Class Notes

Estimation in High Dimensionality Spaces - ECON231C

Mauricio Vargas-Estrada
Master in Quantitative Economics
University of California - Los Angeles

1 Markov Inequality

Being X a random variable such that $X \geq 0$, then:

$$\mathbb{P}\left(X \geq t\right) \leq \frac{\mathbb{E}\left[X\right]}{t}, \quad \forall t > 0$$

Proof. Whe can rewrite the left-hand side of the inequality using the indicator function:

$$X \geq X\mathbf{1}_{\{X \geq t\}}$$

The left-hand side would be greater when $X < t$ and equal when $X \geq t$. Given that:

$$X \geq X\mathbf{1}_{\{X \geq t\}} \geq t\mathbf{1}_{\{X \geq t\}}$$

In this case, $X\mathbf{1}_{\{X \geq t\}} > t\mathbf{1}_{\{X \geq t\}}$ when $X > t$, and $X\mathbf{1}_{\{X \geq t\}} = t\mathbf{1}_{\{X \geq t\}}$ when $X \leq t$ because $X = t$ or the indicator function is zero.

Taking the expectation of the inequality:

$$\begin{aligned}
\mathbb{E}\left[X\right] \geq \mathbb{E}\left[X\mathbf{1}_{\{X \geq t\}}\right] \geq \mathbb{E}\left[t\mathbf{1}_{\{X \geq t\}}\right] \\
\mathbb{E}\left[X\right] \geq \mathbb{E}\left[X\mathbf{1}_{\{X \geq t\}}\right] \geq t\mathbb{E}\left[\mathbf{1}_{\{X \geq t\}}\right] \\
\frac{\mathbb{E}\left[X\right]}{t} \geq \frac{\mathbb{E}\left[X\mathbf{1}_{\{X \geq t\}}\right]}{t} \geq \mathbb{E}\left[\mathbf{1}_{\{X \geq t\}}\right]
\end{aligned}$$

But $\mathbb{E}\left[\mathbf{1}_{\{X \geq t\}}\right] = \mathbb{P}\left(X \geq t\right)$, so:

$$\frac{\mathbb{E}\left[X\right]}{t} \geq \mathbb{P}\left(X \geq t\right)$$

$\square$

2 Chevyshev Inequality

Given a random variable X with mean μ and variance σ^2, then:

$$\mathbb{P}\left(|X-\mu|\geq t\right) \leq \frac{\sigma^2}{t^2}, \quad \forall t > 0$$

Proof. We are going to use the fact that a strictly increasing function of a random variable does not change the probability of an event. Let $Y = (|X-\mu|)^2$. Then,

$$\mathbb{P}\left(|X-\mu|^2\geq t^2\right) = \mathbb{P}\left(Y \geq t^2\right)$$

Using Markov's inequality, we have:

$$\mathbb{P}\left(Y \geq t^2\right) \leq \frac{\mathbb{E}\left[Y\right]}{t^2}$$

Given a random variable Z, the variance of Z is $\mathrm{Var}\left(Z\right) = \mathbb{E}\left[(Z-\mathbb{E}\left[Z\right])^2\right]$. Therefore,

$$\begin{aligned}
\mathbb{P}\left((X-\mu)^2 \geq t^2\right) &\leq \frac{\mathbb{E}\left[(X-\mu)^2\right]}{t^2} \\
\mathbb{P}\left((X-\mu)^2 \geq t^2\right) &\leq \frac{\mathrm{Var}\left(X\right)}{t^2} \\
\mathbb{P}\left((X-\mu)^2 \geq t^2\right) &\leq \frac{\sigma^2}{t^2}
\end{aligned}$$

□

3 Weak Law of Large Numbers

Given a collection of i.i.d. random variables $\{X_i\}_{i=1}^n$, with mean μ and variance σ^2.

Defining $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, the weak law of large numbers states that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(|\bar{X}_n - \mu|\geq \epsilon\right) = 0$$

or equivalently,

$$\bar{X}_n \xrightarrow{\mathbb{P}} \mu$$

Proof. Calculating the variance of $\bar{X}_n$,

$$\begin{aligned}
\operatorname{Var}\left(\bar{X}_n\right) &= \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var}\left(X_i\right) \\
&= \frac{1}{n^2} \cdot n\sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Then, by Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\left(|\bar{X}_n - \mu| \geq \epsilon\right) &\leq \frac{\operatorname{Var}\left(\bar{X}_n\right)}{\epsilon^2} \\
&= \frac{\sigma^2}{n\epsilon^2}
\end{aligned}$$

Taking the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(|\bar{X}_n - \mu| \geq \epsilon\right) &\leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} \\
\lim_{n\to\infty} \mathbb{P}\left(|\bar{X}_n - \mu| \geq \epsilon\right) &\leq 0
\end{aligned}$$

$\square$