Redovi brojeva

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Dakle $\sum_{n=1}^{\infty} \frac{1}{n^r}$ konvergira za $r > 1$ i divergira za $r \leq 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^r}$ je tzv. poopćeni harmonijski red → Dirichletov red

Ispitati konvergenciju reda:

<u>ZADATAK 4</u>

$\sum_{n=1}^{\infty} \frac{1}{n \cdot \ln n}$

S obzirom da je $a(x) = \frac{1}{x \cdot \ln x}$ pozitivna, neprekinuta i padajuća funkcija za

$x =< 2, \infty >$ možemo primjeniti integralni kriterij tj.

$\sum_{n=1}^{\infty} \frac{1}{n \cdot \ln n} \sim \int_2^{\infty} \frac{1}{x \cdot \ln x} dx = \left| \ln x = t\, ,\ \frac{dx}{x} = dt,\ \ x = 2,\ \ t = \ln 2 \right|$

$= \int_{ln2}^{\infty} \frac{dt}{t} = \ln t \begin{cases} \infty \\ ln2 \end{cases} = \ln \infty - \ln \ln 2 = \infty$ → Divergira

<u>ZADATAK 5</u>

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \ldots + \frac{1}{n^2} + \ \ldots$ → Dirichletov red

$r = 2 > 1$ → Konvergira

<u>ZADATAK 6</u>

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ \ldots + \frac{1}{\sqrt{n}} + \ \ldots$ → Dirichletov red

$r = \frac{1}{2} \leq 1$ → Divergira

<u>ZADATAK 7</u>

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3n+5}} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ → Harmonijski red → Divergira

ZADATAK 8

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+n^3+1}} \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}} \rightarrow$ Dirichletov red

$r = \frac{4}{5} \leq 1 \rightarrow$ Divergira

ZADATAK 9

$\sum_{n=1}^{\infty} \frac{\sqrt{n}+\sqrt[3]{n}+\sqrt[4]{n}}{n^2+n} \sim \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \rightarrow$ Dirichletov red

$r = \frac{3}{2} > 1 \rightarrow$ Konvergira

Napomena: sve ove $\sim$ su moguće zbog poredbenog kriterija (limes varijanta).

STAVAK 2: D'Alemberteov kriterij

Neka je red $\sum a_n$ s pozitivnim članovima. Ako postoji limes

$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = q$

onda za:

$q < 1 \rightarrow$ red konvergira

$q > 1 \rightarrow$ red divergira

$q = 1 \rightarrow$ nema odluke

DOKAZ

$\boxed{q < 1}$

Po definiciji limesa niza $\forall \varepsilon > 0$ svaki član $\left(\frac{a_{n+1}}{a_n}\right)$ je u intervalu $(q-\varepsilon, q+\varepsilon)$ za $n \geq n_0$. Odaberemo takav ε da je $q+\varepsilon < 1$ onda vrijedi $\frac{a_{n+1}}{a_n} \leq q+\varepsilon$ za $n \geq n_0$.

$\frac{a_n}{a_{n-1}} \leq q + \varepsilon / ^{a_{n-1}}$

$$a_n \leq (q+\varepsilon)a_{n-1} \leq (q+\varepsilon)^2 a_{n-2} \leq (q+\varepsilon)^3 a_{n-3} \leq \cdots$$

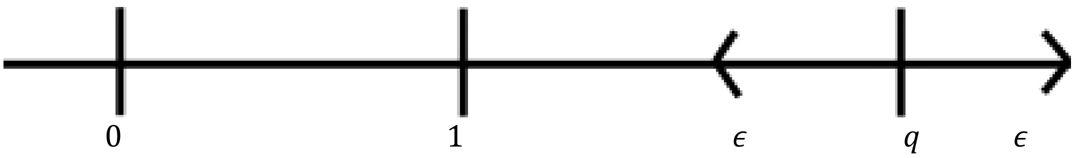
$$\leq (q+\varepsilon)^{n-n_0} a_{n_0} = \frac{a_{n_0}}{(q+\varepsilon)^{n_0}} \cdot (q+\varepsilon)^n \rightarrow a_n \leq \frac{a_{n_0}}{(q+\varepsilon)^{n_0}} \cdot (q+\varepsilon)^n$$

$\sum_{n=0}^{\infty}(q+\varepsilon)^n$ → majoranta → konvergentan

geometrijski red jer $Q < 1$ → $\sum a_n$ konvergira

DOKAZ

$\boxed{q > 1}$



$\forall\varepsilon > 0 \left(\frac{a_{n+1}}{a_n}\right) \epsilon (q-\varepsilon, q+\varepsilon)$ za $n \geq n_0$

Odaberemo ε takav da je $q - \varepsilon > 1$. Sada je $\frac{a_{n+1}}{a_n} \geq q - \varepsilon$ za $n \geq n_0$

$a_{n+1} \geq (q-\varepsilon)\, a_n \geq a_n$

Pa je dakle počevši od nekog člana niz (a_n) monotono rastući pa ne može biti ispunjen nuždan uvjet konvergencije reda.

$\lim_{n\to\infty} a_n \leq 0$ → divergentan

STAVAK 2: (Cauchyev kriterij)

Neka je red $\sum a_n$ s pozitivnim članovima. Ako postoji limes $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ onda za:

$q < 1$ → red konvergira

$q > 1$→ red divergira

$q = 1$→ nema odluke

DOKAZ

$\boxed{q < 1}$

$\sqrt[n]{a_n} \leq q + \varepsilon$ za $n \geq n_0$ $\qquad a_n \leq (q + \varepsilon)^n$

$\boxed{q > 1}$ ↑Majoranta

$\sqrt[n]{a_n} \geq q - \varepsilon$ za $n \geq n_0$ $\qquad a_n \geq (q - \varepsilon)^n$

↑Minoranta

U knjižici postoje još dva obrađena kriterija, tko želi može pročitati, no ne treba ih znati.

Ispitati konvergenciju reda:

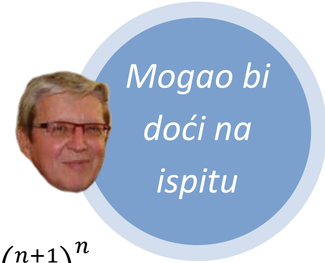
ZADATAK 1

$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{n!}$ Koristimo D'Alemberteov kriterij $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = q$.

$\lim_{n\to\infty} \frac{2^{2n+3}}{(n+1)!} \cdot \frac{n!}{2^{2n+1}} = \lim_{n\to\infty} \frac{2^2}{n+1} = 0 \quad q = 0 < 1 \rightarrow$ konvergentan

ZADATAK 2

$\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ Koristimo D'Alemberteov kriterij.



$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1)}{(n+1)! \cdot (n+1)!} \cdot \frac{n! \cdot n!}{n^n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{n+1}{n}\right)^n =$$

$\lim_{n\to\infty} \frac{1}{n+1} \cdot \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{e}{n+1} = 0 \quad q = 0 < 1 \rightarrow$ konvergentan

ZADATAK 3

$\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+2}\right)^{4n+3}$ Koristimo Cauchyev kriterij $\lim_{n\to\infty} \sqrt[n]{a_n} = q$.

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{3n+2}\right)^{4n+3}} = \lim_{n\to\infty} \left(\frac{2n+1}{3n+2}\right)^{\frac{4n+3}{n}} = \lim_{n\to\infty} \left(\frac{2+\frac{1}{n}}{3+\frac{2}{n}}\right)^{4+\frac{3}{n}} = \left(\frac{2}{3}\right)^4$$

$q = \frac{16}{81} < 1 \rightarrow$ konvergentan

ZADATAK 4

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ Koristimo D'Alemberteov kriterij.

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \sim \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$$

$q = 1 \rightarrow$ nema odluke, no: $n > \ln n \rightarrow \frac{1}{n} < \frac{1}{\ln n} \quad \sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{1}{\ln n}$

Divergentna minoranta $\uparrow \rightarrow$ divergentan

ZADATAK 5

$\sum_{n=1}^{\infty} \frac{3^n}{n}$ Koristimo Cauchyev kriterij $\lim_{n\to\infty} \sqrt[n]{a_n} = q$.

$\lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}} = \lim_{n\to\infty} \left(\frac{3^n}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{\sqrt[n]{n}} = 3 \quad q = 3 > 1 \rightarrow$ divergentan

ZADATAK 6

$$\sum_{n=1}^{\infty} \frac{2n+3}{\sqrt[4]{n^7+1}} \sim \sum_{n=1}^{\infty} \frac{2n}{\sqrt[4]{n^7}} = 2 \cdot \sum_{n=1}^{\infty} \frac{n}{n^{\frac{3}{4}}} \rightarrow \text{Dirichletov red}$$

$r = \frac{3}{4} \rightarrow$ divergentan

ZADATAK 7

$$\sum_{n=1}^{\infty} \frac{2n+3}{\sqrt[3]{n^7+1}} \sim \sum_{n=1}^{\infty} \frac{2n}{\sqrt[3]{n^7}} = 2 \cdot \sum_{n=1}^{\infty} \frac{n}{n^{\frac{4}{3}}} \rightarrow \text{Dirichletov red}$$

$r = \frac{3}{4} \rightarrow$ konvergentan

ZADATAK 8

$$\sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}}$$

$\sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}} > \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}} \quad r = \frac{1}{2} < 1 \rightarrow$ divergentan jer ima divergentnu minorantu

ZADATAK 9

$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2+1}$ (apsolutno jer gledamo samo pozitivne članove)

Teorem o apsolutnoj konvergenciji $\rightarrow$ Svaki apsolutno konvergentan red je konvergentan.

$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \quad r = 2 > 1 \rightarrow$ konvergentan jer ima konvergentnu majorantu

ZADATAK 10

$\sum_{n=4}^{\infty} \frac{1}{n \cdot \ln n \cdot \ln \ln n}$ Koristimo integralni kriterij.

$$\sum_{n=4}^{\infty} \frac{1}{n \cdot \ln n \cdot \ln \ln n} \sim \int_4^{\infty} \frac{dx}{x \cdot \ln x \cdot \ln \ln x} = \left| \ln \ln \mathrm{x} = t\ ,\ \frac{1}{\ln x} \cdot \frac{dx}{x} = dt,\ \ x = 4,\ \ t = \ln \ln 4 \right|$$

$$= \int_{\ln \ln 4}^{\infty} \frac{dt}{t} = \ln t \begin{cases} \infty \\ \ln \ln 4 \end{cases} = \infty \rightarrow \text{divergentan}$$

- Ako ima grešaka (matematičkih ili gramatičkih, kako koga smeta :D) ili nešto nedostaje (moguće da nije sve zapisano) ili imate neku ideju, javite mi na PM ili direktno mailom na Telefunken@fer2.net