

Controls Notes

M516

May 1, 2023

This document is an ongoing collection of symbols, theorems, tools, and terms I have found useful for studying control theory, available as [PDF](#), [HTML](#), and [L^AT_EX source code](#).

Variables and Symbols

- x
- state vector (state space) ([W](#))
- Type: \mathbb{R}^n

- H
- Hamiltonian matrix
 - Hamiltonian (Hamiltonian mechanics) ([W](#))
- type: $\mathbb{R}^n \rightarrow \mathbb{R}$
- Assuming discrete time linear system:

$$H_k = L(x_k, u_k, k) + p_{k+1}^T f(x_k, u_k, k)$$

- \mathcal{L}
- the Lagrangian
- type: $\mathbb{R}^n \rightarrow \mathbb{R}$

- B
- A ball, defined as

$$B(x_0, \epsilon) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq \epsilon\}$$

- C
- C^1 = Continuously differentiable, i.e. the first derivative is continuous.
 - C^n = The n^{th} derivative is continuous.
 - \mathbb{C} : the set of all complex numbers $a + bi$ where a and b are real and $i = \sqrt{-1}$

- $e_{\#}$
- The $\#$ th unit vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots$$

∇ The del operator, which represents one of many long but similar operators on a vector field $v \in \mathbb{R}^n$.

- ∇f : Gradient of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, returning an n -dimensional vector. (W)
This vector points in the direction of the greatest increase, and its magnitude is the slope.

For example, a mountain climber could approximate the shape of a convex mountain as a function f_{mountain} that computes the altitude given some latitude and longitude (assuming a very small mountain very far from the poles). In other words, $f_{\text{mountain}}: \mathbb{R}^2 \rightarrow \mathbb{R}$. The climber could know which direction to climb to summit the peak: it's the direction ∇f , and the grade or slope of the mountain is $|\nabla f|$

Note that if $n = 1$, ∇f is the standard derivative of f .

Formally speaking:

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$

- $\nabla \cdot \vec{v}$: The divergence of a vector field \vec{v}
- $\nabla \times \vec{v}$: The curl of a vector field \vec{v}
- Δf : the Laplace operator on a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, equivalent to the divergence of the gradient of f , i.e.

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f$$

J • Cost to go function
type: $\mathbb{R}^n \rightarrow \mathbb{R}$

p • Lagrange multiplier (W)

$\langle \text{expr} \rangle$ • Lie bracket notation (W)
 $\langle a, b \rangle = b^T a$

$\|\text{expr}\|$ • Vector norm (TODO)
• Matrix norm (TODO)
• Functional norm (TODO)
• **Norm of a system** $y(u, t) = h(u(t))$, where y is an n -dimensional the output of the system, u is an m -dimensional control vector (TODO)

Named Theorems and Conditions

Poincaré-Bendixson the- • TODO
orem

Small-gain Theorem

Given

- H_1 : an Input-Output System with input e_1 and output y_1 that is finite-gain \mathcal{L}_p -stable
- H_2 : an Input-Output System with input e_2 and output y_2 that is finite-gain \mathcal{L}_p -stable
- $y_1 = H_1 e_1$
- $y_2 = H_2 e_2$
- $e_1 = u_1 - y_2$
- $e_2 = u_2 - y_1$

By the definition of finite-gain \mathcal{L}_p -stable,

$$\|y_{1\tau}\|_{\mathcal{L}_p} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}_p} + \beta_1$$

(The \mathcal{L}_p norm of the y_1 is truncated by τ , i.e. the system response is zero when $t > \tau$. This is less than or equal to The \mathcal{L}_p norm of the e_1 truncated by $t < \tau$, multiplied by some gain value γ_1 , plus some bias β_1)

As long as a system does not have a finite escape time, we can compute the \mathcal{L}_p norm of the system.

Likewise,

$$\|y_{2\tau}\|_{\mathcal{L}_p} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}_p} + \beta_2$$

The Small-gain Theorem tells us,

$$\begin{Bmatrix} y_{1\tau} \\ y_{2\tau} \end{Bmatrix}_{\mathcal{L}_p} \leq \frac{1}{1 - \gamma_1 \gamma_2} \left(\|u_{1\tau}\|_{\mathcal{L}_p} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}_p} + \gamma_2 \beta_1 + \beta_2 \right) = \gamma_3 \left(\text{some } \mathcal{L}_p\text{-stable system} \right) \quad (1)$$

Therefore, if γ_1 and γ_2 are less than one, the feedback connection is input/output stable (finite-gain \mathcal{L}_p -stable)

Terms

Classes of Systems Given a dynamic system

- **Dynamic system**

$$\dot{x} = f(x, u, t)$$

- **Time-invariant system** is a dynamic system

$$\dot{x} = f(x, u)$$

- **Autonomous system** is a dynamic system

$$\dot{x} = f(x, t)$$

- **Linear system** ([W](#)) is a dynamic system

$$\dot{x} = f(x, u, t) = A(t)x + B(t)u$$

- **Linear time-invariant system** is a linear system and a time-invariant system

$$\dot{x} = f(x, u) = Ax + Bu$$

Lipschitz Continuity
 (W,) (UC Berkley)

Lipschitz continuous functions are continuous and differentiable almost anywhere in a domain.

Given a domain D and a function $f : D \rightarrow \mathbb{R}, D \in \mathbb{R}^n$,
 f is Lipschitz continuous if $\exists L > 0$ such that $|f(x) - f(y)| < L||x - y|| \forall x, y \in D$

Hessian
 (W), (Kahn Academy),
 (Wolfram)

- A $n \times n$ matrix of all 2nd order partial derivatives of some function $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$Hf(\vec{x}) = f''(\vec{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \quad (2)$$

- The determinant of a Hessian matrix
-

definite (W) *Warning: this definition does not appear to be common outside of controls*

Given a real-valued, continuously differentiable function $V(x) : \mathbb{R} \rightarrow \mathbb{R}$
 $V(x)$ can be classified as

- **(globally) positive semidefinite** if

$$V(x) \geq 0 \quad \forall x \in \mathbb{R}$$

(v is greater than or equal to 0 regardless of x)

- **(globally) positive definite** if positive semidefinite AND

$$V(x) = 0 \iff x = 0$$

(V(x) is zero if and only if x is zero)

- **(globally) negative semidefinite** if

$$V(x) \leq 0 \quad \forall x \in \mathbb{R}$$

(v is less than or equal to 0 regardless of x)

- **(globally) negative definite** if negative semidefinite AND

$$V(x) = 0 \iff x = 0$$

(V(x) is zero if and only if x is zero)

- **locally positive definite (l.p.d)** if

$$V(x) \geq 0 \quad \forall x \in N$$

where N is a small open neighborhood containing $\vec{0}$

(v is greater than or equal to 0 regardless of x in some small open neighborhood N that contains the zero vector)

AND

$$V(x) = 0 \iff x = 0$$

(V(x) is zero if and only if x is zero)

Note that the criteria for a function to be locally positive definite are similar, but more relaxed than, those for globally positive definite functions.

- **positive definite on some domain** $D \in \mathbb{R}^n$ if

we only care if the conditions for positive definite functions hold for all x in D .

Given an autonomous system

$$\dot{x} = f(x, t)$$

and some open connected region \mathcal{D} containing $\vec{0}$

Stability is usually used to describe trajectories around the origin of a system.

- **Stability**

The equilibrium point $x = 0$ is stable if $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \implies \|x(t)\| < \epsilon$

- **In the sense of Lyapunov**

If there exists a scalar, continuously-differentiable function $V(x)$ such that

$$V(x) > 0 \quad \forall x \in \mathcal{D} \setminus \{\vec{0}\}, \quad V(\vec{0}) = 0$$

($V(x)$ is a locally positive definite function)

AND

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \forall x \in \mathcal{D} \setminus \{\vec{0}\}, \quad V(\vec{0}) = 0$$

($\dot{V}(x)$ is a locally negative semidefinite function)

then the origin is stable in the sense of Lyapunov, and $V(x)$ is a Lyapunov function of $f(x)$.

- **Instability**

The equilibrium point $x = 0$ is unstable if it is not stable

- **Asymptotic stability**

The equilibrium point $x = 0$ is asymptotically stable if it is stable and $\exists \delta_1$ such that $\|x(0)\| < \delta_1 \implies \lim_{t \rightarrow \infty} x(t) = 0$

- **In the sense of Lyapunov**

The origin is asymptotically stable in the sense of Lyapunov if stable AND

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \quad \forall x \in \mathcal{D} \setminus \{\vec{0}\}$$

($\dot{V}(x)$ is a locally negative definite function)

- **Exponential stability**

- **In the sense of Lyapunov**

The origin is exponentially stable in the sense of Lyapunov if stable AND

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -\alpha V(x) \quad \forall x \in \mathcal{D} \setminus \{\vec{0}\}$$

- **Uniform stability**

The equilibrium point $x = 0$ is uniformly stable if it is stable and, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, independent of t_0 .

Stability
(continued)

- **Global asymptotic stability**

- **In the sense of Lyapunov**

If the origin is globally asymptotically stable in the sense of Lyapunov if is asymptotically stable and

$$\|x\| \rightarrow \infty \implies V(x) \rightarrow \infty$$

($V(x)$ is radially unbounded)

- **L-stability**
(TODO)
- **I/O L-stability**
(TODO)
- **Finite-gain L-stability**
(TODO)
- **Small-signal I/O L-stability**
(TODO)
- **Small-signal finite-gain L-stability**
(TODO)

Class κ A continuous scalar function on \mathbf{R}^+ is
function

- **class κ** if it is:
 - zero at zero
 - strictly increasing
 - continuous
- **class κ_∞** if it is:
 - zero at zero
 - strictly increasing
 - continuous
 - ∞ at ∞

Radially
Unbounded
function

A function $V(x)$ is radially unbounded if

$$\|x\| \rightarrow \infty \implies \|V(x)\| \rightarrow \infty$$

sup (supre-
mum)

Like a maximum of a functions, but includes limits that aren't necessarily a part of the domain of the function.
(TODO)

Hurwitz

- **Hurwitz (polynomial):**
A polynomial whose roots that are all in the left-half plane. (In other words, the real part of every root is strictly negative)
- **Hurwitz (matrix) (W):**
A square matrix whose characteristic polynomial is Hurwitz, meaning all eigenvalues are in the left-half plane. (In other words, the real part of every eigenvalue is strictly negative)
- **Routh-Hurwitz stability criterion (IEEE):**
TODO

Any hyperbolic fixed point (or equilibrium point) of a continuous dynamical system is locally asymptotically stable if and only if the Jacobian of the dynamical system is Hurwitz stable at the fixed point.

A system is stable if its control matrix is a Hurwitz matrix.

The negative real components of the eigenvalues of the matrix represent negative feedback. Similarly, a system is inherently unstable if any of the eigenvalues have positive real components, representing positive feedback.

Zero-state
observable

A time-invariant system of the form

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$$

is zero-state observable if

$$\begin{cases} y \equiv 0 \\ u \equiv 0 \end{cases} \implies x \equiv 0$$

In other words, when $u = 0$, any nonzero state behavior will be observed at the output ($y \neq 0$)

Sets

- **Invariant Set**

A set of vectors M is invariant with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M, \quad \forall t \in \mathbb{R}$$

(if a solution belongs to M at some time instant, then it belongs to M for all future and past time)

- **Positively Invariant Set**

A set of vectors M is positively invariant with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M, \quad \forall t \geq 0$$

(if a solution belongs to M at some time instant, then it belongs to M for all future time)

- **Open Set**

A set $D \subset \mathbb{R}^n$ (D , which is a set of real vectors) is an **open set** if

$$\forall x \in D, \quad \exists \epsilon > 0 \quad \text{such that} \quad B(x, \epsilon) \subset D$$

(for all vectors x in the domain D , there exists a real scalar ϵ such that we can create a ball around x with radius ϵ , and that whole ball is in D)

- **Closed Set**

A set $D \subset \mathbb{R}^n$ (D , which is a set of real vectors) is a **closed set** if

$$\mathbb{R}^n \setminus D \quad \text{is an open set}$$

(everywhere outside of D is open)

- **Bounded Set**

A set $D \subset \mathbb{R}^n$ (D , which is a set of real vectors) is a **bounded set** if

$$\exists \epsilon > 0 \quad \text{such that} \quad D \subset B(0, \epsilon)$$

(D fits in a ball with a finite, constant radius ϵ)

- **Compact Set**

A set $D \subset \mathbb{R}^n$ (D , which is a set of real vectors) is a **compact set** if it is closed and bounded.

Passivity

For a system $y = h(u, t)$, $h : \mathbb{R}^m \times [0, \infty) \rightarrow \mathbb{R}^n$

(output state y (an n -dimensional vector) is a function of the input state u (an m -dimensional vector) and time t)

- **Invariant Set**

A set of vectors M is invariant with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M, \quad \forall t \in \mathbb{R}$$

(if a solution belongs to M at some time instant, then it belongs to M for all future and past time)

Adjoint

- The **{adjoint or Hermitian transpose}** of a matrix A ([Wolfram](#)) is its conjugate transpose, denoted as A' , A^* , A^H , or A^\dagger i.e.

$$A^H = \overline{A}^T$$

Interesting properties of adjoint matrices:

- $A^H = \overline{A}^T = \overline{A^T}$
- If a matrix is its own conjugate transpose, that matrix is called **self-adjoint** or **Hermetian**
- If A is a real matrix, $A^H = A^T$

Warning: In some older literature, the "adjoint of a matrix" may mean the **adjunct matrix of a square matrix** ([W](#))

- The **adjoint representation of a vector space** ([Wolfram](#))
(TODO)
 - The **adjoint equation** ([W](#))
(TODO)
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