Instituto de Computação
UNIVERSIDADE ESTADUAL DE CAMPINAS

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Realizing Graphs with Cut Constraints[a]

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Classic Problem

Old Graph Realization Problem

GRAPH REALIZATION (GR)

Input: A non-decreasing sequence $\mathtt{d} = (d_1, \ldots, d_n)$ of natural numbers.

Question: Is $\mathtt{d}$ a graphic sequence?

Example

Is $\mathtt{d} = (3, 2, 2, 2, 1)$ graphic?

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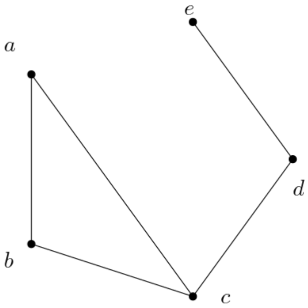

Graphic Sequences Characterization

Theorem (Erdős and Gallai [EG60])

A non-decreasing sequence $\mathbb{d} = (d_1, \ldots, d_n)$ *of natural numbers is graphic if and only if*

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A non-decreasing sequence $d = (d_1, \ldots, d_n)$ *of natural numbers is graphic if and only if*

1. $\sum_{i=1}^{n} d_i$ *is even, and*
2. *For every* $1 \leq k \leq n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}$$

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- Erdős–Gallai gives a simple poly-time criterion;
- Many variations have been considered;
- Vertex degree = size of trivial edge cut;
- We generalize this idea by adding nontrivial constraints.

Nontrivial Cut Constraints

Preliminaries

For a fixed graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$:

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- *G realizes* $\mathcal{L}$ if $|\partial(S_j)| = \ell_j$ for every $(S_j, \ell_j) \in \mathcal{L}$;
- By $w(\mathcal{L})$ we denote $\max_j |S_j|$.

New Graph Realization Problem

Graph Realization with Cut Constraints (GR-C)

Input: A cut list $\mathcal{L}$ for a set of vertices $V = \{v_1, \ldots, v_n\}$, and a non-decreasing sequence $\mathtt{d} = (d_1, \ldots, d_n)$ of natural numbers.

Question: Does there exist a (labeled) simple graph $G = (V, E)$ such that, for every j, $d(v_j) = d_j$ and G realizes $\mathcal{L}$?

Example

Consider $\mathtt{d} = (d_1 = 3, d_2 = 2, d_3 = 2, d_4 = 2, d_5 = 1)$ and
$\mathcal{L} = \{(\{v_2, v_3\}, 4), (\{v_1, v_2, v_5\}, 2)\}$:

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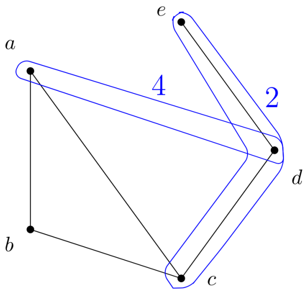

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- If $\mathcal{L} = \emptyset$, we have GR;
- Assume that $2 \leq |S_j| \leq n - 2$ for every j;
- If $\binom{V}{2} \subseteq \{S_j \mid (S_j, \ell_j) \in \mathcal{L}\}$ then the problem becomes trivial;
- GR-C can be seen as a consistency check for cut-queries.

Important Remark

For $S \subseteq V$ let $d(S) = \sum_{v_j \in S} d_j$.

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Remark (1)

An instance $(\mathbb{d}, \mathcal{L})$ *is true only if, for each* $(S, \ell) \in \mathcal{L}$,
$\ell \in \{d(S) - 2k \mid 0 \leq k \leq \binom{|S|}{2}\}$.

Nontrivial Cut Constraints

Small Cuts

Fixed Edges

If $(\{u, v\}, d_u + d_v - 2) \in \mathcal{L}$ then $uv \in E(G)$.

Forbidden Edges

If $(\{u, v\}, d_u + d_v) \in \mathcal{L}$ then $uv \notin E(G)$.

A Single Case

Replace $(\{u, v\}, d_u + d_v - 2)$ by $(\{u, v\}, d_u + d_v)$ while updating `d`.

Possibility Graph

Let F be the set of forbidden edges.

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Then we call $\mathcal{G} = K_n - F$ the *possibility graph*.

Size 2 Cut Constraints

We can reduce GR-C to f-factor!

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Lemma (1)

An instance $(d, \mathcal{L})$ of GR-C can be solved in polynomial time whenever $w(\mathcal{L}) = 2$.

Size 3 Cut Constraints

We can reduce to the previous case!

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Theorem (2)

An instance $(\text{đ}, \mathcal{L})$ *of GR-C can be solved in polynomial time whenever* $w(\mathcal{L}) = 3$.

Proof Sketch

Consider a cut $(S, \ell) \in \mathcal{L}$ where $S = \{u, v, w\}$.

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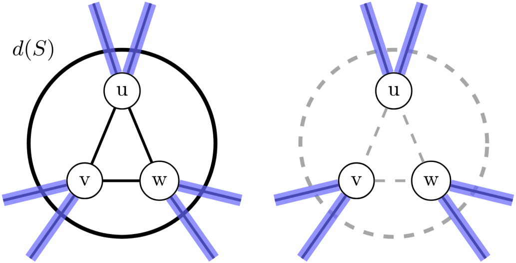
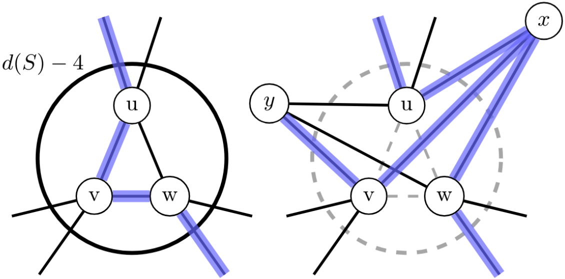
Consider a cut $(S, \ell) \in \mathcal{L}$ where $S = \{u, v, w\}$.

As an example, let $d_u = d_v = d_w = 2$ (so $d(S) = 6$).

Case $\ell = d(S) = 6$



Case $\ell = d(S) - 4 = 2$

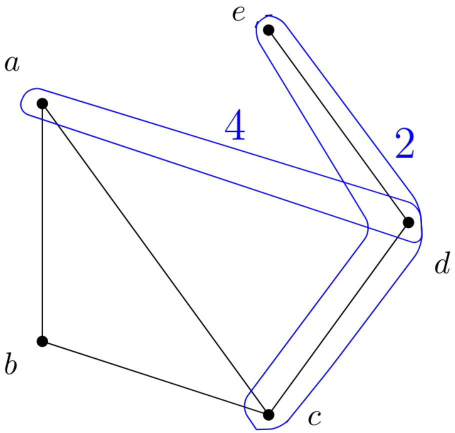


Other Cases

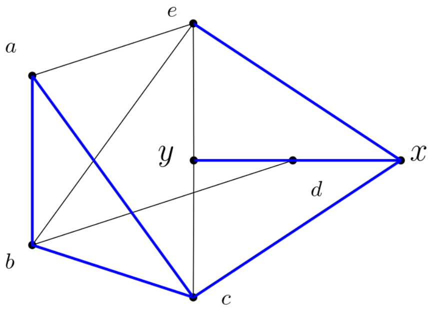
$$\begin{aligned} \ell &= d(S) - 2 = 4 \\ \text{and } \ell &= d(S) - 6 = 0 \end{aligned}$$

Running Example

$$\mathtt{d} = (d_1 = 3, d_2 = 2, d_3 = 2, d_4 = 2, d_5 = 1) \text{ and}$$
$$\mathcal{L} = \{(\{v_2, v_3\}, 4), (\{v_1, v_2, v_5\}, 2)\}:$$



Equivalent f-factor instance:



Nontrivial Cut Constraints

Large Cuts

Size 4 Cut Constraints

Can we keep doing a case-by-case analysis?

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No, we cannot, and the GR-C becomes hard!

Intuition

For $S \in \binom{V}{3}$, $E[S]$ determines how degrees change.

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In contrast, for $S \in \binom{V}{4}$, this claim no longer holds.

Hardness

Theorem (3)

The GR-C problem cannot be solved in polynomial time unless $P = NP$ even when $w(\mathcal{L}) = 4$ and all degrees in the degree sequence $\mathtt{d}$ are 1.

Proof

Reduction from k-True 1-in-3-SAT$_{(2,1)}$

1-IN-3-SAT$_{(2,1)}$

Input: A set of variables X and a formula ϕ in conjunctive normal form over X such that:

- each variable of X occurs twice as a positive literal and once as a negative literal;
- each clause of ϕ has two or three literals.

Question: Is there a truth assignment of X such that exactly one literal in every clause of ϕ is true?

Lemma (2)

1-in-3-SAT$_{(2,1)}$ is NP-complete.

k-True 1-in-3-SAT$_{(2,1)}$

k-TRUE 1-IN-3-SAT$_{(2,1)}$

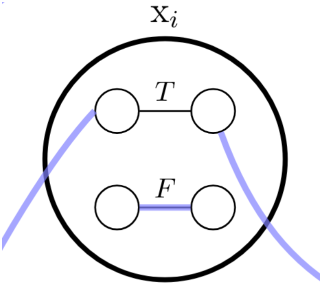
Input: A tuple (X, ϕ, k), where (X, ϕ) is an instance of 1-in-3-SAT$_{(2,1)}$ and k is a nonnegative integer.

Question: Is there a feasible solution to (X, ϕ) in which exactly k variables are assigned to true?

Lemma (3)

k-True 1-in-3-SAT$_{(2,1)}$ cannot be solved in polynomial time unless $P = NP$.

Variable Gadget



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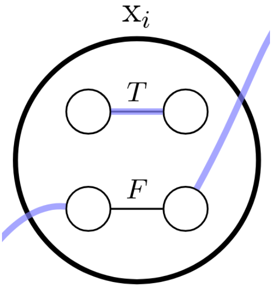
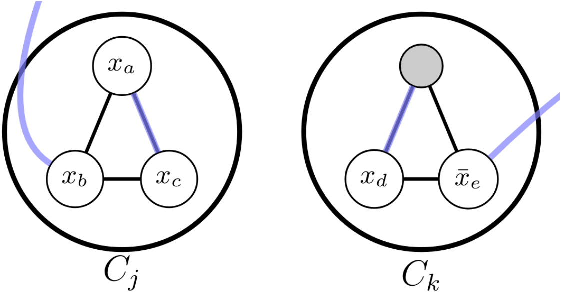


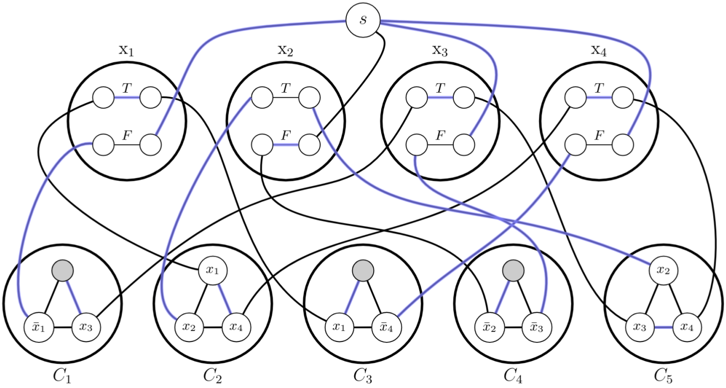
Clause Gadget

$$C_j = (x_a + x_b + x_c) \text{ and } C_k = (x_d + \bar{x}_e)$$



Complete Example

$(\bar{x}_1 + x_3)(x_1 + x_2 + x_4)(x_1 + \bar{x}_4)(\bar{x}_2 + \bar{x}_3)(x_2 + x_3 + x_4)$ and $k = 1$:



Conclusion

Tree Possibility Graph

Proposition (1)

Given an instance $(d, \mathcal{L})$ *of GR-C with a tree possibility graph* $\mathcal{G}$*, we can decide if there is a solution in polynomial time.*

Bipartite Possibility Graph

Theorem (4)

The GR-C problem is NP-complete when the possibility graph $\mathcal{G}$ is subcubic and bipartite, even when $w(\mathcal{L}) = 6$ and $\mathbb{d}$ is a sequence of ones.

$\mathcal{G}$ is planar or has bounded treewidth

Open Problems

The size of $\mathcal{L}$ is small $(|\mathcal{L}| = 1?)$

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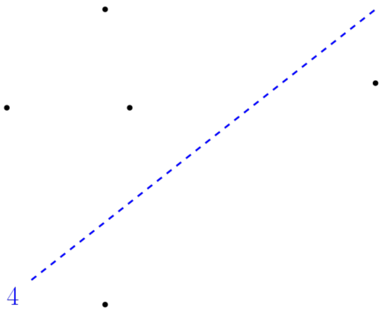
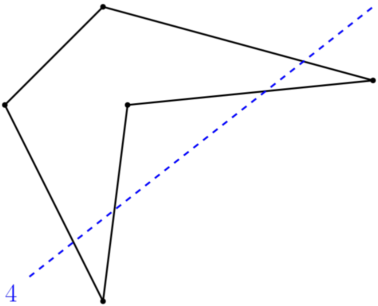
Complexity of 1-in-3 $\mathrm{SAT}_{(2,2)}$

Geometric version of GR-C



Polygon Realization with Cut Constraints



Thank you all for the attention...

The End

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