Exploring the Exponential Nature of Imaginary Powers: A Case Study of i^i

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To calculate i^i , we can begin by employing Euler's formula[1].

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Know we assign θ an angle

$$\theta = \frac{\pi}{2}$$

For $\theta = \frac{\pi}{2}$, this simplifies to

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i \cdot \sin\left(\frac{\pi}{2}\right)$$

Evaluating $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$

When evaluating $\cos\left(\frac{\pi}{2}\right)$, we rely on the unit circle definition where the cosine $\frac{\pi}{2}$ radians corresponds to the x-coordinate of the point $(0,1)$, the graphical representation where the cosine function intersects the x-axis at $\frac{\pi}{2}$, and the right-angled triangle definition where the adjacent side becomes zero for an angle of $\frac{\pi}{2}$ radians.

$$\cos\left(\frac{\pi}{2}\right) = 0 \approx 6{,}123234 \cdot 10^{-17}$$

When evaluating $\sin\left(\frac{\pi}{2}\right)$, we find that it holds true due to the unit circle definition, where the sine of $\frac{\pi}{2}$ radians correspond to the y-coordinate of the point $(0,1)$.

$$\sin\left(\frac{\pi}{2}\right) = 1$$

Now we can simplify it to

$$e^{i\frac{\pi}{2}} = 0 + i \cdot 1 \;\; or \;\; i = e^{i\frac{\pi}{2}}$$

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[1] https://www.ncl.ac.uk/webtemplate/ask-assets/external/maths-resources/core-mathematics/pure-maths/algebra/euler-s-formula-and-euler-s-identity.html

Now that we have established the polar form of i, we can proceed to raise it to the power of i.

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i$$

Now applying the rule $(a^b)^c = a^{bc}$

It can be rewritten as

$$i^i = e^{i \cdot i\frac{\pi}{2}}$$

Since $i \cdot i$ means i^2 and i is defined as $\sqrt{-1}$, squaring i cancels out the square root, leaving the value -1

$$i \cdot i = -1$$

Now it can be simplified to

$$i^i = e^{-(\frac{\pi}{2})}$$

The derivation leads us to the expression $e^{-(\frac{\pi}{2})}$, which can be calculated numerically.

$$e^{-\left(\frac{\pi}{2}\right)} \approx 0{,}2078796$$

$$i^i \approx 0{,}2078796$$

I have now arrived at the conclusion that the value of i^i is approximately $0{,}2078796$. This result demonstrates the intriguing nature of complex numbers, where raising an imaginary unit to the power of itself yields a real number.