Controls Notes

M516

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This document is an ongoing collection of symbols, theorems, tools, and terms I have found useful for studying control theory, available as PDF, HTML, and LaTeXsource code.

Variables and Symbols

x

- state vector (state space) (W)
  Type: $\mathbb{R}^n$

H

- Hamiltonian matrix
- Hamiltonian (Hamiltonian mechanics) (W)
  type: $\mathbb{R}^n \to \mathbb{R}$
  - Assuming discrete time linear system:

$$H_k = L(x_k, u_k, k) + p_{k+1}^T f(x_k, u_k, k)$$

$\mathcal{L}$

- the Lagrangian
  type: $\mathbb{R}^n \to \mathbb{R}$

$e_\#$

- The #th unit vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots$$

∇ — The del operator, which represents one of many long but similar operators on a vector field $v \in \mathbb{R}^n$.

- ∇f: Gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$, returning an n-dimensional vector. (W)
  This vector points in the direction of the greatest increase, and its magnitude is the slope.

  For example, a mountain climber could approximate the shape of a convex mountain as a function $f_{mountain}$ that computes the altitude given some latitude and longitude (assuming a very small mountain very far from the poles). In other words, $f_{mountain} : \mathbb{R}^2 \to \mathbb{R}$. The climber could know which direction to climb to summit the peak: it's the direction ∇f, and the grade or slope of the mountain is $|\nabla f|$

  Note that if $n = 1$, ∇f is the standard derivative of f.
  Formally speaking:

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i$$

- $\nabla \cdot \vec{v}$: The divergence of a vector field $\vec{v}$
- $\nabla \times \vec{v}$: The curl of a vector field $\vec{v}$
- Δf: the Laplace operator on a function $f : \mathbb{R}^n \to \mathbb{R}$, equivalent to the divergence of the gradient of f, i.e.

$$\delta f = \nabla^2 f = \nabla \cdot \nabla f$$

J

- Cost to go function
  type: $\mathbb{R}^n \to \mathbb{R}$

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p

- Lagrange multiplier (W)

$<>$

- Lie bracket notation (W)
  $< a, b >= b^T a$

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norm

- Vector norm (TODO)
- Matrix norm (TODO)
- Functional norm (TODO)

Named Theorems and Conditions

Poincaré-Bendixson theorem

- TODO

Terms

Lipschitz Continuity

W, UC Berkley

Lipschitz continuous functions are continuous and differentiable almost anywhere in a domain.

Given a domain D and a function $f : D \to \mathbb{R}, D \in \mathbb{R}^n$,
f is Lipschitz continuous if $\exists L > 0$ such that $|f(x) - f(y)| < L||(x - y)|| \forall x, y \in D$

Hessian

W, Kahn Academy, Wolfram

- A $2n$ x $2n$ matrix of all 2nd order partial derivatives of some function $f : \mathbb{R}^n \rightarrow \mathbb{R}$
- The determinant of a Hessian matrix

definite

W

Warning: this definition does not appear to be common outside of controls

Given a real-valued, continuously differentiable function $V(x) : \mathbb{R} \rightarrow \mathbb{R}$
$V(x)$ can be classified as

- **(globally) positive semidefinite** if

$$V(x) \geq 0 \qquad \forall x \in \mathbb{R}$$

(v is greater than or equal to 0 regardless of x)

- **(globally) positive definite** if positive semidefinite AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

- **(globally) negative semidefinite** if

$$V(x) \leq 0 \qquad \forall x \in \mathbb{R}$$

(v is less than or equal to 0 regardless of x)

- **(globally) negative definite** if negative semidefinite AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

- **locally positive definite (l.p.d)** if

$$V(x) \geq 0 \qquad \forall x \in N$$

where N is a small open neighborhood containing $\vec{0}$

(v is greater than or equal to 0 regardless of x in some small open neighborhood N that contains the zero vector)

AND

$$V(x) = 0 \iff x = 0$$

($V(x)$ is zero if and only if x is zero)

Note that the criteria for a function to be locally positive definite are similar, but more relaxed than, those for globally positive definite functions.

- **positive definite on some domain** $D \in \mathbb{R}^n$ if
we only care if the conditions for positive definite functions hold for all x in D.

Stability	• (Lyapunov) stability (TODO) • Asymptotic stability (TODO) • Exponential stability (TODO) • Uniform stability (TODO) • Global stability (TODO) • L-stability (TODO) • I/O L-stability (TODO) • Small-signal I/O L-stability (TODO) • Small-signal finite-gain L-stability (TODO)
Class K function	• (TODO)
Radially Unbounded function	• (TODO)
$\sup$ (supremum)	Like a maximum of a functions, but includes limits that aren't necessarily a part of the domain of the function. (TODO)