1. 3B $z^5 - 2z^3 + 2z = 0$

$z(z^4 - 2z^2 + 2) = 0$

$\downarrow \qquad \searrow$

$z_1 = 0 \qquad z^4 - 2z^2 + 2 = 0$ uvedemo $z^2 = t$

$\Rightarrow t^2 - 2t + 2 = 0 \quad t_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

- $z^2 = 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4} \Rightarrow z = \sqrt[4]{2} \operatorname{cis} \frac{\frac{\pi}{4} + 2k\pi}{2}, k = 0, 1$

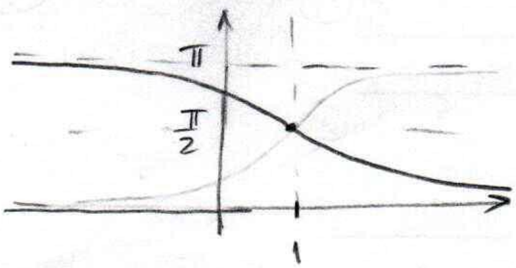
$\Rightarrow z_2 = \sqrt[4]{2} \operatorname{cis} \frac{\pi}{8} \qquad z_3 = \sqrt[4]{2} \operatorname{cis} \frac{9\pi}{8}$

- $z^2 = 1 - i = \sqrt{2} \operatorname{cis} \frac{7\pi}{4} \Rightarrow z = \sqrt[4]{2} \operatorname{cis} \frac{\frac{7\pi}{4} + 2k\pi}{2} \quad k = 0, 1$

$z_4 = \sqrt[4]{2} \operatorname{cis} \frac{7\pi}{8} \qquad z_5 = \sqrt[4]{2} \operatorname{cis} \frac{15\pi}{8}$

2. 5B $f(x) = \frac{\pi}{2} - \operatorname{arctg}(x-1), \quad D(\operatorname{arctg}) = \mathbb{R} \Rightarrow D(f) = \mathbb{R}$

a) 3B $\operatorname{Im}(\operatorname{arctg}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \operatorname{Im}(f) = (0, \pi)$



tj. $f: \mathbb{R} \to (0, \pi)$

b) 2B f je kompozicija bijekcija pa je i sama bijekcija

$f^{-1}: (0, \pi) \to \mathbb{R} \Rightarrow D(f^{-1}) = (0, \pi), \quad \operatorname{Im}(f^{-1}) = \mathbb{R}$

$y = \frac{\pi}{2} - \operatorname{arctg}(x-1) \Rightarrow \operatorname{arctg}(x-1) = \frac{\pi}{2} - y \Rightarrow x - 1 = \operatorname{tg}(\frac{\pi}{2} - y)$

$\Rightarrow x = \operatorname{tg}(\frac{\pi}{2} - y) + 1 \Rightarrow f^{-1}(x) = \operatorname{tg}(\frac{\pi}{2} - x) + 1$

3. a) Neka je $(a_n)_{n\in\mathbb{N}}$ niz. Limes niza (a_n) je $a\in\mathbb{R}$ u oznaci $\lim_n a_n = a$ 1B
ako $(\forall\varepsilon>0)(\exists n_0\in\mathbb{N})$ $n\in\mathbb{N}, n\geq n_0 \Rightarrow |a_n - a|<\varepsilon$ 1B

6B b) Neka je $\{a_n\}$ rastući niz, omeđen odozgo te neka
2B je $a = \sup_n \{a_n\}$.

Tada $(\forall\varepsilon>0)(\exists n_0\in\mathbb{N})$ takav da $a-\varepsilon < a_{n_0} \leq a$

U suprotnom a ne bi bio supremum.

Kako je niz rastući vrijedi $a-\varepsilon < a_{n_0} \leq a_n \leq a$ za $n\geq n_0$
tj. $|a_n - a| \leq \varepsilon$ za $n \geq n_0$

c) 3B $a_1 = 1$, $a_2 = \frac{1}{4}+1 = \frac{5}{4}$ $a_3 = \frac{5}{16}+1 = \frac{21}{16}$ ...

Niz je rastući:

BAZA: $a_2 \geq a_1$ PRETPOSTAVKA: $a_{n+1} \geq a_n$ za neki $n\in\mathbb{N}$

$a_{n+1} \geq a_n \Rightarrow \frac{1}{4}a_{n+1} \geq \frac{1}{4}a_n \Rightarrow \frac{1}{4}a_{n+1}+1 \geq \frac{1}{4}a_n + 1$ tj. $a_{n+2} \geq a_{n+1}$

Niz je omeđen:

BAZA: $a_1 < 2$ PRETPOSTAVKA: $a_n \leq 2$ za neki $n\in\mathbb{N}$

KORAK: $a_{n+1} = \frac{1}{4}a_n + 1 \leq \frac{1}{4}\cdot 2 + 1 = \frac{1}{2}+1 = \frac{3}{2} \leq 2$

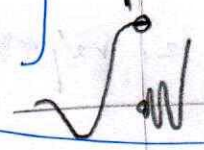
$\Rightarrow$ Niz $\{a_n\}$ je omeđen i monoton pa je primjenom tvrdnje pod b) i konvergentan

$\lim a_{n+1} = \frac{1}{4}\lim a_n + 1$ uz $L = \lim_n a_n \Rightarrow L = \frac{1}{4}L + 1 \Rightarrow L = \frac{4}{3}$

4. 5B

a) (2) $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} \frac{\sin(x)}{x} = \lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$

$\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} x\cdot\sin\left(\frac{1}{x}\right) = 0$



i to dvoje ne možemo spojiti s $f(0)=a$

Vidimo da je $\boxed{\lim\limits_{x\to 0-} f(x) \neq \lim\limits_{x\to 0+} f(x) \text{ pa ne postoji limes u } x_0 = 0}$

b) (1) Funkcija $f: D(f)\to \mathrm{Im}(f)$ je neprekidna u $x_0 \in D(f)$ ako $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

c) (2) $\lim\limits_{x\to 1-} f(x) = \lim\limits_{x\to 1-} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} \overset{L'H}{=} \lim\limits_{x\to 1-} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{3}\frac{1}{\sqrt[3]{x^2}}} = \lim\limits_{x\to 1-} \frac{3}{2}\frac{\sqrt[3]{x^2}}{\sqrt{x}} = \frac{3}{2}$

$\lim\limits_{x\to 1+} f(x) = b\lim\limits_{x\to 1+} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} \overset{L'H}{=} b\lim\limits_{x\to 1+} \frac{\frac{1}{3}\frac{1}{\sqrt[3]{x^2}}}{\frac{1}{4}\frac{1}{\sqrt[4]{x^3}}} = \frac{4}{3}b\lim\limits_{x\to 1+} \frac{\sqrt[4]{x^3}}{\sqrt[3]{x^2}} = \frac{4}{3}b$

Da bi f bila neprekidna mora vrijediti $\frac{3}{2} = a = \frac{4}{3}b$

$$\Rightarrow a = \frac{3}{2},\ b = \frac{9}{8} \Rightarrow f(x) = \begin{cases} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}, & x<1 \\ \frac{3}{2}, & x=1 \\ \frac{9}{8}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}, & x>1 \end{cases}$$

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5. 5B

a) 1 $f'(x_0) = \lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$

b) 2 $(\sqrt{x})' = \lim\limits_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim\limits_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim\limits_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

c) 2 $f:\mathbb{R}\to\mathbb{R}$, $f(x) = |x-2|$

$\lim\limits_{h\to 0+} \frac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0+} \frac{|2+h-2|-|2-2|}{h} = \lim\limits_{h\to 0+} 1 = 1$

$\lim\limits_{h\to 0-}$ — " — $= \lim\limits_{h\to 0-} \frac{|h|}{h} = \lim\limits_{h\to 0-} -1 = -1$

Vidimo $\lim\limits_{h\to 0+} \frac{f(2+h)-f(2)}{h} \neq \lim\limits_{h\to 0-} \frac{f(2+h)-f(2)}{h}$

Dakle $f'(2)$ ne ...

6. To zadovoljava krivulju :-)

4B

$$y - y_0 = y'(x_0)(x - x_0) \qquad T(0,1)$$

$$e^{x^2} - x^2 y + 2\ln y = 1 \quad / \frac{d}{dx}$$

$$e^{x^2} \cdot 2x - 2xy - x^2 y' + 2\frac{1}{y} \cdot y' = 0$$

$$\Rightarrow y'\left(\frac{2}{y} - x^2\right) = 2x\left(y - e^{x^2}\right) \Rightarrow y' = \frac{2x\left(y - e^{x^2}\right)}{\frac{2}{y} - x^2}$$

$$y'(1) = \frac{2 \cdot 0\,(1 - e^0)}{2 - 0} = 0$$

$$\Rightarrow y - 1 = 0 \cdot (x - 0) \Rightarrow \boxed{y = 1} \text{ TANGENTA}$$

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7. 7B

b) (T1) obrat po kontrapoziciji Fermatovog teorema (točna tvrdnja)

(T2) uzmemo $f(x) = x^3 \Rightarrow f'(0) = 0$, a f nema ekstrem u $x = 0$

a) Fermatov teorem:

Neka je $(a,b) \subseteq \mathbb{R}$ otvoren te neka funkcija $f : (a,b) \to \mathbb{R}$ poprima svoju najmanju ili najveću vrijednost u točki $c \in (a,b)$. Ako derivacija u točki c postoji tada je $f'(c) = 0$.

Dokaz:

Neka f u c poprima svoj maksimum. Ako $f'(c)$ postoji tada postoje i

$$\lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0, \quad a \quad \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0$$

$\boxed{8.}$ $D(f)=\mathbb{R}\setminus\{0,4\}$

$\boxed{5B}$

$$\lim_{x\to 0+}\frac{16}{x^2(x-4)}=-\infty \qquad \lim_{x\to 0-}\frac{16}{x^2(x-4)}=-\infty$$

$$\lim_{x\to 4+}\frac{16}{x^2(x-4)}=+\infty \qquad \lim_{x\to 4-}\frac{16}{x^2(x-4)}=-\infty$$

$\boxed{\text{V.A } x=0,\ x=4}$

Asimptote:

$$\lim_{x\to+\infty}\frac{f(x)}{x}=0=\lim_{x\to-\infty}\frac{f(x)}{x}$$

Ali je i $l=0$ jer je $\lim\limits_{x\to\pm\infty} f(x)=0$

$\Rightarrow$ $y=0$ je lijeva i desna horizontalna asimptota

$$f'(x)=16\cdot\frac{-1}{[x^2(x-4)]^2}\cdot\left(2x(x-4)+x^2\right)=-\frac{16}{x^4(x-4)^2}\cdot\left(2x^2-8x+x^2\right)$$

$$=\frac{-16}{x^4(x-4)^2}\cdot x(3x-8)=0 \to x=0,\quad x=\frac{8}{3}$$

	$-\infty$	0	$\frac{8}{3}$	4	$+\infty$
f'	$-$	$+$	$-$	$-$	
f	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	

~~lokalni min~~ (kod 0)   lokalni max (kod $\frac{8}{3}$)

ali 0 nije u domeni!!

$$f\left(\frac{8}{3}\right)=\frac{16}{\frac{64}{9}\left(\frac{8}{3}-4\right)}=\frac{16}{\frac{64}{9}\cdot\frac{-4}{3}}$$

$$=-\frac{4\cdot 3\cdot 9}{4\cdot 16}=-\frac{27}{16}$$