Lattice

枫聆

2021 年 3 月 10 日

目录

The Elements of Universal Algebra

One of the aims of universal algebra is to extract, whenever possible, the common elements of several seemingly different types of algebraic structures.

Definition 1.1. For A a nonempty set and n a nonnegative integer we define $A^0 = \{\emptyset\}$, and, for $n > 0$, A^n is the set of n-tuples of elements from A. An n-ary operation (or function) on A is any function f from A^n to A; n is the arity (or rank) of f. A finitary operation is an n-ary operation, for some n. The image of $\langle a_1, \cdots, a_n \rangle$ under an n-ary operation f is denoted by $f(a_1, \cdots, a_n)$. An operation f on A is called a nullary operation (or constant) if its arity is zero; it is completely determinded by the image $f(\emptyset)$ in A of the only element $\emptyset$ in A^0, and as such it is convenient to identify it with the element $f(\emptyset)$. Thus a nullary operation is thought of as an element of A. An operation f on A is unary, binary, or ternay if its arity is 1,2, or 3, respectively.

Definition 1.2. A language (or type) of algebras is a set $\mathcal{F}$ of function symbols such a nonnegative integer n is assgined to each member f of $\mathcal{F}$. This integer is called the arity (or rank) of f, and f is said to be an n-ary function symbol. The subset of n-ary function symbols in $\mathcal{F}$ is denoted by $\mathcal{F}_n$.

Definition 1.3. If $\mathcal{F}$ is a language of algebras then an algebra $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair $\langle A, F \rangle$ where A is a nonempty set and F is a family of finitary operations on A indexed by the language $\mathcal{F}$ such that corresponding to each n-ary function symbol f in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the universe (全域) (or underlying set) of $\mathbf{A} = \langle A, F \rangle$, and the $f^{\mathbf{A}}$ ' s are called the fundamental operations (基本运算) of $\mathbf{A}$. (In practice we prefer to write just f for $f^{\mathbf{A}}$ —this convention creates an ambiguity which seldom causes a problem. However, in this chapter we will be unusually careful.) If $\mathcal{F}$ is finite, say $\mathcal{F} = \{f_1, \cdots, f_k\}$, we often write $\langle A, f_1, \cdots, f_k \rangle$ for $\langle A, F \rangle$, usually adopting the convention:

$$arity f_1 \geq \cdots \geq arity f_k.$$

An algebra $\mathbf{A}$ is unary if all of its operations are unary, and it is mono-unary if it has just on unary operation.

抽象来说一个 algebra 就是一个集合和一堆 operations 构成的, 在目前已经学到的代数中 operation 的 arity 大多数不会超过 2(够 modern).

Example 1.4. $\mathbf{A}$ is a groupoid if it has just one binary operation; this operation is usually denoted by $+$ or $\cdot$, and we write $a + b$ or $a \cdot b$ for the image of $\langle a, b \rangle$ under this operation, and call it the sum or product of a and b, respectively.

Definition 1.5. An alegebra $\mathbf{A}$ is finite if $|\mathbf{A}|$ is finite, and trivial if $|\mathbf{A}| = 1$.

Example 1.6. Some well-known algebras

- Group
- Semigroup (半群) and Monoid (幺半群)