Home Assignment 1

Econometrics of High-Dimensional Models - ECON231C

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Problem 1

Prove Hoeffding's lemma: for any mean-zero random variable Z satisfying $|Z|\leq a$ almost surely,

$$\mathbb{E}\left[\exp\left(\lambda Z_i\right)\right] \leq \exp\left(\frac{\lambda^2 a^2}{2}\right),$$

for all $\lambda > 0$.

Proof. Since $\exp\left(\lambda Z_i\right)$ is a convex function of Z_i, for all $z_i \in [-a, a]$,

$$\begin{aligned}\exp\left(\lambda z_i\right) &\leq \frac{a - z_i}{a - (-a)} \exp\left(-a\lambda\right) + \frac{z_i - (-a)}{a - (-a)} \exp\left(a\lambda\right) \\ &\leq \frac{a - z_i}{2a} \exp\left(-a\lambda\right) + \frac{z_i + a}{2a} \exp\left(a\lambda\right)\end{aligned}$$

taking expectation on both sides and knowing that Z is mean-zero, we have,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda Z_i\right)\right] &\leq \mathbb{E}\left[\frac{a - Z_i}{2a} \exp\left(-a\lambda\right)\right] + \mathbb{E}\left[\frac{Z_i + a}{2a} \exp\left(a\lambda\right)\right] \\ &\leq \frac{a - \mathbb{E}\left[Z_i\right]}{2a} \exp\left(-a\lambda\right) + \frac{\mathbb{E}\left[Z_i\right] + a}{2a} \exp\left(a\lambda\right) \\ &\leq \frac{a}{2a} \exp\left(-a\lambda\right) + \frac{a}{2a} \exp\left(a\lambda\right) \\ &\leq \frac{1}{2} \exp\left(-a\lambda\right) + \frac{1}{2} \exp\left(a\lambda\right) \\ &\leq \exp\left(-a\lambda\right) + \exp\left(a\lambda\right)\end{aligned}$$

Calculating the Taylor's expansion of the right-hand side of the inequality over λ and simplifying,

$$
\begin{aligned}
\exp(-a\lambda) + \exp(a\lambda) &= \left(1 - a\lambda + \frac{a^2\lambda^2}{2} - \frac{a^3\lambda^3}{6} + \dots\right) + \left(1 + a\lambda + \frac{a^2\lambda^2}{2} + \frac{a^3\lambda^3}{6} + \dots\right) \\
&= 1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{24} \dots
\end{aligned}
$$

We recognize that the Taylor's expansion of $\exp\left(\frac{a^2\lambda^2}{2}\right)$ is:

$$
\exp\left(\frac{a^2\lambda^2}{2}\right) = 1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{8} \dots
$$

meaning that:

$$
\begin{aligned}
1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{24} \dots &\leq 1 + \frac{a^2\lambda^2}{2} + \frac{a^4\lambda^4}{8} \dots \\
\exp(-a\lambda) + \exp(a\lambda) &\leq \exp\left(\frac{a^2\lambda^2}{2}\right)
\end{aligned}
$$

Therefore

$$
\mathbb{E}\left[\exp(\lambda Z_i)\right] \leq \exp\left(\frac{\lambda^2 a^2}{2}\right)
$$

$\square$

Problem 2

While proving the Hoeffding inequality, we said that two probabilities,

$$
P\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \geq t\right)
$$

and

$$
P\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \leq -t\right),
$$

can be bounded in the same way and did the derivation only for the former probability. Show that the latter probability is indeed bounded by the same quantity.

Problem 3

(Tricky) While proving the Hoeffding inequality, we have used the following bound:

$$P(X > \delta) \leq \frac{E[e^{\lambda X}]}{e^{\lambda \delta}}, \quad \lambda > 0.$$

An alternative could be

$$P(X > \delta) \leq \frac{E[|X|^k]}{\delta^k}, \quad k \geq 0.$$

Show that if $X \geq 0$ a.s., then

$$\inf_{k=0,1,2,\ldots} \frac{E[|X|^k]}{\delta^k} \leq \inf_{\lambda>0} \frac{E[e^{\lambda X}]}{e^{\lambda \delta}}.$$

Problem 4

While proving the maximal inequality, i.e., a bound on

$$P\left(\max_{1 \leq j \leq p} \left| \frac{1}{n} \sum_{i=1}^{n} X_{ij} - \mu_j \right| \geq t \right),$$

we applied the union bound followed by the Hoeffding inequality. Show what happens if we replace the Hoeffding inequality by the Chebyshev inequality.