrow_D(x)\} \\
&= \uparrow_D(\bigvee X)
\end{aligned}
$$

Among them, $\uparrow_D(\bigvee X)$ means the upper set of $\bigvee X$ in D. On the other hand,

$$
\begin{aligned}
\overline{D}_Y &= \{a \in P_0 | Y \subseteq C_a\} \\
&= \{a \in P_0 | \forall y \in Y, y \leq a\} \\
&= \{a \in K(D) | \forall y \in Y, a \sqsubseteq y\} \\
&= \{a \in K(D) | \forall y \in Y, a \in \downarrow_{K(D)}(y)\} \\
&= \bigcap_{y \in Y} \{\downarrow_{K(D)}(y)\} \\
&= \downarrow_{K(D)}(\bigwedge Y)
\end{aligned}
$$

Therefore, $I \in \mathcal{F}(P)$ iff $\overline{D}_{\overline{C}_X} \subseteq I$ for any finite subset $X \subseteq^{fin} I$, or

$$
\downarrow_{K(D)}(\bigwedge(\uparrow_D(\bigvee X))) \subseteq I
$$

for each $X \subseteq^{fin} I$. Since

$$
\downarrow_{K(D)}(\bigwedge(\uparrow_D(\bigvee X))) = \downarrow_{K(D)}(\bigvee X),
$$

this is equivalent to say that I is a downward closed, directed subset of compact elements of D. A downward closed, directed subset is called an ideal. Hence, a subset of P_0 belongs to $\mathcal{F}(P)$ if and only if it is an ideal.

At last, by the classical result about algebraic domains [3]: an algebraic domain is isomorphic to the ideal completion of the poset of its compact elements through the isomorphism

$$
d \mapsto \{a \in K(D) | a \sqsubseteq d\}
$$

that is, $\mathcal{F}(P)$ is isomorphic to D.   □

Fig. 3. The figure in Example 4.4
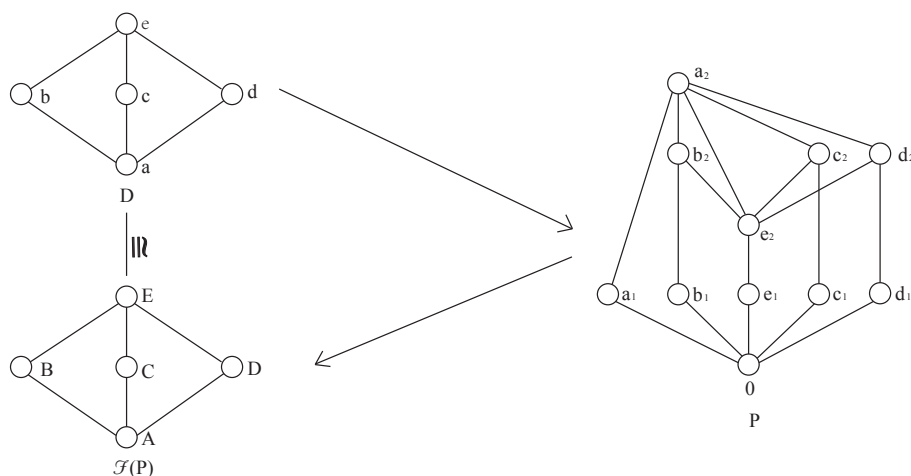
Example 4.4. *Let $D = \{a, b, c, d, e\}$ be an alegraic lattice, the Hasse diagram of D is illustrated by Figure 3. We can get an atomic poset P by Theorem 4.3, whose Hasse diagram is illustrated by Figure 3, and can also get an alegraic lattice $\mathcal{F}(P)$ which is isomorphic to D by Theorem 4.1. In Figure 3, a_1 and a_2 in P is a in D, b_1 and b_2 in P is b in D, c_1 and c_2 in P is c in D, d_1 and d_2 in P is d in D. In $\mathcal{F}(P)$, $A = \{a_2\}$, $B = \{a_2, b_2\}$, $C = \{a_2, c_2\}$, $D = \{a_2, d_2\}$, $E = \{a_2, b_2, c_2, d_2, e_2\}$.*

4.2 Algebraic lattices via $(C_A, D_B^o, \subseteq)$

Let P be an atomic poset. A set A satisfies $A \subseteq P_0$ and for every finite subset $X \subseteq A$, $D_{C_X}^o \subseteq A$. The set of all those sets of P is denoted by $\mathcal{F}^o(P)$.

For A_1, A_2 in $\mathcal{F}^o(P)$. We write $A_1 \leq A_2$ if $A_1 \subseteq A_2$. We can see easily that the relation $\leq$ is an order on $\mathcal{F}^o(P)$. As we can see in Theorem 4.5, $< \mathcal{F}^o(P); \leq>$ is an algebraic lattice.

Theorem 4.5. *Let P be an atomic poset. Then $< \mathcal{F}^o(P); \leq>$ forms an algebraic lattice.*

Proof. We first show that $< \mathcal{F}^o(P); \leq>$ is a complete lattice. To show that $< \mathcal{F}^o(P); \leq>$ is a complete lattice it suffices to show that $< \mathcal{F}^o(P); \leq>$ is a topped intersection structure by Lemma 2.7. Given any subset $T \subseteq \mathcal{F}^o(P)$, it suffices to show that $\bigcap T \in \mathcal{F}^o(P)$. Suppose X is a finite subset of $\bigcap T$. Then $X \subseteq t$ for each $t \in T$. Since each $t \in \mathcal{F}^o(P)$, we have $D_{C_X}^o \subseteq t$ for each $t \subseteq T$. This implies $D_{C_X}^o \subseteq \bigcap T$ and so $\bigcap T \in \mathcal{F}^o(P)$. It is easy to see $P_0 \in \mathcal{F}^o(P)$ and so $< \mathcal{F}^o(P); \leq>$ is a topped intersection structure.

To show that $< \mathcal{F}^o(P); \leq>$ is algebraic, note that $D_{C_X}^o$ is a compact element for each finite X in P_0. To see this, we first show $D_{C_X}^o \in \mathcal{F}^o(P)$. Let X_1 be a finite subset of $D_{C_X}^o$. Then $D_{C_{X_1}}^o \subseteq D_{C_{(D_{C_X}^o)}}^o = D_{C_X}^o$ by Proposition 2.16, which implies $D_{C_X}^o \in \mathcal{F}^o(P)$. Then let $\{A_i | i \in I\}$ be a directed subset of $\mathcal{F}^o(P)$ such that

$$D_{C_X}^o \subseteq \bigcup_{i \in I} A_i.$$

By Proposition 2.16, $X \subseteq D_{C_X}^o$. Therefore $X \subseteq \bigcup_{i \in I} A_i$. Since X is finite and $\{A_i | i \in I\}$ is directed, $X \subseteq A_k$ for some $k \in I$. But $A_k \in \mathcal{F}^o(P)$, therefore $D_{C_X}^o \subseteq A_k$. By Definition 2.9 and Lemma 2.10, $D_{C_X}^o$ is a compact element for each finite X.

Next we will show that for any $T \in \mathcal{F}^o(P)$, $T = \bigcup\{D^o_{C_X}|X \subseteq^{fin} T\}$. For any $X \subseteq^{fin} T$, as $T \in \mathcal{F}^o(P)$, we have $D^o_{C_X} \subseteq T$. Then $\bigcup\{D^o_{C_X}|X \subseteq^{fin} T\} \subseteq T$. As $X \subseteq D^o_{C_X}$, So $T = \bigcup X \subseteq \bigcup\{D^o_{C_X}|X \subseteq^{fin} T\}$. Therefore $T = \bigcup\{D^o_{C_X}|X \subseteq^{fin} T\}$. Therefore $< \mathcal{F}^o(P); \leq>$ forms an algebraic lattice by Definition 2.11. $\quad\square$

Corollary 4.6. *Let P be a finite atomic poset. Then $\mathcal{F}^o(P) = \{D^o_{C_X}|X \subseteq L\}$.*

Proof. First we will show $\mathcal{F}^o(P) \subseteq \{D^o_{C_X}|X \subseteq P\}$. $\forall A \in \mathcal{F}^o(P)$, we have $A \subseteq P_0$ and for every finite subset $X \subseteq A$, $D^o_{C_X} \subseteq A$. As P is finite, we have that A is finite and $D^o_{C_A} \subseteq A$. As $A \subseteq D^o_{C_A}$ by Proposition 2.16. Therefore $A = D^o_{C_A}$. So $\mathcal{F}^o(P) \subseteq \{D^o_{C_X}|X \subseteq P\}$. Then we will show $\{D^o_{C_X}|X \subseteq P\} \subseteq \mathcal{F}^o(P)$. As we show in Theorem 4.5, $D^o_{C_X}$ is a compact element in $< \mathcal{F}^o(P); \leq>$ for each finite X in P_0. As P is finite, so $\{D^o_{C_X}|X \subseteq L\} \subseteq \mathcal{F}^o(P)$. Therefore $\mathcal{F}^o(P) = \{D^o_{C_X}|X \subseteq P\}$. $\quad\square$

Theorem 4.7. *For every algebraic lattice D, there is an atomic poset P such that D is order-isomorphic to $\mathcal{F}^o(P)$.*

Proof. Suppose $(D, \leq)$ is an algebraic lattice. Define a poset $(P, \leq) = \mathcal{A}(P) \bigcup P_0$ with $\mathcal{A}(P) = D$, $P_0 = K(D)$, where $K(D)$ stands for the set of compact elements of D. Further, in P,

(1) let $a\|b$ for $\forall a, b \in \mathcal{A}(P)$;

(2) let $a \leq b$ for $\forall a \in \mathcal{A}(P), b \in P_0$ iff $a = b$ in the case that a is the least in D, $a \leq b$ in the case that b is the largest element in L, in other cases $a \not\geq b$ in L;

(3) let $a \leq b$ for $\forall a, b \in P_0$ iff $a \leq b$ in D.

As D is an algebraic lattice, it is easy to see that P is an atomic poset. We want to show that $(D, \leq)$ is order-isomorphic to $\mathcal{F}^o(P)$.

First note that for any $X(\subseteq P_0)$ which does not contain the least and largest elements, we have

$$
\begin{aligned}
C_X &= \{b \in \mathcal{A}(P)|\exists x \in X, b \leq x\} \\
&= \{b \in D|\exists x \in X, b \not\geq x\} \\
&= D\backslash\{b \in D|\forall x \in X, b \geq x\} \\
&= D\backslash\bigcap_{x \in X}\uparrow_D(x) \\
&= D\backslash\uparrow_D(\bigvee X)
\end{aligned}
$$

Among them, $\uparrow_D(x)$ means the upper set of x in D. On the other hand,

$$
\begin{aligned}
D^o_{C_X} &= \{a \in P_0|C_a \subseteq C_X\} \\
&= \{a \in K(D)|D\backslash\uparrow a \subseteq D\backslash\uparrow(\bigvee X)\} \\
&= \{a \in K(D)|\uparrow a \supseteq \uparrow(\bigvee X)\} \\
&= \{a \in K(D)|a \leq \bigvee X\} \\
&= \downarrow_{K(D)}(\bigvee X)
\end{aligned}
$$

If $X(\subseteq P_0)$ which contains the least or largest element, we can easily check that $D^o_{C_X} = \downarrow(\bigvee X)$. Therefore, $I \in \mathcal{F}^o(P)$ iff $D^o_{C_X} \subseteq I$ for any finite subset $X \subseteq^{fin} I$, or

$$
\downarrow_{K(D)}(\bigvee X) \subseteq I
$$

this is equivalent to say that I is a downward closed, directed subset of compact elements of D. A downward closed, directed subset is called an ideal. Hence, a subset of P_0 belongs to $\mathcal{F}^o(P)$ if and only if it is an ideal.

At last, by the classical result about algebraic domains [3]: an algebraic domain is isomorphic to the ideal completion of the poset of its compact elements through the isomorphism

$$
d \mapsto \{a \in K(D)|a \leq d\}
$$

that is, $\mathcal{F}^o(P)$ is isomorphic to D. $\quad\square$

Fig. 4. The figure in Example 4.8
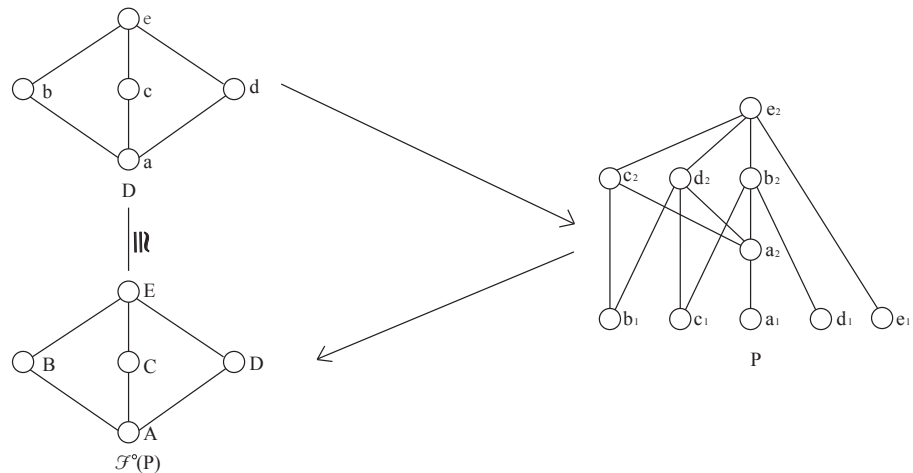
Example 4.8. *Let $D = \{a, b, c, d, e\}$ be an alegraic lattice, the Hasse diagram of D is illustrated by Figure 4. We can get an atomic poset P by Theorem 4.7, whose Hasse diagram is illustrated by Figure 4, and can also get an alegraic lattice $\mathcal{F}^o(P)$ which is isomorphic to L by Theorem 4.5. In Figure 4, a_1 and a_2 in P is a in D, b_1 and b_2 in P is b in D, c_1 and c_2 in P is c in D, d_1 and d_2 in P is d in D. In $\mathcal{F}^o(P)$, $A = \{a_2\}$, $B = \{a_2, b_2\}$, $C = \{a_2, c_2\}$, $D = \{a_2, d_2\}$, $E = \{a_2, b_2, c_2, d_2, e_2\}$.*

5 Conclusions

In this paper, to promote the research and development of completion of poset, we thoroughly study C-operators and D-operators. It is aiming at illustrating fresh methodological achievement in lattice which will also be of soaring importance in the future. We have defined C-operators and D-operators. Next, we investigate some related properties. A distinctive completion of lattice via C-operators and D-operators is followed. Our future work on this topic will focus on studying of completion and algebraization using C-operators and D-operators in poset.

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