

# Exploring the Exponential Nature of Imaginary Powers: A Case Study of $i^i$

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To calculate  $i^i$ , we can begin by employing Euler's formula<sup>1</sup>.

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Now we assign  $\theta$  an angle

$$\theta = \frac{\pi}{2}$$

For  $\theta = \frac{\pi}{2}$ , this simplifies to

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i \cdot \sin\left(\frac{\pi}{2}\right)$$

Evaluating  $\cos\left(\frac{\pi}{2}\right)$  and  $\sin\left(\frac{\pi}{2}\right)$

When evaluating  $\cos\left(\frac{\pi}{2}\right)$ , we rely on the unit circle definition where the cosine  $\frac{\pi}{2}$  radians corresponds to the x-coordinate of the point (0,1), the graphical representation where the cosine function intersects the x-axis at  $\frac{\pi}{2}$ , and the right-angled triangle definition where the adjacent side becomes zero for an angle of  $\frac{\pi}{2}$  radians.

$$\cos\left(\frac{\pi}{2}\right) = 0$$

When evaluating  $\sin\left(\frac{\pi}{2}\right)$ , we find that it holds true due to the unit circle definition, where the sine of  $\frac{\pi}{2}$  radians correspond to the y-coordinate of the point (0,1).

$$\sin\left(\frac{\pi}{2}\right) = 1$$

Now we can simplify it to

$$e^{i\frac{\pi}{2}} = 0 + i \cdot 1 \text{ or } i = e^{i\frac{\pi}{2}}$$

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<sup>1</sup> <https://www.ncl.ac.uk/webtemplate/ask-assets/external/maths-resources/core-mathematics/pure-maths/algebra/euler-s-formula-and-euler-s-identity.html>

Now that we have established the polar form of  $i$ , we can proceed to raise it to the power of  $i$ .

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i$$

Now applying the rule  $(a^b)^c = a^{bc}$

It can be rewritten as

$$i^i = e^{i \cdot i \frac{\pi}{2}}$$

Since  $i \cdot i$  means  $i^2$  and  $i$  is defined as  $\sqrt{-1}$ , squaring  $i$  cancels out the square root, leaving the value  
-1

$$i \cdot i = -1$$

Now it can be simplified to

$$i^i = e^{-\left(\frac{\pi}{2}\right)}$$

The derivation leads us to the expression  $e^{-\left(\frac{\pi}{2}\right)}$ , which can be calculated numerically.

$$e^{-\left(\frac{\pi}{2}\right)} \approx 0,2078796$$

$$i^i \approx 0,2078796$$

I have now arrived at the conclusion that the value of  $i^i$  is approximately 0,2078796. This result demonstrates the intriguing nature of complex numbers, where raising an imaginary unit to the power of itself yields a real number.