Control Theory Quick Reference

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The study of many kinds of systems, control policies, matrices, and vector spaces (including functions) comprises control theory. This document is a free, community-driven growing dictionary of these terms, designed for quickly finding more information about them, rather than hunting down definitions online or in a textbook.

It is available in a few formats: PDF, HTML, and LATEXsource code.

Contributions are not just welcome, they are necessary to keep this project going strong. If you find potential for this project to help someone else, please consider generously donating 15 minutes of your time to describe one term that might confuses someone else, refine an inaccurate/incorrect definition, or fix that one design issue that bothers you. Little contributions like these will add up to make a lasting impact on control theory studies.

Here are a few ways to donate to this project:

- Submit an issue that describes your proposed changes.
- Fork this repo, modify the code as desired, and submit a pull request.
- Email me to gain access to the GitHub repo and/or its corresponding Overleaf project.

Variables and Symbols

x

- state vector (state space) (W)
  Type: $\mathbb{R}^n$

H

- Hamiltonian matrix
- Hamiltonian (Hamiltonian mechanics) (W)
  type: $\mathbb{R}^n \rightarrow \mathbb{R}$
  - Assuming discrete time linear system:

$$H_k = L(x_k, u_k, k) + p_{k+1}^T f(x_k, u_k, k)$$

$\mathcal{L}$

- the Lagrangian
  type: $\mathbb{R}^n \rightarrow \mathbb{R}$

B

- A ball, defined as

$$B(x_0, \epsilon) = \left\{ x \in \mathbb{R}^n : \|x - x_0\| \leq \epsilon \right\}$$

C

- C^1 = Continuously differentiable, i.e. the first derivative is continuous.
- C^n = The n^{th} derivative is continuous.
- $\mathbb{C}$: the set of all complex numbers $a + bi$ where a and b are real and $i = \sqrt{-1}$

$e_\#$

- The #th unit vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots$$

∇

The del operator, which represents one of many long but similar operators on a vector field $v \in \mathbb{R}^n$.

- ∇f: Gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$, returning an n-dimensional vector. (W)
  This vector points in the direction of the greatest increase, and its magnitude is the slope.

  For example, a mountain climber could approximate the shape of a convex mountain as a function $f_{mountain}$ that computes the altitude given some latitude and longitude (assuming a very small mountain very far from the poles). In other words, $f_{mountain} : \mathbb{R}^2 \to \mathbb{R}$. The climber could know which direction to climb to summit the peak: it's the direction ∇f, and the grade or slope of the mountain is $|\nabla f|$

  Note that if $n = 1$, ∇f is the standard derivative of f.
  Formally speaking:

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i$$

- $\nabla \cdot \vec{v}$: The divergence of a vector field $\vec{v}$
- $\nabla \times \vec{v}$: The curl of a vector field $\vec{v}$
- Δf: the Laplace operator on a function $f : \mathbb{R}^n \to \mathbb{R}$, equivalent to the divergence of the gradient of f, i.e.

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f$$

J

- Cost to go function
  type: $\mathbb{R}^n \to \mathbb{R}$

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p

- Lagrange multiplier (W)

$\langle$expr$\rangle$

- Lie bracket notation (W)
  $\langle a, b \rangle = b^T a$

$||$expr$||$

- Vector norm (TODO)
- Matrix norm (TODO)
- Functional norm (TODO)
- **Norm of a system** $y(u,t) = h(u(t))$, where y is an n-dimensional the output of the system, u is an m-dimensional control vector (TODO)

Named Theorems and Conditions

Poincaré-Bendixson theorem	• TODO
Small-gain Theorem	Given • H_1: an Input-Output System with input e_1 and output y_1 that is finite-gain $\mathcal{L}_p$-stable • H_2: an Input-Output System with input e_2 and output y_2 that is finite-gain $\mathcal{L}_p$-stable • $y_1 = H_1 e_1$ • $y_2 = H_2 e_2$ • $e_1 = u_1 - y_2$ • $e_2 = u_2 - y_1$ By the definition of finite-gain $\mathcal{L}_p$-stable, $$\|y_{1\tau}\|_{\mathcal{L}_p} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}_p} + \beta_1$$ *(The $\mathcal{L}_p$ norm of the y_1 is truncated by τ, i.e. the system response is zero when $t > \tau$. This is less than or equal to The $\mathcal{L}_p$ norm of the e_1 truncated by $t < \tau$, multiplied by some gain value γ_1, plus some bias β_1)* As long as a system does not have a finite escape time, we can compute the $\mathcal{L}_p$ norm of the system. Likewise, $$\|y_{2\tau}\|_{\mathcal{L}_p} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}_p} + \beta_2$$ The Small-gain Theorem tells us, $$\left\| \begin{matrix} y_{1\tau} \\ y_{2\tau} \end{matrix} \right\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_1\gamma_2} \left(\|u_{1\tau}\|_{\mathcal{L}_p} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}_p} + \gamma_2\beta_1 + \beta_2 \right) = \gamma_3 \left(\text{some } \mathcal{L}_p\text{-stable system} \right) \quad (1)$$ Therefore, if γ_1 and γ_2 are less than one, the feedback connection is input/output stable (finite-gain $\mathcal{L}_p$-stable)

Terms

Classes of Systems	Given a dynamic system • **Dynamic system** $$\dot{x} = f(x, u, t)$$ • **Time-invariant system** is a dynamic system $$\dot{x} = f(x, u)$$ • **Autonomous system** is a dynamic system $$\dot{x} = f(x, t)$$ • **Linear system** (W) is a dynamic system $$\dot{x} = f(x, u, t) = A(t)x + B(t)u$$ • **Linear time-invariant system** is a linear system and a time-invariant system $$\dot{x} = f(x, u) = Ax + Bu$$
Lipschitz Continuity (W,) (UC Berkley)	Lipschitz continuous functions are continuous and differentiable almost anywhere in a domain. Given a domain D and a function $f : D \to \mathbb{R}, D \in \mathbb{R}^n$, f is Lipschitz continuous if $\exists L > 0$ such that $\lvert f(x) - f(y) \rvert < L \lvert\lvert (x-y) \rvert\rvert \forall x, y \in D$
Hessian (W), (Kahn Academy), (Wolfram)	• A n x n matrix of all 2nd order partial derivatives of some function $f : \mathbb{R}^n \to \mathbb{R}$ $$Hf(\vec{x}) = f''(\vec{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \tag{2}$$ • The determinant of a Hessian matrix

definite (W) *Warning: this definition does not appear to be common outside of controls*

Given a real-valued, continuously differentiable function $V(x) : \mathbb{R} \to \mathbb{R}$
$V(x)$ can be classified as

- **(globally) positive semidefinite** if

$$V(x) \geq 0 \qquad \forall x \in \mathbb{R}$$

(v is greater than or equal to 0 regardless of x)

- **(globally) positive definite** if positive semidefinite AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

- **(globally) negative semidefinite** if

$$V(x) \leq 0 \qquad \forall x \in \mathbb{R}$$

(v is less than or equal to 0 regardless of x)

- **(globally) negative definite** if negative semidefinite AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

- **locally positive definite (l.p.d)** if

$$V(x) \geq 0 \qquad \forall x \in N$$

  where N is a small open neighborhood containing $\vec{0}$

  (v is greater than or equal to 0 regardless of x in some small open neighborhood N that contains the zero vector)

  AND

$$V(x) = 0 \iff x = 0$$

  ($V(x)$ is zero if and only if x is zero)

  Note that the criteria for a function to be locally positive definite are similar, but more relaxed than, those for globally positive definite functions.

- **positive definite on some domain** $D \in \mathbb{R}^n$ if
  we only care if the conditions for positive definite functions hold for all x in D.

Stability (MIT)

Given an autonomous system

$$\dot{x} = f(x, t)$$

and some open connected region $\mathcal{D}$ containing $\vec{0}$

Stability is usually used to describe trajectories around the origin of a system.

- **Stability**
  The equilibrium point $x = 0$ is stable if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \implies \|x(t)\| < \epsilon$
  - **In the sense of Lyapunov**
    If there exists a scalar, continuously-differentiable function $V(x)$ such that

    $$V(x) > 0 \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}, \qquad V(\vec{0}) = 0$$

    ($V(x)$ is a locally positive definite function)

    AND

    $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}, \qquad V(\vec{0}) = 0$$

    ($\dot{V}(x)$ is a locally negative semidefinite function)

    then the origin is stable in the sense of Lyapunov, and $V(x)$ is a Lyapunov function of $f(x)$.

- **Instability**
  The equilibrium point $x = 0$ is unstable if it is not stable

- **Asymptotic stability**
  The equilibrium point $x = 0$ is asymptotically stable if it is stable and $\exists \delta_1$ such that $\|x(0)\| < \delta_1 \implies \lim_{t \to \infty} x(t) = 0$
  - **In the sense of Lyapunov**
    The origin is asymptotically stable in the sense of Lyapunov if stable AND

    $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}$$

    ($\dot{V}(x)$ is a locally negative definite function)

- **Exponential stability**
  - **In the sense of Lyapunov**
    The origin is exponentially stable in the sense of Lyapunov if stable AND

    $$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -\alpha V(x) \quad \forall x \in \mathcal{D} \setminus \left\{\vec{0}\right\}$$

- **Uniform stability**
  The equilibrium point $x = 0$ is uniformly stable if it is stable and, for each $epsilon > 0$, there exists a $\delta(\epsilon) > 0$, independent of t_0.

Stability (continued)	• **Global asymptotic stability** – **In the sense of Lyapunov** If the origin is globally asymptotically stable in the sense of Lyapunov if is asymptotically stable and $$\|x\| \to \infty \implies V(x) \to \infty$$ *($V(x)$ is radially unbounded)* • **L-stability** (TODO) • **I/O L-stability** (TODO) • **Finite-gain L-stability** (TODO) • **Small-signal I/O L-stability** (TODO) • **Small-signal finite-gain L-stability** (TODO)
Class κ function	A continuous scalar function on $\mathbf{R}^+$ is • **class** κ if it is: – zero at zero – strictly increasing – continuous • **class** κ_∞ if it is: – zero at zero – strictly increasing – continuous – ∞ at ∞
Radially Unbounded function	A function $V(x)$ is radially unbounded if $$\|x\| \to \infty \implies \|V(x)\| \to \infty$$
$\sup$ (supremum)	Like a maximum of a functions, but includes limits that aren't necessarily a part of the domain of the function. (TODO)
Hurwitz	• **Hurwitz (polynomial)**: A polynomial whose roots that are all in the left-half plane. (In other words, the real part of every root is strictly negative) • **Hurwitz (matrix)** (W): A square matrix whose characteristic polynomial is Hurwitz, meaning all eigenvalues are in the left-half plane. (In other words, the real part of every eigenvalue is strictly negative) • **Routh-Hurwitz stability criterion** (IEEE): TODO Any hyperbolic fixed point (or equilibrium point) of a continuous dynamical system is locally asymptotically stable if and only if the Jacobian of the dynamical system is Hurwitz stable at the fixed point. A system is stable if its control matrix is a Hurwitz matrix. The negative real components of the eigenvalues of the matrix represent negative feedback. Similarly, a system is inherently unstable if any of the eigenvalues have positive real components, representing positive feedback.

Zero-state observable

A time-invariant system of the form

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

is zero-state observable if

$$\begin{cases} y \equiv 0 \\ u \equiv 0 \end{cases} \implies x \equiv 0$$

In other words, when $u = 0$, any nonzero state behavior will be observed at the output ($y \neq 0$)

Sets

- **Invariant Set**
  A set of vectors M is invariant with respect to $\dot{x} = f(x)$ if

  $$x(0) \in M \implies x(t) \in M, \qquad \forall t \in \mathbb{R}$$

  (if a solution belongs to M at some time instant, then it belongs to M for all future and past time)

- **Positively Invariant Set**
  A set of vectors M is positively invariant with respect to $\dot{x} = f(x)$ if

  $$x(0) \in M \implies x(t) \in M, \qquad \forall t \geq 0$$

  (if a solution belongs to M at some time instant, then it belongs to M for all future time)

- **Open Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is an **open set** if

  $$\forall x \subset D, \quad \exists \epsilon > 0 \quad \text{such that} \quad B(x, \epsilon) \subset D$$

  (for all vectors x in the domain D, there exists a real scalar ϵ such that we can create a ball around x with radius ϵ, and that whole ball is in D)

- **Closed Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is a **closed set** if

  $$\mathbb{R}^n \setminus D \quad \text{is an open set}$$

  (everywhere outside of D is open)

- **Bounded Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is a **bounded set** if

  $$\exists \epsilon > 0 \quad \text{such that} \quad D \subset B(0, \epsilon)$$

  (D fits in a ball with a finite, constant radius ϵ)

- **Compact Set**
  A set $D \subset \mathbb{R}^n$ (D, *which is a set of real vectors*) is a **compact set** if it is closed and bounded.

Passivity

For a system $y = h(u, t), \quad h : \mathbb{R}^m \times [0, \infty) \rightarrow \mathbb{R}^n$
(output state y (an n-dimensional vector) is a function of the input state u (an m-dimensional vector) and time t)

- **Invariant Set**
  A set of vectors M is invariant with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \implies x(t) \in M, \qquad \forall t \in \mathbb{R}$$

(if a solution belongs to M at some time instant, then it belongs to M for all future and past time)

Adjoint

- The $\{$**adjoint or Hermitian transpose**$\}$ **of a matrix** A (Wolfram) is its conjugate transpose, denoted as A', A^*, A^H, or $A^\dagger$ i.e.

$$A^H = \overline{A}^T$$

  Interesting properties of ajoint matrices:
  - $A^H = \overline{A}^T = \overline{A^T}$
  - If a matrix is its own conjugate transpose, that matrix is called **self-adjoint** or **Hermetian**
  - If A is a real matrix, $A^H = A^T$

  Warning: In some older literature, the "adjoint of a matrix" may mean the **adjunct matrix of a square matrix** (W)
- The **adjoint representation of a vector space** (Wolfram)
  (TODO)
- The **adjoint equation** (W)
  (TODO)
