3. DZ

1. riješen u Malom Ivici, str. 48.

2. riješen u Malom Ivici, str. 48.

3. riješen u Malom Ivici, str. 48./49.

4. riješen u Malom Ivici, str. 49.

11. riješen u Malom Ivici, str. 51.

12. riješen u Malom Ivici, str. 51.

15. riješen u Malom Ivici, str. 52.

19. riješen u Malom Ivici, str. 56.

20. riješen u Malom Ivici, str. 57.

Zadaci 5., 6., 7., 8., 9., 10., 13., 14., 16., 17. i 18. su riješeni u nastavku.

5.

. Neka je $\mathbf{A}=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ i $P(x)=x^2+2x$. Naći sve realne brojeve a i b takve da je takve da je $P(\mathbf{A})=0$.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}+2\begin{bmatrix} a & b \\ b & a \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}+\begin{bmatrix} 2a & 2b \\ 2b & 2a \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^2+b^2+2a & 2ab+2b \\ 2ab+2b & a^2+b^2+2a \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^2+b^2+2a=0$$
$$2ab+2b=0$$

Iz druge jednadžbe: $2a=-2 \Leftrightarrow a=-1$

Uvrstimo taj a u prvu jednadžbu:

$$1+b^2-2=0$$
$$b^2=1$$
$$b_1=-1$$
$$b_2=1$$

Iz druge jednadžbe: $b(a+1)=0 \Leftrightarrow b=0, a=-1$

$$a^2+2a=0$$
$$a(a+2)=0$$
$$a_1=0$$
$$a_2=-2$$

Znači dobili smo sljedeće: $a\in\{-1,0,-2\}$ i $b\in\{-1,0,1\}$

Ukombinirajmo sva rješenja:

$$A_1=\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad A_4=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad A_7=\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

$$A_2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_5=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_8=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A_3=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad A_6=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_9=\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Od svih matrica, samo A_1, A_3, A_5 i A_8 zadovoljavaju uvjet $P(A)=0$.

Rješenje:

$$A_1=\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad A_2=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad A_3=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_4=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

6.

. Odredi sve matrice koje komutiraju s matricom $\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

I vrijedi: $A \cdot X = X \cdot A$

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a-2c & b-2d \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} a-3b & -2a+4b \\ c-3d & -2c+4d \end{bmatrix}$$

$$a-2c = a-3b \Leftrightarrow b = \frac{2}{3}c$$
$$b-2d = -2a+4b \Leftrightarrow -2a+3b = -2d \Leftrightarrow 2a-3b = 2d$$
$$-3a+4c = c-3d \Leftrightarrow a-c = d$$
$$-3b+4d = -2c+4d$$

7.

Odredi sve matrice koje komutiraju s matricom $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

I vrijedi: $A \cdot X = X \cdot A$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} a & a+2b & b+c \\ d & d+2e & e+f \\ g & g+2h & h+i \end{bmatrix} = \begin{bmatrix} a+d & b+e & c+f \\ 2d+g & 2e+h & 2f+i \\ g & h & i \end{bmatrix}$$

$$a = a+d \Leftrightarrow d = 0$$
$$a+2b = b+e \Leftrightarrow a-b = e$$
$$b+c = c+f \Leftrightarrow b = f$$
$$d = 2d+g \Leftrightarrow -d = g \Leftrightarrow d = 0$$
$$d+2e = 2e+h \Leftrightarrow d = h$$
$$e+f = 2f+i \Leftrightarrow e-f = i \Leftrightarrow e-b = i \Leftrightarrow a-b-b = i \Leftrightarrow i = a-2b$$
$$g = g$$
$$g+2h = h \Leftrightarrow g = -h \Leftrightarrow g = 0$$
$$h+i = i \Leftrightarrow h = 0$$

$$X = \begin{bmatrix} a & b & c \\ 0 & a-b & b \\ 0 & 0 & a-2b \end{bmatrix}$$

8.

Odredi sve matrice drugog reda čiji je kvadrat jednak nul-matrici.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^2 + bc = 0 \Leftrightarrow a^2 = -bc$$
$$ab + bd = 0 \Leftrightarrow b(a+d) = 0$$
$$ac + cd = 0 \Leftrightarrow c(a+d) = 0$$
$$bc + d^2 = 0 \Leftrightarrow -a^2 + d^2 = 0 \Leftrightarrow a^2 - d^2 = 0 \Leftrightarrow (a-d)(a+d) = 0 \Leftrightarrow d_1 = -a, d_2 = a$$

Ako je $d = -a$, onda imamo: $b \cdot 0 = 0$ i $c \cdot 0 = 0$, što znači da b i c mogu biti bilo koji brojevi.

Ako je $d = a$, onda imamo: $-2ab = 0$ i $-2ac = 0$, što znači da su ili a ili b jednaki 0, odnosno, ili a ili c jednaki 0.

$$A_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$
$$A_5 = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$$

Matrica $A_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ zadovoljava uvjete, pri čemu vrijedi $bc = -a^2$.

Za mtaricu $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ besmisleno govoriti jer je očigledno da je sama ta matrica nul-matrica, pa je očigledno i njen kvadrat jednak nul-matrici.

Kvadrat matrice A_3 izgleda ovako: $A_3 = \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix}$, pa da bi to bila nul-matrica, onda mora biti $b = c = 0$, pa se to opet svodi na matricu A_2.

Kod matrice A_4 će morati biti za kvadrat zadovoljeno $a = 0$ (opet se svodi na A_2.)

Matrica A_5 ne zadovoljava uvjete. **Rješenje:** $A_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

9.

. Izračunaj $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^n$, $n \in \mathbf{N}$.

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^3 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^2\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^4 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^3\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & (-1)^{n+1} n \\ 0 & (-1)^n \end{bmatrix}$$

Dokažimo to indukcijom:

1. za n = 1 vrijedi $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^1 = \begin{bmatrix} (-1)^1 & (-1)^{1+1} 1 \\ 0 & (-1)^1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

2. pretpostavka da za n vrijedi $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & (-1)^{n+1} n \\ 0 & (-1)^n \end{bmatrix}$

3. dokazati da za n + 1 vrijedi $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^{n+1} = \begin{bmatrix} (-1)^{n+1} & (-1)^{n+2}(n+1) \\ 0 & (-1)^{n+1} \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^{n+1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^n\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (-1)^n & (-1)^{n+1} n \\ 0 & (-1)^n \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (-1)^{n+1} & (-1)^{n+2}(n+1) \\ 0 & (-1)^{n+1} \end{bmatrix}$$

Ovime je dokaz izvršen.

Rješenje:

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & (-1)^{n+1} n \\ 0 & (-1)^n \end{bmatrix}$$

10.

. Izračunaj $\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^n$, $n \in \mathbf{N}$.

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^1 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3\lambda & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^2\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3\lambda & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7\lambda & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^3\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7\lambda & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 15\lambda & 16 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ (2^n - 1)\lambda & 2^n \end{bmatrix}$$

Dokažimo to indukcijom:

1. za n = 1 vrijedi $\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^1 = \begin{bmatrix} 1 & 0 \\ (2^n - 1)\lambda & 2^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}$

2. pretpostavka da za n vrijedi $\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ (2^n - 1)\lambda & 2^n \end{bmatrix}$

3. dokazati da za n + 1 vrijedi $\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ (2^{n+1} - 1)\lambda & 2^{n+1} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^n\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (2^n - 1)\lambda & 2^n \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (2^{n+1} - 1)\lambda & 2^{n+1} \end{bmatrix}$$

Ovime je dokaz izvršen.

Rješenje:

$$\begin{bmatrix} 1 & 0 \\ \lambda & 2 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ (2^n - 1)\lambda & 2^n \end{bmatrix}$$

13.

. Dokaži ili opovrgni protuprimjerom tvrdnju: Za sve matrice $\mathbf{A}, \mathbf{B} \in M_n$ vrijedi $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$$

Kako *AB* nije isto što i *BA*, onda ova **tvrdnja ne vrijedi za sve matrice**, nego samo u slučaju kada je *AB* = *BA*.

Npr., za matrice $A = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$ i $B = \begin{bmatrix} 0 & 1 \\ 2 & -6 \end{bmatrix}$ dobijemo da je

$$AB = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 6 & -18 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 8 & -20 \end{bmatrix}$$

14.

. Ako su $\mathbf{A}$ i $\mathbf{B}$ simetrične matrice, mora li tada $\mathbf{AB}$ biti simetrična matrica ? Obrazloži !

AB će biti simetrična onda i samo onda ako matrice *A* i *B* komutiraju, odnosno ako je $AB = BA$.

16. Izračunaj determinante: **a)** $\begin{vmatrix} 1 & 0 & 4 \\ 2 & 0 & -3 \\ -1 & 2 & 2 \end{vmatrix}$ **b)** $\begin{vmatrix} 2 & 3 & 1 & -2 \\ 1 & 0 & 2 & 0 \\ -1 & 1 & 2 & -5 \\ 4 & -2 & 1 & 3 \end{vmatrix}$.

17. Izračunaj determinante: $\begin{vmatrix} 2 & 1 & 1 & -1 & 1 \\ 1 & -2 & 1 & 2 & -1 \\ -1 & 1 & 2 & -2 & 1 \\ 1 & -1 & 1 & -2 & 1 \\ 2 & -1 & 1 & 2 & -1 \end{vmatrix}$.

18. Izračunaj determinante:

a) $\begin{vmatrix} 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{vmatrix}$ **b)** $\begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{vmatrix}$.

16. *a)* $\left|\begin{matrix} 1 & 0 & 4 \\ 2 & 0 & -3 \\ -1 & 2 & 2 \end{matrix}\right| \begin{matrix} 1 & 0 \\ 2 & 0 \\ -1 & 2 \end{matrix} = 16 + 6 = 22$

b) $\begin{vmatrix} 2 & 3 & 1 & -2 \\ 1 & 0 & 2 & 0 \\ -1 & 1 & 2 & -5 \\ 4 & -2 & 1 & 3 \end{vmatrix} = -1\left|\begin{matrix} 3 & 1 & -2 \\ 1 & 2 & -5 \\ -2 & 1 & 3 \end{matrix}\right| \begin{matrix} 3 & 1 \\ 1 & 2 \\ -2 & 1 \end{matrix} - 2\left|\begin{matrix} 2 & 3 & -2 \\ -1 & 1 & -5 \\ 4 & -2 & 3 \end{matrix}\right| \begin{matrix} 2 & 3 \\ -1 & 1 \\ 4 & -2 \end{matrix} =$

$= -1(18 + 10 - 2 - 8 + 15 - 3) - 2(6 - 60 - 4 + 8 - 20 + 9) = -30 + 122 = 92$