Lattice

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目录

Ordered Sets

Definition 1.1. Partially ordered set is a system $\mathcal{P} = (P, \leq)$ where P is a nonempty set and $\leq$ is a binary relation on P satisfying，for all $x, y, z \in P$,

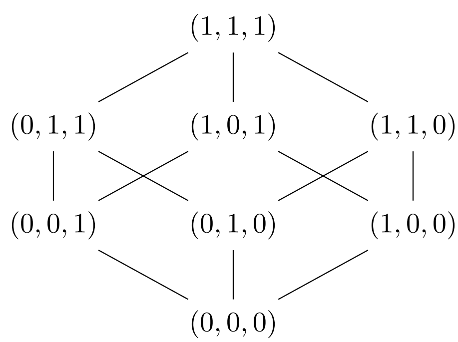
1. $x \leq x$, (reflexivity)
2. if $x \leq y$ and $y \leq x$,then $x = y$, (antisymmetry)
3. if $x \leq y$ and $y \leq z$,then $x \leq z$. (transitivity)

Definition 1.2. $\mathcal{C}$ is a chain if for every $x, y \in \mathcal{C}$，either $x \leq y$ or $y \leq x$.

chain 上的元素都可以相互比较.

Definition 1.3. We say that x is covered by y in $\mathcal{P}$，written $x \prec y$，if $x \leq y$ and there is no $z \in P$ with $x \leq z \leq y$.

Definition 1.4. Hasse diagram for a finite partially order set $\mathcal{P}$: the elements of P are represented by points in the plane, and a line is drawn from a up to b precisely when $a \prec b$.



Definition 1.5. Given a partially order set，f is a order preserving map satisfying the condition $x \leq y$ implies $f(x) \leq f(y)$.

Definition 1.6. Given two posets $(P, \leq_S)$ and $(Q, \leq_Q)$，an order isomorphism from $(P, \leq_S)$ to $(Q, \leq_Q)$ is a bijective order preserving map.

Definition 1.7. An ideal I of a partially ordered set $\mathcal{P}$ is a subset of the elements of P which satisfy the property that if $x \in \mathcal{P}$ and exists $y \in I$ with $x \leq y$，then $x \in I$.

衍生自 the ideal of ring，后面我们将会看见 the ideal of lattice.

Definition 1.8. Given an ordered set $\mathcal{P} = (P, \leq)$. The dual of P is another poset $\mathcal{P}^d = (P, \leq^d)$ with the order relation defined by $x \leq^d y \iff y \leq x$.

Definition 1.9. The dual notion of an ideal is called a filter that F is a subset of P such $x \geq y \in F$ implies $x \in F$

类似的还有 principle ideal 和 principle filter. 就是通过一个元素生成的.

Definition 1.10. The poset $\mathcal{P}$ has a maximum(element) if there exists $x \in P$ such that $y \leq x$ for all $x \in P$.

An element $x \in P$ is maximal if there is no element $y \in P$ with $x \leq y$ and $x \neq y$.

maximum 是一个名词表示最大值 (greatest)，maximal 是一个形容词表示极大的意思. 在 poset 中可能不只有一个 maximal element.

Lemma 1.11. The following are equivalent for an poset $\mathcal{P}$.

1. Every nonempty subset $S \subseteq P$ contains an element minimal in S.
2. $\mathcal{P}$ contains no infinite descending chain

$$a_0 > a_1 > a_2 > \cdots.$$

这里去掉等号是指 $a_0 \neq a_1 \neq a_2 \neq \cdots$

3. If

$$a_0 \geq a_1 \geq a_2 \geq \cdots$$

in $\mathcal{P}$， then there exists k such that $a_n = a_k$ for all $n \geq k$.

这个 lemma 被称为 descending chain condition. 对偶地也有 ascending chain condition. original 'a partially ordered set $\mathcal{P}$ requires that all decreasing sequences in $\mathcal{P}$ become eventually constant'.

证明. (2) $\Rightarrow$ (3) 前提只存在 finite descending chain. 假设 (3) 不成立，且 $a_0 \geq a_1 \geq a_2 \geq \cdots$ 是 infinite chain. 则对于任意的 k，都能找到 $n \geq k$ 使得 $a_n \neq a_k$ 且 $a_k \geq a_n$，那么 $a_k > a_n$. 这样从 $k = 0, 1, 2, \cdots$ 开始我们每次都可以找到 $a_{n_0} > a_{n_1} > \cdots$. 这样我们实际构造了一个 infinite descending chain，这是和前提矛盾的. 若 $a_0 \geq a_1 \geq a_2 \geq \cdots$ 是一个 finite chain，它的最后一个元素显然是满足 (3)，这和假设是矛盾的.

(3) $\Rightarrow$ (2) 也是分 infinite chain 和 finite chain 来讨论，finite 是显然的，infinite 的时候可以把它变成 finite.

(1) $\Rightarrow$ (2) (1) 前提满足下，假设 (2) 不成立，即 $\mathcal{P}$ 存在 infinite descending chain. 把这个 chain 上的元素取出来组成一个 subset S，那么任取 a_k 都有 $a_{k+1} \leq a_k$. 即找不到 minimal.

(2) $\Rightarrow$ (1) (2) 前提满足下，假设 (1) 不成立. 这里需要用一下选择公理了，定义 S 上一个选择函数 $f\colon S \to T$，其中 $T \subseteq S$. 让 $a_0 = f(S)$，递归地定义对任意的 $i \in \omega$ 有 $a_{i+1} = f(\{\, s \in S \mid s < a_i \,\})$. 接下来让这个 definition

make sense，(2) 前提下 S 是没有 minimal, 所以 $\{ s \in S \mid s \leq a_i \}$ 不是 empty set. 这样就找到了一个 infinite descending chain，与假设矛盾. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) (3) $\Rightarrow$ (2) $\Rightarrow$ (1)

done well! $\square$