Genuine Cohomology Theories

Marin Janssen (she/they)

University of Münster

May 6, 2024

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $\dots, \quad i \in \mathbb{Z}$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, ..., $\quad i \in \mathbb{Z}$

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, ..., $\quad i \in \mathbb{Z}$

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, ..., $\quad i \in \mathbb{Z}$

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, ..., $\quad i \in \mathbb{Z}$

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $\ldots$, $\quad i \in \mathbb{Z}$

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Mayer-Vietoris: $H^*(A \coprod_C B) \to H^*(A) \oplus H^*(B) \to H^*(C)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $\dots,\quad i \in \mathbb{Z}$

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Mayer-Vietoris: $H^*(A \coprod_C B) \to H^*(A) \oplus H^*(B) \to H^*(C)$

Dimension: $H^i(\mathrm{pt}) = \mathbb{Z}^{\delta_{i,0}}$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...
Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, ..., $\quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for cohomology theories:

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Mayer-Vietoris: $H^*(A \coprod_C B) \to H^*(A) \oplus H^*(B) \to H^*(C)$

Dimension: $H^i(\mathrm{pt}) = \mathbb{Z}^{\delta_{i,0}}$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...
Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, ..., $\quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for cohomology theories:

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Mayer-Vietoris: $H^*(A \coprod_C B) \to H^*(A) \oplus H^*(B) \to H^*(C)$

Dimension: $H^i(\mathrm{pt}) = \mathbb{Z}^{\delta_{i,0}}$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $K^i(X), \pi_{\mathrm{s}}^i(X), \Omega_G^i(X), K(n,p)^i(X), \ldots, \quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for cohomology theories:

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Mayer-Vietoris: $H^*(A \coprod_C B) \to H^*(A) \oplus H^*(B) \to H^*(C)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $K^i(X), \pi_{\mathrm{s}}^i(X), \Omega_G^i(X), K(n,p)^i(X)$, ..., $\quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for cohomology theories:

Functoriality: $f : X \to Y$ induces map $f^* : H^*(Y) \to H^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{H}^{*+1}(\Sigma X) \simeq \tilde{H}^*(X)$

Additivity: $H^*(\coprod X_\alpha) = \prod H^*(X_\alpha)$

Mayer-Vietoris: $H^*(A \coprod_C B) \to H^*(A) \oplus H^*(B) \to H^*(C)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $K^i(X), \pi_{\mathrm{s}}^i(X), \Omega_G^i(X), K(n,p)^i(X), \ldots, \quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for (generalized) cohomology theories on compact anima:

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $K^i(X), \pi^i_{\mathrm{s}}(X), \Omega^i_G(X), K(n,p)^i(X), \ldots, \quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for (generalized) cohomology theories on compact anima:

Functoriality: $f : X \to Y$ induces map $f^* : E^*(Y) \to E^*(X)$

Homotopy Invariance: homotopic maps induce the same map in cohomology

Suspension: $\tilde{E}^{*+1}(\Sigma X) \simeq \tilde{E}^*(X)$

Mayer-Vietoris: $E^*(A \coprod_C B) \to E^*(A) \oplus E^*(B) \to E^*(C)$

Let X be a topological space, CW-complex, simplicial set, anima, G-anima, manifold, ...

Invariants $F(X)$ of X, e.g.: $\chi(X)$, $\beta_i(X)$, $\pi_i(X)$, $H_i(X)$, $H^i(X)$, $K^i(X), \pi_s^i(X), \Omega_G^i(X), K(n,p)^i(X), \ldots, \quad i \in \mathbb{Z}$

Eilenberg-Steenrod Axioms for (generalized) cohomology theories on compact anima:

Functor $E\colon \mathrm{Sp}^\omega \to \mathrm{Ab}^{op}$

Mayer-Vietoris: $E(A \coprod_C B) \to E(A) \oplus E(B) \to E(C)$ is exact

Motivation

Let M^n closed orientable manifold,

Motivation

Let M^n closed orientable manifold,

H^* be singular cohomology,

Motivation

Let M^n closed orientable manifold,

H^* be singular cohomology,

H_* be singular homology, then

Motivation

Let M^n closed orientable manifold,

H^* be singular cohomology,

H_* be singular homology, then

Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,

H^* be singular cohomology,

H_* be singular homology, then

Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,

E^* be a generalized cohomology theory,

H_* be singular homology, then

Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,

E^* be a generalized cohomology theory,

E_* associated homology theory, then

Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,

E^* be a generalized cohomology theory,

E_* associated homology theory, then

Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,

$E^* \simeq [\Sigma^\infty_+(-), \mathcal{E}]_{-*}$ be a generalized cohomology theory,

E_* associated homology theory, then

Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,
$E^* \simeq [\Sigma^\infty_+(-), \mathcal{E}]_{-*}$ be a generalized cohomology theory,
$E_* := [\mathbb{S}, \Sigma^\infty_+(-) \otimes \mathcal{E}]_*$ associated homology theory, then
Poincaré-Duality:

$$H_*(M) \simeq H^{n-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,

$E^* \simeq [\Sigma^\infty_+(-), \mathcal{E}]_{-*}$ be a generalized cohomology theory,

$E_* := [\mathbb{S}, \Sigma^\infty_+(-) \otimes \mathcal{E}]_*$ associated homology theory, then

Poincaré-Duality:

$$E_*(M) \simeq E^{TM-*}(M)$$

Motivation

Let M^n closed not necessarily orientable manifold,
$E^* \simeq [\Sigma^\infty_+(-), \mathcal{E}]_{-*}$ be a generalized cohomology theory,
$E_* := [\mathbb{S}, \Sigma^\infty_+(-) \otimes \mathcal{E}]_*$ associated homology theory, then
Poincaré-Duality:

$$E_*(M) \simeq E^{TM-*}(M) := [M^{-TM}, \mathcal{E}]_*$$

Motivation

Let M^n closed not necessarily orientable manifold,
$E^* \simeq [\Sigma^\infty_+(-), \mathcal{E}]_{-*}$ be a generalized cohomology theory,
$E_* := [\mathbb{S}, \Sigma^\infty_+(-) \otimes \mathcal{E}]_*$ associated homology theory, then
Poincaré-Duality:

$$E_*(M) \simeq E^{TM-*}(M) := [M^{-TM}, \mathcal{E}]_*$$

Problem

Extend the ES-axioms to include the groups $E^{V-*}(X) = [X^{-V}, \mathcal{E}]_*$ where (X, V) run through all pairs of $X \in \mathrm{An}^\omega$ and V a vector bundle over X.

Motivation

Let M^n closed not necessarily orientable manifold,
$E^* \simeq [\Sigma^\infty_+(-), \mathcal{E}]_{-*}$ be a generalized cohomology theory,
$E_* := [\mathbb{S}, \Sigma^\infty_+(-) \otimes \mathcal{E}]_*$ associated homology theory, then
Poincaré-Duality:

$$E_*(M) \simeq E^{TM-*}(M) := [M^{-TM}, \mathcal{E}]_*$$

Problem

Extend the ES-axioms to include the groups $E^{V-*}(X) = [X^{-V}, \mathcal{E}]_*$ where (X, V) run through all pairs of $X \in \mathrm{An}^\omega$ and V a vector bundle over X.

Inspiration

$RO(G)$-graded cohomology theories. Graded over G-representation which can be viewed as equivariant vector bundles over G/G.

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

for each pair (X, V) an abelian group $E^V(X)$,

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f : X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^* W} X$

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f: X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^*W} X$
- for each affine linear map $\alpha: U \to V$ of VB over X an induced map $E^U X \xrightarrow{E(\alpha)} E^V X$ s.t.

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f\colon X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^*W} X$
- for each affine linear map $\alpha\colon U \to V$ of VB over X an induced map $E^U X \xrightarrow{E(\alpha)} E^V X$ s.t.
- Functoriality and Homotopy Invariance

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f\colon X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^*W} X$
- for each affine linear map $\alpha\colon U \to V$ of VB over X an induced map $E^U X \xrightarrow{E(\alpha)} E^V X$ s.t.
- Functoriality and Homotopy Invariance
- Beck-Chevalley $E(f^*\beta) \circ E(f) = E(f) \circ E(\beta)$

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f\colon X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^*W} X$
- for each affine linear map $\alpha\colon U \to V$ of VB over X an induced map $E^U X \xrightarrow{E(\alpha)} E^V X$ s.t.
- Functoriality and Homotopy Invariance
- Beck-Chevalley $E(f^*\beta) \circ E(f) = E(f) \circ E(\beta)$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f\colon X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^*W} X$
- for each affine linear map $\alpha\colon U \to V$ of VB over X an induced map $E^U X \xrightarrow{E(\alpha)} E^V X$ s.t.
- Functoriality and Homotopy Invariance
- Beck-Chevalley $E(f^*\beta) \circ E(f) = E(f) \circ E(\beta)$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism
- Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact.

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- for each pair (X, V) an abelian group $E^V(X)$,
- for each map $f: X \to Y$ of anima an induced map $E^W Y \xrightarrow{E(f)} E^{f^* W} X$
- for each affine linear map $\alpha: U \to V$ of VB over X an induced map $E^U X \xrightarrow{E(\alpha)} E^V X$ s.t.
- Functoriality and Homotopy Invariance
- Beck-Chevalley $E(f^*\beta) \circ E(f) = E(f) \circ E(\beta)$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V \oplus W}(S^W) \to E^{V \oplus W}(S^W, X)$ is an isomorphism
- Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact.

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- Functor $E\colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathrm{Ab}^{op}$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism
- Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact.

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- Functor $E\colon \mathcal{C}_{\mathsf{An,AffLin}} \to \mathrm{Ab}^{op}$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism
- Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact.

$\mathrm{Gen}(\mathcal{C}_{\mathsf{An,AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories.

Genuine Cohomology Theories

Definition

A genuine Cohomology Theory E consists of the following data

- Functor $E\colon \mathcal{C}_{\mathsf{An,AffLin}} \to \mathrm{Ab}^{op}$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism
- Mayer-Vietoris $E^V(B\coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact.

$\mathrm{Gen}(\mathcal{C}_{\mathsf{An,AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories.

$\mathrm{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An,AffLin}}, \mathcal{D})$ is the ∞-category of split genuine homology theories.

Genuine Cohomology Theories

Definition

Definition: A split genuine Homology Theory E with coefficients in a pointed ∞-category $\mathcal{D}$ consists of the following data

$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories.
$\mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathcal{D})$ is the ∞-category of split genuine homology theories.

Genuine Cohomology Theories

Definition

Definition: A split genuine Homology Theory E with coefficients in a pointed ∞-category $\mathcal{D}$ consists of the following data

- Functor $E\colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathcal{D}$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism

- $\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories.
- $\mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathcal{D})$ is the ∞-category of split genuine homology theories.

Genuine Cohomology Theories

Definition

Definition: A split genuine Homology Theory E with coefficients in a pointed ∞-category $\mathcal{D}$ consists of the following data

- Functor $E\colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathcal{D}$
- Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism
- Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact whenever $A \to C$ or $A \to B$ admits a retraction.

$\mathrm{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories. $\mathrm{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathcal{D})$ is the ∞-category of split genuine homology theories.

Genuine Cohomology Theories

Definition

Definition: A split genuine Homology Theory E with coefficients in a pointed ∞-category $\mathcal{D}$ consists of the following data

Functor $E\colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathcal{D}$

Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism

Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact whenever $A \to C$ or $A \to B$ admits a retraction.

$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories.
$\mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathcal{D})$ is the ∞-category of split genuine homology theories.
$\mathsf{Gen}(\mathcal{C}_{\mathsf{An}_G,\underline{\mathsf{AffLin}}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of G-equivariant genuine cohomology theories.

Genuine Cohomology Theories

Definition

Definition: A split genuine Homology Theory E with coefficients in a pointed ∞-category $\mathcal{D}$ consists of the following data

Functor $E\colon \mathcal{C}_{\mathrm{An},\mathrm{AffLin}} \to \mathcal{D}$

Thom-Isomorphism The map $E^V(X) \to E^V(S^W) \to E^{V\oplus W}(S^W) \to E^{V\oplus W}(S^W, X)$ is an isomorphism

Mayer-Vietoris $E^V(B \coprod_A C) \to E^V(B) \oplus E^V(C) \to E^V(A)$ exact whenever $A \to C$ or $A \to B$ admits a retraction.

$\mathrm{Gen}(\mathcal{C}_{\mathrm{An},\mathrm{AffLin}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of genuine cohomology theories.
$\mathrm{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathrm{An},\mathrm{AffLin}}, \mathcal{D})$ is the ∞-category of split genuine homology theories.
$\mathrm{Gen}(\mathcal{C}_{\mathrm{An}_G,\underline{\mathrm{AffLin}}}, \mathrm{Ab}^{\mathrm{op}})$ is the category of G-equivariant genuine cohomology theories.
$\mathrm{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathrm{An}_G,\underline{\mathrm{AffLin}}}, \mathcal{D})$ is the ∞-category of G-equivariant split genuine homology theories.

Examples and Main Results

$(X, V) \mapsto X^{-V}$ defines a functor $\mathrm{Th}^{-} \colon \mathcal{C}_{\mathrm{An},\mathrm{AffLin}} \to \mathrm{Sp}^{\omega}$

Examples and Main Results

$(X,V) \mapsto X^{-V}$ defines a functor $\mathrm{Th}^{-} \colon \mathcal{C}_{\mathrm{An,AffLin}} \to \mathrm{Sp}^{\omega}$

Example

Let $E \colon \mathrm{Sp}^{\omega} \to \mathrm{Ab}^{op}$ be a cohomology theory, then

$E \circ Th^{-} \in \mathrm{Gen}(\mathcal{C}_{\mathrm{An,AffLin}}, \mathrm{Ab}^{op})$

Examples and Main Results

$(X, V) \mapsto X^{-V}$ defines a functor $\mathsf{Th}^{-}\colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathsf{Sp}^{\omega}$

Example

Let $E\colon \mathsf{Sp}^{\omega} \to \mathrm{Ab}^{op}$ be a cohomology theory, then

$E \circ Th^{-} \in \mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathrm{Ab}^{op})$

Theorem

The functor $E \mapsto E \circ \mathsf{Th}^{-}$ defines an equivalence of categories

$$\{\text{cohomology theories } E\colon \mathsf{Sp}^{\omega} \to \mathrm{Ab}^{\mathrm{op}}\} \simeq \mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathrm{Ab}^{\mathrm{op}})$$

Examples and Main Results

$(X, V) \mapsto X^{-V}$ defines a functor $\mathrm{Th}^{-}\colon \mathcal{C}_{\mathsf{An,AffLin}} \to \mathsf{Sp}^{\omega}$

Example

Let $E\colon \mathsf{Sp}^{\omega} \to \mathrm{Ab}^{op}$ be a cohomology theory, then $E \circ Th^{-} \in \mathsf{Gen}(\mathcal{C}_{\mathsf{An,AffLin}}, \mathrm{Ab}^{op})$

Theorem

The functor $E \mapsto E \circ \mathrm{Th}^{-}$ defines an equivalence of categories

$$\{\text{cohomology theories } E\colon \mathsf{Sp}^{\omega} \to \mathrm{Ab}^{\mathrm{op}}\} \simeq \mathsf{Gen}(\mathcal{C}_{\mathsf{An,AffLin}}, \mathrm{Ab}^{\mathrm{op}})$$

Theorem

For a fixed compact Lie group G we have an equivalence of ∞-categories

$$\{RO(G)\text{-graded cohomology theories }\} \simeq \mathsf{Gen}(\mathcal{C}_{\mathsf{An}_G,\underline{\mathsf{AffLin}}}, \mathrm{Ab}^{\mathrm{op}})$$

Main Theorem

Main Theorem

Main Theorem

Let $\mathcal{D}$ be a pointed ∞-category with all split pushouts.

Main Theorem

Main Theorem

Let $\mathcal{D}$ be a pointed ∞-category with all split pushouts.

The pullback along $\mathsf{Th}^- \colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathsf{Sp}^\omega$ induces an equivalence of ∞-categories

$$\mathrm{Exc}^{\mathrm{split}}(\mathsf{Sp}^\omega, \mathcal{D}) \simeq \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathcal{D})$$

Main Theorem

Main Theorem

Let $\mathcal{D}$ be a pointed ∞-category with all split pushouts.

The pullback along $\mathsf{Th}^{-}\colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}} \to \mathsf{Sp}^{\omega}$ induces an equivalence of ∞-categories

$$\mathrm{Exc}^{\mathrm{split}}(\mathsf{Sp}^{\omega}, \mathcal{D}) \simeq \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathcal{D})$$

Slogan

The functor Th^{-} is the universal (split) genuine homology theory.

Main Theorem

Main Theorem

Let $\mathcal{D}$ be a pointed ∞-category with all split pushouts.

The pullback along $\mathsf{Th}^{-}\colon \mathcal{C}_{\mathsf{An}_G,\underline{\mathsf{AffLin}}} \to \mathsf{Sp}_G^{\omega}$ induces an equivalence of ∞-categories

$$\mathrm{Exc}^{\mathrm{split}}(\mathsf{Sp}_G^{\omega}, \mathcal{D}) \simeq \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An}_G,\underline{\mathsf{AffLin}}}, \mathcal{D})$$

Slogan

The functor Th^{-} is the universal (split) genuine homology theory.

Proof Sketch of the Main Theorem

Prove an unstable version of the theorem.

Proposition

The pullback along $\mathsf{Th}^{-} \colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V) \to \mathsf{An}_*^{\omega}$ induces an equivalence of ∞-categories

$$\mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*, \mathcal{D}) \simeq \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V), \mathcal{D})$$

Proof Sketch of the Main Theorem

Prove an unstable version of the theorem.

Proposition

The pullback along $\mathsf{Th}^{-} \colon \mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V) \to \mathsf{An}_*^{\omega}$ induces an equivalence of ∞-categories

$$\mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*, \mathcal{D}) \simeq \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V), \mathcal{D})$$

Reduce the stable version to the unstable version.

Lemma

We have equivalences of ∞-categories

$$\mathsf{colim}_V\, \mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V) \simeq \mathcal{C}_{\mathsf{An},\mathsf{AffLin}}$$

$$\mathsf{colim}_V\, \mathsf{An}_*^{\omega} \simeq \mathsf{Sp}^{\omega}$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^{-})^{*}$ as follows:

$$\begin{aligned} \mathsf{res}\colon\ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V), \mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_{*}^{\omega}, \mathcal{D}) \\ E \mapsto \mathsf{res}\, E\colon X &\mapsto \mathsf{cof}[E^{V}(\mathrm{pt}) \to E^{r^{*}V}(X)] \end{aligned}$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^{-})^{*}$ as follows:

$$\begin{aligned} \mathsf{res}\colon\ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V), \mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_{*}^{\omega}, \mathcal{D}) \\ E \mapsto \mathsf{res}\, E\colon X &\mapsto \mathsf{cof}[E^{V}(\mathrm{pt}) \to E^{r^{*}V}(X)] \end{aligned}$$

We compute:

$$\mathsf{res}(E \circ \mathsf{Th}^{-})(X) \quad \simeq \quad \mathsf{cof}[E(\mathrm{pt}_{+}) \to E(X_{+})]$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^-)^*$ as follows:

$$\begin{aligned} \mathsf{res}\colon \ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V), \mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*^{\omega}, \mathcal{D}) \\ E \mapsto \mathsf{res}\, E\colon X &\mapsto \mathsf{cof}[E^V(\mathrm{pt}) \to E^{r^*V}(X)] \end{aligned}$$

We compute:

$$\begin{aligned} \mathsf{res}(E \circ \mathsf{Th}^-)(X) &\simeq \mathsf{cof}[E(\mathrm{pt}_+) \to E(X_+)] \\ &\simeq E(X) \quad (\text{since } E \text{ is split excisive}) \end{aligned}$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^-)^*$ as follows:

$$\begin{aligned}\mathsf{res}\colon\ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V),\mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*^{\omega},\mathcal{D})\\ E \mapsto \mathsf{res}\,E\colon\ X &\mapsto \mathsf{cof}[E^V(\mathrm{pt}) \to E^{r^*V}(X)]\end{aligned}$$

We compute:

$$\begin{aligned}\mathsf{res}(E\circ\mathsf{Th}^-)(X) &\simeq \mathsf{cof}[E(\mathrm{pt}_+) \to E(X_+)]\\ &\simeq E(X) \quad (\text{since } E \text{ is split excisive})\end{aligned}$$

and

$$((\mathsf{res}\,E)\circ\mathsf{Th}^-)^U(X) \simeq \tilde{E}^V(\mathsf{Th}_X(U^{\perp_V}))$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^-)^*$ as follows:

$$\begin{aligned} \mathsf{res}\colon\ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V),\mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*^{\omega},\mathcal{D}) \\ E \mapsto \mathsf{res}\,E\colon X &\mapsto \mathsf{cof}[E^V(\mathrm{pt}) \to E^{r^*V}(X)] \end{aligned}$$

We compute:

$$\begin{aligned} \mathsf{res}(E\circ\mathsf{Th}^-)(X) &\simeq \mathsf{cof}[E(\mathrm{pt}_+) \to E(X_+)] \\ &\simeq E(X) \quad (\text{since } E \text{ is split excisive}) \end{aligned}$$

and

$$\begin{aligned} ((\mathsf{res}\,E)\circ\mathsf{Th}^-)^U(X) &\simeq \tilde{E}^V(\mathsf{Th}_X(U^{\perp_V})) \\ &\simeq E^V(S_X^{U^{\perp_V}},X) \quad (\text{Mayer-Vietoris Axiom}) \end{aligned}$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^-)^*$ as follows:

$$\begin{aligned} \mathsf{res}\colon\ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V), \mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*^{\omega}, \mathcal{D}) \\ E &\mapsto \mathsf{res}\, E\colon X \mapsto \mathsf{cof}[E^V(\mathrm{pt}) \to E^{r^*V}(X)] \end{aligned}$$

We compute:

$$\begin{aligned} \mathsf{res}(E \circ \mathsf{Th}^-)(X) &\simeq \mathsf{cof}[E(\mathrm{pt}_+) \to E(X_+)] \\ &\simeq E(X) \quad (\text{since } E \text{ is split excisive}) \end{aligned}$$

and

$$\begin{aligned} ((\mathsf{res}\, E) \circ \mathsf{Th}^-)^U(X) &\simeq \tilde{E}^V(\mathsf{Th}_X(U^{\perp V})) \\ &\simeq E^V(S_X^{U^{\perp V}}, X) \quad (\text{Mayer-Vietoris Axiom}) \\ &\simeq E^{U \oplus U^{\perp V}}(S_X^{U^{\perp V}}, X) \end{aligned}$$

Proof Sketch of the Main Unstable Theorem

We can define the inverse functor of $(\mathsf{Th}^-)^*$ as follows:

$$\begin{aligned} \mathsf{res}\colon\ \mathsf{Gen}^{\mathrm{split}}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}(V),\mathcal{D}) &\to \mathrm{Exc}^{\mathrm{split}}(\mathsf{An}_*^\omega,\mathcal{D}) \\ E \mapsto \mathsf{res}\,E\colon X &\mapsto \mathsf{cof}[E^V(\mathrm{pt}) \to E^{r^*V}(X)] \end{aligned}$$

We compute:

$$\begin{aligned} \mathsf{res}(E\circ \mathsf{Th}^-)(X) &\simeq \mathsf{cof}[E(\mathrm{pt}_+)\to E(X_+)] \\ &\simeq E(X) \quad (\text{since } E \text{ is split excisive}) \end{aligned}$$

and

$$\begin{aligned} ((\mathsf{res}\,E)\circ \mathsf{Th}^-)^U(X) &\simeq \tilde{E}^V(\mathsf{Th}_X(U^{\perp V})) \\ &\simeq E^V(S_X^{U^{\perp V}},X) \quad (\text{Mayer-Vietoris Axiom}) \\ &\simeq E^{U\oplus U^{\perp V}}(S_X^{U^{\perp V}},X) \\ &\simeq E^U(X) \quad (\text{Thom-Isomorphism Axiom}) \quad \square \end{aligned}$$

Application

Let $\mathcal{D} = \mathsf{An}_*$, then

$$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*) \simeq \mathrm{Exc}(\mathsf{Sp}^{\omega}, \mathsf{An}_*) \simeq \mathsf{Sp}$$

Application

Let $\mathcal{D} = \mathsf{An}_*$, then

$$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*) \simeq \mathrm{Exc}(\mathsf{Sp}^{\omega}, \mathsf{An}_*) \simeq \mathsf{Sp}$$

Let $E \in \mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*)$, then we can compute

Application

Let $\mathcal{D} = \mathsf{An}_*$, then

$$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*) \simeq \mathrm{Exc}(\mathsf{Sp}^\omega, \mathsf{An}_*) \simeq \mathsf{Sp}$$

Let $E \in \mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*)$, then we can compute

$$E^V(\mathrm{pt}) \quad \simeq \quad E^{V \oplus W}(S^W, \mathrm{pt}) \quad \text{Thom Isomorphism}$$

Application

Let $\mathcal{D} = \mathsf{An}_*$, then

$$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*) \simeq \mathrm{Exc}(\mathsf{Sp}^{\omega}, \mathsf{An}_*) \simeq \mathsf{Sp}$$

Let $E \in \mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*)$, then we can compute

$$\begin{aligned} E^V(\mathrm{pt}) &\simeq E^{V\oplus W}(S^W, \mathrm{pt}) && \text{Thom Isomorphism} \\ &\simeq \Omega^W E^{V\oplus W}(\mathrm{pt}) && \text{Mayer-Vietoris} \end{aligned}$$

Application

Let $\mathcal{D} = \mathsf{An}_*$, then

$$\mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*) \simeq \mathrm{Exc}(\mathsf{Sp}^{\omega}, \mathsf{An}_*) \simeq \mathsf{Sp}$$

Let $E \in \mathsf{Gen}(\mathcal{C}_{\mathsf{An},\mathsf{AffLin}}, \mathsf{An}_*)$, then we can compute

$$\begin{aligned} E^V(\mathrm{pt}) &\simeq E^{V\oplus W}(S^W, \mathrm{pt}) \quad \text{Thom Isomorphism} \\ &\simeq \Omega^W E^{V\oplus W}(\mathrm{pt}) \quad \text{Mayer-Vietoris} \end{aligned}$$

$X(V) := E^V(\mathrm{pt})$ defines an orthogonal spectrum.