f, ZZV

Zadatak 1.

VRIJEDI ZA ZADATKE POD a, b, c

a) Dokaži $\nabla f(\|\vec{r}-\vec{a}\|) = f'(\|\vec{r}-\vec{a}\|)\cdot \frac{\vec{r}-\vec{a}}{\|\vec{r}-\vec{a}\|}$

$$\boxed{\begin{array}{l} \vec{r} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n \ , \|\vec{r}\| = \sqrt{x_1^2 + \dots + x_n^2} \\ \vec{a} \in \mathbb{R}^n \longrightarrow \vec{a} = y_1\vec{e}_1 + \dots + y_n\vec{e}_n \ , \ \|\vec{a}\| = \sqrt{y_1^2+\dots+y_n^2} \end{array}}$$

Radi jednostavnosti: $\|\vec{r}-\vec{a}\| = r-a$, $\|\vec{r}\| = r$, $\|\vec{a}\| = a$

$$\nabla f(r-a) = \sum_{i=1}^{n} \frac{df(r-a)}{d(x_i-y_i)}\vec{e_i} \quad \sum_{i=1}^{n} \frac{df(r-a)}{d(r-a)} \cdot \frac{d(r-a)}{d(x_i-y_i)}\vec{e_i} =$$

$$= \sum_{i=1}^{n} f'(r-a) \cdot \frac{d(r-a)}{d(x_i-y_i)} \vec{e_i} =$$

$$= f'(r-a) \cdot \frac{(x_1-y_1)\vec{e}_1 + \dots + (x_n-y_n)\vec{e}_n}{\sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}} =$$

$$= f'(r-a)\frac{\vec{r}-\vec{a}}{r-a}$$

$$\Rightarrow f'(\|\vec{r}-\vec{a}\|) \cdot \frac{\vec{r}-\vec{a}}{\|\vec{r}-\vec{a}\|} \qquad \underline{\underline{\text{QED}}}$$

b) Dokaži $\nabla\|\vec{r}-\vec{a}\|^2 = 2(\vec{r}-\vec{a})$

Radi jednostavnosti: $\|\vec{r}-\vec{a}\|^2 = (r-a)^2$

$$\nabla(r-a)^2 = \sum_{i=1}^{n} \frac{d[(r-a)^2]}{d(x_i-y_i)}\vec{e_i} = \sum_{i=1}^{n} \frac{d[(r-a)^2]}{d(r-a)} \cdot \frac{d(r-a)}{d(x_i-y_i)}\vec{e_i} =$$

$$= \sum_{i=1}^{n} [(r-a)^2]' \cdot \frac{d(r-a)}{d(x_i-y_i)}\vec{e_i} =$$

$$= 2(r-a) \cdot \frac{(x_1-y_1)\vec{e}_1 + \dots + (x_n-y_n)\vec{e}_n}{\sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}} =$$

$$= 2\cancel{(r-a)} \cdot \frac{\vec{r}-\vec{a}}{\cancel{r-a}} = 2(\vec{r}-\vec{a}) \qquad \underline{\underline{\text{QED}}}$$

c) $\nabla e^r = ?$

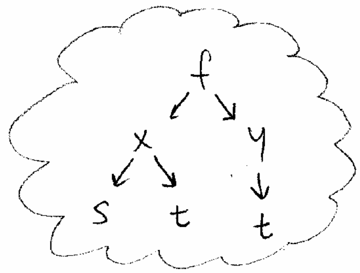
$$\nabla e^r = \sum_{i=1}^{n} \frac{\partial (e^r)}{\partial x_i} e_i = \sum_{i=1}^{n} \frac{\partial (e^r)}{\partial r} \cdot \frac{\partial r}{\partial x_i} \vec{e}_i = (e^r)' \cdot \frac{x_1 \vec{e}_1 + \dots x_n \vec{e}_n}{\sqrt{x_1^2 + \dots + x_n^2}} =$$

$$= e^r \cdot \frac{\vec{r}}{r}$$ QED.

Zadatak 2.

$f(x,y) = x^2 - \ln y$

$x = s \sin t, \ y = t^2$



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$\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} = ?$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} = 2x \cdot \sin t = 2 s \sin^2 t$$

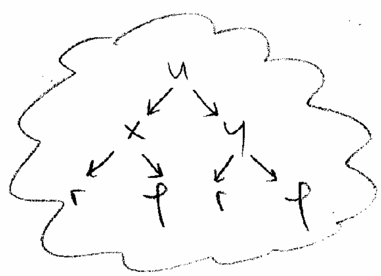
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = 2x \cdot s \cdot \cos t - \frac{1}{y} \cdot 2t = s^2 \sin 2t - \frac{2}{t}$$

Zadatak 3.

$u = x^2 y - \sin(x-y)$

$x = r \cos \varphi$

$y = r \sin \varphi$



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$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \varphi} = ?$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} =$$

$$= [2xy - \cos(x-y)] \cdot \cos \varphi + [x^2 - \cos(x-y)] \sin \varphi =$$

$$= r^2 \sin 2\varphi \cdot \cos \varphi - \cos(r^2 \sin 2\varphi) \cdot \cos \varphi + r^2 \cos^2 \varphi \sin \varphi - \cos(r^2 \sin 2\varphi) \sin \varphi$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \varphi} = [2xy - \cos(x-y)] \cdot (-r \sin \varphi) + [x^2 - \cos(x-y)] \cdot r \cos \varphi =$$

$$= -r^3 \sin 2\varphi \cdot \sin \varphi + r \cos(r^2 \sin 2\varphi) \sin \varphi + r^3 \cos^3 \varphi - r \cos(r^2 \sin 2\varphi) \cos \varphi$$

Zadatak 4.

$f(x,y) = x^2 - y^4$

$A(2,1)$

Tangenta na nivo-krivulju?

Nivo-krivulja: $\quad x^2 - y^4 = c$, točk A leži na krivulji:

$2^2 - 1^4 = c$

$c = 3$

$x^2 - y^4 = 3 \quad \rightarrow$ nivo-krivulja

Tangenta na krivulju $x^2 - y^4 - 3 = 0$, odn $f(x,y) = x^2 - y^4 - 3$:

$(f'_x)_A (x - x_A) + (f_y')_A (y - y_A) = 0$

$2x_A (x - x_A) + (-4y_A^3)(y - y_A) = 0$

$4(x-2) - 4(y-1) = 0$

$4x - 4y - 8 + 4 = 0$

$x - y - 1 = 0$

$t \ldots \; y = x - 1$

Zadatak 5.

$f(x,y,z) = x^2 \sin(yz) - y \ln(x+z)$

$A(1,1,1)$

Tang. ravnina na nivo-plohu?

$x^2 \sin(yz) - y \ln(x+z) = c$, c određen točkom A, ali je nebitan jer se izgubi deriviranjem

$\pi \equiv (f_x')_A (x - x_A) + (f_y')_A (y - y_A) + (f_z')_A (z - z_A) = 0$

$(f'_x)_A = 2x_A \sin(y_A z_A) - y_A \dfrac{1}{x_A + z_A} = 2\sin 1^\circ - \dfrac{1}{2} \quad \rightarrow \quad \sin \dfrac{\pi}{180}$

$(f'_y)_A = x_A^2 \cos(y_A z_A) \cdot z_A - \ln(x_A + z_A) = \cos 1^\circ - \ln 2 \quad \rightarrow \quad \cos \dfrac{\pi}{180}$

$(f'_z)_A = x_A^2 \cos(y_A z_A) \cdot y_A - y_A \dfrac{1}{x_A + z_A} = \cos 1^\circ - \dfrac{1}{2} \quad \rightarrow \quad \cos \dfrac{\pi}{180}$

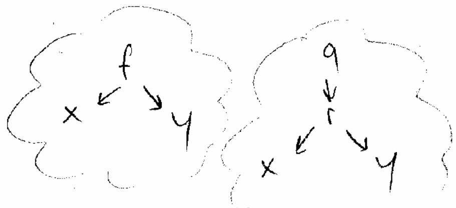
$(\pi) \equiv (2\sin 1^\circ - \frac{1}{2})(x-1) + (\cos 1^\circ - \ln 2)(y-1) + (\cos 1^\circ - \frac{1}{2})(z-1) = 0$

$\to (2\sin 1^\circ - \frac{1}{2})x + (\cos 1^\circ - \ln 2)y + (\cos 1^\circ - \frac{1}{2})z = 2\cos 1^\circ + 2\sin 1^\circ - \ln 2 - 1$

Zadatak 6.

$f(x,y) = q(r)$

$r = \sqrt{x^2+y^2}$



Dokaži $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = q''(r) + \frac{1}{r} q'(r)$

$$\frac{\partial f}{\partial x} = \frac{\partial q}{\partial x} = \frac{\partial q}{\partial r} \cdot \frac{\partial r}{\partial x} = q'(r) \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = q'(r) \cdot \frac{x}{r}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(q'(r) \cdot \frac{x}{r}\right) = \frac{\partial}{\partial x}(q'(r)) \cdot \frac{x}{r} + q'(r) \cdot \frac{\partial}{\partial x}\left(\frac{x}{r}\right) =$$

$$= \frac{\partial[q'(r)]}{\partial r} \cdot \frac{\partial r}{\partial x} \cdot \frac{x}{r} + q'(r) \cdot \frac{\frac{\partial x}{\partial x} \cdot r - x \cdot \frac{\partial r}{\partial x}}{r^2} =$$

$$= q''(r) \cdot \frac{x}{r} \cdot \frac{x}{r} + q'(r) \cdot \frac{r - x \cdot \frac{x}{r}}{r^2} = \underline{q''(r) \cdot \frac{x^2}{r^2} + q'(r) \cdot \frac{r^2 - x^2}{r^3}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial q(r)}{\partial y} = \frac{\partial q(r)}{\partial r} \cdot \frac{\partial r}{\partial y} = q'(r) \cdot \frac{y}{r}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(q'(r) \cdot \frac{y}{r}\right) = \frac{\partial[q'(r)]}{\partial y} \cdot \frac{y}{r} + q'(r) \cdot \frac{\partial}{\partial y}\left(\frac{y}{r}\right) =$$

$$= q''(r) \cdot \frac{y^2}{r^2} + q'(r) \cdot \frac{\frac{\partial y}{\partial y} \cdot r - y \cdot \frac{\partial r}{\partial y}}{r^2} =$$

$$= q''(r) \cdot \frac{y^2}{r^2} + q'(r) \cdot \frac{r - y \cdot \frac{y}{r}}{r^2} = \underline{q''(r) \cdot \frac{y^2}{r^2} + q'(r) \cdot \frac{r^2 - y^2}{r^3}}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = q''(r) \underbrace{\left[\frac{\overset{r^2}{x^2 + y^2}}{r^2}\right]}_{1} + q'(r) \cdot \underbrace{\left[\frac{r^2 - x^2 + r^2 - y^2}{r^3}\right]}_{1/r} =$$

$$= q''(r) + q'(r) \cdot \frac{1}{r}$$

QED.

zadatak 7.

$$f(x,y) = e^{-x^2-y^2}$$

DOKAŽI : $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4f(x,y)(x^2+y^2-1)$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2} \cdot (-2x) = -2xe^{-x^2-y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-2xe^{-x^2-y^2}\right) = -2\left[\frac{\partial x}{\partial x} \cdot e^{-x^2-y^2} + x \cdot \frac{\partial (e^{-x^2-y^2})}{\partial x}\right] =$$

$$= -2 \cdot \left[\underbrace{e^{-x^2-y^2}}_{f(x,y)} + x \cdot \underbrace{e^{-x^2-y^2}}_{f(x,y)} \cdot (-2x)\right] =$$

$$= \underline{-2f(x,y) + 4x^2 f(x,y)}$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2} \cdot (-2y) = -2ye^{-x^2-y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-2ye^{-x^2-y^2}\right) = -2\left[\frac{\partial y}{\partial y}\, e^{-x^2-y^2} + y \cdot \frac{\partial (e^{-x^2-y^2})}{\partial y}\right] =$$

$$= -2\left[\underbrace{e^{-x^2-y^2}}_{f(x,y)} + y \cdot \underbrace{e^{-x^2-y^2}}_{f(x,y)} \cdot (-2y)\right] =$$

$$= \underline{-2f(x,y) + 4y^2 f(x,y)}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4f(x,y) + 4f(x,y)\left[x^2+y^2\right] =$$

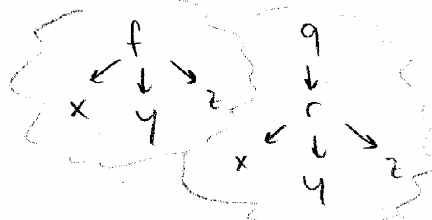
$$= 4f(x,y)\left[x^2+y^2-1\right]$$

QED

Zadatak 8.

$f(x,y,z) = g(r)$

$r = \sqrt{x^2+y^2+z^2}$



Dokaži : $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = g''(r) + \frac{2}{r} g'(r)$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial r} \cdot \frac{\partial r}{\partial x} = \underbrace{g'(r) \cdot \frac{x}{r}}, \text{ analogno: } \frac{\partial f}{\partial y} = \underbrace{g'(r) \cdot \frac{y}{r}}, \quad \frac{\partial f}{\partial z} = \underbrace{g'(r) \cdot \frac{z}{r}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(g'(r) \cdot \frac{x}{r}\right) = \frac{\partial g'(r)}{\partial x} \cdot \frac{x}{r} + g'(r) \cdot \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{\partial g'(r)}{\partial r} \cdot \frac{\partial r}{\partial x} \cdot \frac{x}{r} + g'(r) \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= g''(r) \cdot \frac{x^2}{r^2} + g'(r) \cdot \frac{r - x \cdot \frac{x}{r}}{r^2} = \underbrace{g''(r) \cdot \frac{x^2}{r^2} + g'(r) \cdot \frac{r^2 - x^2}{r^3}}$$

Analogno: $\frac{\partial^2 f}{\partial y^2} = \underbrace{g''(r) \cdot \frac{y^2}{r^2} + g'(r) \cdot \frac{r^2 - y^2}{r^3}}$

$\frac{\partial^2 f}{\partial z^2} = \underbrace{g''(r) \cdot \frac{z^2}{r^2} + g'(r) \cdot \frac{r^2 - z^2}{r^3}}$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = g''(r) \frac{x^2+y^2+z^2}{r^2} + g'(r) \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3} =$$

$$= g''(r) \frac{r^2}{r^2} + g'(r) \frac{3r^2 - r^2}{r^3} =$$

$$= g''(r) + g'(r) \cdot \frac{2}{r} \quad \text{QED.}$$

Zadatak 9.

$u = u(r, \varphi)$

$x = r\cos\varphi$

$y = r\sin\varphi$

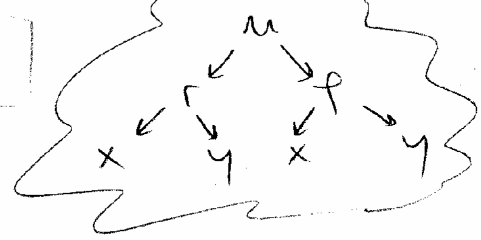
Dokazati : $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2}$

$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Treba primijetiti:

$x^2 + y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2 \Rightarrow \boxed{r = \sqrt{x^2+y^2}}$

$\frac{x}{y} = \frac{\cos\varphi}{\sin\varphi} \Rightarrow \operatorname{tg}\varphi = \frac{x}{y} \Rightarrow \boxed{\varphi = \operatorname{arctg}\frac{x}{y}}$



$$\frac{1}{1+\frac{x^2}{y^2}} = \frac{1}{\frac{y^2+x^2}{y^2}} = \frac{y^2}{y^2+x^2} = \frac{y^2}{r^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x} =$$

$$= u'_r \cdot \frac{x}{r} + u'_\varphi \cdot \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y} =$$

$$= \underline{u'_r \cdot \frac{x}{r} + u'_\varphi \cdot \frac{y}{r^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(u'_r \frac{x}{r} + u'_\varphi \frac{y}{r^2}\right) = \underline{u''_r \cdot \frac{x^2}{r^2} + u'_r \cdot \frac{r - x\frac{x}{r}}{r^2} + u''_\varphi \frac{y^2}{r^4} + u'_\varphi \cdot \frac{0 - y\cdot 2r\frac{x}{r}}{r^4}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} =$$

$$= u'_r \cdot \frac{y}{r} + u'_\varphi \cdot \frac{\cancel{y^2}}{r^2}\cdot\frac{-x}{\cancel{y^2}} =$$

$$= \underline{u'_r \cdot \frac{y}{r} - u'_\varphi \cdot \frac{x}{r^2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(u'_r \cdot \frac{y}{r} - u'_\varphi \cdot \frac{x}{r^2}\right) = \underline{u''_r \cdot \frac{y^2}{r^2} + u'_r \cdot \frac{r - y\cdot\frac{y}{r}}{r^2} + u''_\varphi \cdot \frac{x^2}{r^4} - u'_\varphi \cdot \frac{0 - x\cdot 2\frac{y}{r}}{r^4}}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u''_r \underbrace{\frac{\overbrace{x^2+y^2}^{r^2}}{r^2}}_{=1} + u'_r \underbrace{\frac{2r^2 - \overbrace{(x^2+y^2)}^{r^2}}{r^3}}_{=\frac{1}{r}} + u''_\varphi \underbrace{\frac{\overbrace{x^2+y^2}^{r^2}}{r^4}}_{=\frac{1}{r^2}} + u'_\varphi \underbrace{\frac{-2xy+2xy}{r^5}}_{=0} =$$

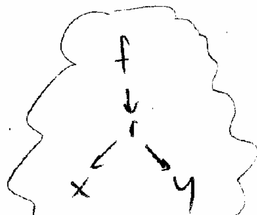
$$= u''_r + \frac{1}{r}u'_r + \frac{1}{r^2}u''_\varphi =$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \varphi^2}$$

Zadatak 10.

a) $f(x,y) = \ln r$

$r = \sqrt{x^2+y^2}$



$$\Delta f = \frac{2r^2 - 2\overbrace{(x^2+y^2)}^{r^2}}{r^4} = 0$$

QED

Treba pokazati: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{1}{r}\cdot\frac{x}{r} = \frac{x}{r^2},\quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{r^2 - x\cdot 2\cancel{r}\cdot\frac{x}{\cancel{r}}}{r^4} = \frac{r^2 - 2x^2}{r^4}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} = \frac{1}{r}\cdot\frac{y}{r} = \frac{y}{r^2},\quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) = \frac{r^2 - y\cdot 2r\cdot\frac{y}{r}}{r^4} = \frac{r^2 - 2y^2}{r^4}$$

b) $f(x,y) = x^2 - y^2$

$\Delta f = 0 \rightarrow$ treba dokazati

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2 \\ \frac{\partial f}{\partial y} &= -2y, \quad \frac{\partial^2 f}{\partial y^2} = -2 \end{aligned}\right\} \quad \Delta f = 2 + (-2) = 0 \quad \underline{\underline{QED}}$$

c) $f(x,y) = xy$

$\Delta f = 0 \rightarrow$ treba dokazati

$$\left.\frac{\partial f}{\partial x} = y, \quad \frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial^2 f}{\partial y^2} = 0 \right\} \quad \Delta f = 0 + 0 = 0 \quad \underline{\underline{QED.}}$$

d) $u(x,y) = Re(z^3)$

$z = x + iy$

$$z^3 = ? \rightarrow z^3 = (x+iy)^3 = x^3 + 3x^2 \cdot iy + 3x(iy)^2 + (iy)^3 =$$

$$= \underbrace{x^3 - 3xy^2}_{Re(z^3)} + 3x^2 y i - y^3 i$$

$\Rightarrow u(x,y) = x^3 - 3xy^2$

Treba dokazati $\Delta u = 0$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x \\ \frac{\partial u}{\partial y} &= -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x \end{aligned}\right\} \quad \Delta u = 6x - 6x = 0 \quad \underline{\underline{QED}}$$

e) $v(x,y) = Im(z^3)$

$z = x + iy$

$$z^3 = x^3 - 3xy^2 + \underbrace{(3x^2 y - y^3)}_{Im(z^3)} i$$

$$\left.\begin{aligned} \frac{\partial v}{\partial x} &= 6xy \longrightarrow \frac{\partial^2 v}{\partial x^2} = 6y \\ \frac{\partial v}{\partial y} &= 3x^2 - 3y^2 \longrightarrow \frac{\partial^2 v}{\partial y^2} = -6y \end{aligned}\right\} \quad \Delta v = 6y - 6y = 0 \quad \underline{\underline{QED}}$$

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f) $u(x,y) = \mathrm{Re}(z^n)$

$v(x,y) = \mathrm{Im}(z^n)$

$n \in \mathbb{N}$

$z = x + iy$

$$z^n = (x+iy)^n = \underbrace{\binom{n}{0}x^n}_{\mathrm{Re}} + \underbrace{\binom{n}{1}x^{n-1}(iy)}_{\mathrm{Im}} + \underbrace{\binom{n}{2}x^{n-2}(iy)^2}_{\mathrm{Re}} + \dots + \binom{n}{n-1}x(iy)^{n-1} + \binom{n}{n}(iy)^n$$

$\Rightarrow$ parne potencije : Re

$\Rightarrow$ neparne potencije: Im

$$u(x,y) = \sum_{k=0}^{n}\binom{n}{2k}x^{n-k}y^{k}(-1)^k$$

$$v(x,y) = \sum_{k=0}^{n}\binom{n}{2k+1}x^{n-2k-1}y^{2k+1}(-1)^k$$

$$\frac{\partial u}{\partial x} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k}(n-k)x^{n-k-1}y^k$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k}(n-k)(n-k-1)x^{n-k-2}y^k$$

$$\frac{\partial u}{\partial y} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k}x^{n-k}\cdot ky^{k-1}$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k}x^{n-k}\cdot k(k-1)y^{k-2}$$

$$\frac{\partial v}{\partial x} = \sum_{k=0}^{n}\binom{n}{2k+1}(n-2k-1)x^{n-2k-2}y^{2k+1}(-1)^k$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k+1}(n-2k-1)(n-2k-2)x^{n-2k-3}y^{2k+1}$$

$$\frac{\partial v}{\partial y} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k+1}x^{n-2k-1}(2k+1)y^{2k}$$

$$\frac{\partial^2 v}{\partial y^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{2k+1}x^{n-2k-1}(2k+1)2k\,y^{2k-1}$$

?

9) $f(x,y,z) = \frac{1}{r}$

$r = \sqrt{x^2+y^2+z^2}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} = -\frac{1}{r^2} \cdot \frac{x}{r} = -\frac{x}{r^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = -\frac{r^3 - x \cdot 3r^2 \cdot \frac{x}{\cancel{r}}}{r^6} = -\frac{r^3 - 3x^2 r}{r^6}$$

Analogno za y i z

$$\Delta f = -\frac{r^2-3x^2}{r^5} - \frac{r^2-3y^2}{r^5} - \frac{r^2-3z^2}{r^5} =$$

$$= \frac{-r^2+3x^2-r^2+3y^2-r^2+3z^2}{r^5} =$$

$$= \frac{3(x^2+y^2+z^2) - 3r^2}{r^5} =$$

$$= \frac{3r^2 - 3r^2}{r^5} = 0 \qquad \underline{\underline{QED}}$$

Zadatak 11.

a) $f(t,x) = \sin(x+2t) + \cos(x-2t)$

$$\frac{\partial^2 f}{\partial t^2} = 4\frac{\partial^2 f}{\partial x^2} \quad \leftarrow \text{treba dokazati}$$

$$\frac{\partial f}{\partial t} = \cos(x+2t)\cdot 2 - \sin(x-2t)\cdot(-2) = 2\left[\cos(x+2t) + \sin(x-2t)\right]$$

$$\frac{\partial^2 f}{\partial t^2} = 2\left[-\sin(x+2t)\cdot 2 + \cos(x-2t)\cdot(-2)\right] = -4f(t,x)$$

$$\frac{\partial f}{\partial x} = \cos(x+2t) - \sin(x-2t)$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+2t) - \cos(x-2t) = -f(t,x)$$

$$\frac{\partial^2 f}{\partial t^2} = 4\cdot\frac{\partial^2 f}{\partial x^2}$$

$\underline{\underline{QED,}}$

b) $f(t,x) = q(x+ct) + h(x-ct)$ $\quad \to \quad u = x+ct \Rightarrow q(u)$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad v = x-ct \Rightarrow v(v)$

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial q(u)}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial h(v)}{\partial v}\cdot\frac{\partial v}{\partial t} = q'(u)\cdot c - h'(v)\cdot c$$

$$\frac{\partial^2 f}{\partial t^2} = c\left[\frac{\partial q'(u)}{\partial t} - \frac{\partial h'(v)}{\partial t}\right] = c\cdot\left[\frac{\partial q'(u)}{\partial u}\cdot\frac{\partial u}{\partial t} - \frac{\partial h'(v)}{\partial v}\cdot\frac{\partial v}{\partial t}\right] =$$

$$= c\cdot q''(u)\cdot c + c\cdot h''(u)\cdot c = \underline{c^2\left(q''(u) + h''(u)\right)}$$

$$\frac{\partial f}{\partial x} = \frac{\partial q(u)}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial h(v)}{\partial v}\cdot\frac{\partial v}{\partial x} = q'(u)\cdot 1 + h'(v)\cdot 1$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial q'(u)}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial h'(v)}{\partial v}\cdot\frac{\partial v}{\partial x} = \underline{q''(u) + h''(v)}$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = c^2\cdot\frac{\partial^2 f}{\partial x^2} \qquad \underline{\underline{\text{QED}}}$$

<u>Zadatak 12.</u>

$u(r,\varphi) = r^n\cos(n\varphi)$
$v(r,\varphi) = r^n\sin(n\varphi)$ $\Big\}$ harmonijske?

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos(n\varphi), \quad \frac{\partial^2 u}{\partial r^2} = n(n-1)r^{n-2}\cos(n\varphi)$$

$$\frac{\partial u}{\partial \varphi} = -r^n\sin(n\varphi)\cdot n, \quad \frac{\partial^2 u}{\partial \varphi^2} = -r^n\cos(n\varphi)n^2$$

$\to$ Vrijedi: $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\cdot\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial^2 \varphi} =$

$$= n(n-1)r^{n-2}\cos(n\varphi) + \frac{1}{r}\cdot nr^{n-1}\cos(n\varphi) + \frac{1}{r^2}\left(-r^n\cos(n\varphi)n^2\right) =$$

$$= n(n-1)r^{n-2}\cos(n\varphi) + nr^{n-2}\cos(n\varphi) - r^{n-2}\cos(n\varphi)n^2 =$$

$$= nr^{n-2}\cos(n\varphi)\left[n-1+1-n\right] = 0 \qquad \underline{\underline{\text{QED}}}$$

Analogno za $v(r,\varphi)$.

Zadatak 13.

$$\operatorname{div}\vec{a} = \nabla\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\operatorname{rot}a = \nabla\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

a) $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$

$\operatorname{div}\vec{a}, \operatorname{rot}\vec{a} = ?$

$$\operatorname{div}\vec{a} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right) =$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\operatorname{rot}\vec{a} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\times\left(x\vec{i} + y\vec{j} + z\vec{k}\right) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \vec{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \vec{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) =$$

$$= 0$$

b) $\operatorname{div}(\operatorname{rot}\vec{a}) = 0$

$\nabla \operatorname{rot}\vec{a} = \nabla\cdot\nabla\times\vec{a} = \nabla\cdot 0 = 0$

$\operatorname{rot}(\operatorname{rot}\vec{a}) = 0$

$\nabla\times\operatorname{rot}\vec{a} = \nabla\times\nabla\times\vec{a} = \nabla\times 0 = 0$

$\operatorname{rot}(\operatorname{grad} f) = 0$

$\nabla\times\operatorname{grad} f = \nabla\times\nabla f = 0$

Zadatak 14.

$$\vec{a}(x,y) = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

$$\vec{a}(x,y)\cdot(x\vec{i}+y\vec{j}) = 0 \implies \vec{a}(\vec{r}) \perp \vec{r}$$

$$\|\vec{a}(\vec{r})\| = \frac{1}{r}$$

a) $f(x,y) = \operatorname{arctg}\frac{y}{x}$, $x \neq 0$ potencijal ?

Ako je $f(x,y)$ potencijal onda će mi polje biti $\nabla f(x,y)$!

$$\frac{\partial f}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{-y}{\cancel{x^2}} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^{\cancel{2}}}{x^2+y^2}\cdot\frac{1}{\cancel{x}} = \frac{x}{x^2+y^2}$$

$$\Rightarrow \nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} = \underbrace{\vec{a}(x,y)}_{\text{polje!}}$$

b) $\operatorname{div}\vec{a} = 0$?

$$\operatorname{div}\vec{a} = \nabla\cdot\vec{a} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right)\left(-\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}\right) =$$

$$= -\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) =$$

$$= \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

QED.

Zadatak 15.

$$f(x,y) = x^2 + 3y^2 - 2xy$$

$$d^2 f = ?$$

$$\frac{\partial f}{\partial x} = 2x - 2y, \quad \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial f}{\partial y} = 6y - 2x, \quad \frac{\partial^2 f}{\partial y^2} = 6; \quad \frac{\partial^2 f}{\partial x\,\partial y} = -2$$

$$d^2 f = d(dz) = \frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}(dy)^2 =$$

$$= 2(dx)^2 - 4\,dx\,dy + 6(dy)^2$$

Definitnost?

$$d^2f = 2\left[(dx)^2 - 2\,dx\,dy + (dy)^2 + 2(dy)^2\right] =$$

$$= 2\Big[\underbrace{(dx-dy)^2 + 2(dy)^2}_{\text{zbroj kvadrata } > 0}\Big]$$

$\Rightarrow$ POZITIVNO DEFINITNA FORMA!

Zadatak 16.

$f(x,y) = \underbrace{x^2 + xy - x + y - 1}$ po potencijama od $(x-1)$, $(y+2)$

- polinom 2. stupnja $\Rightarrow$ Taylor 2. stupnja!

$\Rightarrow R_2(T_0) = 0$ (jer je 3. derivacija 0)

$T_0(1,-2)$

$$f(x,y) = \underset{\underbrace{1-1\cdot2-1-2-1}_{-5}}{f(T_0)} + (x-1)\underset{\underbrace{2x_0+y_0=1}_{-1}}{\left(\frac{\partial f}{\partial x}\right)_{T_0}} + (y+2)\underset{\underbrace{x_0+1}_{2}}{\left(\frac{\partial f}{\partial y}\right)_{T_0}} + \frac{1}{2}\left[(x-1)^2\underset{2}{\left(\frac{\partial^2 f}{\partial x^2}\right)_{T_0}} + 2(x-1)(y+2)\underset{1}{\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{T_0}} + (y+2)^2\underset{0}{\left(\frac{\partial^2 f}{\partial y^2}\right)_{T_0}}\right.$$

$$= -5 - (x-1) + 2(y+2) + (x-1)^2 + (x-1)(y+2)$$