Controls Notes

M516

May 1, 2023

This document is an ongoing collection of symbols, theorems, tools, and terms I have found useful for studying control theory, available as PDF, HTML, and LaTeXsource code.

Variables and Symbols

x	• state vector (state space) (W) Type: $\mathbb{R}^n$
H	• Hamiltonian matrix • Hamiltonian (Hamiltonian mechanics) (W) type: $\mathbb{R}^n \to \mathbb{R}$ – Assuming discrete time linear system: $$H_k = L(x_k, u_k, k) + p_{k+1}^T f(x_k, u_k, k)$$
$\mathcal{L}$	• the Lagrangian type: $\mathbb{R}^n \to \mathbb{R}$
B	• A ball, defined as $$B(x_0, \epsilon) = \{x \in \mathbb{R}^n : \lVert x - x_0 \rVert \leq \epsilon\}$$
C	• C^1 = Continuously differentiable, i.e. the first derivative is continuous. • C^n = The n^{th} derivative is continuous. • $\mathbb{C}$: the set of all complex numbers $a + bi$ where a and b are real and $i = \sqrt{-1}$
$e_\#$	• The #th unit vector $$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots$$

∇ The del operator, which represents one of many long but similar operators on a vector field $v \in \mathbb{R}^n$.

- ∇f: Gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$, returning an n-dimensional vector. (W)
  This vector points in the direction of the greatest increase, and its magnitude is the slope.

  For example, a mountain climber could approximate the shape of a convex mountain as a function $f_{mountain}$ that computes the altitude given some latitude and longitude (assuming a very small mountain very far from the poles). In other words, $f_{mountain} : \mathbb{R}^2 \to \mathbb{R}$. The climber could know which direction to climb to summit the peak: it's the direction ∇f, and the grade or slope of the mountain is $|\nabla f|$

  Note that if $n = 1$, ∇f is the standard derivative of f.
  Formally speaking:

  $$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i$$

- $\nabla \cdot \vec{v}$: The divergence of a vector field $\vec{v}$
- $\nabla \times \vec{v}$: The curl of a vector field $\vec{v}$
- Δf: the Laplace operator on a function $f : \mathbb{R}^n \to \mathbb{R}$, equivalent to the divergence of the gradient of f, i.e.

  $$\delta f = \nabla^2 f = \nabla \cdot \nabla f$$

J

- Cost to go function
  type: $\mathbb{R}^n \to \mathbb{R}$

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p

- Lagrange multiplier (W)

$\langle \text{expr} \rangle$

- Lie bracket notation (W)
  $\langle a, b \rangle = b^T a$

$||\text{expr}||$

- Vector norm (TODO)
- Matrix norm (TODO)
- Functional norm (TODO)
- **Norm of a system** $y(u, t) = h(u(t))$, where y is an n-dimensional the output of the system, u is an m-dimensional control vector (TODO)

Named Theorems and Conditions

Poincaré-Bendixson theorem

- TODO

Small-gain Theorem

Given

- H_1: an Input-Output System with input e_1 and output y_1 that is finite-gain $\mathcal{L}_p$-stable
- H_2: an Input-Output System with input e_2 and output y_2 that is finite-gain $\mathcal{L}_p$-stable
- $y_1 = H_1 e_1$
- $y_2 = H_2 e_2$
- $e_1 = u_1 - y_2$
- $e_2 = u_2 - y_1$

By the definition of finite-gain $\mathcal{L}_p$-stable,

$$\|y_{1\tau}\|_{\mathcal{L}_p} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}_p} + \beta_1$$

(The $\mathcal{L}_p$ norm of the y_1 is truncated by τ, i.e. the system response is zero when $t > \tau$. This is less than or equal to The $\mathcal{L}_p$ norm of the e_1 truncated by $t < \tau$, multiplied by some gain value γ_1, plus some bias β_1)

As long as a system does not have a finite escape time, we can compute the $\mathcal{L}_p$ norm of the system.

Likewise,

$$\|y_{2\tau}\|_{\mathcal{L}_p} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}_p} + \beta_2$$

The Small-gain Theorem tells us,

$$\left\| \begin{matrix} y_{1\tau} \\ y_{2\tau} \end{matrix} \right\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_1\gamma_2} \left(\|u_{1\tau}\|_{\mathcal{L}_p} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}_p} + \gamma_2\beta_1 + \beta_2 \right) = \gamma_3 \left(\text{some } \mathcal{L}_p\text{-stable system} \right) \quad (1)$$

Therefore, if γ_1 and γ_2 are less than one, the feedback connection is input/output stable (finite-gain $\mathcal{L}_p$-stable)

Terms

Classes of Systems

Given a dynamic system

- **Dynamic system**

$$\dot{x} = f(x, u, t)$$

- **Time-invariant system** is a dynamic system

$$\dot{x} = f(x, u)$$

- **Autonomous system** is a dynamic system

$$\dot{x} = f(x, t)$$

- **Linear system** (W) is a dynamic system

$$\dot{x} = f(x, u, t) = A(t)x + B(t)u$$

- **Linear time-invariant system** is a linear system and a time-invariant system

$$\dot{x} = f(x, u) = Ax + Bu$$

Lipschitz Continuity (W,) (UC Berkley)	Lipschitz continuous functions are continuous and differentiable almost anywhere in a domain. Given a domain D and a function $f : D \rightarrow \mathbb{R}, D \in \mathbb{R}^n$, f is Lipschitz continuous if $\exists L > 0$ such that $\lvert f(x) - f(y) \rvert < L\lvert\lvert(x-y)\rvert\rvert \forall x, y \in D$
Hessian (W), (Kahn Academy), (Wolfram)	• A n x n matrix of all 2nd order partial derivatives of some function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ $$Hf(\vec{x}) = f''(\vec{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \quad (2)$$ • The determinant of a Hessian matrix

definite (W) *Warning: this definition does not appear to be common outside of controls*

Given a real-valued, continuously differentiable function $V(x) : \mathbb{R} \to \mathbb{R}$
$V(x)$ can be classified as

- **(globally) positive semidefinite** if

$$V(x) \geq 0 \qquad \forall x \in \mathbb{R}$$

(v is greater than or equal to 0 regardless of x)

- **(globally) positive definite** if positive semidefinite AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

- **(globally) negative semidefinite** if

$$V(x) \leq 0 \qquad \forall x \in \mathbb{R}$$

(v is less than or equal to 0 regardless of x)

- **(globally) negative definite** if negative semidefinite AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

- **locally positive definite (l.p.d)** if

$$V(x) \geq 0 \qquad \forall x \in N$$

  where N is a small open neighborhood containing $\vec{0}$

  (v is greater than or equal to 0 regardless of x in some small open neighborhood N that contains the zero vector)

  AND

$$V(x) = 0 \iff x = 0$$

  ($V(x)$ is zero if and only if x is zero)

  Note that the criteria for a function to be locally positive definite are similar, but more relaxed than, those for globally positive definite functions.

- **positive definite on some domain $D \in \mathbb{R}^n$** if
  we only care if the conditions for positive definite functions hold for all x in D.

Stability ([MIT]) Given an autonomous system

$$\dot{x} = f(x,t)$$

and some open connected region $\mathcal{D}$ containing $\vec{0}$

Stability is usually used to describe trajectories around the origin of a system.

- **Stability**
  The equilibrium point $x = 0$ is stable if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \implies \|x(t)\| < \epsilon$
  - **In the sense of Lyapunov**
    If there exists a scalar, continuously-differentiable function $V(x)$ such that

    $$V(x) > 0 \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}, \qquad V(\vec{0}) = 0$$

    ($V(x)$ is a locally positive definite function)

    AND

    $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}, \qquad V(\vec{0}) = 0$$

    ($\dot{V}(x)$ is a locally negative semidefinite function)

    then the origin is stable in the sense of Lyapunov, and $V(x)$ is a Lyapunov function of $f(x)$.

- **Instability**
  The equilibrium point $x = 0$ is unstable if it is not stable

- **Asymptotic stability**
  The equilibrium point $x = 0$ is asymptotically stable if it is stable and $\exists \delta_1$ such that $\|x(0)\| < \delta_1 \implies \lim_{t \to \infty} x(t) = 0$
  - **In the sense of Lyapunov**
    The origin is asymptotically stable in the sense of Lyapunov if stable AND

    $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}$$

    ($\dot{V}(x)$ is a locally negative definite function)

- **Exponential stability**
  - **In the sense of Lyapunov**
    The origin is exponentially stable in the sense of Lyapunov if stable AND

    $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -\alpha V(x) \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}$$

- **Uniform stability**
  The equilibrium point $x = 0$ is uniformly stable if it is stable and, for each $epsilon > 0$, there exists a $\delta(\epsilon) > 0$, independent of t_0.

Stability (continued)

- **Global asymptotic stability**
  - **In the sense of Lyapunov**
    If the origin is globally asymptotically stable in the sense of Lyapunov if is asymptotically stable and

$$\|x\| \to \infty \implies V(x) \to \infty$$

($V(x)$ is radially unbounded)

- **L-stability**
  (TODO)
- **I/O L-stability**
  (TODO)
- **Finite-gain L-stability**
  (TODO)
- **Small-signal I/O L-stability**
  (TODO)
- **Small-signal finite-gain L-stability**
  (TODO)

Class κ function

A continuous scalar function on $\mathbf{R}^+$ is

- **class** κ if it is:
  - zero at zero
  - strictly increasing
  - continuous
- **class** κ_∞ if it is:
  - zero at zero
  - strictly increasing
  - continuous
  - ∞ at ∞

Radially Unbounded function

A function $V(x)$ is radially unbounded if

$$\|x\| \to \infty \implies \|V(x)\| \to \infty$$

$\sup$ (supremum)

Like a maximum of a functions, but includes limits that aren't necessarily a part of the domain of the function. (TODO)

Hurwitz

- **Hurwitz (polynomial)**:
  A polynomial whose roots that are all in the left-half plane. (In other words, the real part of every root is strictly negative)
- **Hurwitz (matrix)** (W):
  A square matrix whose characteristic polynomial is Hurwitz, meaning all eigenvalues are in the left-half plane. (In other words, the real part of every eigenvalue is strictly negative)
- **Routh-Hurwitz stability criterion** (IEEE):
  TODO

Any hyperbolic fixed point (or equilibrium point) of a continuous dynamical system is locally asymptotically stable if and only if the Jacobian of the dynamical system is Hurwitz stable at the fixed point.
A system is stable if its control matrix is a Hurwitz matrix.
The negative real components of the eigenvalues of the matrix represent negative feedback. Similarly, a system is inherently unstable if any of the eigenvalues have positive real components, representing positive feedback.

Zero-state observable

A time-invariant system of the form

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

is zero-state observable if

$$\begin{cases} y \equiv 0 \\ u \equiv 0 \end{cases} \implies x \equiv 0$$

In other words, when $u = 0$, any nonzero state behavior will be observed at the output ($y \neq 0$)

Sets

- **Invariant Set**
  A set of vectors M is invariant with respect to $\dot{x} = f(x)$ if

  $$x(0) \in M \implies x(t) \in M, \qquad \forall t \in \mathbb{R}$$

  (if a solution belongs to M at some time instant, then it belongs to M for all future and past time)

- **Positively Invariant Set**
  A set of vectors M is positively invariant with respect to $\dot{x} = f(x)$ if

  $$x(0) \in M \implies x(t) \in M, \qquad \forall t \geq 0$$

  (if a solution belongs to M at some time instant, then it belongs to M for all future time)

- **Open Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is an **open set** if

  $$\forall x \subset D, \quad \exists \epsilon > 0 \quad \text{such that} \quad B\left(x, \epsilon\right) \subset D$$

  (for all vectors x in the domain D, there exists a real scalar ϵ such that we can create a ball around x with radius ϵ, and that whole ball is in D)

- **Closed Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is a **closed set** if

  $$\mathbb{R}^n \setminus D \quad \text{is an open set}$$

  (everywhere outside of D is open)

- **Bounded Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is a **bounded set** if

  $$\exists \epsilon > 0 \quad \text{such that} \quad D \subset B\left(0, \epsilon\right)$$

  (D fits in a ball with a finite, constant radius ϵ)

- **Compact Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is a **compact set** if it is closed and bounded.

Passivity

For a system $y = h(u, t)$, $\quad h : \mathbb{R}^m \times [0, \infty) \to \mathbb{R}^n$
(output state y (an n-dimensional vector) is a function of the input state u (an m-dimensional vector) and time t)

- **Invariant Set**
  A set of vectors M is invariant with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M, \qquad \forall t \in \mathbb{R}$$

(if a solution belongs to M at some time instant, then it belongs to M for all future and past time)

Adjoint

- The $\{$**adjoint or Hermitian transpose**$\}$ **of a matrix** A (Wolfram) is its conjugate transpose, denoted as A', A^*, A^H, or $A^\dagger$ i.e.

$$A^H = \overline{A}^T$$

  Interesting properties of ajoint matrices:
  - $A^H = \overline{A}^T = \overline{A^T}$
  - If a matrix is its own conjugate transpose, that matrix is called **self-adjoint** or **Hermetian**
  - If A is a real matrix, $A^H = A^T$

  Warning: In some older literature, the "adjoint of a matrix" may mean the **adjunct matrix of a square matrix** (W)
- The **adjoint representation of a vector space** (Wolfram)
  (TODO)
- The **adjoint equation** (W)
  (TODO)
