MATH 220 – Revision about Chapter 9

Section 9.1
Reading: Examples 1, 2, 5, 7, 4, Theorem 6 and, from your lectures notes, formula 5 of Theorem 5;
Exercises: p.501 #1, 15, 27, 31, 35, 39, 43, 45, 49, 53, 69, 87;
Learning outcome: Identify sequences, general terms, and indices. Find limits of sequences by replacing n with x, when this replacement gives an answer. Establish convergence of sequences by the Sandwich Theorem ~~and the Bounded Monotonic Sequences theorem.~~

Sequence: It is an infinite list of numbers in a certain rank, with possible repetitions:

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots\}.$$

The rank n is called the *index*.

Convergence & divergence: $\lim_{n\to\infty} a_n = L$ means that a_n gets arbitrarily close to L when n is large enough. If L exists and is finite, the sequence *converges* to L. Otherwise, the sequence *diverges.*

Techniques for computing $\lim_{n\to\infty} a_n$:

- Treat n as a real variable (it's valid each time we get an answer): #27, 29.
- In particular, you may use l'Hospital's rule: #49.
- Sandwich theorem: shrink $\{a_n\}$ between two sequences converging to L: #45.

Convergence and Divergence

Which of the sequences $\{a_n\}$ in Exercises 27–90 converge, and which diverge? Find the limit of each convergent sequence.

27. $a_n = 2 + (0.1)^n$

29. $a_n = \dfrac{1 - 2n}{1 + 2n}$ **49.** $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$ **45.** $a_n = \dfrac{\sin n}{n}$

Section 9.2

Reading: Examples 1, 2, 5, 7, 9, 10;
Exercises: p.511 #9, 13, 27, 31, 41, 69, 71;
Learning outcome: Identify series, general terms, and indices of summation. Study the convergence of geometric series ~~and telescopic series~~, and find their sums. Identify divergent series whose general term does not go to 0. Re-index series.

Series: Expression of the form $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$

Convergence: $\sum_{n=0}^{\infty} a_n = L$ means that

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = L,$$

in other words, the *sequence of partial sums* $\{\sum_{n=0}^{N} a_n\}_{N=0}^{\infty}$ converges to L.

Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$. It diverges otherwise. See #13.

Red signal: If a_n does not go to 0, then $\sum a_n$ diverges. See #27.

If a series converges, find its sum.

13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$ **27.** $\sum_{n=1}^{\infty} \frac{n}{n+10}$

~~**Section 9.3**~~

~~Reading: Examples 1, 2, 3, 4, 5;~~
~~Exercises: p.511 #7, 11-25, 55, 56;~~
~~Learning outcome: Establish the convergence or divergence of some series by comparison with an integral when the general term *decreases* to 0.~~

Section 9.4
Reading: Examples 1, 2, 3;
Exercises covered in class: p.523 #25, 26, 33 to 41;
Exercises: p.523 #17, 19, 21, 23, 43, 53;
Learning outcome: Establish the convergence or divergence of series by comparison (or limit-comparison) with well-known series (geometric, harmonic, p-series...)

<u>Well-known series</u>:

- *Geometric series*: $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$. It diverges otherwise.
- *Harmonic series*: $\sum_{n=1}^{\infty} 1/n$ diverges.
- *p-series:* $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$. It diverges otherwise.

<u>Limit-comparison</u>: If $a_n \approx b_n$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

- The idea is to remove negligible parts in a_n, so after deletion $\sum a_n$ becomes a well-known series $\sum b_n$, see #17, 21.
- We need a_n and b_n to be positive.
- We justify $a_n \approx b_n$ by proving that $\lim\limits_{n\to\infty} a_n/b_n$ exists and is neither 0 nor ∞.

<u>Comparison</u>: Sometimes to alter a_n affects the way $\{a_n\}$ evolves, but

- $0 < a_n < b_n$ and $\sum b_n < \infty$ imply $\sum a_n < \infty$, see #19.
- $0 < b_n < a_n$ and $\sum b_n = \infty$ imply $\sum a_n = \infty$, see #27.

Determining Convergence or Divergence

17. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$ **21.** $\sum_{n=1}^{\infty} \frac{2n}{3n - 1}$ **19.** $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ **27.** $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

Section 9.5
Reading: Examples 1, 2;
~~Supplemental reading (to discuss briefly next time): Theorem 14 (the root test);~~
Exercises: p.529 #1, 3, 5, 7, 23, 25, 37;
Learning outcomes: Identify absolutely convergent sequences and conclude in their convergence. Establish convergence or divergence of series by the ratio test.

<u>Ratio test</u>: For studying $\sum a_n$, where $a_n > 0$, compute $\lim_{n\to\infty} a_{n+1}/a_n = \rho$.

- If $\rho < 1$, then the series converges absolutely.
- If $\rho > 1$, then the series diverges.
- If $\rho = 1$ or does not exist, then use another strategy.

- Very useful when dealing with factorials, see #55.
- For doing cancellations, observe that $(n+1)! = (n+1)n!$, also $\big(2(n+1)\big)! = (2n+2)! = (2n+2)(2n+1)(2n)!$, etc.

Convergence or Divergence

55. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$

Section 9.6
Reading: Examples 1, 2, 4;
Exercises: p.535 #15, 17, 19, 23, 33, 35;
Learning outcomes: Identify alternating series whose general term (in absolute value) *decreases* to 0, and conclude in their convergence. Distinguish between absolute and conditional convergence.

Absolute convergence: If $\sum |a_n| < \infty$, then $\sum a_n$ converges (see #22). Indeed, it converges *absolutely* (we can reorder the terms arbitrarily without changing the result).

Conditional convergence: If $\sum |a_n| = \infty$, then $\sum a_n$ either diverges, or converges *conditionally* (we cannot reorder the terms arbitrarily).

Alternating series: Series of the form $\sum (-1)^n a_n$, where $a_n > 0$.

Alternating series test: If a_n decreases to 0 when $n \to \infty$, then the alternating series $\sum (-1)^n a_n$ converges.

- a_n may start to decrease after a certain n (see #8).
- In most cases, a_n is not oscillatory, so the condition $\lim\limits_{n\to\infty} a_n = 0$ suffices (#29).

Determining Convergence or Divergence

22. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$ **8.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ **29.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$

Beginning of Section 9.7
Reading: Example 1, 2, 3;
Supplemental reading (to discuss next time): Example 3(b, c, d);
Exercises: p.544 #9, 11, 15, 31, 41, 47;
Learning outcomes: Identify power series. Compute the sum of a power series when it is also geometric. Use the ratio test to determine the interval where a power series converges absolutely, and study the convergence at the endpoints.

<u>Reduction to a geometric series</u>: $\sum_{n=0}^{\infty}(f(x))^n = \frac{1}{1-f(x)}$ for $|f(x)| < 1$, see #3.

<u>Power series centered at a</u>: $\sum_{n=0}^{\infty} c_n(x-a)^n$

<u>Interval of convergence:</u>

- Use the ratio test on $\sum|c_n||x-a|^n$. It gives:
  - When $\lim_{n\to\infty}| c_{n+1}/c_n | |x-a| < 1$, the series converges absolutely.
  - When $\lim_{n\to\infty}| c_{n+1}/c_n | |x-a| > 1$, the series diverges.
- Conclude that the radius of convergence is $R = 1/\lim_{n\to\infty}|\frac{c_{n+1}}{c_n}|$.
- Hence, the series converges absolutely in $(a-R, a+R)$, while it diverges outside $[a-R, a+R]$.
- At the endpoint $x = a - R$, check if $\sum_{n=0}^{\infty} c_n(-R)^n$ converges.
- At the endpoint $x = a + R$, check if $\sum_{n=0}^{\infty} c_n(R)^n$ converges.
- Conclude.
- N.B.: It will come for sure in all exams. See #9.

Intervals of Convergence

(a) find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally?

3. $\sum_{n=0}^{\infty}(-1)^n(4x+1)^n$ **9.** $\sum_{n=1}^{\infty}\frac{x^n}{n\sqrt{n}\,3^n}$

End of Section 9.7
Reading: Examples 4, 5, 6;
Exercises: p.545 #53, 54;
Learning outcomes: Differentiate and integrate a power series. Find a formula for some power series, using well-known series and termwise differentiation/integration.

Differentiation: $(\sum c_n(x-a)^n)' = \sum(c_n(x-a)^n)' = \sum c_n\, n(x-a)^{n-1}$

Integration: $\int \sum c_n(x-a)^n \, \mathrm{d}x = \sum \int c_n(x-a)^n \mathrm{d}x = \sum(\frac{c_n}{n+1})(x-a)^{n+1} + C$

- The radius of convergence does not change.
- It helps when it's time to find a formula for a series, related by differentiation or integration to a well-known series.

Exercise: Find a formula for $\sum_{n=0}^{\infty} nx^n$, then for $\sum_{n=0}^{\infty} nx^{2n}$.

Exercise: Expand $\ln(1-x)$ in series about $x = 0$.

Section 9.8
Reading: Examples 1, 2, 3;
Exercises: p.550 #11, 13, 15, 27, 29;
Learning outcomes: Use Taylor's formula to expand a real-analytic function in a power series centered at a given point.

<u>Taylor's expansion of $f(x)$, centered at a</u>:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n$$

- The equality holds on the interval of convergence for the *usual* (real-analytic) functions.
- See #27 for practicing the formula.

<u>Well-known expansions (provided in the test)</u>:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{for } |x| < 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \qquad (\text{for } |x| < 1)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n}$$

- Keeping in mind that $f(x)$ may replace x on both sides of the equations, it helps to find a formula for many series.

Finding Taylor and Maclaurin Series

find the Taylor series generated by f at $x = a$.

27. $f(x) = 1/x^2, \quad a = 1$

Exercise: Find a formula for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

More about Taylor series
Reading: Section 9.10, Examples 3, 4
Problems solved in class: p.545 #50, 51, 52; p.551 #41, 43; p.564 #47, 49, 50;
Exercises: p.551 #42; p.564 #20 (keep a few terms and do not worry about the error);
Learning outcomes: Find the quadratic approximation of a function about a given point. ~~Express in series the solutions of non-elementary integrals.~~

<u>Quadratic approximation</u>: Keeping only the first 3 terms of Taylor's expansion gives

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

for $x \approx a$.

Exercise: Find the quadratic approximation of e^{x^2} for $x \approx 0$.