Solving the integral given to Howard Wolowitz by Sheldon Cooper during the Series 08 Episode 02 – The Junior Professor Solution, The Big Bang Theory.

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Sheldon: Hmm. Do you know how to integrate X squared times E to the minus X, without looking it up?

Howard: I'd use Feynman's trick, differentiate under the integral sign.

The integral given by Sheldon:

$$\int_0^\infty x^2 \cdot e^{-x} dx$$

I have chosen to make it an improper integral, otherwise it would be way to easy.

Since Howard chose to use the Feynman's trick, let's use it also.
First let's define a known integral as I

$$I = \int_0^\infty e^{-x} dx$$

Now let's define the parameter alpha, and calculate the integral

$$I(\alpha) = \int_0^\infty e^{-\alpha x} dx = \left[-\frac{1}{\alpha} e^{-\alpha x} \right]_0^\infty = \lim_{x \to \infty} \frac{-1}{\alpha} e^{-\alpha x} + \frac{1}{\alpha} e^{-\alpha \cdot 0} = 0 + \frac{1}{\alpha} e^0 = \frac{1}{\alpha} \cdot 1 = \frac{1}{\alpha}$$

Now we can use the trick by differentiating under the integral sign, and we are differentiating with the respect to alpha.

$$I'(\alpha) = \frac{d}{d\alpha} \int_0^\infty e^{-\alpha x} dx$$

When moving the derivative inside the integral sign, it changes to a partial derivative

$$I'(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha} e^{-\alpha x} dx$$

Now calculating the partial derivative, and then calculating the integral after that we get

$$I'(\alpha) = \int_0^\infty -x \cdot e^{-\alpha x}\, dx = \left[-x \cdot \frac{1}{-\alpha} e^{-\alpha x} + 1 \cdot -\frac{e^{-\alpha x}}{\alpha^2} \right]_0^\infty$$

$$= \int_0^\infty -x \cdot e^{-\alpha x}\, dx = \left[\frac{-x}{-\alpha} e^{-\alpha x} + -\frac{e^{-\alpha x}}{\alpha^2} \right]_0^\infty$$

$$= \lim_{x \to \infty} \frac{-x}{-\alpha} e^{-\alpha x} + -\frac{e^{-\alpha x}}{\alpha^2} - \left(\frac{-0}{-\alpha} e^{-\alpha \cdot 0} + -\frac{e^{-\alpha \cdot 0}}{\alpha^2} \right) = -\left(-\frac{e^{-\alpha \cdot 0}}{\alpha^2} \right) = -\frac{1}{\alpha^2}$$

By taking the derivative once more we get to the integral given by Sheldon

$$I''(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha} - x \cdot e^{-\alpha x}\, dx$$

Now calculating this and calculating the integral

$$I''(\alpha) = \int_0^\infty x^2 \cdot e^{-\alpha x}\, dx = \left[x^2 \cdot \frac{1}{-\alpha} \cdot e^{-\alpha x} - 2x \cdot -\frac{e^{-\alpha x}}{\alpha^2} + 2 \cdot \frac{e^{-\alpha x}}{\alpha^3}\right]_0^\infty$$

$$= \lim_{x\to\infty} x^2 \cdot \frac{1}{-\alpha} \cdot e^{-\alpha x} - 2x \cdot -\frac{e^{-\alpha x}}{\alpha^2} + 2 \cdot \frac{e^{-\alpha x}}{\alpha^3}$$

$$-\left(0^2 \cdot \frac{1}{-\alpha} \cdot e^{-\alpha \cdot 0} - 2 \cdot 0 \cdot -\frac{e^{-\alpha \cdot 0}}{\alpha^2} + 2 \cdot \frac{e^{-\alpha \cdot 0}}{\alpha^3}\right) = 2 \cdot \frac{e^{-\alpha \cdot 0}}{\alpha^3} = \frac{2}{\alpha^3}$$

Now taking the third derivative, we should start to see a pattern of the solutions

$$I'''(\alpha) = \int_0^\infty \frac{\partial}{\partial \alpha} x^2 \cdot e^{-\alpha x}\, dx$$

$$I'''(\alpha) = \int_0^\infty -x^3 \cdot e^{-\alpha x}\, dx$$

$$= \left[-x^3 \cdot \frac{1}{-\alpha} \cdot e^{-\alpha x} + 3x^2 \cdot -\frac{e^{-\alpha x}}{\alpha^2} - 6x \cdot \frac{e^{-\alpha x}}{\alpha^3} + 6 \cdot -\frac{e^{-\alpha x}}{\alpha^4}\right]_0^\infty$$

$$= \lim_{x\to\infty} -x^3 \cdot \frac{1}{-\alpha} \cdot e^{-\alpha x} + 3x^2 \cdot -\frac{e^{-\alpha x}}{\alpha^2} - 6x \cdot \frac{e^{-\alpha x}}{\alpha^3} + 6 \cdot -\frac{e^{-\alpha x}}{\alpha^4}$$

$$-\left(-0^3 \cdot \frac{1}{-\alpha} \cdot e^{-\alpha \cdot 0} + 3 \cdot 0^2 \cdot -\frac{e^{-\alpha \cdot 0}}{\alpha^2} - 6 \cdot 0 \cdot \frac{e^{-\alpha \cdot 0}}{\alpha^3} + 6 \cdot -\frac{e^{-\alpha \cdot 0}}{\alpha^4}\right)$$

$$= 6 \cdot -\frac{e^{-\alpha \cdot 0}}{\alpha^4} = -\frac{6}{\alpha^4}$$

When taking the partial derivative of the integrand and then solving the improper integral we see this pattern in the solution:

$$\frac{n!}{\alpha^{n+1}}$$

So going back and looking at the given integral from Sheldon we can use this pattern to calculate it

We found out by taking the double derivative of I we get the integral from Sheldon

$$I''(\alpha) = \int_0^\infty x^2 \cdot e^{-\alpha x}\, dx$$

Now using the pattern, we can calculate it. First, we need to define the alpha variable. By looking where the alpha variable is located, we can see that, if the variable is equal to 1 we get the original integrand so we know that alpha needs to be equal to 1

$$\alpha = 1$$

$$I''(\alpha) = \int_0^\infty x^2 \cdot e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}, where\ n\ is\ the\ exponent\ in\ x^2$$

$$I''(1) = \int_0^\infty x^2 \cdot e^{-1 \cdot x}\, dx = \frac{2!}{1^{2+1}} = \frac{2!}{1^3} = \frac{2!}{1} = 2! = 2$$

$$\int_0^\infty x^2 \cdot e^{-x} dx = 2$$

Now we have solved the improper integral as I assumed Sheldon would give to Howard instead of the indefinite integral.

The assumed integral given to Howard by Sheldon have we now solved, using the Feynman's trick as Howard told he would use to solve it.