Homework 1 - Algorithm Design

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Exercise 1

Problem

Solution

Let's define a matrix M ($k \times n$), in which we have the expected value opening the i^{th} box having j as current best reward ($i \in [0, k-1]$, $j \in [0, n]$)

Algorithm 1 Populate the matrix

```
for i in (0, ..., k − 1) do
    for j in (0, ..., n) do
        M[i, j] ← − Σ_{h=1}^{i} c_h + j/(n+1) (j − 1) + 1/(n+1) Σ_{h=j}^{n} h
```

$$M[i,j] \leftarrow -\sum_{h=1}^{i} c_h + \frac{j}{n+1}(j-1) + \frac{1}{n+1}\sum_{h=j}^{n} h$$

Now, for each box we can calculate the "*a priori*" expected value as follow:

$$\mathbf{\bar{v_k} = \sum_{i=0}^{n} P_{k-1}(i) v_k(i)}$$

where $\mathbf{P_{k-1}(i)}$ is the probability of having reward i at least in one of the $k-1$ boxes, while $\mathbf{v_k(i)}$ is the expected value opening the k^{th} box having as current best reward i.

$$\mathbf{P_{k-1}(i) = (\tfrac{1}{n+1})(\tfrac{i+1}{n+1})^{k-2}(k-1)}, \qquad \mathbf{v_k(i) = M[k,i]}$$

So we have:

$$\mathbf{\bar{v_k} = \sum_{i=0}^{n} (\tfrac{1}{n+1})(\tfrac{i+1}{n+1})^{k-2}(k-1) M[k,i] = \tfrac{k-1}{n+1} \sum_{i=0}^{n} (\tfrac{i+1}{n+1})^{k-2} M[k,i]}$$

Now we have to calculate $\mathbf{\bar{v_k}}$ for each box and return the max:

Algorithm 2 Find expected optimal reward

```
k_max ← 0
v[k] ← ∅
for i in (0, ..., k − 1) do
    for j in (0, ..., n) do
        v[i] += (k−1)/(n+1) ((i+1)/(n+1))^(k−2) M[k, i]
    if v[i] > v[k_max] then
        k_max ← i
return v[k_max]
```

Exercise 2

First problem

Given a weighted tree T with n nodes, find the complete graph G of minimum weight such that T $\subseteq$ G and T is the unique minimum spanning tree of G.

Solution

Idea: Insert edges that are not in the tree so as to obtain the complete graph. These edges must have a greater weight than those of the tree, so that T is the only MST of G.

Algorithm 3 Find complete graph

```
for u in V do
    for v in V do
        e ← (u, v)
        if e not in E then
            weight ← max(u.getHeaviestEdge().getWeight(), v.getHeaviestEdge().getWeight())
            e.setWeight(weight + 1)
            G.addEdge(e)
```

Cost: $\mathbf{O}(|E|)$, but $|E| = \frac{|V|(|V|-1)}{2}$ and $|V| = n$, so $|E| = \frac{n^2-n}{2}$ and the cost is $\mathbf{O}(n^2)$

Second problem

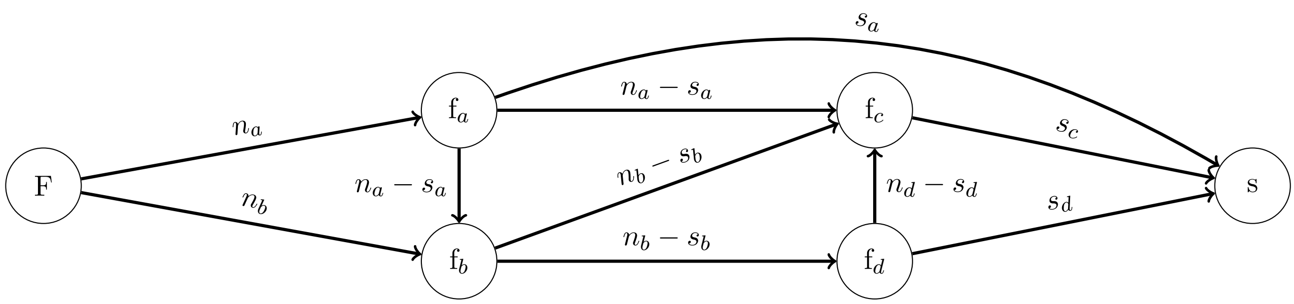
Solution

Hint: Since *Kruskal*'s algorithm uses the *Union-Find* structures for representing the cuts, we use this structures to solve the problem.

Exercise 3

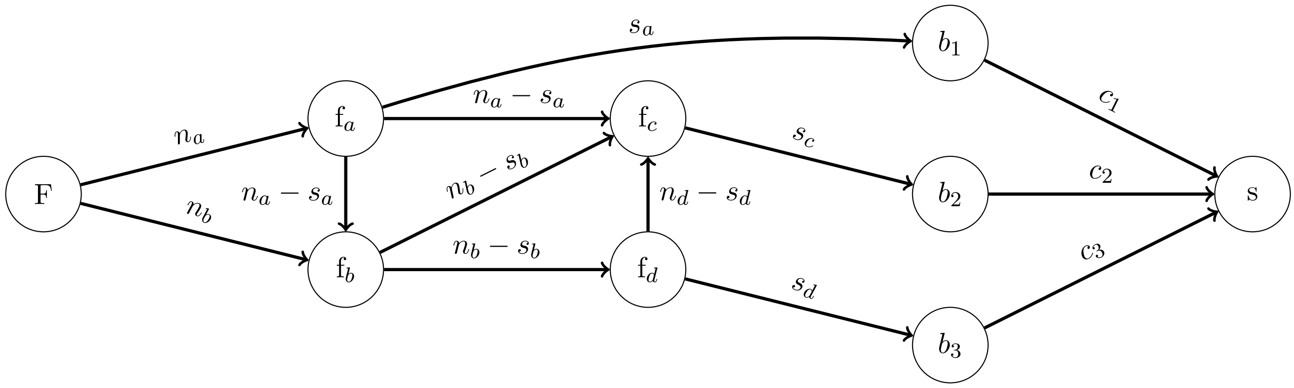
First Problem

Solution



Second Problem

Solution



Exercise 4

Problem

Solution