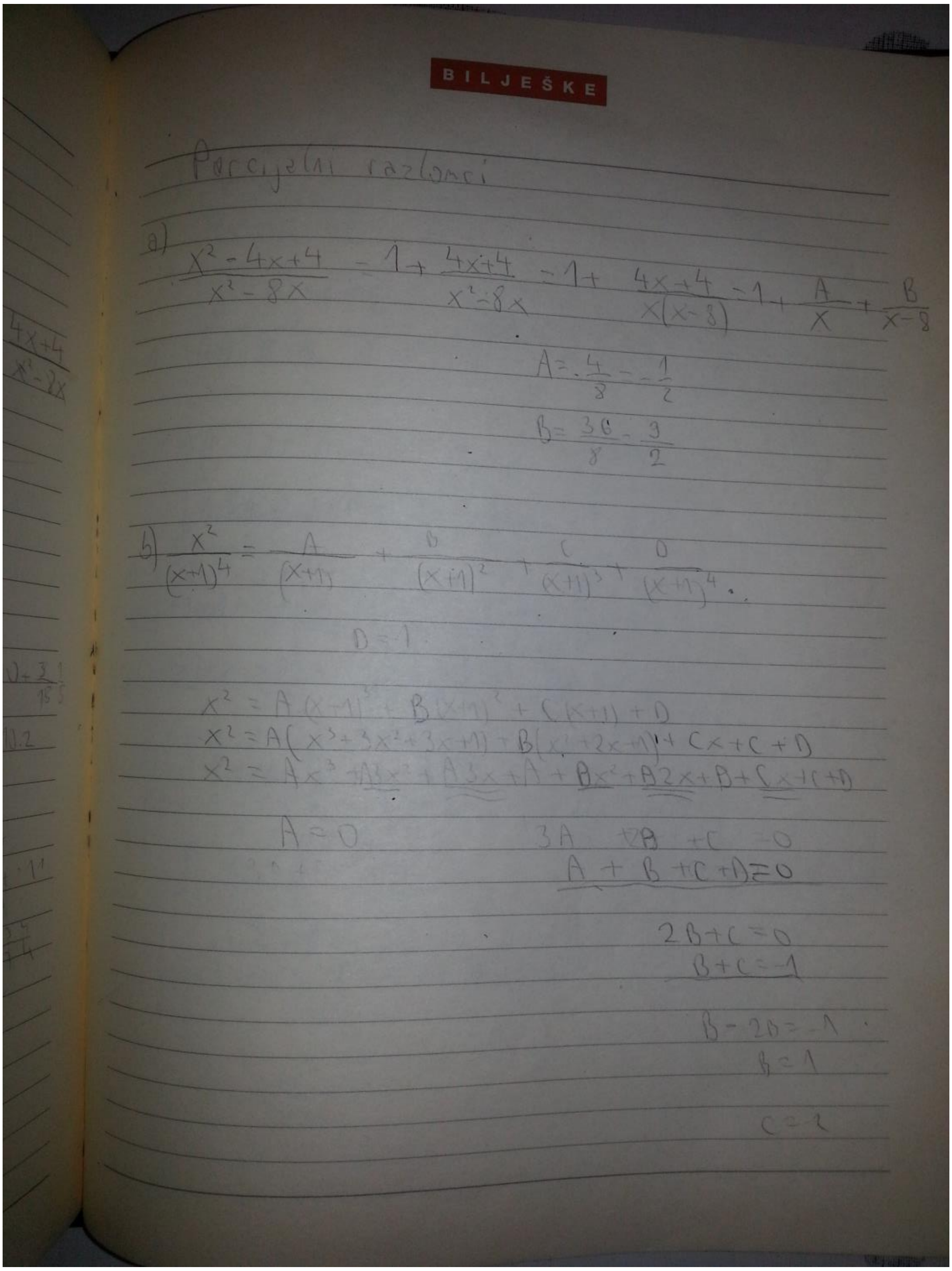
MK

Ako nekom ne idu parcijalni razlomci tu sam stavio par zadataka da to malo izvježbate.



BILJEŠKE

Parcijalni razlomci

a) $\frac{x^2-4x+4}{x^2-8x} = 1 + \frac{4x+4}{x^2-8x} = 1 + \frac{4x+4}{x(x-8)} = 1 + \frac{A}{x} + \frac{B}{x-8}$

$A = -\frac{4}{8} = -\frac{1}{2}$

$B = \frac{36}{8} = \frac{9}{2}$

b) $\frac{x^2}{(x+1)^4} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}$

$D = 1$

$x^2 = A(x+1)^3 + B(x+1)^2 + C(x+1) + D$

$x^2 = A(x^3+3x^2+3x+1) + B(x^2+2x+1) + Cx + C + D$

$x^2 = Ax^3 + A3x^2 + A3x + A + Bx^2 + B2x + B + Cx + C + D$

$A = 0$

$3A + 2B + C = 0$

$A + B + C + D = 0$

$2B + C = 0$

$B + C = -1$

$B - 2B = -1$

$B = 1$

$C = -2$

d) $\frac{x^3-3x^2+7x-1}{(x^2-1)^2} = \frac{x^3-3x^2+7x-1}{(x-1)^2(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$

$x^3-3x^2+7x-1 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)(x+1) + D(x-1)^2$

$= A(x^3-2x^2+x-x^2-2x-1)+Bx^2+2Bx+B$

$B=1$

$+C(x^3-2x^2+x+x^2-2x+1)+Dx^2-2Dx+D$

$= Ax^3-2Ax^2+Ax-Ax^2-2Ax-A+Bx^2+2Bx+B$

$D=-3$

$+Cx^3-2Cx^2+Cx+Cx^2-2Cx+C+Dx^2-2Dx+D$

$A+C=1$

$-2A-A+B-2C+C+D=-3$

$-3A+B-2(1-A)+1-A-3=-3$

$-A+B+C+D=-1$

$-3A+B-2+2A-A=-1$

$-A+B+1-A=2$

$-2A+B=1$

$B-2A=1$

$-2A+B=1$

$A-2A+2B+C-2C-2D=7$

$-A+2B-C=1$

$A=0$ $C=1$

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e)

$$\frac{5x^2-2x+3}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$5x^2-2x+3 = (Ax+B)(x-1) + C(x^2+1)$$

$$x=1 \qquad \underline{Ax^2} - \underline{Ax} + \underline{Bx} - \underline{B} + \underline{Cx^2} + \underline{C}$$

$$C = 3 \qquad A+C=5$$

$$-B+C=3 \qquad A=2$$

$$B=0$$

$$\int_{-1}^{0} \frac{5x^2-2x+3}{(x^2+1)(x-1)}\,dx = \int_{-1}^{0} \frac{2x}{x^2+1}\,dx + \int_{-1}^{0} \frac{3}{x-1}\,dx$$

$$\int_{-1}^{0} \frac{du}{u} + 3\int \ln|x-1|\,dx$$

$$\ln(x^2+1)\Big|_{-1}^{0} + 3(\ln|x-1|)\Big|_{-1}^{0}$$

$$0 - \ln 2 + 3(0 - \ln 2) = 4\ln(2)$$

f).

$$\int \frac{dx}{(x+1)\sqrt{-x^2-2x}} = \int \frac{dx}{(x+1)\sqrt{1-(x+1)^2}} \qquad \sqrt{-(x^2+2x)} = \sqrt{-(x^2+2x+1)+1}$$

$$\begin{bmatrix} \cos t = x+1 \\ -\sin t\,dt = dx \end{bmatrix} = \int \frac{-\sin t}{\cos t\,\sin t}\,dt = -\int \frac{1}{\cos t}\,dt$$

Izračunati sljedeće integrale:

1. $\int \frac{\sqrt{x}+1}{\sqrt[3]{x}} dx$,

2. $\int_0^1 \frac{x\,dx}{\sqrt{x+1}}$,

3. $\int_0^{\sqrt{3}} x\sqrt{x^2+1} dx$,

4. $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x dx$,

5. $\int_0^1 \frac{dx}{x^2+x+1}$,

6. $\int \frac{x\,dx}{\sqrt{2+x-x^2}}$,

7. $\int \frac{dx}{3e^x+e^{-x}+2}$,

8. $\int \operatorname{tg}(3x) dx$,

9. $\int_0^1 \arcsin x\, dx$,

10. $\int_0^1 x \operatorname{arc\,tg} x\, dx$

11. $\int \sin(\sqrt{x}) dx$,

12. $\int_0^1 \sqrt{x} e^{-\sqrt{x}} dx$,

13. $\int_0^{\frac{\pi}{2}} e^{\sin x} \sin(2x) dx$,

14. $\int e^{-x} \sin x\, dx$,

15. $\int \frac{x \cos x}{\sin^2 x}\, dx$.

8.6. Zadatci za vježbu

1.
$$\int \frac{\sqrt{x}+1}{\sqrt[3]{x}}\,dx = \int \frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{3}}}\,dx = \int \left(x^{\frac{1}{6}}+x^{-\frac{1}{3}}\right)dx$$

$$= \frac{6}{7}x^{\frac{7}{6}} + \frac{3}{2}x^{\frac{2}{3}} + C$$

2.
$$\int_0^1 \frac{x\,dx}{\sqrt{x+1}} \quad \left|\begin{array}{l} t = x+1 \\ dt = dx \\ 1 \to 2 \\ 0 \to 1 \end{array}\right| = \int_1^2 \frac{t-1}{t^{\frac{1}{2}}}\,dt = \int_1^2 t^{\frac{1}{2}} - t^{-\frac{1}{2}}\,dt$$

$$\frac{2}{3}t^{\frac{3}{2}}\Big|_1^2 - 2t^{\frac{1}{2}}\Big|_1^2 = \frac{2}{3}\sqrt[2]{8} - \frac{2}{3} - \left(2\sqrt{2}-2\right)$$

$$= \frac{4\sqrt{2}}{3} - 2\sqrt{2} + \frac{4}{3} = \frac{4\sqrt{2}-6\sqrt{2}+4}{3} = \frac{-2\sqrt{2}+4}{3}$$

3.
$$\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx = \left|\begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \\ \sqrt{3} \to 4 \\ 0 \to 1 \end{array}\right| = \int_1^4 \sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int_1^4 t^{\frac{1}{2}}\,dt$$

$$\frac{1}{2}\left(\frac{2}{3}t^{\frac{3}{2}}\Big|_1^4\right) = \frac{1}{2}\left(\frac{2}{3}\sqrt{4^3} - \frac{2}{3}\right) = \frac{1}{2}\cdot\frac{14}{3} = \frac{7}{3}$$

4.
$$\int_0^{\frac{\pi}{4}} \sin x^3 \cos x\,dx = \left|\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \\ \frac{\pi}{4} \to \frac{\sqrt{2}}{2} \\ 0 \to 0 \end{array}\right| = \int_0^{\frac{\sqrt{2}}{2}} t^3\,dt$$

$$\frac{t^4}{4}\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{\frac{4}{16}}{4} = \frac{1}{16}$$

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5.
$$\int_0^1 \frac{dx}{x^2+x+1} = \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \left| \begin{array}{l} u = x+\frac{1}{2} \\ du = dx \\ 1 \to \frac{3}{2} \\ 0 \to \frac{1}{2} \end{array} \right| = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{du}{u^2+\underbrace{\frac{3}{4}}_{\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$\frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arctg} \frac{u}{\frac{\sqrt{3}}{2}} \Bigg|_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{2}{\sqrt{3}}\left(\operatorname{arctg}\left(\frac{3}{\sqrt{3}}\right) - \operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{2}{\sqrt{3}} \operatorname{arctg}\sqrt{3} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

7.
$$\int \frac{dx}{\underbrace{3e^x+e^{-x}+2}_{\frac{3e^{2x}+1+2e^x}{e^x}}} \left| \begin{array}{l} t = e^x \\ dt = e^x\,dx \end{array} \right| = \int \frac{dt}{3t^2+2t+1} = \int \frac{dt}{\left(\sqrt{3}\,t+\frac{1}{\sqrt{3}}\right)^2+\frac{2}{3}}$$

$$\left| \begin{array}{l} \sqrt{3}\,t+\frac{1}{\sqrt{3}} = u \\ \sqrt{3}\,dt = du \end{array} \right| = \int \frac{\frac{du}{\sqrt{3}}}{u^2+\frac{2}{3}} = \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{2}} \operatorname{arctg} \frac{u}{\sqrt{\frac{2}{3}}} =$$

$$= \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{\sqrt{3}\,t+\frac{1}{\sqrt{3}}}{\sqrt{\frac{2}{3}}}\right) = \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{\frac{3e^x+1}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}}}\right) = \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{3e^x+1}{\sqrt{2}}\right) + C$$

6. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)+C$

$$\int \frac{x\,dx}{\sqrt{2+x-x^2}} = \int \frac{x\,dx}{\sqrt{\frac{9}{4}-(x-\frac{1}{2})^2}} = \quad \Big| \; u=x-\frac{1}{2}, \quad du=dx \; \Big|$$

(do potpunog kvadrata)

$$\int \frac{u+\frac{1}{2}}{\sqrt{\frac{9}{4}-u^2}}\,du \quad \Big| \; u=\frac{3\sin t}{2}, \quad du=\frac{3}{2}\cos t\,dt \; \Big| \quad = \int \frac{\frac{3\sin t}{2}+\frac{1}{2}}{\sqrt{\frac{9}{4}-\frac{9}{4}\sin^2 t}}\,\frac{3}{2}\cos t\,dt$$

$$\int \frac{\frac{3\sin t}{2}+\frac{1}{2}}{\cancel{\frac{3\cos t}{2}}}\,\cancel{\frac{3}{2}\cos t}\,dt \qquad \frac{2u}{3}=\sin t \qquad t=\arcsin\left(\frac{2u}{3}\right)$$

$$\int \frac{3\sin t}{2}+\frac{1}{2}\,dt = \int \frac{3\sin t}{2}\,dt+\int \frac{1}{2}\,dt = \frac{3}{2}-\cos+\frac{1}{2}t$$

$$=\frac{1}{2}t-\frac{3\cos t}{2} = \frac{1}{2}\arcsin\frac{2u}{3}-\frac{3\cos(\arcsin\frac{2u}{3})}{2}$$

$$\frac{1}{2}\arcsin\frac{2u}{3}-\frac{3}{2}\sqrt{1-\frac{4u^2}{9}} = \frac{1}{2}\arcsin\left(\frac{2x-1}{3}\right)-\frac{3}{2}\sqrt{1-\frac{4(x-\frac{1}{2})^2}{9}}$$

$$=\frac{1}{2}\arcsin\left(\frac{2x-1}{3}\right)-\frac{1}{2}\sqrt{9\left(1-\frac{4(x-\frac{1}{2})^2}{9}\right)}$$

$$=\frac{1}{2}\arcsin\left(\frac{2x-1}{3}\right)-\frac{1}{2}\sqrt{9-4(x-\frac{1}{2})^2}$$

$$=\frac{1}{2}\arcsin\left(\frac{2x-1}{3}\right)-\underbrace{\frac{1}{2}\sqrt{9-4x^2+4x-1}}_{\frac{1}{2}\sqrt{4(2-x^2+x)}}$$

$$-||- \qquad -\sqrt{2-x^2+x} \qquad +C$$

x^2 $\qquad \frac{x^3}{3}$

8. $\int tg(3x)\,dx = \left| \begin{matrix} t=3x \\ dt=3\,dx \end{matrix} \right| = \frac{1}{3}\int \frac{\sin t}{\cos t}\,dt \; \left| \begin{matrix} u=\cos t \\ du=-\sin t\,dt \end{matrix} \right|$

$-\frac{1}{3}\int \frac{\sin t}{u}\frac{du}{\sin t} = -\frac{1}{3}\ln|u| + c = -\frac{1}{3}\ln|\cos(3x)| + c$

9. $\int_0^1 \operatorname{arc\,ctg} x\,dx = \left| \begin{matrix} u=\operatorname{arc\,ctg} x, & du=-\frac{1}{x^2+1}dx \\ dv=dx, & v=x \end{matrix} \right|$

$x\operatorname{arcctg}x\Big|_0^1 + \int_0^1 \frac{x}{x^2+1}\,dx \; \left| \begin{matrix} u=x^2+1 \\ du=2x\,dx \\ 1\to 2 \\ 0\to 1 \end{matrix} \right|$

$-||- + \frac{1}{2}\int_1^2 \frac{1}{u}\,du$

$-||- + \frac{1}{2}\left(\ln|u|\Big|_1^2\right)$

$\frac{\pi}{4} + \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} + \frac{1}{2}\ln 2 + c$

10. $\int_1^e x\ln x\,dx = \left| \begin{matrix} t=\ln x \\ dt=\frac{1}{x}dx \end{matrix} \right| = \int_0^1 t e^{2t}\,dt = \left| \begin{matrix} u=t & du=dt \\ dv=e^{2t}dt, & v=\frac{1}{2}e^{2t} \end{matrix} \right|$

$e\to 1$

$1\to 0$

$t\frac{1}{2}e^{2t}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2t}\,dt \Rightarrow$

$\frac{e^2}{2} - \frac{1}{2}\int_0^1 e^{2t} = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^{2t}}{2}\Big|_0^1\right) = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$

$\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$

10. $\int_1^e x \ln x \, dx \quad \left| \begin{array}{ll} u = \ln x, & du = \frac{1}{x} dx \\ dv = x\,dx, & v = \frac{x^2}{2} \end{array} \right|$

$\frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{\cancel{x}} \, dx$

$= \frac{x^2}{2} \ln x \Big|_1^e - \frac{1}{2} \left(\frac{x^2}{2} \Big|_1^e \right)$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$

11.

$\int \sin(\sqrt{x}) \, dx \quad \left| \begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}} dx \end{array} \right| = \int \sin(u) \, 2u \, du$

$\left| \begin{array}{l} u = u \to du = du \\ dv = \sin(u) \to v = -\cos(u) \end{array} \right| = -2u\cos(u) - 2\int -\cos(u) \, du$

$= -2\sqrt{x} \cos\sqrt{x} + 2 \sin\sqrt{x} + C$

12. $\int_0^1 \sqrt{x} \, e^{\sqrt{x}} \, dx = \left| \begin{array}{l} t = \sqrt{x} \\ dt = \frac{1}{2t} dx \\ 1 \to 1 \\ 0 \to 0 \end{array} \right| = \int_0^1 t e^t \, 2t \, dt$

$2\int_0^1 t^2 e^t \, dt \quad \left| \begin{array}{l} u = t^2 \to du = 2t\,dt \\ dv = e^t dt \to v = e^t \end{array} \right|$

$2\left(t^2 \cdot e^t \Big|_0^1 - \int_0^1 e^t \, 2t \, dt \right) \to \left| \begin{array}{ll} u = t, & du = dt \\ dv = e^t dt, & v = e^t \end{array} \right| = e^t t \Big|_0^1 - \int_0^1 e^t \, dt =$

$2(e-2) = \underline{\underline{2e-4}} \qquad = e - \left(e^t \Big|_0^1\right) = e - e + 1$

13. $\int_0^{\frac{\pi}{2}} e^{\sin x} \sin(2x)\,dx = \int_0^{\frac{\pi}{2}} e^{\sin x}\, 2\sin x \cos x\, dx$

$2\int_0^{\frac{\pi}{2}} e^{\sin x} \sin x \cos x\, dx = \left| \begin{array}{l} z=\sin x \\ dz=\cos x\, dx \\ \frac{\pi}{2} \to 1 \\ 0 \to 0 \end{array} \right| = 2\int_0^1 e^z z\, dz$

$\left| \begin{array}{ll} u=z & du=dz \\ dv=e^z dz & v=e^z \end{array} \right| = 2\left(z e^z \Big|_0^1 - \int_0^1 e^z\, dz \right) = 2(e-(e-1)) = 2$

14.

$\overbrace{\int e^{-x} \sin x\, dx}^{I} = \left| \begin{array}{ll} u=\sin x, & du=\cos x\, dx \\ dv=\frac{1}{e^x}\, dx, & v=-e^{-x} \end{array} \right|$

$= -e^{-x}\sin x - \int -e^{-x}\cos x\, dx$

$\int -e^{-x}\cos x\, dx = \left| \begin{array}{ll} u=\cos x, & du=-\sin x\, dx \\ dv=-e^{-x}dx, & v=+e^{-x} \end{array} \right|$

$+e^{-x}\cos x - \int -e^{-x}\sin x\, dx$

$I = -e^{-x}\sin x - (e^{-x}\cos x + I)$

$2I = -e^{-x}\sin x - e^{-x}\cos x$

$I = \frac{-e^{-x}(\sin x + \cos x)}{2} + C$

BILJEŠKE

Podsjetnik

$$\boxed{\sin^2 x = \frac{1}{2}(1-\cos 2x)}$$

? 15.

$$Z = \int e^{-x}\sin^2 x\,dx = \left| \begin{matrix} u=\sin^2 x, & du = 2\sin x\cos x\,dx \\ dv = e^{-x}dx, & v=-e^{-x} \end{matrix} \right|$$

$$-e^{-x}\sin^2 x + \int +e^{-x}\,2\sin x\cos x\,dx$$

$$1 \rightarrow \int e^{-x}\sin 2x\,dx \left| \begin{matrix} t=2x \\ dt=2dx \end{matrix} \right| = \int e^{-\frac{t}{2}}\sin t\,\frac{dt}{2}$$

$$\frac{1}{2}\int e^{-\frac{t}{2}}\sin t\,dt = \left| \begin{matrix} u=\sin t, & du=\cos t\,dt \\ dv=e^{-\frac{t}{2}}dt, & v=-2e^{-\frac{t}{2}} \end{matrix} \right|$$

- parcijalna ?
zamjena $t \rightarrow x$
natrag

$$-2e^{-x}\sin(2x) - 2\int -e^{-\frac{t}{2}}\cos t\,dt. \left| \begin{matrix} r=\cos t & dr=-\sin t\,dt \\ dc=-e^{-\frac{t}{2}}dt, & c=2e^{-\frac{t}{2}} \end{matrix} \right|$$

$$-2e^{-x}\sin(2x) - 2\left(2e^{-\frac{t}{2}}\cos t - 2\int e^{-\frac{t}{2}} - \sin t\,dt.\right)$$

$$= -2e^{-x}\sin(2x) - 4e^{-x}\cos(2x) - 4\int e^{-x}\sin(2x)\,2\,dx$$

$$= -e^{-x}\sin(2x) - 4e^{-x}\cos(2x) - 4\underbrace{\int e^{-x}\sin(2x)\,dx}_{1}$$

$$5 1 = -e^{-x}\sin(2x) - 4e^{-x}\cos(2x)$$

$$1 = \frac{-e^{-x}(\sin(2x)+4\cos(2x))}{5}$$

$$Z = -e^{-x}\sin^2 x + \frac{-e^{-x}(\sin(2x)+4\cos(2x))}{5} + C$$

Tu sam krivo dobio,probajte ga rješavati na ovaj način što je pisano drugim rukopisom (to je profesorica pisala ☺ pa bi trebalo ispasti dobro)

BILJEŠKE

16.

$$I=\int \frac{x\cos x}{\sin^2 x}\,dx = \left| \begin{matrix} t=\sin x \\ dt=\cos x\,dx \\ dx=\frac{dt}{\cos x} \\ x=\arcsin t\end{matrix}\right| = \int \frac{\arcsin t}{t^2}\,dt$$

$$\left| \begin{matrix} u=\arcsin t, & du=\frac{1}{\sqrt{1-t^2}}\,dt \\ dv=\frac{1}{t^2}\,dt, & v=-\frac{1}{t} \end{matrix}\right|$$

$$u=x,\quad dv=\frac{\cos x\,dx}{\sin^2 x}$$
$$du=dx \qquad v=\int\frac{\cos x\,dx}{\sin^2 x}=\left|\, t=\sin x \,\right|$$

$$= -\frac{\arcsin t}{t} - \int -\frac{1}{t\sqrt{1-t^2}}\,dt$$

$$\int \frac{1}{t\sqrt{1-t^2}}\,dt \left| \begin{matrix} t=\sin(x) \\ dt=\cos x\,dx \end{matrix}\right| = \int\frac{\cos x\,dx}{\sin x\cos x} = \int\frac{dt}{\sin x}$$

$$\int\frac{dx}{\sin x} = \int\frac{\sin x\,dx}{\sin^2 x} = \int\frac{\sin x\,dx}{1-\cos^2 x} = \left| \begin{matrix} t=\cos x \\ dt=-\sin x\,dx \\ dx=-\frac{1}{\sin x}\,dt \end{matrix}\right|$$

$$= -\int\frac{dt}{1-t^2} = \int\frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= -\frac{\arcsin \sin x}{\sin x} + \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C = -\frac{x}{\sin x} + \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|$$

$$\int\frac{dx}{\sin x} \qquad t=\operatorname{tg}\frac{x}{2} \quad \text{univ. supst.}$$

$$\int\frac{dx}{\cos x} = \int\frac{\cos x\,dx}{\cos^2 x} = \left|\, u=\sin x \,\right|$$

$$\int\frac{\sin x\,dx}{\sin^2 x}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\cos^2\frac{x}{2}\cdot\operatorname{tg}\frac{x}{2} = 2t\cdot\frac{1}{1+t^2} = \frac{2t}{1+t^2}$$

$$\cos^2 x = \frac{1}{1+\operatorname{tg}^2 x}$$