A Short Proof of König's Matching Theorem

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Definition: Matching

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Maximum Matching

A maximum matching M_G of G is a matching whose cardinality is largest. Denote $\nu(G) = |M_G|$.

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Minimum Vertex Cover

A minimum vertex cover W_G of G is a vertex cover whose cardinality is smallest. Denote $\tau(G) = |W_G|$.

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  - Question: Is it possible to select the same vertex? Impossible
- There may be $E(G \setminus W) \neq \emptyset$; therefore $|W| \leq |W_G|$, a minimum vertex cover of G.
- Finally, $\nu(G) = |M_G| = |W| \leq |W_G| = \tau(G)$

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Conclusion

We can only consider "connected bipartite graph" without loss of generality.

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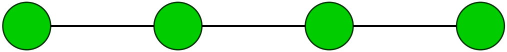
- If G is a path with n vertices ...
- When n is even, $\nu(G) = \frac{n}{2} = \tau(G)$



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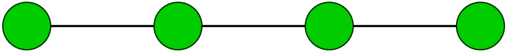
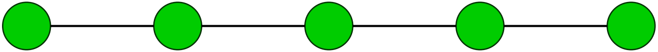


- When n is odd, $\nu(G) = \lfloor \frac{n}{2} \rfloor = \tau(G)$



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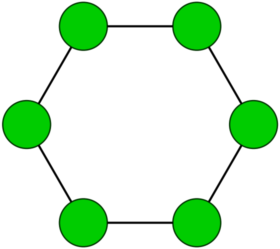

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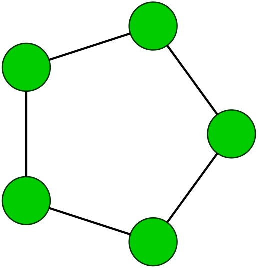

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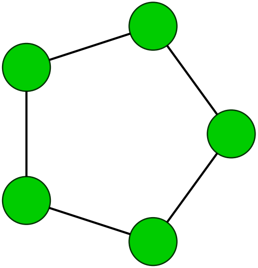

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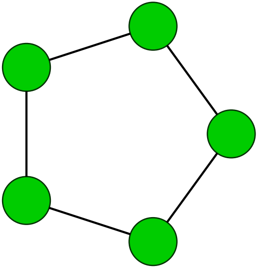

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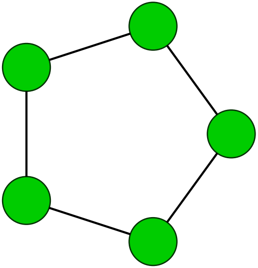

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- Did we find the new world?

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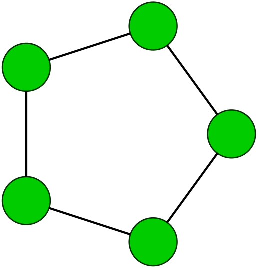

- $\nu(G) = 2$, but $\tau(G) = 3$, YOOOOOOOOOOOOOO!
- Did we find the new world? NO

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Property

If a graph G is bipartite graph iff G is bicolouring.

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Conclusion

Thus, G is neither a path nor a cycle, and then it is useful that G exist some vertices whose degree is larger than 3.

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Observations and Lemma

- Observation 1 (done): Given a bipartite graph $G = (V, E)$, if $\nu(G) < \tau(G)$, then there exists a component $G^{'}$ in G where $\nu(G^{'}) < \tau(G^{'})$.

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- Observation 2 (done): Given a connected bipartite graph $G = (V, E)$, if $\nu(G) < \tau(G)$, then G is not a path nor a cycle.
- Lemma (todo): Given a connected bipartite graph G, G is neither a path nor a cycle, then $\nu(G) = \tau(G)$.

Idea of Proof (cont.)

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- By the Fact, we know that $\nu(G) \leq \tau(G)$ for all graph G
- By Observation 1 & 2 and Lemma, we prove the theorem.

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Technique

Suppose a minimal counterexample G, i.e. for all subgraph H of G hold $\nu(H) = \tau(H)$ except G which holds $\nu(G) < \tau(G)$.

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  1. $\nu(H) < \nu(G) \Rightarrow e \in M_G$ (Haha I'm smart)
  2. $e \notin M_G$

Lemma cont.

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