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Genuine Cohomology Theories

–2024–

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Universität Münster

Mathematics

Genuine Cohomology Theories

Inaugural dissertation for the award of a
doctoral degree (Dr. rer. nat.)

in the field of Mathematics and Computer Science from
the Faculty of Mathematics and Natural Sciences of the
University of Münster, Germany

submitted
by
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–2024–

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Genuine Cohomology Theories

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April 11, 2024

Abstract

We prove a new universal property of the ∞-category Sp^ω of compact spectra. In its simplest form, this universal property states an equivalence between the ∞-category of additive functors $\mathrm{Sp}^\omega \to \mathcal{D}$ to a certain subcategory of functors $\mathcal{C} \to \mathcal{D}$ out of a specific ∞-category $\mathcal{C}$. The objects of $\mathcal{C}$ are pairs (X, V) where X is a compact homotopy type and V is a vector bundle over X. This allows us to rephrase the Eilenberg-Steenrod axioms in terms of cohomology theories graded by vector bundles instead of integers. We call such cohomology theories genuine cohomology theories. Inspired by this we prove a similar result for the ∞-category of compact genuine G-spectra for a compact Lie group G and recover axioms for genuine cohomology theories which are very similar to $RO(G)$-graded cohomology theories.

Contents

Introduction

Main results Cohomology theories arise in various contexts in mathematics and are a powerful invariant of the objects that one wants to study. In their book [ES52] Eilenberg and Steenrod axiomatized cohomology theories that appear in classical homotopy theory. The Eilenberg-Steenrod axioms are quite simple so that one can easily generalize them to different ∞-categories. For example, functors out of the ∞-category of G-anima satisfying the Eilenberg-Steenrod axioms are represented by so-called naive G-spectra. In practice, people often use cohomology theories that are not naive, but rather genuine in the sense that they have better properties than naive cohomology theories, e.g. Poincaré duality for G-manifolds. In that sense, the Eilenberg-Steenrod axioms are not sufficient to capture the essence of interesting cohomology theories in every setting.

In this paper we give a set of axioms for cohomology theories, that is equivalent to the Eilenberg-Steenrod axioms for the ∞-category of anima, but models $RO(G)$-graded cohomology theories in the ∞-category of G-anima.

We call these cohomology theories *genuine cohomology theories* (GCTs). We want to give a quick description of what a genuine cohomology theory essentially is for a detailed description we refer to Section 3.4.

A *genuine cohomology theory (in the ∞-category of anima with coefficients in abelian groups)* consists of a collection $E^V(X)$ of abelian groups for each compact anima X and vector bundle V over X [1], morphisms

$$f^* := E^W(f)\colon E^W(Y) \to E^{f^*W}(X)$$

for each map $f\colon X \to Y$ of anima and vector bundle W over Y, and

$$\alpha_* := E^\alpha(X)\colon E^V(X) \to E^W(X)$$

for each generalized map $\alpha\colon V \to W$ between vector bundles V, W over X, that satisfy various compatibility requirements, most importantly a certain map

$$E^V(X) \to E^{V\oplus W}(S^W, X)$$

is an equivalence, which we will call the *Thom isomorphism.*

Essential to GCTs is the grading over vector bundles, but if we restrict a GCT E only to trivial bundles we obtain a (classical) cohomology theory

$$E^n(X) := E^{\mathbb{R}^n}(X).$$

[1] A vector bundle V over the anima X is a map $V\colon X \to \coprod_n BO(n)$.

One of our results states that every (classical) cohomology theory arises this way. That is, it extends to a GCT and this extension to a GCT is unique and given by twisted cohomology

$$E^V(X) := [X^{-V}, E].$$[2]

In other words:

Theorem A. *Restriction of a genuine cohomology theory to a (classical generalized) cohomology theory is an equivalence of categories*

$$\{\text{GCTs on } \mathrm{An}^{\omega} \text{ with coefficients in Ab}\} \simeq$$
$$\{\text{Cohomology Theories on } \mathrm{An}^{\omega} \text{ with coefficients in Ab}\}.$$

The inverse functor is given by the extension of a cohomology theory to twisted cohomology.

For more details, we refer to Proposition 3.4.6 and Theorem 4.

A key insight in this paper is that we generalize GCTs so that we allow them to take values in any arbitrary pointed ∞-category instead of abelian groups. To keep track of the additional coherences that appear when working with ∞-categories, we model GCTs as certain functors out of an ∞-category

$$\int_{X \in \mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}^{X}$$

that keeps track of the combinatorics of vector bundles and the objects they live over. We will give more details about the above ∞-category later in the introduction.

In this greater generality of working with arbitrary pointed ∞-category as the target of GCTs, we are able to prove a much stronger result than Theorem A.

Theorem B. *There exists a universal GCT*

$$\mathrm{Th}^{-} \colon \int_{X \in \mathrm{An}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}^{X} \to \mathrm{Sp}^{\omega}.$$

That is, for every GCT $E \colon \int \underline{\mathrm{Sph}^{\mathrm{op}}} \to \mathcal{D}$ there exists a unique functor $\mathcal{E} \colon \mathrm{Sp}^{\omega} \to \mathcal{D}$ such that we have a factorization

$$\begin{tikzcd} \int_{X \in \mathrm{An}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}^{X} \arrow[rr, "\mathrm{Th}^{-}"] \arrow[dr, "E"'] & & \mathrm{Sp}^{\omega} \arrow[dl, dashed, "\mathcal{E}"] \\ & \mathcal{D}. & \end{tikzcd}$$

A more detailed version of Theorem B is given by Theorem 3.

The notion of a GCT only depends on the notion of a vector bundle over an object. Therefore, the axioms of genuine cohomology theories are so formal that they carry over word by word to other settings, for example to the topos of G-anima for a compact Lie group G. There one has a natural candidate for the notion of a vector bundle over a G-anima, by saying that a vector bundle over G/H is nothing else than a representation of H. Thus we define in Section 4.4 G-equivariant genuine cohomology theories and prove

[2] X^{-V} is the Thom spectrum of the dual of V and E is the spectrum representing the cohomology theory E^*.

Theorem C. *The categories of G-equivariant genuine cohomology theories and $RO(G)$-graded cohomology theories are equivalent.*

For more details see Theorem 6. Again we show a deeper result, namely

Theorem D. *There exists a universal GCT*

$$\mathrm{Th}^{-}\colon \int_{X\in \mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}}^X \to \mathrm{Sp}_G^{\omega}\,.$$

That is, for every GCT $E\colon \int \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathcal{D}$ there exists a unique functor $\mathcal{E}\colon \mathrm{Sp}_G^{\omega} \to \mathcal{D}$ such that we have a factorization

$$\begin{tikzcd}
\int_{X\in \mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}}^X \arrow[rr, "\mathrm{Th}^{-}"] \arrow[dr, "E"'] & & \mathrm{Sp}_G^{\omega} \arrow[dl, dashed, "\mathcal{E}"] \\
& \mathcal{D}. &
\end{tikzcd},$$

see Theorem 5.

Motivation and Background Cohomology theories are a powerful tool to study various objects of interest in different fields of mathematics. For example, singular cohomology is one of the go-to invariants to study manifolds. Though singular cohomology is easy to compute, more information about the underlying homotopy type is revealed when one considers generalized cohomology theories. These theories have been axiomatized by Eilenberg and Steenrod in [ES52]. By the famous theorem of Brown [Bro62], cohomology theories are represented by spectra a notion that has first been introduced in [Lim58]. It didn't take much time until people tried to use cohomology theories to understand equivariant phenomena. First people tried to understand global actions on homotopy types via Borel-cohomology introduced in [Bor74]. This only depends on the homotopy orbits of the action, and thus does not take into account the finer equivariant structure of the underlying G-CW complex. Hence Bredon-cohomology was introduced in [Bre67]. Though Bedon-cohomology was capable of distinguishing between equivariant and non-equivariant homotopies, it lacked some crucial properties like Poincaré duality for G-manifolds. Finally, $RO(G)$-graded cohomology theories have been invented. Their additional grading over representations of G was the nodular point that allows Poincaré-duality for G-manifolds. It is these further properties of $RO(G)$-graded cohomology theories that give their representing spectra the name genuine G-spectra. The term *genuine* emphasizes the importance of certain properties of the nature of cohomology theories which are not necessarily captured by Eilenberg-Steenrod-like axioms.

Cohomology theories do not stop becoming useful at the case of global actions. To study orbifolds and topological stacks with cohomology theories people began to let the, G in genuine G-spectra vary through all compact Lie groups. This is necessary to adapt to the fact that isotropy groups of orbifolds do not sit inside one big compact Lie group. This was the birth hour of global homotopy theory (see for example [GH], [Sch18] and [Jur]). Though with orthogonal spectra one had a suitable model for stable global homotopy types at hand many questions about global homotopy theory arose. Which universal properties does the ∞-category of global spectra suffice? Is there a set of Eilenberg-Steenrod-like axioms for global cohomology theories?

While the former question drew a lot of attention in the recent past (see for example [LNP22] and [CLL]), it is the latter question that motivated this paper and the introduction of what we call *genuine cohomology theories*.

The typical strategy to establish axioms for cohomology theories is to prove a Brown-representability theorem for the homotopy category of the representing stable homotopy types. The modern formulation of Brown representability by Lurie [Lur16, Theorem 1.4.1.2] seems to fail for global homotopy types as there are simply not enough cogroup objects in the homotopy category of global homotopy types. Some authors call these cogroup objects generalized spheres. It is these spheres that become invertible in the stabilization process and are responsible for the additional grading appearing for example in the formulation of $RO(G)$-graded cohomology theories. The lack of generalized spheres can be fixed by considering the ∞-category of global homotopy types as a parametrized category. More precisely one has to invert generalized spheres in all of the slices $\mathrm{An}_{\mathrm{glo}}/_{\mathbb{B}G}$ at the same time. The analogous task for the ∞-category of genuine G-spectra would translate as: 'invert all H-representation spheres' at the same time. But this task is redundant as H-representations spheres sit inside G-representation spheres and thus become automatically invertible. Globally, we do not have those lucky circumstances. As spheres in a slice category $\mathcal{X}/_X$ can be viewed as sphere bundles over X, it is natural to try to find a formulation of cohomology theories that are graded by the sphere bundles over objects. This is what we call genuine cohomology theories.

History and Related Work One can think of the notion of genuine cohomology theories, as an attempt to axiomatize line bundle twisted cohomology, where the line bundle comes from an unstable spherical fibration. There is a vast literature on twisted cohomology theories with a good general treatment developed in [ABG$^+$14a, ABG18, ABG$^+$14b]. For an overview of the connection of twisted cohomology with physics, we refer to [Ros24].

Before twisted cohomology became the generic term, (because of the popularity of the term twisted K-theory), it was known as cohomology with local coefficients. The first occurrence of cohomology with local coefficients in the literature can be traced back to [Rei53] as the cohomology in terms of cochains of the universal cover and the action of the deck transformation group. Independently, [Ste43] introduced the notion of cohomology with local coefficients. Larmore used the term twisted cohomology in [Lar72] and considered the cohomology with coefficients in what later in[MS06] became known as parametrized spectrum.

Though definitions of twisted cohomology groups are found in abundance in the literature, there is a very limited supply of axiomatic descriptions of twisted cohomology theories. What seems to be closely related to our work is the notion of bivariant theory introduced by Fulton and MacPherson in [FM81]. They axiomatically introduce bivariant homology theories and prove, though not in our language, that for a homotopy ring spectrum E the groups

$$\mathbb{B}^*(Y \to X) := \pi_{-*} \lim_{x \in X} (\Sigma^\infty_+(Y \times_X \{x\}) \otimes E)$$

define a bivariant homology theory. In that sense, the groups

$$\mathbb{B}^*(S^V_X \to X)$$

are closely related to our genuine cohomology groups

$$E^V(X).$$

This is also captured by the fact, that Fulton and MacPherson only require the theories $\mathbb{B}^*$ to be covariant functorial in what they call *confined maps* $S^V \to S^W$ for which our *affine linear morphisms* $V \to W$ are a special case. Fulton and MacPherson do not classify bivariant homology theories. Their interest was to use them as an organization principle for Riemann-Roch theorems in algebraic geometry. In contrast to that, we are interested in the classification of genuine cohomology theories in the form of Brown representability theorems.

Outline of the paper We want to give a quick overview of the main ideas and the structure of the paper.

Consider the functor

$$\mathrm{Th}^-\colon\ \mathrm{An}^\omega/_{\coprod_n BO(n)} \to \mathrm{Sp}^\omega$$

that sends a vector bundle

$$V\colon X \to \coprod_n BO(n)$$

to the Thom spectrum of its dual

$$\mathrm{Th}^-(X,V) := X^{-V} := \operatorname*{colim}_{x\in X} \mathbb{D}\Sigma^\infty S^{V_x}.$$

Let $E\colon\ \mathrm{Sp}^\omega \to \mathrm{Ab}^{\mathrm{op}}$ be any cohomology theory. The functor $E\circ\mathrm{Th}^-$ not only computes the E-cohomology groups of every compact anima but also the E-homology groups of every smooth closed manifold by Atiyah duality.

By this observation, we are motivated to model genuine cohomology theories as certain functors

$$\mathrm{An}^\omega/_{\coprod_n BO(n)} \to \mathrm{Ab}^{\mathrm{op}}.$$

Ideally, we want to intrinsically describe the category of functors

$$E\colon\ \mathrm{An}^\omega/_{\coprod_n BO(n)} \to \mathrm{Ab}^{\mathrm{op}}$$

that factorize through Th^- and a cohomology theory $\tilde{E}\colon\ \mathrm{Sp}^\omega \to \mathrm{Ab}^{\mathrm{op}}$, i.e.

$$\begin{tikzcd} \mathrm{An}^\omega/_{\coprod_n BO(n)} \arrow[rr, "\mathrm{Th}^-"] \arrow[dr, "E"'] & & \mathrm{Sp}^\omega \arrow[dl, dotted, "\tilde{E}"] \\ & \mathrm{Ab}^{\mathrm{op}} & \end{tikzcd}$$

Our methods to prove the universal property for the functor Th^- requires that we enlarge its source category

$$\mathrm{An}^\omega/_{\coprod_n BO(n)} \simeq \int_{\mathrm{An}^\omega} \mathrm{Fun}(X, \coprod_n BO(n))$$

to an ∞-category

$$\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}X}}$$

where the parametrized ∞-categories $\underline{\mathrm{Sph}^{\mathrm{op}X}}$ sit (non fully-faithful) in between the ∞-categories

$$\mathrm{Fun}(X, \coprod_n BO(n)) \subset \underline{\mathrm{Sph}^{\mathrm{op}X}} \subset \mathrm{Fun}(X, \mathrm{An}^\omega_*)^{\mathrm{op}},$$

but very close to the former. This is because our methods require the source of Th^{-} to contain morphisms

$$f\colon V \to W$$

between vector bundles V, W over X. We even require the ∞-category $\underline{\mathrm{Sph}}^{\mathrm{op}X}$ to contain more morphisms than linear maps between vector bundles over X. Crucial, to our approach, is that the ∞-category $\underline{\mathrm{Sph}}^{\mathrm{pt}}$ contains more than one morphism from the 0-vector space to another vector space V. We require

$$\mathrm{Map}_{\underline{\mathrm{Sph}}^{\mathrm{pt}}}(0, V) \simeq S^V.$$

This leads us to the notion of an *affine linear morphisms* which we define in Subsections 3.1 and 4.1. We formalize this requirement in Section 2. There we explain what the parametrized categories $\underline{\mathrm{Sph}^{\mathrm{op}}}$ needs to satisfy to get a grasp on the category of functors

$$\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}X}} \to \mathrm{Sp}^{\omega} \xrightarrow{\text{cohomology theory } E} \mathrm{Ab}^{\mathrm{op}}$$

See Definitions 2.2.1,2.2.8 and 2.3.4.

Our methods are quite formal so they work in a larger generality than just for anima. We consider a large class of topoi $\mathcal{X}$ in Section 2 that include the ∞-categories of anima and G-anima for a compact Lie group G. With Theorem 1 we prove an unstable version of Theorem B that is completely formal and works for any topos $\mathcal{X}$. The parametrized category

$$\underline{\mathrm{Sph}^{\mathrm{op}}(W)}$$

appearing in Theorem 1 is a construction that we explain in detail in Subsection 2.1. It is a category that parametrizes the data of maps of vector bundles

$$\begin{array}{rcl} f\colon U & \to & V \\ g\colon V^{\perp_W} & \to & U^{\perp_W} \end{array}$$

that are unstable representatives of the pair of dual morphisms after stabilization

$$\begin{gathered} \mathbb{D}[\Sigma^{\infty} f\colon \Sigma^{\infty} S^U \to \Sigma^{\infty} S^V] \simeq \\ [\Sigma^{\infty - W} g\colon \Sigma^{\infty - W} S^{V^{\perp_W}} \to \Sigma^{\infty - W} S^{U^{\perp_W}}] \end{gathered}$$

It is the key insight that allows us to deduce stable versions, namely Theorem 3 and Theorem 5 from Theorem 2. It does that by replacing the functor

$$\mathrm{Th}^{-}\colon (X; U, V) \mapsto X^{-U}$$

by its unstable version

$$\mathrm{Th}^{\perp}\colon (X; U, V) \mapsto \mathrm{Th}_X(V) := S^V/X$$

which sends a pair of vector bundles (U, V) over X to the Thom construction of V over X. See Lemma 3.3.9. This leads us to Theorem 1 which essentially states that

$$\begin{array}{rrcl} \mathrm{Th}^{\perp}\colon & \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} & \to & \mathcal{X}_* \\ & (X; U, V) & \mapsto & \mathrm{Th}_X(V) \end{array}$$

is the universal functor among those functors with source

$$\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}$$

that send certain squares

$$\begin{array}{ccccc}
(\mathrm{pt}; W, 0) & \longleftarrow & (X; W, 0) & \longrightarrow & (X, \emptyset, \emptyset) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathrm{Th}\, V; W, 0) & \longleftarrow & (S^V; W, 0) & \xrightarrow[\theta]{} & (X, U, V)
\end{array}$$

to pushouts and $(X, \emptyset, \emptyset)$ to the zero object. Noteworthy is that θ arises as a counit of an adjunction (see Proposition 2.2.7 and Corollary 2.2.10) and plays a crucial role in the proof of the universal property of $\mathrm{Th}^{\perp}$ as it will later induce the Thom isomorphism on genuine cohomology theories.

In analogy to Theorem 1 one would expect that Th^{-} is the universal functor among those functors with source

$$\int_X \underline{\mathrm{Sph}^{\mathrm{op}}}$$

that sends squares of the form

$$\begin{array}{ccccc}
(\mathrm{pt}, W) & \longleftarrow & (X, W) & \longrightarrow & (X, \emptyset) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathrm{Th}\, V, W) & \longleftarrow & (S^V, W) & \longrightarrow & (X, U)
\end{array}$$

to pushouts and $(X, \emptyset)$ to the zero object. Indeed, we verify this in Theorem 3 and Theorem 5 over the ∞-category of anima and G-anima respectively. But with the minor difference, that we require the objects X to be compact. The reason why we only verify the universal property for Th^{-} in these cases lies in our methods. We require the squares above to depend functorially in the data $(X; U, V)$ and $(X; U)$ respectively. For the squares in the ∞-category

$$\int \underline{\mathrm{Sph}^{\mathrm{op}}(W)}^X$$

this is always the case but for the squares inside the ∞-category

$$\int \underline{\mathrm{Sph}^{\mathrm{op}}}^X$$

this is false. This has something to do with the fact that inside the data of an object of $\int \underline{\mathrm{Sph}^{\mathrm{op}}(W)}^X$ we chose a complement V to U inside W, i.e.

$$U \oplus V \simeq W$$

which makes the construction of the squares functorial. In contrast to that, the data of an object of $\int \underline{\mathrm{Sph}^{\mathrm{op}}}$ does not contain the data of a complement to U inside W. Since data of a lift of an object of $\int \underline{\mathrm{Sph}^{\mathrm{op}}}$ to an object of $\int \underline{\mathrm{Sph}^{\mathrm{op}}(W)}$ is essentially a choice of a complement to U inside W, one can hope that the anima of choices becomes contractible for growing codimension of U inside W. Indeed, we show in two cases that the functors

$$\int \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \int \underline{\mathrm{Sph}^{\mathrm{op}}(W \oplus V')} \colon (X; U, V) \mapsto (X; U, V \oplus V')$$

induce an equivalence of parametrized categories

$$\operatorname*{colim}_W \int \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \simeq \int \underline{\mathrm{Sph}^{\mathrm{op}}}.$$

This is the content of lemma 3.3.2 in the case of anima and vector bundles, where we let W run through all trivial n-dimensional vector bundles. In the case of G-anima and G-equivariant vector bundles, we let W run through all finite-dimensional subrepresentations of a fixed universe for G, and consider them as trivial G-equivariant vector bundles (see Lemma 4.2.3). In order for the equivalence $\mathrm{colim}_W \int \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \simeq \int \underline{\mathrm{Sph}^{\mathrm{op}}}$ to hold, we need to restrict the unstraightenings

$$\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \quad \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}}$$

to compact objects of $\mathcal{X}$

$$\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \quad \int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}.$$

This manifests itself in the fact that over non-compact anima not every bundle has a finite-dimensional complement bundle. For this reason we introduce Definition 2.3.4 and prove Theorem 2, which essentially states the aforementioned universal property for the functor

$$\mathrm{Th}^\perp \colon \int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \mathcal{X}^\omega_*$$

which is a version restricted to compact objects. Inspired by the commutative square

$$\begin{tikzcd}
\int_{\mathcal{X}^\omega} \mathrm{Sph}^{\mathrm{op}}(W) \arrow[r, "\mathrm{Th}^\perp"] \arrow[d] & \mathcal{X}^\omega_* \arrow[d, "- \wedge S^{V'}"] \\
\int_{\mathcal{X}^\omega} \mathrm{Sph}^{\mathrm{op}}(W \oplus V') \arrow[r, "\mathrm{Th}^\perp"'] & \mathcal{X}^\omega_*
\end{tikzcd}$$

we study the functoriality of

$$\mathrm{Th}^\perp \colon \int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \mathcal{X}^\omega_*$$

in the variable W in Subsection 2.1 and 2.4.

The goal of Section 2 is to prove Proposition 2.4.4. Everything in this section is formal and works for a large class of topoi. Sections 3 and 4 then take this result and specialize it to the topos of anima and the topos of G-anima for a compact Lie group G respectively. Strictly speaking, the results of Section 4 imply the results of Sections 3 and do not depend on them. We still decided to spell out the non-equivariant case, because it is more accessible and shows the main ideas of the paper more transparently. Also, our method to induce the universal property for Th^- from the universal property for $\mathrm{Th}^\perp$ do not depend on the choice of a universe for $G = \{e\}$. Instead of the finite-dimensional subspaces of a universe we can use the natural numbers for the parametrization appearing for example in Lemma 3.3.2. This is closely related to the fact, that one can model the same ∞-category of spectra as either orthogonal or sequential spectra.

In these sections, more specifically in subsections 3.1 and 4.1 respectively, we define suitable categories of compact spheres (see Definition 2.3.4)

$$\underline{\mathrm{Sph}}.$$

We then prove our main results

Theorem E.
$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^\omega, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}).$$

(see Theorem 3) and

Theorem F.
$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}).$$

(see Theorem 5). In Subsections 3.4 and 4.3 we specialize these results to the category $\mathcal{D} = \mathrm{Ab}^{\mathrm{op}}$ of abelian groups. We observe that inside the categories

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^\omega, \mathrm{Ab}^{\mathrm{op}})$$

and

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathrm{Ab}^{\mathrm{op}})$$

lie the full subcategories of cohomology theories

$$\{\text{cohomology theories}\} \subset \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^\omega, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}$$

and G-equivariant genuine cohomology theories

$$\{\text{genuine } G\text{-equivariant cohomology theories}\} \subset \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}$$

respectively. We then identify which subcategories of

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}.$$

and

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}.$$

correspond to the categories of cohomology theories (see Definition 3.4.5 and 4.3.1). We denote these categories by

$$\mathrm{Gen}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})$$

and

$$\mathrm{Gen}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})$$

respectively. Since there is an axiomatic description of cohomology theories, we also give an axiomatic description of the categories above in Subsection 3.5 and 3.5.

Finally in Section 5 we give a quick sketch of how we think one can extend our results of Section 3 and 4 to the general context of a strongly compact topos and a category of compact spheres.

Notation and conventions Throughout this paper we freely make use of the language of higher category theory. We will generally follow a model-independent approach to higher categories. We consider every ordinary 1-category as an ∞-category via the nerve construction, but we do not write $N(\mathcal{C})$ for the nerve of a category $\mathcal{C}$, instead, we simply write $\mathcal{C}$.

Table 1.1: Common notation

$\mathcal{C}\mathrm{at}_\infty$	Ambient ∞-category of ∞-categories
An	∞-category of anima or ∞-groupoids
$\int_{\mathcal{C}} \mathcal{F} \to \mathcal{C}$	Cartesian unstraightening of a functor $\mathcal{F} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$
$\mathcal{C}^{\simeq}$	The anima given by the core of the ∞-category $\mathcal{C}$
$\mathcal{C}^{\Delta^1}$	Arrow category of $\mathcal{C}$
$\mathcal{C}_*$	∞-category of pointed objects in $\mathcal{C}$
$\mathcal{C}^\omega$	∞-category of compact objects in $\mathcal{C}$
$\underline{\mathcal{C}}$	Parametrized ∞-category over some topos
Pr^L	∞-category of presentable ∞-categories and colimit preserving functors
Pr^R	∞-category of presentable ∞-categories and right adjoint functors
$f^* \colon \underline{\mathcal{C}}^Y \to \underline{\mathcal{C}}^X$	pullback functor between the fibers of the Cartesian unstraightening $\int \underline{\mathcal{C}}$ over an edge $f \colon X \to Y$
$f_! \colon \underline{\mathcal{C}}^X \to \underline{\mathcal{C}}^Y$	left adjoint of f^*
$f_* \colon \underline{\mathcal{C}}^X \to \underline{\mathcal{C}}^Y$	right adjoint of f^*

The Abstract Machinery of Genuine Homology Theories

2.1 Complements and Lax Action Groupoids

In this subsection, we use the machinery of span categories which became popular in [Bar17]. We mainly rely on some results of [HHLN23] that we quickly recall here.

Definition 2.1.1 (Definition 2.1. of [HHLN23])**.** *An* adequate triple $(\mathcal{C}, b.w., f.w.)$ *consists of an ∞-category $\mathcal{C}$ and two wide subcategories $b.w.$ and $f.w.$, whose morphisms are called* backwards *and* forwards *respectively, such that*

1. *for any forward morphism $f\colon Y \to X$ and any backward morphism $g\colon X' \to X$ there exists a pullback*

$$\begin{array}{ccc} Y' & \xrightarrow[g']{} & Y \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ X' & \xrightarrow{g} & X, \end{array}$$

2. *and in any such pullback, f' is again forward and g' is backward.*

Squares whose horizontal arrows are backwards and whose vertical arrows are forwards are called ambigressive*, and* ambigressive cartesian *if they are furthermore pullbacks diagrams. A functor*

$$F\colon (\mathcal{C}, b.w., f.w.) \to (\mathcal{D}, b.w.', f.w.')$$

of adequate triples is given by a functor $F\colon \mathcal{C} \to \mathcal{D}$ which preserves ambigressive pullbacks.

Remark 2.1.2. *The notion of an adequate triple is useful because one can associate a span category*

$$\mathrm{Span}(\mathcal{C}, b.w., f.w.)$$

with it. (see [HHLN23])

Informally, $\mathrm{Span}(\mathcal{C}, b.w., f.w.)$ *is constructed from $\mathcal{C}$ by restricting the morphisms to those that are either backwards or forwards and then reversing the direction of the backwards morphisms. An important example (Example 2.3 of [HHLN23]) is*

$$\mathrm{Span}(\mathcal{C}, \mathcal{C}, \mathcal{C}^{\simeq})$$

which is equivalent to $\mathcal{C}^{\mathrm{op}}$ (Proposition 2.15 of [HHLN23]).

For the rest of this subsection, we fix a symmetric monoidal pointed ∞-category $(\mathcal{C}, \oplus)$, that is

- a symmetric monoidal ∞-category $(\mathcal{C}, \oplus)$,
- with a zero object $\emptyset \in \mathcal{C}$,
- such that $\emptyset \oplus c \simeq \emptyset$ for all $c \in \mathcal{C}$.[1]

We need the machinery of span categories to define the ∞-category $\mathcal{C}^{\mathrm{op}}(c)$; a gadget that will keep track of the data of pairs of objects (a, b) and an equivalence $\alpha\colon a \oplus b \to c$ in $\mathcal{C}$, such that b behaves like the monoidal dual of a when c is the unit of $\mathcal{C}$.

Construction 2.1.3. *Let $\oplus/c$ be the pullback of ∞-categories*

$$\begin{tikzcd} \oplus/c \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}/c \arrow[d, "\mathrm{fgt}"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\oplus"'] & \mathcal{C} \end{tikzcd}$$

The ∞-category $\mathcal{C} \times \mathcal{C}$ carries a natural adequate triple stricture given by

$$b.w._{\mathcal{C}\times\mathcal{C}} := \mathcal{C} \times \mathcal{C}^{\simeq}$$

and

$$f.w._{\mathcal{C}\times\mathcal{C}} := \mathcal{C}^{\simeq} \times \mathcal{C}.$$

We invoke Proposition 2.6 of [HHLN23] to find that $\oplus/c$ inherits the structure of an adequate triple:

$$\begin{aligned} b.w._{\oplus/c} &:= q^{-1}(b.w._{\mathcal{C}\times\mathcal{C}}) \\ f.w._{\oplus/c} &:= q^{-1}(f.w._{\mathcal{C}\times\mathcal{C}}), \end{aligned}$$

since q is a right fibration. Let $\mathcal{C}^{\mathrm{op}}(c)$ be the full subcategory of

$$\mathrm{Span}(\oplus/c, b.w._{\oplus/c}, f.w._{\oplus/c})$$

on those objects $(a, b, \alpha\colon a \oplus b \xrightarrow{\simeq} c)$ where α is an equivalence or $a = b = \emptyset$.

Remark 2.1.4. *By Proposition 2.15 of [HHLN23] we find*

$$\mathrm{Span}(\mathcal{C} \times \mathcal{C}, b.w._{\mathcal{C}\times\mathcal{C}}, f.w._{\mathcal{C}\times\mathcal{C}}) \simeq \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$

and by Proposition 2.16 of [HHLN23] q induces a functor

$$\mathcal{C}^{\mathrm{op}}(c) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

A typical morphism in $\mathcal{C}^{\mathrm{op}}(c)$ is of the form

$$(a, b, \alpha\colon a \oplus b \xrightarrow{\simeq} c) \xleftarrow{f} (A, B, \beta\colon A \oplus B \xrightarrow{\simeq} c) \xrightarrow{g} (a', b', \alpha'\colon a' \oplus b' \xrightarrow{\simeq} c)$$

[1] Though we denote the tensor product with the $\oplus$ symbol, it is not a biproduct, since the zero object is not necessarily the unit of $\mathcal{C}$. We use this notation because we want to think of elements of $\mathcal{C}$ as vector spaces. Then $\emptyset$ should be an empty vector space, which makes sense the moment we allow affine linear maps since then 0 is not initial and terminal anymore.

In particular, it gives us a commutative diagram

$$\begin{tikzcd} & A \oplus B \arrow[dl, "f=f_1\oplus f_2"'] \arrow[d, "\beta"] \arrow[dr, "g=g_1\oplus g_2"] & \\ a \oplus b \arrow[r, "\alpha"'] & c & a' \oplus b' \arrow[l, "\alpha'"] \end{tikzcd}$$

where f_2 and g_1 are equivalences or $A = B = \emptyset$. Hence, we can think of the data of a morphism in $\mathcal{C}^{\mathrm{op}}(c)$ to consist of

- *A map $\phi = (g_1)^{-1} \circ f_1 : a' \to a$,*
- *A map $\psi = g_2 \circ (f_2)^{-1} : b \to b'$,*
- *A commutative square*

$$\begin{tikzcd} a' \oplus b \arrow[r, "a'\oplus\psi"] \arrow[d, "\phi\oplus b"'] & a' \oplus b' \arrow[d, "\alpha'"] \\ a \oplus b \arrow[r, "\alpha"] & c \end{tikzcd}$$

The next lemma makes this idea precise.

Lemma 2.1.5. *On mapping anima, we have a natural pullback*

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}^{\mathrm{op}}(c)}((a,b,\alpha),(a',b',\alpha')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}(b,b') \arrow[d, "\alpha'\circ(a'\oplus -)"] \\ \mathrm{Map}(a',a) \arrow[r, "\alpha\circ(-\oplus b)"'] & \mathrm{Map}(a'\oplus b, c) \end{tikzcd}$$

Proof. Let

$$M := \mathrm{Map}_{\mathcal{C}^{\mathrm{op}}(c)}((a,b,\alpha),(a',b',\alpha')).$$

In the case $\emptyset \in \{a, b, a', b'\}$ we easily compute $M \simeq \mathrm{pt} \simeq \mathrm{Map}_{\mathcal{C}}(b, \emptyset) \simeq \mathrm{Map}_{\mathcal{C}}(\emptyset, a')$ and the claim follows from the fact that $\emptyset$ is an absorbing element with respect to $\oplus$. So suppose $a, b, a', b' \neq \emptyset$. By unraveling the construction of $\mathcal{C}^{\mathrm{op}}(c)$ and the main result of [HHLN23] we have that M is equivalent to the core of the full subcategory of

$$\mathrm{Fun}(\Lambda_0^2, \oplus/c) \times_{(\mathrm{ev}_1,\mathrm{ev}_2),\oplus/c\times\oplus/c} \{(a,b,\alpha),(a',b',\alpha')\}$$

on those wedges

$$(a,b,\alpha) \xleftarrow{f} (A,B,\beta) \xrightarrow{g} (a',b',\alpha')$$

for which f is a right equivalence and g is a left equivalence. By writing $\mathrm{Fun}(\Lambda_0^2, \oplus/c)$ as the pullback $\oplus/c^{\Delta^1} \times_{s,\oplus/c,t} \oplus/c^{\Delta^1}$ and commuting limits with limits we find that M is equivalent to the core of

$$r.e./(a,b,\alpha) \times_{\oplus/c} l.e./(a',b',\alpha').$$

The ∞-category of right equivalences, for example, is itself a pullback of ∞-categories. Therefore, we can compute $r.e./(a,b,\alpha)$, after commuting limits again, as the pullback of

$$\mathcal{C}/a \times (\mathcal{C})^{\simeq}/b \xrightarrow{(\oplus,c)} \mathcal{C}/a\oplus b \times \mathcal{C}/c \xleftarrow{(s,t)} (\mathcal{C})^{\Delta^1}/\alpha$$

The ∞-category $(\mathcal{C})^{\simeq}/b$ is contractible, so we can omit it in the above wedge. Moreover, the above wedge is of the form $A \times_{\mathrm{pt}} \mathrm{pt} \to B \times_{\mathrm{pt}} C \leftarrow D \times_D D$, so we can compute its pullback P as the pullback of $A \times_B D \to D \leftarrow \mathrm{pt} \times_C D$. Let us first compute

$$\begin{aligned}
\mathrm{pt} \times_C D &= \{\mathrm{id}_c\} \times_{\mathcal{C}/c,t} (\mathcal{C})^{\Delta^1}/\alpha \\
&\simeq \{\mathrm{id}_c\} \times_{(\mathcal{C})^{\Delta^1}, \mathrm{ev}_{\{1\}\subset\{1,2\}}} \mathrm{Fun}(\mathrm{P}(\langle 2\rangle), \mathcal{C}) \times_{(\mathcal{C})^{\Delta^1}, \mathrm{ev}_{\{2\}\subset\{1,2\}}} \{\alpha\} \\
&\simeq \mathcal{C}/a \oplus b = B.
\end{aligned}$$

One can check that the composition $B = \mathrm{pt} \times_C D \to D \to B$ is the identity, and that makes the map from P to $A = \mathcal{C}/a$ an equivalence. Similarly one can compute that $l.e./(a', b', \alpha')$ is equivalent to $\mathcal{C}/a'$. If we put these results together we can identify M with the core of the pullback

$$\mathcal{C}/a \to \oplus/c \leftarrow \mathcal{C}/b'$$

where the left map sends a map $f \colon U \to a$ to the triple $(U, b, U \oplus b \xrightarrow{f \oplus b} a \oplus b \xrightarrow{\alpha} c)$ and the right map sends $g \colon U \to b'$ to the triple $(a', U, a' \oplus U \xrightarrow{a' \oplus g} a' \oplus b' \xrightarrow{\alpha'} c)$. There is an evident map from $\mathrm{Map}(a', a) \times_{\mathrm{Map}(a' \oplus b, c)} \mathrm{Map}(b, b')$ into $\mathcal{C}/a \times_{\oplus/c} \mathcal{C}/b'$. It is obtained as the pullback:

$$\begin{tikzcd}
\mathrm{Map}(a', a) \times_{\mathrm{Map}(a' \oplus b, c)} \mathrm{Map}(b, b') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}/a \times_{\oplus/c} \mathcal{C}/b \arrow[d, "s \times_s s"] \\
\mathrm{pt} \times_{\mathrm{pt}} \mathrm{pt} \arrow[r, "(a', b)"] & \mathcal{C} \times_{\mathrm{id} \times \{a'\}, \mathcal{C} \times \mathcal{C}, \{b\} \times \mathrm{id}} \mathcal{C}
\end{tikzcd}$$

After taking cores the lower map becomes an equivalence, which shows that M is equivalent to $\mathrm{Map}(a', a) \times_{\mathrm{Map}(a' \oplus b, c)} \mathrm{Map}(b, b')$. □

Now we want to study the functoriality in c of the construction $\mathcal{C}^{\mathrm{op}}(c)$.

Definition 2.1.6. *Let M be a symmetric monoidal groupoid, i.e. an $\mathbb{E}_\infty$-monoid in* An. *The* lax action groupoid *of M on itself is the ∞-category*

$$M//^{\mathrm{lax}}M := \int_{BM} M$$

where M acts on itself via left translation.

Remark 2.1.7. *One can think of an object of $M//^{\mathrm{lax}}M$ as an object m of M and a morphism exists from m into n if n can be written as the sum $n \simeq m \oplus k$ for some k and the morphism remembers the specific choice of k.*

Definition 2.1.8. *Let M be a symmetric monoidal groupoid. And let $\mathcal{D}$ be an ∞-category. A* fanned out M-action *is a functor $d \colon M//^{\mathrm{lax}}M \to \mathcal{D}$. An* equivariant map *between fanned out M-actions is a natural transformation between functors $M//^{\mathrm{lax}}M \to \mathcal{D}$. A fanned out M-action d encodes in particular for every pair of objects $m, n \in M$ a morphism*

$$d(m) \to d(m \oplus n)$$

in $\mathcal{D}$.

Example 2.1.9. *Via the pullback along the functor $M//^{\mathrm{lax}}M \to BM$ every object d with an M-action defines a fanned out M-action. Where the pair of objects $m, n \in M$ gets sent to the action of n on d:*

$$d(m) = d \xrightarrow{\text{action of } n} d = d(m \oplus n).$$

Construction 2.1.10. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let M be its core. The assignment $d \mapsto \mathcal{C}/d$ defines a fanned out M-action via the obvious functors*

$$\mathcal{C}/d \xrightarrow{e\oplus -} \mathcal{C}/e \oplus d.$$

We want to make this precise. The ∞-category $\mathcal{C}$ and its arrow category $\mathcal{C}^{\Delta^1}$ carry a natural M action, such that the functors

$$\mathrm{target}\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$$

and

$$M \to \mathcal{C}$$

are equivariant. Hence, we can form the pullback

$$\begin{tikzcd}
\int_{BM} \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int_{BM} \mathcal{C}^{\Delta^1} \arrow[d] \\
M//^{\mathrm{lax}}M \arrow[r] & \int_{BM} \mathcal{C}
\end{tikzcd}$$

The left vertical map now is the fibration classifying the functor $d \mapsto \mathcal{C}/d$.

We let M act on $\mathcal{C}$ via left translation and on $\mathcal{C} \times \mathcal{C}$ by letting it act trivially on the left factor and via left translation on the right factor, that is $e \in M$ acts as

$$(c, d) \mapsto (c, e \oplus d).$$

These actions define a fanned-out M-action on the pullback

$$\begin{tikzcd}
\oplus/d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}/d \arrow[d] \\
\mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

This action is also compatible with the adequate triple structure. Hence we obtain an induced fanned-out M-action on the span category and also on

$$(\mathcal{C}^{\mathrm{op}}(d))_{d \in M}.$$

Let M act trivially on $\mathcal{C}^{\mathrm{op}}$. The forgetful functors

$$\mathcal{C}^{\mathrm{op}}(d) \to \mathcal{C}^{\mathrm{op}} = \mathrm{Span}(\mathcal{C}, b.w. = \mathcal{C}, f.w. = \mathcal{C}^{\simeq})$$

induced from the projection to the first coordinate

$$\oplus/d \xrightarrow{q} \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

are equivariant maps of fanned-out M-actions. Hence, we obtain a functor

$$\operatorname*{colim}_{d \in M//^{\mathrm{lax}}M} \mathcal{C}^{\mathrm{op}}(d) \to \mathcal{C}^{\mathrm{op}}.$$

Conjecture 2.1.11. *The functor*

$$\operatorname*{colim}_{d \in M//^{\mathrm{lax}}M} \mathcal{C}^{\mathrm{op}}(d) \xrightarrow{\simeq} \mathcal{C}^{\mathrm{op}}$$

is an equivalence.

Remark 2.1.12. *In Lemma 3.3.2 and 4.2.3 we prove this conjecture for certain symmetric monoidal ∞-categories $\mathcal{C}$ and subcategories of the $M//^{\mathrm{lax}}M$, which remain conjecturally cofinal in $M//^{\mathrm{lax}}M$.*

2.2 Genuine Homology Theories with fixed Trivialization

In this section, we use the lingo of categories internal to a topos. We refer the interested reader to [Mar22] for a detailed introduction to this subject.

Definition 2.2.1. *Let $\mathcal{X}$ be a topos. An* $\mathcal{X}$-category $\underline{\mathcal{C}}$ *is a limit-preserving functor*

$$\mathcal{X}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty \colon X \mapsto \mathcal{C}^X.$$

An $\mathcal{X}$-functor *is then a natural transformation of such functors. Likewise one defines a* symmetric monoidal $\mathcal{X}$-category $\mathcal{C}$ *to be a limit-preserving functor*

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty) \colon X \mapsto \mathcal{C}^X.$$

A pointed symmetric monoidal $\mathcal{X}$-category *is a symmetric monoidal $\mathcal{X}$-category $\mathcal{C}$ such that $\mathcal{C}^X$ is a symmetric monoidal pointed ∞-category for all $X \in \mathcal{X}$ and the pullback functors preserve the zero object.*

Though [Mar22] introduce $\mathcal{X}$-categories as complete Segal objects in $\mathcal{X}$, we will work with the equivalent notion of sheaves of ∞-categories on $\mathcal{X}$ as is proven in Proposition 3.5 of [Mar22].

Example 2.2.2. *For every topos $\mathcal{X}$, we have the $\mathcal{X}$-categories $\underline{\mathcal{X}}, \underline{\mathcal{X}_*}$ defined via*

$$\underline{\mathcal{X}} \colon X \mapsto \mathcal{X}_{/X}$$
$$\underline{\mathcal{X}_*} \colon X \mapsto \mathcal{X}_{X/\cdot/X}.$$

If $\underline{\mathcal{C}}$ is an $\mathcal{X}$-category, then for every $X \in \mathcal{X}$ we have $\mathcal{X}_{/X}$-categories $\underline{\mathcal{C}^X}$ defined via

$$(Y \to X) \mapsto \mathcal{C}^Y.$$

Construction 2.2.3. *Let $\underline{\mathcal{C}}$ be an $\mathcal{X}$-category. Then $\mathcal{C}^{\mathrm{pt}}$ is an $\mathcal{X}$-enriched category; we present a construction for the $\mathcal{X}$-mapping objects. Since $\mathcal{X}$ is a topos, it is equivalent to the ∞-category of limit preserving functors $\mathcal{X}^{\mathrm{op}} \to \mathrm{An}$. For every pair of objects $c, d \in \mathcal{C}^{\mathrm{pt}}$, we obtain a limit preserving functor*

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{An} \colon X \mapsto \mathcal{C}^X((X \to \mathrm{pt})^*c, (X \to \mathrm{pt})^*d).$$

The resulting functor $(\mathcal{C}^{\mathrm{pt}})^{\mathrm{op}} \times \mathcal{C}^{\mathrm{pt}} \to \mathcal{X}$ is denoted by $\mathrm{Map}_{\underline{\mathcal{C}}}(-,-)$. Applying this construction to the $\mathcal{X}_{/X}$-categories $\underline{\mathcal{C}^X}$, we obtain a functor of $\mathcal{X}$-categories

$$\begin{aligned} \underline{\mathcal{C}}(-,-) \colon \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}} &\to \underline{\mathcal{X}} \\ \underline{\mathcal{C}}^X(-,-) = \mathrm{Map}_{\underline{\mathcal{C}^X}}(-,-) \colon (\mathcal{C}^X)^{\mathrm{op}} \times \mathcal{C}^X &\to \mathcal{X}_{/X}. \end{aligned}$$

Every object $c \in \mathcal{C}^{\mathrm{pt}}$ defines a functor of $\mathcal{X}$-categories

$$\underline{\mathcal{C}}(c,-)\colon \underline{\mathcal{C}} \to \underline{\mathcal{X}}$$

via

$$X \mapsto [\mathcal{C}^X \to \mathcal{X}_{/X}\colon d \mapsto \mathrm{Map}_{\underline{\mathcal{C}}^X}((X \to \mathrm{pt})^*c, d)].$$

We say that an $\mathcal{X}$-functor $\underline{\mathcal{C}} \to \underline{\mathcal{X}}$ is corepresentable if it is equivalent to $\underline{\mathcal{C}}(c,-)$ for some $c \in \mathcal{C}^{\mathrm{pt}}$.

Definition 2.2.4. *For every object $X \in \mathcal{X}$ let $r_X\colon X \to \mathrm{pt}$ be the unique morphism. There is an induced adjunction*

$$r_{X,!}\colon \mathcal{X}_{/X} \rightleftarrows \mathcal{X}\colon r_X^*.$$

In fact, $r_{X,!}$ is just the functor that sends $Y \to X$ to Y. We also want to give a different perspective on the adjunction $r_{X,!} \dashv r_X^*$. Since topoi are equivalent to their ∞-categories of sheaves[2]*, we have an adjunction*

$$r_{X,!}\colon \mathrm{Sh}(\mathcal{X}_{/X}) \rightleftarrows \mathrm{Sh}(\mathcal{X})\colon r_X^*.$$

The functor r_X^* sends a sheaf $\mathcal{F}$ on $\mathcal{X}$ to the sheaf $r_X^*\mathcal{F}$ on $\mathcal{X}_{/X}$ defined by $(Y \to X) \mapsto \mathcal{F}(Y)$. The formula for $r_{X,!}$ is more complicated, but it is still given by the sheaf that sends

$$Y \mapsto \operatorname*{colim}_{f \in \mathrm{Map}(Y,X)} \mathcal{F}(Y \xrightarrow{f} X).$$

Remark 2.2.5. *The construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}(c)$ of Subsection 2.1 generalizes to symmetric monoidal $\mathcal{X}$-categories, as follows: Let $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{X}$-category. Then for every object $c \in \mathcal{C}^{\mathrm{pt}}$ we have a symmetric monoidal $\mathcal{X}$-category $\underline{\mathcal{C}^{\mathrm{op}}(c)}$, whose value on an object $X \in \mathcal{X}$ is the ∞-category*

$$(\underline{\mathcal{C}}^X)^{\mathrm{op}}(r_X^*c).$$

This is again a limit preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ as the construction of $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}(c)$ is limit preserving, as one can check by going carefully through our construction and using Lemma 2.4 of [HHLN23], which says that limits of adequate triples are computed underlying, and using the main result of [HHLN23], which establishes the functor Span *as a right adjoint functor.*

Proposition 2.2.6. *The composite of $\mathcal{X}$-functors*

$$\underline{\mathcal{C}^{\mathrm{op}}(c)} \xrightarrow{\mathrm{fgt}} \underline{\mathcal{C}} \xrightarrow{\underline{\mathcal{C}}(0,-)} \underline{\mathcal{X}}$$

is corepresented by

$$(c, 0, \mathrm{triv}) \in \mathcal{C}^{\mathrm{pt,op}}(c)$$

Proof. Since we have a pullback of enriched mapping objects

$$\begin{tikzcd}
\underline{\mathcal{C}^{\mathrm{op}}(c)}((c,0,\mathrm{triv}),(a,b,\alpha\colon a \oplus b \simeq c)) \arrow[r, "\mathrm{fgt}"] \arrow[d, "\mathrm{fgt}"'] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathcal{C}}(0,b) \arrow[d, "\alpha\circ(-\oplus a)"] \\
\underline{\mathcal{C}}(a,c) \arrow[r, "\mathrm{triv}\circ(-\oplus 0)"'] & \underline{\mathcal{C}}(a \oplus 0, c)
\end{tikzcd}$$

and the lower map is an equivalence, the claim follows. $\square$

[2]sheaf here means presheaf that is a limit preserving functor to anima

Proposition 2.2.7. *Let $\mathcal{X}$ be a topos and let $\underline{\mathcal{C}}$ be an $\mathcal{X}$-category. Then for every object $c \in \mathcal{C}^{\mathrm{pt}}$ we have a pair of adjoint functors*

$$\iota_c(-)\colon \mathcal{X} \rightleftarrows \int_{\mathcal{X}} \underline{\mathcal{C}} \colon r_!(\underline{\mathcal{C}}(c,-)).$$

Proof. We compute:

$$\mathrm{Map}_{\int_{\mathcal{X}} \underline{\mathcal{C}}}(\iota_c X, (Y,W)) \quad \simeq \quad \operatorname*{colim}_{f \in \mathrm{Map}_{\mathcal{X}}(X,Y)} \underline{\mathcal{C}}^X(r_X^* c, f^* W)$$

Likewise, we have

$$\begin{aligned}
\mathrm{Map}_{\mathcal{X}}(X, r_!(\underline{\mathcal{C}}(c,-))(Y,W)) &\simeq (r_!(\underline{\mathcal{C}}(c,-))(Y,W))(X) \\
&\simeq \operatorname*{colim}_{f \in \mathcal{X}_{/Y} \times_{\mathcal{X}} \{X\}} \underline{\mathcal{C}}^X(r_X^* c, f^* W) \\
&\simeq \operatorname*{colim}_{f \in \mathrm{Map}_{\mathcal{X}}(X,Y)} \underline{\mathcal{C}}^X(r_X^* c, f^* W). \quad \square
\end{aligned}$$

Definition 2.2.8. *Let $\mathcal{X}$ be a topos. A* category of $\mathcal{X}$-spheres *is a pointed symmetric monoidal $\mathcal{X}$-category $\underline{\mathrm{Sph}}$ such that $\underline{\mathrm{Sph}}(0,-)\colon \underline{\mathrm{Sph}} \to \underline{(\mathcal{X}_*, \wedge)}$ is a strong monoidal $\mathcal{X}$-functor.*

Definition 2.2.9. *Let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. Let*

$$S\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}} \to \mathcal{X}$$

be the functor $r_!(\underline{\mathrm{Sph}}(0,-))$. We call $S^V := S(V)$ the spherical fibration *associated to $V \in \underline{\mathrm{Sph}}^X$. Let*

$$\mathrm{Th}\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}} \to \mathcal{X}_*$$

be the functor

$$\mathrm{cof}\, r_!(\underline{\mathrm{Sph}}(\emptyset,-)) \to r_!(\underline{\mathrm{Sph}}(0,-)).$$

We call $\mathrm{Th}_X(V)$ the Thom construction *associated to $V \in \underline{\mathrm{Sph}}^X$. It is equivalently given by $\underline{\mathrm{Sph}}(0,-)$ followed by the left adjoint of*

$$\mathcal{X}_* \hookrightarrow \int \underline{\mathcal{X}_*}$$

and evaluates on (X,V) as the cofiber of the map

$$X \xrightarrow{\sigma^\infty} S_X^V.$$

Let

$$S^\perp\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(V)} \to \mathcal{X}$$

be the composite of the functors $S \circ \int \mathrm{fgt}$. And

$$\mathrm{Th}^\perp\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(V)} \to \mathcal{X}_*$$

be the composite of the functors $\mathrm{Th} \circ \int \mathrm{fgt}$.

Corollary 2.2.10. *Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. Then for every object $V \in \mathrm{Sph}^{\mathrm{pt}}$ we have a pair of adjoint functors*

$$\iota_{(V,0,\mathrm{triv})}(-)\colon \mathcal{X} \rightleftarrows \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(V)}\colon S^{\perp}.$$

Lemma 2.2.11. *Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. Then*

- *$S(X,0) \simeq X \coprod X$ and*
- *$\mathrm{Th}(X,0) \simeq X_+$ for every $X \in \mathcal{X}$.*

Proof. Because of the strong monoidality of $\underline{\mathrm{Sph}}(0,-)$, we have $\underline{\mathrm{Sph}}(0,0) \simeq S^0 \in \mathcal{X}$. We then compute

$$\begin{aligned}
\mathrm{Map}(Y, S(X,0)) &\simeq \mathrm{Map}((Y,0),(X,0)) \\
&\simeq \mathrm{Map}(Y,X) \times \mathrm{Sph}^{Y}(0,0) \\
&\simeq \mathrm{Map}(Y,X) \times \mathcal{X}(Y,S^0) \\
&\simeq \mathrm{Map}(Y, X \times S^0) \\
&\simeq \mathrm{Map}(Y, X \coprod X).
\end{aligned}$$

Moreover, we have

$$\mathrm{Th}(X,0) \simeq S(X,0)/X \simeq (X \coprod X)/X \simeq X_+.$$

□

Lemma 2.2.12. *Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. There exists an up to contractible choice unique natural map*

$$\iota_0 X \to (X,W)$$

inside the category $\int_{\mathcal{X}} \underline{\mathrm{Sph}}$. As a consequence, we have a unique natural square in $\int_{\mathcal{X}} \underline{\mathrm{Sph}}$

$$\begin{tikzcd}
(X,\emptyset) \arrow[r, "(\mathrm{id}_X,\infty)"] & (X,W) \\
\iota_0 X \arrow[u, "(\mathrm{id}_X,\infty)"] \arrow[r, "\iota_0\sigma_\infty"'] & \iota_0 S^W_X \arrow[u, "\theta_{X,W}"']
\end{tikzcd}$$

Proof. Such a natural map $\iota_0 X \to (X,W)$ is equivalent to a natural transformation between the functors $(X,W) \mapsto X$ and $(X,W) \mapsto S^W_X$. Let $\emptyset \in \mathrm{Sph}^{\mathrm{pt}}$ be the initial and terminal object and let $0 \in \mathrm{Sph}^{\mathrm{pt}}$ be the monoidal unit. Then the functors above are corepresented by $\emptyset$ and 0 respectively. Therefore such a natural transformation is a map $\emptyset \to 0$ in $\mathrm{Sph}^{\mathrm{pt}}$. □

Lemma 2.2.13. *Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. Under the functor* $\mathrm{Th}\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}} \to \mathcal{X}_*$ *the square*

$$\begin{tikzcd}
(X,\emptyset) \arrow[r, "(\mathrm{id}_X,\infty)"] & (X,W) \\
\iota_0 X \arrow[u, "(\mathrm{id}_X,\infty)"] \arrow[r, "\iota_0\sigma_\infty"'] & \iota_0 S^W_X \arrow[u, "\theta_{X,W}"']
\end{tikzcd}$$

gets sent to the cofiber sequence

$$\begin{tikzcd} \mathrm{pt} \arrow[r] & \mathrm{Th}_X(W) \\ X_+ \arrow[u] \arrow[r, "(\sigma_\infty)_+"'] & (S_X^W)_+ \arrow[u] \end{tikzcd}$$

Proof. It follows immediately from the construction that $\mathrm{Th}_X(\emptyset) \simeq \mathrm{pt}$. Therefore, we have to show that the composition

$$S_X^W \to (S_X^W)_+ \xrightarrow{\mathrm{Th}(\theta)} \mathrm{Th}_X(W)$$

is the canonical quotient map. By definition the map factors as

$$S_X^W \to (S_X^W)_+ \xrightarrow{S(\theta)} (S_X^W) \to \mathrm{Th}_X(W).$$

By the zigzag identities for adjunctions, we only have to show that the inclusion

$$S_X^W \to (S_X^W)_+$$

is the unit of the adjunction $\iota_0 \dashv S$. The unit of the adjunction $\iota_0 \dashv S$ is a natural transformation

$$\mathrm{id} \to (-)_+$$

between endofunctors of the topos $\mathcal{X}$. By the Yoneda-Lemma internal to the topos $\mathcal{X}$ this anima of natural transfromations is equivalent to the anima

$$\mathrm{Map}_{\mathcal{X}}(\mathrm{pt}, \mathrm{pt}_+)$$

via evaluation at the point. We show that the unit of the adjunction $\iota_0 \dashv S$ corresponds to the inclusion $\mathrm{pt} \hookrightarrow \mathrm{pt}_+$. Consider the following chain of equivalences

$$\begin{array}{rl} \mathrm{unit} \in & \mathrm{Map}_{\mathcal{X}}(\mathrm{pt}, \mathrm{pt}_+) \\ & \simeq \mathrm{Map}(\mathrm{pt}, S(\iota_0 \mathrm{pt})) \\ \mathrm{id} \in & \simeq \mathrm{Map}(\iota_0 \mathrm{pt}, \iota_0 \mathrm{pt}) \\ \mathrm{id} \in & \simeq \mathrm{Sph}(0, 0) \\ \mathrm{id} \in & \simeq \mathcal{X}_*(S^0, S^0) \\ \mathrm{incl} \in & \simeq \mathcal{X}(\mathrm{pt}, S^0). \end{array}$$

□

Definition 2.2.14. *A square in the ∞-category $\int_{\mathcal{X}} \underline{\mathrm{Sph}}$ is called* distinguished *if it is of the form*

$$\begin{tikzcd} (X, \emptyset) \arrow[r, "(\mathrm{id},\infty)"] & (X, W) \\ (X, 0) \arrow[u, "(\mathrm{id},\infty)"] \arrow[r, "\iota_0 \sigma_\infty"'] & (S_X^W, 0) \arrow[u, "\theta_X^W"'] \end{tikzcd}$$

for some $X \in \mathcal{X}$ and $W \in \mathrm{Sph}^X$. A square in the ∞-category $\int_{\mathcal{X}} \underline{\mathrm{Sph}}^{\mathrm{op}}(W)$ is called distinguished *if its image in $\int_{\mathcal{X}} \underline{\mathrm{Sph}}$ under the functor $\int \mathrm{fgt}$ is distinguished. A square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & D \end{tikzcd}$$

in a category $\mathcal{D}$ is called a split pushout square *if it is a pushout square and either f or g admit a left inverse. The image of a split pushout square under the functor $\iota_{(W,0,\mathrm{triv})}$ is called a* split pushout with trivial coefficients. *Let $\mathcal{D}$ be a pointed ∞-category. Let ϕ be a functor $\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \mathcal{D}$. We say*

- *ϕ is* reduced *if it sends objects of the form $(X, \emptyset, \emptyset, \emptyset)$ to the zero object of $\mathcal{D}$,*
- *ϕ* realizes split Mayer-Vietoris sequences *if it sends split pushouts with trivial coefficients to pushout squares in $\mathcal{D}$,*
- *ϕ* realizes Thom-isomorphisms *if it sends distinguished squares to pushouts in $\mathcal{D}$.*

A reduced functor $\phi\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \mathcal{D}$ that realizes split Mayer-Vietoris sequences and Thom-isomorphisms is called a split genuine homology theory with coefficients in $\mathcal{D}$. *We denote the ∞-category of split genuine homology theories with coefficients in $\mathcal{D}$ by $\mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$.*

Example 2.2.15. *Lemmata 2.2.11 and 2.2.13 show that*

$$\mathrm{Th}^{\perp} \in \mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{X}_*).$$

In fact, $\mathrm{Th}^{\perp}$ is the universal split genuine homology theory as our main result of this chapter shows.

Example 2.2.16. *If ϕ is a split genuine homology theory with coefficients in $\mathcal{D}$ and $f\colon \mathcal{D} \to \mathcal{D}'$ is a reduced functor that preserves split pushouts, then $f \circ \phi$ is a split genuine homology theory with coefficients in $\mathcal{D}'$. We call functors* split excisive *if they preserve split pushouts and denote the ∞-category of split excisive functors by* $\mathrm{Exc}^{\mathrm{split}}(\mathcal{D}, \mathcal{D}')$. *Suppose we introduced a notion of a cohomology theory for functors $E\colon \mathcal{X}_* \to \mathrm{Ab}^{\mathrm{op}}$, that in particular requires them to be reduced split excisive functors. Then we can apply the previous observation to obtain a genuine cohomology theory $E \circ \mathrm{Th}^{\perp}$ with coefficients in $\mathrm{Ab}^{\mathrm{op}}$.*

Definition 2.2.17. *A* valid coefficient category *$\mathcal{D}$ is a pointed ∞-category such that every morphism $f\colon X \to Y$ in $\mathcal{D}$ that admits a right inverse $g\colon Y \to X$ has a cofiber in $\mathcal{D}$. For example, all pointed ∞-categories with cofibers for all maps are valid coefficient categories.*

Definition 2.2.18. *Let $\mathcal{D}$ be a valid coefficient category. Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. We define a functor*

$$\mathrm{res}|_{\mathcal{X}_*}\colon \mathrm{Fun}(\int_{\mathcal{X}} \underline{\mathrm{Sph}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{X}_*, \mathcal{D}).$$

by the formula

$$\mathrm{res}|_{\mathcal{X}_*}(\phi)(\mathrm{pt} \to X) = \mathrm{cof}(\phi(\iota_{(W,0,\mathrm{triv})}\mathrm{pt}) \to \phi(\iota_{(W,0,\mathrm{triv})}X)).$$

Theorem 1. *Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. Let $\mathcal{D}$ be a valid coefficient category. The functors*

$$\mathrm{res}|_{\mathcal{X}_*}\colon \mathrm{Fun}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{X}_*, \mathcal{D}),$$

and

$$(\mathrm{Th}^{\perp})^*\colon \operatorname{Fun}(\mathcal{X}_*, \mathcal{D}) \to \operatorname{Fun}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$$

restrict to mutually inverse equivalences of ∞-categories

$$\operatorname{Gen}^{\mathrm{split}}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}) \simeq \operatorname{Exc}_*^{\mathrm{split}}(\mathcal{X}_*, \mathcal{D})$$

Proof. Example 2.2.16 shows that the functor

$$(\mathrm{Th}^{\perp})^*\colon \operatorname{Fun}(\mathcal{X}_*, \mathcal{D}) \to \operatorname{Fun}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$$

restricts to a functor between the full subcategories

$$\operatorname{Exc}_*^{\mathrm{split}}(\mathcal{X}_*, \mathcal{D})$$

and

$$\operatorname{Gen}^{\mathrm{split}}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}).$$

The restriction functor $\mathrm{res}\,|_{\mathcal{X}_*}$ automatically sends every functor in $\operatorname{Fun}(\int_{\mathcal{X}} \underline{\mathrm{Sph}}, \mathcal{D})$ to a reduced functor in $\operatorname{Fun}(\mathcal{X}_*, \mathcal{D})$. The split Mayer-Vietoris axiom ensures that the restriction of a split genuine homology theory is a functor $\mathcal{X}_* \to \mathcal{D}$ that sends split pushouts to pushouts in $\mathcal{D}$. To show that both functors are mutually inverse equivalences on these subcategories we compute their composites: By unraveling the definitions we find

$$(\mathrm{Th}^{\perp})^* \circ \mathrm{res}\,|_{\mathcal{X}_*}(\phi)(X; U, V, \alpha) \simeq \operatorname{cof}(\phi(\mathrm{pt}; W, 0, \mathrm{triv}) \to \phi(\mathrm{Th}_X(V); W, 0, \mathrm{triv})).$$

and by Lemma 2.2.11 we have

$$\mathrm{res}\,|_{\mathcal{X}_*} \circ (\mathrm{Th}^{\perp})^*(\psi)(\mathrm{pt} \to X) \simeq \operatorname{cof}(\psi(\mathrm{pt}_+) \to \psi(X_+)).$$

We show that both formulas are naturally equivalent to the identity on both subcategories. Consider the following natural split pushout square in $\mathcal{X}_*$:

$$\begin{tikzcd} \mathrm{pt}_+ \arrow[r] \arrow[d] & X_+ \arrow[d] \\ \mathrm{pt} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \end{tikzcd}$$

If ψ preserves split pushouts, then we obtain a natural equivalence

$$\operatorname{cof}(\psi(\mathrm{pt}_+) \to \psi(X_+)) \simeq \operatorname{cof}(\psi(\mathrm{pt}) \to \psi(X)).$$

If ψ is additionally reduced, then we obtain a natural equivalence

$$\operatorname{cof}(\psi(\mathrm{pt}) \to \psi(X)) \simeq \psi(X).$$

This shows that

$$\mathrm{res}\,|_{\mathcal{X}_*} \circ (\mathrm{Th}^{\perp})^*$$

is equivalent to the identity on the subcategory $\mathrm{Exc}_*^{\mathrm{split}}(\mathcal{X}_*, \mathcal{D})$. The other composite is equivalent to the identity by a similar sequence of arguments. Consider the natural split pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S_X^V \arrow[d] \\ \mathrm{pt} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Th}_X(V) \end{tikzcd}$$

in $\mathcal{X}$. By the split Mayer-Vietoris axiom, we have a natural equivalence

$$\begin{aligned} \mathrm{cof}(\phi(\mathrm{pt}; W, 0, \mathrm{triv}) &\to \phi(\mathrm{Th}_X(V); W, 0, \mathrm{triv})) \simeq \\ \mathrm{cof}(\phi(X; W, 0, \mathrm{triv}) &\to \phi(S_X^V; W, 0, \mathrm{triv})). \end{aligned}$$

Now, consider the natural distinguished square in $\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}$:

$$\begin{tikzcd} (X; \emptyset, \emptyset, \infty) \arrow[r, "(\mathrm{id},\infty)"] & (X; U, V, \alpha) \\ (X; W, 0, \mathrm{triv}) \arrow[u, "(\mathrm{id}_X,\infty)"] \arrow[r, "(W,0,\mathrm{triv})\sigma_\infty"'] & (S_X^V; W, 0, \mathrm{triv}) \arrow[u, "\theta"'] \end{tikzcd}$$

that by the Thom-isomorphism axiom gets sent to a natural pushout square, which induces a natural equivalence

$$\mathrm{cof}(\phi(X; W, 0, \mathrm{triv}) \to \phi(S_X^V; W, 0, \mathrm{triv})) \simeq \mathrm{cof}(\phi(X; \emptyset, \emptyset, \infty) \to \phi(X; U, V, \alpha)).$$

If ϕ is reduced, then we obtain a natural equivalence

$$\mathrm{cof}(\phi(X; \emptyset, \emptyset, \infty) \to \phi(X; U, V, \alpha)) \simeq \phi(X; U, V, \alpha).$$

This sequence of natural equivalence defines a natural equivalence between the composite $(\mathrm{Th}^{\perp})^* \circ \mathrm{res}\,|_{\mathcal{X}_*}$ and the identity on the subcategory $\mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$.
$\square$

Remark 2.2.19. *We want to emphasize the importance of the naturality of the squares appearing in the previous proof. The bulk of this chapter is devoted to making the above squares natural, and working around the fact that we do not have a natural distinguished squares in the vertical opposite category*

$$\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}}$$

2.3 Genuine Homology Theories on the ∞-category of Compact Objects

Lemma 2.3.1. *Let $\mathcal{C}$ be an ∞-category and $X \in \mathcal{C}$ fixed. Then the forgetful functor*

$$\mathcal{C}_{X/} \to \mathcal{C}$$

preserves compact objects if and only if X is compact in $\mathcal{C}$. In that case, it induces an equivalence of ∞-categories

$$(\mathcal{C}_{X/})^{\omega} \simeq (\mathcal{C}^{\omega})_{X/}.$$

Proof. The object $\mathrm{id}\colon X \to X$ is initial in $\mathcal{C}_{X/}$, so it is compact in it. If the forgetful functor preserves compact objects, then X is compact. Conversely, if X is compact, then we claim that $X \to Y$ is compact in $\mathcal{C}_{X/}$ if and only if Y is compact in $\mathcal{C}$. The claim follows from the fact that we have a fiber sequence of corepresented functors

$$\mathrm{Map}_{X/}(X \to Y, -) \to \mathrm{Map}_{\mathcal{C}}(Y, -) \to \mathrm{Map}_{\mathcal{C}}(X, -)$$

where depending on the assumptions the first or the second and always the third functor commutes with filtered colimits. And therefore all three functors. □

Our main interest is when $X = \mathrm{pt}$ is the terminal object of $\mathcal{C}$.

Definition 2.3.2. *Let $\mathcal{X}$ be a topos. We call $\mathcal{X}$* strongly compact *if the terminal object* $\mathrm{pt} \in \mathcal{X}$ *is compact.*

Example 2.3.3. *Suppose $\mathcal{X}$ is the presheaf topos of a small ∞-category $\mathcal{C}$ that has a terminal object. Then via the Yoneda embedding every object of $\mathcal{C}$ becomes a compact object of $\mathcal{X}$. The image of* $\mathrm{pt} \in \mathcal{C}$ *is* $\mathrm{pt} \in \mathcal{X}$. *Therefore $\mathcal{X}$ is strongly compact. For example* $\mathrm{An} = \mathcal{P}(\{\mathrm{pt}\})$ *and* $\mathrm{pt} = G/G \in \mathrm{An}_G = \mathcal{P}(\mathrm{Orb_G})$ *are strongly compact topoi. A counterexample is for example* An^{BC_2} *as homotopy C_2-fixed points do not preserve filtered colimits in general.*

Definition 2.3.4. *Let $\mathcal{X}$ be a strongly compact topos. A* category of compact $\mathcal{X}$-spheres $\underline{\mathrm{Sph}}$ *is a category of spheres such that the functor* $\underline{\mathrm{Sph}}(0,-)\colon \underline{\mathrm{Sph}}|_{\mathcal{X}^\omega} \to \underline{\mathcal{X}_*}|_{\mathcal{X}^\omega}$ *takes only values in the full subcategories of compact objects, that is for every $X \in \mathcal{X}^\omega$ we have a factorization*

$$\underline{\mathrm{Sph}}(0,-)\colon \underline{\mathrm{Sph}}^X \to \mathcal{X}^\omega_{X/\cdot/X} \subset \mathcal{X}_{X/\cdot/X}.$$

Lemma 2.3.5. *Let $\underline{\mathrm{Sph}}$ be a category of compact $\mathcal{X}$-spheres. The functors*

1. $S\colon \int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}} \to \mathcal{X}$
2. $\mathrm{Th}\colon \int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}} \to \mathcal{X}_*$

all take values in the ∞-category of compact objects.

Proof. We have to show that the objects $S(X, V)$ and $\mathrm{Th}(X, V)$ are compact for every $X \in \mathcal{X}^\omega$ and $V \in \mathrm{Sph}^X$. By definition S and Th are composites of functors that preserve compact objects. Namely

$$S\colon \underline{\mathrm{Sph}}^X \xrightarrow{\underline{\mathrm{Sph}}(0,-)} \mathcal{X}_{X/\cdot/X} \xrightarrow{\mathrm{fgt}} \mathcal{X}_{/X} \xrightarrow{r_{X,!}} \mathcal{X}$$

and

$$\mathrm{Th}\colon \underline{\mathrm{Sph}}^X \xrightarrow{\underline{\mathrm{Sph}}(0,-)} \mathcal{X}_{X/\cdot/X} \xrightarrow{r_{X,!}} \mathcal{X}_*.$$

□

Theorem 2. *[Addendum to Theorem 1] Let $\mathcal{X}$ be a topos and let $\underline{\mathrm{Sph}}$ be a category of compact $\mathcal{X}$-spheres. Let $\mathcal{D}$ be a valid coefficient category. The functors*

$$\mathrm{res}\,|_{\mathcal{X}^\omega_*}\colon \mathrm{Fun}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{X}^\omega_*, \mathcal{D}),$$

and

$$(\mathrm{Th}^\perp)^*\colon \mathrm{Fun}(\mathcal{X}^\omega_*, \mathcal{D}) \to \mathrm{Fun}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$$

restrict to mutually inverse equivalences of ∞-categories

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}) \simeq \mathrm{Exc}^{\mathrm{split}}_*(\mathcal{X}^\omega_*, \mathcal{D})$$

Proof. We observe that the proof of Theorem 1 carries over word by word since all constructions in the proof lie inside the ∞-categories of compact objects by Lemma 2.3.5 and Lemma 2.3.1. $\square$

2.4 Functoriality

We observe that in Theorems 1 and 2 in the stated equivalence of ∞-categories

$$\mathrm{Gen}^{\mathrm{split}}(\int \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}) \simeq \mathrm{Exc}_*^{\mathrm{split}}(\mathcal{X}_*, \mathcal{D})$$

the left-hand side depends on the choice of $W \in \mathrm{Sph}^{\mathrm{pt}}$, while the right-hand side does not. We want to explore this further by analyzing the functoriality of this equivalence in W.

Example 2.4.1. *Let $\underline{\mathrm{Sph}}$ be a category of $\mathcal{X}$-spheres. Let M be the core of $\mathrm{Sph}^{\mathrm{pt}}$. From Construction 2.1.10, we know that $W \mapsto \underline{\mathrm{Sph}^{\mathrm{op}}}(W)$ defines a fanned out M-action in $\mathcal{X}$-categories. Also the $\mathcal{X}$-category $\underline{\mathrm{Sph}}$ receives an M-action where $W \in M$ acts via the functors*

$$\mathrm{Sph}^X \to \mathrm{Sph}^X, \quad V \mapsto V \oplus r_X^* W.$$

We denote this action by $\underline{\mathrm{Sph}} \xrightarrow{-\oplus W} \underline{\mathrm{Sph}}$. Since $\underline{\mathrm{Sph}}(0,-)$ is a symmetric monoidal $\mathcal{X}$-functor $\underline{\mathcal{X}_*}$ enherits an M-action

$$\mathcal{X}_{X/\cdot/X} \to \mathcal{X}_{X/\cdot/X}, \quad U \mapsto U \wedge_X \underline{\mathrm{Sph}}(0, r_X^* W).$$

In particular, the ordinary ∞-category $\mathcal{X}_*$ receives an M-action, and since the functor

$$\mathcal{X}_{X/\cdot/X} \to \mathcal{X}_*, \quad U \mapsto U/X$$

is symmetric monoidal for the smash products, we find that

$$\mathrm{Th}\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}} \to \mathcal{X}_*$$

is equivariant for the M-actions on both sides.

Proposition 2.4.2. *The forgetful functors $\underline{\mathrm{Sph}^{\mathrm{op}}}(W) \to \underline{\mathrm{Sph}}, \quad W \in M$ assemble into an equivariant functor between fanned out M-actions. And thus the functors*

$$\mathrm{Th}^{\perp}\colon \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \mathcal{X}_*$$

also, assemble into an equivariant functor between fanned out M-actions. In particular, we obtain commutative diagrams

$$\begin{tikzcd}
\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \arrow[r, "\mathrm{Th}^{\perp}"] \arrow[d, "-\oplus \overrightarrow{W'}"'] & \mathcal{X}_* \arrow[d, "-\wedge S^{W'}"] \\
\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W \oplus W')} \arrow[r, "\mathrm{Th}^{\perp}"'] & \mathcal{X}_*
\end{tikzcd}$$

We observe that the functors

$$\begin{gathered}\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)} \to \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W \oplus W')} \\ (X; U, V, \alpha \colon U \oplus V \simeq r_X^* W) \mapsto (X; U, V \oplus r_X^* W', \alpha \oplus \mathrm{id}_{r_X^* W'})\end{gathered}$$

send split pushouts with trivial coefficients to split pushouts with trivial coefficients, distinguished squares to distinguished squares and objects of the form $(X; \emptyset, \emptyset, \infty)$ to objects of the form $(X; \emptyset, \emptyset, \infty)$. Also the functors $- \wedge S^{W'} \colon \mathcal{X}_* \to \mathcal{X}_*$ preserve split pushouts therefore

Proposition 2.4.3. *The equivalences of ∞-categories*

$$(\mathrm{Th}^{\perp})^* \colon \mathrm{Exc}_*^{\mathrm{split}}(\mathcal{X}_*, \mathcal{D}) \to \mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$$

assemble into an equivariant equivalence of fanned-out M-actions.

Proposition 2.4.4 (Addendum to Proposition 2.4.3)**.** *Let $\underline{\mathrm{Sph}}$ be a category of compact $\mathcal{X}$-spheres. Then we have an equivariant equivalence of fanned out M-actions*

$$(\mathrm{Th}^{\perp})^* \colon \mathrm{Exc}_*^{\mathrm{split}}(\mathcal{X}_*^{\omega}, \mathcal{D}) \to \mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$$

Remark 2.4.5. *With the appropriate definitions of split genuine homology theories on the colimit the last proposition yields equivalences*

$$\mathrm{Exc}_*^{\mathrm{split}}(\operatorname*{colim}_{W \in M//^{\mathrm{lax}}M} \mathcal{X}_*, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\operatorname*{colim}_{W \in M//^{\mathrm{lax}}M} \int_{\mathcal{X}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D})$$

and in the case that $\underline{\mathrm{Sph}}$ is a category of compact spheres we have

$$\mathrm{Exc}_*^{\mathrm{split}}(\operatorname*{colim}_{W \in M//^{\mathrm{lax}}M} \mathcal{X}_*^{\omega}, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\operatorname*{colim}_{W \in M//^{\mathrm{lax}}M} \int_{\mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}, \mathcal{D}).$$

In its current form, these equivalences are not very useful. We have to identify the colimits

$$\operatorname*{colim}_{W \in M//^{\mathrm{lax}}M} \mathcal{X}_*^{\omega}$$

and

$$\operatorname*{colim}_{W \in M//^{\mathrm{lax}}M} \int_{\mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}$$

with categories we know. Therefore it makes sense to prove Conjecture 2.1.11 for interesting cases. We do something similar in the next chapters. More specifically in Section 3.3 and in Section 4.2 we restrict the fanned-out M-action to non-full filtered subcategories $\mathcal{U} \to M//^{\mathrm{lax}}M$. We are then able to identify

$$\operatorname*{colim}_{W \in \mathcal{U}} \mathcal{X}_*^{\omega}$$

with $\mathcal{X}_*^{\omega}[(\mathrm{Sph}^{\mathrm{pt}})^{-1}]$ and

$$\operatorname*{colim}_{W \in \mathcal{U}} \int_{\mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}(W)}$$

with

$$\int_{\mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}$$

for the topoi $\mathcal{X} = \mathrm{An}$ and $\mathcal{X} = \mathrm{An}_G$.

GCTs on Anima and Affine Linear Vector Bundles

We want to apply the ideas and results of the previous section to the setting of ordinary cohomology theories on the ∞-category of anima.

We consider the topos $\mathcal{X} = \mathrm{An}$ of anima. The universal property of $\mathrm{An} = \mathcal{P}(\mathrm{pt})$ (see Theorem 5.1.5.6 of [Lur09]) implies that the ∞-category of An-categories is equivalent to the ∞-category of ∞-categories via the evaluation at the point

$$\mathrm{Fun}^{\lim}(\mathrm{An}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow{\simeq} \mathcal{C}\mathrm{at}_\infty \colon \underline{\mathcal{C}} \mapsto \mathcal{C}^{\mathrm{pt}}.$$

The inverse equivalence is given by the functor that associates to an ∞-category $\mathcal{C}$ the functor

$$\mathrm{An}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty \colon X \mapsto \mathrm{Fun}(X, \mathcal{C}) =: \mathcal{C}^X.$$

To fix a notion of genuine homology theories, we need to fix a category of spheres. As an example for a category of spheres we could for example take the full subcategory

$$\{S^\emptyset := \mathrm{pt}, S^0, S^1, S^2, \dots\} \subset \mathrm{An}_*$$

with the restriction of the smash product as a symmetric monoidal structure. This is a completely valid choice for a category of spheres, but we can work more efficiently by passing to non-full subcategory of $\{S^i\}_i$ to model the same ∞-category of genuine homology theories. Optimally, we would like to take the ∞-category

$$\mathrm{Vect}^{\simeq} = \coprod_n BO(n)$$

of finite-dimensional vector spaces over $\mathbb{R}$ with linear isomorphisms as our category of spheres, where the functor to An_* is given by sending a vector space V to its one-point compactification S^V. But this category is not a category of spheres as the functor $\mathrm{Map}(0, -)\colon \mathrm{Vect}^{\simeq} \to \mathrm{An}_*$ is not strong symmetric monoidal. The problem is that there are simply not enough morphisms in the ∞-category $\mathrm{Vect}^{\simeq}$. Even if we add non-invertible linear morphisms to this ∞-category, we still only have one morphism

$$0 \to V.$$

We can fix this by considering a larger class of morphisms, namely affine linear morphisms between vector bundles.

3.1 Affine Linear Morphisms

In this section, we introduce the ∞-category of vector spaces and affine linear maps between them. It is the ∞-category, that is often used in the parametrization of (non-equivariant) orthogonal spectra. For more details, we refer to [MMSS01].

Definition 3.1.1. *Let V, W be two finite-dimensional Euclidean vector spaces. An* affine linear morphism $f\colon V \to W$ *is a pointed continuous map between the one-point compactifications $S^V \to S^W$ which is either*

- *the constant map to the added point at infinity or*
- *a map that arises from a proper continuous map $V \to W$ which itself is the composition of a linear isometric embedding $\phi\colon V \hookrightarrow W$ followed by a translation $W \to W\colon w \mapsto w + w_0$ for some $w_0 \in \phi(V)^\perp$.*

Let us denote the subspace of affine linear morphisms between V and W by

$$\mathrm{AffLin}(V, W) \subset \mathrm{Map}_*(S^V, S^W).$$

Remark 3.1.2. *The topological space* $\mathrm{AffLin}(V, W)$ *is homeomorphic to the Thom space of the complementary image bundle*

$$\{(\phi, w_0) | \phi \in \mathrm{LinIsoEmb}(V, W), w_0 \in \phi(V)^\perp\} \to \mathrm{LinIsoEmb}(V, W)$$

over the space of linear isometric embeddings. It is possible to construct a pointed topologically enriched symmetric monoidal category TopAffLin *whose objects are finite-dimensional Euclidean vector spaces and whose morphism spaces are given by* $\mathrm{AffLin}(V, W)$*. For technical purposes to have a pointed category, we also consider another object $\emptyset \in$* TopAffLin *which is strictly speaking not a vector space, but we call it the* empty vector space*. We can think of the empty vector space in terms of its one-point compactification $S^\emptyset :=$* pt*. An affine linear morphism into or out of it is simply given by the unique map to the point at infinity.*

Definition 3.1.3. *Let* AffLin *be the underlying pointed symmetric monoidal ∞-category of* TopAffLin*. The inclusion* $\mathrm{AffLin}(V, W) \subset \mathrm{Map}_*(S^V, S^W)$ *refines to a symmetric monoidal functor*

$$(\mathrm{AffLin}, \oplus) \to (\mathrm{An}_*, \wedge)\colon V \mapsto S^V$$

Lemma 3.1.4. *The functor* $\mathrm{AffLin} \to \mathrm{An}_*$ *is corepresended by the unit* $0 \in \mathrm{AffLin}$ *and makes* AffLin *into a category of spheres.*

Proof. The complementary image bundle over the unique linear isometric morphism $0 \to V$ is given by

$$V \to \mathrm{pt}.$$

Therefore $\mathrm{AffLin}(0, V)$ is the one-point compactification of V. By construction $V \mapsto S^V$ is strong symmetric monoidal. $\square$

3.2 The ∞-category of Compactified Affine Spaces

We want to give a further example of a category of spheres that is a little bit larger than the ∞-category AffLin that we previously introduced. It has the same objects, but we also allow non-isometric embeddings as well as translations in all directions.

Definition 3.2.1. *Let $V \curvearrowright (X, \infty)$ be a continuous action of a finite-dimensional real vector space V on a pointed compact Hausdorff space (X, ∞). We call (V, X, ∞) a* compactified affine space *if $X \setminus \infty$ is a torsor over V. That means that $X = \infty$ or $X \setminus \infty \simeq V$ via the action of V and a choice of a point $x \in X \setminus \infty$. In particular, $\infty, X \setminus \infty$ are invariant subspaces. An* affine linear morphism *between two compactified affine spaces (V, X, ∞) and (W, Y, ∞) is a continuous map $f : X \to Y$ such that the subspace $f^{-1}(\{\infty\})$ is*

- *either the entire space X or*
- *$\{\infty\}$, so that the resulting function $f : X \setminus \{\infty\} \to Y \setminus \{\infty\}$ is an affine linear morphism between torsors over V and W, i.e. if $\exists x_0 \in X \setminus \{\infty\}$, then the function*
$$f(x_0 + \bullet) - f(x_0) : V \to W$$
is a linear morphism.

Remark 3.2.2. *If V has dimension 0, then the above condition reads, that $X = \infty$ or $X = S^0$. If V is positive dimensional, then $X \simeq (X \setminus \infty)^+ \simeq V^+ = S^V$ is non-canonically equivalent to the one-point compactification S^V of V. If $X = \infty$, then all objects $(V, \{\infty\}, \infty)$ are canonically isomorphic. On the other hand, if $X \neq \infty$, then one can recover X from V. Let us write ∞ for any object $(V, \{\infty\}, \infty)$ and S^V for the object (V, S^V, ∞). Then the full subcategory on the objects*
$$\{\infty, S^0, S^{\mathbb{R}^1}, S^{\mathbb{R}^2}, \ldots\}.$$
is a skeleton of the ∞-category of compactified affine spaces.

Proposition 3.2.3. *Via*
$$(V, X, \infty) \oplus (W, Y, \infty) := (V \oplus W, X \wedge Y, \infty)$$
the ∞-category of compactified affine spaces is a symmetric monoidal ∞-category. Moreover, the functor
$$(V, X, \infty) \mapsto (X, \infty) \in \mathrm{Top}_*$$
is symmetric monoidal and corepresented by the unit $(0, S^0, \infty)$.

Proof. Since the forgetful functor from the category of compactified affine spaces to the category of pointed topological spaces is faithful, the symmetric monoidal structure restricts to the category of compactified affine spaces, when its values do not leave the category of compactified affine spaces. On objects, it is clear that
$$S^V \wedge S^W \simeq S^{V \oplus W}$$
and
$$\{\infty\} \wedge X \simeq \{\infty\}$$

holds. While on morphisms, one has that the one-point compactification of the cartesian product of two affine linear morphisms is the smash product of the one-point compactifications of the morphisms. Now, we also see that the restricted tensor product is given by the tensor product defined in the proposition. And this makes the forgetful functor canonically strong symmetric monoidal.

Let $S^0 = \{0, \infty\}$ then a map

$$(0, S^0, \infty) \to (V, X, \infty)$$

is determined by its value $x_0 \in X$ of 0. But any such point of X defines a valid map of compactified affine spaces. Hence, the forgetful functor is corepresented by S^0. $\square$

Corollary 3.2.4. *The limit preserving extension of the category of compactified affine spaces to a functor*

$$\mathrm{An}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

is a category of compact spheres.

3.3 Genuine Homology Theories on Compact Anima

Let $\underline{\mathrm{Sph}}$ one of the categories of compact spheres in anima, that we introduced in this chapter, so its objects (over the point) are either pointed spheres (including the point) or vector spaces (including the "empty vector space") and one of the notions of affine morphism that we introduced in this chapter. In any case, we can define a functor

$$(\mathbb{N}, \leq) \to M//^{\mathrm{lax}}M,$$

where M is the core of $\mathrm{Sph}^{\mathrm{pt}}$. This functor sends n to the respective n-dimensional object, that is if Sph is the full subcategory of pointed spheres then n gets sent to S^n. If Sph is one of the categories AffLin then n gets sent to $\mathbb{R}^n$. If $n \leq m$ then the functor sends this to the morphism

$$S^n \wedge S^{m-n} \simeq S^m$$

or

$$\mathbb{R}^n \oplus \mathbb{R}^{m-n} \simeq \mathbb{R}^m,$$

where S^n (or $\mathbb{R}^n$ respectively) are included into S^m (or $\mathbb{R}^m$) as the first n coordinates.

Applying Proposition 2.4.4 to this we obtain an equivalence of towers

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_*^\omega, \mathcal{D}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}(1)^{\mathrm{op}}}, \mathcal{D}) \arrow[d] \\
\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_*^\omega, \mathcal{D}) \arrow[r, "\simeq"] & \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}(0)^{\mathrm{op}}}, \mathcal{D})
\end{tikzcd}$$

The limit of the left-hand side can be identified with a full subcategory of functors

$$\operatorname*{colim}_n(\mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \cdots) \to \mathcal{D}.$$

Lemma 3.3.1. *We have an equivalence of ∞-categories*

$$\operatorname*{colim}_n(\mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \cdots) \simeq \mathrm{Sp}^\omega .$$

and under this equivalence the full subcategory

$$\lim_n \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_*^\omega, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Sp}^\omega, \mathcal{D})$$

corresponds to those functors which are reduced and split excisive.

Proof. The ∞-category Sp is the filtered colimit

$$\mathrm{colim}(\mathrm{An}_* \xrightarrow{\Sigma} \mathrm{An}_* \xrightarrow{\Sigma} \cdots)$$

in the ∞-category Pr^L. This filtered colimit is in the image of the left adjoint functor

$$\mathrm{Ind}\colon \mathcal{C}\mathrm{at}_\infty^{\mathrm{Rex}} \to \mathrm{Pr}^L$$

of the diagram

$$\mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \cdots$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Rex}}$. But since filtered colimits in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Rex}}$ are computed underlying, i.e. in $\mathcal{C}\mathrm{at}_\infty$, we conclude

$$\mathrm{Sp}^\omega \simeq \operatorname*{colim}_n(\mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \mathrm{An}_*^\omega \xrightarrow{-\wedge S^1} \cdots)$$

as both sides are idempotent complete; the left-hand side is a category of compact objects of a presentable ∞-category and the right-hand side as a filtered colimit of such. We claim that a functor $\mathcal{F}\colon \mathrm{Sp}^\omega \to \mathcal{D}$ is reduced and split excisive if and only if each of its restrictions

$$\mathcal{F} \circ \Sigma^{\infty-n}\colon \mathrm{An}_*^\omega \to \mathcal{D}$$

is. Since $\Sigma^{\infty-n}$ preserves finite colimits $\mathcal{F} \circ \Sigma^{\infty-n}$ is reduced and split excisive when $\mathcal{F}$ is. On the other hand, every split pushout square in Sp^ω is a finite diagram and therefore in the image of one of the functors $\Sigma^{\infty-n}$, and after a potential finite amount of suspensions again a pushout, i.e. the image of a split pushout under a functor $\Sigma^{\infty-N}$ for some larger N. Likewise, the zero object is in the image. This shows the other direction. $\square$

Lemma 3.3.2. *The forgetful functors*

$$\underline{\mathrm{Sph}(n)}^{\mathrm{op}} \to \underline{\mathrm{Sph}}^{\mathrm{op}}, \quad n \geq 0$$

induce a natural transformation

$$\operatorname*{colim}_n \underline{\mathrm{Sph}(n)}^{\mathrm{op}} \to \underline{\mathrm{Sph}}^{\mathrm{op}}$$

between functors $\mathrm{An}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, *that restricts to an equivalence between functors* $\mathrm{An}^{\omega,\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$.[1]

[1]The colimit is taken pointwise, that is in the ∞-category of all functors $\mathrm{An}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$.

Proof. The right-hand side of the stated equivalence preserves all limits. The left-hand side is a filtered colimit of limit-preserving functors so it preserves finite limits. Therefore we need to check the stated equivalence only on the point, as $\mathrm{An}^{\omega,\mathrm{op}}$ is generated under retracts of finite limits of the point. That means we have to show that the functor

$$\mathcal{F}\colon \operatorname*{colim}_n \mathrm{Sph}(n)^{\mathrm{pt},\mathrm{op}} \to \mathrm{Sph}^{\mathrm{pt},\mathrm{op}}.$$

is an equivalence. We show that $\mathcal{F}$ is fully faithful. The essential surjectivity is clear. As a filtered colimit of ∞-categories, each pair of objects a, b in the colimit

$$\mathcal{C} := \operatorname*{colim}_n \mathrm{Sph}(n)^{\mathrm{pt},\mathrm{op}}$$

is in the image of the functor of some $\mathrm{Sph}(n)^{\mathrm{pt},\mathrm{op}} \to \mathcal{C}$. That is $a = (U, V, \alpha\colon U \oplus V \to \mathbb{R}^n)$ and $b = (U', V', \alpha'\colon U' \oplus V' \to \mathbb{R}^n)$. And the mapping anima

$$\mathrm{Map}_{\mathcal{C}}(a, b)$$

is the filtered colimit

$$\operatorname*{colim}_k \mathrm{Map}_{\mathrm{Sph}(n+k)^{\mathrm{pt},\mathrm{op}}}((U, V \oplus \mathbb{R}^k, \alpha \oplus \mathbb{R}^k), (U', V' \oplus \mathbb{R}^k, \alpha' \oplus \mathrm{id}_{\mathbb{R}^k}))$$

of the mapping anima inside the ∞-categories $\mathrm{Sph}(n+k)^{\mathrm{op}}$. To take advantage of that we invoke Lemma 2.1.5 to find this filtered colimit is the pullback of the filtered colimits

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(a, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathrm{Sph}^{\mathrm{pt}}}(U', U) \arrow[d] \\
\operatorname*{colim}_k \mathrm{Map}_{\mathrm{Sph}^{\mathrm{pt}}}(V \oplus \mathbb{R}^k, V' \oplus \mathbb{R}^k) \arrow[r] & \operatorname*{colim}_k \mathrm{Map}_{\mathrm{Sph}^{\mathrm{pt}}}(U' \oplus V \oplus \mathbb{R}^k, \mathbb{R}^n \oplus \mathbb{R}^k)
\end{tikzcd}$$

Note that the top arrow in that diagram is the effect of the functor $\mathcal{F}$ on mapping anima. We show that it is an equivalence by rewriting the pullback as the pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(a, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathrm{Sph}^{\mathrm{pt}}}(U', U) \arrow[d] \\
\operatorname*{colim}_k \mathrm{Map}_{\mathrm{Sph}^{\mathrm{pt}}}(V \oplus \mathbb{R}^k, V' \oplus \mathbb{R}^k) \arrow[r] & \operatorname*{colim}_k \mathrm{Map}_{\mathrm{Sph}^{\mathrm{pt}}}(U' \oplus V \oplus \mathbb{R}^k, U' \oplus V' \oplus \mathbb{R}^k)
\end{tikzcd}$$

by conjugating the bottom right corner with α'. Now the bottom arrow becomes an equivalence as it is given by a shift $k \to U' \oplus k$ in the index variable of the colimit diagram. □

Corollary 3.3.3.

$$\operatorname*{colim}_n \int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}(n)^{\mathrm{op}}} \simeq \int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}.$$

Proof. As the unstraightening functor

$$\int\colon \mathrm{Fun}(\mathrm{An}^\omega, \mathcal{C}\mathrm{at}_\infty) \to \mathrm{CartFib}(\mathrm{An}^\omega)$$

preserves all colimits and since the forgetful functor

$$\mathrm{CartFib}(\mathrm{An}^\omega) \to \mathcal{C}\mathrm{at}_\infty$$

preserves all colimits, we obtain the stated equivalence, by the previous lemma. $\square$

We want to define genuine homology theories out of the ∞-category

$$\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}.$$

Because we do not have an adjunction between the vertical opposite

$$\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}$$

and the ∞-category of compact pointed anima the next definitions require more effort. Nonetheless, they are quite similar to Definition 2.2.14.

Definition 3.3.4. *Let $\underline{\mathrm{Sph}}$ be a category of spheres. Let θ be the counit of the adjunction $\iota_0 \dashv S^0$ introduced in Proposition 2.2.7. evaluated on an object (X,V) we obtain a morphism*

$$(S^V_X, 0) \xrightarrow{\theta_{(X,V)}} (X,V).$$

It factors into a morphism in $\theta' \in \mathrm{Sph}^{S^V}(0, p^*V)$ followed by a Cartesian morphism

$$(S^V_X, p^*V) \to (X,V).$$

The morphism θ' gives us a map in $\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}$

$$(S^V, p^*V) \xrightarrow{\theta'} (S^V, 0)$$

Consider the composite with a Cartesian arrow

$$(S^V, p^*V) \to (S^V, 0) \to (X, 0).$$

Let $\alpha\colon U \oplus V \simeq W$ an equivalence in Sph^X, where $W \simeq r^*\mathbb{R}^n$ for some $n \in \mathbb{N}$. Together with the morphism above this gives us morphism

$$\Theta\colon (S^V, p^*W) \xrightarrow{\alpha_*} (S^V, p^*U \oplus p^*V) \xrightarrow{\theta'} (X, U).$$

We call a square in $\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}$ distinguished *if it is of the form*

$$\begin{tikzcd}
(X,\emptyset) \arrow[r, "(\mathrm{id}_X{,}\infty)"] & (X,U) \\
(X,W) \arrow[u, "(\mathrm{id}_X{,}\infty)"] \arrow[r, "(\sigma_\infty{,}\mathrm{id}_W)"'] & (S^V,W) \arrow[u, "\Theta=(p{,}\Theta')"']
\end{tikzcd}$$

Lemma 3.3.5. *A square in $\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}$ is distinguished if and only if it is the image of a distinguished square under a functor*

$$\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(n)} \to \int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}$$

Proof. To avoid bloating of notation we will write V instead of p^*V, etc. for the remainder of the proof.

We want to show that any such distinguished square

$$\begin{tikzcd}
(X,\emptyset) \arrow[r, "(\mathrm{id},\infty)"] & (X,U) \\
(X,W) \arrow[u, "(\mathrm{id}_X,\infty)"] \arrow[r, "(\sigma_\infty,\mathrm{id}_W)"'] & (S^V,W) \arrow[u, "\Theta"']
\end{tikzcd}$$

can be lifted to a distinguished square

$$\begin{tikzcd}
(X,\emptyset,\emptyset,\infty) \arrow[r, "(\mathrm{id},\infty,\infty,\infty)"] & (X,U,V,\alpha\colon U\oplus V\simeq W) \\
(X,W,0,\mathrm{triv}) \arrow[u, "(\mathrm{id}_X,\infty,\infty,\infty)"] \arrow[r, "(\sigma_\infty,\mathrm{id}_W,\mathrm{id}_0,\mathrm{triv})"'] & (S^V,W,0,\mathrm{triv}) \arrow[u, "(p,\theta')"']
\end{tikzcd}$$

in $\int_{\mathrm{An}^\omega}\underline{\mathrm{Sph}}^{\mathrm{op}}(n)$; here (p,θ') is the counit of the adjunction $\iota_{(W,0,\mathrm{triv})} \dashv S^{\perp}$ by Proposition 2.2.6. The obvious difficulty shows itself in lifting the right arrow. For that, we note that both diagrams are lifts of the square

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] & X \\
X \arrow[u, "\mathrm{id}"] \arrow[r, "\sigma_\infty"'] & S^V \arrow[u, "p"']
\end{tikzcd}$$

in An^ω. We want to analyze the effect of the forgetful functor on mapping anima between the objects on the right-hand side of the distinguished squares over the morphism $p\colon S^V\to X$. That is the morphism

$$\mathrm{Sph}(W)^{\mathrm{op},S^V}((W,0,\mathrm{triv}),(U,V,\alpha)) \to \mathrm{Sph}^{S^V}(U,W)$$

By Lemma 2.1.5 this is the top arrow of the pullback

$$\begin{tikzcd}
\mathrm{Sph}(W)^{\mathrm{op},S^V}((W,0,\mathrm{triv}),(U,V,\alpha)) \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sph}^{S^V}(U,W) \arrow[d, "\simeq"] \\
\mathrm{Sph}^{S^V}(0,V) \arrow[r] & \mathrm{Sph}^{S^V}(U,W)
\end{tikzcd}$$

We constructed Θ' so that it is the image of θ' under the map

$$\mathrm{Sph}^{S^V}(0,V) \to \mathrm{Sph}^{S^V}(U,W).$$

□

Definition 3.3.6. *The image of a split pushout square under one of the functors* $\iota_n\colon \mathrm{An}^\omega \to \int_{\mathrm{An}^\omega}\underline{\mathrm{Sph}}^{\mathrm{op}}$ *is called a* split pushout with trivial coefficients. *Let* $\mathcal{D}$ *be a pointed* ∞*-category. Let* ϕ *be a functor* $\int_{\mathrm{An}^\omega}\underline{\mathrm{Sph}}^{\mathrm{op}} \to \mathcal{D}$*. We say*

- ϕ *is* reduced *if it sends objects of the form* $(X,\emptyset)$ *to the zero object of* $\mathcal{D}$*,*

- ϕ realizes split Mayer-Vietoris sequences *if it sends split pushouts with trivial coefficients to pushout squares in* $\mathcal{D}$*,*

- ϕ realizes Thom-isomorphisms *if it sends distinguished squares to pushouts in* $\mathcal{D}$*.*

A reduced functor $\phi\colon \int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}} \to \mathcal{D}$ *that realizes split Mayer-Vietoris sequences and Thom-isomorphisms is called* a split genuine homology theory with coefficients in $\mathcal{D}$*. We denote the* ∞*-category of split genuine homology theories with coefficients in* $\mathcal{D}$ *by* $\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D})$*.*

Lemma 3.3.7. *The limit*

$$\lim_n \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}(n)^{\mathrm{op}}}, \mathcal{D})$$

is a full subcategory of the functor category

$$\mathrm{Fun}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D})$$

and coincides with the ∞*-category*

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D})$$

of split genuine homology theories.

Proof. It is clear by Corollary 3.3.3 that the limit can be identified with a full subcategory of the functor category

$$\mathrm{Fun}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}).$$

The forgetful functors

$$\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(n)} \to \int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}$$

send

- split pushouts with trivial coefficients to split pushouts with trivial coefficients,
- distinguished squares to distinguished squares.
- objects of the form $(X, \emptyset, \emptyset, \infty)$ to $(X, \emptyset)$.

Therefore the restriction of a split genuine homology theory to $\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(n)}$ is a split genuine homology theory. On the other hand, by Lemma 3.3.5 a functor $\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}} \to \mathcal{D}$ realizes Thom-isomorphisms when all of it restrictions do. The same is true for realizations of split Mayer-Vietoris sequences because of the following commuting diagrams

$$\begin{tikzcd}
\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(n)} \arrow[rr, "\int \mathrm{fgt}"] & & \int_{\mathrm{An}^\omega} \mathrm{Sph}^{\mathrm{op}} \\
& \mathrm{An}^\omega \arrow[ul, "\iota_{(n,0,\mathrm{triv})}"] \arrow[ur, "\iota_n"'] &
\end{tikzcd}$$

and finally also for reducedness as one easily checks. $\square$

Definition 3.3.8. *The functors of* An*-categories*

$$\underline{\mathrm{Sph}^{\mathrm{op}}} \to \underline{\mathrm{An}_*^{\omega,\mathrm{op}}} \xrightarrow{\Sigma^\infty} \underline{\mathrm{Sp}^{\omega,\mathrm{op}}} \xrightarrow{\mathbb{D}} \underline{\mathrm{Sp}^{\omega}}$$

[2] *define a functor on unstraightenings*

$$\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}} \to \int_{\mathrm{An}^\omega} \underline{\mathrm{Sp}^{\omega}}.$$

Let the Thom spectrum of the inverse bundle *be the composite of the functors*

$$\mathrm{Th}^-\colon \int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}} \to \int_{\mathrm{An}^\omega} \underline{\mathrm{Sp}^{\omega}} \xrightarrow{r_{X,!}} \mathrm{Sp}^\omega .$$

The value $\mathrm{Th}^-(X,V)$ *is the colimit over* X *of the functor*

$$X \to \mathrm{Sp}^\omega \colon x \mapsto \mathbb{D}(\Sigma^\infty S^{V_x})$$

Lemma 3.3.9. *The following square of functors commute*

$$\begin{tikzcd}
\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}} \arrow[r, "\mathrm{Th}^-"] & \mathrm{Sp}^\omega \\
\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(n)} \arrow[u, "\int \mathrm{fgt}"] \arrow[r, "\mathrm{Th}^\perp"'] & \mathrm{An}_*^\omega \arrow[u, "\Sigma^{\infty-n}"']
\end{tikzcd}$$

Proof. By unwinding the definitions of the functors involved. We see that the top composite sends an object $(X; U, V, \alpha\colon U \oplus V \simeq \mathbb{R}^n)$ to the colimit

$$\operatorname*{colim}_{x\in X} \mathbb{D}(\Sigma^\infty S^{U_x})$$

while the bottom composite sends the same object to the colimit

$$\operatorname*{colim}_{x\in X} \Sigma^{\infty-n} S^{V_x}.$$

The stabilization of α gives us a natural equivalence

$$\Sigma^\infty \alpha_x \colon \Sigma^\infty S^{U_x} \otimes \Sigma^\infty S^{V_x} \xrightarrow{\simeq} \Sigma^\infty S^n, \quad x \in X.$$

Which gives us a natural equivalence

$$\mathbb{D}(\Sigma^\infty S^{U_x}) \xrightarrow{\simeq} \Sigma^{\infty-n} S^{V_x}, \quad x \in X.$$

After taking colimits, we obtain the stated natural equivalence. $\square$

Theorem 3. *Let* $\mathcal{D}$ *be a valid coefficient category. The restriction along the functor* Th^- *induces an equivalence of* ∞*-categories*

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^\omega, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}).$$

[2]The category $\underline{\mathrm{An}_*^{\omega,\mathrm{op}}}$ evaluates on X as the opposite of the category $\mathrm{An}^\omega_{X/\cdot/X}$.

Proof. The restriction along the functor Th^{-} is a functor

$$\mathrm{Fun}(\mathrm{Sp}^{\omega}, \mathcal{D}) \to \mathrm{Fun}(\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}).$$

that by Lemma 4.2.8 becomes the functor

$$\lim_n (\mathrm{Th}^{\perp})^* \colon \lim_n \mathrm{Fun}(\mathrm{An}_*^{\omega}, \mathcal{D}) \to \lim_n \mathrm{Fun}(\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}(n)^{\mathrm{op}}}, \mathcal{D})$$

under the identifications of Corollary 3.3.3 and by Part 1 of Lemma 3.3.1. Under this identifications the full subcategory $\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^{\omega}, \mathcal{D})$ corresponds to the limit

$$\lim_n \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_*^{\omega}, \mathcal{D}) \subset \lim_n \mathrm{Fun}(\mathrm{An}_*^{\omega}, \mathcal{D})$$

by Part 2 of Lemma 3.3.1. By Lemma 3.3.7 the full subcategory $\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D})$ corresponds to the limit

$$\lim_n \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}(n)^{\mathrm{op}}}, \mathcal{D}) \subset \lim_n \mathrm{Fun}(\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}(n)^{\mathrm{op}}}, \mathcal{D}).$$

and by Proposition 2.4.4 we have an equivalence of limits

$$\lim_n (\mathrm{Th}^{\perp})^* \colon \lim_n \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_*^{\omega}, \mathcal{D}) \xrightarrow{\simeq} \lim_n \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^{\omega}} \underline{\mathrm{Sph}(n)^{\mathrm{op}}}, \mathcal{D}).$$

The result follows. □

3.4 Genuine Cohomology Theories with Values in Ab

We quickly recall the definition of cohomology theories and their relation to stable homotopy theory via Brown representability.

Definition 3.4.1. *A* (generalized reduced) cohomology theory on An_* with values in Ab *is a family* $\{h^n \colon \mathrm{An}_*^{\mathrm{op}} \to \mathrm{Ab}\}_{n \in \mathbb{N}}$ *of functors together with natural isomorphisms* $\sigma_n \colon h^n(X) \to h^{n+1}(\Sigma X)$ *for all* $X \in \mathrm{An}_*$*, such that the following axioms are satisfied for each* h^n*:*

- *(Wedge Axiom) For a wedge* $X = \bigvee_{i \in I} X_i$ *and inclusions* $\iota_i \colon X_i \to X$ *the map*

$$\prod_i \iota_i^* \colon h^n(X) \to \prod_{i \in I} h^n(X_i)$$

  is an isomorphism

- *(Excision Axiom) For a cofiber sequence* $X \to Y \to Z$ *the sequence*

$$h^n(Z) \to h^n(Y) \to h^n(X)$$

  is exact.

A (generalized reduced) cohomology theory on An_*^{ω} with values in Ab *is a family* $\{h^n \colon \mathrm{An}_*^{\omega,\mathrm{op}} \to \mathrm{Ab}\}_{n \in \mathbb{N}}$ *of functors together with natural isomorphisms* $\sigma_n \colon h^n(X) \to h^{n+1}(\Sigma X)$ *for all* $X \in \mathrm{An}_*^{\omega}$*, such that the following axioms are satisfied for each* h^n*:*

- *(Wedge Axiom)* $h^n(\mathrm{pt}) = 0$ *and for each* $X, Y \in \mathrm{An}_*^\omega$ *the product map*

$$h^n(X \vee Y) \to h^n(X) \oplus h^n(Y)$$

is an isomorphism.

- *(Excision Axiom) For a cofiber sequence* $X \to Y \to Z$ *the sequence*

$$h^n(Z) \to h^n(Y) \to h^n(X)$$

is exact.

A cohomology theory on compact spectra with values in Ab *is a functor*

$$E\colon \mathrm{Sp}^{\omega,\mathrm{op}} \to \mathrm{Ab}$$

that sends cofiber sequences

$$X \to Y \to Z$$

to exact sequences

$$E(Z) \to E(Y) \to E(X).$$

In [Bro62] Brown shows that

Proposition 3.4.2 (Brown representability)**.** *For every cohomology theory* h^n *on* An_* *with values in* Ab *there exists a (unique) spectrum* E *and natural equivalences*

$$h^n(X) \simeq [X, \Omega^{\infty-n}E]$$

of functors $\mathrm{An}_*^{\mathrm{op}} \to \mathrm{Ab}$*. The converse also holds, that is for every spectrum* $E = \{\Omega^{\infty-n}E, \sigma_n\colon \Omega^{\infty-n}E \xrightarrow{\simeq} \Omega\Omega^{\infty-n-1}E\}_{n\in\mathbb{N}}$ *the family of functors*

$$h^n(X) := [X, \Omega^{\infty-n}E]$$

and the induced natural equivalences $\sigma_n\colon h^n(-) \to h^{n+1}(\Sigma-)$ *define a cohomology theory on* An_* *with values in* Ab*.*

Under assumptions on the size of the values $h^n(X)$ Brown was able to prove the same result for cohomology theories on An_*^ω. Later Adams showed that these assumptions are unnecessary and proved the following result in [Ada71]

Proposition 3.4.3 (Adams version of Brown representability)**.** *Every cohomology theory on* An_*^ω *with values in* Ab *is equivalent to a functor*

$$[\Sigma^\infty -, E]$$

for a spectrum E*. And any natural transformation*

$$[\Sigma^\infty -, E] \to [\Sigma^\infty -, E']$$

is induced by a (non-unique) map of spectra $E \to E'$*.*

Likewise, every cohomology theory on compact spectra with values in Ab *is equivalent to a functor*

$$[-, E]|_{\mathrm{Sp}^\omega}$$

for a spectrum E*. And any natural transformation*

$$[-, E]|_{\mathrm{Sp}^\omega} \to [-, E']|_{\mathrm{Sp}^\omega}$$

is induced by a (non-unique) map of spectra $E \to E'$*.*

This version of Adams theorem can be found in [Nee97]. We refer the interested reader to [Nee97] for a more detailed discussion on the generality in which Brown and Adams' results hold.

For us important is that

Lemma 3.4.4. *The category of cohomology theories on compact pointed anima (or compact spectra) with values in* Ab *is equivalent to the category of functors*

$$E\colon \mathrm{Sp}^{\omega,\mathrm{op}} \to \mathrm{Ab}$$

- *that send pushouts*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & D \end{tikzcd}$$

to weak pullbacks

$$\begin{tikzcd} E(D) \arrow[r] \arrow[d] & E(B) \arrow[d] \\ E(C) \arrow[r] & E(A) \end{tikzcd},$$

[3]

- *and that send the zero object to the zero object.*

Proof. It's a corollary of Adams' version of Brown representability that the category of cohomology theories on compact pointed anima with values in Ab is equivalent to the category of cohomology theories on compact spectra with values in Ab. It is an easy exercise to see that a functor $E\colon \mathrm{Sp}^{\omega,\mathrm{op}} \to \mathrm{Ab}$ that sends pushouts to weak pullbacks and the zero object to the zero object is precisely a cohomology theory on compact spectra with values in Ab. □

We want to apply the results of the previous section, in particular Theorem 3, to the setting $\mathcal{D} = \mathrm{Ab}^{\mathrm{op}}$. We observe that inside the left-hand side $\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^\omega, \mathrm{Ab}^{\mathrm{op}})$ of the equivalence of Theorem 3 sits the opposite of the category $\mathrm{Coh}(\mathrm{Sp}^{\omega,\mathrm{op}}, \mathrm{Ab})$ of cohomology theories in the sense of Definition 3.4.1. Inspired by this we define:

Definition 3.4.5. *A* genuine cohomology theory *(with values in* Ab*) is a functor* $E \in \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}$ *such that for every pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & D \end{tikzcd}$$

in An^ω *and every natural number* $n \in \mathbb{N}$ *the map*

$$E(\iota_n D) \to E(\iota_n B) \times_{E(\iota_n A)} E(\iota_n C)$$

is surjective. We denote the full subcategory of genuine cohomology theories by

$$\mathrm{Gen}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}}) \subset \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}.$$

[3] That means that the map $E(D) \to E(B) \times_{E(A)} E(C)$ is surjective.

Proposition 3.4.6. *We have an equivalence of* ∞*-categories*

$$\begin{array}{rcl} \{\textit{genuine cohomology theories}\} & \simeq & \{\textit{cohomology theories on compact spectra}\} \\ E & \mapsto & E(\Sigma^{\infty-n}X) := \mathrm{fib}(E^{\mathbb{R}^n}(X) \to E^{\mathbb{R}^n}(\mathrm{pt})) \\ E^V(X) := [X^{-V}, E] & \leftarrow\!\shortmid & E \end{array}$$

[4]

Proof. Theorem 3 shows that the functors in question induce an equivalence of overcategories

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}^\omega, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}.$$

We just have to show that the full subcategory of cohomology theories on compact spectra corresponds to the full subcategory of genuine cohomology theories under this equivalence. Suppose E is a cohomology theory on compact spectra. And

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

a pushout in compact anima. We have to show that the map

$$E(\iota_n D) \to E(\iota_n B) \times_{E(\iota_n A)} E(\iota_n C)$$

is surjective, for the associated genuine cohomology theory $E^V(X) := [X^{-V}, E]$. But this map is given by the map

$$[\Sigma^{\infty-n}D, E] \to [\Sigma^{\infty-n}B, E] \times_{[\Sigma^{\infty-n}A,E]} [\Sigma^{\infty-n}C, E]$$

which is surjective since

$$\begin{tikzcd} \Sigma^{\infty-n}A \arrow[r] \arrow[d] & \Sigma^{\infty-n}B \arrow[d] \\ \Sigma^{\infty-n}C \arrow[r] & \Sigma^{\infty-n}D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout. Conversely, suppose E is a genuine cohomology theory. If

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout of finite spectra, then it lies in the image of a pushout of some functor

$$\mathrm{An}_*^\omega \xrightarrow{\Sigma^{\infty-n}} \mathrm{Sp}^\omega, \quad n \in \mathbb{N}.$$

By assumption

$$E^{\mathbb{R}^n}(D) \to E^{\mathbb{R}^n}(B) \times_{E^{\mathbb{R}^n}(A)} E^{\mathbb{R}^n}(C)$$

is surjective. Also on reductions $\tilde{E}^{\mathbb{R}^n}(X) := \mathrm{fib}(E^{\mathbb{R}^n}(X) \to E^{\mathbb{R}^n}(\mathrm{pt}))$ we have a surjective map

$$\tilde{E}^{\mathbb{R}^n}(D) \to \tilde{E}^{\mathbb{R}^n}(B) \times_{\tilde{E}^{\mathbb{R}^n}(A)} \tilde{E}^{\mathbb{R}^n}(C).$$

Which is what we wanted to show. □

[4] Or more precisely $\mathrm{Gen}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}}) \simeq \mathrm{Coh}(\mathrm{Sp}^{\omega,\mathrm{op}}, \mathrm{Ab})$.

Remark 3.4.7. *The ∞-category of spectra* Sp *is equivalent to the ∞-category of reduced excisive functors*

$$\mathrm{Exc}_*(\mathrm{An}_*^\omega, \mathrm{An}^{\mathrm{op}})^{\mathrm{op}}.$$

By a similar reasoning as in Proposition 3.4.6 we can identify the ∞-category of reduced excisive functors with

$$\mathrm{Gen}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{An}^{\mathrm{op}})$$

the opposite of the full subcategory of

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{An}^{\mathrm{op}})$$

on those functors that send images

$$\begin{tikzcd} \iota_n A \arrow[r] \arrow[d] & \iota_n B \arrow[d] \\ \iota_n C \arrow[r] & \iota_n D \end{tikzcd}$$

of puhsouts

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & D \end{tikzcd}$$

in An^ω *to pushouts in* $\mathrm{An}^{\mathrm{op}}$*, i.e. pullbacks in* An.

3.5 Axioms of Genuine Cohomology Theories

Lastly in this chapter, we want to axiomatize genuine cohomology theory on compact spectra in the sense of the axioms of Eilenberg-Steenrod cohomology theories.

Definition 3.5.1. *An* axiomatized genuine cohomology theory *consists of a collection $E^V(X)$ of abelian groups, where X runs trough all compact anima and V is a vector bundle over X. For each map $f\colon X \to Y$ and a vector bundle W over Y we have induced maps $f^W\colon E^W(Y) \to E^{f^*W}(X)$. For each affine linear map $\phi\colon V \to W$ of vector bundles over X we have induced maps $\phi_X\colon E^V(X) \to E^W(X)$. The usual functoriality conditions hold that is*

- *If f, ϕ above are the identity maps then they induce the identity.*
- *We have $(g \circ f)^V = f^{g^*V} \circ g^V$ and $(\phi \circ \psi)_X = \phi_X \circ \psi_X$.*

We require this collection of abelian groups and induced maps to satisfy some properties

(i) (Homotopy Invariance)
If $\phi \simeq \psi$ are homotopic morphisms of vector bundles $V \to W$ over X, then the induced morphisms $\phi_X = \psi_X$ are equal. If $H\colon f \simeq g$ is a homotopy, then

it induces an equivalence $\phi_H\colon f^*V \simeq g^*V$ between vector bundles over X. We require that $(\phi_H)_X \circ f^V = g^V$, i.e. the following commutes

$$\begin{tikzcd}
 & E^{f^*V}(X) \arrow[dd, "(\phi_H)_X"] \\
E^V(X) \arrow[ur, "f^V"] \arrow[dr, "g^V"'] & \\
 & E^{g^*V}(X)
\end{tikzcd}$$

(ii) (Beck-Chevalley)
Let $f\colon X \to Y$ be a map of anima and $\phi\colon V \to W$ be a map of vector bundles over Y. Let $f^*\phi$ be the induced map $f^*V \to f^*W$ of vector bundles over X. Then the following square commutes

$$\begin{tikzcd}
E^V(Y) \arrow[r, "f^V"] \arrow[d, "\phi_Y"'] & E^{f^*V}(X) \arrow[d, "(f^*\phi)_X"] \\
E^W(Y) \arrow[r, "f^W"'] & E^{f^*W}(X)
\end{tikzcd}$$

(iii) (Reducedness)
Let X be a compact anima and let $\emptyset$ be the empty bundle over X. Then $E^{\emptyset}(X) \cong 0$.

(iv) (Mayer-Vietoris)
For each pushout square of anima

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_2 \arrow[d, "f_3"] \\
X_3 \arrow[r, "f_4"'] \arrow[ur, phantom, "\overset{H}{\Rightarrow}"] & X_4
\end{tikzcd}$$

and vector bundle V over X_4 the induced sequence

$$E^V(X_4) \xrightarrow{(f_3^V, f_4^V)} E^{f_3^*V}(X_2) \oplus E^{f_4^*V}(X_3) \xrightarrow{f_1^{f_3^*V} - (\phi_H)_X \circ f_2^{f_4^*V}} E^{f_1^* f_3^* V}(X_1)$$

is exact in the middle.

(v) (The Thom Isomorphism)
Let V, W be two vector bundles over X. Let

$$E^{V\oplus W}(S_X^V, X) = E^{p^*(V\oplus W)}(S_X^V) \ominus E^{V\oplus W}(X)$$

be the complement of the direct summand inside $E^{p^*(V\oplus W)}(S_X^V)$ of $E^{V\oplus W}(X)$ due to the retraction $\sigma^{p^*(V\oplus W)} \circ p^{V\oplus W} = \mathrm{id}$. Then the map

$$E^W(X) \xrightarrow{p^W} E^{p^*W}(S_X^V) \xrightarrow{(\theta_{(X)}^{(V,W)})_{S_X^V}} E^{p^*(V\oplus W)}(S_X^V) \to E^{V\oplus W}(S_X^V, X)$$

is an isomorphism.

A morphism of axiomatized genuine cohomology theories $\zeta\colon E \to F$ *is a collection of group homomorphisms*

$$\zeta_X^V\colon E^V(X) \to F^V(X)$$

that commute with the induced maps f^W, ϕ_X.

We want to make the above definition rigorous, by observing that some of the stated axioms of a genuine cohomology theory are implemented by a functor out of $\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}}^{\mathrm{op}}$. In general, we can describe the homotopy category of the total category of a Cartesian fibration as follows:

Remark 3.5.2. *Let* $p\colon \mathcal{C} \to \mathcal{I}$ *be a Cartesian fibration. We want to give a 'generators and relations' description of the homotopy category of* $\mathcal{C}$ *in terms of the functor* $\mathrm{d}p\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ *that classifies* p. *First, consider the functor* $h\mathrm{d}p\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, *that we obtain by postcomposing* $\mathrm{d}p$ *with the functor* $h\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ *that sends an* ∞*-category* $\mathcal{D}$ *to its homotopy category* $h\mathcal{D}$. *The natural transformation* $\mathrm{id}_{\mathcal{C}\mathrm{at}_\infty} \to h$ *induces a functor* $h\mathcal{C} \to h\int_{\mathcal{I}} h\mathrm{d}p$. *This functor is an equivalence of* ∞*-categories; which is a consequence for example of the main result of [GHN17], explicitly this follows from the universal property of the unstraightening as a lax colimit which is preserved by the functor* h. *Since* $h\mathrm{d}p$ *maps into the full subcategory* $\mathcal{C}\mathrm{at}_1$ *of* $\mathcal{C}\mathrm{at}_\infty$ *of ordinary categories, which is a* $(2,1)$*-category, the functor* $h\mathrm{d}p$ *factorizes over the homotopy-2 category* $\tau_{\leq 2}\mathcal{I}$. *We conclude, that we can compute the homotopy category* $h\mathcal{C}$ *by computing* $h\int_{\tau_{\leq 2}} h\mathrm{d}p$, *i.e. the homotopy category of the Grothendieck construction of the* 2*-functor from the homotopy-2 category of* $\mathcal{I}^{\mathrm{op}}$ *to the* $(2,1)$*-category of* 1*-categories, given by the fiber-wise homotopy category of the straightening of* p. *Luckily, there is a formula for that construction (compare with [Bak]):*

An object of $h\mathcal{C}$ *is a pair* (i, c) *consisting of an object* $i \in \mathcal{I}$ *and an object* $c \in h\mathrm{d}p(i)$.

A morphism $(i, c) \to (j, d)$ *is an equivalence class represented by a pair* $(f, [\phi])$ *consisting of a morphism* $f\colon j \to i$ *in* $\mathcal{I}$ *and homotopy class of morphisms* $[\phi\colon c \to f^*d]$. *Two of such pairs* $(f, [\phi])$ *and* $(g, [\psi])$ *get identified if there exists a homotopy* $H\colon f \to g$ *in* $\mathcal{I}$ *such that with the induced natural isomorphism* $H^*\colon f^* \simeq g^*$ *between functors* $h\mathrm{d}p(j) \to h\mathrm{d}p(i)$ *the post composition* $c \xrightarrow{[\phi]} f^*d \xrightarrow{H^*} g^*d$ *is equal to the class* $[\psi]$. *We can even further simplify the description of* $h\mathcal{C}$ *by abusing the fact that we can factorize every morphism in* $\mathcal{C}$ *in a fiber-wise morphism followed by a Cartesian morphism. Let* $(f, [\phi])\colon (i, c) \to (j, d)$ *represent a morphism in* $h\mathcal{C}$. *Then we can write it as the composite* $(f, [\phi]) = (f, [\mathrm{id}]) \circ (\mathrm{id}, [\phi])$

$$\begin{tikzcd}
(i,c) \arrow[rr, "{(f,[\phi])}"] \arrow[dr, "{(\mathrm{id},[\phi])}"'] & & (j,d) \\
& (i, f^*d). \arrow[ur, "{(f,\mathrm{id})}"'] &
\end{tikzcd}$$

The composition of two such decompositions $(f, [\phi]) = (f, [\mathrm{id}]) \circ (\mathrm{id}, [\phi])$ *and* $(g, [\psi]) = (g, [\mathrm{id}]) \circ (\mathrm{id}, [\psi])$ *is given by the decomposition* $(\mathrm{id}, [\psi\phi]) \circ (\mathrm{id}, gf)$

$$\begin{tikzcd}
(i,c) \arrow[d, "{(\mathrm{id},[\phi])}"'] & & \\
(i, f^*d) \arrow[r, "{(f,\mathrm{id})}"'] \arrow[d, dashed, "{(\mathrm{id},[\psi])}"'] & (j,d) \arrow[d, "{(\mathrm{id},[\psi])}"] & \\
(i, f^*g^*e) \arrow[r, dashed, "{(f,\mathrm{id})}"'] & (j, g^*e) \arrow[r, "{(g,\mathrm{id})}"'] & (k,b)
\end{tikzcd}$$

. *Thus a functor* $F\colon h\mathcal{C} \to \mathcal{D}$ *into an ordinary category consists of the data*

- *A specified object* $F(i,c) \in \mathcal{D}$ *for each pair* $i \in \mathcal{I}$ *and* $c \in h\mathrm{dp}(i)$*.*
- *A morphism* $F(f)\colon F(i, f^*d) \to F(j,d)$ *for every morphism* $f\colon i \to j$ *in* $\mathcal{C}$ *and fixed object* $d \in h\mathrm{dp}(j)$*.*
- *A morphism* $F([\phi])\colon F(i,c) \to F(i,d)$ *for every homotopy class of morphisms* $[\phi\colon c \to d]$ *in* $h\mathrm{dp}(i)$*.*
- *A specified natural isomorphism* $\epsilon_{H,d}\colon F(i,f^*d) \xrightarrow{\cong} F(i,g^*d)$ *for every homotopy* $H\colon f \simeq g$ *between morphisms* $i \to j$ *in* $\mathcal{C}$ *and object* $d \in h\mathrm{dp}(j)$

such that

- *(Beck-Chevalley) Both composites agree* $F(\phi) \circ F(f) = F(f) \circ F(f^*\phi)$

$$\begin{tikzcd} F(i,f^*c) \arrow[r, "F(f)"] \arrow[d, "F([f^*\phi])"'] & F(j,c) \arrow[d, "F([\phi])"] \\ F(i,f^*d) \arrow[r, "F(f)"'] & F(j,d) \end{tikzcd}$$

- $F(f) = F(g) \circ \epsilon_{H,d}$

$$\begin{tikzcd} F(i,f^*d) \arrow[dr, "F(f)"] \arrow[dd, "\epsilon_{H,d}"'] & \\ & F(j,d) \\ F(i,g^*d) \arrow[ur, "F(g)"'] & \end{tikzcd}$$

Theorem 4. *The categories of axiomatized genuine cohomology theories and cohomology theories on compact anima are equivalent.*

Proof. The last remark shows that an axiomatized genuine cohomology theory is the same data as a functor

$$E\colon h(\int_{\mathrm{An}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}) \to \mathrm{Ab}^{\mathrm{op}}$$

that satisfies the axioms of Definition 3.4.5. By Proposition 3.4.6 we have an equivalence of categories between the category of genuine cohomology theories and the category of cohomology theories on compact spectra. The rest is Lemma 3.4.4. $\square$

Equivariant Genuine Cohomology Theories

Now we want to apply our results to the topos of G-anima for a fixed compact Lie group G. We follow the strategy of Chapter 3 by first defining $\underline{\mathrm{AffLin}}$, a model for a category of compact G-spheres. We invoke our main result Theorem 2, which states the equivalence

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_{G,*}^{\omega}, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D})$$

and exploit the functoriality of this equivalence akin to Proposition 2.4.4 by letting W vary in a G-universe $\mathcal{U}$. In the limit for $W \in \mathcal{U}$ the left-hand side becomes a full subcategory of functors

$$\operatorname*{colim}_{\mathcal{U}} \mathrm{An}_{G,*}^{\omega} \simeq \mathrm{Sp}_G^{\omega} \to \mathcal{D}$$

and the right-hand side becomes a full subcategory of functors

$$\operatorname*{colim}_{W \in \mathcal{U}} \int_{\mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \simeq \int_{\mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathcal{D}.$$

This will lead us to the equivalence

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^{\omega}, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D}).$$

We will then analyze this equivalence for specific choices of $\mathcal{D}$. Our main interest will be in the case $\mathcal{D} = \mathrm{Ab}^{\mathrm{op}}$, where we will obtain a new set of axioms for equivariant cohomology theories.

4.1 Equivariant Affine Linear Morphisms

Construction 4.1.1. *We consider the ∞-category of G-anima which is defined as the presheaf category*

$$\mathrm{An}_G := \mathcal{P}(\mathrm{Orb}_G).$$

For a closed subgroup $H \leq G$ we let AffLin^H be the underlying ∞-category of the topological functor category

$$\mathrm{Fun}(\mathbb{B}H, \mathrm{TopAffLin}).$$

It is the category consisting of finite-dimensional Euclidean vector spaces with an H-action through linear isometries. A morphism $f\colon V \to W$ is an affine linear morphism that decomposes as an H-equivariant linear isometric embedding $\iota\colon V \hookrightarrow W$ followed by a translation along a vector $w \in (\iota(V)^{\perp})^H$ out of the H fixed points of the orthogonal complement of $\iota(V)$. Thus $\mathrm{AffLin}^{\{e\}} \simeq \mathrm{AffLin}$. The assignment $G/H \mapsto \mathrm{AffLin}^H$ makes AffLin^- into a G-category (short for An_G-category), that is a functor $\mathrm{Orb}_G^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ or equivalently a limit preserving functor

$$\underline{\mathrm{AffLin}}\colon \mathrm{An}_G^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty.$$

Lemma 4.1.2. *For a closed subgroup $H \leq G$ let $\mathrm{Rep}^{H,\simeq}$ be the core of the category of finite-dimensional orthogonal H-representations. Then the functor*

$$(\mathrm{Rep}^{H,\simeq})_+ \to \mathrm{AffLin}^H\colon V \mapsto V, \quad + \mapsto \emptyset$$

induces an equivalence onto the core of AffLin^H.

Proof. The core of AffLin^H is given by the underlying ∞-category of the core of the topological functor category

$$\mathrm{Fun}(\mathbb{B}H, \mathrm{TopAffLin}),$$

which is the underlying ∞-category of

$$\mathrm{Fun}(\mathbb{B}H, \mathrm{TopAffLin}^{\simeq}).$$

We have an equivalence

$$\mathrm{TopAffLin}^{\simeq} \simeq (\coprod_n \mathbb{B}O(n))_+$$

and $\mathrm{Fun}(\mathbb{B}H, \mathbb{B}O(n))$ is equivalent to the category of orthogonal H-representations of dimension n. □

It follows that the core of the G-category $\underline{\mathrm{AffLin}}$ is the G-category given by $\mathrm{Rep}^{-,\simeq}$. It is also straightforward to check that the G-category $\underline{\mathrm{AffLin}}$ is a category of compact An_G-spheres.

4.2 G-Universes and Genuine Homology Theories on Compact G-Anima

Recall, that with view towards Lemma 4.1.2 we have for every finite-dimensional G-representation W an equivalence of ∞-categories

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_{G,*}^{\omega}, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^{\omega}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D})$$

by Theorem 2, which is natural in

$$W \in M//^{\mathrm{lax}}M$$

due to Proposition 2.4.4. Here M is the core of the category of finite-dimensional orthogonal G-representations.

This motivates the following definition.

Definition 4.2.1. *A* complete G-universe *is a countably infinite dimensional G-representation $\mathcal{U}$ with an G-invariant inner product such that for every finite-dimensional G-representation V there is a G-equivariant linear embedding*

$$V \hookrightarrow \mathcal{U}.$$

We view $\mathcal{U}$ as a partially ordered set by letting its objects be finite-dimensional sub-representations $V \subset \mathcal{U}$ and letting $V \leq W$ whenever $V \subset W$. By mapping

$$\mathcal{U} \ni V \mapsto V \in M//^{\mathrm{lax}}M$$

and the morphism $V \leq W$ to the arrow in $M//^{\mathrm{lax}}M$ corresponding to the representation given by the orthogonal complement

$$V^{\perp_W}$$

of V inside of W we obtain a functor

$$\mathcal{U} \to M//^{\mathrm{lax}}M.$$

Lemma 4.2.2. *The forgetful functors*

$$\mathrm{AffLin}^{G,\mathrm{op}}(W) \to \mathrm{AffLin}^{G,\mathrm{op}}$$

induce an equivalence of ∞-categories

$$\operatorname*{colim}_{W \in \mathcal{U}} (\mathrm{AffLin}^G)^{\mathrm{op}}(W) \simeq \mathrm{AffLin}^{G,\mathrm{op}}.$$

Proof. This is essentially the same proof as in Lemma 3.3.2. From Lemma 2.1.5, we find that we can compute the mapping anima of $(\mathrm{AffLin}^G)^{\mathrm{op}}$ via the pullback

$$\begin{tikzcd}
(\mathrm{AffLin}^G)^{\mathrm{op}}(W)((U,V,\alpha),(U',V',\alpha')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{AffLin}^G(U',U) \arrow[d] \\
\mathrm{AffLin}^G(V,V') \arrow[r] & \mathrm{AffLin}^G(V \oplus U', W)
\end{tikzcd}.$$

By manipulating the lower right corner of the pullback with the isomorphisms α, α' we can reformulate the pullback like

$$\begin{tikzcd}
(\mathrm{AffLin}^G)^{\mathrm{op}}(W)((U,V,\alpha),(U',V',\alpha')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{AffLin}^G(U',U) \arrow[d] \\
\mathrm{AffLin}^G(V,V') \arrow[r] & \mathrm{AffLin}^G(V \oplus U', V' \oplus U')
\end{tikzcd}$$

Let

$$M_W(U,V;U',V') := (\mathrm{AffLin}^G)^{\mathrm{op}}(W)((U,V,\alpha),(U',V',\alpha')).$$

By passing to the colimit over $W \in \mathcal{U}$ we have

$$\begin{tikzcd}
\operatorname*{colim}_{W' \in \mathcal{U}} M_{W'}(U, V \oplus W^{\perp}; U', V' \oplus W^{\perp}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{AffLin}^G(U',U) \arrow[d] \\
\operatorname*{colim}_{W' \in \mathcal{U}} \mathrm{AffLin}^G(V \oplus W^{\perp}, V' \oplus W^{\perp}) \arrow[r] & \operatorname*{colim}_{W' \in \mathcal{U}} \mathrm{AffLin}^G(V \oplus W^{\perp} \oplus U', V' \oplus W^{\perp} \oplus U')
\end{tikzcd}$$

The lower map is an equivalence as it is induced by a shift in the indexing variable $W' \rightsquigarrow W' \oplus U'$. Therefore the upper map is an equivalence as well. The formula for the filtered colimit of a diagram of ∞-categories proves that the functor out of the colimit is fully faithful. It is also easy to see that the functor is essentially surjective. $\square$

The functors

$$\mathrm{AffLin}^{G,\mathrm{op}}(W) \to \mathrm{AffLin}^{G,\mathrm{op}}$$

from Lemma 4.2.2 come from An_G-functors

$$\underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \to \underline{\mathrm{AffLin}^{\mathrm{op}}}$$

and define a natural transformation of functors $\mathrm{An}_G^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ out of the pointwise colimit

$$\operatorname*{colim}_{W \in \mathcal{U}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \to \underline{\mathrm{AffLin}^{\mathrm{op}}}.$$

Lemma 4.2.3. *The natural transformation*

$$\operatorname*{colim}_{W \in \mathcal{U}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \to \underline{\mathrm{AffLin}^{\mathrm{op}}}$$

of functors $\mathrm{An}_G^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ restricts to a natural equivalence

$$\operatorname*{colim}_{W \in \mathcal{U}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}|_{\mathrm{An}^\omega} \xrightarrow{\simeq} \underline{\mathrm{AffLin}^{\mathrm{op}}}|_{\mathrm{An}^\omega}$$

of finite limit preserving functors $\mathrm{An}_G^{\omega,\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$.

Proof. The functor

$$\operatorname*{colim}_{W \in \mathcal{U}} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}|_{\mathrm{An}^\omega}$$

is finite limit preserving as a filtered colimit of limit preserving functors. We have to show that for each sub-group $H \leq G$ the colimit

$$\operatorname*{colim}_{W \in \mathcal{U}} \mathrm{AffLin}^{H,\mathrm{op}}(\mathrm{res}^G_H W)$$

is still equivalent to $\mathrm{AffLin}^{H,\mathrm{op}}$. By Theorem 4.5 of [BtD85] we know that $\mathrm{res}^G_H \mathcal{U}$ is a complete H-universe. Therefore we can invoke the Lemma 4.2.2 to find that we have an equivalence

$$\operatorname*{colim}_{W' \in \mathrm{res}^G_H \mathcal{U}} \mathrm{AffLin}^{H,\mathrm{op}}(W') \simeq \mathrm{AffLin}^{H,\mathrm{op}}.$$

where the colimit runs over all finite-dimensional sub-representations of $\mathrm{res}^G_H \mathcal{U}$. There is an obvious map of diagrams $\{\mathrm{AffLin}^{H,\mathrm{op}}(\mathrm{res}^G_H W)\}_W \to \{\mathrm{AffLin}^{H,\mathrm{op}}(W')\}_{W'}$. and to see that this is cofinal, we need that every finite-dimensional H-representation $W' \subset \mathrm{res}^G_H \mathcal{U}$ sits equivariantly inside a finite-dimensional G-representation $W \subset \mathcal{U}$. This is also true because of Theorem 4.5 of [BtD85]. $\square$

Definition 4.2.4 (G-Spectra)**.** *We define the ∞-category of G-spectra Sp_G to be the universal commutative algebra in Pr^L under $\mathrm{An}_{G,*}$ in which the image of every representation sphere S^V becomes invertible.*

Proposition 4.2.5. *The assignment $G/H \mapsto \mathrm{Sp}_H$ extends to a functor*

$$\mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Pr}^R$$

Definition 4.2.6. *Let $\underline{\mathrm{Sp}_G}$ be the limit preserving extension*

$$\mathrm{An}_G^{\mathrm{op}} \to \mathrm{Pr}^R$$

of the functor $G/H \mapsto \mathrm{Sp}_H$. Let Sp_G^X be its value on an object $X \in \mathrm{An}_G$. The unique morphism $r_X \colon X \to G/G$ induces a functor

$$r_{X,!} \colon \mathrm{Sp}_G^X \to \mathrm{Sp}_G\,.$$

Furthermore, consider the sequence of An_G^ω-functors

$$\underline{\mathrm{AffLin}^{\mathrm{op}}} \to \underline{\mathrm{An}_{G,*}^{\omega,\mathrm{op}}} \to \underline{\mathrm{Sp}_G^{\omega,\mathrm{op}}} \xrightarrow{\mathbb{D}} \underline{\mathrm{Sp}_G^\omega}$$

and the functor

$$\mathrm{Th}^- \colon \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sp}_G^\omega} \xrightarrow{r_!} \mathrm{Sp}_G^\omega\,.$$

We write

$$X^{-V} := \mathrm{Th}^-(X, V)$$

and call X^{-V} the Thom spectrum *of the inverse bundle of V.*

Through our construction of the $M//^{\mathrm{lax}}M$-functoriality of $\underline{\mathrm{AffLin}}$ in subsection 2.4 the ∞-category $\mathrm{An}_{G,*}^\omega$ becomes functorial in the universe $\mathcal{U}$ by assigning it to each $W \in \mathcal{U}$ and the morphism $W \leq W'$ to the functor

$$\mathrm{An}_{G,*}^\omega \xrightarrow{-\wedge S^{W^\perp W'}} \mathrm{An}_{G,*}^\omega\,.$$

Lemma 4.2.7. *We have an equivalence of ∞-categories*

$$\operatorname*{colim}_{W \in \mathcal{U}} \mathrm{An}_{G,*}^\omega \simeq \mathrm{Sp}_G^\omega\,.$$

Proof. This is discussed in Appendix C. The ∞-category of G-spectra of [GM23]. $\square$

The rest of this subsection is essentially identical to the subsection 3.3

Lemma 4.2.8. *The following square of functors commute*

$$\begin{tikzcd}
\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \arrow[r, "\mathrm{Th}^-"] & \mathrm{Sp}_G^\omega \\
\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \arrow[u, "\int \mathrm{fgt}"] \arrow[r, "\mathrm{Th}^\perp"'] & \mathrm{An}_{G,*}^\omega \arrow[u, "\Omega^W\Sigma^\infty"']
\end{tikzcd}$$

Proof. We give a similar proof as in Lemma 3.3.9. Again by unwinding the definitions we find that the composite of the top and left functor sends an object $(X; U, V, \alpha \colon U \oplus V \to W)$ to the object

$$r_{X,!}\mathbb{D}(\Sigma^\infty S^U)$$

where $\Sigma^\infty S^U$ is the image of $S^U \in \underline{\mathrm{An}_{G,*}^\omega}^X$ under the parametrized stabilization $\Sigma^\infty \colon \underline{\mathrm{An}_{G,*}^\omega}^X \to \underline{\mathrm{Sp}_G^\omega}^X$. Likewise, the composite of the bottom and right functors sends the same object to

$$r_{X,!}\Omega^W\Sigma^\infty S^V.$$

The stabilization of α gives us a natural equivalence

$$\Sigma^\infty \alpha \colon \Sigma^\infty S^U \otimes \Sigma^\infty S^V \xrightarrow{\simeq} \Sigma^\infty S^W.$$

in the ∞-category $\underline{\mathrm{Sp}_G^{\omega}}^X$. Hence an equivalence

$$\mathbb{D}(\Sigma^\infty S^U) \simeq \Omega^W \Sigma^\infty S^V$$

The result follows. $\square$

Definition 4.2.9. *We copy Definition 3.3.4 and call a square in*

$$\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}$$

distinguished *if it is of the form*

$$\begin{tikzcd}
(X, \emptyset) \arrow[r, "(\mathrm{id}_X, \infty)"] & (X, U) \\
(X, W) \arrow[u, "(\mathrm{id}_X, \infty)"] \arrow[r, "(\sigma_\infty, \mathrm{id}_W)"'] & (S^V, W) \arrow[u, "(p, \Theta')"']
\end{tikzcd}$$

where $\Theta' \in \underline{\mathrm{AffLin}}^{S^V}(U, W)$ *is the morphism corresponing to the morphism* $\theta' \in \underline{\mathrm{AffLin}}^{S^V}(0, V)$*, which is explained in Lemma 3.3.5. Moreover,* θ' *is part of the counit* $(p, \theta') \colon (S^V, 0) \to (X, V)$ *of the adjunction*

$$\iota_0 \colon \mathrm{An}_G^\omega \rightleftarrows \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}} \colon S$$

As in Lemma 3.3.5 one can verbatim prove the following lemma

Lemma 4.2.10. *Every distinguished square in* $\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}$ *lifts to a distinguished square under a functor*

$$\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \to \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}.$$

With this, we obtain a better description of the subcategory

$$\lim_{W \in \mathcal{U}} \mathrm{Gen}^{\mathrm{split}}\Big(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D}\Big) \subset \mathrm{Fun}\Big(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D}\Big)$$

Definition 4.2.11. *The image of a split pushout square under one of the functors* $\iota_W \colon \mathrm{An}_G^\omega \to \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}$ *is called a* split pushout with trivial coefficients*. Let* $\mathcal{D}$ *be a pointed* ∞*-category. Let* ϕ *be a functor* $\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathcal{D}$*. We say*

- ϕ *is* reduced *if it sends objects of the form* $(X, \emptyset)$ *to the zero object of* $\mathcal{D}$*,*
- ϕ realizes split Mayer-Vietoris sequences *if it sends split pushouts with trivial coefficients to pushout squares in* $\mathcal{D}$*,*
- ϕ realizes Thom-isomorphisms *if it sends distinguished squares to pushouts in* $\mathcal{D}$*.*

A reduced functor $\phi\colon \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathcal{D}$ *that realizes split Mayer-Vietoris sequences and Thom-isomorphisms is called* a split genuine homology theory with coefficients in $\mathcal{D}$. *We denote the ∞-category of split genuine homology theories with coefficients in $\mathcal{D}$ by* $\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D})$.

Lemma 4.2.12. *The limit*

$$\lim_{W \in \mathcal{U}} \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D})$$

is a full subcategory of the functor category

$$\mathrm{Fun}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D})$$

and coincides with the ∞-category

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D})$$

of split genuine homology theories.

Proof. It is clear by Lemma 4.2.3 that the limit can be identified with a full subcategory of the functor category

$$\mathrm{Fun}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D}).$$

The forgetful functors

$$\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \to \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}$$

send

- split pushouts with trivial coefficients to split pushouts with trivial coefficients,
- distinguished squares to distinguished squares.
- objects of the form $(X, \emptyset, \emptyset, \infty)$ to $(X, \emptyset)$.

Therefore the restriction of a split genuine homology theory to $\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}$ is a split genuine homology theory. On the other hand, by Lemma 3.3.5 a functor $\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathcal{D}$ realizes Thom-isomorphisms when all of it restrictions do. The same is true for realizations of split Mayer-Vietoris sequences because of the following commuting diagrams

$$\begin{tikzcd}
\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)} \arrow[rr, "\int \mathrm{fgt}"] & & \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \\
& \mathrm{An}_G^\omega \arrow[ul, "\iota_{(W,0,\mathrm{triv})}"] \arrow[ur, "\iota_W"'] &
\end{tikzcd}$$

and finally also for reducedness as one easily checks. $\square$

Theorem 5. *Let $\mathcal{D}$ be a valid coefficient theory. Then the restriction along the functor* Th^- *induces an equivalence of ∞-categories*

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathcal{D}) \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D}).$$

Proof. The proof is essentially the same as in Theorem 3. The restriction along the functor Th^- is a functor

$$\mathrm{Fun}(\mathrm{Sp}_G^\omega, \mathcal{D}) \to \mathrm{Fun}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D}).$$

that by Lemma 4.2.8 becomes the functor

$$\lim_{W \in \mathcal{U}} (\mathrm{Th}^\perp)^* \colon \lim_{W \in \mathcal{U}} \mathrm{Fun}(\mathrm{An}_{G,*}^\omega, \mathcal{D}) \to \lim_{W \in \mathcal{U}} \mathrm{Fun}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D})$$

under the identifications of Lemma 4.2.3 and by Lemma 4.2.7. Under this identifications the full subcategory $\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathcal{D})$ corresponds to the limit

$$\lim_{W \in \mathcal{U}} \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_{G,*}^\omega, \mathcal{D}) \subset \lim_{W \in \mathcal{U}} \mathrm{Fun}(\mathrm{An}_{G,*}^\omega, \mathcal{D})$$

by an identical argument as in the proof of Lemma 3.3.7. By Lemma 4.2.12 the full subcategory $\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathcal{D})$ corresponds to the limit

$$\lim_{W \in \mathcal{U}} \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D}) \subset \lim_{W \in \mathcal{U}} \mathrm{Fun}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D}).$$

and by Proposition 2.4.4 we have an equivalence of limits

$$\lim_{W \in \mathcal{U}} (\mathrm{Th}^\perp)^* \colon \lim_{W \in \mathcal{U}} \mathrm{Exc}_*^{\mathrm{split}}(\mathrm{An}_{G,*}^\omega, \mathcal{D}) \xrightarrow{\simeq} \lim_{W \in \mathcal{U}} \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}(W)}, \mathcal{D}).$$

The result follows. □

4.3 Equivariant Cohomology Theories

Now we want to follow Subsection 3.4 and define equivariant genuine cohomology theories. For this, we want to apply Theorem 5 to the coefficient theory $\mathcal{D} = \mathrm{Ab}^{\mathrm{op}}$. Again, we observe that inside the left-hand side $\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathrm{Ab}^{\mathrm{op}})$ of the equivalence of Theorem 5 sits the opposite of the category $\mathrm{Coh}(\mathrm{Sp}_G^{\omega,\mathrm{op}}, \mathrm{Ab})$ of G-equivariant cohomology theories, that is functors

$$E \colon \mathrm{Sp}_G^{\omega,\mathrm{op}} \to \mathrm{Ab}$$

that send pushouts

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

to weak pullbacks and the zero object to the zero object.

Definition 4.3.1. *A G-*equivariant genuine cohomology theory *(with values in* Ab*) is a functor $E \in \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}$ such that for every pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & D \end{tikzcd}$$

in An_G^ω and every finite-dimensional G-representation W the map

$$E(\iota_W D) \to E(\iota_W B) \times_{E(\iota_W A)} E(\iota_W C)$$

is surjective. We denote the full subcategory of G-equivariant genuine cohomology theories by

$$\mathrm{Gen}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}}) \subset \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}.$$

Proposition 4.3.2. *Pullback along the functor* Th^- *induces an equivalence of categories*

$$\begin{array}{rcl} \{G\text{-equivariant GCTs}\} & \simeq & \{\text{cohomology theories on compact } G\text{-spectra}\}^{1} \\ E & \mapsto & E(\Omega^W \Sigma^\infty X) := \mathrm{fib}(E^W(X) \to E^W(\mathrm{pt})) \\ E^V(X) := [X^{-V}, E] & \leftarrow\!\shortmid & E \end{array}$$

Proof. The proof is essentially the same as in Proposition 3.4.6. That means under the equivalence

$$\mathrm{Exc}_*^{\mathrm{split}}(\mathrm{Sp}_G^\omega, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}^{\mathrm{op}})^{\mathrm{op}}$$

obtained from Theorem 5 the condition for a functor to lie in the full subcategory of cohomology theories on compact G-spectra corresponds to the condition for the associated split genuine homology theory to lie in the full subcategory of genuine cohomology theories. □

Remark 4.3.3. *The ∞-category Sp_G of G-spectra is equivalent to the ∞-category of reduced excisive functors $\mathrm{Exc}_*(\mathrm{Sp}_G^\omega, \mathrm{An}_*^{\mathrm{op}})^{\mathrm{op}}$ as both categories are simply equivalent to $\mathrm{Ind}(\mathrm{Sp}_G^\omega)$. Hence, we can recover the entire ∞-category of G-spectra as a full subcategory of*

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathrm{An}_*^{\mathrm{op}})^{\mathrm{op}}.$$

By a similar argument as in Subsection 3.4 it consists precisely of those functors ϕ that send the images of every pushout square in An_G^ω under the functor ι_W, $W \in \mathrm{Rep}_G$ to a pullback in anima. That is

$$\begin{tikzcd} \phi(D,W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \phi(B,W) \arrow[d] \\ \phi(C,W) \arrow[r] & \phi(A,W) \end{tikzcd}$$

is a pullback in An *for every pushout square in*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in An_G^ω *and every finite-dimensional* G*-representation* W*. Let us call the* ∞*-category of such functors* $\mathrm{Gen}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathrm{An}_*)$*.* Genuine G-Cohomology theories *with values in* An_**. There is a straightforward equivalence between*

$$\mathrm{Gen}(\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}, \mathrm{An}_*)$$

and the ∞*-category of* G*-equivariant orthogonal spectra as follows. Let*

$$\phi\colon \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathrm{An}_*^{\mathrm{op}}$$

be a genuine G*-cohomology theory. Then one can check that the condition that* ϕ *sends the images of every pushout square in* An_G^ω *under the functor* $\iota_W, \quad W \in \mathrm{Rep}_G$ *to a pullback in anima is equivalent to the condition that* ϕ *sends every finite colimit with constant coefficients to a limit in* An_**. This condition is equivalent to the fact that* ϕ *is right Kan extended from its restriction to the full subcategory*

$$\phi|_{\mathrm{Orb}_G}\colon \int_{\mathrm{Orb}_G} \underline{\mathrm{AffLin}^{\mathrm{op}}} \subset \int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}.$$

Consider the precomposition of the $\phi|_{\mathrm{Orb}_G}$ *with the functor*

$$\begin{array}{rcl} \mathrm{Orb}_G^{\mathrm{op}} \times \mathrm{AffLin}_G^{\mathrm{op}} & \to & \int_{\mathrm{Orb}_G} \underline{\mathrm{AffLin}^{\mathrm{op}}} \\ (G/H, V) & \mapsto & (G/H, r_{G/H}^* V) \end{array}$$

It defines a functor

$$\mathrm{Orb}_G^{\mathrm{op}} \times \mathrm{AffLin}_G^{\mathrm{op}} \to \mathrm{An}_*^{\mathrm{op}}$$

or equivalently a functor

$$X\colon \mathrm{AffLin}_G \to \mathrm{An}_{G,*}$$

where $X(V)$ *is the* G*-anima with* H *fixed points given by*

$$X(V)^H = \phi(G/H, r_{G/H}^* V).$$

The condition that ϕ *is right Kan extended from its values on orbits forces*

$$\phi(S^V, U \oplus V) \simeq \mathrm{Map}_{\mathrm{An}_G}(S^V, X(U \oplus V))$$

therefore the fiber

$$\phi(S^V, U \oplus V) \to \phi(G/G, U \oplus V)$$

is equivalent to

$$\Omega^V X(U \oplus V)$$

and the Thom-isomorphism axioms for ϕ *gives us an equivalence*

$$X(U) \xrightarrow{\simeq} \Omega^V X(U \oplus V).$$

Hence X *is a* G*-equivariant orthogonal spectrum.*

4.4 Axiomatization of G-equivariant genuine Cohomology Theories

Definition 4.4.1. *An* axiomatized G-equivariant genuine cohomology theory *is a collection $E^V(X)$ of abelian groups, where X runs trough all compact G-anima and V is an isomorphism class of an G-equivariant vector bundle over X. For each map $f\colon X \to Y$ and a vector bundle W over Y we have induced maps $f^W\colon E^W(Y) \to E^{f^*W}(X)$. For each map $\phi\colon V \to W$ of vector bundles over X we have induced maps $\phi_X\colon E^V(X) \to E^W(X)$. The usual functoriality conditions hold that is*

- *If f, ϕ above are the identity maps then they induce the identity.*
- *We have $(g \circ f)^V = f^{g^*V} \circ g^V$ and $(\phi \circ \psi)_X = \phi_X \circ \psi_X$.*

We require this collection of abelian groups and induced maps to satisfy some properties

(i) (Homotopy Invariance)
If $\phi \simeq \psi$ are homotopic morphisms of vector bundles $V \to W$ over X, then the induced morphisms $\phi_X = \psi_X$ are equal. If $H\colon f \simeq g$ is a homotopy, then it induces an equivalence $\phi_H\colon f^*V \simeq g^*V$ between vector bundles over X. We require that $(\phi_H)_X \circ f^V = g^V$, i.e. the following commutes

$$\begin{tikzcd} & E^{f^*V}(X) \arrow[dd, "(\phi_H)_X"] \\ E^V(X) \arrow[ur, "f^V"] \arrow[dr, "g^V"'] & \\ & E^{g^*V}(X) \end{tikzcd}$$

(ii) (Beck-Chevalley)
Let $f\colon X \to Y$ be a map of G-anima and $\phi\colon V \to W$ be a map of vector bundles over Y. Let $f^*\phi$ be the induced map $f^*V \to f^*W$ of vector bundles over X. Then the following square commutes

$$\begin{tikzcd} E^V(Y) \arrow[r, "f^V"] \arrow[d, "\phi_Y"'] & E^{f^*V}(X) \arrow[d, "(f^*\phi)_X"] \\ E^W(Y) \arrow[r, "f^W"'] & E^{f^*W}(X) \end{tikzcd}$$

(iii) (Reducedness)
Let X be a compact G-anima and let $\emptyset$ be the empty bundle over X. Then $E^{\emptyset}(X) \cong 0$.

(iv) (Mayer-Vietoris)
For each pushout square of G-anima

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_2 \arrow[d, "f_3"] \\ X_3 \arrow[r, "f_4"'] \arrow[ur, phantom, "\overset{H}{\Rightarrow}"] & X_4 \end{tikzcd}$$

and vector bundle V *over* X_4 *the induced sequence*

$$E^V(X_4) \xrightarrow{(f_3^V, f_4^V)} E^{f_3^*V}(X_2) \oplus E^{f_4^*V}(X_3) \xrightarrow{f_1^{f_3^*V} - (\phi_H)_X \circ f_2^{f_4^*V}} E^{f_1^* f_3^* V}(X_1)$$

is exact in the middle.

(v) (The Thom Isomorphism)
Let V, W *be two vector bundles over* X*. Let*

$$E^{V\oplus W}(S_X^V, X) = E^{p^*(V\oplus W)}(S_X^V) \ominus E^{V\oplus W}(X)$$

be the complement of the direct summand inside $E^{p^*(V\oplus W)}(S_X^V)$ *of* $E^{V\oplus W}(X)$ *due to the retraction* $\sigma^{p^*(V\oplus W)} \circ p^{V\oplus W} = \mathrm{id}$*. Then the map*

$$E^W(X) \xrightarrow{p^W} E^{p^*W}(S_X^V) \xrightarrow{(\theta_{(X)}^{(V,W)})_{S_X^V}} E^{p^*(V\oplus W)}(S_X^V) \to E^{V\oplus W}(S_X^V, X)$$

is an isomorphism.

Proposition 4.4.2. *The category of axiomatized* G*-equivariant genuine cohomology theories is equivalent to the category of* G*-equivariant genuine cohomology theories with values in* Ab*.*

Proof. The collection of abelian groups $E^V(X)$ together with the morphisms associated to maps between G-anima and vector bundles so that the axioms (i) and (ii) hold, is the same data as a functor

$$\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}} \to \mathrm{Ab}^{\mathrm{op}}$$

by Remark 3.5.2. The axioms (iii), (iv), and (v) are precisely the axioms for a G-equivariant genuine cohomology theory. □

We recall the definition of an $RO(G)$-graded cohomology theory first introduced in [tD79]. We use the more precise definition suggested in Paragraph XIII. of [May96].

Definition 4.4.3 ($RO(G)$-graded cohomology theory)**.** *A* G*-cohomology theory* E *is a functor* $E\colon ((\mathrm{An}_G)_*^\omega)^{\mathrm{op}} \times \mathrm{AffLin}_G^{\simeq} \to \mathrm{Ab}$ *together with natural equivalences*

$$\sigma_V\colon E(X, W) \to E(S^V \wedge X, V \oplus W)$$

such that $E(-, \mathbb{R}^\bullet_{\mathrm{triv}} \oplus V)$ *is a cohomology theory for every* V *and the following diagrams commute*

$$\begin{tikzcd}
 & E(X,V) \arrow[dl, "\sigma_W"'] \arrow[dr, "\sigma_U"] & \\
E(S^W \wedge X, W \oplus V) \arrow[d, "\sigma_U"] & & E(S^U \wedge X, U \oplus V) \arrow[d, "\sigma_W"'] \\
E(S^U \wedge S^W \wedge X, U \oplus W \oplus V) \arrow[rr, "\simeq"] & & E(S^W \wedge S^U \wedge X, W \oplus U \oplus V)
\end{tikzcd},$$

$$\begin{tikzcd}
E(X;W) \arrow[r, "\sigma_V"] \arrow[d, "\sigma_{U\oplus V}"'] & E(S^V \wedge X; V \oplus W) \arrow[d, "\sigma_U"] \\
E(S^{U\oplus V} \wedge X; U \oplus V \oplus W) \arrow[r, "\simeq"'] & E(S^U \wedge S^V \wedge X; U \oplus V \oplus W).
\end{tikzcd}$$

Moreover, if $\alpha\colon W \to W'$ is a morphism in $\mathrm{AffLin}_G^{\simeq}$, then the following diagram commutes

$$\begin{tikzcd}
E(X,V) \arrow[r, "\sigma^W"] \arrow[d, "\sigma^{W'}"'] & E(\Sigma^W X, V \oplus W) \arrow[d, "E(\Sigma^W X, V\oplus\alpha)"] \\
E(\Sigma^{W'} X, V \oplus W') \arrow[r, "(\Sigma^\alpha \mathrm{id})^*"'] & E(\Sigma^W X, V \oplus W')
\end{tikzcd}$$

By combining the results of Theorem 3.4 (Adams) in Paragraph XIII. of [May96] and Proposition 4.3.2 we obtain a proof that the category of $RO(G)$-graded cohomology theories is equivalent to the category of G-equivariant genuine cohomology theories. By Remark 3.5.2 the category of G-equivariant genuine cohomology theories is equivalent to the category of axiomatized G-equivariant genuine cohomology theories. We summarize this in the following theorem.

Theorem 6. *The categories of axiomatized G-equivariant genuine cohomology theories and $RO(G)$-graded cohomology theories are equivalent.*

We want to dedicate the rest of the paper to giving more direct proof of this fact. Or at least we directly show how a genuine G-cohomology theory gives rise to an $RO(G)$-graded cohomology theory.

Definition 4.4.4. *Let E be a G-equivariant genuine cohomology theory, $Y \to Z$ a map of G-anima and fix $V \in \mathrm{AffLin}^G$, then we define the V-th E-cohomology group of Z relative to Y as*

$$E^V(Z,Y) := E(Z,Y;V) := \ker(E(\iota_V Z \to \iota_V Y)).$$

For a pointed G-anima X we set

$$\tilde{E}^V(X) := \tilde{E}(X;V) := E(X, \mathrm{pt}; V).$$

Lemma 4.4.5. *The diagram*

$$\begin{tikzcd}
E(\mathrm{pt}, W) \arrow[d] & E(X,W) \arrow[l] \arrow[d] \\
E(S^V \vee X, V \oplus W) & E(S^V \times X, V \oplus W) \arrow[l] \\
E(\mathrm{pt}, V \oplus W) \arrow[u] & E(S^V \wedge X, V \oplus W) \arrow[l] \arrow[u]
\end{tikzcd}$$

induces on horizontal kernels a zigzag

$$\tilde{E}(X,W) \to E(S^V \times X, S^V \vee X; V \oplus W) \leftarrow \tilde{E}(S^V \wedge X, V \oplus W),$$

where both morphisms are isomorphisms.

Proof. By the snake lemma the following ladder diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & E(S^V \times X; V \oplus W) \arrow[r] \arrow[d] & E(S^V \times X; V \oplus W) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \tilde{E}(S^V; V \oplus W) \arrow[r] & E(S^V \vee X; V \oplus W) \arrow[r] & E(X; V \oplus W) \arrow[r] & 0
\end{tikzcd}$$

induces a long exact sequence

$$0 \to E(S^V \times X, S^V \vee X; V \oplus W) \to E(X, W) \to \tilde{E}(S^V; V \oplus W) \to \dots$$

The Thom-isomorphism axiom shows that $\tilde{E}(S^V; V \oplus W) \simeq E(\mathrm{pt}, W)$. Hence

$$E(S^V \times X, S^V \vee X; V \oplus W) \simeq \tilde{E}(X; W).$$

A careful examination shows that the left arrow of the span establishes this isomorphism. The right arrow of the span is an isomorphism because of the Mayer-Vietoris axiom. □

Definition 4.4.6. *We define*

$$\sigma_V \colon \tilde{E}^W(X) \to \tilde{E}^{V \oplus W}(S^V \wedge X)$$

to be the resulting isomorphism.

Proposition 4.4.7. *Let* $E\colon (\int_{\mathrm{An}_G^\omega} \underline{\mathrm{AffLin}^{\mathrm{op}}}) \to \mathrm{Ab}^{\mathrm{op}}$ *be a genuine cohomology theory. Then* $\tilde{E}$ *is a* $RO(G)$*-graded cohomology theory.*

Proof. By definition of the Mayer-Vietoris axiom we have that $\tilde{E}(-, \mathbb{R}^\bullet \oplus V)$ is an ordinary cohomomology theory for every V. We will shows that $\sigma_{U \oplus V} \simeq \sigma_U \circ \sigma_V$ holds. Then we are finished since the axiom $\sigma_U \circ \sigma_V \simeq \sigma_V \circ \sigma_U$ then follows from this and the functoriality of σ_- in AffLin^G. The essential part in proving $\sigma_{U \oplus V} \simeq \sigma_U \circ \sigma_V$ comes from the commutativity of the square

$$\begin{tikzcd}
(X; W) & (S^V \times X; V \oplus W) \arrow[l, "{(\mathrm{pr}, \theta_V)}"'] \\
(S^{U \oplus V} \times X; U \oplus V \oplus W) \arrow[u, "{(\mathrm{pr}, \theta_{U \oplus V})}"] & (S^U \times S^V \times X; U \oplus V \oplus W) \arrow[l, "{(q \times X; \mathrm{id})}"] \arrow[u, "{(\mathrm{pr}, \theta_U)}"']
\end{tikzcd}$$

where $q\colon S^U \times S^V \to S^{U \oplus V}$ is the quotient map. The compatibility of the suspension with the morphisms on vector bundles follows from the Beck-Chevalley axiom. □

Outlook

We introduced the notion of a genuine cohomology theory in this paper (Definitions 3.4.5, 4.3.1, 3.5.1 and 4.4.1) and we have seen that it models the arguably interesting cohomology theories in the context of non-equivariant homotopy theory (Proposition 3.4.6 and Theorem 4) and equivariant homotopy theory for a fixed ambient compact Lie group (Proposition 4.3.2 and Theorem 6). A goal of this project was to establish Brown representability theorems in contexts where the theorems of [Lur16] (Theorem 1.4.1.2) and [Nee96] (Theorem 3.1) are not applicable or produce unsatisfactory results. Our approach was to change the axioms of Eilenberg-Steenrod. We did this by enlarging the amount of 'generators and relations' appearing in the axioms. To ensure a built-in Poincare duality for our cohomology theories we needed to keep track of the data of vector bundle twisted cohomology groups and Thom-isomorphism in the definition of genuine cohomology theories. As our formulation of genuine cohomology theories is easy to generalize, one can ask what type of objects they represent in different contexts. For example, let $\underline{\mathrm{Sph}}$ be a category of compact $\mathcal{X}$-spheres for a strongly compact topos $\mathcal{X}$. We introduced the categories of genuine cohomology theories with values in Ab

$$\mathrm{Gen}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab})$$

and with values in An_*

$$\mathrm{Gen}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{An}_*).$$

For a couple of categories of compact spheres over the topoi of An and An_G we have shown that genuine cohomology theories with values in Ab are nothing else than cohomology theories defined on suitable stabilizations of the ∞-categories of compact objects of An and An_G respectively, i.e. we have shown that the categories above are equivalent to the categories of cohomology theories

$$\mathrm{Coh}(\mathrm{Sp}, \mathrm{Ab})$$

and

$$\mathrm{Coh}(\mathrm{Sp}_G, \mathrm{Ab})$$

respectively. We conjecture that genuine cohomology theories are cohomology theories defined on the correct stabilization of the ∞-category of compact objects. We want to make this precise.

Definition 5.0.1. *A* presentable $\mathcal{X}$-category $\underline{\mathcal{Y}}$ *is a limit preserving functor*

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{Pr}^R.$$

A presentably symmetric monoidal $\mathcal{X}$-category *is a limit preserving functor*

$$\mathcal{X}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^R).$$

Let $\underline{\mathrm{Sph}}$ be a category of compact $\mathcal{X}$-spheres. Let $\underline{\mathcal{Y}}$ be the initial symmetric monoidal presentable $\mathcal{X}$-category under $\underline{\mathcal{X}_*}$ in which the image of every object of $\underline{\mathrm{Sph}}$ is invertible, that is for all $X \in \mathcal{X}$ the functor

$$\mathrm{Sph}^X \to \mathcal{X}_{X/-/X} \to \mathcal{Y}^X$$

lies inside the full subcategory of invertible objects of $\mathcal{Y}^X$.

Conjecture 5.0.2. *We have an equivalence of ∞-categories*

$$\mathrm{Gen}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{An}_*) \simeq \mathcal{Y}^{\mathrm{pt}}$$

Conjecture 5.0.3.

$$\mathrm{Gen}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathrm{Ab}) \simeq \mathrm{Coh}((\mathcal{Y}^{\mathrm{pt}})^\omega, \mathrm{Ab})$$

These conjectures follow from a deeper conjecture namely that there exists a universal genuine cohomology and it is given by the following functor

Definition 5.0.4. *Consider the functor*

$$\mathrm{Th}^-\colon \int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}} \to (\mathcal{Y}^{\mathrm{pt}})^\omega$$

that is defined as follows. Let

$$\mathbb{S}^{-\bullet}\colon \underline{\mathrm{Sph}^{\mathrm{op}}} \to \underline{\mathcal{Y}}$$

be the $\mathcal{X}$-functor that sends $V \in \mathrm{Sph}^{X,\mathrm{op}}$ to the inverse of the image of V under the functor

$$\mathrm{Sph}^X \to \mathcal{X}_{X/-/X} \to \mathcal{Y}^X.$$

Let

$$r_!\colon \int \underline{\mathcal{Y}} \to \mathcal{Y}^{\mathrm{pt}}$$

be the left adjoint to the inclusion. Then we define

$$\mathrm{Th}^- := r_! \circ \int \mathbb{S}^{-\bullet}$$

and conjecture that it takes values only in compact objects.

Conjecture 5.0.5. *For a valid coefficient category $\mathcal{D}$ pullback along the functor* Th^- *induces an equivalence of ∞-categories*

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}) \simeq \mathrm{Exc}_*^{\mathrm{split}}((\mathcal{Y}^{\mathrm{pt}})^\omega, \mathcal{D})$$

As it stands, our methods to prove these conjectures in the cases $\mathcal{X} = \mathrm{An}$ and $\mathcal{X} = \mathrm{An}_G$ do not apply in this generality. For once we do not have a model for $\underline{\mathcal{Y}}$ to compare our ∞-categories to. In fact it follows from one of our conjectures that

$$X \mapsto \mathrm{Gen}(\int_{(\mathcal{X}/_X)^\omega} \underline{\mathrm{Sph}^{\mathrm{op},X}}, \mathrm{An}_*)$$

is a model for $\underline{\mathcal{Y}}$. Moreover, in our examples, we did not have to work parametrized. What we mean by this, is that we obtained $\mathcal{Y}^{\mathrm{pt}}$ as the stabilization of the ∞-category $\mathcal{X}^{\mathrm{pt}}$ with respect to the spheres $\mathrm{Sph}^{\mathrm{pt}}$. For the model of this stabilization, we used a filtered colimit

$$(\mathcal{Y}^{\mathrm{pt}})^\omega := (\mathrm{colim}\, \mathcal{X}_*^\omega \xrightarrow{-\wedge S^{V_i}} \mathcal{X}_*^\omega \xrightarrow{-\wedge S^{V_{i+1}}} \mathcal{X}_*^\omega \xrightarrow{-\wedge S^{V_{i+2}}} \cdots)$$

over smashing with spheres S^{V_i} parametrized by the posets of the natural numbers and the subrepresentations of a complete G-universe respectively. We mirrored this process by constructing parametrized categories

$$\underline{\mathrm{Sph}^{\mathrm{op}}(V_i)}$$

and proved Theorem 2, an unstable version of the conjecture above

$$\mathrm{Gen}^{\mathrm{split}}(\int_{\mathcal{X}^\omega} \underline{\mathrm{Sph}^{\mathrm{op}}(V_i)}, \mathcal{D}) \simeq \mathrm{Exc}_*^{\mathrm{split}}(\mathcal{X}_*^\omega, \mathcal{D})$$

as well as a compatible colimit description

$$\operatorname*{colim}_i \underline{\mathrm{Sph}^{\mathrm{op}}(V_i)} \simeq \underline{\mathrm{Sph}^{\mathrm{op}}},$$

see Lemma 3.3.2 and 4.2.3. We want to give some ideas on how to salvage this proof strategy. Let us look at the equivariant case, more specifically suppose we try to stabilize the parametrized category $\underline{\mathrm{An}_{G,*}}$ with respect to representation spheres. As the H-fixed points of $\underline{\mathrm{An}_{G,*}}$ are given by the ∞-category $\mathrm{An}_{H,*}$ of pointed H-anima, we are inclined to think that one should stabilize the H-fixed points of $\underline{\mathrm{An}_{G,*}}$ with respect to H-representation spheres. This is not how we approached the stabilization of An_G in this paper (see Lemma 4.2.3 and 4.2.7). We stabilized the H-fixed points of $\mathrm{An}_{G,*}$ with respect to the restricted G-representation spheres. This is not a problem for a fixed compact Lie group G as every finite-dimensional H-representation embedds into a restricted finite-dimensional G-representation. But we think that there is a more general approach to stabilization. Fix a complete G-universe $\mathcal{U} =: \mathcal{U}^G$ and consider the An_G-category, which we also denote by $\mathcal{U}$, whose H-fixed points is the poset of H-subrepresentations of $\mathrm{res}^G_H \mathcal{U}^G$. Note that $\mathrm{res}^G_H \mathcal{U}^G$ is a complete H-universe and that $\mathcal{U}^H$ also contains H-representations which are not restricted from G-representations. There is a functor

$$\mathcal{F}\colon \int_{\mathrm{An}_G} \mathcal{U} \to \mathcal{C}\mathrm{at}_\infty$$

that sends an H-representation V to the ∞-category of $\mathrm{An}_{H,*}$, and the fiber-wise morphisms

$$V \leq W$$

to the functor

$$S^{V^{\perp W}} \wedge -\colon \mathrm{An}_{H,*} \to \mathrm{An}_{H,*}$$

and a cartesian arrow over $G/K \to G/H$ to the functor

$$\mathrm{res}^H_K \colon \mathrm{An}_{H,*} \to \mathrm{An}_{K,*}\,.$$

We define the *parametrized colimit*

$$\operatorname*{colim}_{\mathcal{U}} \mathcal{F} \colon \mathrm{An}_G^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

to be the reflection to the full subcategory of limit-preserving functors

$$\mathrm{Fun}^{\lim}(\mathrm{An}_G^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \subset \mathrm{Fun}(\mathrm{An}_G^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty)$$

applied to the left Kan extension

$$\begin{tikzcd} \int \mathcal{U} \arrow[r, "\mathcal{F}"] \arrow[d, "p"'] & \mathcal{C}\mathrm{at}_\infty \\ \mathrm{An}_G^{\mathrm{op}}. \arrow[ur, "Lan_p(\mathcal{F})"'] & \end{tikzcd}$$

By Lemma 4.2.7 we have

$$(\operatorname*{colim}_{\mathcal{U}} \mathcal{F}^\omega)^H \simeq \operatorname*{colim}_{\mathcal{U}^H} \mathrm{An}^\omega_{H,*} \simeq \mathrm{Sp}^\omega_H$$

The statement for non-compact objects

$$(\operatorname*{colim}_{\mathcal{U}} \mathcal{F})^H \simeq \operatorname*{colim}_{\mathcal{U}^H} \mathrm{An}_{H,*} \simeq \mathrm{Sp}_H$$

is also true if the parametrized colimit is taken in the category of presentable An_G-categories. We hope that the parametrized colimit is a sufficient tool to apply our methods to the general case. We give a sketch of how to prove the conjectures above in this more general setting. For every pair $X \in \mathcal{X}$ and $W \in \mathrm{Sph}^X$ we constructed the ∞-category

$$\mathrm{Sph}^{X,\mathrm{op}}(W)$$

together with forgetful functors

$$\mathrm{Sph}^{X,\mathrm{op}}(W) \to \mathrm{Sph}^{X,\mathrm{op}}$$

We conjecture that these ∞-categories assemble into a functor

$$\begin{aligned} \underline{\mathrm{Sph}^{\mathrm{op}}(-)} \colon \int_{X \in \mathcal{X}} (\mathrm{Sph}^X)^{\simeq} //^{\mathrm{lax}} (\mathrm{Sph}^X)^{\simeq} &\to \mathcal{C}\mathrm{at}_\infty \\ (X, W) &\mapsto \mathrm{Sph}^{X,\mathrm{op}}(W). \end{aligned}$$

By possibly passing to a suitable (non-full) $\mathcal{X}$-subcategory

$$\mathcal{U} \to \underline{(\mathrm{Sph})^{\simeq} //^{\mathrm{lax}} \mathrm{Sph}^{\simeq}}$$

we hope to prove

Conjecture 5.0.6. *The $\mathcal{X}$-functor*

$$\operatorname*{colim}_{\mathcal{U}} \underline{\mathrm{Sph}^{\mathrm{op}}(-)} \to \underline{\mathrm{Sph}^{\mathrm{op}}}$$

restricts to an equivalence of functors

$$\mathcal{X}^{\omega,\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty.$$

and

Conjecture 5.0.7.
$$\operatorname*{colim}_{\mathcal{U}} \underline{\mathcal{X}^{\omega}_{*}} \simeq \underline{\mathcal{Y}^{\omega}}$$

Assuming these conjectures, we hope one can make the following computation rigorous.

$$\begin{aligned}
\mathrm{Gen}^{\mathrm{split}}\Big(\int_{\mathcal{X}^{\omega}} \underline{\mathrm{Sph}^{\mathrm{op}}}, \mathcal{D}\Big) &\simeq \mathrm{Gen}^{\mathrm{split}}\Big(\int_{\mathcal{X}^{\omega}} \operatorname*{colim}_{\mathcal{U}} \underline{\mathrm{Sph}^{\mathrm{op}}(-)}, \mathcal{D}\Big) \\
&\simeq \left(\lim_{\mathcal{U}} \underline{\mathrm{Gen}^{\mathrm{split}}}\Big(\int \underline{\mathrm{Sph}^{\mathrm{op}}(-)}, \mathcal{D}\Big)\right)^{\mathrm{pt}} \\
&\simeq \left(\lim_{\mathcal{U}} \underline{\mathrm{Exc}^{\mathrm{split}}_{*}}(\underline{\mathcal{X}^{\omega}_{*}}, \mathcal{D})\right)^{\mathrm{pt}} \\
&\simeq \left(\underline{\mathrm{Exc}^{\mathrm{split}}_{*}}(\operatorname*{colim}_{\mathcal{U}} \underline{\mathcal{X}^{\omega}_{*}}, \mathcal{D})\right)^{\mathrm{pt}} \\
&\simeq \mathrm{Exc}^{\mathrm{split}}_{*}(\mathcal{Y}^{\mathrm{pt},\omega}, \mathcal{D})
\end{aligned}$$

Acknowledgements The author would like to thank Prof. Dr. Thomas Nikolaus, for his guidance, support and the general idea of this project, Prof. Dr. Fabian Hebestreit for the many lengthy discussions and the many ideas that flowed into Section 2, Prof. Dr. Achim Krause for clarifying to the author some subtilities in the theory of equivariant homotopy theory, the working group of Prof. Dr. Thomas Nikolaus for the many discussions and the general atmosphere, Sil Linskens, Branko Juran and Andres Mejia for fruitful discussions and my colleagues Florian Riedel, Max Tönies, Dominik Winkler, Dr. Julian Kranz for the interesting discussions during coffee breaks.

Finally, I would like to thank my family and friends for their support and encouragement during a rough time.

The author was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 –390685587, Mathematics Münster: Dynamics–Geometry–Structure

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