Homework 1 - Algorithm Design

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Exercise 1

Problem

Given as input the number of boxes k, the number of different rewards n as well as the cost c_i (which is guaranteed to be integer) of every box, design an algorithm to find the expected optimal reward.

Solution

Let's define a matrix M $(k \times n)$, in which we have the expected value opening the i^{th} box having j as current best reward ($i \in [0, k-1]$, $j \in [0, n]$)

Algorithm 1 Populate the matrix

1: **for** i in $(0, ..., k-1)$ **do**
2: **for** j in $(0, ..., n)$ **do**
3: $M[i,j] \leftarrow -\sum_{h=1}^{i} c[h] + \frac{j}{n+1}(j-1) + \frac{1}{n+1}\sum_{h=j}^{n} h$

Now, for each box we can calculate the expected value as follow:

$$\bar{\mathbf{v_k}} = \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{P_{k-1}(i)v_k(i)}$$

where $\mathbf{P_{k-1}(i)}$ is the probability of having reward i at least in one of the $k-1$ boxes, while $\mathbf{v_k(i)}$ is the expected value opening the k^{th} box having as current best reward i.

$$\mathbf{P_{k-1}(i)} = (\tfrac{\mathbf{1}}{\mathbf{n+1}})(\tfrac{\mathbf{i+1}}{\mathbf{n+1}})^{\mathbf{k-2}}(\mathbf{k-1}), \qquad \mathbf{v_k(i)} = \mathbf{M[k,i]}$$

So we have:

$$\bar{\mathbf{v_k}} = \sum_{\mathbf{i=0}}^{\mathbf{n}} (\tfrac{\mathbf{1}}{\mathbf{n+1}})(\tfrac{\mathbf{i+1}}{\mathbf{n+1}})^{\mathbf{k-2}}(\mathbf{k-1})\mathbf{M[k,i]} = \tfrac{\mathbf{k-1}}{\mathbf{n+1}} \sum_{\mathbf{i=0}}^{\mathbf{n}} (\tfrac{\mathbf{i+1}}{\mathbf{n+1}})^{\mathbf{k-2}}\mathbf{M[k,i]}$$

Now we have to calculate $\bar{\mathbf{v_k}}$ for each box and return the max:

Algorithm 2 Find expected optimal reward

1: $k_{max} \leftarrow 0$
2: $v[k] \leftarrow \emptyset$
3: **for** i in $(0, ..., k-1)$ **do**
4: **for** j in $(0, ..., n)$ **do**
5: $v[i] \mathrel{+}= \frac{k-1}{n+1}(\frac{i+1}{n+1})^{k-2} M[i,j]$
6: **if** $v[i] > v[k_{max}]$ **then**
7: $k_{max} \leftarrow i$
8: return $v[k_{max}]$

Exercise 2

First problem

Given a weighted tree T with n nodes, find the complete graph G of minimum weight such that T $\subseteq$ G and T is the unique minimum spanning tree of G.

Solution

Idea: Insert edges that are not in the tree so as to obtain the complete graph. These edges must have a greater weight than those of the tree, so that T is the only MST of G.

Algorithm 3 Find complete graph

```
1: for u in V do
2:     for v in V do
3:         e ← (u, v)
4:         if e not in E then
5:             weight ← max(u.getHeaviestEdge().getWeight(), v.getHeaviestEdge().getWeight())
6:             e.setWeight(weight + 1)
7:             G.addEdge(e)
```

Cost: $\mathbf{O}(|E|)$, but $|E| = \frac{|V|(|V|-1)}{2}$ and $|V| = n$, so $|E| = \frac{n^2-n}{2}$ and the cost is $\mathbf{O}(n^2)$

Second problem

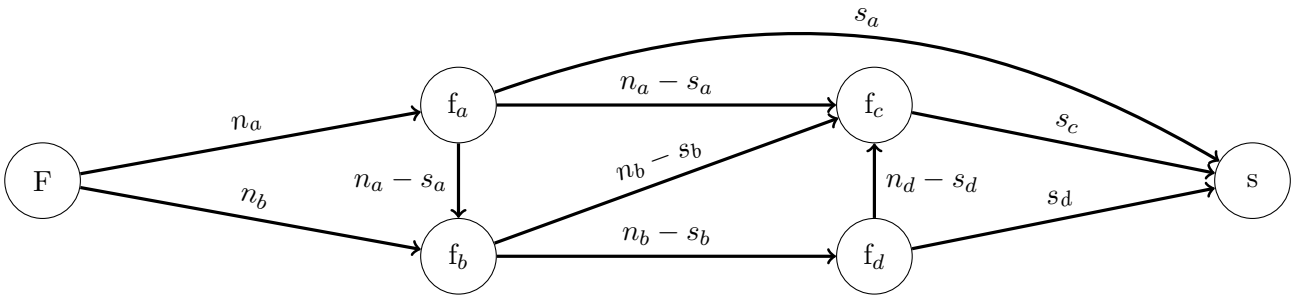
Solution

Hint: Since *Kruskal*'s algorithm uses the *Union-Find* structures for representing the cuts, we use this structures to solve the problem.

Exercise 3

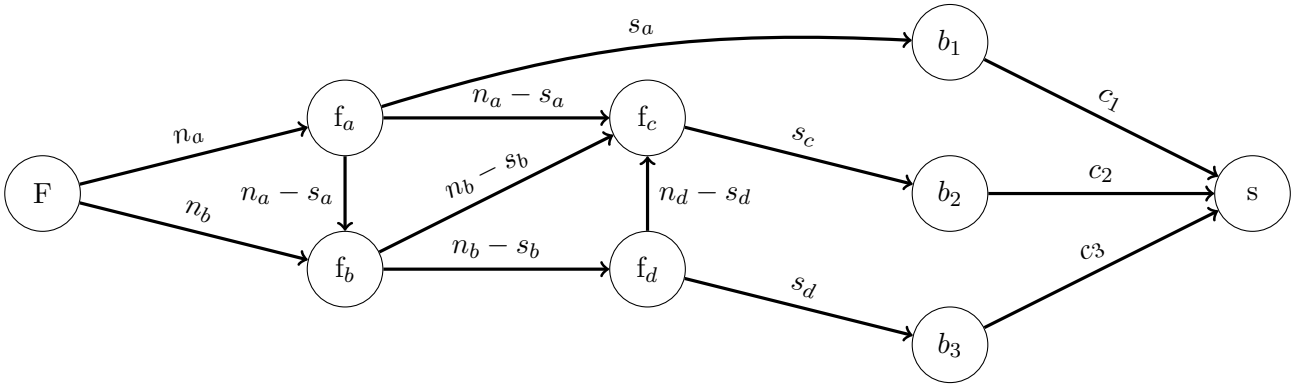
First Problem

Solution



Second Problem

Solution



Exercise 4

Problem

Solution