# The homomorphism equation on semilattices

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# The homomorphism equation on semilattices

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**Abstract**

Several results concerning the homomorphism functional equation

$$f(x \vee y) = f(x) \vee f(y),$$

in the class of semilattices, as well as other similar equations, are presented in the paper.

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## 1 Introduction

The standard concepts from order and semilattice theories, which this paper deals with are covered by the initial chapters of virtually every text on these matters: [1], Chap. 1; [2], Chaps. 1 and 2; [3], Chaps. 1 and 2; [4], Chap. 1; [6], Chaps. 1-3; and [7], Chaps. 1 and 2 is a fairly incomplete list of them. However, a few conventions are in order on the terminology and notation as used below.

Two elements $x$, $y$ belonging to a partially ordered set (poset) $(P, \leq)$ are said to be *comparable* when $x \leq y$ or $y \leq x$. Otherwise we write $x \parallel y$ and the elements $x$, $y$ are said to be *parallel*. We write $x < y$ to specify that $x \leq y$ but $x \neq y$. A poset $C$ is called a *chain* when every pair of elements of $C$ is comparable. Let us denote by $C_n$ the finite chain of $n+1$ elements: $C_n = \{a_0, a_1, \ldots, a_n\}$ with the linear order $a_0 < a_1 < \cdots < a_n$. If $P$ is a poset and $a \in P$, then we write $\downarrow a := \{x \in P : x \leq a\}$, $\uparrow a := \{x \in P : a \leq x\}$ and $\parallel a = \{x \in P : x \parallel a\}$.

A function $f : P_1 \to P_2$ between the posets $(P_1, \leq)$ and $(P_2, \preceq)$ is said to be *isotone* when $f(x) \preceq f(y)$ provided that $x \leq y$, while it is called *antitone* in the case that $f(y) \preceq f(x)$. Thus, the order is preserved by isotone functions and it is reversed by antitone ones. A function which is isotone or antitone is said to be *monotone*. For example, given two semilattices $(S_1, \circ)$ and $(S_2, *)$, a solution $f$ to the *homomorphism functional equation*

$$f(x \circ y) = f(x) * f(y), \ x, y \in S_1, \tag{1}$$

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is always a monotone function. Indeed, $f$ will be isotone or antitone depending on how orders on the semilattices $S_1$ and $S_2$ are considered. When $S_1$ and $S_2$ are both considered as *join-semilattices*; i.e., when the order on $S_1$ is defined by $x \leq y$ if and only if $x \circ y = y$ and analogously for the order of $S_2$, then

$$f(y) = f(x \circ y) = f(x) * f(y)$$

provided that $x \leq y$, whence $f(x) \preceq f(y)$ and $f$ turns out to be isotone. The same holds when $S_1$ and $S_2$ are both considered as *meet-semilattices* (which means that the order is defined by $x \leq y$ if and only if $x \circ y = x$), while a similar reasoning shows that $f$ must be antitone when $S_1$ is a join-semilattice but $S_2$ is a meet-semilattice (or vice versa).

In what follows, the operation of a general semilattice $S$ will be denoted by "$\vee$" or "$\wedge$" depending on whether $S$ is a join or meet-lattice, and the use of the same symbol "$\vee$" to denote the operation in two different join-semilattices $S_1$ and $S_2$ should not cause confusions. Further simplifications of the notation will be introduced when needed. Thus, equation (1) written in the form

$$f(x \vee y) = f(x) \vee f(y), \; x, y \in S_1, \tag{2}$$

means that joins are preserved by the function $f : (S_1, \vee) \to (S_2, \vee)$. When $S$ is a meet-semilattice, $\downarrow a$ is the *principal ideal generated by* $a$ and $\downarrow a = a \wedge S_1 = \{a \wedge x : x \in S_1\}$ is a sub-semilattice of $S$. Dually, $\uparrow a$ is the *principal filter generated by* $a$ when $S_1$ is a join-semilattice and clearly $\uparrow a = a \vee S_1 = \{a \vee x : x \in S_1\}$ is also a sub-semilattice of $S$ (for the corresponding concepts in lattices, see [4], pg. 32-33, or [6], pg. 14).

This paper deals with functional equations in semilattices, mainly with (1), an equation whose solutions are the homomorphisms of the semilattices $S_1$ and $S_2$. In this way, the problem of finding the general solution to (1) is equivalent to determining the class $Hom(S_1, S_2)$ of semilattice homomorphisms of $S_1$ into $S_2$. On the other hand, the class $Hom(P_1, P_2)$ of order homomorphisms of the posets $P_1$ and $P_2$ is merely the class constituted by all isotone functions $f : P_1 \to P_2$. Thus, if $PO(S)$ denotes the join-semilattice $S$ when considered as a poset (under the order defined by $x \leq y$ if and only if $x \vee y = y$), it was previously shown that

$$Hom(S_1, S_2) \subseteq Hom(PO(S_1), PO(S_2)).$$

Simple examples of (finite) semilattices in which this inclusion is strict are presented in ppg. 30-31 of [4], so that it makes sense to say that a pair $(S_1, S_2)$ of join-semilattices is *order dominated* when

$$Hom(S_1, S_2) = Hom(PO(S_1), PO(S_2)) \tag{3}$$

or, in other words, when the class of isotone functions $f : S_1 \to S_2$ is the general solution to the functional equation (2). A complete characterization of all order dominated pairs $(S_1, S_2)$ of join-semilattices is given in Section 2. The family of all symmetric homomorphisms $\phi : C \times C \to S$, where $C$ is a chain is typified in Section 3. An infinite set of functional equations which are equivalent to the homomorphism equation is presented in Section 4. In the final Section 5, some homomorphism-like equations involving constants are studied.

## 2 Chains and order dominated pairs

Let us begin by observing that for an isotone function $f \in Hom\left(PO\left(S_1\right), PO\left(S_2\right)\right)$, equation (2) holds for every pair $x, y$ of comparable elements: by eventually interchanging $x$ and $y$, it can be assumed that $x \leq y$, whence $f(x) \leq f(y)$, and therefore $f(x \vee y) = f(y) = f(x) \vee f(y)$. As a consequence, every pair $(C, S)$ of join-semilattices in which $C$ is a chain turns out to be order dominated. Moreover, for a pair of the form $(S, C_0)$ (where $C_0$ is the unitary chain), every function $f : S \to C_0$ reduces to a constant, and then equality (2) holds in a trivial way. In this way, $(S, C_0)$ is order dominated whichever the semilattice $S$. As formally stated below, there are no other types of dominated pairs $(S_1, S_2)$ of join-semilattices than these two.

**Theorem 1** *A pair $(S_1, S_2)$ of join-semilattices is order dominated if and only if $S_1$ is a chain or $S_2$ is the unitary chain $C_0$.*

**Proof.** Sufficiency was shown in the initial paragraph of the section. To prove the necessity, suppose that $(S_1, S_2)$ is an order dominated pair of join-semilattices such that $S_1$ is not a chain and, at the same time, $S_2$ is not the unitary chain $C_0$. In this case, there exists a pair $a$, $b \in S_1$ such that $a \parallel b$, and the set $E = \{a, b, a \vee b\}$ is a sub-semilattice of $S_1$. Furthermore, since $S_2 \neq C_0$, $S_2$ must contain a pair of elements $\alpha$, $\beta$ such that $\alpha < \beta$ (or, in other words, $S_1$ must have a sub-semilattice isomorphic to $C_1$, the chain with two elements). The isotone function $f : E \to S_1$ defined by

$$f(x) = \begin{cases} \alpha, & x = a, b \\ \beta, & x = a \vee b \end{cases},$$

can be extended to an isotone function $\overline{f}$ defined on the whole $S_1$ as follows:

$$\overline{f}(x) = \begin{cases} \alpha, & x \in A \\ \beta, & x \in B \end{cases},$$

where

$$A := \downarrow (a \vee b) \setminus \{a \vee b\}$$

and

$$B := \uparrow (a \vee b) \cup \parallel (a \vee b).$$

First of all note that, due to the fact that exactly one of the three alternatives $x < a \vee b$, $x \geq a \vee b$ or $x \parallel (a \vee b)$ holds for every $x \in S_1$, the function $\overline{f}$ is really well defined on $S_1$ and $A \cap B = \varnothing$. Moreover, since $a$, $b < a \vee b$, and $a \vee b \in \uparrow (a \vee b)$, it turns out that $\overline{f}(x) = f(x)$, $x \in E$. Now, to prove that $\overline{f}$ is isotone, consider $x, y \in S_1$ such that $x \leq y$. Since $\overline{f}$ reduces to a constant on $A$ and $B$, it is true that $\overline{f}(x) \leq \overline{f}(y)$ in the event that $x, y \in A$ or $x, y \in B$. Furthermore, $\overline{f}(x) = \alpha < \beta = \overline{f}(y)$ provided that $x \in A$ and $y \in B$. It is easy to see that the remaining case in which $x \in B$ and $y \in A$ can not occur, so that $\overline{f}$ is isotone. Now, observing that

$$\overline{f}(\alpha \vee b) = \beta < \alpha = \alpha \vee \alpha = \overline{f}(a) \vee \overline{f}(b),$$

we conclude that $\overline{f}$ is not a semilattice homomorphism, in contradiction to the fact that the pair $(S_1, S_2)$ is order dominated. This contradiction originates from the assumption that $S_1$ is not a chain and $S_2$ is not the unitary chain $C_0$, which finishes the proof. ■

A restatement of Theorem 1 characterizing chains among other semilattices is now given.

**Theorem 2** *A join-semilattice $S_1$ is a chain if and only if there exists a join-semilattice $S_2 \neq C_0$ such that the pair $(S_1, S_2)$ is order dominated.*

**Proof.** The result is a direct consequence of Theorem 1. ■

# 3 Direct product of chains

This section deals with the problem of solving equation (2) in the case that $S_1$ is a direct product of chains. The simple case of two identical factors is considered in the following result.

**Theorem 3** *Let $C$, $S$ be two join-semilattices such that $C$ is a chain. A function $\phi : C \times C \to S$ is a symmetric homomorphism if and only if*

$$\phi(x, y) = f(x) \vee f(y), \quad x, y \in C, \tag{4}$$

*for any isotone $f : C \to S$.*

Recall that, given two sets $X$ and $Y$, a function $f : X^n \to Y$ is said to be *symmetric* when $f(x_1, \ldots, x_n)$ remains invariant under permutations of the variables $x_1, \ldots, x_n$.

**Proof.** A function $\phi : C \times C \to S$ of the form (4) is clearly symmetric. Moreover, since the pair $(C, S)$ is order dominated by Theorem 1, and $f$ is isotone, $f \in Hom(C, S)$ and then

$$\begin{aligned}
\phi((x_1, y_1) \vee (x_2, y_2)) &= \phi(x_1 \vee x_2, y_1 \vee y_2) \\
&= f(x_1 \vee x_2) \vee f(y_1 \vee y_2) \\
&= f(x_1) \vee f(x_2) \vee f(y_1) \vee f(y_2) \\
&= f(x_1) \vee f(y_1) \vee f(x_2) \vee f(y_2) \\
&= \phi(x_1, y_1) \vee \phi(x_2, y_2)
\end{aligned}$$

for every $x_i$, $y_i \in C$, $i = 1, 2$. Thus, $\phi$ turns out to be a symmetric homomorphism.

Conversely, assume that $\phi : C \times C \to S$ is a symmetric homomorphism. Then,

$$\phi\left((x_1, y_1) \vee (x_2, y_2)\right) = \phi\left(x_1, y_1\right) \vee \phi\left(x_2, y_2\right), \tag{5}$$

for every $x_i$, $y_i \in C$, $i = 1, 2$, and taking into account that

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_1, y_1 \vee y_2) \vee (x_2 \vee x_2, y_1 \vee y_2),$$

we can write

$$\begin{aligned}
\phi\left((x_1, y_1) \vee (x_2, y_2)\right) &= \phi\left((x_1 \vee x_1, y_1 \vee y_2) \vee (x_2 \vee x_2, y_1 \vee y_2)\right) \\
&= \phi\left(x_1 \vee x_1, y_1 \vee y_2\right) \vee \phi\left(x_2 \vee x_2, y_1 \vee y_2\right) \\
&= \phi\left((x_1, y_1) \vee (x_1, y_2)\right) \vee \phi\left((x_2, y_1) \vee (x_2, y_2)\right) \\
&= \phi\left(x_1, y_1\right) \vee \phi\left(x_1, y_2\right) \vee \phi\left(x_2, y_1\right) \vee \phi\left(x_2, y_2\right). \qquad (6)
\end{aligned}$$

From (5) and (6) it is deduced that

$$\phi\left(x_1, y_1\right) \vee \phi\left(x_1, y_2\right) \vee \phi\left(x_2, y_1\right) \vee \phi\left(x_2, y_2\right) = \phi\left(x_1, y_1\right) \vee \phi\left(x_2, y_2\right),$$

or, in an equivalent way

$$\phi\left(x_1, y_2\right) \vee \phi\left(x_2, y_1\right) \leq \phi\left(x_1, y_1\right) \vee \phi\left(x_2, y_2\right) \tag{7}$$

for every $x_i$, $y_i \in C$, $i = 1, 2$. Interchanging the role of $y_1$ and $y_2$ in the last inequality, we derive

$$\phi\left(x_1, y_1\right) \vee \phi\left(x_2, y_2\right) \leq \phi\left(x_1, y_2\right) \vee \phi\left(x_2, y_1\right), \tag{8}$$

the reverse inequality of (7). It is concluded that

$$\phi\left(x_1, y_2\right) \vee \phi\left(x_2, y_1\right) = \phi\left(x_1, y_1\right) \vee \phi\left(x_2, y_2\right) \tag{9}$$

for every $x_i$, $y_i \in C$, $i = 1, 2$. Setting $x_1 = y_1 = x$ and $x_2 = y_2 = y$ in (9) and taking the symmetry of $\phi$ into account, yields

$$\phi\left(x, y\right) = \phi\left(x, y\right) \vee \phi\left(y, x\right) = \phi\left(x, x\right) \vee \phi\left(y, y\right)$$

for every $x$, $y \in C$, and therefore (4) holds for the isotone function $f : C \to S$ given by $f(x) = \phi(x, x)$. ■

**Remark 4** *Theorem 3 can be generalized to the direct product of any number of chains. Concretely, if $C$, $S$ are two join-semilattices such that $C$ is a chain and $n \geq 2$ is an integer number, then a function $\phi : C^n \to S$ is a symmetric homomorphism if and only if*

$$\phi\left(x_1, \ldots, x_n\right) = \bigvee_{i=1}^{n} f\left(x_i\right), \ x_i \in C, \, i = 1, \ldots, n, \tag{10}$$

*for any isotone $f : C \to L$. The simple inductive proof of this result is omitted. Observe that the equality $f(x) = \phi(x, \ldots, x)$, $x \in C$, is derived by setting $x_1 = \cdots = x_n = x$ in (10), so that $\phi$ is determined by its values on the diagonal of $C^n$.*

## 4 A class of equations solved by homomorphisms

For $n \in \mathbb{N}$ an *n-ary term* in a (join-)semilattice $(S, \vee)$ is an expression of the form $x_{i_1} \vee \cdots \vee x_{i_n}$, where, for every $j = 1, \ldots, n$ we have $i_j \in \mathbb{N}$ and $x_i$ is a variable symbol (cf. [4], pg. 66 for the corresponding concept in lattices, or [2], pg. 6, for the general case of the concept in algebras). We abbreviate $x_{i_1} \vee \cdots \vee x_{i_n}$ by $x_{i_1} \cdots x_{i_n}$.

For a function $f : S_1 \to S_2$, where $S_1$ and $S_2$ are (join-)semilattices we define an *insertion term*

$$T(f; x_{i_1}, \ldots, x_{i_n}) = f\left(x_{i_1} \cdots x_{i_{k_1}}\right) f\left(x_{i_{k_1+1}} \cdots x_{i_{k_2}}\right) \cdots f\left(x_{i_{k_r+1}} \cdots x_{i_n}\right), \tag{11}$$

where the symbol $T$ covers the information on the ordered partition (also named composition; see [5], Chap. 1) of the integer $n = k_1 + (k_2 - k_1) + \cdots + (n - k_r)$, which corresponds to the number of variables of $T$.

Every function $f\left(x_{i_{k_j+1}} \cdots x_{i_{k_{j+1}}}\right)$ appearing in (11) is said to be a *component* of the insertion term. The first and last components are called *head* and *tail*, respectively, while a component is said to be *internal* when it is not the head or the tail. The number of variables in a component is said to be its *size*. Note that the function $f$ can be recovered from whatever insertion term by identifying the variables: $f(x) = T(f; x, \ldots, x)$, $x \in S_1$. Clearly, the number of possible insertions of a functional symbol in an $n$-ary term is $2^{n-1}$, the same as the number of compositions of $n$. Furthermore, the number $\binom{n-1}{k-1}$ of insertions with exactly $k$ components coincides with the number of compositions of $n$ with exactly $k$ parts ([5], Theorem 1.3, pg. 2).

Let $n_1, n_2 \in \mathbb{N}$ with $n_1 \le n_2$ and consider the insertion terms $T_i(f; x_1, \ldots, x_{n_i})$, $i = 1, 2$. In this section let us consider functional equations of the form

$$T_1(f; x_1, \ldots, x_{n_1}) = T_2(f; x_1, \ldots, x_{n_2}), \; x_1, \ldots, x_{n_2} \in S_1, \tag{12}$$

with the implicit assumption that $T_1 \neq T_2$.

To begin with, the case in which $n_1 < n_2$ will be studied. In this case, setting $x_1 = \cdots = x_{n_1} = x$ and $x_{n_1+1} = \cdots = x_{n_2} = y$, the right hand side, $T_1$ of equation (12) becomes merely $f(x)$, while the left hand side, $T_2$ may assume a few different values depending on the form of its components. After analyzing the various possibilities, these forms of the components of an $n_2$-ary insertion term $T$ can be typified as follows:

$$T(f; x_1, \ldots, x_{n_2}) = \begin{cases} \textbf{i)} & f(x_1, \ldots, x_{n_2}), \\ \textbf{ii)} & T'(f; x_1, \ldots, x_{n_1}) T''(f; x_{n_1+1}, \ldots, x_{n_2}), \\ \textbf{iii)} & T'(f; x_1, \ldots, x_k) T''(f; x_{k+1}, \ldots, x_{n_2}) \text{ for any } 1 \le k < n_1, \\ \textbf{iv)} & T'(f; x_1, \ldots, x_k) T''(f; x_{k+1}, \ldots, x_{n_2}) \text{ for any } n_1 < k < n_2, \\ \textbf{v)} & \begin{array}{l} T'(f; x_1, \ldots, x_j) T''(f; x_{j+1}, \ldots, x_k) T'''(f; x_{k+1}, \ldots, x_{n_2}) \\ \text{for any } 1 < j < n_1 < k < n_2, \end{array} \end{cases},$$

where $T'$, $T''$ and $T'''$ denote generic insertion terms (a notation often used in the sequel). Clearly, the cases **i)-v)** in the list are exhaustive and mutually exclusive: given $n_1 < n_2$, the components of an $n_2$-ary insertion term $T$ are of a unique type of the five specified ones. Observe that, after the formerly specified substitution of variables, the values assumed by $T_2(f; x_1, \ldots, x_{n_2})$ are given by

$$T_2\left(f; \overbrace{x, \ldots, x}^{n_1}, \overbrace{y, \ldots, y}^{n_2 - n_1}\right) = \begin{cases} f(xy) & \text{in case } \mathbf{i)}, \\ f(x) f(y) & \text{in case } \mathbf{ii)}, \\ f(x) f(xy) & \text{in case } \mathbf{iii)}, \\ f(xy) f(y) & \text{in case } \mathbf{iv)}, \\ f(x) f(xy) f(y) & \text{in case } \mathbf{v)}. \end{cases} .$$

At this point, we can state the following:

**Theorem 5** *Let $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$. Then, the general solution to the functional equation (12) is the family*

$$\mathcal{C}(S_1, S_2) = \{f : S_1 \to S_2 : \ f = \alpha \text{ for any } \alpha \in S_2\}$$

*of constant functions.*

**Proof.** Clearly, a constant function is always a solution to equation (12). After the discussion in the paragraphs preceding the statement of the theorem, to see that there are no other solutions, it is sufficient to show that the five equations

$$f(x) = \begin{cases} f(xy) \\ f(x) f(y) \\ f(x) f(xy) \\ f(xy) f(y) \\ f(x) f(xy) f(y) \end{cases} , \ x, y \in S_1,$$

all possess the family of constants as their general solution. This is a simple issue in those cases when the right hand side of the equation is a symmetric function $F(x, y)$ of $x$, $y$: if $x, y \in S_1$, then $f(x) = F(x, y) = F(y, x) = f(y)$. In this way, the general solution to the equations coincide with the family of constants in cases **i)**, **ii)** and **v)**.

In the remaining cases, let us prove that the equations can be ultimately reduced to the cases with a symmetric second member. For the equation corresponding to case **iii)**, observe that if $x \leq y$, then $f(x) = f(x) f(xy) = f(x) f(y) \geq f(y)$. This shows that $f$ must be antitone and the inequality

$$f(xy) \geq f(x) f(y) \tag{13}$$

quickly follows from this fact. On the other hand, from the equation itself we derive

$$f(x) f(y) = f(x) f(xy) f(y) = f(x) f(y) f(xy) \geq f(xy), \tag{14}$$

and therefore, from (13) and (14) we obtain

$$f(xy) = f(x) f(y);$$

i.e., $f$ must be a homomorphism. But then

$$f(x) = f(x) f(xy) = f(x) f(x) f(y) = f(x) f(y),$$

and the equation is reduced to the equation of case **ii)**.

Finally, let us show that the equation corresponding to case **iv)** can be reduced to that of case **iii)**. In fact, interchanging the role of the variables $x$ and $y$ in the equation we obtain $f(y) = f(xy) f(x)$, and a replacement of this value of $f(y)$ in the second member of the equation yields

$$f(x) = f(xy) f(y) = f(xy) f(xy) f(xy) f(x) = f(x) f(xy),$$

as asserted. This finishes the proof. ■

In the remainder of this section, instances of equation (12) in which $n_1 = n_2 = n$ will be studied. When $n = 1$, the implicit assumption $T_1 \neq T_2$ can not be satisfied, so let us consider $n \geq 2$. Since 2 is precisely the number of insertion terms when $n = 2$, there is essentially a unique equation (12) in this case, namely, the homomorphism equation $f(xy) = f(x) f(y)$. Therefore, $Hom(S_1, S_2)$ is the general solution when $n = 2$. As stated in the following result, the same is true for every $n \geq 2$.

**Theorem 6** *Let $n_1, n_2 \in \mathbb{N}$ with $n_1 = n_2 \geq 2$. Then, the general solution to the functional equation (12) is $Hom(S_1, S_2)$, the class of all homeomorphisms from $S_1$ to $S_2$.*

**Proof.** The proof proceeds by induction on $n = n_1 = n_2$. The case $n = 2$ was earlier shown to be true. Assuming that the theorem is true for every $2 \leq k \leq n$, let us prove that it is also true for $k = n+1$. To this end, first consider the case in which the insertion terms in the equation

$$T_1(f; x_1, \ldots, x_{n+1}) = T_2(f; x_1, \ldots, x_{n+1}) \tag{15}$$

have a common component; i.e., when the equation can be written in the form

$$\begin{aligned} & T_1'(f; x_1, \ldots, x_{k-1}) f(x_k \cdots x_l) T_1''(f; x_{l+1}, \ldots, x_{n+1}) \\ = \; & T_2'(f; x_1, \ldots, x_{k-1}) f(x_k \cdots x_l) T_2''(f; x_{l+1}, \ldots, x_{n+1}), \end{aligned}$$

with $f(x_k \cdots x_l)$ being the common component of $T_1$ and $T_2$. To obtain from this equation another equation with fewer variables, two cases will be considered. In the first one of these, the size of the common component is greater than or equal to 2; i.e., $k < l$, and after the identification of variables given by $x_k = \cdots = x_l$ the equation reads as

$$\begin{aligned} & T_1'(f; x_1, \ldots, x_{k-1}) f(x_k) T_1''(f; x_{l+1}, \ldots, x_{n+1}) \\ = \; & T_2'(f; x_1, \ldots, x_{k-1}) f(x_k) T_2''(f; x_{l+1}, \ldots, x_{n+1}). \end{aligned}$$

After replacing $x_i$ by $x_{i-l+k}$ for every $i = l+1, \ldots, n+1$ in this last equation, we obtain

$$\begin{aligned} & T_1'(f; x_1, \ldots, x_{k-1}) f(x_k) T_1''(f; x_{k+1}, \ldots, x_{n+1-l+k}) \\ = \; & T_2'(f; x_1, \ldots, x_{k-1}) f(x_k) T_2''(f; x_{k+1}, \ldots, x_{n+1-l+k}), \end{aligned}$$

which turns out to be an equation in $n+1-l+k \leq n$ variables.

In the second case the size of the component is 1; i.e., the equation has the form

$$\begin{aligned} & T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k)\,T_1''(f;x_{k+1},\ldots,x_{n+1}) \\ = \; & T_2'(f;x_1,\ldots,x_{k-1})\,f(x_k)\,T_2''(f;x_{k+1},\ldots,x_{n+1}), \end{aligned}$$

for any $1 \leq k \leq n+1$, and there is no loss of generality in supposing that $k = n+1$ (so that both insertion terms share the same tail of size 1). In fact, if $1 \leq k \leq n$, then the replacement $x_k = x_{n+1}$ and $x_i = x_{i-1}$ for every $i = k+1,\ldots,n+1$, gives

$$\begin{aligned} & T_1'(f;x_1,\ldots,x_{k-1})\,f(x_{n+1})\,T_1''(f;x_k,\ldots,x_n) \\ = \; & T_2'(f;x_1,\ldots,x_{k-1})\,f(x_{n+1})\,T_2''(f;x_k,\ldots,x_n) \end{aligned}$$

or, after rearranging the components,

$$\begin{aligned} & T_1'(f;x_1,\ldots,x_{k-1})\,T_1''(f;x_k,\ldots,x_n)\,f(x_{n+1}) \\ = \; & T_2'(f;x_1,\ldots,x_{k-1})\,T_2''(f;x_k,\ldots,x_n)\,f(x_{n+1}), \end{aligned}$$

an equation whose insertion terms both have a tail of size 1. It is asserted that the equation can be reduced in this case, too. In fact, it can be assumed that the component immediately preceding the common tail $(f(x_{n+1}))$ is $f(x_k \cdots x_n)$ for the first term and $f(x_l \cdots x_n)$ for the second one, and that $k \leq l$. In other words, without limiting generality, it can be supposed that the equation has the form

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k \cdots x_n)\,f(x_{n+1}) = T_2'(f;x_1,\ldots,x_{l-1})\,f(x_l \cdots x_n)\,f(x_{n+1}) \tag{16}$$

with $k \leq l$. Making the substitution $x_{n+1} = x_k \cdots x_n$ in (16), we get

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k \cdots x_n) = T_2'(f;x_1,\ldots,x_{l-1})\,f(x_l \cdots x_n)\,f(x_k \cdots x_n)$$

or,

$$T_1''(f;x_1,\ldots,x_n) = T_2''(f;x_1,\ldots,x_n)\,f(x_k \cdots x_n) \tag{17}$$

where

$$T_1''(f;x_1,\ldots,x_n)\,f(x_k \cdots x_n) = T_1''(f;x_1,\ldots,x_n). \tag{18}$$

Making instead the substitution $x_{n+1} = x_l \cdots x_n$ in (16) we have

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k \cdots x_n)\,f(x_l \cdots x_n) = T_2'(f;x_1,\ldots,x_{l-1})\,f(x_l \cdots x_n),$$

or

$$T_1''(f;x_1,\ldots,x_n)\,f(x_l \cdots x_n) = T_2''(f;x_1,\ldots,x_n), \tag{19}$$

where

$$T_2''(f;x_1,\ldots,x_n)\,f(x_l \cdots x_n) = T_2''(f;x_1,\ldots,x_n). \tag{20}$$

Thus, from (17), (18), (19) and (20) we deduce

$$\begin{aligned}
T_1''(f;x_1,\ldots,x_n) &= T_2''(f;x_1,\ldots,x_n)\,f(x_k\cdots x_n)\\
&= T_2''(f;x_1,\ldots,x_n)\,f(x_l\cdots x_n)\,f(x_k\cdots x_n)\\
&= T_2''(f;x_1,\ldots,x_n)\,f(x_k\cdots x_n)\,f(x_l\cdots x_n)\\
&= T_1''(f;x_1,\ldots,x_n)\,f(x_l\cdots x_n)\\
&= T_2''(f;x_1,\ldots,x_n),
\end{aligned}$$

which is an equation in $n$ variables.

In the second place, let us consider equation (15) with the insertion terms $T_1$ and $T_2$ having no common components and show that it can be reduced. Indeed, by specifying the form of the tail of $T_1$ and $T_2$, the equation can be written as:

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k\cdots x_{n+1}) = T_2'(f;x_1,\ldots,x_{l-1})\,f(x_l\cdots x_{n+1})$$

with $k<l\le n+1$. Let us distinguish two cases depending on whether $l<n+1$ or $l=n+1$. In the first one, the substitution $x_i=x_l$, $i=l+1,\ldots,n+1$, transforms the equation into

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k\cdots x_l) = T_2'(f;x_1,\ldots,x_{l-1})\,f(x_l),$$

an equation in $l\le n$ variables. Now, if $l=n+1$, the equation can be rewritten as

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k\cdots x_{n+1}) = T_2'(f;x_1,\ldots,x_{j-1})\,f(x_j\cdots x_n)\,f(x_{n+1}),$$

with $k$, $j\le n$. Two other cases must be considered depending on whether $k\le j$ or $k>j$. If $k\le j$, setting $x_{n+1}=x_j\cdots x_n$ in the equation yields

$$T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k\cdots x_n) = T_2'(f;x_1,\ldots,x_{j-1})\,f(x_j\cdots x_n),$$

an equation in $n$ variables. A reduction to another equation in $n$ variables is obtained in the case $k>j$ and $k<n$ after applying the substitutions $x_i=x_k$, $i=k+1,\ldots,n$ and $x_{n+1}=x_{k+1}\cdots x_n$:

$$\begin{aligned}
&T_1'(f;x_1,\ldots,x_{k-1})\,f(x_k\cdots x_n)\\
=\ &T_2'(f;x_1,\ldots,x_{j-1})\,f(x_j\cdots x_k)\,f(x_{k+1}\cdots x_n).
\end{aligned}$$

Finally, when $n=k>j$, i.e., when the equation can be written in the form

$$T_1'(f;x_1,\ldots,x_{n-1})\,f(x_nx_{n+1}) = T_2'(f;x_1,\ldots,x_{j-1})\,f(x_j\cdots x_n)\,f(x_{n+1}),$$

the substitutions $x_i=x_1$, $i=1,\ldots,n$ and $x_{n+1}=x_2$ give

$$f(x_1)\,f(x_1x_2) = f(x_1)\,f(x_2). \tag{21}$$

To finish the inductive proof, let us see that a solution to equation (21) must be a homomorphism. In fact, if $x_2\le x_1$, then

$$f(x_2)\le f(x_1)\,f(x_2) = f(x_1)\,f(x_1x_2) = f(x_1),$$

so that a solution $f$ to (21) is an isotone function. Thus $f(x_1) \leq f(x_1 x_2)$, and the first member $f(x_1) f(x_1 x_2)$ of the equation turns out to be simply equal to $f(x_1 x_2)$. This finishes the proof. ■

**Remark 7** *Theors. 5 and 6 can be suitably reformulated when $S_1$ or $S_2$ is a meet-semilattice while there are no significant changes to be introduced in the proofs of the corresponding results.*

# 5 Equations involving constants

Constant terms were no part of the functional equations so far considered. In this section, the equations

$$f(xy) = \alpha f(x) f(y), \ x, y \in S_1, \tag{22}$$

and

$$f(axy) = \alpha f(x) f(y), \ x, y \in S_1, \tag{23}$$

which are slight modifications of the homomorphism equation involving certain constants $a \in S_1$ and $\alpha \in S_2$, will be studied. Given two constants $a \in S_1$, $\alpha \in S_2$ and a class of functions $\mathcal{F} \subseteq S_2^{S_1}$ let us denote by $\mathcal{F}_{\uparrow\alpha}$ and $\mathcal{F}_{Per(a)}$ the subclasses of $\mathcal{F}$ respectively defined by

$$\mathcal{F}_{\uparrow\alpha} = \{f \in \mathcal{F} : f(x) \geq \alpha, \, x \in S_1\},$$

and

$$\mathcal{F}_{Per(a)} = \{f \in \mathcal{F} : f(ax) = f(x), \, x \in S_1\}.$$

In particular,

$$Hom_{\uparrow\alpha}(S_1, S_2) \left(\sim (Hom(S_1, S_2))_{\uparrow\alpha}\right) = \{f \in Hom(S_1, S_2) : f(x) \geq \alpha, \, x \in S_1\}$$

and

$$Hom_{Per(a)}(S_1, S_2) = \{f \in Hom(S_1, S_2) : f(ax) = f(x), \, x \in S_1\}.$$

**Theorem 8** *$Hom_{\uparrow\alpha}(S_1, S_2)$ is the general solution of the functional equation (22).*

**Proof.** If $f$ solves equation (22), then

$$f(x) = \alpha f(x) f(x) = \alpha f(x) \geq \alpha$$

for every $x \in S_1$. As a consequence,

$$f(xy) = \alpha f(x) f(y) = f(x) f(y)$$

for every $x$, $y \in S_1$, and thus $f \in Hom_{\uparrow\alpha}(S_1, S_2)$. Now, when $f \in Hom_{\uparrow\alpha}(S_1, S_2)$, we can write

$$f(xy) = f(x) f(y) = \alpha f(x) f(y), \ x, \, y \in S_1,$$

so that equation (22) is solved by $f$. This proves the theorem. $\blacksquare$

When $\alpha = f(a)$ equation (22) has additional properties. The substitutions $y = a$ and $y = x$ respectively yield

$$f(ax) = f(a) f(x)$$

and

$$f(x) = f(a) f(x),$$

for every $x \in S_1$. Thus $f(ax) = f(x)$, $x \in S_1$, and therefore

$$f(xy) = f(a) f(x) f(y) = f(x) f(y)$$

for every $x$, $y \in S_1$, which means that $f \in Hom_{Per(a)}(S_1, S_2)$. Conversely, a homomorphism $f \in Hom_{Per(a)}(S_1, S_2)$ satisfies

$$f(xy) = f(x) f(y) = f(ax) f(y) = f(a) f(x) f(y)$$

for every $x$, $y \in S_1$ and then, it is a solution to equation (22) with $\alpha = f(a)$. Another way of expressing this fact is the following: the functional equation (22) with $\alpha = f(a)$ is equivalent to the system of equations

$$\begin{cases} f(xy) = f(x) f(y) \\ f(ax) = f(x) \end{cases}, \ x, y \in S_1.$$

**Remark 9** *As a consequence of Theor. 8 and the previous discussion, for every $a \in S_1$ we can write*

$$Hom_{\uparrow f(a)}(S_1, S_2) = Hom_{Per(a)}(S_1, S_2)$$

*(where a forcing of the notation occurs since it is understood that $Hom_{\uparrow f(a)}(S_1, S_2) = \{f \in Hom(S_1, S_2) : f(x) \geq f(a), x \in S_1\}$). For example, when $S_1$ is a chain, $Hom_{\uparrow f(a)}(S_1, S_2)$ is the family of isotone functions such that $f(x) \geq f(a)$, $x \in S_1$, so that $f(x) = f(a)$ when $x \leq a$. On the other side, $Hom_{Per(a)}(S_1, S_2)$ is the family of isotone functions satisfying $f(a) f(x) = f(ax) = f(a)$, $x \in S_1$, and once more $f(x) \geq f(a)$, $x \in S_1$, and $f(x) = f(a)$ when $x \leq a$.*

Regarding equation (23), note that $axy \in \uparrow a$ for every $x, y \in S_1$, so that no restriction on $f$ is imposed by (23) when $x$ or $y$ are not members of the principal filter $\uparrow a$. Moreover, like in the case of equation (22), the values assumed by $f$ are always greater than $\alpha$. The following result is a development of these simple observations.

**Theorem 10** *If a function $f : S_1 \to S_2$ is a solution to the functional equation (23), then the restriction $f|_{\uparrow a} \in Hom_{\uparrow \alpha}(\uparrow a, S_2)$. Furthermore, every homomorphism $g \in Hom_{\uparrow \alpha}(\uparrow a, S_2)$ is extended to a solution $f$ to equation (23) by defining*

$$f(x) = g(ax), \ x \in S_1. \tag{24}$$

**Proof.** If $f$ solves equation (23) and $x \in\uparrow a$, then

$$f(x) = f(axx) = \alpha f(x) f(x) = \alpha f(x) \geq \alpha, \tag{25}$$

and thus, for every $x$, $y \in\uparrow a$,

$$f(xy) = f(axy) = \alpha f(x) f(x) = f(x) f(x). \tag{26}$$

(25) and (26) show that $f|_{\uparrow a} \in Hom_{\uparrow_\alpha}(\uparrow a, S_2)$. To prove the remaining assertion of the theorem, note that if $g \in Hom_{\uparrow_\alpha}(\uparrow a, S_2)$ and a function $f : S_1 \to S_2$ is defined by (24), then $f(x) = g(x)$ for every $x \in\uparrow a$; i.e., $f$ is an extension of $g$. Furthermore,

$$f(xy) = g(axy) = g((ax)(ay)) = g(ax) g(ay) = \alpha g(ax) g(ay) = \alpha f(x) f(y),$$

for every $x$, $y \in S_1$. Thus, $f$ turns out to be an extension of the homomorphism $g$, which solves equation (23). ■

**Remark 11** *Let $a \in S_1$ and consider the map $F_a : Hom(\uparrow a, S_2) \to Hom(S_1, S_2)$ defined by*

$$F_a(f)(x) = f(ax), \ x \in S_1.$$

*Clearly $F_a$ is injective and, since $F_a(fg)(x) = fg(ax) = f(ax) g(ax) = F_a(f)(x) F_a(g)(x)$ for every $x \in S_1$ (i.e., $F_a$ is a homomorphism when $Hom(\uparrow a, S_2)$ and $Hom(S_1, S_2)$ are endowed with the natural semilattice structure), we can write*

$$Hom(\uparrow a, S_2) \overset{F_a}{\approx} F_a(Hom(\uparrow a, S_2)) \subseteq Hom(S_1, S_2).$$

*In a similar way it can be shown that*

$$Hom_{\uparrow\alpha}(\uparrow a, S_2) \overset{F_a}{\approx} F_a(Hom_{\uparrow\alpha}(\uparrow a, S_2)) \subseteq Hom_{\uparrow\alpha}(S_1, S_2).$$

At the end of the section, let us observe that other simple equations involving constants like, for instance

$$f(abxy) = f(ax) f(by), \ x, y \in S_1,$$

or

$$f(axy) = f(bx) f(y), \ x, y \in S_1,$$

(where $a, b \in S_1$ are constants) can be treated similarly to what was done to (22) and (23).

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