Pricing Options with Mathematical Models

1. OVERVIEW

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

- What we want to accomplish:

Learn the basics of option pricing so you can:

- (i) continue learning on your own, or in more advanced courses;

- (ii) prepare for graduate studies on this topic, or for work in industry, or your own business.

- The prerequisites we need to know:

- (i) Calculus based probability and statistics, for example computing probabilities and expected values related to normal distribution.
- (ii) Basic knowledge of differential equations, for example solving a linear ordinary differential equation.
- (iii) Basic programming or intermediate knowledge of Excel

- A rough outline:

- Basic securities: stocks, bonds
- Derivative securities, options
- Deterministic world: pricing fixed cash flows, spot interest rates, forward rates

- A rough outline (continued):

- Stochastic world, pricing options:

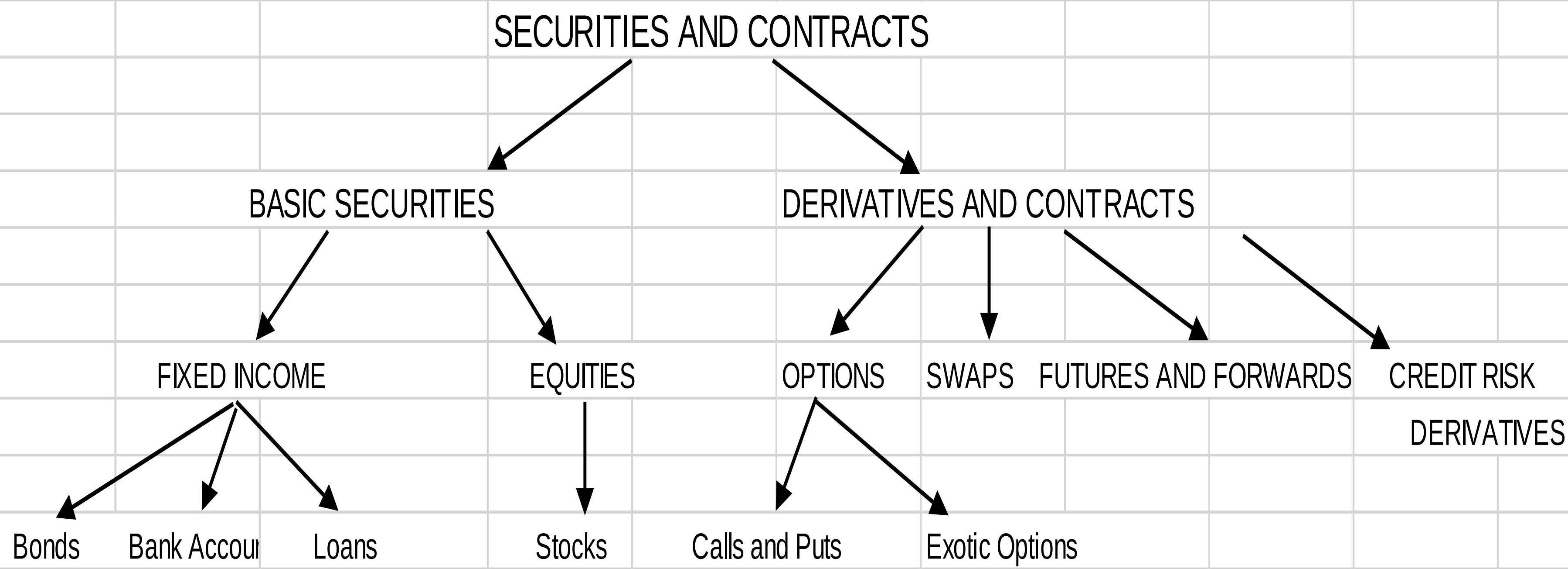
  - Pricing by no-arbitrage
  - Binomial trees
  - Stochastic Calculus, Ito’s rule, Brownian motion
  - Black-Scholes formula and variations
  - Hedging
  - Fixed income derivatives

Pricing Options with Mathematical Models

2. Stocks, Bonds, Forwards

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

A Classification of Financial Instruments



Stocks

- Issued by firms to finance operations
- Represent ownership of the firm
- Price known today, but not in the future
- May or may not pay dividends

Bonds

- Price known today
- Future payoffs known at fixed dates
- Otherwise, the price movement is random
- Final payoff at maturity: face value/nominal value/principal
- Intermediate payoffs: coupons
- Exposed to default/credit risk

Derivatives

- Sell for a **price/value/premium** today.
- Future value **derived** from the value of the underlying securities (as a function of those).
- Traded at exchanges – standardized contracts, no credit risk;
- or, over-the-counter (OTC) – a network of dealers and institutions, can be non-standard, some credit risk.

Why derivatives?

- To hedge risk
- To speculate
- To attain “arbitrage” profit
- To exchange one type of payoff for another
- To circumvent regulations

Forward Contract

- An agreement to buy **(long)** or sell (**short**) a given **underlying** asset S:
  - At a predetermined future date T (**maturity**).
  - At a predetermined price F (**forward price**).
- F is chosen so that the contract has zero value today.
- Delivery takes place at maturity T:
  - Payoff at maturity: S(T) - F or F - S(T)
  - Price F set when the contract is established.
  - S(T) = **spot (market) price** at maturity.

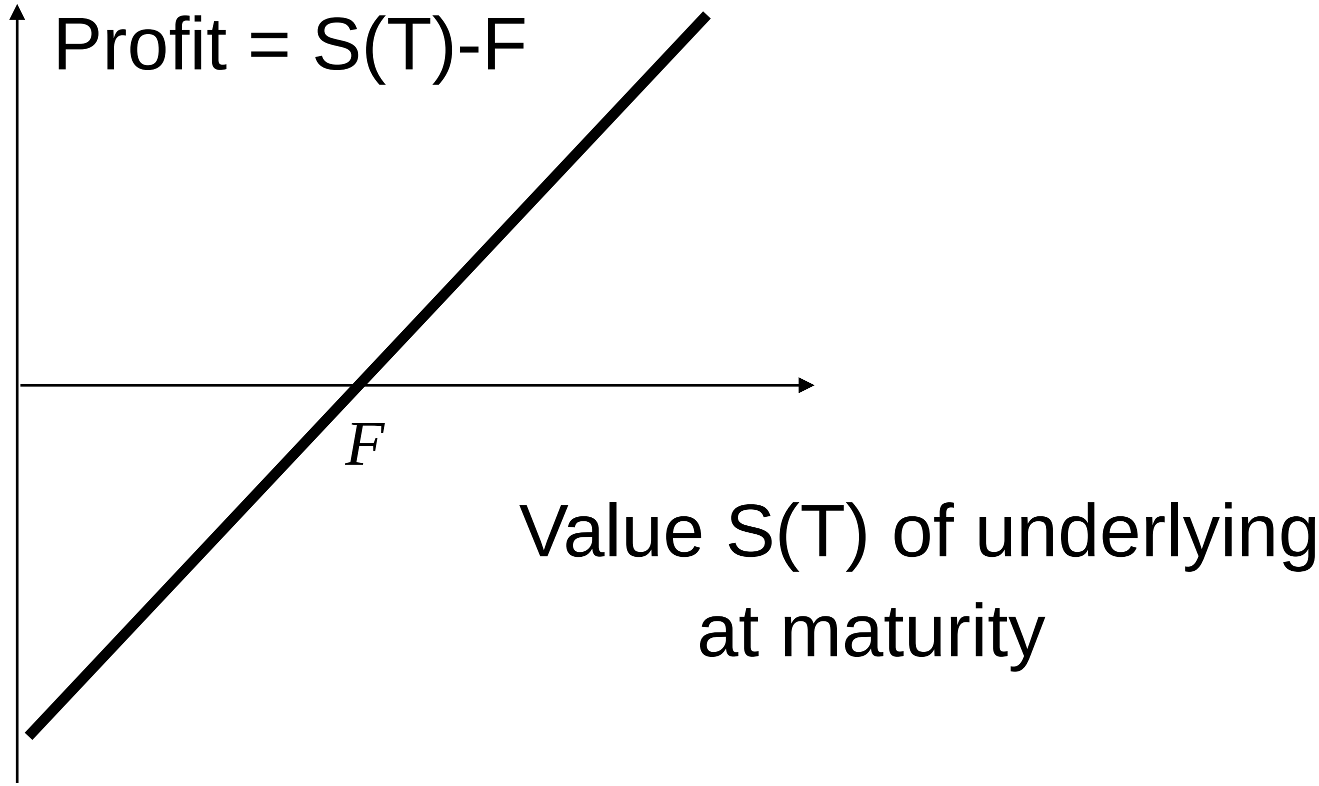
Forward Contract (continued)

- Long position: obligation to buy
- Short position: obligation to sell
- Differences with options:
  - Delivery has to take place.
  - Zero value today.

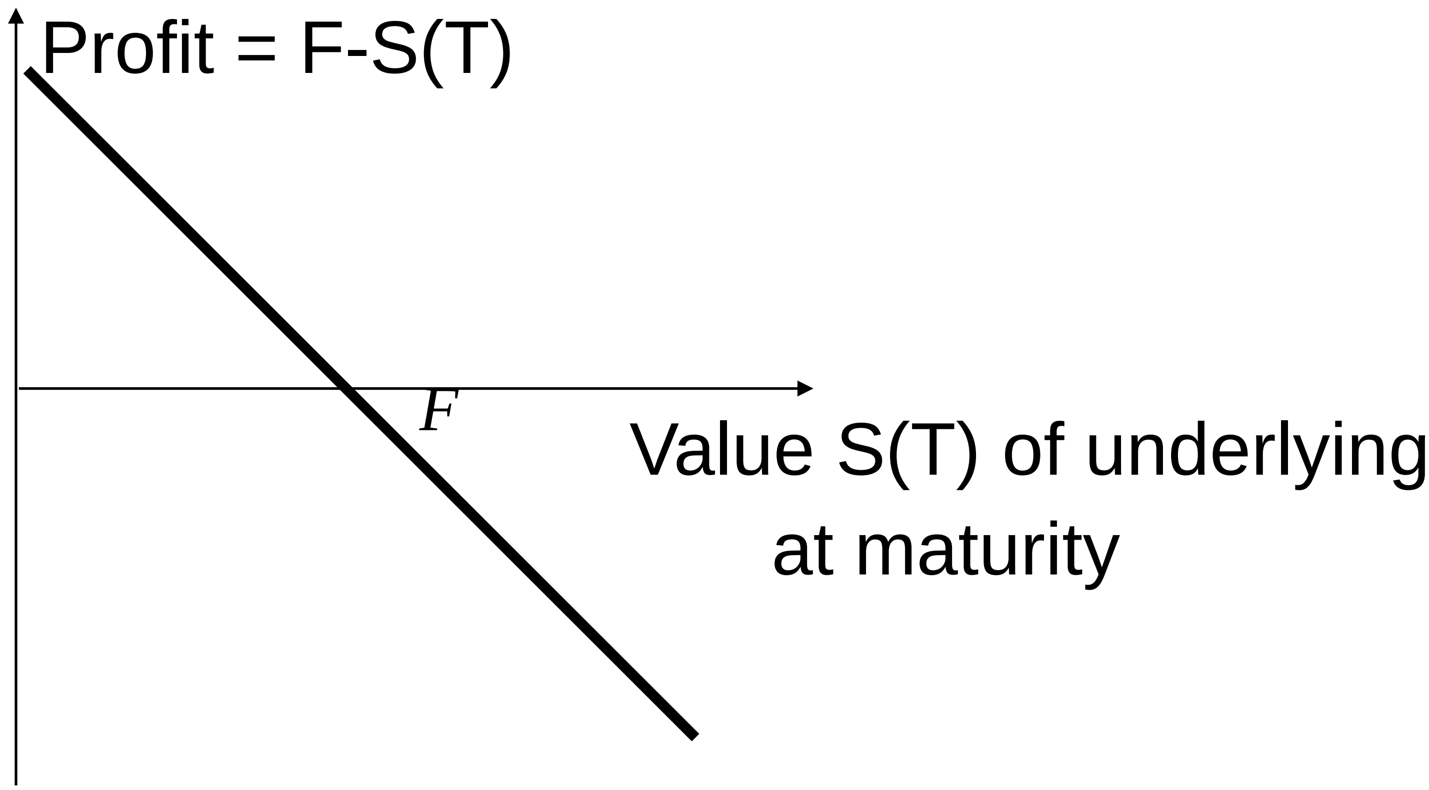
Example

- On May 13, a firm enters into a long forward contract to buy one million euros in six months at an exchange rate of 1.3
- On November 13, the firm pays F=$1,300,000 and receives S(T)= one million euros.
- How does the payoff look like at time T as a function of the dollar value of S(T) spot exchange rate?

Profit from a long forward position



Profit from a short forward position



Pricing Options with Mathematical Models

3. Swaps

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Swaps

- Agreement between two parties to exchange two series of payments.
- Classic interest rate swap:
  - One party pays **fixed** interest rate payments on a notional amount.
  - Counterparty pays **floating** (random) interest rate payments on the same notional amount.
- Floating rate is often linked to LIBOR (London Interbank Offer Rate), reset at every payment date.

Motivation

- The two parties may be exposed to different interest rates in different markets, or to different institutional restrictions, or to different regulations.

A Swap Example

- New pension regulations require higher investment in fixed income securities by pension funds, creating a problem: liabilities are long-term while new holdings of fixed income securities may be short-term.
- Instead of selling assets such as stocks, a pension fund can enter a swap, exchanging returns from stocks for fixed income returns.
- Or, if it wants to have an option not to exchange, it can buy **swaptions** instead.

Swap Comparative Advantage

- US firm B wants to borrow AUD, Australian firm A wants to borrow USD
- Firm B can borrow at 5% in USD, 12.6% AUD
- Firm A can borrow at 7% USD, 13% AUD
- Expected gain = (7-5) – (13-12.6) = 1.6%
- Swap:

	← USD5%		←USD6.3%		
← Firm B		BANK		Firm A→	
5%	→AUD11.9%		→AUD13%		13%

- Bank gains 1.3% on USD, loses 1.1% on AUD, gain=0.2%
- Firm B gains (12.6-11.9) = 0.7%
- Firm A gains (7-6.3) = 0.7%
- Part of the reason for the gain is credit risk involved

A Swap Example: Diversifying

- Charitable foundation CF receives 50mil in stock X from a privately owned firm.
- CF does not want to sell the stock, to keep the firm owners happy
- Equity swap: pays returns on 50mil in stock X, receives return on 50mil worth of S&P500 index.
- A bad scenario: S&P goes down, X goes up; a potential cash flow problem.

Swap Example: Diversifying II

- An executive receives 500mil of stock of her company as compensation.
- She is not allowed to sell.
- Swap (if allowed): pays returns on a certain amount of the stock, receives returns on a certain amounts of a stock index.
- Potential problems: less favorable tax treatment; shareholders might not like it.

Pricing Options with Mathematical Models

4. Call and Put Options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Vanilla Options

- **Call** option: a right to buy the underlying
- **Put** option: a right to sell the underlying
- **European** option: the right can be **exercised** only at **maturity**
- **American** option: can be exercised at any time before maturity

Various underlying variables

- Stock options
- Index options
- Futures options
- Foreign currency options
- Interest rate options
- Credit risk derivatives
- Energy derivatives
- Mortgage based securities
- Natural events derivatives …

Exotic options

- **Asian options**: the payoff depends on the average underlying asset price
- **Lookback options**: the payoff depends on the maximum or minimum of the underlying asset price
- **Barrier options**: the payoff depends on whether the underlying crossed a barrier or not
- **Basket options**: the payoff depends on the value of several underlying assets.

Terminology

- ***Writing*** *an option*: selling the option
- ***Premium***: price or value of an option
- Option **in/at/out of the money**:
  - *At*: strike price equal to underlying price
  - *In*: immediate exercise would be profitable
  - -*Out*: immediate exercise would not be profitable

Long Call

Outcome at maturity

	$S(\mathrm{T}) \leq K$	$S(\mathrm{T}) > K$
Payoff:	0	$S(\mathrm{T}) - K$
Profit:	$-C(\mathrm{t}, K, \mathrm{T})$	$S(\mathrm{T}) - K - C(\mathrm{t}, K, \mathrm{T})$

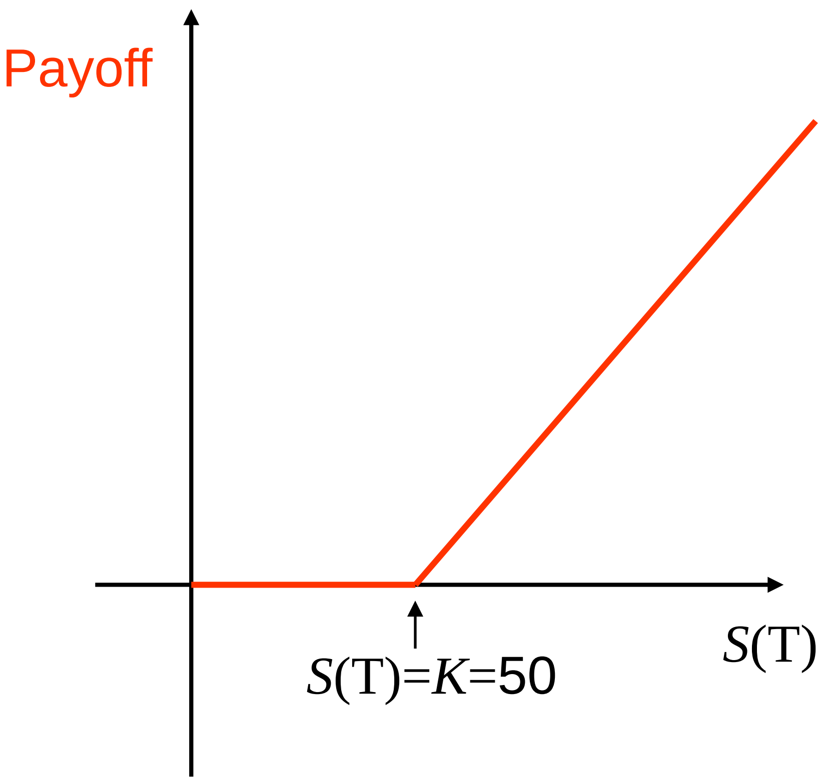
A more compact notation:

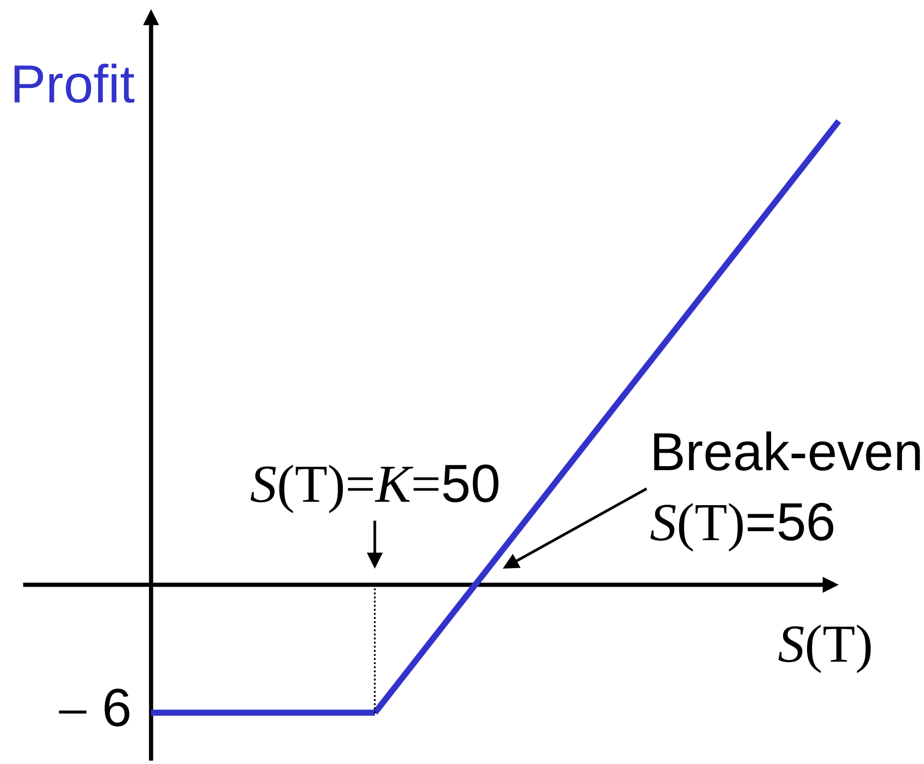
Payoff: $\mathit{max}\,[\mathrm{S(T)} - K,\, 0] = (\mathrm{S(T)\text{-}K})_{+}$

Profit: $\mathit{max}\,[\mathrm{S(T)} - K,\, 0] - C(\mathrm{t},K,\mathrm{T})$

Long Call Position

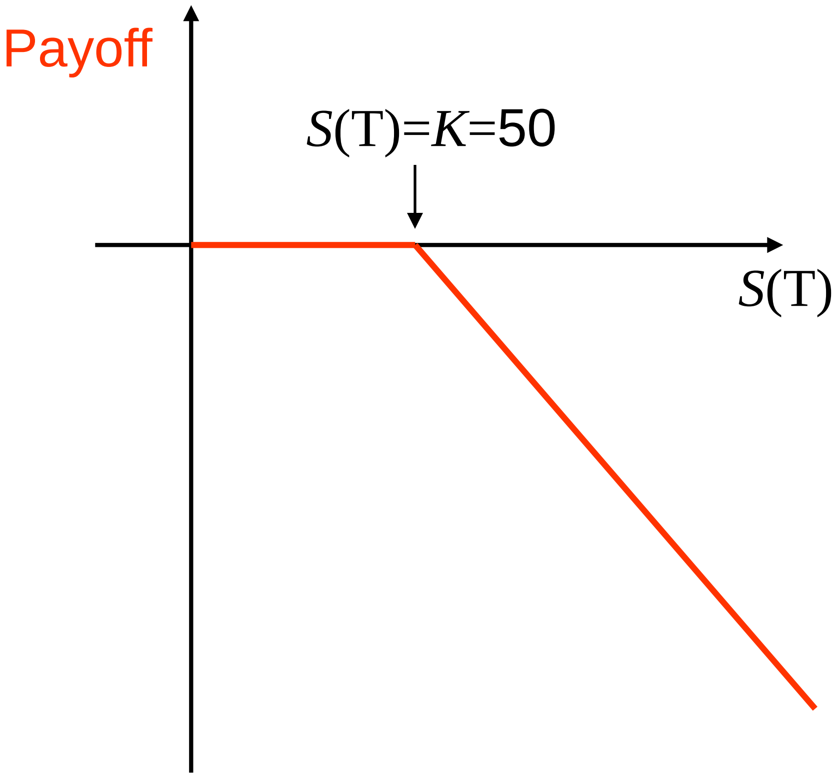
- Assume K = \$50, C(t,K,T) = \$6
- Payoff: $\max\,[S(T) - 50, 0]$
- Profit: $\max\,[S(T) - 50, 0] - 6$

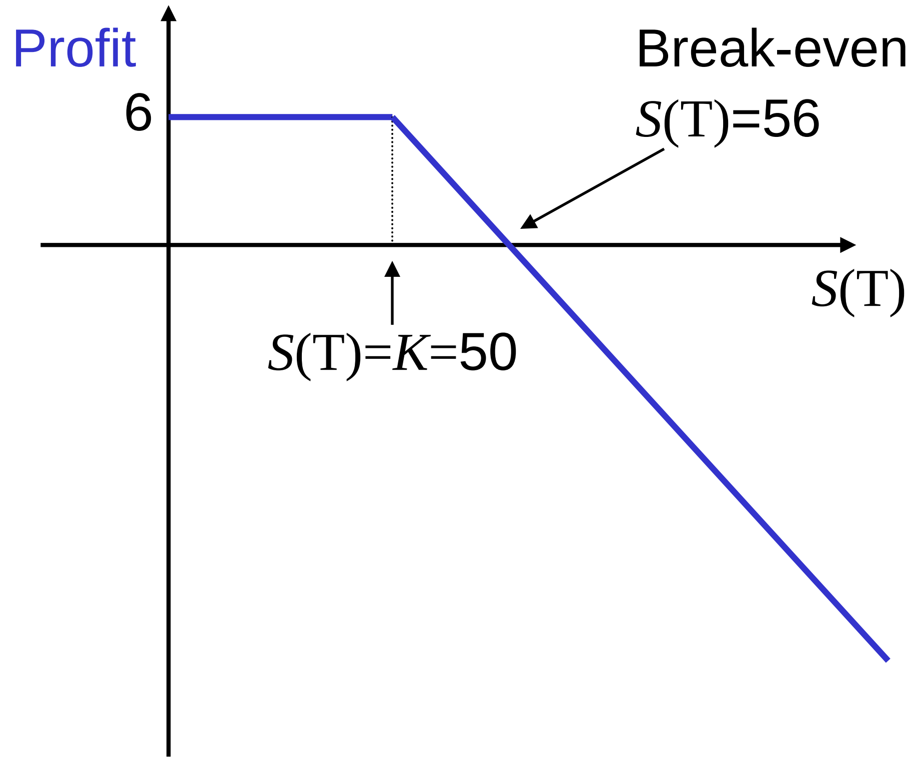




Short Call Position

- K = \$50, C(t,K,T) = \$6
- Payoff: $-\,max\,[S(T) - 50, 0]$
- Profit: $6 - max\,[S(T) - 50, 0]$





Long Put

	Outcome at maturity	
	$S(\mathrm{T}) \leq K$	$S(\mathrm{T}) > K$
Payoff:	$K - S(\mathrm{T})$	0
Profit:	$K - S(\mathrm{T}) - P(\mathrm{t}, K, \mathrm{T})$	$-P(\mathrm{t}, K, \mathrm{T})$

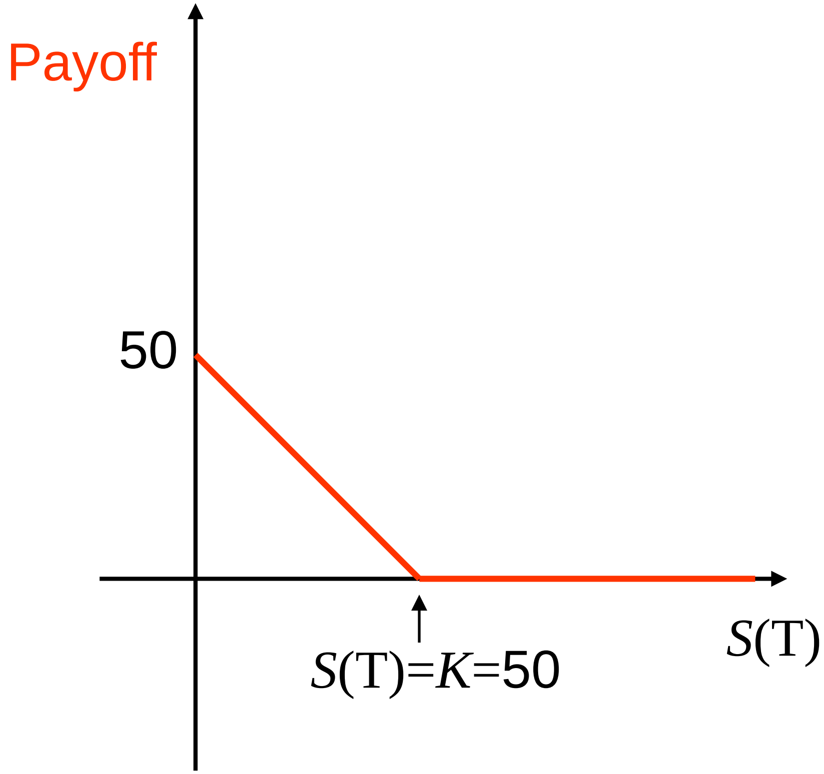
A more compact notation:

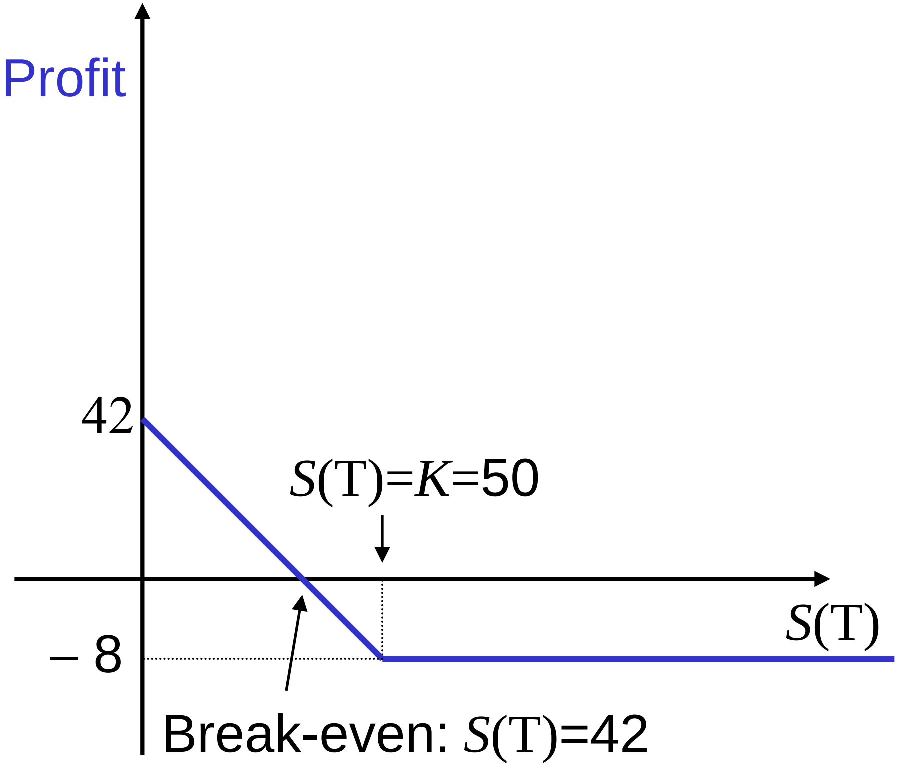
Payoff: $\quad max\,[K - \mathrm{S(T)}, 0] = (\mathrm{K{-}S(T)})_+$

Profit: $\quad max\,[K - \mathrm{S(T)}, 0] - P(\mathrm{t},K,\mathrm{T})$

Long Put Position

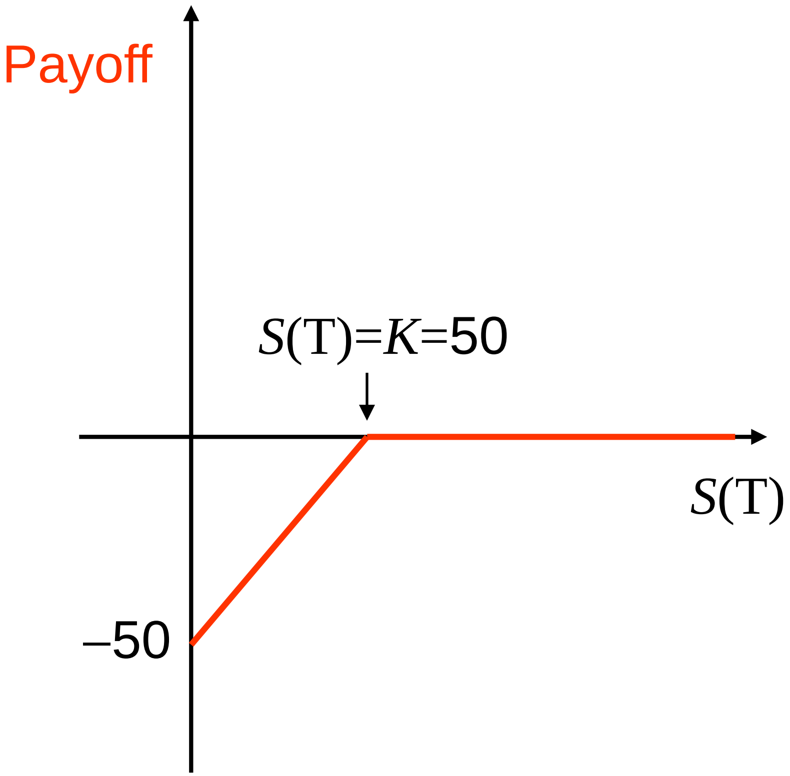
- Assume K = \$50, $P(t,K,T)$ = \$8
- Payoff: $max\ [50 - S(T), 0]$
- Profit: $max\ [50 - S(T), 0] - 8$

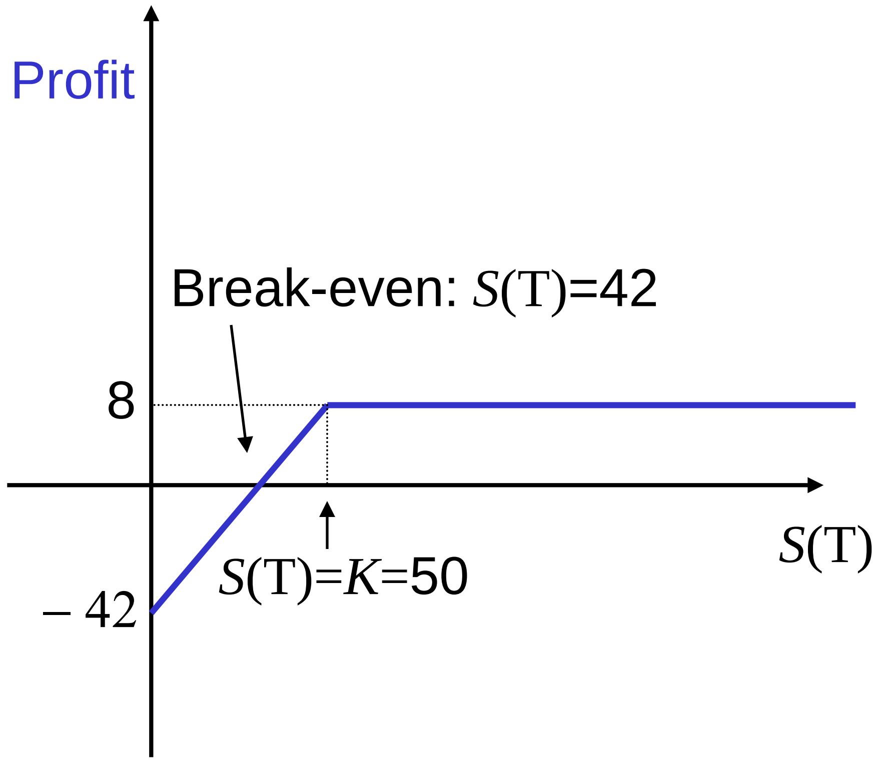




Short Put Position

- K = \$50, $P(t,K,T)$ = \$8
- Payoff: $- \mathit{max}\,[50 - S(T), 0]$
- Profit: $8 - \mathit{max}\,[50 - S(T), 0]$





Implicit Leverage: Example

- Consider two securities
  - Stock with price S(0) = \$100
  - Call option with price C(0) = \$2.5 (K = \$100)
- Consider three possible outcomes at t=T:
  - Good: S(T) = \$105
  - Intermediate: S(T) = \$101
  - Bad: S(T) = \$98

Implicit Leverage: Example (continued)

Suppose we plan to invest $100

Invest in:	Stocks	Options
Units	1	40
Return in:		
Good State	5%	100%
Mid State	1%	-60%
Bad State	-2%	-100%

EQUITY LINKED BANK DEPOSIT

- Investment =10,000
- Return = 10,000 if an index below the current value of 1,300 after 5.5 years
- Return = 10,000 × (1+ 70% of the percentage return on index)
- Example: Index=1,500. Return = $=10{,}000\cdot(1+(1{,}500/1{,}300-1)\cdot 70\%)=11{,}077$
- Payoff = Bond + call option on index

HEDGING EXAMPLE

Your bonus compensation: 100 shares of the company, each worth $150.

Your hedging strategy: buy 50 put options with strike K = 150

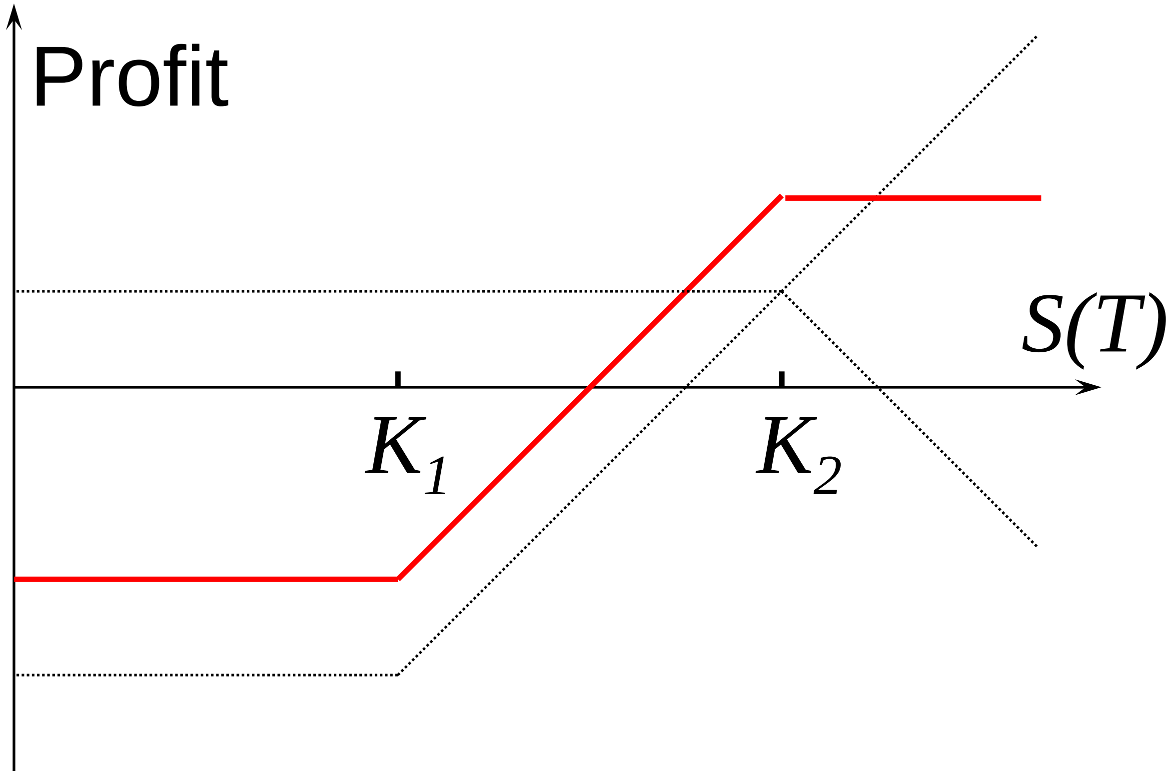
If share value falls to $100: you lose $5,000 in stock, win $2,500 minus premium in options

Pricing Options with Mathematical Models

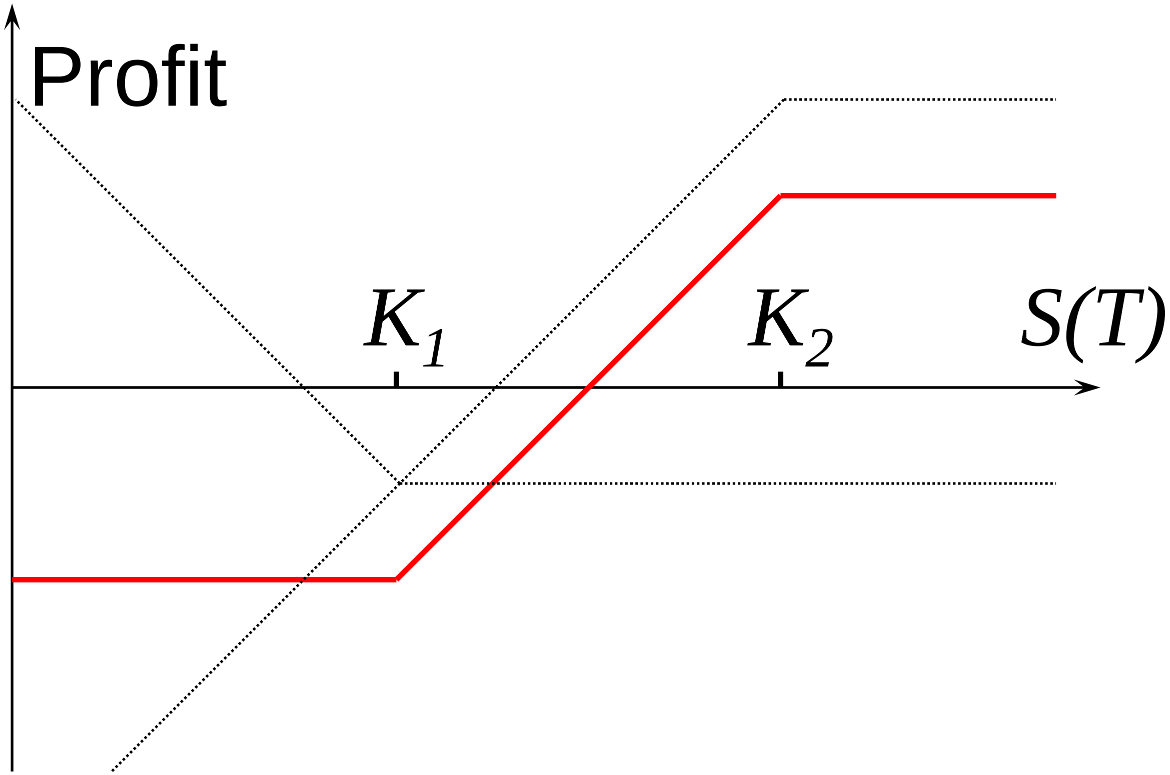
5. Options Combinations

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

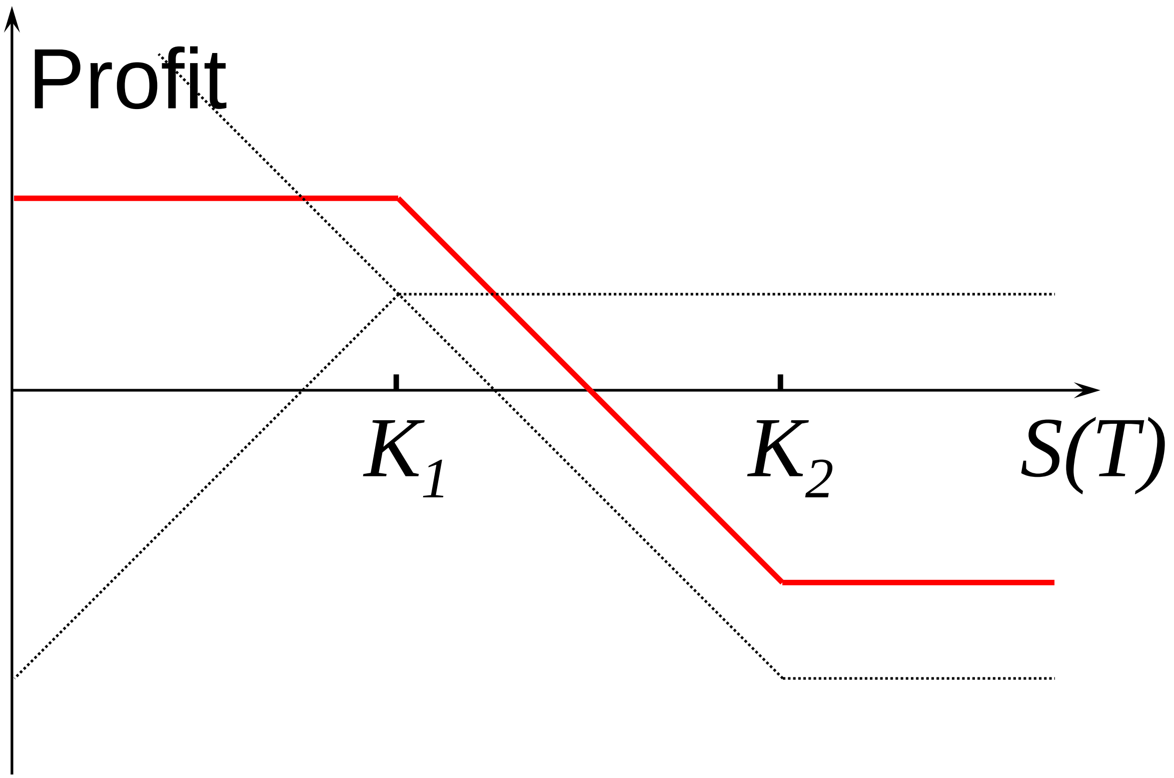
Bull Spread Using Calls



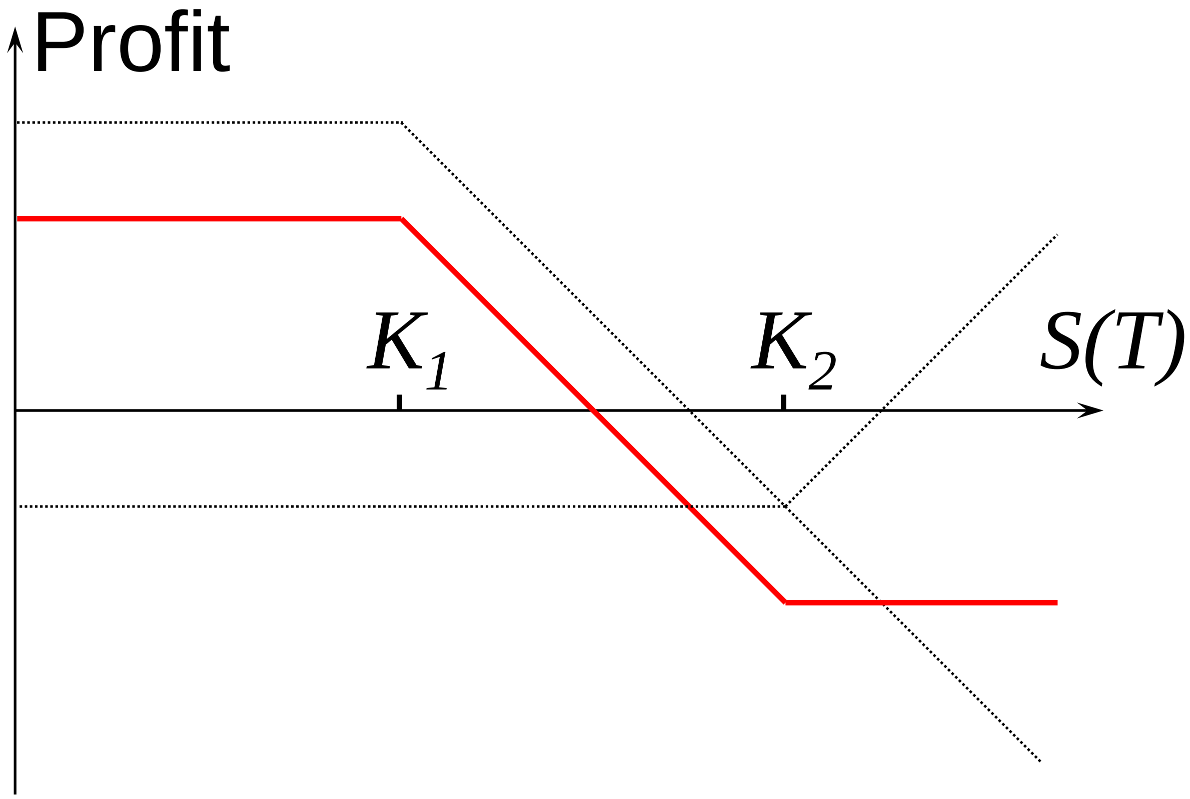
Bull Spread Using Puts



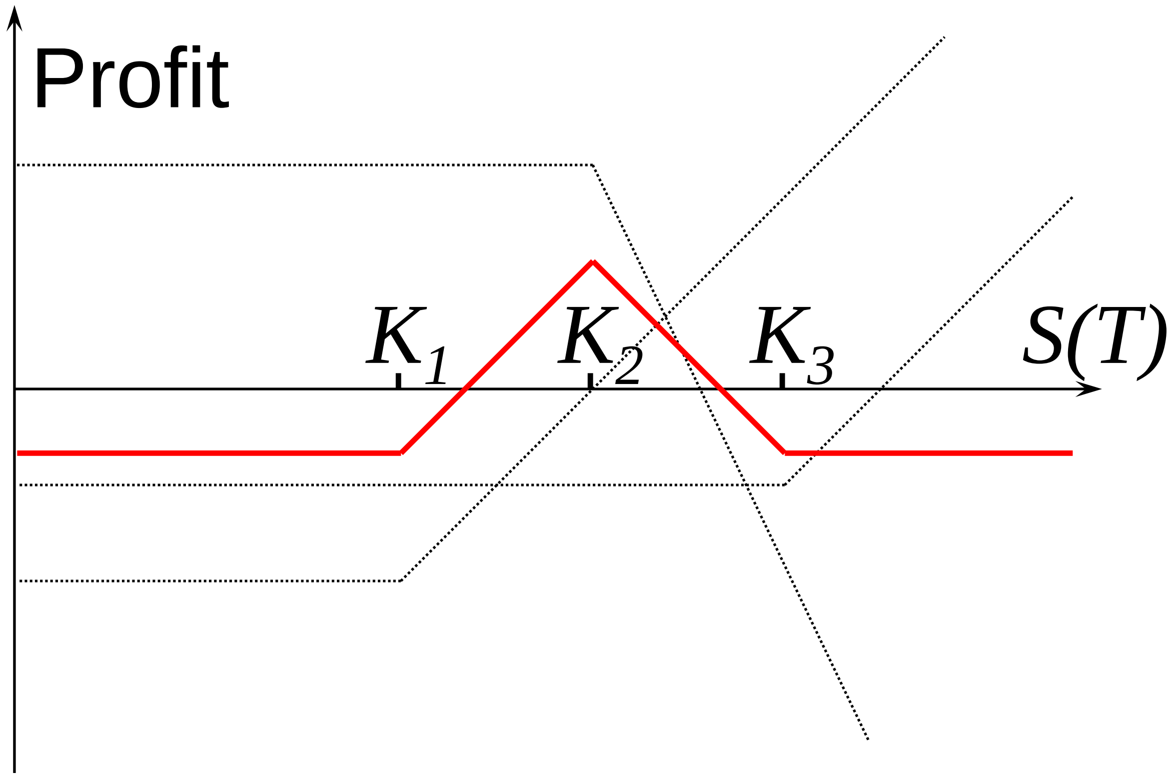
Bear Spread Using Puts



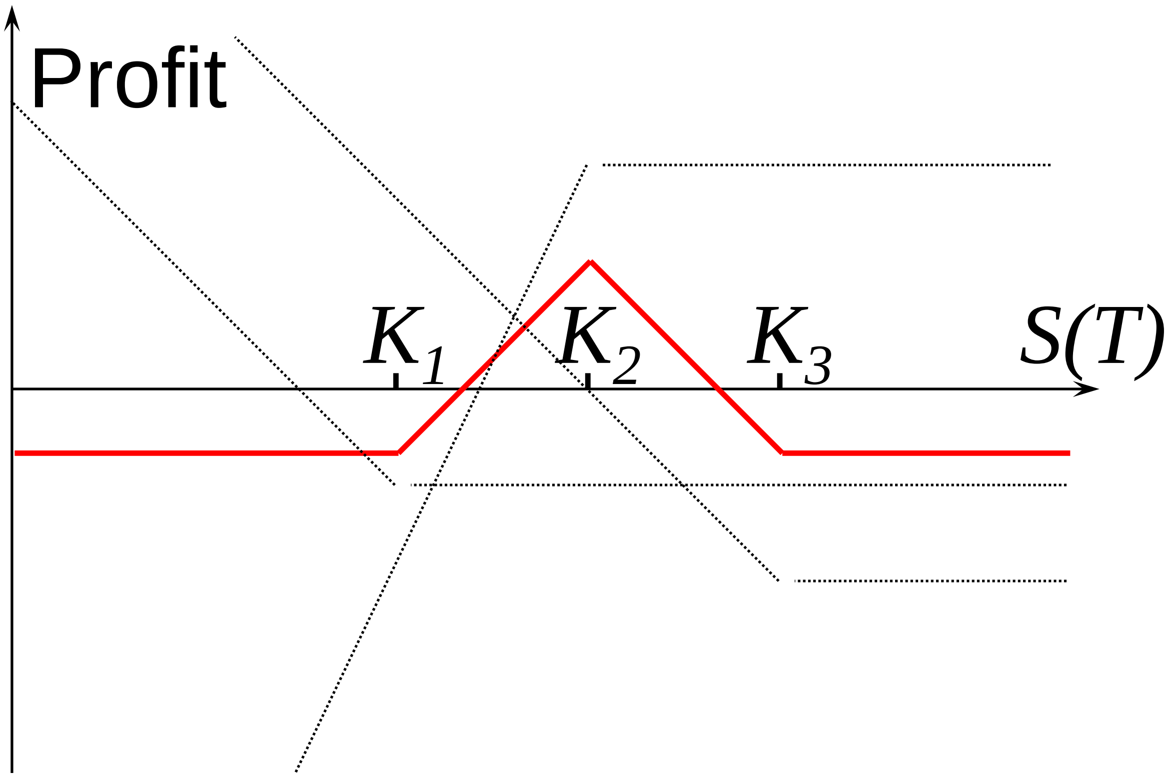
Bear Spread Using Calls



Butterfly Spread Using Calls



Butterfly Spread Using Puts



Bull Spread (Calls)

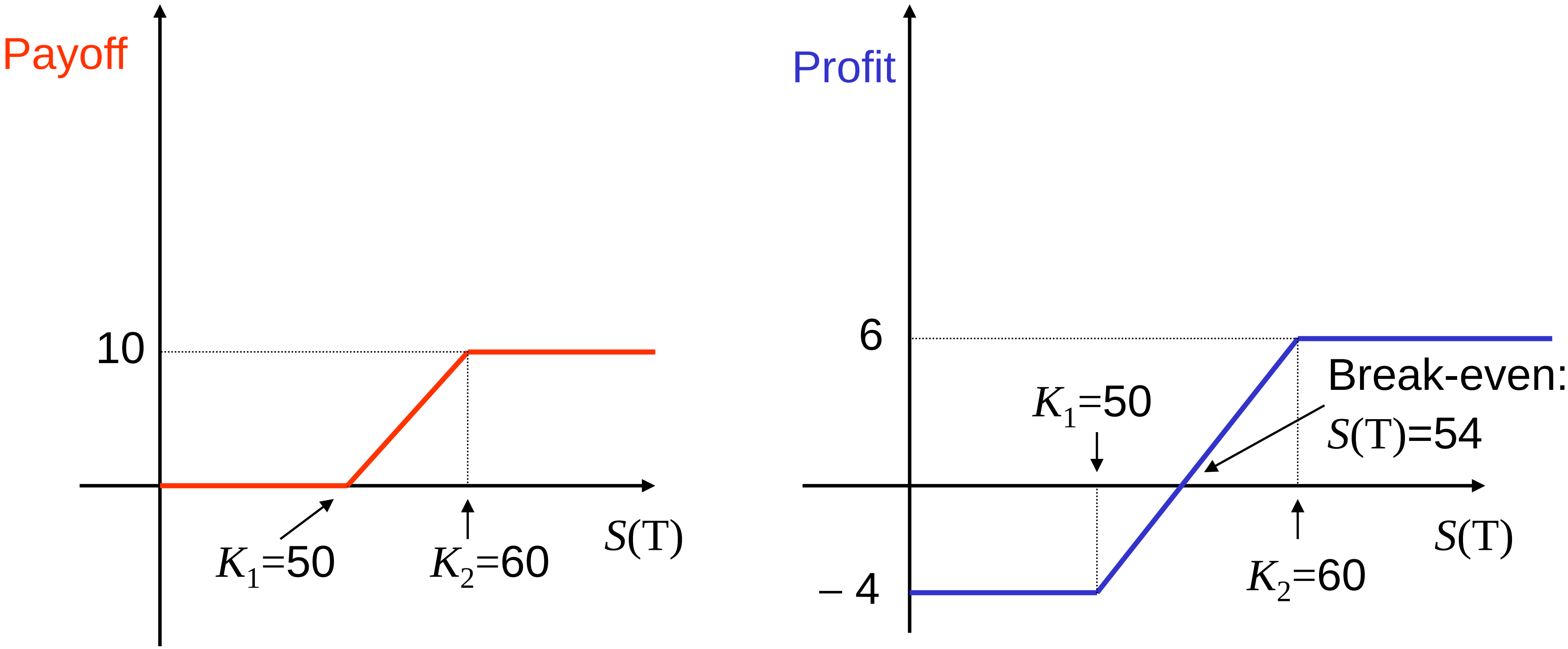
- Two strike prices: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $C(K_1)$, $C(K_2)$

Outcome at Expiration

	$S(\mathrm{T}) \le K_1$	$K_1 < S(\mathrm{T}) \le K_2$	$S(\mathrm{T}) > K_2$
Payoff:	0	$S(\mathrm{T}) - K_1$	$S(\mathrm{T}) - K_1 - (S(\mathrm{T}) - K_2) =$ $= K_2 - K_1$
Profit:	$C(K_2) - C(K_1)$	$C(K_2) - C(K_1)$ $+ S(\mathrm{T}) - K_1$	$C(K_2) - C(K_1) + K_2 - K_1$

Bull Spread (Calls)

- Assume K_1 = \$50, K_2 = \$60, $C(K_1)$ = \$10, $C(K_2)$ = \$6
- Payoff: *max* [S(T) – 50, 0] – *max* [S(T) – 60, 0]
- Profit: (6–10) + *max* [S(T)–50,0] – *max* [S(T)–60,0]



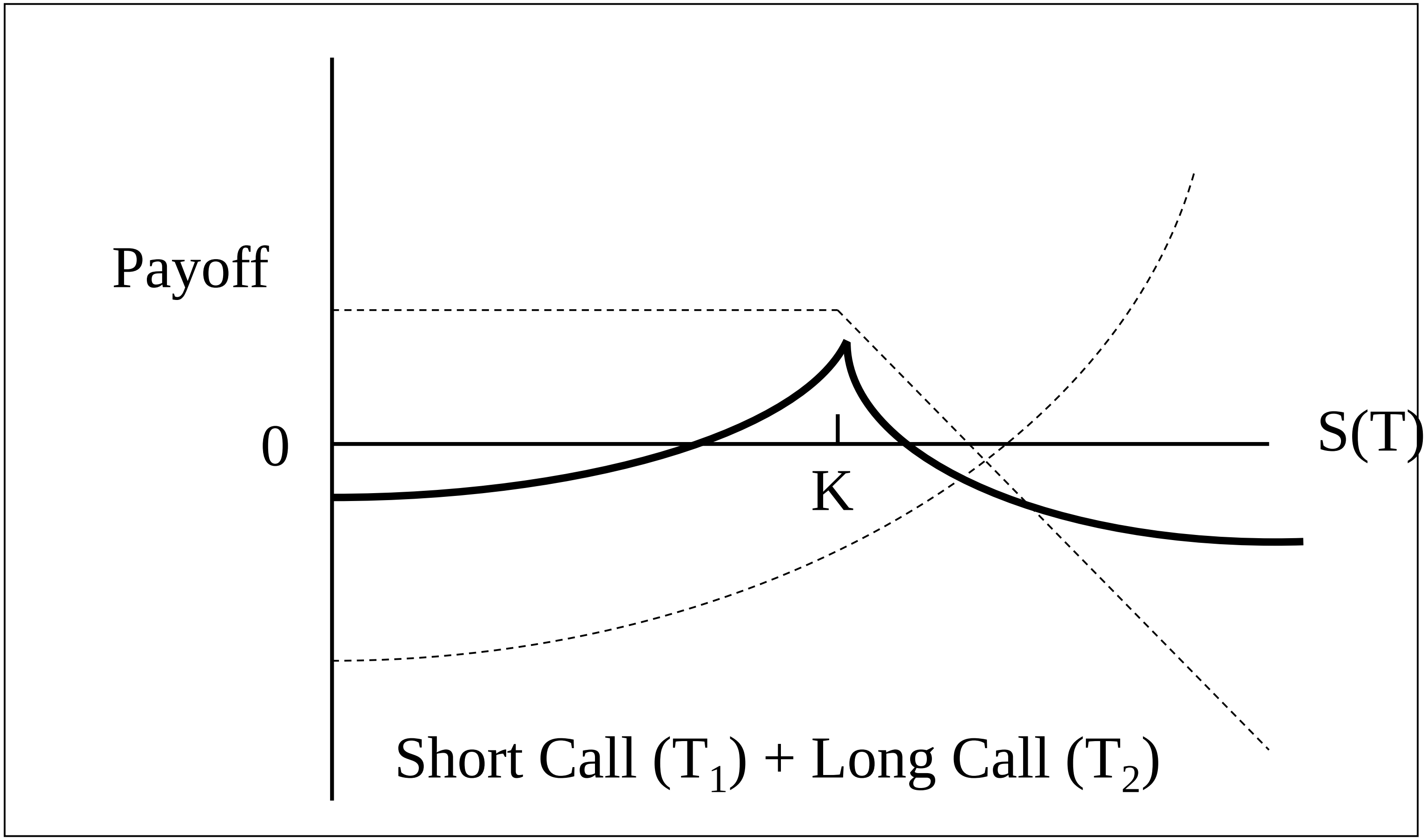
Bear Spread (Puts)

- Again two strikes: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $P(K_1)$, $P(K_2)$

Outcome at Expiration

	$S(\mathrm{T}) \le K_1$	$K_1 < S(\mathrm{T}) \le K_2$	$S(\mathrm{T}) > K_2$
Payoff:	$K_2 - S(\mathrm{T}) - (K_1 - S(\mathrm{T})) =$ $= K_2 - K_1$	$K_2 - S(\mathrm{T})$	0
Profit:	$P(K_1) - P(K_2) + K_2 - K_1$	$P(K_1) - P(K_2) +$ $+ K_2 - S(\mathrm{T})$	$P(K_1) - P(K_2)$

Calendar Spread



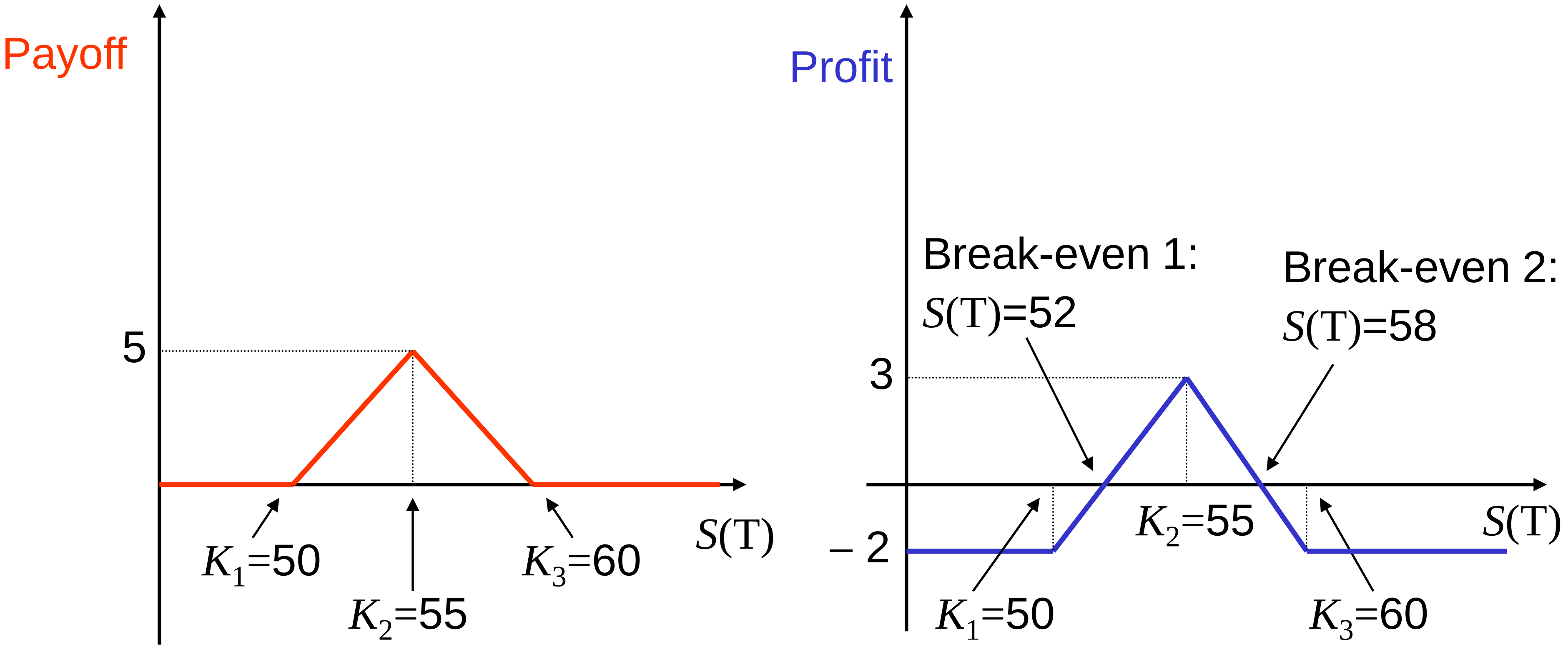
Butterfly Spread

- Positions in **three** options of the same class, with same maturities but different strikes K_1, K_2, K_3
  - Long butterfly spreads: buy one option each with strikes K_1, K_3, sell two with strike K_2
- $K_2 = (K_1 + K_3) /2$

$\Leftrightarrow$

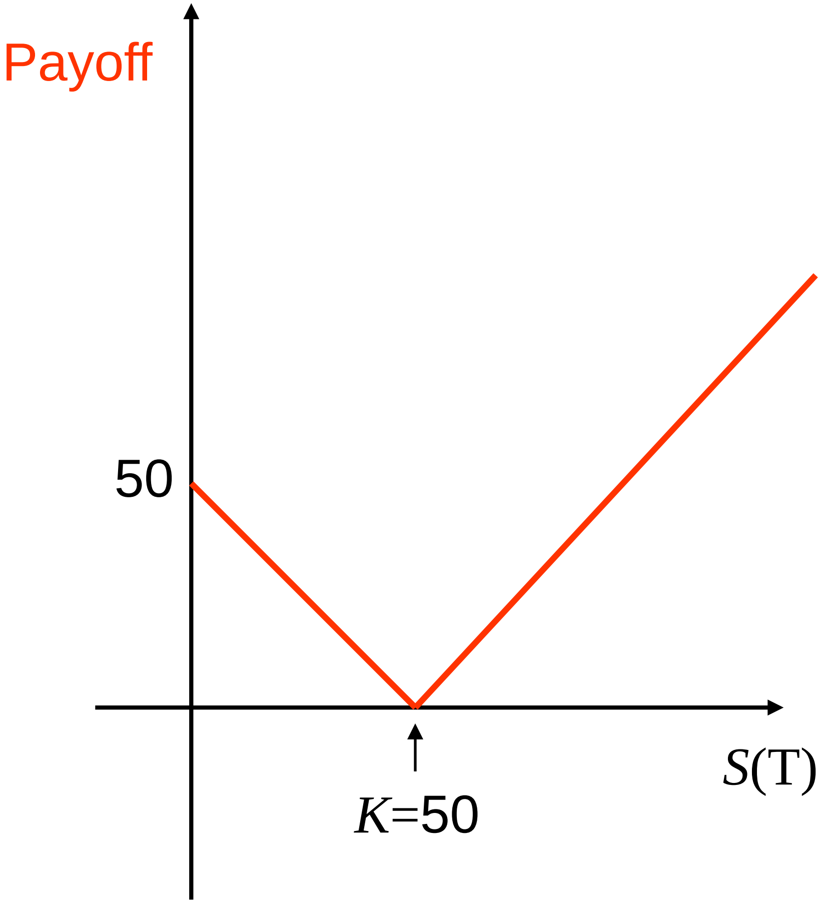
Long Butterfly Spread (Puts)

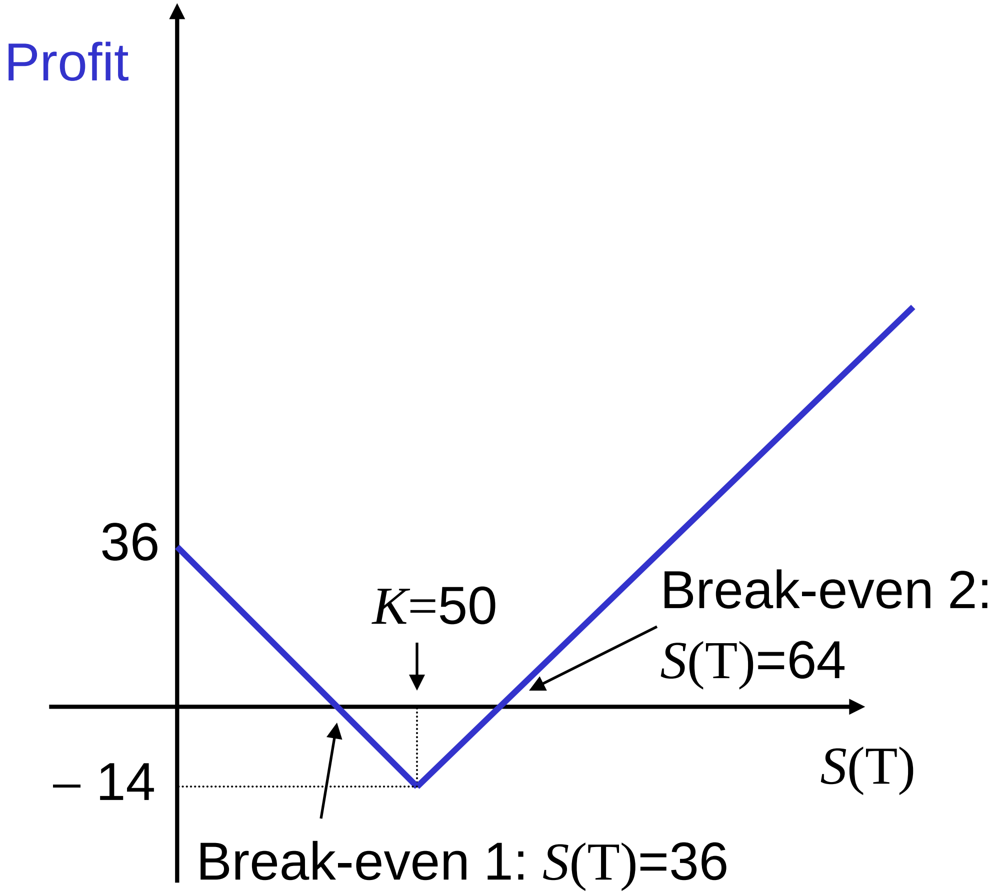
- K_1 = \$50, K_2 = \$55, K_3 = \$60
- $P(K_1)$ = \$4, $P(K_2)$ = \$6, $P(K_3)$ = \$10



Bottom Straddle

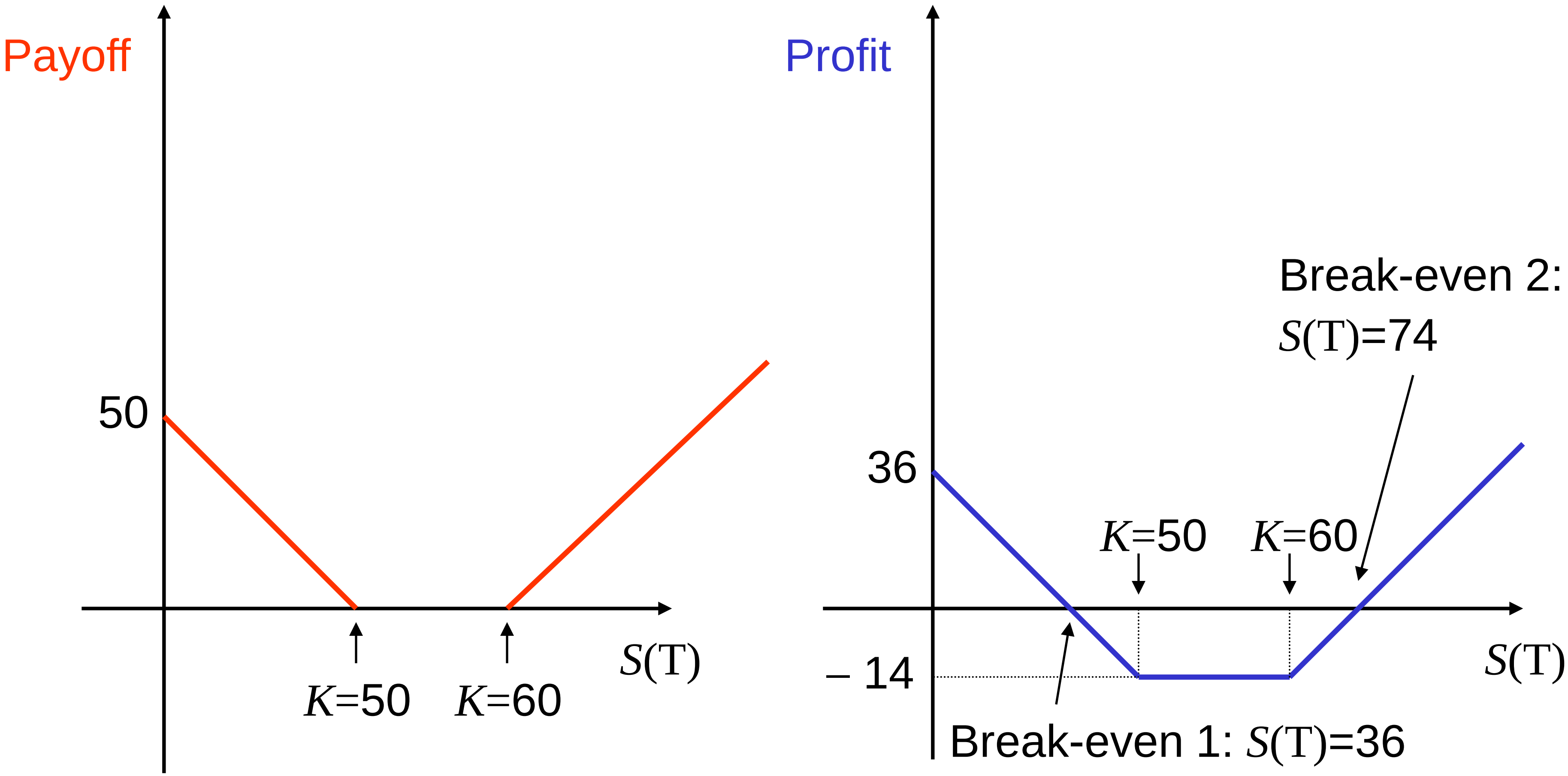
Assume K = \$50, $P(K)$ = \$8, $C(K)$ = \$6





Bottom Strangle

Assume K_1 = \$50, K_2 = \$60, $P(K_1)$ = \$8, $C(K_2)$ = \$6



Arbitrary payoff shape

- Suppose we want to have a payoff of the form

f (S(T)) for some function f ( ). Assume that call options written on S(T) are traded for all possible strike values K.

- CLAIM: If f ( ) is smooth and $f'(\infty) \bullet 0 = 0$, then

- $f(s) = f(0) + f'(0)s + \int_0^\infty f''(K) \max(S - K, 0)\, dK$

Proof sketch

$$\int_0^\infty f''(K)\max(s-K,0)\,dK$$

= (integration by parts) =

$$= f'(\infty) \bullet 0 - f'(0) \bullet s - \int_0^\infty f'(K)\,\mathrm{d}[\max(s-K,0)]$$

$$= - f'(0) \bullet s + \int_0^s f'(K)dK$$

$$= - f'(0) \bullet s + f(s) - f(0)\,.$$