

# Controls Notes

M516

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This document is an ongoing collection of symbols, theorems, tools, and terms I have found useful for studying control theory, available as [PDF](#), [HTML](#), and [L<sup>A</sup>T<sub>E</sub>X source code](#).

## Variables and Symbols

$x$

- state vector (state space) ([W](#))

Type:  $\mathbb{R}^n$

$H$

- Hamiltonian matrix
- Hamiltonian (Hamiltonian mechanics) ([W](#))

type:  $\mathbb{R}^n \rightarrow \mathbb{R}$

- Assuming discrete time linear system:

$$H_k = L(x_k, u_k, k) + p_{k+1}^T f(x_k, u_k, k)$$

$\mathcal{L}$

- the Lagrangian

type:  $\mathbb{R}^n \rightarrow \mathbb{R}$

$C$

- $C^1$  = Continuously differentiable, i.e. the first derivative is continuous.
- $C^n$  = The  $n^{\text{th}}$  derivative is continuous.
- $\mathbb{C}$ : the set of all complex numbers  $a + bi$  where  $a$  and  $b$  are real and  $i = \sqrt{-1}$

$e_{\#}$

- The  $\#$ th unit vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots$$

$\nabla$  The del operator, which represents one of many long but similar operators on a vector field  $v \in \mathbb{R}^n$ .

- $\nabla f$ : Gradient of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , returning an  $n$ -dimensional vector. (W)  
This vector points in the direction of the greatest increase, and its magnitude is the slope.

For example, a mountain climber could approximate the shape of a convex mountain as a function  $f_{\text{mountain}}$  that computes the altitude given some latitude and longitude (assuming a very small mountain very far from the poles). In other words,  $f_{\text{mountain}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ . The climber could know which direction to climb to summit the peak: it's the direction  $\nabla f$ , and the grade or slope of the mountain is  $|\nabla f|$

Note that if  $n = 1$ ,  $\nabla f$  is the standard derivative of  $f$ .  
Formally speaking:

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$

- $\nabla \cdot \vec{v}$ : The divergence of a vector field  $\vec{v}$
- $\nabla \times \vec{v}$ : The curl of a vector field  $\vec{v}$
- $\Delta f$ : the Laplace operator on a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , equivalent to the divergence of the gradient of  $f$ , i.e.

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f$$

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$J$  • Cost to go function  
type:  $\mathbb{R}^n \rightarrow \mathbb{R}$

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$p$  • Lagrange multiplier (W)

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$\langle \rangle$  • Lie bracket notation (W)  
 $\langle a, b \rangle = b^T a$

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norm • Vector norm (TODO)  
• Matrix norm (TODO)  
• Functional norm (TODO)

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## Named Theorems and Conditions

Poincaré-Bendixson the-  
orem • TODO

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## Small-gain Theorem

Given

- $H_1$ : an Input-Output System with input  $e_1$  and output  $y_1$  that is finite-gain  $\mathcal{L}_p$ -stable
- $H_2$ : an Input-Output System with input  $e_2$  and output  $y_2$  that is finite-gain  $\mathcal{L}_p$ -stable
- $y_1 = H_1 e_1$
- $y_2 = H_2 e_2$
- $e_1 = u_1 - y_2$
- $e_2 = u_2 - y_1$

By the definition of finite-gain  $\mathcal{L}_p$ -stable,

$$\|y_{1\tau}\|_{\mathcal{L}_p} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}_p} + \beta_1$$

(The  $\mathcal{L}_p$  norm of the  $y_1$  is truncated by  $\tau$ , i.e. the system response is zero when  $t > \tau$ . This is less than or equal to The  $\mathcal{L}_p$  norm of the  $e_1$  truncated by  $t < \tau$ , multiplied by some gain value  $\gamma_1$ , plus some bias  $\beta_1$ )

As long as a system does not have a finite escape time, we can compute the  $\mathcal{L}_p$  norm of the system.

Likewise,

$$\|y_{2\tau}\|_{\mathcal{L}_p} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}_p} + \beta_2$$

The Small-gain Theorem tells us,

$$\begin{Bmatrix} y_{1\tau} \\ y_{2\tau} \end{Bmatrix}_{\mathcal{L}_p} \leq \frac{1}{1 - \gamma_1 \gamma_2} \left( \|u_{1\tau}\|_{\mathcal{L}_p} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}_p} + \gamma_2 \beta_1 + \beta_2 \right) = \gamma_3 \left( \text{some } \mathcal{L}_p\text{-stable system} \right) \quad (1)$$

Therefore, if  $\gamma_1$  and  $\gamma_2$  are less than one, the feedback connection is input/output stable (finite-gain  $\mathcal{L}_p$ -stable)

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## Terms

### Lipschitz Continuity

[W](#), [UC Berkley](#)

Lipschitz continuous functions are continuous and differentiable almost anywhere in a domain.

Given a domain  $D$  and a function  $f : D \rightarrow \mathbb{R}, D \in \mathbb{R}^n$ ,  
 $f$  is Lipschitz continuous if  $\exists L > 0$  such that  $|f(x) - f(y)| < L \|x - y\| \forall x, y \in D$

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### Hessian

[W](#), [Kahn Academy](#), [Wolfram](#)

- A  $2n \times 2n$  matrix of all 2nd order partial derivatives of some function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$
- The determinant of a Hessian matrix

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definite

Warning: this definition does not appear to be common outside of controls

Given a real-valued, continuously differentiable function  $V(x) : \mathbb{R} \rightarrow \mathbb{R}$   
 $V(x)$  can be classified as

- **(globally) positive semidefinite** if

$$V(x) \geq 0 \quad \forall x \in \mathbb{R}$$

( $v$  is greater than or equal to 0 regardless of  $x$ )

- **(globally) positive definite** if positive semidefinite AND

$$V(x) = 0 \iff x = 0$$

( $V(x)$  is zero if and only if  $x$  is zero)

- **(globally) negative semidefinite** if

$$V(x) \leq 0 \quad \forall x \in \mathbb{R}$$

( $v$  is less than or equal to 0 regardless of  $x$ )

- **(globally) negative definite** if negative semidefinite AND

$$V(x) = 0 \iff x = 0$$

( $V(x)$  is zero if and only if  $x$  is zero)

- **locally positive definite (l.p.d)** if

$$V(x) \geq 0 \quad \forall x \in N$$

where  $N$  is a small open neighborhood containing  $\vec{0}$

( $v$  is greater than or equal to 0 regardless of  $x$  in some small open neighborhood  $N$  that contains the zero vector)

**AND**

$$V(x) = 0 \iff x = 0$$

( $V(x)$  is zero if and only if  $x$  is zero)

Note that the criteria for a function to be locally positive definite are similar, but more relaxed than, those for globally positive definite functions.

- **positive definite on some domain**  $D \in \mathbb{R}^n$  if  
 we only care if the conditions for positive definite functions hold for all  $x$  in  $D$ .

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#### Stability

- (Lyapunov) stability (TODO)
- Asymptotic stability (TODO)
- Exponential stability (TODO)
- Uniform stability (TODO)
- Global stability (TODO)
- L-stability (TODO)
- I/O L-stability (TODO)
- Small-signal I/O L-stability (TODO)
- Small-signal finite-gain L-stability (TODO)

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#### Class K function

- (TODO)
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Radially  
function

Unbounded

- (TODO)

sup (supremum)

Like a maximum of a functions, but includes limits that aren't necessarily a part of the domain of the function. (TODO)

Hurwitz

- **Hurwitz (polynomial):**

A polynomial whose roots that are all in the left-half plane. (In other words, the real part of every root is strictly negative)

- **Hurwitz (matrix) (W):**

A square matrix whose characteristic polynomial is Hurwitz, meaning all eigenvalues are in the left-half plane. (In other words, the real part of every eigenvalue is strictly negative)

- **Routh-Hurwitz stability criterion (IEEE):**

TODO

Any hyperbolic fixed point (or equilibrium point) of a continuous dynamical system is locally asymptotically stable if and only if the Jacobian of the dynamical system is Hurwitz stable at the fixed point.

A system is stable if its control matrix is a Hurwitz matrix.

The negative real components of the eigenvalues of the matrix represent negative feedback. Similarly, a system is inherently unstable if any of the eigenvalues have positive real components, representing positive feedback.

Zero-state observable

A time-invariant system of the form

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$$

is zero-state observable if

$$\begin{cases} y \equiv 0 \\ u \equiv 0 \end{cases} \implies x \equiv 0$$

In other words, when  $u = 0$ , any nonzero state behavior will be observed at the output ( $y \neq 0$ )