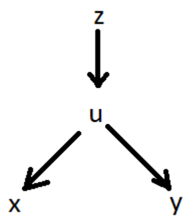
1. Izračunati $\frac{\partial z}{\partial x}$ i $\frac{dz}{dx}$ ako je $z = x^y$, gdje je $y = \varphi(x)$.

Rješenje:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = yx^{y-1} + x^y \ln x \cdot \varphi'(x) = x^y\left[\frac{y}{x} + \varphi'(x)\ln x\right]$$

2. Izračunati $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ ako je $z = f(u)$, gdje je $u = xy + \frac{y}{x}$.

Rješenje:

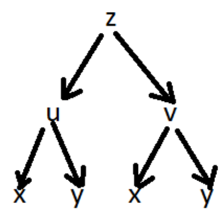


$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u)\left[y - \frac{y}{x^2}\right]$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = f'(u)\left[x + \frac{1}{x}\right]$$

3. Izračunati $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ ako je $z = f(u, v)$, gdje je $u = x^2 - y^2$, $v = e^{xy}$.

Rješenje:

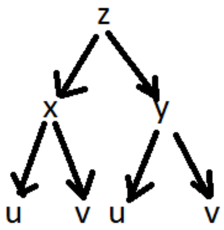


$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2xf_u'(u,v) + ye^{xy}f_v'(u,v)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2yf_u'(u,v) + xe^{xy}f_v'(u,v)$$

4. Izračunati $\frac{\partial z}{\partial u}$ i $\frac{\partial z}{\partial v}$ ako je $z = \operatorname{arctg}\left(\frac{x}{y}\right)$, gdje je $x = u\sin v$, $y = u\cos v$.

Rješenje:



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}\sin v - \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{x}{y^2}\cos v = \frac{\operatorname{tg} v}{u(1+\operatorname{tg}^2 v)} - \frac{\operatorname{tg} v}{u(1+\operatorname{tg}^2 v)} = 0$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}u\cos v + \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{x}{y^2}u\sin v = \frac{1}{1+\operatorname{tg}^2 v} + \frac{\operatorname{tg}^2 v}{1+\operatorname{tg}^2 v} = 1$$

5. Pokazati da je $\frac{\partial u}{\partial \varphi} = 0$ i $\frac{\partial u}{\partial \psi} = 0$ ako je $u = \Phi(x^2+y^2+z^2)$, gdje je $x = R\cos\varphi\cos\psi$, $y = R\cos\varphi\sin\psi$ i $z = R\sin\varphi$.

Rješenje:

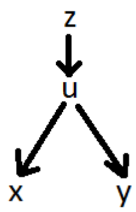
$$x^2+y^2+z^2 = R^2\cos^2\varphi\cos^2\psi + R^2\cos^2\varphi\sin^2\psi + R^2\sin^2\varphi = R^2$$

$$u = \Phi(R^2) \to \frac{\partial u}{\partial \varphi} = 0, \quad \frac{\partial u}{\partial \psi} = 0$$

6. Pokazati da je $\frac{\partial z}{\partial y} = a\frac{\partial z}{\partial x}$ ako je $z = f(x+ay)$, gdje je f derivabilna funkcija.

Rješenje:

$$x + ay = u \to z = f(u)$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = af'(u), \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u) \to \frac{\partial z}{\partial y} = a\frac{\partial z}{\partial x}$$

7. Stranica pravokutnika x = 20 m produljuje se brzinom od 5 m/s, druga stranica y = 30 m skraćuje se brzinom od 4 m/s. Kojom brzinom se mijenja opseg i ploština pravokutnika?

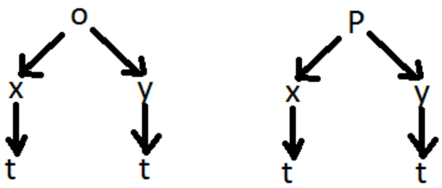
Rješenje:

$$x = \varphi(t) = 20 + 5t$$

$$y = \psi(t) = 30 - 4t$$

$$o = o(x, y) = 2(x + y)$$

$$P = P(x, y) = xy$$



U konačnici opseg i površina ovise samo o varijabli t pa pišemo:

$$\frac{do}{dt} = \frac{\partial o}{\partial x}\frac{dx}{dt} + \frac{\partial o}{\partial y}\frac{dy}{dt} = 2 \cdot 5 + 2 \cdot (-4) = 10 - 8 = 2\left[\frac{m}{s}\right]$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} = y \cdot 5 + x \cdot (-4) = 5(30 - 4t) - 4(20 + 5t) = 70 - 40t\left[\frac{m^2}{s}\right]$$

8. Vektorska jednadžba gibanja točke u prostoru je $\vec{r}(t) = t^2\vec{\imath} + \sin t\,\vec{\jmath} + e^t\vec{k}$.
(a) Odrediti iznos brzine točke u trenutku t = 0, tj. izračunati $\|\vec{r}\,'(0)\|$.
(b) Odrediti jednadžbu tangente na krivulju u točki koja odgovara parametru t = 0.

Rješenje:

(a)

$$\|\vec{r}\,'(0)\| = \left\|\left(2t\vec{\imath} + \cos t\,\vec{\jmath} + e^t\vec{k}\right)_0\right\| = \|\vec{\jmath} + \vec{k}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

(b)

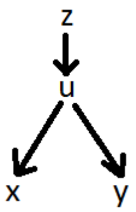
$$t \equiv \frac{x - x_0}{x'(0)} = \frac{y - y_0}{y'(0)} = \frac{z - z_0}{z'(0)}$$

$$t \equiv \frac{x}{0} = \frac{y}{1} = \frac{z - 1}{1}$$

9. Pokazati da funkcija $z = f\left(\sqrt{x^2+y^2}\right)$ zadovoljava jednadžbu $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$.

Rješenje:

$$\sqrt{x^2+y^2} = u \to z = f(u)$$



$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} - y\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = xf'(u)\frac{y}{\sqrt{x^2+y^2}} - yf'(u)\frac{x}{\sqrt{x^2+y^2}} = 0$$

10. Odrediti točke na krivulji

$$C \ldots \begin{cases} x = t^2 \\ y = t^3 \\ z = t^2 - 2t \end{cases}$$

u kojima je tangenta na krivulju paralelna s ravninom $2x - y + 2z - 1 = 0$.

Rješenje:

Vektor smjera tangente:

$$\vec{t} = x'(t_0)\vec{i} + y'(t_0)\vec{j} + z'(t_0)\vec{k} = 2t_0\vec{i} + 3t_0^2\vec{j} + (2t_0 - 2)\vec{k}$$

Normala ravnine:

$$\vec{n} = 2\vec{i} - \vec{j} + 2\vec{k}$$

Ako je tangenta paralelna s ravninom, onda je vektor smjera tangente okomit na normalu. Iz uvjeta okomitosti dobijemo:

$$\vec{t}\cdot\vec{n} = 0 \to 4t_0 - 3t_0^2 + 4t_0 - 4 = 0 \to 3t_0^2 - 8t_0 + 4 = 0 \to t_{01} = \frac{2}{3}, \qquad t_{02} = 2$$

$$T_1\left(\frac{4}{9}, \frac{8}{27}, -\frac{8}{9}\right) \qquad T_2(4,8,0)$$

11. Odrediti točku na krivulji

$$C \ldots \begin{cases} x = t^2 \\ y = 3t \\ z = 3t^3 + 2t \end{cases}$$

u kojoj je tangenta paralelna s pravcem $p \equiv \frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-1}{-11}$.

Rješenje:

Vektor smjera tangente:

$$\vec{t} = x'(t_0)\vec{\imath} + y'(t_0)\vec{\jmath} + z'(t_0)\vec{k} = 2t_0\vec{\imath} + 3\vec{\jmath} + (9t_0^2 + 2)\vec{k}$$

Vektor smjera pravca:

$$\vec{c} = 2\vec{\imath} - 3\vec{\jmath} - 11\vec{k}$$

Vektor smjera tangente i vektor smjera pravca moraju biti kolinearni:

$$\vec{t} = \tau\vec{c} \rightarrow 2t_0\vec{\imath} + 3\vec{\jmath} + (9t_0^2 + 2)\vec{k} = 2\tau\vec{\imath} - 3\tau\vec{\jmath} - 11\tau\vec{k}$$

$$3 = -3\tau \rightarrow \tau = -1$$

pa imamo:

$$2t_0\vec{\imath} + 3\vec{\jmath} + (9t_0^2 + 2)\vec{k} = -2\vec{\imath} + 3\vec{\jmath} + 11\vec{k}$$

$$2t_0 = -2 \rightarrow t_0 = -1$$

$$3 = 3$$

$$9t_0^2 + 2 = 11 \rightarrow t_0 = \pm 1 \rightarrow t_0 = -1$$

Slijedi da je točka:

$$T(t_0^2, 3t_0, 3t_0^3 + 2t_0) \rightarrow T(1, -3, -5)$$

12. Dokazati da funkcija

$$y(x)=\int_0^\infty \frac{e^{-xz}}{z^2+1}dz\,,\ \ x>0$$

zadovoljava diferencijalnu jednadžbu $y''+y=\frac{1}{x}$.

Rješenje:

$$y'(x)=\int_0^\infty \frac{\partial}{\partial x}\left(\frac{e^{-xz}}{z^2+1}\right)dz=\int_0^\infty \frac{-ze^{-xz}}{z^2+1}dz$$

$$y''(x)=\int_0^\infty \frac{\partial}{\partial x}\left(\frac{-ze^{-xz}}{z^2+1}\right)dz=\int_0^\infty \frac{z^2e^{-xz}}{z^2+1}dz$$

$$y''(x)+y(x)=\int_0^\infty \frac{z^2e^{-xz}}{z^2+1}dz\int_0^\infty \frac{e^{-xz}}{z^2+1}dz=\int_0^\infty \frac{(z^2+1)e^{-xz}}{z^2+1}dz=\int_0^\infty e^{-xz}dz=$$

$$=-\frac{1}{x}e^{-xz}\Big|_0^\infty=-\frac{1}{x}(0-1)=\frac{1}{x}$$

13. Koristeći deriviranje integrala po parametru izračunati

$$F(\alpha,\beta)=\int_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}dx\,,\ \ \alpha\geq 0.$$

Rješenje:

$$\frac{\partial F}{\partial \beta}=\int_0^\infty xe^{-\alpha x}\frac{\cos(\beta x)}{x}dx=\int_0^\infty e^{-\alpha x}\cos(\beta x)\,dx$$

$$I=\int_0^\infty e^{-\alpha x}\cos(\beta x)\,dx=\begin{bmatrix} u=e^{-\alpha x} & dv=\cos(\beta x)\,dx \\ du=-\alpha e^{-\alpha x}dx & v=\dfrac{\sin(\beta x)}{\beta}\end{bmatrix}=\frac{e^{-\alpha x}\sin(\beta x)}{\beta}+$$

$$+\frac{\alpha}{\beta}\int_0^\infty e^{-\alpha x}\sin(\beta x)\,dx=\begin{bmatrix} u=e^{-\alpha x} & dv=\sin(\beta x)\,dx \\ du=-\alpha e^{-\alpha x}dx & v=-\dfrac{\cos(\beta x)}{\beta}\end{bmatrix}=\frac{e^{-\alpha x}\sin(\beta x)}{\beta}+$$

$$+\frac{\alpha}{\beta}\left(-\frac{e^{-\alpha x}\cos(\beta x)}{\beta}-\frac{\alpha}{\beta}\int_0^\infty e^{-\alpha x}\cos(\beta x)\,dx\right)$$

$$I=\frac{e^{-\alpha x}\sin(\beta x)}{\beta}-\frac{\alpha e^{-\alpha x}\cos(\beta x)}{\beta^2}-\frac{\alpha^2}{\beta^2}I$$

$$I\left(1+\frac{\alpha^2}{\beta^2}\right)=\frac{\beta e^{-\alpha x}\sin(\beta x)-\alpha e^{-\alpha x}\cos(\beta x)}{\beta^2}$$

$$I=\frac{\beta e^{-\alpha x}\sin(\beta x)-\alpha e^{-\alpha x}\cos(\beta x)}{\alpha^2+\beta^2}$$

$$\frac{\partial F}{\partial \beta}=\frac{\beta e^{-\alpha x}\sin(\beta x)-\alpha e^{-\alpha x}\cos(\beta x)}{\alpha^2+\beta^2}\Bigg|_0^{\infty}=\frac{\alpha}{\alpha^2+\beta^2}$$

$$F(\alpha,\beta)=\int\frac{\alpha}{\alpha^2+\beta^2}d\beta=\frac{1}{\alpha}\int\frac{d\beta}{1+\left(\frac{\beta}{\alpha}\right)^2}=\operatorname{arctg}\left(\frac{\beta}{\alpha}\right)+C$$

$$F(\alpha,0)=\operatorname{arctg}\left(\frac{0}{\alpha}\right)+C=C$$

$$F(\alpha,0)=\int_0^{\infty}e^{-\alpha x}\frac{\sin(0)}{x}dx=0\to C=0$$

$$F(\alpha,\beta)=\operatorname{arctg}\left(\frac{\beta}{\alpha}\right)$$

14. Izračunajte $F'\left(\frac{1}{2}\right)$ ako je

$$F(\alpha)=\int_{\pi}^{2\pi}\frac{\sin(\alpha x)}{x}dx.$$

Rješenje:

$$F'(\alpha)=\int_{\pi}^{2\pi}\frac{\partial}{\partial\alpha}\left(\frac{\sin(\alpha x)}{x}\right)dx=\int_{\pi}^{2\pi}\cos(\alpha x)\,dx=\frac{1}{\alpha}\sin(\alpha x)\Bigg|_{\pi}^{2\pi}=\frac{\sin(2\pi\alpha)-\sin(\pi\alpha)}{\alpha}$$

$$F'\left(\frac{1}{2}\right)=\frac{\sin\left(\frac{2\pi}{2}\right)-\sin\left(\frac{\pi}{2}\right)}{\frac{1}{2}}=\frac{0-1}{\frac{1}{2}}=-2$$