Categorical Semantics for STLC

枫聆

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目录

First Try

Category of Baby Type Theory

Definition 1.1. A *baby type system* with only atomic types and where typing context are singletons.

Definition 1.2. A interpretion of baby type system is category C_{bT} such that objects are intrepretions of types and morphism are intrepretions of term-in-context(sequent).

- *objects*: $[\![A]\!]$ where A is atomic type;
- *morphism*: $[\![\Gamma \vdash E : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$, abbreviated to $[\![E]\!] : [\![\Gamma]\!] \to [\![\Gamma]\!]$;
- *identity*: $[\![x : A \vdash x : A]\!] = 1_{[\![A]\!]} : [\![A]\!] \to [\![A]\!]$, it corresponds to

$$\frac{}{x : A \vdash x : A}\ \textsc{Var}$$

- *composition*: $[\![E_2[y \to E_1]]\!] = [\![E_2]\!] \circ [\![E_1]\!] : [\![A]\!] \to [\![C]\!]$, it corresponds to

$$\frac{x : A \vdash E_1 : B \quad y : B \vdash E_2 : C}{x : A \vdash E_2[y \to E_1]}\ Sub$$

- *unit law*:

$$[\![x : A \vdash E[x \to x]]\!] = [\![x : A \vdash E : B]\!] \circ [\![x : A \vdash x : A]\!]$$
$$[\![x : A \vdash E[y \to E]]\!] = [\![y : B \vdash y : B]\!] \circ [\![x : A \vdash E : B]\!]$$

- *associative law*: Given $x : A \vdash E_1 : B, y : B \vdash E_2 : C, z : C \vdash E_3 : D$, we have

$$[\![E_3[z \to E_2[y \to E_1]]]\!] = [\![E_3[z \to E_2][y \to E_1]]\!] : [\![A]\!] \to [\![D]\!]$$

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Category of Propositions and Derivations

Definition 1.3. A netural deduction system corresponds to category C_{ND} such that objects are interpretations of propositions and morphisms are interpretations of derivations.

1. *morphism*: a morphism $[\![\mathcal{D}]\!] : [\![\Gamma]\!] \to [\![A]\!]$ corresponds

$$\frac{\frac{\Gamma}{\mathcal{D}}}{A}$$

2. *identity*: $1_{[\![A]\!]}$ corresponds identity derivation $\frac{A}{A}$.

Unit Type and Void Type

Definition 1.4. The unit type is interpreted by the terminal object in categorical representation

$$[\![\top]\!] := 1$$

where $\top$ is actually unit type[1] and 1 represents terminal object. The introduction rule for unit type

$$\frac{}{\Gamma \vdash * : \top} \ \top I$$

is interpreted by the unique map to terminal object

$$[\![*]\!] :=!([\![\Gamma]\!]) : \Gamma \to 1.$$

Definition 1.5. The void type is interpreted by the initial object in categorical representation

$$[\![\bot]\!] := 0$$

where $\bot$ is actually unit type and 1 represents initial object. The elimintaion rule for void type

[1] Though $\top$ is described in subtyping

Truth and Falsehood

Definition 1.6. Logical propositional constant truth is interpreted by the terminal object in categorical representation

$$[\![\top]\!] := 1$$

where 1 represents terminal object. The natural deduction introduction rule for truth

$$\frac{}{\top}\ \top I$$

is interpreted by the unique map to termninal object

$$[\![\top I]\!] :=!([\![A]\!]) : [\![A]\!] \to 1.$$

Definition 1.7. Logical propositional constant falsehood is interpreted by the initial object in categorical representation

$$[\![\perp]\!] := 0$$

where 0 represents initial object. The natural deduction elimination rule for falsehood

$$\frac{\perp}{A}\ \perp E$$

is interpreted by the unique map from the initial object:

$$[\![\perp E]\!] := ¡([\![A]\!]) : 0 \to [\![A]\!]$$

参考文献

[1] Edward Morehouse. Basic Category Theory. OPLSS, 2016. https://www.ioc.ee/~ed/research/notes/intro_categorical_semantics.pdf

[2] Edward Morehouse. Basic Category Theory. OPLSS, 2015.