Algorithm Design - Homework 2

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Exercise 1

First Problem

We need to model Michele's problem as a integer linear problem, and relax this to a corresponding LP.

Solution

The problem can be modeled as an *ILP* as follow:

$$\begin{cases} minimize & \sum_{i \in F} \sum_{j \in O} x_{ij} w(i,j) \\ s.t. & \sum_{j \in O} x_{ij} = 1, \forall i \in F \\ & \sum_{i \in F} x_{ij} \leq 1, \forall j \in O \\ & x_{ij} \in \{0,1\} \end{cases}$$

For the relaxation of this problem to a LP problem, we need simply to change the last condition in $x_{ij} \in [0,1]$ $(0 \leq x_{ij} \leq 1)$.

Second Problem

Give a polynomial-time algorithm that, from any given optimal LP-solution, computes an optimal integer assignment, knowing that there is no *integrality gap* and for every fractional LP-solution, there exists an integral feasible solution with the same cost.

Solution

Exercise 2

Problem

Solution

Exercise 3

Problem

Given $\alpha \geq 0$ calculate Philips payoff at some mixed Nash equilibrium, for $\alpha < 1$ and $\alpha \geq 1$.

Solution for $\alpha < 1$

Let's define as q_r the probability that the opponent plays R, q_p the probability that the opponent plays P and $q_s = 1 - q_r - q_p$ the probability that the opponent plays S. Now we compute Philip's expected payoffs:

- Philip's expected payoff playing R: $\alpha \cdot q_r + (\alpha - 1) \cdot q_p + (\alpha + 1) \cdot (1 - q_r - q_p)$
- Philip's expected payoff playing P: $1 \cdot q_r + 0 \cdot q_p - 1 \cdot (1 - q_r - q_p) = 2q_r + q_p - 1$
- Philip's expected payoff playing S: $-1 \cdot q_r + 1 \cdot q_p + 0 \cdot (1 - q_r - q_p) = -q_r + q_p$

Since the strategy must be best responses, the payoffs must be equal, and we obtain:

$$\begin{cases} \alpha \cdot q_r + (\alpha - 1) \cdot q_p + (\alpha + 1) \cdot (1 - q_r - q_p) = 2q_r + q_p - 1 \\ \alpha \cdot q_r + (\alpha - 1) \cdot q_p + (\alpha + 1) \cdot (1 - q_r - q_p) = -q_r + q_p \\ 2q_r + q_p - 1 = -q_r + q_p \end{cases} \implies \begin{cases} q_r = \frac{1}{3} \\ q_p = \frac{\alpha+1}{3} \\ q_s = 1 - q_r - q_p = \frac{1-\alpha}{3} \end{cases}$$

Since Philip is mixing in a Nash Equilibrium, the opponent must mix with weights $\left(\frac{1}{3}, \frac{\alpha+1}{3}, \frac{1-\alpha}{3}\right)$.
Now let's do the same for opponent's expected payoffs, defining as q'_r, q'_p and $q'_s = 1 - q'_r - q'_p$ the probability that Philip plays R, P or S respectively:

- Opponent expected payoff playing R: $0 \cdot q'_r - 1 \cdot q'_p + 1 \cdot (1 - q'_r - q'_p) = -2q'_p - q'_r + 1$
- Opponent expected payoff playing S: $1 \cdot q'_r + 0 \cdot q'_p - 1 \cdot (1 - q'_r - q'_p) = 2q'_r + q'_p - 1$
- Opponent expected payoff playing P: $-1 \cdot q'_r + 1 \cdot q'_p + 0 \cdot (1 - q'_r - q'_p) = -q'_r + q'_p$

Since the strategy must be best responses, the payoffs must be equal, and we obtain:

$$\begin{cases} -2q'_p - q'_r + 1 = 2q'_r + q'_p - 1 \\ -2q'_p - q'_r + 1 = -q'_r + q'_p \\ -2q'_r + q'_p - 1 = -q'_r + q'_p \end{cases} \implies \begin{cases} q'_r = \frac{1}{3} \\ q'_p = \frac{1}{3} \\ q'_s = 1 - q_r - q_p = \frac{1}{3} \end{cases}$$

Since the opponent is mixing in a Nash Equilibrium, Philip must mix with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.
So the MNE is: $\left[\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{\alpha+1}{3}, \frac{1-\alpha}{3}\right)\right]$.
Finally we can calculate Philip's payoff as follow:

$$\alpha \cdot q_r \cdot q'_r + (\alpha-1) \cdot q_p \cdot q'_r + (\alpha+1) \cdot q_s \cdot q'_r + 1 \cdot q_r \cdot q'_p + 0 \cdot q_p \cdot q'_p + (-1) \cdot q_s \cdot q'_p + (-1) \cdot q_r \cdot q'_s + 1 \cdot q_p \cdot q'_s + 0 \cdot q_s \cdot q'_s = \frac{\alpha}{3}$$

Solution for $\alpha \geq 1$

In this case, looking at the table it's possible to see that simplifications can be made: for Philip, P is dominated by R because of $\alpha \geq 1$, so we can delete the relative row. After that, for the opponent S is dominated by R, so we can delete the relative column.

Ph/Op	R	P	S
R	α, 0	$(\alpha - 1)$, 1	$(\alpha + 1)$, -1
P	1, -1	0 , 0	-1 , 1
S	-1, 1	1 , -1	0 , 0

Let's define as q_r and $q_p = 1 - q_r$ the probability that the opponent plays R or P respectively, while as q'_r and $q'_s = 1 - q'_r$ the probability that Philip plays R or S respectively.

We have the following payoffs:

	Philip	Opponent
R	$\alpha \cdot q_r + (\alpha - 1) \cdot (1 - q_r) = q_r + \alpha - 1$	$0 \cdot q'_r + 1 \cdot (1 - q'_r) = 1 - q'_r$
P	none	$1 \cdot q'_r - 1 \cdot (1 - q'_r) = 2q'_r - 1$
S	$-1 \cdot q_r + 1 \cdot (1 - q_r) = -2q_r + 1$	none

Since the strategy must be best responses, the payoffs of each player must be equal, and we obtain: $q_r = \frac{2-\alpha}{3}$, $q_p = \frac{\alpha+1}{3}$, $q'_r = \frac{2}{3}$ and $q'_s = \frac{1}{3}$.
So the MNE is: $\left[\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2-\alpha}{3}, \frac{\alpha+1}{3}\right)\right]$.
Finally we can calculate Philip's payoff as follow:
$\alpha \cdot q_r \cdot q'_r + (\alpha - 1) \cdot q_p \cdot q'_r + (-1) \cdot q_r \cdot q'_s + 1 \cdot q_p q'_s = \frac{2\alpha-1}{3}$.

Exercise 4

Problem

Solution

Exercise 5

Problem

Solution