Međuispit iz Matematike 2

28. travnja 2014.

1. (5 bodova)

a) Iskažite i dokažite Cauchyjev kriterij za konvergenciju redova realnih brojeva s pozitivnim članovima.

b) Ispitajte konvergenciju redova

$$\sum_{n=1}^{\infty} \frac{2n-1}{3n+1}, \quad \sum_{n=1}^{\infty} \frac{2n-1}{(3n+1)^2}, \quad \sum_{n=1}^{\infty} \frac{2n-1}{(3n+1)^3}.$$

Što možete reći o konvergenciji reda $\sum_{n=1}^{\infty} \frac{2n-1}{(3n+1)^p}$ za svaki $p > 3$? Obrazložite sve tvrdnje.

2. (5 bodova)

a) Razvijte u Taylorov red oko $c = 0$ funkciju

$$f(x) = \frac{x}{(1-x)^2},$$

te odredite područje konvergencije dobivenog reda.

b) Koju od sljedećih dviju suma

$$\sum_{n=1}^{\infty} (-1)^n n \left(\frac{1}{3}\right)^n, \quad \sum_{n=1}^{\infty} (-1)^n n 5^n$$

možemo izračunati koristeći razvoj pod a). Obrazložite sve tvrdnje, te izračunajte sumu.

3. (5 bodova) Zadani su vektori $\vec{a}$ i $\vec{b}$ takvi da vrijedi

$$\|\vec{a}\| = 2, \ \|\vec{b}\| = 3, \quad \angle(\vec{a}, \vec{b}) = \frac{\pi}{6}.$$

a) Izračunajte skalarni produkt vektora $4\vec{a} + 5\vec{b}$ i $\vec{a} - \vec{b}$.

b) Izračunajte površinu paralelograma razapetog s vektorima $2\vec{a} - \vec{b}$ i $\vec{a} + \vec{b}$.

4. (5 bodova) Odredite kanonsku jednadžbu pravca p koji je presjek ravnina

$$\pi_1 \ldots . \ x - y + 2z - 1 = 0, \quad \pi_2 \ldots . \ 2x + y - z + 2 = 0.$$

Napišite jednadžbu ravnine koja sadrži pravac p i točku $A(-1, 1, 0)$.

5. (5 bodova) Odredite i skicirajte domenu funcije $z(x, y) = \ln(\arcsin \frac{x}{y})$.

6. (5 bodova)

a) Skicirajte i imenujte plohu $z(x, y) = 6 - 2x^2 - 2y^2$.

b) Nađite točku na plohi pod a) u kojoj je tangencijalna ravnina okomita na tangentu krivulje

$$\mathcal{C} \ \ldots . \ \begin{cases} x(t) = t \\ y(t) = t^2 + 1 \\ z(t) = -3t, \end{cases}$$

za $t = 2$.

OKRENI!!!!!

7. (5 bodova)

a) Napišite formulu za izračunavanje približne vrijednosti funkcije $f : D_f \to R,\ D_f \subseteq R^n$.

b) Koristeći formulu pod a) izračunajte približnu vrijednost izraza

$$A = \sqrt{(2.95)^2 + 2 \cdot (2.01)^3}.$$

8. (5 bodova) Neka je $u(x,y,z) = x^2 - y^2 + z^2 - xyz$.

a) Izračunajte $\frac{\partial u}{\partial \overrightarrow{s}}(1,2,1)$, ako je $\overrightarrow{s} = \overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$.

b) Nađite jedinični vektor $\overrightarrow{a} \in V^3$ takav da je

$$-\frac{\partial u}{\partial \overrightarrow{a}}(1,2,1) \leq \frac{\partial u}{\partial \overrightarrow{b}}(1,2,1) \leq \frac{\partial u}{\partial \overrightarrow{a}}(1,2,1), \quad \forall \overrightarrow{b} \in V^3.$$

c) Koristeći lančano pravilo izračunajte

$$\frac{d}{dt}\Big(u(x(t), y(t), z(t))\Big),$$

gdje su

$$\begin{cases} x(t) = 3t^2 \\ y(t) = 2t \\ z(t) = 1. \end{cases}$$

Vrijeme pisanja ispita je **120min**.

Nije dozvoljena uporaba računala. Dozvoljena je isključivo uporaba službenih formula.

fra

(1) [5 bodova]

2 boda (a) $\sum a_n$ red s poz. čl. (1.) ako $\exists q<1$ t.d. $\sqrt[n]{a_n} \le q$, $\forall n \ge n_0 \Rightarrow$ red KVG. (2.) ako je $\sqrt[n]{a_n} \ge 1$, $\forall n \ge n_0 \Rightarrow$ RED DIV

Ako $\exists q = \lim_{n\to\infty} \sqrt[n]{a_n}$, tada: za $q<1$ (KVG.), za $q>1$ (DIV.), $q=1$ (nema odluke) Dokaz: KNJIŽICA 1, str. 18.

Iskaz (1 b) Dokaz (1 b)

3 boda (b) $\sum_{n=1}^{\infty} \frac{2n-1}{3n+1}$, DIVERG. (nije zadovoljen $\lim_n a_n = 0$, $\lim_n \frac{2n-1\,/:n}{3n+1\,/:n} = \frac{2}{3} \neq 0$) 1 b

$\sum_{n=1}^{\infty} \frac{2n-1}{(3n+1)^2}$, DIVERG. (uspor. krit. sa $\sum_{n=1}^{\infty} \frac{1}{n}$, $\lim_n \frac{\frac{2n-1}{(3n+1)^2}}{\frac{1}{n}} = \lim_n \frac{2n^2-n}{(9n^2+6n+1)} = \frac{2}{9}$

$\sum_{n=1}^{\infty} \frac{2n-1}{(3n+1)^3}$, KVG. jer $\sim \sum_{n=1}^{\infty} \frac{1}{n^2}$; $\lim_n \frac{\frac{2n-1}{(3n+1)^3}}{\frac{1}{n^2}} = \frac{2}{27}$.

$\rightarrow \sum_{n=1}^{\infty} \frac{2n-1}{(3n+1)^p} \sim \sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \rightarrow$ KVG, za $p-1>1$ tj. $p>2$ (KVG. UVJET!)

---

(2) [5 bodova]

(a) $f(x) = \frac{x}{(1-x)^2} = x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \cdot \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = x \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} = \boxed{\sum_{n=1}^{\infty} n \cdot x^n}$, za $|x|<1$. 2 b

Područje kvg.: $|x|<1$, tj. $x \in \langle -1, 1\rangle$ 1 b

(b) Suma $\sum_{n=1}^{\infty} (-1)^n \cdot n \cdot \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} n \cdot \left(-\frac{1}{3}\right)^n = \left\{ \left|-\frac{1}{3}\right| < 1 \right\} = f\left(-\frac{1}{3}\right) = -\frac{3}{16}$ 1 b

Red $\sum_{n=1}^{\infty} (-1)^n \cdot n \cdot 5^n$, jer dani red od $f(x)$ kvg. samo za $|x|<1$. 1 b

---

(3) [5 bodova]

(a) $\|\vec{a}\| = 2$, $\|\vec{b}\| = 3$, $\sphericalangle(\vec{a},\vec{b}) = \frac{\pi}{6}$

Skal. pr.: $(4\vec{a}+5\vec{b})\cdot(\vec{a}-\vec{b}) = 4|\vec{a}|^2 - 4\vec{a}\cdot\vec{b} + 5\underbrace{\vec{b}\cdot\vec{a}}_{=\vec{a}\cdot\vec{b}} - 5|\vec{b}|^2 = 4\overbrace{|\vec{a}|^2}^{4} + \vec{a}\cdot\vec{b} - 5\overbrace{|\vec{b}|^2}^{9} = 4\cdot4 + \vec{a}\cdot\vec{b} - 5\cdot9 = 16 + 3\sqrt{3} \ldots$

$= \boxed{-29+3\sqrt{3}}$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\frac{\pi}{6} = 2\cdot3\cdot\frac{\sqrt{3}}{2}$

(b) Pov. paral. razap. sa $\underbrace{2\vec{a}-\vec{b}}_{\vec{m}}$, $\underbrace{\vec{a}+\vec{b}}_{\vec{n}}$

$P = |\vec{m}\times\vec{n}| = |\vec{m}|\cdot|\vec{n}|\cdot\sin\sphericalangle(\vec{m},\vec{n})$

$\vec{m}\times\vec{n} = (2\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{a}) + 2(\vec{a}\times\vec{b}) - \vec{b}\times\vec{a} - \vec{b}\times\vec{b} = 3(\vec{a}\times\vec{b})$

$P = |\vec{m}\times\vec{n}| = 3\cdot|\vec{a}\times\vec{b}| = 3|\vec{a}|\cdot|\vec{b}|\cdot\sin\sphericalangle(\vec{a},\vec{b}) = 3\cdot2\cdot3\cdot\frac{1}{2} = \boxed{9 \text{ kv. jed.}}$

---

(4) [5 bodova]

$\Pi_1 \ldots\ x - y + 2z - 1 = 0 \quad |\cdot(-2)$

$\Pi_2 \ldots\ 2x + y - z + 2 = 0$

$\Rightarrow$ $x - y + 2z - 1 = 0$, $3y - 5z + 4 = 0$ $\Rightarrow$ $z = t$, $3y = 5z - 4$, $y = \frac{5}{3}t - \frac{4}{3}$ $\Rightarrow$ $x = y - 2z + 1 = \frac{5}{3}t - \frac{4}{3} - 2t + 1$

$\Rightarrow x = -\frac{1}{3}t - \frac{1}{3}$

$\Rightarrow \boxed{p \ldots \frac{x+\frac{1}{3}}{-\frac{1}{3}} = \frac{y+\frac{4}{3}}{\frac{5}{3}} = \frac{z}{1}.}$

Ravnina koja sadrži p, sadrži i dvije točke na toj pravcu:

za $t=2 \Rightarrow T_1(-1, 2, 2)$

za $t=5 \Rightarrow T_2(-2, 7, 5)$

$\Rightarrow \Pi \ldots \begin{vmatrix} x+1 & y-1 & z \\ -1+1 & 2-1 & 2 \\ -2+1 & 7-1 & 5 \end{vmatrix} = 0 \Rightarrow \Pi \ldots -7x - 2y + z - 5 = 0$

tj. $\boxed{\Pi \ldots 7x + 2y - z + 5 = 0}$

(5.) [5 bodova]

$z(x,y) = \ln\left(\arcsin \frac{x}{y}\right)$

1. $y \neq 0$
2. $-1 \le \frac{x}{y} \le 1$
3. $\arcsin \frac{x}{y} > 0 \iff \frac{x}{y} > 0$

$\frac{x}{y} > 0$: $x>0$ & $y>0$ $\vee$ $x<0$ & $y<0$

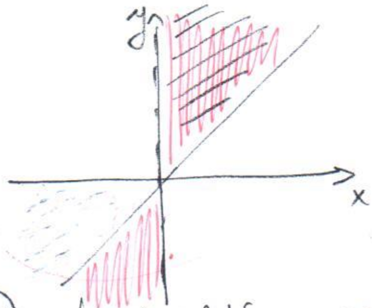
$-1 \le \frac{x}{y}$ & $\frac{x}{y} \le 1$

$0 \le \frac{x}{y} + 1$ $\qquad$ $\frac{x-y}{y} \le 0$

$\frac{x+y}{y} \ge 0$

$y<0$ & $x+y \le 0$ & $x-y \ge 0$ $\qquad$ $y>0$ & $x+y \ge 0$ & $x-y \le 0$

$\boxed{y>0 \ \& \ y \le x \le -y}$ $\qquad$ $\boxed{y>0 \ \& \ -y \le x \le y}$



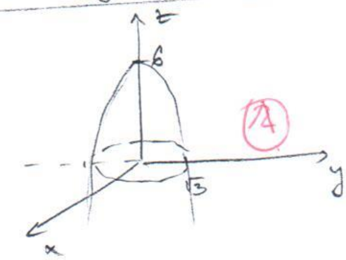
$\Rightarrow D_z = \{(x,y) \in \mathbb{R}^2 \mid \{x>0, y>0, -y \le x \le y\} \vee \{x<0, y<0, y \le x \le -y\}\}.$

(6.) [5 bodova]

(a) $z(x,y) = 6 - 2x^2 - 2y^2$

$6 - z = 2x^2 + 2y^2$

$\frac{6-z}{2} = x^2 + y^2$ ROT. PARABOLOID (1)



(1)

(b) $t=2$

$$\left.\begin{array}{l} x(t) = t \Rightarrow x'(t) = 1 \\ y(t) = t^2+1 \Rightarrow y'(t) = 2t \overset{t=2}{=} 4 \\ z(t) = -3t \Rightarrow z'(t) = -3 \end{array}\right\} \Rightarrow \text{vekt. smjera tangente u točki za } t=2: (1,4,-3)$$

Vektor normale tang. ravnine: $\left(\left(\frac{\partial z}{\partial x}\right)_0, \left(\frac{\partial z}{\partial y}\right)_0, -1\right) = (-4x_0, -4y_0, -1)$

Mora vrijediti: $(-4x_0, -4y_0, -1) = \lambda \cdot (1, 4, -3) \Rightarrow \boxed{\lambda = \frac{1}{3}}$

$$\left.\begin{array}{l} \frac{1}{3} = -4x_0 \Rightarrow x_0 = -\frac{1}{12} \\ \frac{4}{3} = -4y_0 \Rightarrow y_0 = -\frac{1}{3} \end{array}\right\} \Rightarrow \boxed{T_0\left(-\frac{1}{12}, -\frac{1}{3}, \frac{415}{72}\right)}.$$

$z_0 = 6 - 2\cdot\left(-\frac{1}{12}\right)^2 - 2\cdot\left(-\frac{1}{3}\right)^2 = 6 - \frac{2}{144} - \frac{2}{9} = \frac{415}{72}$

(7.) [5 bodova]

(a) $f(\vec{x}+\vec{h}) \approx f(\vec{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i$, za $\|\vec{h}\|$ mali.

(b) $f(x,y) = \sqrt{x^2+2y^3}$, $T_0(3,2)$, $h = (\Delta x, \Delta y) = (-0.05, 0.01)$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$\frac{\partial f}{\partial x} = \left.\frac{x}{\sqrt{x^2+2y^3}}\right|_{(3,2)} = \frac{3}{5}$, $\frac{\partial f}{\partial y} = \left.\frac{3y^2}{\sqrt{x^2+2y^3}}\right|_{(3,2)} = \frac{12}{5}$

$df(3,2) = \frac{3}{5}\cdot(-0.05) + \frac{12}{5}\cdot(0.01) = -0.006$ //

$A = \sqrt{(2.95)^2 + 2\cdot(2.01)^3} \approx f(3,2) + df(3,2) = 5 - 0.006 = \boxed{4.994}$

(8.) [5 bodova]

(a) $\frac{\partial u}{\partial \vec{s}}(1,2,1) = Du(1,2,1)\cdot \vec{s_0} = (0,-6,0)\cdot\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) =$

$= \frac{6}{\sqrt{3}} = \cancel{\boxed{2\sqrt{3}}} \quad = \frac{5}{\sqrt{3}}$

$\vec{s_0} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$

$Du = (2x - yz, -2y - x\cancel{z}, 2z - xy)$

$Du(1,2,1) = (0, -6, 0)$

(b) /str. 7, knjiž. 7/

Jedinični vektor u smjeru gradijenta

$\operatorname{grad} u \cdot \vec{a_0} \geq \operatorname{grad} u \cdot \vec{b_0} \geq -\operatorname{grad} u \cdot \vec{a_0} \iff |\cancel{\operatorname{grad} u}|\cdot \underbrace{|\vec{a_0}|}_{=1}\cdot\cos(\operatorname{grad} u, \vec{a_0}) \geq$

$\geq |\cancel{\operatorname{grad} u}|\cdot \underbrace{|\vec{b_0}|}_{=1}\cdot \cos(\operatorname{grad} u, \vec{b_0}) \geq -|\cancel{\operatorname{grad} u}|\cdot \underbrace{|\vec{a_0}|}_{=1}\cdot\cos(\operatorname{grad} u, \vec{a_0})$

$\iff \underbrace{\cos(\operatorname{grad} u, \vec{a_0})}_{=1} \geq \cos(\operatorname{grad} u, \vec{b_0}) \geq -\underbrace{\cos(\operatorname{grad} u, \vec{a_0})}_{=1} \Rightarrow \boxed{\vec{a} = \operatorname{grad} u}$

$\vec{a_0} = \vec{j}$

(c) /Lančano pravilo/

$\frac{d}{dt}\left(u(x(t), y(t), z(t))\right) = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} = (2x - yz)\cdot 6t - (2y + xz)\cdot 2 + \cancel{(2z - xy)}\cdot 0$

$= \text{uvrštavanje} = \boxed{38t^3 - 18t^2 - 8t}$