Home Assignment 1

Econometrics of High-Dimensional Models - ECON231C

Mauricio Vargas-Estrada
Master in Quantitative Economics
University of California - Los Angeles

Problem 1

Prove Hoeffding's lemma: for any mean-zero random variable Z satisfying $|Z|\leq a$ almost surely,

$$\mathbb{E}\left[\exp\left(\lambda Z_i\right)\right] \leq \exp\left(\frac{\lambda^2 a^2}{2}\right),$$

for all $\lambda > 0$.

Proof. Since $\exp\left(\lambda Z_i\right)$ is a convex function of Z_i, for all $z_i \in [-a, a]$,

$$\begin{aligned}\exp\left(\lambda z_i\right) &\leq \frac{a - z_i}{a-(-a)}\exp\left(-a\lambda\right) + \frac{z_i - (-a)}{a-(-a)}\exp\left(a\lambda\right)\\ &\leq \frac{a - z_i}{2a}\exp\left(-a\lambda\right) + \frac{z_i + a}{2a}\exp\left(a\lambda\right)\end{aligned}$$

taking expectation on both sides and knowing that Z is mean-zero, we have,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda Z_i\right)\right] &\leq \mathbb{E}\left[\frac{a - Z_i}{2a}\exp\left(-a\lambda\right)\right] + \mathbb{E}\left[\frac{Z_i + a}{2a}\exp\left(a\lambda\right)\right]\\ &\leq \frac{a - \mathbb{E}\left[Z_i\right]}{2a}\exp\left(-a\lambda\right) + \frac{\mathbb{E}\left[Z_i\right] + a}{2a}\exp\left(a\lambda\right)\\ &\leq \frac{a}{2a}\exp\left(-a\lambda\right) + \frac{a}{2a}\exp\left(a\lambda\right)\\ &\leq \frac{1}{2}\exp\left(-a\lambda\right) + \frac{1}{2}\exp\left(a\lambda\right)\\ &\leq \exp\left(-a\lambda\right) + \exp\left(a\lambda\right)\end{aligned}$$

Calculating the Taylor's expansion of the right-hand side of the inequality over λ and simplifying,

$$
\begin{aligned}
\exp(-a\lambda) + \exp(a\lambda) &= \left(1 - a\lambda + \frac{a^2\lambda^2}{2} - \frac{a^3\lambda^3}{6} + \dots\right) + \left(1 + a\lambda + \frac{a^2\lambda^2}{2} + \frac{a^3\lambda^3}{6} + \dots\right) \\
&= 1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{24} \dots
\end{aligned}
$$

We recognize that the Taylor's expansion of $\exp\left(\frac{a^2\lambda^2}{2}\right)$ is:

$$
\exp\left(\frac{a^2\lambda^2}{2}\right) = 1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{8} \dots
$$

meaning that:

$$
\begin{aligned}
1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{24} \dots &\leq 1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{8} \dots \\
\exp(-a\lambda) + \exp(a\lambda) &\leq \exp\left(\frac{a^2\lambda^2}{2}\right)
\end{aligned}
$$

Therefore

$$
\mathbb{E}\left[\exp(\lambda Z_i)\right] \leq \exp\left(\frac{\lambda^2 a^2}{2}\right)
$$

$\square$

Problem 2

While proving the Hoeffding inequality, we said that two probabilities,

$$
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \geq t\right)
$$

and

$$
\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \leq -t\right),
$$

can be bounded in the same way and did the derivation only for the former probability. Show that the latter probability is indeed bounded by the same quantity.

Proof. Lets define $Z_i = X_i - \mu, \forall i = 1, 2, \dots, n$. Then, we have:

$$
\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i = \bar{X}_n - \mu
$$

and,

$$|Z_i|\leq a, \forall i = 1, 2, \dots, n$$

then, whe can rewrite the former probability as:

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \leq -t\right) = \mathbb{P}\left(\bar{Z}_n \geq -t\right)$$

Expanding the average $\bar{Z}_n$:

$$\mathbb{P}\left(\bar{Z}_n \geq -t\right) = \mathbb{P}\left(\sum_{i=1}^{n} Z_i \leq -nt\right)$$

for any $\lambda > 0$, we have [1]:

$$\mathbb{P}\left(\bar{Z}_n \geq -t\right) = \mathbb{P}\left(\lambda\sum_{i=1}^{n} Z_i \leq -\lambda nt\right)$$

and, by Markov's inequality:

$$\mathbb{P}\left(\lambda\sum_{i=1}^{n} Z_i \leq -\lambda nt\right) \leq \frac{\mathbb{E}\left[\exp(\lambda \sum_{i=1}^{n} Z_i)\right]}{\exp(-\lambda nt)}$$

since Z_i are independent and identically distributed, we can write:

$$\begin{aligned}
\frac{\mathbb{E}\left[\exp(\lambda \sum_{i=1}^{n} Z_i)\right]}{\exp(-\lambda nt)} &= \frac{\prod_{i=1}^{n} \mathbb{E}\left[\exp(-\lambda Z_i)\right]}{\exp(-\lambda nt)} \\
&= \frac{\prod_{i=1}^{n} \exp(-1)\mathbb{E}\left[\exp(\lambda Z_i)\right]}{\exp(-\lambda nt)} \\
&= \frac{\prod_{i=1}^{n} \mathbb{E}\left[\exp(\lambda Z_i)\right]}{\exp(\lambda nt)}
\end{aligned}$$

Applying the Hoeffding's lemma [2] to the above expression, we have:

---

1. Given that $f(x) = \lambda x$ is a monotonically increasing function when $\lambda > 0$
2. If X is a random variable such that $X \leq a$, then for any $\lambda > 0$, we have:

$$\mathbb{E}\left[\exp(\lambda X)\right] \leq \exp\left(\frac{\lambda^2 a^2}{2}\right)$$

$$
\begin{aligned}
\frac{\prod_{i=1}^n \mathbb{E}\left[\exp(\lambda Z_i)\right]}{\exp(\lambda nt)} &\leq \frac{\prod_{i=1}^n \exp\left(\frac{\lambda^2 a^2}{2}\right)}{\exp(\lambda nt)} \\
&\leq \frac{\exp\left(\frac{n\lambda^2 a^2}{2}\right)}{\exp(\lambda nt)} \\
&\leq \exp\left(\frac{n\lambda^2 a^2}{2} - \lambda nt\right)
\end{aligned}
$$

Because the above inequality holds for any $\lambda > 0$, we can optimize the right-hand side with respect to λ.

$$
\lambda^* = \arg\min_{\lambda>0} \left\{\frac{n\lambda^2 a^2}{2} - \lambda nt\right\}
$$

Calculating the F.O.C. with respect to λ, we get:

$$
na^2\lambda^* - nt = 0 \Rightarrow \lambda^* = \frac{t}{a^2}
$$

Substituting λ^* back into the inequality:

$$
\begin{aligned}
\mathbb{P}\left(\bar{Z}_n \geq -t\right) &\leq \exp\left(\frac{n\left(\frac{t}{a^2}\right)^2 a^2}{2} - \frac{t}{a^2}nt\right) \\
&\leq \exp\left(\frac{nt^2}{2a^2} - \frac{nt^2}{a^2}\right) \\
&\leq \exp\left(-\frac{nt^2}{2a^2}\right)
\end{aligned}
$$

replacing $\bar{Z}_n$ by $\bar{X}_n - \mu$:

$$
\mathbb{P}\left(|\bar{X}_n - \mu| \geq t\right) \leq \exp\left(-\frac{nt^2}{2a^2}\right)
$$

$\square$

Problem 3

(Tricky) While proving the Hoeffding inequality, we have used the following bound:

$$P(X > \delta) \le \frac{E[e^{\lambda X}]}{e^{\lambda\delta}}, \quad \lambda > 0.$$

An alternative could be

$$P(X > \delta) \le \frac{E[|X|^k]}{\delta^k}, \quad k \ge 0.$$

Show that if $X \ge 0$ a.s., then

$$\inf_{k=0,1,2,\ldots} \frac{E[|X|^k]}{\delta^k} \le \inf_{\lambda>0} \frac{E[e^{\lambda X}]}{e^{\lambda\delta}}.$$

Problem 4

While proving the maximal inequality, i.e., a bound on

$$P\left(\max_{1\le j\le p} \left|\frac{1}{n}\sum_{i=1}^{n} X_{ij} - \mu_j\right| \ge t\right),$$

we applied the union bound followed by the Hoeffding inequality. Show what happens if we replace the Hoeffding inequality by the Chebyshev inequality.