16.05.2011.

10. Diferencijalne jednadžbe (prvog reda)

POJAM DIF. JED. (n-tog reda)

(*) $F(x, y, y', \dots, y^{n}) = 0$ DIF. JED. n-tog REDA

Rješenje od (*) je svaka funkc. $y = f(x)$ definirana na nekom intervalu koja ima sve potrebne derivacije na tom intervalu i koja uvrštena u jed. (*) nju identički zadovoljava.

Primjer:

1) $y' = f(x)$

$\frac{dy}{dx} = y'$

$dy = f(x)dx \;/\int$

$y = \int f(x)\,dx + C$

2) $y' = x + e^{2x}$

$y = \frac{x^2}{2} + \frac{1}{2}e^{2x} + C$

3) $y'' + y = 0$

1. $y_1 = \cos x \quad (-\cos x + \cos x = 0)$
2. $y_2 = \sin x \quad (-\sin x + \sin x = 0)$
3. $y_3 = C_1 \cos x + C_2 \sin x$

4) $y'' = 1$

$y = \frac{x^2}{2} + C_1 x + C_2$

provjera: $y' = x + C_1$

$y'' = 1$

Rješenje svake dif. jed. n-tog reda može se napisati u obliku:

$y = f(x, C_1, \dots, C_n)$ OPĆE RJEŠENJE

ili

$\Phi(x, y, C_1, \dots, C_n)$ OPĆI INTEGRAL DIF. JED.

OBRAT:

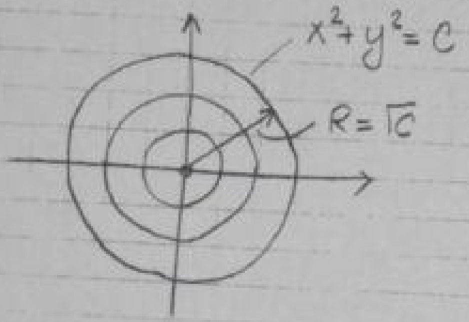
Svakoj familiji krivulja $\Phi(x,y,C_1,\dots,C_n)=0$ možemo pridružiti dif. jed. koju sve te krivulje zadovoljavaju.

ALGORITAM:

Zadanu jed. familije treba derivirati n puta i iz dobivenog sustava jed. ELIMINIRATI $C_1,\dots,C_n$

Naći dif. jed. zadane familije krivulja:

① $x^2+y^2=C$



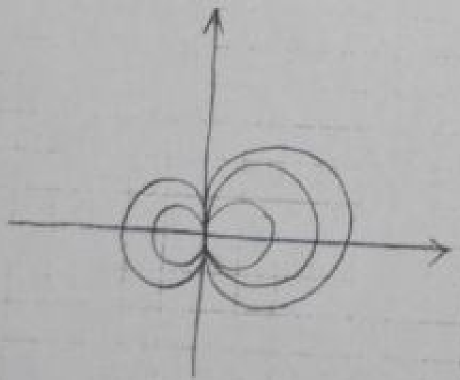
1) $x^2+y^2=C \quad / \frac{d}{dx}$

2) $2x+2yy'=0$

$x+yy'=0$

$y'=-\frac{x}{y}$

② $(x-C)^2+y^2=C^2$



1) $(x-C)^2+y^2=C^2 \quad / \frac{d}{dx}$

2) $\cancel{2}(x-C)+\cancel{2}yy'=0 \Rightarrow x-C=-yy'$

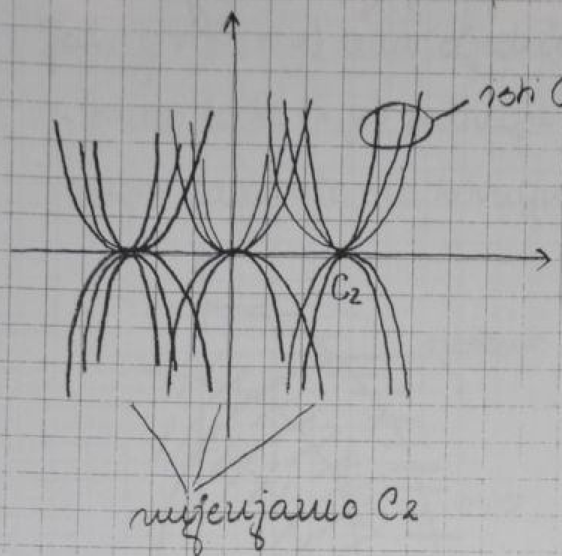
eliminacija C

$(-yy')^2+y^2=(x+yy')^2$

$\cancel{y^2y'^2}+y^2=x^2+2xyy'+\cancel{y^2y'^2}$

$y'=\frac{y^2-x^2}{2xy}$

③ $y = C_1(x-C_2)^2$ DVOPARAMETARSKA FAMILIJA



$T(C_2, 0)$

$\Rightarrow$ sve parabole čije tjeme leži na Ox

1) $y = C_1(x-C_2)^2 \quad / \frac{d}{dx}$

2) $y' = 2C_1(x-C_2)$

3) $y'' = 2C_1 \quad \Rightarrow C_1 = \frac{y''}{2}$

$$y = \frac{y''}{2}(x-C_2)^2$$

$$y' = y''(x-C_2)$$

$$y = \frac{y''}{2}\left(\frac{y'}{y''}\right)^2 = \frac{(y')^2}{2y''}$$

$$2yy'' - (y')^2 = 0$$

* CAUCHY-EV PROBLEM *

DEF.

Određivanje funkc. $y = y(x)$ koja zadovoljava dif. jed.
i početni uvjet:

$$y' = f(x,y)$$
$$y(x_0) = y_0$$

naziva se Cauchyev problem 1. reda.

Primjedba:

Analogno je Cauchyev problem rješenje dif. jed. n-tog reda sa n početnih uvjeta.

Riješi Cauchyev problem:

① $\begin{cases} y' = 2x \\ y(1) = 2 \end{cases}$

$y' = \frac{dy}{dx} = 2x \quad / dx$

$dy = 2x\,dx \quad / \int$

$y = x^2 + C_1$ opće rješenje

poč. uvjet $x = 1,\ y = 2$

$2 = 1^2 + C_1 \Rightarrow C_1 = 1$

$\boxed{y = x^2 + 1}$ RJEŠENJE CAUCHYEVOG PROBLEMA

② $\begin{cases} y'' = x + 1 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

$y'' = \frac{dy'}{dx} = x + 1$

$dy' = (x+1)dx \quad / \int$

$y' = \frac{x^2}{2} + x + C_1$

poč. uvjet: $x = 0,\ y = 1$

$1 = \frac{0^2}{2} + 0 + C_1 \Rightarrow C_1 = 1$

$y' = \frac{x^2}{2} + x + 1$

$dy = \left(\frac{x^2}{2} + x + 1\right) dx \quad / \int$

$y = \frac{x^3}{6} + \frac{x^2}{2} + x + C_2$

poč. uvjet: $x = 0,\ y = 1$

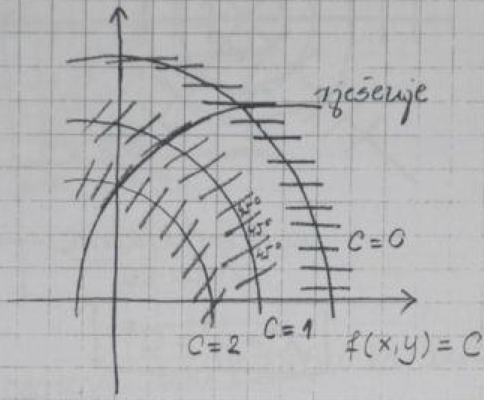
$1 = \frac{0^3}{6} + \frac{0^2}{2} + 0 + C_2$

$C_2 = 1$

$\boxed{y = \frac{x^3}{6} + \frac{x^2}{2} + x + 1}$

* POLJE SMJEROVA I IZOKLINE *

Jed. $y' = f(x,y)$ ima sljedeću geometrijsku interpretaciju: u svakoj točki (x,y) područja definicije funk. f određen je smjer tangente na krivulju y u toj točki (tg kuta tangente $= f(x,y)$)



$C=1 \Rightarrow tg\alpha = 1$

$\alpha = 45^\circ$

Ako ucrtamo sve smjerove u svim točkama područja def. oni će nam dati POLJE SMJEROVA iz kojeg možemo uočiti grafove integralnih krivulja. Zbog lakšeg crtanja određujemo IZOKLINE: krivulje u čijim se točkama podudaraju smjerovi. To su krivulje $f(x,y)=C$ jer je na njima $y'=C$. Njih crtamo obično za cjelobrojne vrijednosti konst. C.

Koristeći izokline nađite grafičko rj. dif. jed.

1. $y' = x$

$y' = f(x,y) = x$

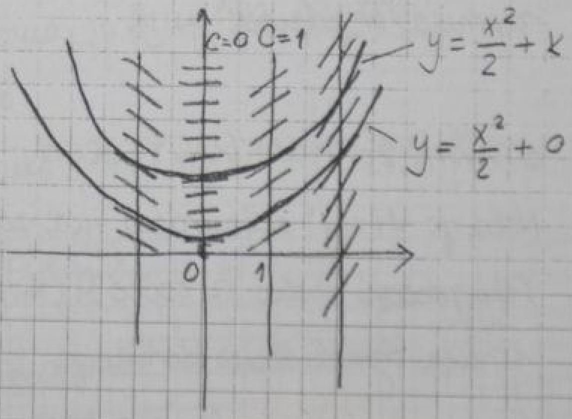
jed. izokline $y' = C \Rightarrow x = C$

$x=1$, $y'=1 \Rightarrow \alpha = 45^\circ$

$x=0$ $y'=0 \Rightarrow \alpha = 0^\circ$

$x=-1$ $C=-1$ $y=-1 \Rightarrow \alpha = 135^\circ$

$C=2 \Rightarrow \alpha = 70^\circ$



② $y' = -\frac{x}{y}$

$y' = f(x,y) = -\frac{x}{y}$

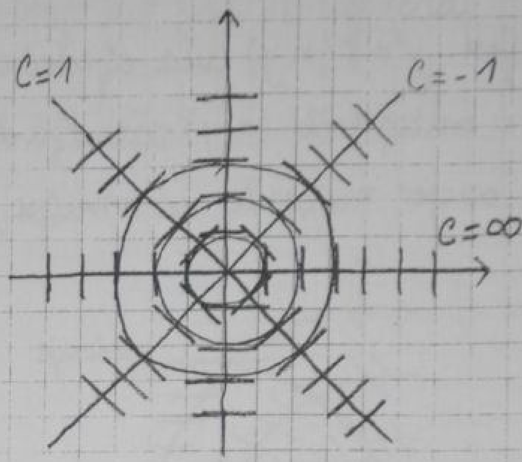
jed. izokline: $-\frac{x}{y} = C$

$y = -\frac{1}{C}x$

$C = 1 \Rightarrow y = -x \quad \alpha = 45^\circ$

$C = -1 \Rightarrow y = x \quad \alpha = 135^\circ$

$C = \infty \Rightarrow y = 0$



KONCENTRIRANE KRUŽNICE

$\boxed{x^2 + y^2 = C^2}$

OSNOVNI PROBLEMI:

1) egzistencija rješenja promatrane dif. jed.
2) pronalaženje svih ili samo nekih rješenja
3) jednoznačnost rješenja uz zadani početni uvjet

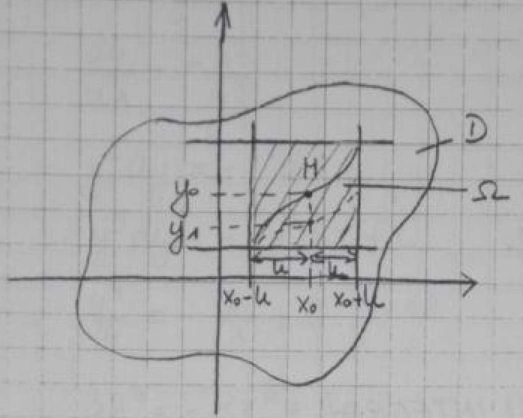
* STAVAK 1 – PEANOV TEOREM (1)

Ako je funk. $f(x,y)$ neprekinuta u okolišu točke (x_0, y_0) onda jed. $y' = f(x,y)$ ima bar jedno rješenje koje u točki (x_0, y_0) poprima vrijednost $y = y_0$.

STAVAK 2 – PICARDOV TEOREM (3)

Neka je $f(x,y)$ definirana na području D ravnine xOy.
Ako postoji okoliš Ω točke $M(x_0, y_0) \in D$ u kojem je f neprekinuta i ima ograničenu derivaciju $\frac{\partial f}{\partial y}$, onda postoji interval $(x_0 - h, x_0 + h)$ na Ox na kojem postoji i jednoznačno je

rješenje $y = f(x)$ Cauchyevog problema: $y' = f(x,y)$
$y(x_0) = y_0$



Picardov teorem ima lokalni karakter: on garantira jednoznačnost rješenja samo u malom okolišu točke x_0. Sama jed. $y' = f(x,y)$ ima beskonačno mnogo rj. Jedno prolazi kroz točku (x_0, y_0), drugo kroz blisku točku (x_0, y_1) itd.

18.05.2011.

* OPĆI I PARTIKULARNI INTEGRAL *

DEF 2.

Opće rješenje dif. jed. $y' = f(x,y)$ u nekom području Ω u kojem su zadovoljeni uvjeti Picardovog teorema je familija funkcija $y = f(x, C)$ koja zadovoljava uvjete:

1) za svaku moguću vrijednost C funkc. $y = f(x,C)$ zadovoljava jed.:

$$f'_x(x,C) \equiv f(x, f(x,C))$$

2) za proizvoljnu točku $(x_0, y_0) \in \Omega$ možemo odrediti konst. $C = C_0$ tako da bude $f(x_0, C_0) = y_0$

$\Rightarrow$ * uvrštavanjem neke konkretne vrijed. za konst. C dobivamo neko partikularno rješenje. Opće rješenje možemo shvatiti kao skup ovih partikularnih rješenja.

ANALOGNO:

$\underbrace{\Phi(x, y, C) = 0}_{\text{iz } y = f(x, C)}$ OPĆI INTEGRAL

$\Phi(x, y, C_0) = 0$ PARTIKULARNI INTEGRAL

* TIPOVI DIF. JED. PRVOG REDA *

1. Dif. jed. sa SEPARIRANIM varijablama

opći oblik: $f(y)\,dy = g(x)\,dx$

ili

$$y' = \frac{g(x)}{f(y)}$$

rješava se neposrednim integriranjem tj.

$$\boxed{\int f(y)\,dy = \int g(x)\,dx + C}$$ OPĆE RJEŠENJE

$\Rightarrow$ Cauchyev problem $y' = \dfrac{g(x)}{f(y)}$

$y(x_0) = y_0$

$$\int_{y_0}^{y} f(y)\,dy = \int_{x_0}^{x} g(x)\,dx$$

Nađite opće rješenje:

① $y' = \frac{-x^2}{y^3}$

$\frac{dy}{dx} = -\frac{x^2}{y^3}$

$y^3 dy = -x^2 dx$

$\int y^3 dy = -\int x^2 dx + C$

$\frac{y^4}{4} = -\frac{x^3}{3} + C \quad / \cdot 12$

$3y^4 = -4x^3 + C_1$

$3y^4 + 4x^3 + C_1 = 0$

② $e^{x^2} dx = \frac{dy}{\ln y} \quad / \int$

$\int e^{x^2} dx = \int \frac{dy}{\ln y} + C$

opći integral

③ $xyy' = 1 - x^2$

$xy\frac{dy}{dx} = 1 - x^2$

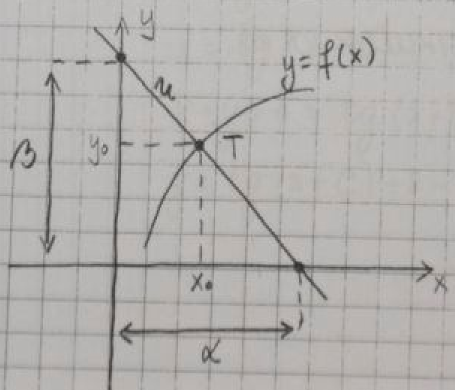
$y\,dy = \frac{1-x^2}{x} dx \quad / \int$

$\int y\,dy = \int \frac{1-x^2}{x} dx + C$

$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C$

$y^2 + x^2 - 2\ln|x| + C_1 = 0$

④ U po volji odabranoj točki T neke krivulje povučena je normala. Naći krivulju ako je poznato da se odrezak normale između koordinatnih osi raspolavlja u točki T.



$n \equiv y - y_0 = -\frac{1}{y_0'}(x - x_0)$

⇒

- sjecišta normale s koordinatnim osima

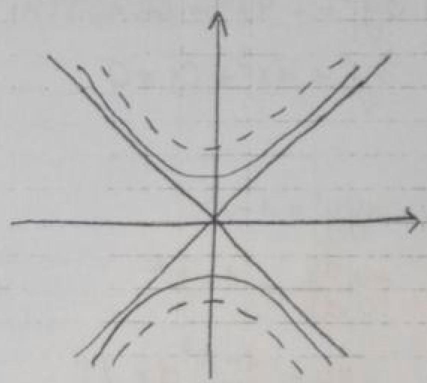
$$x=0 \rightarrow y = y_0 + \frac{x_0}{y_0'} = \beta$$

$$y=0 \rightarrow x = x_0 + y_0 y_0' = \alpha$$

uvjet zadatka $T(x_0, y_0) = S(\overline{AB}) = \left(\frac{\alpha+0}{2}, \frac{0+\beta}{2}\right) = \left(\underset{x_0}{\frac{\alpha}{2}}, \underset{y_0}{\frac{\beta}{2}}\right)$

↳ središte

$$\alpha = 2x_0$$



$x_0 + y_0 y_0' = 2x_0$ dif. jed.

$x_0 \rightarrow x \quad y_0 \rightarrow y \quad y_0' \rightarrow y'$

$x + yy' = 2x$

$ydy = xdx \quad /\int$

$\frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2} \quad / \cdot 2$

$y^2 - x^2 = C$ FAMILIJA HIPERBOLA

(5) Naći krivulju koja prolazi točkom $T(2,4)$, a u svakoj njenoj točki je raspolovljen odrezak normale između koordinatnih osi.

$\Rightarrow$ koristimo rješenje iz prošlog zadatka

$T(2,4) \in \gamma \Rightarrow 4^2 - 2^2 = C$

$C = 12$

$y^2 - x^2 = 12$ PARTIKULARNO Rj.

2. Dif. jed. oblika $y' = f(ax+by+c)$, $a,b,c \in \mathbb{R}$

- rješava se supstitucijom $z = ax+by+c$ gdje je z nova zavisna varijabla

$$z = ax+by+c \quad / \frac{d}{dx}$$

$$z' = \frac{dz}{dx} = a + b\frac{dy}{dx}$$

$$\frac{dy}{dx} = y' = \frac{1}{b}\left[\frac{dz}{dx} - a\right]$$

$$\frac{1}{b}\left[\frac{dz}{dx} - a\right] = f(z)$$

$$\frac{dz}{dx} - a = bf(z)$$

$$\frac{dz}{dx} = bf(z) + a$$

$$dx = \frac{dz}{bf(z)+a}$$

(1) Naći partik. rj. koje prolazi točkom $T(0,0)$ za dif. jed. $y' = (x+y+1)^2$

SUPST. $x+y+1 = z \quad / \frac{d}{dx}$

$$1 + y' = z'$$

$$y' = z' - 1$$

$$z' - 1 = z^2$$

$$\frac{dz}{dx} = z^2 + 1$$

$$\frac{dz}{z^2+1} = dx \quad / \int$$

$$\operatorname{arctg} z = x + C$$

$$z = \operatorname{tg}(x+C)$$

$x+y+1 = \operatorname{tg}(x+C)$ opće rj.

$$y = \operatorname{tg}(x+C) - x - 1$$

$$0 = \operatorname{tg} C - 0 - 1$$

$$\operatorname{tg} C = 1$$

$$C = \frac{\pi}{4}$$

$$\boxed{y = \operatorname{tg}\left(x + \frac{\pi}{4}\right) - x - 1}$$

3. Homogene dif. jed. prvog reda

Za funkc. $M(x,y)$ kažemo da je homogena funkc. u varijablama x,y ako za svaki $t>0$ vrijedi:

$$M(tx,ty) = t^{\alpha} M(x,y)$$

α - STUPANJ HOMOGENOSTI OD M

Nađite stupanj homogenosti:

1. $M(x,y) = \sqrt{x^2+y^2} - 2x$

$\alpha = 1$

$$M(tx,ty) = \sqrt{(tx)^2+(ty)^2} - 2(tx) = t\left[\sqrt{x^2+y^2} - 2x\right] = t^1 M(x,y)$$

2. $M(x,y) = x^3 \ln\left(\frac{y}{x}\right) + xy^2$

$\alpha = 3$

3. $M(x,y) = x^2y + xy^2 + 2$ NIJE HOMOGENA

4. $M(x,y) = f\left(\frac{y}{x}\right)$

$$M(tx,ty) = f\left(\frac{ty}{tx}\right) = f\left(\frac{y}{x}\right) = M(x,y) = t^0 M(x,y)$$

$\alpha = 0$

DEF. 3.

Za dif. jed. kažemo da je homogena ako je možemo svesti na oblik:

$$y' = f\left(\frac{y}{x}\right)$$

ALGORITAM RJEŠAVANJA:

SUPST. $\frac{y}{x} = z$, z - nova zavisna var.

$$y = xz \quad \Big/ \frac{d}{dx}$$

$$y' = z + xz'$$

$$z + xz' = f(z)$$

$$x\frac{dz}{dx} = f(z) - z$$

$$\frac{dz}{f(z)-z} = \frac{dx}{x} \quad \Big/ \int$$

$$\int \frac{dz}{f(z)-z} = \ln|x| + C_1$$

Nađite opće rješenje zadanih jed.

① $y' = \frac{y}{x} + tg\left(\frac{y}{x}\right)$ homogena

$$z = \frac{y}{x}$$

$$y = xz$$

$$y' = z + xz'$$

$$z + xz' = z + tg\, z$$

$$\frac{dz}{tg z} = \frac{dx}{x} \quad \Big/ \int \qquad \left| \frac{dz}{tg z} = \frac{\cos z\, dz}{\sin z} \right|$$

$$\ln|\sin z| = \ln|x| + \ln C$$

$$\ln|\sin z| = \ln|xC|$$

$$\sin z = \pm xC$$

$$\sin\left(\frac{y}{x}\right) = xC$$

$$y = x \arcsin(xC)$$

② $(x^2+y^2)dx - xydy = 0$

$$y' = \frac{dy}{dx} = \frac{x^2+y^2}{xy} \quad \begin{matrix} /:x^2 \\ /:x^2 \end{matrix}$$

$$y' = \frac{1+\left(\frac{y}{x}\right)^2}{\frac{y}{x}} \quad \text{homogena}$$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = z + xz'$$

$$z + xz' = \frac{1+z^2}{z} = \frac{1}{z} + z$$

$$xz' = \frac{1}{z}$$

$$zdz = \frac{dx}{x} \quad /\int$$

$$\frac{z^2}{2} = \ln|x| + \ln C = \ln|Cx|$$

$$z^2 = 2\ln|Cx|$$

$$\frac{y^2}{x^2} = 2\ln|Cx|$$

$$y^2 = 2x^2\ln|Cx| \quad \text{ili} \quad y^2 = x^2\ln(Cx^2)$$

③ Dokazati da je jed. $M(x,y)\,dx + N(x,y)\,dy = 0$ homogena ako su M i N homogene funkc. istog stupnja homogenosti.

$$y' = \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{M\left(x, \frac{y}{x}\cdot x\right)}{N\left(x, \frac{y}{x}\cdot x\right)} =$$

$$= -\frac{M\left(1\cdot x, \frac{y}{x}\cdot x\right)}{N\left(1\cdot x, \frac{y}{x}\cdot x\right)} = -\frac{x^\alpha M\left(1, \frac{y}{x}\right)}{x^\alpha N\left(1, \frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$

Q.E.D.

4. Dif. jed. koje se svode na homogenu

opći oblik: $y' = f\left(\dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$

Moguća su dva slučaja:

1) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ tj. pravci predstavljeni jed.

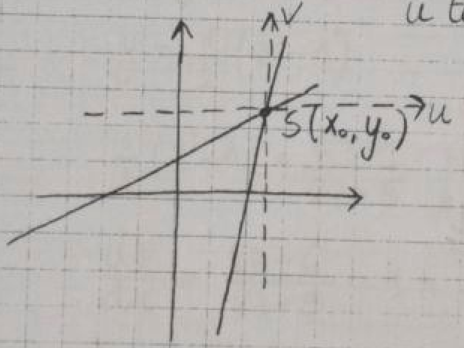
$$\left.\begin{array}{l} p_1 \equiv a_1x + b_1y + c_1 \\ p_2 \equiv a_2x + b_2y + c_2 \end{array}\right\} \text{PARALELNI}$$

SUPST. $a_1x + b_1y = z$, z-nova zav. var.

$a_1 + b_1y' = z'$

$y' = \dfrac{z' - a_1}{b_1} = f\left(\dfrac{z + c_1}{kz + c_2}\right)$ itd. (separacija)

2) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ tj. pravci predst. jed. se SIJEKU u točki $S(x_0, y_0)$



nova nezav. var.

SUPST. $\boxed{\begin{array}{l} x = u + x_0 \\ y = v + y_0 \end{array}}$

nova zav. var.

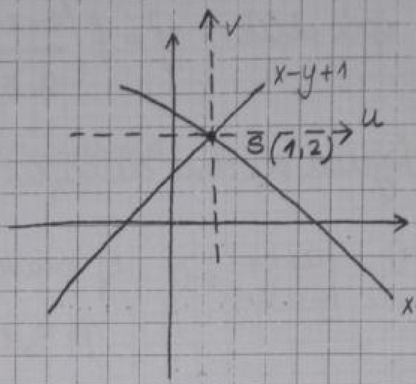
$$y' = \frac{dy}{dx} = \frac{d(v + y_0)}{d(u + x_0)} = \frac{dv}{du}$$

$$\frac{dv}{du} = \frac{a_1u + b_1v}{a_2u + b_2v} = \frac{a_1 + b_1\frac{v}{u}}{a_2 + b\frac{v}{u}}$$

20.05.2011.

Naći opće rješenje

(1) $y' = \frac{x-y+1}{x+y-3}$



NOVI KOORDINATNI SUSTAV

$\begin{cases} x-y+1=0 \\ x+y-3=0 \end{cases}$ neparalelni pravci

SUPST. $x = u+1$

$y = v+2$

$$y' = \frac{dy}{dx} = \frac{d(v+2)}{d(u+1)} = \frac{dv}{du} = \frac{\overbrace{u+1}^{x} - \overbrace{(v+2)}^{y} + 1}{u+1+v+2-3} = \frac{u-v}{u+v} = \frac{1-\frac{v}{u}}{1+\frac{v}{u}}$$

homogena u var. u, v

SUPST. $\frac{v}{u} = z$

$v = uz$

$v' = z + uz'$

$z + uz' = \frac{1-z}{1+z}$

$uz' = \frac{1-z}{1+z} - z = \frac{1-2z-z^2}{1+z}$

$u\frac{dz}{dx} = -\frac{z^2+2z-1}{z+1}$

$\frac{z+1}{z^2+2z-1}\,dz = -\frac{du}{u} \quad /\int$

$\int \frac{z+1}{(z+1)^2-2}\,dz = -\int \frac{du}{u}$

$\left| \begin{matrix} z+1=t \\ dz=dt \end{matrix} \right|$

$\frac{1}{2}\ln|(z+1)^2-2| = -\ln|u| + \ln C$

$\ln|z^2+2z-1| = 2\ln\left|\frac{C}{u}\right|$

$\ln|z^2+2z-1| = \ln\left|\frac{C_1}{u^2}\right|$

$$|z^2+2z-1| = \left|\frac{C_1}{u^2}\right|$$

$$z^2+2z-1 = \pm\frac{C_1}{u^2} = \frac{C_2}{u^2}$$

$$\frac{v^2}{u^2}+2\frac{v}{u}-1 = \frac{C_2}{u^2} \quad / \cdot u^2$$

$$v^2+2vu-u^2 = C_2$$

$$(y-2)^2+2(x-1)(y-2)-(x-1)^2 = C_2$$

$$\boxed{y^2-x^2+2xy-2x-6y = C}$$ opći integral

(2) $$y' = \frac{x+2y+3}{2x+4y+5}$$

$$\left.\begin{aligned} p_1 &\equiv x+2y+3=0 \\ p_2 &\equiv 2x+4y+5=0 \end{aligned}\right\} \quad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0 \Rightarrow p_1 \parallel p_2$$

SUPST. $x+2y = z \quad / \frac{d}{dx}$

$$1+2y' = z'$$

$$y' = \frac{1}{2}(z'-1)$$

$$\frac{1}{2}(z'-1) = \frac{z+3}{2z+5} \quad / \cdot 2$$

$$\frac{dz}{dx} = \frac{2z+6}{2z+5}+1 = \frac{4z+11}{2z+5}$$

$$\frac{2z+5}{4z+11}\,dz = dx \quad / \cdot 2$$

$$\frac{4z+10}{4z+11}\,dz = 2dx \quad / \int$$

$$\int \frac{4z+11}{4z+11} - \frac{1}{4z+11}\,dz = \int 2dx$$

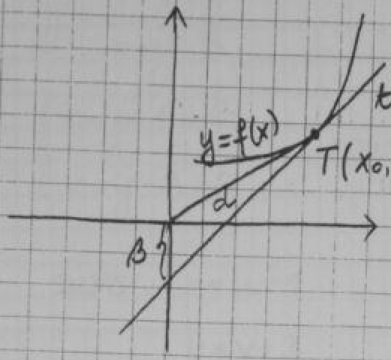
$$z - \frac{1}{4}\ln|4z+11| = 2x+C$$

$$\ln|4z+11| = 4z-8x+C_1$$

$$\ln|4(x+2y)+11| = 4(x+2y)-8x+C_1$$

$$\boxed{\ln|4x+8y+11| = -4x+8y+C_1}$$

③ Naći jed. krivulje koja prolazi točkom $T(1,0)$ i za koju je odsječak tangente na Oy jednak udaljenosti od dirališta tangente do ishodišta.



$t \equiv y - y_0 = y_0'(x - x_0)$

odrezak na $Oy = b$

$b - y_0 = y_0'(0 - x_0)$

$b = y_0 - x_0 y_0'$

$d = d(OT) = \sqrt{x_0^2 + y_0^2}$

$\Rightarrow$ zadatak traži $|b| = d$

$$\sqrt{x_0^2 + y_0^2} = |y_0 - x_0 y_0'|$$

$$x_0 \to x \quad y_0 \to y \quad y_0' \to y'$$

$$\sqrt{x^2 + y^2} = |y - xy'|$$

$$y - xy' = \pm\sqrt{x^2 + y^2}$$

$$y' = y \pm \frac{\sqrt{x^2+y^2}}{x}$$

$$y' = \frac{y}{x} \pm \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

SUPST. $\frac{y}{x} = z$

$y = xz$

$y' = z + xz'$

$$z + xz' = z \pm \sqrt{1 + z^2}$$

$$x\frac{dz}{dx} = \pm\sqrt{1+z^2}$$

$$\frac{dz}{\sqrt{1+z^2}} = \pm\frac{dx}{x} \quad /\int$$

1) $\ln|z + \sqrt{1+z^2}| = \ln|x| + \ln C_1$

$$|z + \sqrt{1+z^2}| = |C_1 x|$$

$$z + \sqrt{1+z^2} = \pm C_1 x = C_2 x$$

$$\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = C_2 x \quad /\cdot x$$

$$y + \sqrt{x^2 + y^2} = C_2 x^2$$

$$\sqrt{x^2+y^2} = C_2 x^2 - y \quad /^2$$

$$\boxed{y = \frac{C_2^2 x^2 - 1}{2C_2} = \frac{C_2}{2}x^2 - \frac{1}{2C_2}}$$

2) analogno $z + \sqrt{1+z^2} = \frac{C_3}{x}$

***

$$\boxed{y = -\frac{1}{2C_3}x^2 + \frac{1}{2}C_3}$$

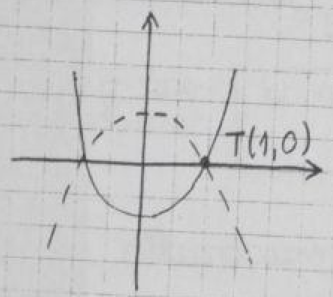
$\Rightarrow$ obzirom da se u rj. 2) postavljanjem $C_3 = \frac{1}{C_2}$ dobije rj. 1) to su ista rj., znači da je opće rj. jed.

$$\boxed{y = \frac{C}{2}x^2 - \frac{1}{2C}}$$

$T(1,0) \in y$

$$0 = \frac{C}{2}1^2 - \frac{1}{2C} \Rightarrow C = \pm 1$$

$\boxed{y = \pm\frac{1}{2}(x^2-1)}$ partikularno rj.



5. linearne dif. jed.

opći oblik: $y' + f(x) \cdot y = g(x)$

$\hookrightarrow$ FUNKCIJA SMETNJE (PERTURBACIJA)

METODA VARIJACIJE KONSTANTE

1) homogena jed. koja se dobiva odbacivanjem lijeve strane

$$y' + f(x) y = 0$$

$$\frac{dy}{y} + f(x) dx = 0 \quad / \int$$

$$\ln|y| + \int f(x) dx = \ln C$$

$$\ln|y| = \ln C - \ln e^{\int f(x) dx}$$

$$|y| = \left| \frac{C}{e^{\int f(x) dx}} \right| \Rightarrow y = \pm C \cdot e^{-\int f(x) dx}$$

$$\boxed{y_0 = C \cdot e^{-\int f(x) dx}}$$ opće rj. skraćene jed.

2) u dobivenom općem rj. konst. C smatramo za funkc. tj. $\boxed{C = C(x)}$

$$y = C(x) e^{-\int f(x) dx}$$ opće rj. zadane jed.

određujemo uvrštavanjem u zadanu jed.

$$\underbrace{C'(x) e^{-\int f(x) dx} + \cancel{C(x) e^{-\int f(x) dx} (-f(x))}}_{y'} + \underbrace{\cancel{f(x) C(x) e^{-\int f(x) dx}}}_{y} = g(x)$$

$$C'(x) \cdot e^{-\int f(x) dx} = g(x)$$

$$C'(x) = \frac{d C(x)}{dx} = g(x) \cdot e^{\int f(x) dx} \quad / \cdot dx$$

$$\int d C(x) = C(x) = \int g(x) e^{\int f(x) dx} dx + C_1$$

STVARNA KONST.

3) ponatak u opće rj. zadane jed.

$$\boxed{y = \left[\int g(x)\, e^{\int f(x)\,dx}\,dx + C_1\right] \cdot e^{-\int f(x)\,dx}}$$ OPĆE RJ. LIN. JED.

⇒ druga mogućnost

- nadi opće rj. lin. jed. → teoretsko pitanje

$$y = \underbrace{C \cdot e^{-\int f(x)\,dx}}_{y_0} + \underbrace{e^{-\int f(x)\,dx} \int g(x) \cdot e^{\int f(x)\,dx}\,dx}_{y_p \text{ (partikularno) ili } y_u}$$

$\boxed{y = y_0 + y_p}$ gdje je y_0 rješenje skraćene (homogene) jed., a y_p je neko partikularno rj. početne jed. uzrokovano funkc. smetnje $g(x)$

(1) Nadi opće rj. dif. jed.

$$xy' - y - x^3 = 0 \quad /:x$$

$$y' - \underbrace{\frac{1}{x}}_{f(x)} y = \underbrace{x^2}_{g(x)} \quad \text{linearna}$$

⇒ metoda varijacije konst.

1) $y' - \frac{1}{x}y = 0$

$$\frac{dy}{y} = \frac{dx}{x} \quad / \int$$

$$\ln|y| = \ln|x| + \ln C$$

$$|y| = |Cx|$$

$$y = \pm Cx$$

$\boxed{y_0 = Cx}$ opće rj. skraćene jed.

2) $C = C(x)$

$y = C(x) \cdot x$

$\underbrace{C'(x) \cdot x + \cancel{C(x) \cdot 1}}_{y'} - \cancel{\frac{1}{x} C(x) \cdot x} = x^2 \quad /:x$

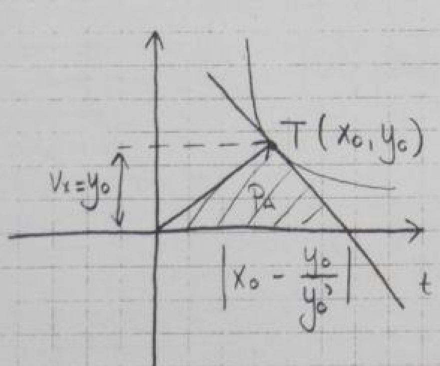
$C'(x) = x \quad / \int$

$C(x) = \int x \, dx = \frac{x^2}{2} + C_1$

3) $y = \left(\frac{x^2}{2} + C_1 \right) x$

$y = \underbrace{C_1 x}_{y_0} + \underbrace{\frac{x^3}{2}}_{y_p}$

(2) Nadi sve krivulje sa svojstvom da je konst. površina trokuta koji tvore Ox, tangenta i radij vektor dirališta po volji odabrane točke na krivulji (i da je $P_\Delta = a^2$)



$t \equiv y - y_0 = y_0' (x - x_0)$

$\frac{y_0 \left(x_0 - \frac{y_0}{y_0'} \right)}{2} = a^2 \qquad x_0 \to x \quad y_0 \to y \quad y_0' \to y'$

$\left| \frac{y \left(x - \frac{y}{y'} \right)}{2} \right| = a^2$

$xy - \frac{y^2}{y'} = \pm 2a^2$ opći oblik

$\Rightarrow$ nije lin. u var. ali jest. u x, x'

$$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'}$$

$$\boxed{y' = \frac{1}{x'}}$$

$$xy - y^2 x' = \pm 2a^2 \quad / : (-y^2)$$

$$\boxed{\underbrace{x'}_{\frac{dx}{dy}} - \underbrace{\frac{1}{y}}_{f(x)} x = \underbrace{\pm \frac{2a^2}{y^2}}_{g(x)}}$$ linearna

$$C = C(y)$$

$\oplus$ $$x = Cy + \frac{a^2}{y}$$

opće rješenje

$\ominus$ $$x = Cy - \frac{a^2}{y}$$