MK

10.3. Zadatci za vježbu

1.

a) $$\int_0^\infty \frac{x\,dx}{x^2+4} \quad \left| \begin{array}{ll} u=x & dv=\frac{dx}{x^2+4} \\ du=dx & v=\frac{1}{2}\,\mathrm{arctg}\,\frac{x}{2} \end{array} \right|$$

$$= \frac{x\,\mathrm{arctg}\left(\frac{x}{2}\right)}{2}\Bigg|_0^\infty - \frac{1}{2}\int_0^\infty \mathrm{arctg}\left(\frac{x}{2}\right)$$

$\Downarrow$

∞ – drugi netreba ni racunati, kad prvi dio divergira onda pocetni sigurno divergira

b) $$\int_0^\infty \frac{dx}{x^2+4} = \frac{\mathrm{arctg}\left(\frac{x}{2}\right)}{2}\Bigg|_0^\infty = \frac{\pi}{4}$$

2. $$\int_a^\infty \frac{dx}{x\ln^2 x} \quad (a>1) \quad \left| \begin{array}{l} u=\ln x \\ \frac{1}{x}dx=du \\ \infty \to \infty \\ a \to \ln a \end{array} \right| = \int_{\ln a}^\infty \frac{du}{u^2}$$

$$-\frac{1}{u}\Bigg|_{\ln a}^\infty = -\left(\frac{1}{\infty} - \frac{1}{\ln a}\right) = \frac{1}{\ln a}$$

3. $$\int_{-\infty}^\infty \frac{dx}{x^2+4x+9} = \int_{-\infty}^\infty \frac{dx}{x^2+4x+4+5} = \int_{-\infty}^\infty \frac{dx}{(x+2)^2+5} \quad \left| \begin{array}{l} t=x+2 \\ dt=dx \end{array} \right|$$

$$\int_{-\infty}^\infty \frac{dt}{t^2+5} = \frac{1}{\sqrt5}\,\mathrm{arctg}\,\frac{t}{\sqrt5}\Bigg|_{-\infty}^{+\infty} = \frac{1}{\sqrt5}\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = \frac{\pi}{\sqrt5}$$

4. $\int_0^\infty \frac{dx}{(2x+1)(x^2+1)}$

$\frac{1}{(2x+1)(x^2+1)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)(2x+1)$

$1 = Ax^2 + A + 2Bx^2 + Bx + 2Cx + C$

$A + 2B = 0 \Rightarrow A = 4C = 4(1-A) = 4-4A$

$B + 2C = 0 \qquad 5A = 4 \Rightarrow A = \frac{4}{5}$

$A + C = 1 \qquad C = \frac{A}{4} = \frac{\frac{4}{5}}{4} = \frac{1}{5}$

$-2B = \frac{B}{2} = 1$

$B = \frac{A}{-2} = -\frac{2}{5}$

$\frac{4}{5}\int_0^\infty \frac{dx}{2x+1} + \int_0^\infty \frac{-\frac{2}{5}x + \frac{1}{5}}{x^2+1}$

$\left| \begin{matrix} t = 2x \\ dt = 2dx \\ \infty \\ 0 \end{matrix} \right| = \frac{4}{5}\int_0^\infty \frac{dt}{2(t+1)} - \frac{2}{5}\int \frac{x\,dx}{x^2+1} + \frac{1}{5}\int_0^\infty \frac{dx}{x^2+1}$

$\left| \begin{matrix} c = x^2 \\ dc = 2x\,dx \end{matrix} \right|$

$= \frac{2}{5}\ln|t+1|\Big|_0^\infty - \frac{2}{5}\int_0^\infty \frac{dc}{2(c+1)} + \frac{1}{5}\int_0^\infty \frac{dx}{x^2+1}$

$= \frac{2}{5}\left(\lim_{t\to\infty}\ln|t+1| - 0\right) - \frac{1}{5}\left(\lim_{t\to\infty}\ln|c+1| - 0\right) + \frac{1}{5}\cdot\frac{\pi}{2} = \left(\begin{matrix} t\to x \\ c\to x\end{matrix}\right)$

$= \frac{1}{5}\lim_{x\to\infty}\ln\frac{(2x+1)^2}{x^2+1} + \frac{\pi}{10} = \frac{1}{5}\ln\left(\lim_{x\to+\infty}\frac{4x^2+4x+1}{x^2+1}\right) + \frac{\pi}{10}$

$= \frac{1}{5}\ln 4 + \frac{\pi}{10} = \frac{1}{5}\ln 2^2 + \frac{\pi}{10} = \frac{2}{5}\ln 2 + \frac{\pi}{10}$

$\boxed{\cos\alpha = \frac{1}{\sqrt{1+tg^2\alpha}}}$

5. $\int_0^\infty \frac{(x+1)\,dx}{\sqrt{(x^2+1)^3}} = \int_0^\infty \frac{(x+1)}{\sqrt{(x^2+1)}\cdot(x^2+1)}\,dx = \left| \begin{matrix} x = tg\,t \\ t = arc\,tg\,x \\ dt = \frac{dx}{1+x^2} \end{matrix} \right| =$

$\int_0^{\frac{\pi}{2}} \frac{(tg\,t+1)\,dt}{\sqrt{tg^2 t+1}} = \int_0^{\frac{\pi}{2}} \cos t\,(tg\,t+1)\,dt = \int \cos t\left(\frac{\sin t}{\cos t}+1\right)dt$

$= \int_0^{\frac{\pi}{2}} (\sin t + \cos t)\,dt = -\cos t\Big|_0^{\frac{\pi}{2}} + \sin t\Big|_0^{\frac{\pi}{2}} = -(0-1)+(1-0) = 2$

6.
$$\int_0^\infty \frac{sh^2x}{ch^4x} = \int_0^\infty \frac{sh^2x}{ch^2x}\cdot\frac{1}{ch^2x}\,dx = \int_0^\infty th^2x\cdot\frac{dx}{ch^2x} \quad \left|\begin{array}{l} t = thx \\ dt = \frac{1}{ch^2x}dx \end{array}\right|$$

$$\int_0^1 t^2\,dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

7.
$$\int_0^\infty x^3 e^{-x^2}dx = \int_0^\infty x\,x^2\,e^{-x^2}dx = \left|\begin{array}{l} t = x^2 \\ dt = 2x\,dx \end{array}\right| = \frac{1}{2}\int_0^\infty t\,e^{-t}\,dt$$

$$\left|\begin{array}{ll} u = t & dv = e^{-t}dt \\ du = dt & v = -e^{-t} \end{array}\right| = \frac{1}{2}\left[-\frac{t}{e^t}\Big|_0^\infty + \int_0^\infty e^{-t}\,dt\right]$$

$$= \frac{1}{2}\left[-\frac{t}{e^t}\Big|_0^\infty - e^{-t}\Big|_0^\infty\right] = \qquad \boxed{\lim_{t\to\infty}\frac{t}{e^t} \overset{L'H}{=} \lim_{t\to\infty}\frac{1}{e^t} = 0}$$

$$= \frac{1}{2}\left[-(0-0)-(0-1)\right] = \frac{1}{2}$$

8.
$$\int_{-\infty}^\infty \frac{e^x}{e^{4x}+3e^{2x}+2}dx = \left|\begin{array}{l} t = e^x \\ dt = e^x dx \\ \infty \to \infty \\ -\infty \to 0 \end{array}\right| = \int_0^\infty \frac{dt}{t^4+3t^2+2}$$

$$(t^4+3t^2+2):(t^2+2) = (t^2+1)$$
$$\underline{t^4+2t^2}$$
$$t^2+2$$
$$\underline{t^2+2}$$
$$0$$

Parcijalni: $\frac{1}{(t^2+2)(t^2+1)} = \frac{At+B}{t^2+2} + \frac{Ct+D}{t^2+1}$

$1 = (At+B)(t^2+1) + (Ct+D)(t^2+2)$

$1 = At^3 + At + Bt^2 + B + Ct^3 + 2Ct + Dt^2 + 2D$

$B+2D = 1$, $A+C=0$, $B+D=0$, $A+2C=0$
$A=-C$

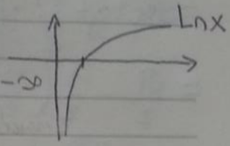
$\boxed{D=1}$ $\boxed{B=-1}$ $\boxed{\begin{array}{l} C=0 \\ A=0 \end{array}}$

$$\int_0^\infty \frac{1\,dt}{t^2+1} - \int_0^\infty \frac{dt}{t^2+2} = arctg\,t\Big|_0^\infty - \left(\frac{1}{\sqrt{2}}arctg\frac{t}{\sqrt{2}}\right)\Big|_0^\infty$$

$$= \frac{\pi}{2} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}\pi - \pi}{2\sqrt{2}} = \frac{\pi(\sqrt{2}-1)}{2\sqrt{2}}$$

9. a) $\int_0^1 \frac{dx}{\sqrt{x}} = \left| \begin{matrix} t=\sqrt{x} \\ dt=\frac{1}{2\sqrt{x}}dx \end{matrix} \right| = \int_0^1 2\,dt = 2t\Big|_0^1 = 2$

b) $\int_{-1}^2 \frac{dx}{x} = \ln|x|\Big|_{-1}^0 + \ln|x|\Big|_0^2 = -\infty + \ln 2 - \infty = -\infty$



c) $\int_0^3 \frac{dx}{(x-1)^3} \left|\begin{matrix} t=x-1\\ dt=dx\end{matrix}\right| = \int_{-1}^2 \frac{dt}{t^3} = -\frac{1}{2t}\Big|_{-1}^0 - \frac{1}{2t}\Big|_0^2 = -(\infty-\frac{1}{2})-(\frac{1}{8}-\infty)$

= divergira

10. $\int_0^{\frac{\pi}{2}} tgx\,dx = \int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x}dx \left|\begin{matrix} t=\cos x\\ dt=-\sin x\,dx\\ \frac{\pi}{2}\to 0\\ 0\to 1\end{matrix}\right| = \int_0^1 \frac{dt}{t}$

$= \ln|t|\Big|_0^1 = (0-(-\infty)) = \infty$

$\left[\cos^2 x = \frac{1}{tg^2x+1}\right]$

11. $\int_0^{\frac{\pi}{2}} \frac{tg^2x}{tg^2x+1}dx \left|\begin{matrix}t=tg\,x\\ dt=\frac{1}{\cos^2x}dx\end{matrix}\right| = \int_0^\infty \frac{t^2}{t^2+1}\cos^2x\,dt$

$\int_0^\infty \frac{t^2}{t^2+1}\,\frac{dt}{t^2+1} = \int_0^\infty \frac{t^2+1-1}{(t^2+1)(t^2+1)}dt = \int_0^\infty\frac{1}{t^2+1}dt - \int_0^\infty \frac{dt}{(t^2+1)^2}$

$m = \operatorname{arctg} t$ $\qquad \cos^2 x = \frac{1}{1+\operatorname{tg}^2 x}$

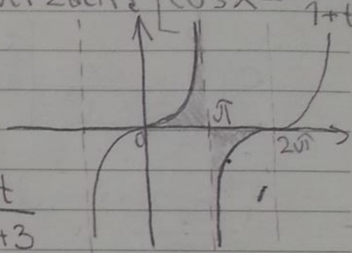
$$= \operatorname{arctg} t \Big|_0^{\infty} - \int_0^{\infty} \frac{dt}{(t^2+1)^2} \left| \begin{array}{l} t = \operatorname{tg} m \\ dt = \frac{1}{\cos^2 m}\, dm \end{array} \right| = \int_0^{\frac{\pi}{2}} \frac{dm}{\cos^2 m(\operatorname{tg}^2 m+1)^2} = \int_0^{\frac{\pi}{2}} \frac{dm}{\cos^2 m \frac{1}{(\cos^2 m)^2}}$$

$$= \operatorname{arctg} t \Big|_0^{\infty} - \int_0^{\frac{\pi}{2}} \cos^2 m \, dm = \left[\cos^2 m = \frac{1+\cos 2m}{2} \right]$$

$$= \operatorname{arctg} t \Big|_0^{\infty} - \left[\int_0^{\frac{\pi}{2}} \frac{1}{2}\, dm + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2m \, dm \right] = \operatorname{arctg} t - \frac{1}{2} m \Big|_0^{\frac{\pi}{2}} - \frac{\sin 2m}{4} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{1}{4}(0-0) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

12. $$\int_0^{2\pi} \frac{dx}{2+\cos x} \left| \begin{array}{l} \operatorname{tg}\frac{x}{2} = t \\ x = 2\operatorname{arctg} t \\ dx = \frac{2\,dt}{1+t^2} \end{array} \right| \rightarrow \text{Univerzalna } \left[\cos x = \frac{1-t^2}{1+t^2} \right]$$



$$= \int \frac{2\,dt}{(1+t^2)\left(2 + \frac{1-t^2}{1+t^2}\right)} = \int \frac{2\,dt}{2(1+t^2) + 1 - t^2} = \int \frac{2\,dt}{t^2+3}$$

$$2\int_0^{\infty} \frac{dt}{t^2+3} + 2\int_{-\infty}^{0} \frac{dt}{t^2+3} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{t}{\sqrt{3}}\right)\Big|_0^{+\infty} + \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{t}{\sqrt{3}}\right)\Big|_{-\infty}^{0}$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - 0\right) + \frac{2}{\sqrt{3}}\left(0 - -\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

13. $$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx = \left[\begin{array}{l} t = \frac{1+x}{1-x} \Rightarrow t - tx = 1 + x \Rightarrow t - 1 = x(t+1) \\ x = \frac{t-1}{t+1} \Rightarrow dx = \frac{(t+1)-(t-1)}{(t+1)^2} = \frac{2\,dt}{(t+1)^2} \end{array} \right]$$

$$2\int_1^{\infty} \frac{\sqrt{t}}{(t+1)^2}\, dt \left| \begin{array}{l} m = \sqrt{t} \Rightarrow t = m^2 \\ dm = \frac{1}{2\sqrt{t}}\, dt \end{array} \right| = 2\int_1^{\infty} \frac{\sqrt{t}\; 2\sqrt{t}\, dm}{(\underbrace{m^2}_{t=m^2}+1)^2} = 4\int_1^{\infty} \frac{m^2}{(m^2+1)^2}\, dm$$

$$= 4\int_1^{\infty} \frac{m^2+1-1}{(m^2+1)^2}\, dm = 4\left[\int_1^{\infty} \frac{1\,dm}{m^2+1} - \int_1^{\infty} \frac{dm}{(m^2+1)^2} \right]$$

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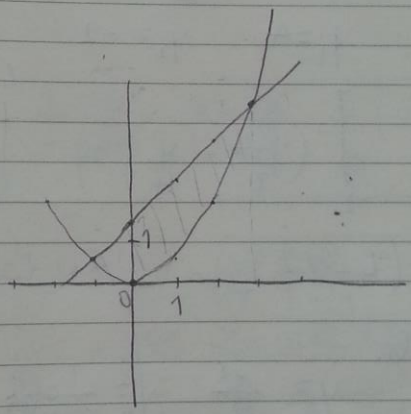
$$= 4\,\mathrm{arctg}\, m \Big|_1^\infty - 4\int_1^\infty \frac{dm}{(m^2+1)^2} \quad \left| \begin{matrix} \infty \Rightarrow \frac{\pi}{2} \\ 1 \to \frac{\pi}{4} \\ m = \mathrm{tg}\, c \\ dm = \frac{1}{\cos^2 c}\, dc \end{matrix} \right| \quad \text{Dalje kao 11. zad.}$$

$$= 4\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - 4\left[\frac{1}{2}c \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{\sin 2c}{4}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \pi - 4\left[\frac{\pi}{8} + \left(0 - \frac{1}{4}\right)\right] = \pi - \frac{\pi}{2} + 1 = \frac{\pi}{2} + 1.$$

14.



$$y = \frac{x^2}{2} \qquad y = x + \frac{3}{2}$$

$$\frac{x^2}{2} = x + \frac{3}{2} \quad / \cdot 2$$

$$x^2 - 2x - 3 = 0$$

$$x_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2}$$

$$x_1 = 3 \qquad x_2 = -1$$

$$\int_{-1}^{3} \left(x + \frac{3}{2} - \frac{x^2}{2}\right) dx = \left(\frac{x^2}{2} + \frac{3}{2}x - \frac{x^3}{6}\right)\Big|_{-1}^{3} = \left(\frac{9}{2} + \frac{9}{2} - \frac{27}{6}\right) - \left(\frac{1}{2} - \frac{3}{2} + \frac{1}{6}\right)$$

$$\frac{18}{2} - \frac{27}{6} + 1 - \frac{1}{6} = \frac{20}{2} - \frac{28}{6} = \frac{60-28}{6} = \frac{32}{6} = \frac{16}{3}$$

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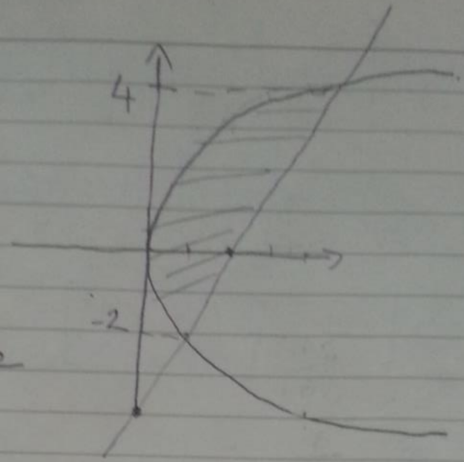
15.

$y^2=4x \qquad y=2x-4$



$\frac{y^2}{4}=\frac{y+4}{2}$

$2y^2-4y-16=0$

$y^2-2y-8=0$

$y_{12}=\frac{2\pm\sqrt{4+32}}{2}=\frac{2\pm6}{2}$

$y_1=4 \qquad y_2=-2$

$$\int_{-2}^{4}\left(\frac{y+4}{2}-\frac{y^2}{4}\right)dy=\int_{-2}^{4}\left(\frac{y}{2}+2-\frac{y^2}{4}\right)dy=\left(\frac{y^2}{4}+2y-\frac{y^3}{12}\right)\Big|_{-2}^{4}.$$

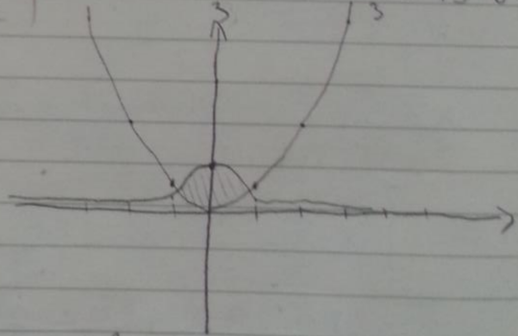
$$\left(4+8-\frac{16}{3}\right)-\left(1-4+\frac{8}{12}\right)=12-\frac{16}{3}+3-\frac{2}{3}=15-6=9$$

16.

$y=\frac{x^2}{2} \qquad y=\frac{1}{x^2+1}$

$yx^2+y=1$

$x^2=\frac{1-y}{y}$



$2y=\frac{1-y}{y}$

$2y^2+y-1=0$

$y_{12}=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm3}{4}$

$y_1=\frac{1}{2} \qquad y_2=-1$

$x=\sqrt{2y}=\pm1$

$$\int_{-1}^{1}\frac{1}{x^2+1}-\frac{x^2}{2}\,dx$$

$$=\operatorname{arctg}x\Big|_{-1}^{1}-\frac{x^3}{6}\Big|_{-1}^{1}$$

$$=\frac{\pi}{4}+\frac{\pi}{4}-\left(\frac{1}{6}+\frac{1}{6}\right)=\frac{\pi}{2}-\frac{1}{3}$$

17.

$P=1 \qquad T(0,0) \qquad y=ax^2 \qquad x=\sqrt{\frac{y}{a}}$

$$P=2\int_0^1 \sqrt{\frac{y}{a}}\,dy \left| \begin{matrix} t=\frac{y}{a} \\ dt=\frac{dy}{a} \end{matrix} \right| =2\int_0^{\frac{1}{a}} \sqrt{t}\,a\,dt=2a\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{\frac{1}{a}}=$$

$$2\,\frac{2}{3}a\left(\left(\frac{1}{a}\right)^{\frac{3}{2}}\right)=1$$

$$\frac{a}{a^{\frac{3}{2}}}=\frac{3}{4}$$

$$\frac{1}{\sqrt{a}}=\frac{3}{4} \qquad 3\sqrt{a}=4 \qquad \sqrt{a}=\frac{4}{3} \qquad a=\frac{16}{9}$$

$$y=\frac{16}{9}x^2$$

18.

$P=3 \qquad y=A\sin(\omega x+\varphi) \qquad y=A\sin(ax+b) \qquad b=0 \leftarrow \varphi=0 \qquad T=\pi$

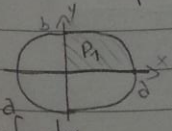
$a=2 \Leftarrow \omega=2\pi\frac{1}{T}=2$

$$P=\int_0^{\frac{\pi}{2}} A\sin 2x\,dx=-\frac{A}{2}\cos 2x\Bigg|_0^{\frac{\pi}{2}}=-\frac{A}{2}(-1-1)=A$$

$$y=3\sin(2x)$$

19. elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{x^2}{a^2} \;/\cdot b^2 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)=\frac{b^2}{a^2}(a^2-x^2)$

$y=\frac{b}{a}\sqrt{a^2-x^2}$



$\cos^2 t+\sin^2 t=1$

$\frac{x}{a}=\cos t \Rightarrow x=a\cos t \Rightarrow dx=-a\sin t\,dt \qquad \frac{y}{b}=\sin t \Rightarrow y=b\sin t$

$P_1=\frac{1}{4}P=\int_0^a y\,dx$

$$P_1=\int_0^a b\sin t(-a\sin t)dt=-ab\int \sin^2 t\,dt=-ab\int\frac{1-\cos 2t}{2}dt=-\frac{ab}{2}\int(1-\cos 2t)dt$$

$$=-\frac{ab}{2}\left(t-\frac{\sin 2t}{2}\right)\Bigg|_{\frac{\pi}{2}}^{0}=\frac{ab}{4}(\sin 2t-2t)\Bigg|_{\frac{\pi}{2}}^{0}=\frac{ab}{4}(\sin 0-2\cdot 0)-\frac{ab}{4}(\sin\pi-\pi)=\frac{ab\pi}{4}$$

$$P=4P_1=4\,\frac{ab\pi}{4}=ab\pi$$

20.

$y = \mathrm{sh}\,x,\ y = \frac{1}{2}e^{x-1},$ os y

$$\boxed{\mathrm{sh}\,x = \frac{e^x - e^{-x}}{2}}$$

$$\frac{e^x - e^{-x}}{2} = \frac{e^{x-1}}{2}$$

$$e^x - e^{-x} = e^x \cdot e^{-1} \ / : e^x$$

$$1 - \frac{1}{e^{2x}} = \frac{1}{e}$$

$$\frac{1}{e^{2x}} = 1 - \frac{1}{e}$$

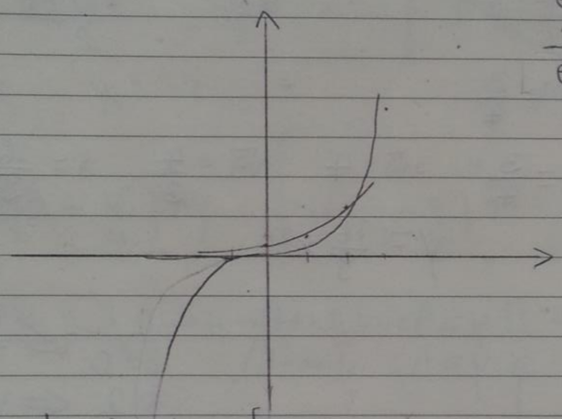
$$\frac{1}{e^{2x}} = \frac{e-1}{e}$$

$$e^{2x} = \frac{e}{e-1} \ / \mathrm{Ln}$$

$$2x = \mathrm{Ln}\left(\frac{e}{e-1}\right)$$

$$x = \frac{\mathrm{Ln}\left(\frac{e}{e-1}\right)}{2}$$

$$x = \frac{1 - \mathrm{Ln}(e-1)}{2}$$



$$\int_0^{\frac{1-\mathrm{Ln}(e-1)}{2}} \left(\frac{1}{2}e^{x-1} - \mathrm{sh}\,x\right) dx = \int \left(\frac{e^x}{2e} - \mathrm{sh}\,x\right) dx$$

$$e^{\frac{1-\mathrm{Ln}(e-1)}{2}} = \sqrt{e^{1-\mathrm{Ln}(e-1)}} = \sqrt{\frac{e}{e^{\mathrm{Ln}(e-1)}}} = \sqrt{\frac{e}{e-1}}$$

$$\frac{1}{2e} e^x \Big|_0^{\frac{1-\mathrm{Ln}(e-1)}{2}} - \mathrm{ch}\,x \Big|_0^{\frac{1-\mathrm{Ln}(e-1)}{2}}$$

$$\frac{1}{2e}\left(\sqrt{\frac{e}{e-1}} - 1\right) - \frac{e^x + e^{-x}}{2}\Big|_0^{\frac{1-\mathrm{Ln}(e-1)}{2}} = \frac{1}{2e}\left(\sqrt{\frac{e}{e-1}} - 1\right) - \left(\frac{\sqrt{\frac{e}{e-1}} + \frac{1}{\frac{\sqrt{e}}{\sqrt{e-1}}}}{2} - 1\right)$$

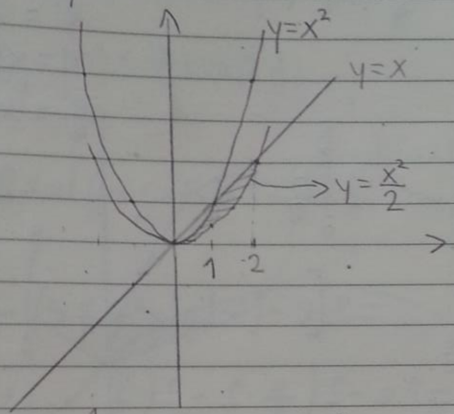
$$\frac{1}{2e}\left(\sqrt{\frac{e}{e-1}} - 1\right) - \left(\frac{\sqrt{\frac{e}{e-1}} + \sqrt{\frac{e-1}{e}} - 2}{2}\right) = \frac{1}{2e}\,\frac{\sqrt{e} - \sqrt{e-1}}{\sqrt{e-1}} - \frac{\frac{e + e - 1}{\sqrt{e-1}\,\sqrt{e}}}{2} + 1$$

$$= \frac{\sqrt{e}}{2e\sqrt{e-1}} - \frac{1}{2e} - \frac{2e-1}{\sqrt{e}\sqrt{e-1}} + 1 = \frac{1}{2\sqrt{e}\sqrt{e-1}} - \frac{1}{2e} - \frac{2e-1}{\sqrt{e}\sqrt{e-1}} + 1$$

$$= \frac{1 - 2(2e-1)}{2\sqrt{e}\sqrt{e-1}} - \frac{1+2e}{2e}$$

21.

$y=x^2$, $y=\frac{x^2}{2}$, $y=x$



$x^2=x$

$x^2-x=0$

$x(x-1)=0$

$x=0 \quad \boxed{x=1}$

$\frac{x^2}{2}=x$

$x^2-2x=0$

$x(x-2)=0$

$x=0 \quad \boxed{x=2}$

$$\int_0^1 \left(x^2-\frac{x^2}{2}\right)dx + \int_1^2 \left(x-\frac{x^2}{2}\right)dx$$

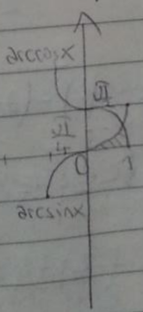
$$=\frac{x^3}{6}\Big|_0^1 + \frac{x^2}{2}\Big|_1^2 - \frac{x^3}{6}\Big|_1^2 = \frac{1}{6}+\frac{1}{2}(4-1)-\frac{1}{6}(8-1)$$

$$=\frac{1}{6}+\frac{3}{2}-\frac{7}{6}=-1+\frac{3}{2}=\frac{1}{2}$$

22.

$y=\arcsin x$, $y=\arccos x$, $\cos x$

$x=\sin y \qquad x=\cos y$



$$\to \int_0^{\frac{\pi}{2}} \cos y\,dy - \left[\int_0^{\frac{\pi}{4}} \sin y\,dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos y\,dy\right]$$

$$=\sin y\Big|_0^{\frac{\pi}{2}} - \left[-\cos y\Big|_0^{\frac{\pi}{4}} + \sin y\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$=1-\left[-\left(\frac{\sqrt{2}}{2}-1\right)+1-\frac{\sqrt{2}}{2}\right]$$

$$=1+\frac{\sqrt{2}}{2}-1-1+\frac{\sqrt{2}}{2}=-1+\sqrt{2}=\sqrt{2}-1$$

23.

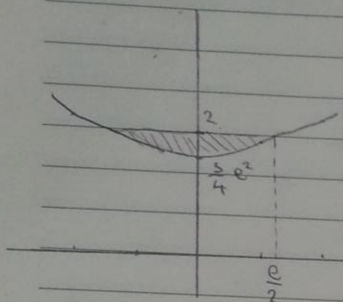
$y = \ln\left(x^2 + \frac{3}{4}e^2\right), \quad y = 2$

$2 = \ln\left(x^2 + \frac{3}{4}e^2\right) \;/e^{\wedge}$

$e^2 = x^2 + \frac{3}{4}e^2 \;/:e^2$

$1 = \frac{x^2}{e^2} + \frac{3}{4}$

$\frac{e^2}{4} = x^2 \qquad x = \pm\frac{e}{2}$



$2\cdot\int_0^{\frac{e}{2}}\left(2 - \ln\left(x^2 + \frac{3}{4}e^2\right)\right)dx = 2\left(e - \int_0^{\frac{e}{2}}\ln\left(x^2 + \frac{3}{4}e^2\right)dx\right)$

$= 2\left[e - \left(x\ln\left(x^2 + \frac{3}{4}e^2\right)\Big|_0^{\frac{e}{2}} - \int_0^{\frac{e}{2}}\frac{2x^2}{x^2 + \frac{3e^2}{4}}dx\right)\right] \qquad \left|\begin{array}{ll} u = \ln\left(x^2 + \frac{3e^2}{4}\right) & dv = dx \\ du = \frac{2x}{x^2 + \frac{3e^2}{4}} & v = x \end{array}\right|$

$= 2e - 2\left(\frac{e}{2}\cdot 2 - 2\left(\frac{e}{2} - \frac{3e^2}{4}\int_0^{\frac{e}{2}}\frac{dx}{x^2 + \frac{3e^2}{4}}\right)\right)$

$= 2e - 2e + 4\left(\frac{e}{2} - \frac{3e^2}{4}\frac{1}{\frac{\sqrt{3}e}{2}}\operatorname{arctg}\left(\frac{x}{\frac{\sqrt{3}e}{2}}\right)\Big|_0^{\frac{e}{2}}\right) = 2e - \frac{3e^2}{\frac{\sqrt{3}e}{2}}\left(\frac{\pi}{6} - 0\right) = 2e - \frac{e\pi}{\sqrt{3}}$

$= 2e\left(1 - \frac{\sqrt{3}}{6}\pi\right)$

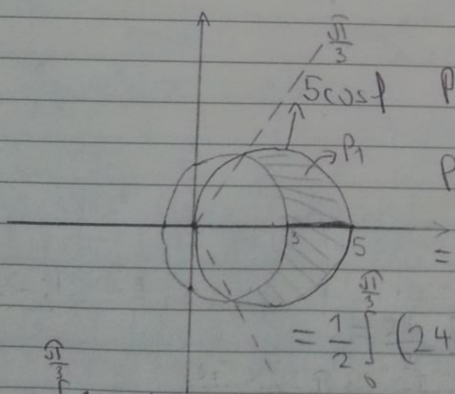
24.

$r = 5\cos\varphi, \quad r = 2 + \cos\varphi \qquad P = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)\,d\varphi$

$5\cos\varphi = 2 + \cos\varphi$

$4\cos\varphi = 2$

$\cos\varphi = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}, -\frac{\pi}{3}$



$P = 2P_1$

$P_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}}\left((5\cos\varphi)^2 - (2 + \cos\varphi)^2\right)d\varphi$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}}\left(25\cos^2\varphi - 4 - 4\cos\varphi - \cos^2\varphi\right)d\varphi$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}}\left(24\cos^2\varphi - 4\cos\varphi - 4\right)d\varphi = 12\int_0^{\frac{\pi}{3}}\cos^2\varphi - 2\sin\varphi\Big|_0^{\frac{\pi}{3}} - 2\varphi\Big|_0^{\frac{\pi}{3}}$

$= 12\int_0^{\frac{\pi}{3}}\frac{1 + \cos 2\varphi}{2}d\varphi - \frac{2\sqrt{3}}{2} - \frac{2\pi}{3} = 12\left(\frac{1}{2}\varphi\Big|_0^{\frac{\pi}{3}} + \frac{\sin 2\varphi}{4}\Big|_0^{\frac{\pi}{3}}\right) - \sqrt{3} - \frac{2\pi}{3} = \frac{12\pi}{6} + \frac{12}{4}\frac{\sqrt{3}}{2} - \sqrt{3} - \frac{2\pi}{3}$

$= 2\pi + \frac{3\sqrt{3}}{2} - \sqrt{3} - \frac{2\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$

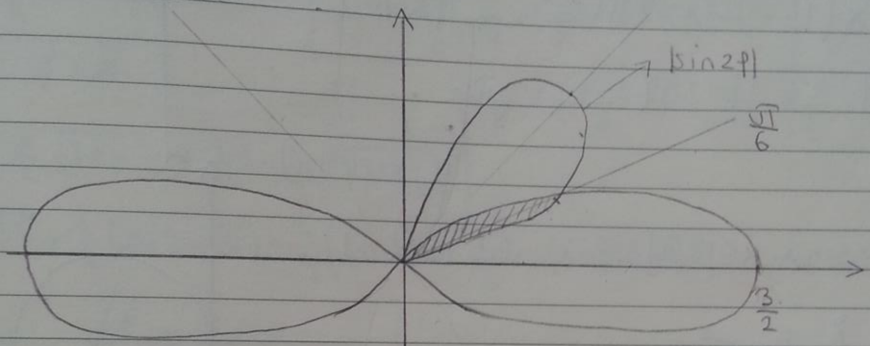
$P = 2P_1 = 2\left(\frac{4\pi}{3} + \frac{\sqrt{3}}{2}\right) = \frac{8\pi}{3} + \sqrt{3}$

25. krivo ispada.

BILJEŠKE

25.

$$r \le |\sin 2\varphi| \qquad r^2 \le \frac{3}{2}\cos(2\varphi)$$



$$\sin 2\varphi = \sqrt{\frac{3}{2}\cos 2\varphi} \quad /^2$$

$$\sin^2 2\varphi = \frac{3}{2}\cos 2\varphi$$

$$\frac{1-\cos 4\varphi}{2} = \frac{3}{2}\cos 2\varphi$$

$$3\cos 2\varphi + \cos 4\varphi = 1$$

$$\varphi = \frac{\pi}{6}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} \left(|\sin 2\varphi|\right)^2 d\varphi + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sqrt{\frac{3}{2}\cos(2\varphi)}\right)^2 d\varphi$$

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} \left(\frac{1-\cos 4\varphi}{2}\right) d\varphi + \frac{1}{2}\,\frac{3}{2}\,\frac{\sin 2\varphi}{2}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{1}{2}\varphi\Big|_0^{\frac{\pi}{6}} - \frac{\sin 4\varphi}{8}\Big|_0^{\frac{\pi}{6}}\right) + \frac{3}{8}\left(0 - \frac{\sqrt{3}}{2}\right)$$

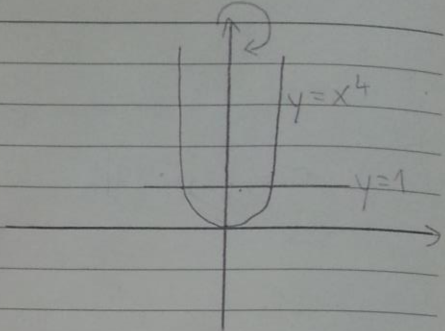
$$= \frac{1}{2}\left(\frac{\pi}{12} - \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)\right) - \frac{3\sqrt{3}}{16} = \frac{\pi}{24} - \frac{\sqrt{3}}{32} - \frac{3\sqrt{3}}{16} = \frac{\pi}{24} - \frac{7\sqrt{3}}{32}$$

Rj: $\frac{\pi}{6} + \frac{3}{2} - \frac{7\sqrt{3}}{8}$

$$2\pi\int_a^b \underbrace{(x)}_{\text{radius}}\underbrace{(f(x)-g(x))}_{\text{visina}}\,dx$$

26.

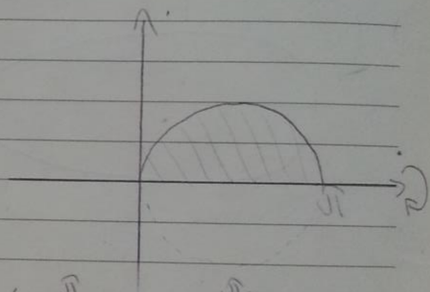
Volumen $y=x^4$, $y=1$, oko osi y



$$2\pi\int_0^1 x(1-x^4)\,dx = 2\pi\left(\frac{x^2}{2}-\frac{x^6}{6}\right)\Big|_0^1$$
$$=2\pi\left(\frac{1}{2}-\frac{1}{6}\right)=2\pi\left(\frac{3-1}{6}\right)=\frac{2\pi}{3}$$

27.

Volumen: $y=\sin x$ $x\in[0,\pi]$ i os x, oko os x

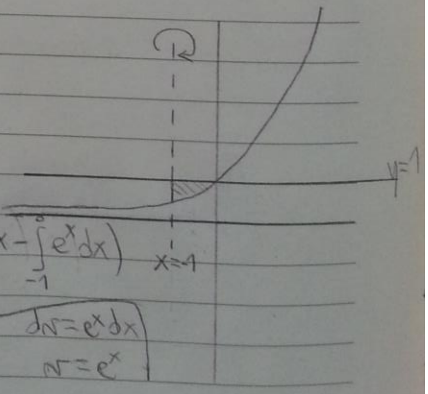


$$\int_a^b \pi\left[f(x)\right]^2 dx$$

$$\pi\int_0^\pi \sin^2 x\,dx=\pi\int_0^\pi \frac{1-\cos 2x}{2}\,dx=\pi\left(\frac{1}{2}x\Big|_0^\pi-\frac{\sin 2x}{4}\Big|_0^\pi\right)=\pi\left(\frac{\pi}{2}-0\right)$$
$$=\frac{\pi^2}{2}$$

28.

Volumen $y=e^x$, $y=1$ i $x=-1$ oko $x=-1$



$$2\pi\int_{-1}^0 (x+1)(1-e^x)\,dx=2\pi\left(\frac{x^2}{2}\Big|_{-1}^0+x\Big|_{-1}^0-\int_{-1}^0 xe^x dx-\int_{-1}^0 e^x dx\right)$$

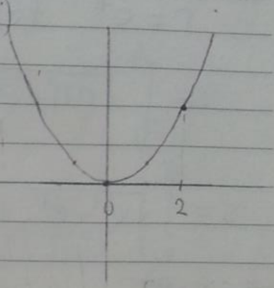
$$\boxed{\begin{array}{ll} u=x & dv=e^x dx \\ du=dx & v=e^x \end{array}}$$

$$=2\pi\left[\left(0-\frac{1}{2}\right)+(0+1)-\left(xe^x\Big|_{-1}^0-\int_{-1}^0 e^x dx\right)-e^x\Big|_{-1}^0\right]$$
$$=2\pi\left[\frac{1}{2}-\left(\left(0+\frac{1}{e}\right)-\left(1-\frac{1}{e}\right)\right)-\left(1-\frac{1}{e}\right)\right]$$
$$=2\pi\left[\frac{1}{2}-\frac{1}{e}+1-\frac{1}{e}-1+\frac{1}{e}\right]=2\pi\left(\frac{1}{2}-\frac{1}{e}\right)=\pi\left(1-\frac{2}{e}\right)$$

BILJEŠKE

29

Duljina Luka parabole $y=\frac{x^2}{2}$ od $(0,0)$ do $(2,2)$



$$\int_a^b \sqrt{1+f'(x)^2}\,dx = \int_0^2 \sqrt{1+x^2}$$

$y'=x$

$$=\left[\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]\Big|_0^2$$

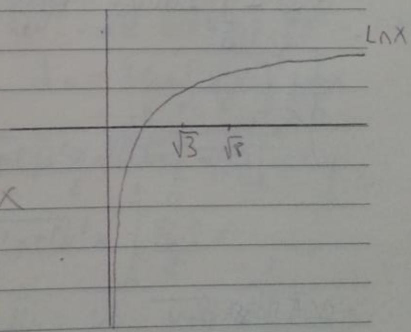
$$=\sqrt{5}+\frac{1}{2}\ln(2+\sqrt{5})-\frac{1}{2}(\ln 1)=\frac{1}{2}\ln(2+\sqrt{5})+\sqrt{5}$$

Pravila: 1) $\int\sqrt{1+x^2}\,dx=\left[\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]$

2) $\int\frac{1}{x}\sqrt{1+x^2}\,dx=\sqrt{1+x^2}-\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|$

30

$y=\ln x$ od $x=\sqrt{3}$ do $x=\sqrt{8}$



$y'=\frac{1}{x}$

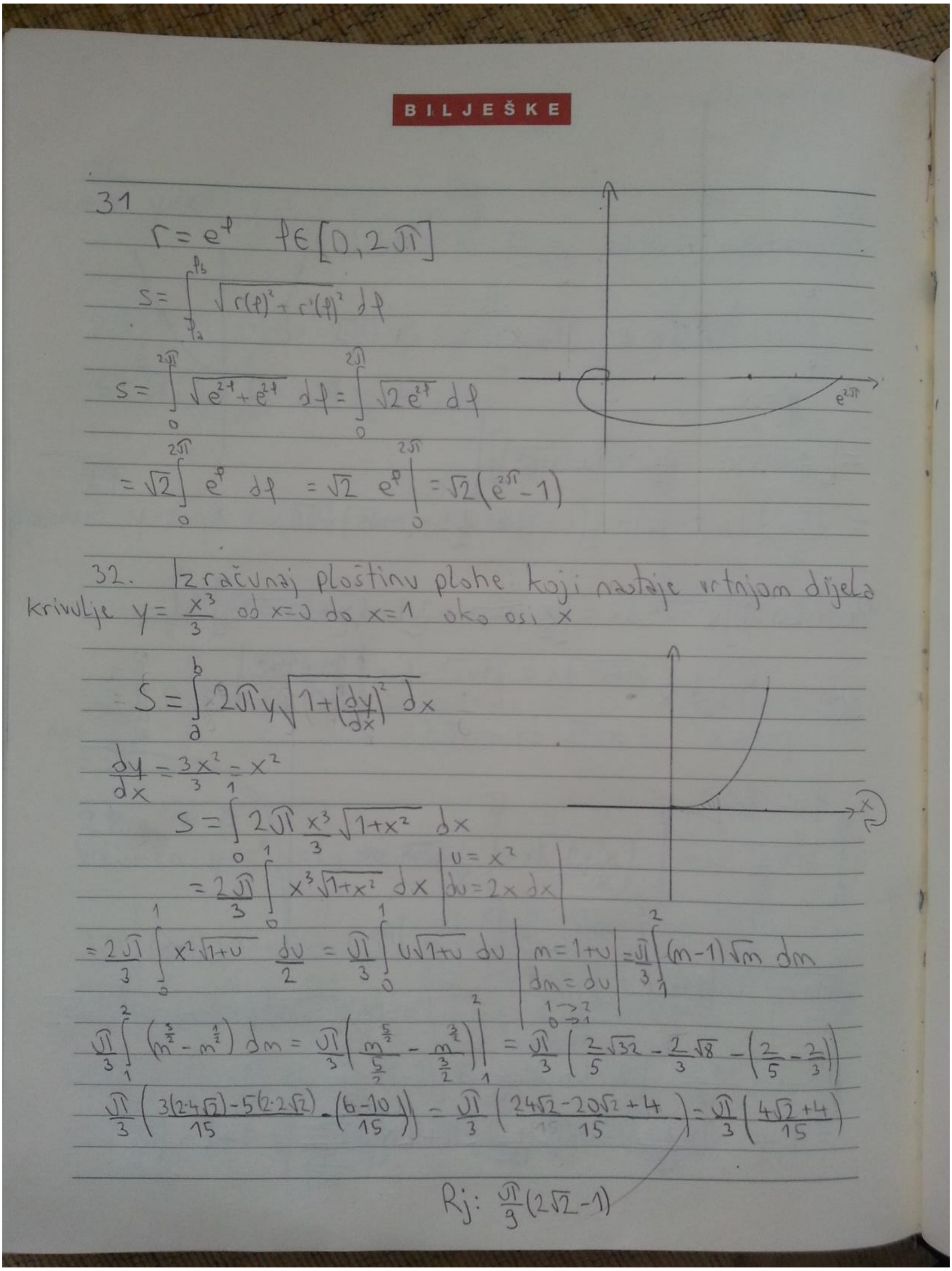
$$\int_{\sqrt{3}}^{\sqrt{8}}\sqrt{1+\frac{1}{x^2}}\,dx=\int_{\sqrt{3}}^{\sqrt{8}}\sqrt{\frac{x^2+1}{x^2}}\,dx=\int_{\sqrt{3}}^{\sqrt{8}}\frac{1}{x}\sqrt{x^2+1}\,dx$$

$$=\left(\sqrt{1+x^2}-\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|\right)\Big|_{\sqrt{3}}^{\sqrt{8}}$$

$$=3-\ln\left|\frac{1+3}{\sqrt{8}}\right|-\left(2-\ln\left|\frac{1+2}{\sqrt{3}}\right|\right)$$

$$=1+\ln\left|\frac{3}{\sqrt{3}}\right|-\ln\left|\frac{4}{\sqrt{8}}\right|=1+\ln\left|\frac{\sqrt{3}}{\frac{4}{2\sqrt{2}}}\right|=1+\ln\left|\frac{\sqrt{6}}{2}\right|$$

32. krivo ispada.



BILJEŠKE

31

$r = e^{\varphi} \quad \varphi \in [0, 2\pi]$

$$S = \int_{\varphi_a}^{\varphi_b} \sqrt{r(\varphi)^2 + r'(\varphi)^2}\, d\varphi$$

$$S = \int_0^{2\pi} \sqrt{e^{2\varphi} + e^{2\varphi}}\, d\varphi = \int_0^{2\pi} \sqrt{2e^{2\varphi}}\, d\varphi$$

$$= \sqrt{2}\int_0^{2\pi} e^{\varphi}\, d\varphi = \sqrt{2}\, e^{\varphi}\Big|_0^{2\pi} = \sqrt{2}\left(e^{2\pi}-1\right)$$

$e^{2\pi}$

32. Izračunaj ploštinu plohe koji nastaje vrtnjom dijela krivulje $y = \frac{x^3}{3}$ od $x=0$ do $x=1$ oko osi x

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{3x^2}{3} = x^2$$

$$S = \int_0^1 2\pi \frac{x^3}{3}\sqrt{1+x^2}\, dx$$

$$= \frac{2\pi}{3}\int_0^1 x^3\sqrt{1+x^2}\, dx \quad \left|\begin{array}{l} u = x^2 \\ du = 2x\, dx \end{array}\right|$$

$$= \frac{2\pi}{3}\int_0^1 x^2\sqrt{1+u}\, \frac{du}{2} = \frac{\pi}{3}\int_0^1 u\sqrt{1+u}\, du \quad \left|\begin{array}{l} m = 1+u \\ dm = du \\ 1 \to 2 \\ 0 \to 1 \end{array}\right| = \frac{\pi}{3}\int_1^2 (m-1)\sqrt{m}\, dm$$

$$\frac{\pi}{3}\int_1^2 \left(m^{\frac{3}{2}} - m^{\frac{1}{2}}\right) dm = \frac{\pi}{3}\left(\frac{m^{\frac{5}{2}}}{\frac{5}{2}} - \frac{m^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^2 = \frac{\pi}{3}\left(\frac{2}{5}\sqrt{32} - \frac{2}{3}\sqrt{8} - \left(\frac{2}{5} - \frac{2}{3}\right)\right)$$

$$\frac{\pi}{3}\left(\frac{3(2\cdot4\sqrt{2}) - 5(2\cdot2\sqrt{2})}{15} - \left(\frac{6-10}{15}\right)\right) = \frac{\pi}{3}\left(\frac{24\sqrt{2} - 20\sqrt{2} + 4}{15}\right) = \frac{\pi}{3}\left(\frac{4\sqrt{2}+4}{15}\right)$$

Rj: $\frac{\pi}{9}\left(2\sqrt{2}-1\right)$

BILJEŠKE

33.

$y=\frac{x^3}{3}$ od $x=0$ do $x=1$ oko osi y

$$S=\int_c^d 2\pi x \underbrace{\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}_{ds}$$

$$=\int_0^1 2\pi x \sqrt{1+x^4}\,dx \left|\begin{matrix} u=x^2 \\ du=2x\,dx \end{matrix}\right| = \int_0^1 \pi\sqrt{1+u^2}\,du$$

$$=\pi\left(\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right)\Bigg|_0^1=\frac{\pi}{2}\left(\sqrt{2}+\ln(1+\sqrt{2})\right)$$

Formule za ploštinu plohe:

oko x-osi: $\int 2\pi y\,ds$

oko y-osi: $\int 2\pi x\,ds$

- ds biramo kako nam paše:

$ds=\sqrt{1+[f'(x)]^2}\,dx$

iLi

$ds=\sqrt{1+[g'(y)]^2}\,dy$

Nešto sam pokušavao, ali dalje mi se nije dalo rješavati ovaj 34. .

BILJEŠKE

19. zadatak iz DZ $ds = 6a$

34. Izračunati ploštinu plohe koja nastaje vrtnjom astroide

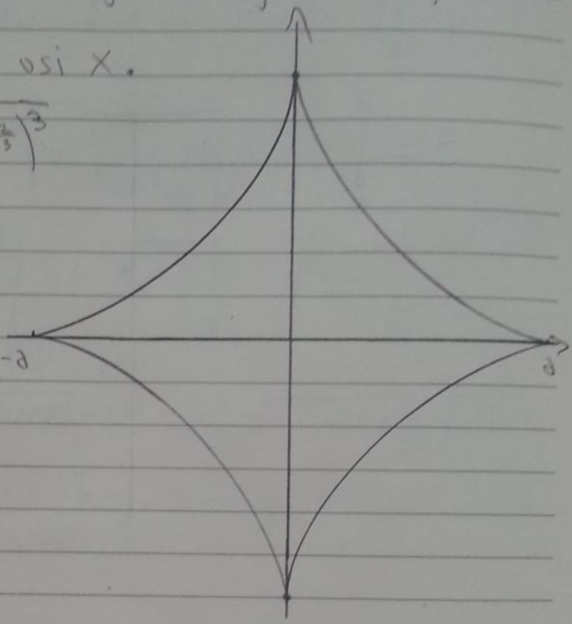
$x = a\cos^3 t$, $y = a\sin^3 t$ oko osi x.

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ $\quad y = \sqrt{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3}$

$ds = 6a$

$= \int_{-a}^{a} 2\pi y \, ds$

$= 12a\pi \int_{-a}^{a} \sqrt{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3} \, dx$



Rj: $\frac{12}{5}\pi a^2$