Gaussian integral

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1 Introduction

One day, while sitting in school feeling bored, I asked a classmate if we should try proving or solving the Gaussian integral just for fun—and he was up for it. I know the Gaussian integral is pretty famous and has been solved countless times on YouTube and other platforms, but surprisingly, I had never actually seen it done. So I figured, why not take it on as a challenge.

At first, I started thinking about possible approaches. I considered doing some algebraic rewriting or maybe even using complex analysis to tackle it. I ended up going with a method I was already familiar with—rewriting it from Cartesian to polar coordinates. And yes, after solving it this way, I found out that this is actually the most common and preferred method for evaluating the Gaussian integral. So I guess I got lucky and hit on the right strategy on my first try—doubt me all you want.

2 Steps

First the Gaussian integral is given

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx \tag{1}$$

Now for the rewriting part. I once read a paper on solving integrals in which the mathematician squared the integral to turn it into a double integral, and that technique had always stuck with me. I had been wanting to try it out for a while, and now I finally had the opportunity. The Gaussian integral seemed like the perfect candidate for it, especially because of the exponential function: Euler's number raised to the power of negative x squared. It just made sense to apply the method here.

$$I^2 = (\int_{-\infty}^{\infty} e^{-x^2}\, dx)^2 \tag{2}$$

Now I just write the integral out, and since the two integrals are independent of each other, I can rename the variable in the second one from x to y.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \tag{3}$$

In Calculus III, we are introduced to multivariable calculus, which covers concepts like double and triple integrals. We then observe that when two integrals are multiplied, we can apply Fubini's Theorem to combine them into a double integral.

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy \tag{4}$$

Now, using basic algebra, we combine the products by adding the exponents.

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy \tag{5}$$

Now comes the tricky part: converting these Cartesian coordinates into polar coordinates. First we just introduce some variables

$$x = r \cdot cos(\theta)$$
$$y = r \cdot sin(\theta)$$

Now to transform the area element dxdy into polar form, we need the Jacobian determinant which measures how much area changes under the coordinate transformation.

$$J(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

now calculating each partial derivative

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r \cdot cos(\theta)) = cos(\theta)$$
$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r \cdot cos(\theta)) = -r \cdot sin(\theta)$$
$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r \cdot sin(\theta)) = sin(\theta)$$
$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r \cdot sin(\theta)) = r \cdot cos(\theta)$$

So the Jacobian matrix becomes

$$J(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Now to compute the Jacobian determinant, we find the scaling factor between dxdy and $drd\theta$, by computing the determinant of the Jacobian matrix.

$$det(J) = cos(\theta) \cdot r \cdot cos(\theta) - (-r \cdot sin(\theta)) \cdot sin(\theta) = r \cdot (cos^2(\theta) + sin^2(\theta)) = r$$

so the absolute value of the determinant is

$$|J| = r$$

Now we just replace dxdy with the polar area element

$$dxdy = |J|drd\theta = r \cdot drd\theta$$

now we need to apply the transformation to the integrand

$$x^2 + y^2 = r^2 \implies e^{-x^2-y^2} = e^{-r^2}$$

the only thing left is to set the new limits of integration. The cartesian domain is given

$$(x, y) \in R^2 \iff x \in (-\infty, \infty), y \in (-\infty, \infty)$$

where in the polar domain, every point in the plane appears exactly once if

$$r \in [0, \infty), \quad \theta \in [0, 2\pi)$$

so the double integral becomes

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} \cdot r\, dr\, d\theta \tag{6}$$

now to calculate the integral we can apply the Fubini's Theorem again to split the integrals

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} \cdot r\, dr\, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^{\infty} e^{-r^2} \cdot r\, dr \tag{7}$$

now we just calculate the first integral with theta

$$\int_0^{2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi - 0 = 2\pi \tag{8}$$

second integral, we can use u-sub to evaluate it

$$\int_0^{\infty} e^{-r^2} \cdot r\, dr \tag{9}$$

$$u = r^2$$

$$\frac{du}{dr} = 2r \implies \frac{du}{2r} = dr$$

$$\int_0^{\infty} e^{-r^2} \cdot r\, dr \implies \int_0^{\infty} e^{-u} \cdot r\, \frac{du}{2r} = \frac{1}{2} \int_0^{\infty} e^{-u}\, du = \lim_{t \to \infty} [\frac{1}{2} \cdot (-e^{-u})]_0^t$$

(10)

$$= 0 - (\frac{1}{2} \cdot (-e^0)) = 0 - (\frac{1}{2} \cdot (-1)) = 0 - (-\frac{1}{2}) = \frac{1}{2} \tag{11}$$

now combining the answers

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} \cdot r \, dr \, d\theta = 2\pi \cdot \frac{1}{2} = \pi \tag{12}$$

and since we squared the original

$$I^2 = (\int_{-\infty}^{\infty} e^{-x^2} \, dx)^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} \cdot r \, dr \, d\theta \tag{13}$$

we just take the square root of the integral to get back to the original integral

$$\sqrt{I^2} = \sqrt{(\int_{-\infty}^{\infty} e^{-x^2} \, dx)^2} = \sqrt{\int_0^{2\pi} \int_0^{\infty} e^{-r^2} \cdot r \, dr \, d\theta}$$

$$I = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\int_0^{2\pi} \int_0^{\infty} e^{-r^2} \cdot r \, dr \, d\theta} = \sqrt{\pi}$$

Therefore

$$\boxed{I = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}}$$