Simple Drawings of K_n from Rotation Systems

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A complete rotation system on n vertices is a collection of n cyclic permutations of the elements $[n] \backslash \{i\}$, for $i \in [n]$. If D is a drawing of a labelled graph, then a rotation at vertex v is the cyclic ordering of the edges at v. In particular, the collection of all vertex rotations of a simple drawing of K_n is a complete rotation system. Can we characterize when a complete rotation system can be represented as a simple drawing of K_n (a.k.a. realizable)?

This thesis is motivated by two specific results on complete rotation systems. The first motivating theorem was published by Kynčl in 2011, who, using homotopy, proved as a corollary that if all complete 6-vertex rotation systems of a complete n-vertex rotation system H are realizable, then H is realizable. Combined with communications with Aichholzer, Kynčl determined that complete realizable n-vertex rotation systems are characterized by their complete 5-vertex rotation systems. The second motivating theorem was published by Gioan in 2005, he proved that if two simple drawings of the complete graph D and D' have the same rotation system, then there is a sequence of Reidemeister III moves that transforms D into D'.

Motivated by these results, we prove both facts combinatorially by sequentially drawing the edge crossings of an edge to form a simple drawing. Such a method can be used to prove both theorems, generate every simple drawing of a complete rotation system, or find a non-realizable complete 5-vertex rotation system in any complete rotation system (when one exists).

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I would like to extend my deepest appreciation to Bruce Richter for spending an uncountable number of hours helping make this thesis what it is today. From placing the initial concept in my head in Osnabrück Germany, to the immeasurable amount of time spent discussing concepts, proofs and theorems in research meetings, to edits and suggestions on the writing drafts. This thesis would not exist without his guidance.

I also wish to thank my partner Jessica Turecek for helping me through the day to day trials and tribulations of thesis writing during a pandemic. I could not have foreseen the amount of help I would require to start and finish the thesis writing at home without an office.

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Table of Contents

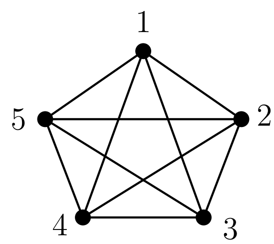
List of Figures

List of Symbols

H_n	5	complete n-vertex rotation system
$\pi(v)$	5	rotation at a vertex i
$V(H_n)$	5	vertex set of H_n
$H - S$	5	Rotation system H without the vertex set S
$\pi_S(v)$	5	$\pi(v)$ restricted to S
$\gamma_{e_1,\ldots,e_k}$	10	unique simple closed curve on $e_1, \ldots, e_k$
$\overrightarrow{(i,j,k)}$	10	an oriented cycle
$\overrightarrow{(i,j,k)}_L$	10	left side of an oriented cycle
$\overrightarrow{(i,j,k)}_R$	10	right side of an oriented cycle
$\triangle_{\{e,f,g\}}$	10	the side of $\gamma_{e,f,g}$ that contains no vertices
$\rho_{\{e,f,g\}}(D)$	11	a Reidemeister III move
$\overrightarrow{(u,v)}$	13	a directed edge
$\prec_e^D$	13	order e crosses edges in D
$G[V]$	13	the induced graph on the vertex set V
$G[E]$	13	the induced graph on the edge set E
$D[S]$	13	the induced drawing on D from the set of edges and vertices S
$\mathcal{B}(R)$	14	boundary of region R
$<_{\wedge}^{e}$	29	edge ordering of two adjacent edges on e
$<_{\parallel}^{e}$	29	edge ordering of two non-crossing and non-adjacent edges on e
$<_{\triangle}^{e}$	29	edge ordering of two crossing edges on e certified by a vertex
$<_{K_6}^{e}$	30	edge ordering of two non-adjacent edges in their K_6 with e
$\mathcal{A}_\ell$	81	the length of a chain or cycle of relations
$\mathcal{A}_\parallel$	81	the number of $<_{\parallel}^{e}$ relations in a chain or cycle of relations
$\mathcal{A}_w$	81	the sum of $\mathcal{A}_\ell$ and $\mathcal{A}_\parallel$

1 Introduction

A *complete rotation system* H on n vertices is a set of n cyclic permutations $\pi(i)$ such that for each i, $\pi(i)$ is a cyclic permutation of $[n]\backslash\{i\}$. Such a structure arises naturally in the area of graph drawings as a drawing of a labelled complete graph in the plane naturally has a clockwise (or counter clockwise) rotation around each vertex inducing an associated complete rotation system for that graph as seen in Figure 1.



1 : (2345)
2 : (3451)
3 : (4512)
4 : (5123)
5 : (1234)

Figure 1: A labelled K_5 and its associated clockwise rotation system.

A drawing D of a graph is *simple* if all pairs of edges intersect in at most one point in D and edges are not self crossing in D.

The Harary-Hill conjecture states that the minimum number of edge crossings in any drawing of the complete graph in the sphere $cr(K_n)$ is equal to

$$H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

This conjecture has been open for over 50 years and has been verified for $n \leq 12$, the most recent results appearing in [**10**]. There exist simple drawings of K_n achieving $H(n)$ crossings (see [**12**]) and so it is left to prove that $H(n) \leq cr(K_n)$. Dan Archdeacon's combinatorial generalization of the Harary-Hill conjecture to rotation systems in [**4**] states that in any n-vertex complete rotation system, the number of induced non-planar K_4's is at least $H(n)$. This generalizes the Harary-Hill conjecture to complete rotation systems and turns a geometric problem into a purely combinatorial problem. Archdeacon wrote a hill-climbing program that found for small values of n the conjecture is true, however the conjecture remains unresolved.

The complete graph is one of the most well studied classes of graphs for simple drawings. In [**13**], Kynčl finds the existence of simple drawings

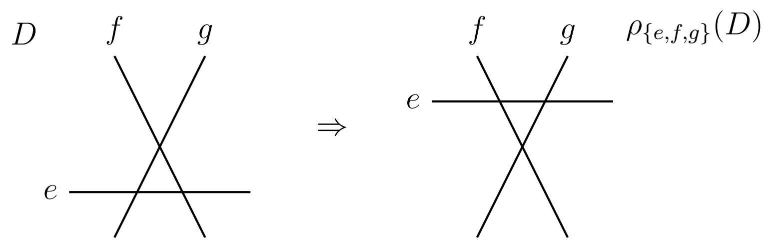


Figure 2: A Reidemeister III move over the edges e, f, g.

of K_n with edges having few crossings. In [**14**], Pach and Tóth estimate the number of drawings of K_n under various restrictions. In [**15**], Pach, Solymosi and Tóth show a Ramsey type theorem on simple drawings of K_n, in particular, for every integer r, there exists an integer n such that every simple drawings of K_n contains one of two simple drawings of K_r. The same analysis on complete rotation systems gives the same result for the two associated rotation systems of the two simple drawings of K_r. In [**16**], Pach, Rubin and Tardos use geometric arguments to show that any simple drawing of K_n with straight line segments has a set of at least $n^{1-o(1)}$ edges that are each pairwise crossing.

If H is a complete n-vertex rotation system, then there is a drawing D_H of a graph K_n in the sphere having H as its rotation system by simply drawing the vertices in the sphere with their associated rotations and connecting the edges. If we restrict the drawings to be *simple* (i.e. drawings with no loops, edges intersecting in at most one point (such a point being a crossing or a vertex), and no three edges crossings at the same point), then it is not as obvious that complete rotation systems have an associated simple drawing. A rotation system is *realizable* if it has an associated simple drawing. An (n, k) *complete rotation system* H is a complete rotation system with n vertices such that every complete rotation system of size k inside H is realizable.

Given a triple of pairwise crossings edges (e, f, g) in a drawing D such that there is a face bounded by exactly e, f and g, then informally a *Reidemeister III move* $\rho_{\{e,f,g\}}$ over edges (e, f, g) is the operation of moving one of the edges over the crossing of the other two without having it pass any other edges of vertices in D as seen in Figure 2.

The main motivation of this thesis is the following result.

Theorem 1.1. *If $n \geq 6$ and H_n is a complete $(n, 5)$-rotation system, then*

H_n is realizable.

In [**1**], Kynčl proves Theorem 1.1 holds for $(n, 6)$-rotation systems by proving this on complete abstract topological graphs. A *complete abstract topological graph* is a tuple (G, X) such that G is a graph and X is a set of pairs of edges from G. A complete abstract topological graph (G, X) is said to be *realizable* if there is a simple drawings of G in which exactly the pairs in X cross. Since every complete rotation system is a complete abstract topological graph, complete abstract topological graphs are more general than complete rotation systems.

Comparatively, Kynčl's methods use homotopy and orderings of edge crossings on a fixed star, whereas ours will use combinatorial arguments and orderings of edge crossings on a fixed edge. Through private communications with Aichholzer (see [**2**],[**8**]), it is known computationally that complete $(6, 5)$-rotation systems are realizable, however such a result has not been published. Combining these two results implies Theorem 1.1.

We fill a hole in literature by giving a formal proof that $(6, 5)$-rotation systems are realizable. Seeing as Kynčl chose to prove Theorem 1.1, we give a different proof using a combinatorial approach. Such a proof gives rise to Theorem 1.2.

Theorem 1.2. *Let $n \geq 4$ and D be a simple drawing of K_n. If c is an edge of K_n and P is a point of D in some face, then either:*

1. *There is a sequence (possibly empty) of Reidemeister III moves on D to a simple drawing D' such that a non-trivial segment of c is on the boundary of the face of D' containing P; or*

2. *There is some drawing $\mathcal{D}$ in D on a K_4 containing c such that no face of $\mathcal{D}$ contains P and has a non-trivial segment of c on its boundary.*

Theorem 1.2 is independent of rotation systems and is a easily stated fact of how edges and faces in simple drawings of K_n are related. Such a fact provides a new structural theorem to a well studied area of graph drawings.

Theorem 1.3. *Let n be a positive integer, H_n be a realizable complete n-vertex rotation system. If D and D' are two simple drawings realizing H_n, then one can be obtained from the other through a series of Reidemeister III moves.*

Gioan presents a sketch of the proof for Theorem 1.3 in [**3**], but a published version of his work has yet to appear. However, a full version of the proof of Theorem 1.3 appears in [**4**]. We present a proof of the same result from the perspective of drawing the crossings sequentially (similar to our proof of Theorem 1.1). All proofs of Theorem 1.3 follow the inductive arguments found in [**3**] and use some of the ideas that are in [**3**].

Our arguments brings an essential simplification to the proof of Theorem 1.3 by noticing that if D is a simple drawing of K_n and L is a subdrawing of D containing a partial edge e_i, then the set of edges that e_i can cross in L that extend that drawing to a simple drawing of K_n with the same associated rotation system as D appear consecutively on the boundary of the face containing e_i in L.

The main new contributions of this thesis are:

- A combinatorial proof of Theorem 1.1, improving Kynčl's results from a result on $(n, 6)$-rotation systems to a result on $(n, 5)$-rotation systems;
- Theorem 1.2, a new structural theorem relating edges and faces in simple drawings of K_n; and
- A simplified perspective on the proof of Theorem 1.3.

We end this section with a description of the thesis. In Section 2, we describe the preliminary work on complete rotation systems to introduce the reader to the literature and basic concepts. Section 3 characterizes how edges and faces interact in simple drawings of K_n. Section 4 is used to prove Theorem 1.1 for $n = 6$, an interesting result in itself as it extends Kynčl's results. Section 5 is used to find orderings of edge crossings on a fixed edge. This section is broken into three smaller sections for the cases $n = 7, n = 8$ and $n \geq 9$.

Sections 6 completes the proof of Theorem 1.1 by using the ordered edge crossings in Section 5 along with the edges and faces theorem in Section 3 to algorithmically draw simple drawings of a realizable complete rotation system. Section 7 offers a closed proof of Gioan's Theorem (Theorem 1.3) originally stated in [**3**].

Appendix A offers an algorithm that takes a complete n-vertex rotation system H for $n \geq 5$ as input, and outputs either a non-realizable complete 5-vertex rotation system in H or a simple drawing D realizing H. Furthermore, given specific choices, the algorithm will output any of the simple drawings realizing H.

2 Groundwork

This section will be dedicated to fundamental definitions and observations made on complete rotation systems and drawings of graphs. Many of the observations found in this section can be found in existing literature like [**1**], [**2**], [**3**], [**5**] and [**7**]. Let us start by defining edges and edge segment topologically in drawings of graphs. For the purposes of this thesis, an *edge* is a homeomorph of the closed compact interval $[0, 1]$ and a *non-trivial segment* of an edge is a closed connected component of an edge that is not a point.

Definition 2.1. A *complete n-vertex rotation system H_n* is a collection of n cyclic permutations such that for each $i \in [n]$, there exists a unique permutation $\pi(i)$ in H_n on the elements of $[n]\backslash\{i\}$. Define the vertices of H_n to be $V(H_n) = [n]$ and H_n^{-1} to be the rotation system H_n with every cyclic permutation reversed.

For the purposes of this thesis, every rotation system will be considered to be a complete rotation system on some set of vertices, we remove complete for simplicity.

Notation 2.2. Let H be a rotation system on vertices $V(H)$. For $S \subseteq V(H)$, let $H - S$ be the rotation system contained in H induced by the vertices $V(H)\backslash S$.

Notation 2.3. Let H_n be a complete n-vertex rotation system. For all $v \in V(H_n)$ and for any subset S of $V(H_n)$ not containing v, let $\pi_S(v)$ be the restriction of $\pi(v)$ to S.

Definition 2.4. A complete n-vertex rotation system H_n is *realizable* if there exists a simple drawing D on a labelled K_n such that the associated rotation system on D is H_n.

Definition 2.5. For positive integers n and k such that $n \geq k$, an *(n, k)-rotation system H* is an n-vertex rotation system such that every rotation system that is a restriction of H to a set of k vertices is realizable.

Observation 2.6 ([**7**], Sec.2)**.** *If H is a 4-vertex realizable rotation system, then the rotation of 3 vertices determines the rotation at the fourth.*

Ábrego et al. use this observation in [**7**] to computationally generate $(n, 4)$-rotation systems. We will use this observation in the end of the proof of

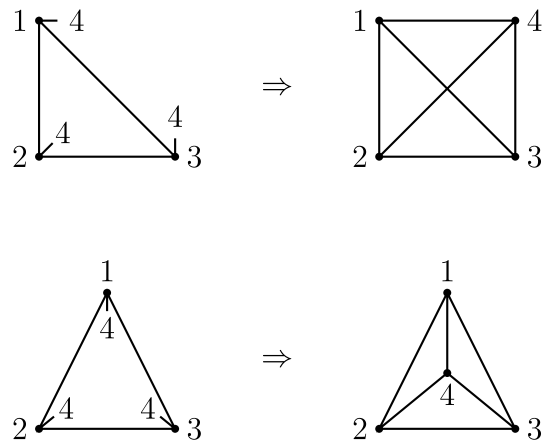


Figure 3: 3-cycles with small edge segments representing realizable rotation systems.

Theorem 1.1 to verify in a generated drawing of the complete graph that the rotation at a specific vertex coincides with the rotation system that produced the drawing.

Observation 2.7. *If H is a realizable 4-vertex rotation system, then there is a unique labelled drawing D that is a realization of H. In particular, H determines the oriented crossings of D.*

There are $2^4 = 16$ 4-vertex rotation systems. By Observation 2.6, half of these rotation systems are not realizable. Let C be a 3-cycle having three of the four vertices of H on it. Draw small segments at each vertex on C to represent the fourth vertex in the rotation of each of the vertices in H. If the small segments to the 4th vertex v all start on one side of C, then connect them all at a point on that side of C and call that point the 4th vertex as seen in Figure 3. This results in two rotation systems (depending on the side of C the edge segments are in) each of which have unique labelled planar representations.

Alternatively, one small segment starts on the opposite side of the other two small segments. There are 6 ways to choose the side of C that contains the single small segment and the starting vertex for the small segment. For each choice, have the single small segment cross C on the edge that is not incident to its starting vertex, then connect all the segments at a point on

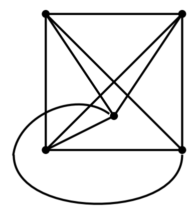


Figure 4: The Harborth drawing of K_5.

that side of C. Each choice produces a unique oriented crossing and a unique rotation system. Portions of this observation have appeared in various forms; for example see [**5**] Lemma 10.

Let us now prove a short lemma of a similar nature.

Lemma 2.8. *Let $n \leq 5$ be a positive integer. If H_n is a realizable rotation system, then there is a unique labelled simple drawing D that realizes H_n.*

Proof. For $n \leq 3$ this is trivial as there is only one rotation system H_n that corresponds to the unique simple drawing of K_n. If $n = 4$, then Observation 2.7 is our desired result. Therefore, assume without loss of generality $n = 5$. Let D be a simple drawing that is a realization of H, $V(H) = [5]$, and D_i be the simple drawing $D - i$. By Observation 2.7, D_5 is uniquely determined. Every 3-cycle in D_j not containing 5 has 5 on a specific side of the cycle, for $j \in [4]$. Observe for each of two simple drawings of K_4, each face is uniquely determined by the intersection of sides of triangles of that K_4.

It follows that the intersecting sides of the 3-cycles containing 5 is a unique face in D_5. Therefore, 5 is contained in a unique face in D_5.

Again observe over all possible simple drawings of D_5, the edge $(u, 5)$ in D is uniquely determined by the location of 5 and the starting of $(u, 5)$ at the rotation of u. This implies that there is a unique labelled simple drawing D that realizes H_n. $\square$

For any induction on $(n, 5)$-rotation systems, this lemma will be very useful for the base case as it allows us to ignore the rotation system itself, and talk about its associated simple drawing.

A common simple drawing of K_5 that appears frequently is the *Harborth* drawing of K_5 (also known as the twisted graph, see [**5**]). Such a drawing can be seen in Figure 4.

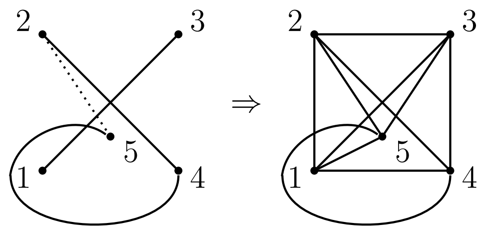


Figure 5: Harborth subdrawings determining vertex rotations.

Observation 2.9. *The Harborth drawing of K_5 is the unique simple drawing of K_5 that has a 3-cycle crossed three times by a single edge. Furthermore, the labelled subdrawing of a 3-cycle crossed three times by a single edge uniquely determines the labelled Harborth drawing.*

To prove this observation, draw a labelled 3-cycle crossed three times by a single labelled edge. Each edge not in the drawing can be added in a unique way to form a simple drawing of K_5. This can be explicitly seen along with the unique rotations in Figure 5 by including the dotted line $(2, 5)$ in the leftmost drawing.

Observation 2.10. *The Harborth drawing of K_5 is the unique simple drawing of K_5 that has an edge e crossed by two edges f and g sharing an endpoint such that f and g cross e from opposite sides when starting at their common endpoint. Furthermore, the labelled subdrawing of e, f and g uniquely determines the labelled Harborth drawing.*

Again, this observations follows from drawing the labelled edges e, f and g, then extending it to a simple drawing of K_5. This can be explicitly seen along with the unique rotations in Figure 5 by excluding the dotted line in the leftmost drawing.

Definition 2.11. Two $(n, 4)$-rotation systems H_1 and H_2 are *weakly isomorphic* if they have the same set of pairwise edge crossings.

Originally, this definition is defined on realizable rotation systems in [**2**, **5**] and is used in [**7**], however, it can be generalized. In particular, most literature apply it in the context of the following proposition:

Proposition 2.12. *Two $(n, 4)$-rotation systems H_1 and H_2 are weakly isomorphic if and only if $H_1 = H_2$ or $H_1 = H_2^{-1}$.*

Proof. An important fact to be applied in this proof is that adjacent transpositions on a totally ordered set generate the symmetric group on that set of elements. In particular, the bubble sort method (a method which takes an element A and compares it too another element B by taking two adjacent entries of A and swaps them if they are in the incorrect order compared to B) shows that every transposition applied is applied at most once.

As a corollary, for two rotation systems A and B on the same point set P, for any $x \in P$ with the rotation of x in A being $\pi^A(x)$ and the rotation of x in B being $\pi^B(x)$, there exists a set of adjacent transpositions from $\pi^A(x)$ to $\pi^B(x)$ such that no transposition is applied twice. We will prove Proposition 2.12 for 5-vertex rotation systems first.

Let A and B be 5-vertex two rotation systems on the same vertex set P (without loss of generality $P = [5]$), with the same crossings. Without loss of generality, assume $B \neq A$. We will show $B = A^{-1}$. Let A_i and B_i be the rotation systems $A - i$ and $B - i$, respectively, for $i \in P$. Since $B \neq A$, without loss of generality $B_1 \neq A_1$. Let $\{e_j\}_{j=1}^k$ be a sequence of adjacent transpositions that sends A to B.

Since $B_1 \neq A_1$ and they have the same crossings, it follows that $B_1 = A_1^{-1}$. Since $B_1 = A_1^{-1}$, four transpositions were applied to A_1 to obtain B_1. Each of these transpositions uniquely apply to another A_i. It follows that every A_i has had at least one transposition applied to it. Since every B_i has the same crossing as A_i, it follows that $B_i = A_i$ or $B_i = A_i^{-1}$. Since every A_i has had at least one transposition applied to it to obtain B_i, it follows that for all $i \in P$, $B_i = A_i^{-1}$, and $B = A^{-1}$.

Comparing the 5-vertex rotation systems H_5 and H_5' of two weakly isomorphic $(n, 4)$-rotation system H and H' on the same vertex set, we have shown that $H_5 = H_5'$ or $H_5 = H_5'^{-1}$. From here, we follow the arguments of Kynčl for Proposition 6 in [**9**]

Let H be an $(n, 4)$ rotation systems and H' be any rotation system weakly isomorphic to H. From our arguments, it is clear that 5-vertex rotation systems of H and H' on common vertex sets are the same or inverses. Following Lemma in Proposition 6 of [**9**], if B' and C' are two 5-vertex rotation systems in H' with exactly 4 common vertices, then B' uniquely determines C'. The proof of this fact is the proof of Lemma in Proposition 6 of [**9**].

As the proof of Proposition 6 in [**9**] states repeated use of this fact results in every 5-vertex rotation system of H' being the same as H or inverse. Since the rotation at a vertex is uniquely determined by its 3-element subsets, it follows that $H' = H$ or $H' = H^{-1}$ □

Kynčl had already considered Proposition 2.12 on realizable rotation system in [**9**] and it was used in various literature (see [**2**, **5**, **9**]). We modify Kynčl's arguments and proposition to extend to $(n, 4)$-rotation systems. As mentioned before, such rotation systems are interesting as counting crossings is still viable, and Archdeacon has suggested in [**4**] that the crossing number of $(n, 4)$-rotation systems is the same as the crossing number of K_n. Theorem 1.1 reduces this question to comparing the crossing number of $(n, 4)$-rotation systems to the crossing number of $(n, 5)$-rotation systems.

Notation 2.13. Let D be a drawing of a graph G, and $E = \{e_1, \dots, e_k\}$ be a set of edges. If there is a unique simple closed curve defined on the edges of E in D, then we define $\gamma_{e_1,\dots,e_k}$ to be that curve.

Notation 2.14. Let $n \geq 3$ and D be a simple drawing of K_n. For distinct vertices $i, j, k \in V(K_n)$, let $\overrightarrow{(i, j, k)}$ be the directed 3-cycle whose labels appear in clockwise order i, then j, then k. The *left side (right side)* $\overrightarrow{(i, j, k)}_L$ ($\overrightarrow{(i, j, k)}_R$) is the region that is on the left side (right side) of $\overrightarrow{(i, j, k)}$.

Notation 2.15. Let D be a simple drawing of a graph G and e, f and g be three edges in G that pairwise cross in D. If $\gamma_{e,f,g}$ has a side that contains no vertices of $V(\{e, f, g\})$, then this side is $\triangle_{\{e,f,g\}}$.

For three edges e, f, g, if $\triangle_{\{e,f,g\}}$ exists and is a face, then there are two drawings D and $\bar{D}$ of $\triangle_{\{e,f,g\}}$ with the same oriented crossings. In both drawings, performing a boundary walk just outside of $\triangle_{\{e,f,g\}}$, we find a closed disc S ($\bar{S}$) that contains $\gamma_{e,f,g}$ on its interior, contains only non-trivial segments of e, f, g, and does not contain any vertices. Both S and $\bar{S}$ are homotopically equivalent.

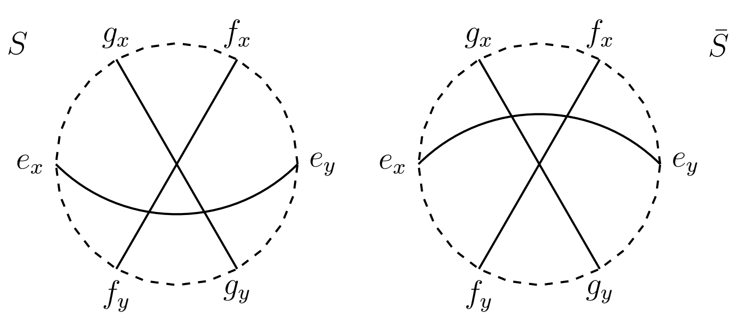


Figure 6: A Reidemeister III move.

Definition 2.16. Let $n \geq 6$, D be a simple drawing of K_n and e, f, g be three pairwise crossings edges. Suppose $\triangle_{\{e,f,g\}}$ exists and is a face. Let S be a closed disc containing $\triangle_{\{e,f,g\}}$ in its interior, contains no vertices, and only contains non-trivial segments of e, f, g. Each of e, f, g intersect the boundary of S at two points $e_x, e_y, f_x, f_y, g_x, g_y$ respectively, such that from e_x to e_y, e crosses f than g.

Let $\bar{S}$ be a closed disc whose boundary is the boundary of S that contains three edge segments $\bar{e}, \bar{f}, \bar{g}$ that pairwise cross once such that the ends of $\bar{e}, \bar{f}, \bar{g}$ are $e_x, e_y, f_x, f_y, g_x, g_y$, respectively, and from e_x to e_y, $\bar{e}$ crosses g than f.

A *Reidemeister III move* $\rho_{\{e,f,g\}}(D)$ is the drawing $(D[K_n]\backslash S) \cup \bar{S}$.

Given a simple drawing D of K_n, and the existence of $\triangle_{\{e,f,g\}}$, we note the following simple facts about Reidemeister III moves:

- When performing the Reidemeister III move $\rho_{\{e,f,g\}}$ on D, we can choose $\bar{S}$ in such a way that exactly one edge changes when compared to S and we can choose which edge changes.
- If $\triangle_{\{e,f,g\}}$ is a face in D, then, $\rho_{\{e,f,g\}}(\rho_{\{e,f,g\}}(D)) = D$;
- For three edges a, b, c in K_n, if $\triangle_{\{a,b,c\}}$ exists in D, then $\triangle_{\{a,b,c\}}$ exists in $\rho_{\{e,f,g\}}(D)$; and
- For three edges a, b, c in K_n, if $\triangle_{\{a,b,c\}}$ exists in D, is not a face in D and is a face in $\rho_{\{e,f,g\}}(D)$, then $\rho_{\{a,b,c\}}(\rho_{\{e,f,g\}}(D))$ is a valid simple drawing.

The label Reidemeister III move originates from Knot Theory. In literature this is also known as a triangle mutation (see [**3**]) or triangle flip (see [**4**]). Since such a move applied to a simple drawing of a graph does not change the rotation at any vertex, it does not change the associated rotation system of a simple drawing. This motivates Theorem 1.3 and plays a crucial role in Theorem 3.8 and implicitly Theorem 1.1.

We show that if $\triangle_{\{e,f,g\}}$ contains no vertices in a simple drawing D of K_n, then $\triangle_{\{e,f,g\}}$ can be emptied of edge segments by applying Reidemeister III moves.

Lemma 2.17. *Let D be a simple drawing of the complete graph, x, y, z be three edges such that $\triangle_{x,y,z}$ exists and contains no vertices, and $\bar{x}, \bar{y}$ and $\bar{z}$*

are the segments of x, y, z, respectively, on the boundary of $\triangle_{x,y,z}$. If there are no edges crossing both $\bar{y}$ and $\bar{z}$, then there exists $\triangle_{x,y_1,z_1}$ contained in $\triangle_{x,y,z}$ that is a face.

Proof. Define $\triangle_{x,y_1,z_1}$ to be a triangle contained inside $\triangle_{x,y,z}$ such that one side of $\triangle_{x,y_1,z_1}$ is $\bar{x_1}$ contained in x, every edge crossing $\triangle_{x,y_1,z_1}$ crosses $\bar{x_1}$, and the number of crossings in $\triangle_{x,y_1,z_1}$ is minimal (including crossings on the boundary). $\triangle_{x,y_1,z_1}$ exists as $\triangle_{x,y,z}$ satisfies the definition containing some finite number of crossings. $\triangle_{x,y_1,z_1}$ is the desired triangle unless it is non-empty.

Assume by way of contradiction that $\triangle_{x,y_1,z_1}$ is not empty and let $\bar{y_1}$ and $\bar{z_1}$ be the segments of y_1 and z_1, respectfully, bounding $\triangle_{x,y_1,z_1}$. Without loss of generality, let $\bar{y_1}$ be a side of $\triangle_{x,y_1,z_1}$ that has a crossing apart from its endpoints.

Let z_2 be the edge that crosses both $\bar{x_1}$ and $\bar{y_1}$ (other than z_1) that is furthest away from the (x, y_1) crossing on $\bar{x_1}$, $\bar{x_2}$ be the segment from the (x, y_1) crossing to z_2 on $\bar{x_1}$, and $\tilde{y_1}$ be the segment from the (x, y_1) crossing to z_2 on $\bar{y_1}$.

By definition of z_2, every edge that crosses $\bar{y_1}$ other than z_1 does so on $\bar{x_2}$. Since $\triangle_{x,y_1,z_2}$ is contained in $\triangle_{x,y,z}$ and $\triangle_{x,y,z}$ contains no vertices, it follows that $\triangle_{x,y_1,z_2}$ contains no vertices. In particular, every edge that crosses $\triangle_{x,y_1,z_2}$ at z_2 also crosses $\bar{x_2}$.

Therefore, every edge that crosses $\triangle_{x,y_1,z_2}$, crosses $\bar{x_2}$. In particular, $\triangle_{x,y_1,z_2}$ contains less crossings than $\triangle_{x,y_1,z_1}$, a contradiction. □

Corollary 2.18. *Let D be a simple drawing of the complete graph. Suppose $\triangle_{x,y,z}$ exists with the boundary consisting of segments x_1, y_1 and z_1 of the edges x, y and z, respectively. If there are no edges crossing both y_1 and z_1, then there exists a series of Reidemeister III moves $\{\rho_{X_i}\}_{i=1}^{k}$ on D with $\rho_{X_i}(D_{i-1}) = D_i$ and $D = D_0$ such that*

- *For all $i \in [k]$, $x \in X_i$;*
- *$x, y, z \in X_k$; and*
- *$\triangle_{X_i} \subset \triangle_{X_k}$ in D_{i-1}.*

This corollary follows by repeatedly applying Reidemeister III moves to the triangles found in Lemma 2.17 until the final Reidemeister III move made is over the triangle on edges x, y and z. We end this section with three final definitions for notational purposes.

Notation 2.19. Let G be a graph. For $V \subseteq V(G)$ and $E \subseteq E(G)$, let $G[V]$ denote the subgraph induced by the vertex set V in G and $G[E]$ the subgraph containing E and its endpoints in G.

Notation 2.20. Let D be a drawing of a graph G, $\bar{V} \subseteq V(G)$, and $\bar{E}$ be a set of edges of G. Let $D[\bar{V} + \bar{E}]$ be the subdrawing of $G[\bar{V}] \cup G[\bar{E}]$ in D.

Notation 2.21. Let G be a graph and (u, v) an edge in $E(G)$. Then $\overrightarrow{(u, v)}$ is the directed edge from u to v.

Notation 2.22. Let D be a simple drawing of a graph G and e, f and g be three edges such that e is a directed edge and both f and g cross e. Define $f \prec_D^e g$ if e crosses f then g in D.

3 Characterizing Edges and Faces

The purpose of this section is to prove Theorem 3.8 which is a more technical result of Theorem 1.2 that relates faces to edges in simple drawings of K_n. In particular, Theorem 3.8 shows for any simple drawing D of K_n, point P not in the K_n and any edge c, either there exists a K_4 drawn in D that separates c from P or there is a series of Reidemeister III moves on D such that the resulting drawing has c on the boundary of the face containing P.

This is crucial to the algorithmic proof of Theorem 1.1 as sequentially drawing an edge e by its crossing segments requires c the next edge crossed to be on the boundary of the appropriate face. If c is on the appropriate face, then we continue algorithmically drawing. If c is not on the appropriate face, then Theorem 3.8 finds a K_4 that separates the edge from the current region which can be used to relate to an associated small non-realizable rotation system or finds a set of Reidemeister III moves that brings the edge we want to the boundary of our desired face.

We start this section by relating faces and sides of 3-cycles. Following this, we will describe how edges intersect boundaries of faces in simple drawings of K_n. Finally we state and prove Theorem 3.8.

Notation 3.1. Let D be a drawing of a graph G and let R be a face in D. Define $\mathcal{B}(R)$ to be the boundary of R.

The following Lemma is a portion of Lemma 4.7 from [**11**] and is known as Carathéodory's Theorem for simple complete topological graphs (for simple drawings of K_n).

Lemma 3.2. *Let D be a simple drawing of K_n and let x be a point in the interior of a bounded face of D. Then there is a 3-cycle (u, v, w) in D containing x in its bounded side.*

An immediate consequence of Lemma 3.2 is Corollary 3.3 by considering two faces F_1 and F_2 one of which is the unbounded face by choice, and the other being bounded. There exists a 3-cycle separating F_1 and F_2. For each pairing (F_i, F_j) we do this comparison and consider the intersections of sides of these 3-cycles.

Corollary 3.3. *Let D be a simple drawing of K_n. If R is a face in D, then R is the unique open intersection of specific sides of all 3-cycles in K_n.*

Proof. Let D be a simple drawing of K_n and let $\mathcal{F} = \{F_1, \dots, F_k\}$ be the set of faces in D. Let F_i and F_j be two different faces in D. Without loss of generality, assume F_i is the unbounded face in D. Let x be a point in F_j. By Lemma 3.2 there is a 3-cycle $C_{i,j}$ in K_n such that F_i and x are on opposite sides. In particular, F_i and F_j are on opposite sides of $C_{i,j}$.

Let $\mathcal{C}$ be the set of such 3-cycles, ones for each pair (i, j). It follows that each face $\mathcal{F}$ is uniquely determined by the intersections of sides of 3-cycles in $\mathcal{C}$. Since $\mathcal{C}$ is a subset of the set of all 3-cycles in K_n, it follows that each face in $\mathcal{F}$ is uniquely determined by the intersection of sides of 3-cycles in K_n. □

As an important note to this corollary, the intersections of sides of 3-cycles in a simple drawing of K_n does not always determine a face. This corollary will be very useful in helping characterize the relation between edges and faces in any simple drawing of K_n.

Lemma 3.4. *Let $n \geq 4$, D be a simple drawing of $K_n + e_i$ where e_i is a partial edge starting at $u \in V(G)$ and has i crossings. If R_{e_i} is a face in $D - e_i$, then e_i has exactly one non-trivial segment in R_{e_i}.*

Proof. Let $n \geq 4$, D be a simple drawing of $K_n + e_i$ where e_i is a partial edge starting at $u \in V(G)$ and has i crossings, and R_{e_i} be a face in $D - e_i$. By way of contradiction, assume e_i has at least two non-trivial segments in R_{e_i}. It follows that there is some non-trivial segment e_c of e_i starting and ending at $\mathcal{B}(R_{e_i})$ and is contained in S the side of $\mathcal{B}(R_{e_i})$ that does not contain R_{e_i}.

Let f be one of the edges on $\mathcal{B}(R_{e_i})$ that e_c crosses. Pick $\mathcal{C}$ to be any closed curved on $\mathcal{B}(R_{e_i})$ and e_c that contains e_c. Since R_{e_i} is a face, it follows that the ends of f are on separate sides of $\mathcal{C}$.

Let u_1 be the end of f that is on the opposite side of $\mathcal{C}$ from u. In D, (u, u_1) does not cross e_c and it does not cross R_{e_i} in D_{e_i}. It follows that (u, u_1) does not cross $\mathcal{C}$, a contradiction with u and u_1 on opposite sides of $\mathcal{C}$. □

This Lemma is a tool to describe how an edge is drawn sequentially in simple drawings. It will be used in the inductive proof of Theorem 1.1 and is not relevant to the results in this section. The following Lemma shows a similar result for edges in a simple drawing of K_n.

Lemma 3.5. Let $n \geq 4$, D be a simple drawing of K_n, R be a face in D and e be an edge of K_n. If e has a non-trivial segment on $\mathcal{B}(R)$, then $e \cap \mathcal{B}(R)$ is exactly one non-trivial segment of e

Proof. Let $n \geq 4$, D be a simple drawing of K_n, R be a face in D and $e = (u, v)$ be an edge of K_n. Suppose e has a non-trivial segment on $\mathcal{B}(R)$. By way of contradiction, assume e intersects $\mathcal{B}(R)$ at two separate connected components. It follows that there is some non-trivial segment e_1 of e starting and ending at $\mathcal{B}(R)$, is contained in S the side of $\mathcal{B}(R)$ that does not contain R, and crosses an edge on at least one end.

Let f be one of the edges on $\mathcal{B}(R)$ that e_1 crosses. Pick $\mathcal{C}$ to be any closed curved on $\mathcal{B}(R)$ and e_1 that uses e_1. Since R is a face, it follows that the ends of f are on separate sides of $\mathcal{C}$.

Let u_1 be the end of f that is on the opposite side of $\mathcal{C}$ from u. In D, (u, u_1) does not cross e_1 and it does not cross R. It follows that (u, u_1) does not cross $\mathcal{C}$, a contradiction with u and u_1 on opposite sides of $\mathcal{C}$. □

Lemma 3.6. *Let $n \geq 4$ and D be a simple drawing of K_n. If c is an edge of K_n, P is a point of D in some face and there is a 3-cycle having c and P on opposite sides, then there is some drawing $\mathcal{D}$ in D on a K_4 containing c such that no face of $\mathcal{D}$ contains P and has a non-trivial segment of c on its boundary.*

Proof. By way of contradiction, assume every drawing in D on a K_4 containing c has a face containing P and has c on the boundary. Let $T = (q_1, q_2, q_3)$ be a 3-cycle in D having c and P on opposite sides. Let S_c be the side of T that contains c and S_P be the side of T that contains P.

If some 3-cycle involving u and two of q_1, q_2, q_3 has c and P on opposite sides, then this 3-cycle along with v will induce a drawing of a K_4 in D that has c not on the boundary of the face containing P, a contradiction.

If the edges of u to the vertices in T are drawn inside S_c, then such a 3-cycle exists. Therefore, one of the edges (u, q_i) must cross T, for $i \in [3]$. Without loss of generality, let (u, q_1) cross (q_2, q_3).

Similarly, c and P are not on opposite sides of (u, q_2, q_3). Therefore, c is in the face bounded by (u, q_1, q_2, q_3). Note the face bounded by $\gamma_{(q_1,q_3),(q_2,q_3),(u,q_1)}$ is symmetric to the face bounded by $\gamma_{(q_1,q_2),(q_2,q_3),(u,q_1)}$ up to relabelling of q_2 and q_3.

In particular, since P is in S_P, it follows that P is in one of these faces. Without loss of generality, assume P is in the face bounded by $\gamma_{(q_1,q_3),(q_2,q_3),(u,q_1)}$ as in Figure 7. It follows that (u, q_1, q_3) separates P from c, and thus this 3-cycle along with v will induce a drawing of a K_4 in D that has c not on the boundary of the face containing P, a contradiction. □

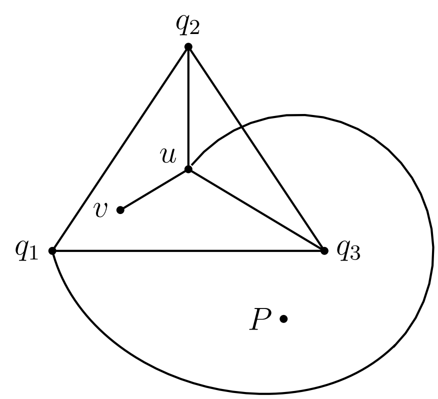


Figure 7: End of proof of Lemma 3.6.

Before stating and proving the main result of this section, we must state one last observation on simple drawings of K_4 to simplify the arguments in the forthcoming proof.

Observation 3.7. *Let D be a simple drawing of a K_4 and let u be a vertex of K_4, then there are three distinct faces in D that each have a distinct pair of edges on the boundary incident with u.*

The proof of Observation 3.7 follows by checking this fact on the two simple drawings K_4 at any vertex.

Theorem 3.8. *Let $n \geq 4$ and D_1 be a simple drawing of K_n. If c is an edge of K_n and P is a point of D_1 in some face, then either:*

1. *There is a sequence (possibly empty) of Reidemeister III moves $\{\rho_{X_i}\}_{i=1}^{k}$ with sets of edges X_i such that $D_{i+1} = \rho_{X_i}(D_i)$ with:*

    i. *A non-trivial segment of c is on the boundary of the face of D_{k+1} containing P;*

    ii. *$P \notin \triangle_{X_i}$, $\forall i \in [k]$; and*

    iii. *For $i \in [k]$, if $c \notin X_i$, then there exists $j > i$ in $[k]$ such that $c \in X_j$ and $\triangle_{X_i} \subset \triangle_{X_j}$ in D_i; or*

2. *There is some drawing $\mathcal{D}$ in D_1 on a K_4 containing c such that no face of $\mathcal{D}$ contains P and has a non-trivial segment of c on its boundary.*

For the proof of Theorem 1.1, we add an edge by successively drawing its segments across a face. To do so, we choose a special edge c that must be crossed in the current drawing on the boundary of some face F. However, it is not guaranteed that c is on the boundary of F. Therefore, Theorem 3.8 offers some structure on the relation of such an edge/face pair in a simple drawing of the complete graph. Note that Theorem 1.2 is a simplification of Theorem 3.8. We end this section with a proof of Theorem 3.8.

Proof. Let $n \geq 4$, $c = (u, v)$ be an edge of K_n, D_1 be a simple drawing of K_n, D_j be an arbitrary simple drawing of K_n derived from applying a sequence of Reidemeister III moves to D_1, and P be a point in some face R_j in D_j such that c does not have a non-trivial segment on R_j.

Our goal is to show that if for all drawings $\mathcal{D}$ in D_1 on a K_4 containing c, some face of $\mathcal{D}$ contains R_1 and has a non-trivial segment of c on its boundary, then there are sets of three edges X_i and a sequence of Reidemeister III moves $\{\rho_{X_i}\}_{i=1}^{k}$ on D_1 to a drawing D_{k+1} such that:

i. D_{k+1} has a non-trivial segment of c on the face containing P;

ii. the intersection of each $\triangle_{X_i}$ with P is empty; and

iii. for each X_i not containing c, there exists a $j > i$ such that $c \in X_j$ and $\triangle_{X_i}$ is contained in $\triangle_{X_j}$ in D_i.

Note that any drawing of a K_4 in D_j is topologically equivalent to the drawing of that K_4 in D_1 and P is on the same side of every cycle 3-cycle in D_1 and D_j (as long as we choose S and $\bar{S}$ from Definition 2.16 carefully as to not contain P). These two facts combined imply that any K_4 in D_j having P in a face and c not on the boundary of the face containing P also has this property in D_1. Without loss of generality, we can assume:

(1) $c \cap \mathcal{B}(R_j) \subseteq \{u, v\}$ for;

(2) for all drawings $\mathcal{D}$ in D_j on a K_4 containing c, some face of $\mathcal{D}$ contains R_j and has a non-trivial segment of c on its boundary.

Claim 1. $c \cap \mathcal{B}(R_j) = \emptyset$.

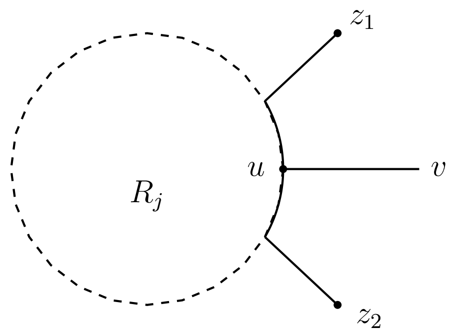


Figure 8: Set-up for Claim 1.

Proof. By way of contradiction, assume $c \cap \mathcal{B}(R_j) \neq \emptyset$. Up to relabelling and by (1), without loss of generality $u \in \mathcal{B}(R_j)$. There are two edges (z_1, u) and (z_2, u) that have non-trivial segments more than just their endpoints on the boundary of R_j as seen in Figure 8. In particular, Observation 3.7 implies that c and R_j are in separate regions in $D_j[\{z_1, z_2, u, v\}]$, a contradiction with (2). ■

Define V_{R_j} to be the vertex set of K_n that induces the face R_j (the set of vertices that are endpoints of edges that have non-trivial segments on R_j).

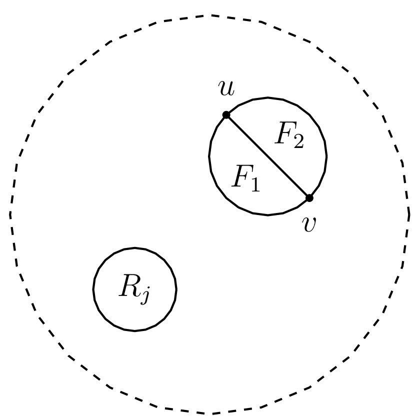


Figure 9: Set-up for Claim 2.

Claim 2. *$D_j[V_{R_j} \cup \{u, v\}]$ has an edge that crosses c.*

Proof. By way of contradiction, assume $D_j[V_{R_j} \cup \{u, v\}]$ has no edge that crosses c. It follows that c bounds two faces F_1 and F_2 in $D_j[V_{R_j} \cup \{u, v\}]$ as seen in Figure 9. Applying Corollary 3.3 to F_i and R_j implies that there is a 3-cycle T_i that separates F_i from R_j. If both T_1 and T_2 contain c, then the union of T_1 and T_2 covers c as F_1 and F_2 are bounded by opposite sides of c. Thus, the simple drawings on the K_4 induced by T_1 and T_2 has R_j separated from c, a contradiction with (2).

Therefore, one of T_1 or T_2 does not contain c. Without loss of generality, assume T_1 does not contain c. Since T_1 does not contain c, c does not cross T_1, and F_1 and R_j are separated by T_1, it follows that c and R_j are separated by T_1, a contradiction with Lemma 3.6 ■

Define $E(\mathcal{B}(R_j))$ to be the set of edges that have non-trivial segments on $\mathcal{B}(R_j)$ and $E(D_j[V_{R_j} \cup \{u, v\}])$ to be the set of edges in the drawing of $D_j[V_{R_j} \cup \{u, v\}]$. Notice that R_j is a face of $D_j[V_{R_j}]$.

Claim 3. *An edge in $E(\mathcal{B}(R_j))$ crosses c.*

Proof. By way of contradiction, assume no edge in $E(\mathcal{B}(R_j))$ crosses c. It follows by Claim 2 that some edge d in $E(D_j[V_{R_j} \cup \{u, v\}]) \backslash E(\mathcal{B}(R_j))$ crosses c.

Subclaim 3.1. *d has an endpoint on $\mathcal{B}(R_j)$.*

Proof. By way of contradiction, assume d does not have an endpoint on $\mathcal{B}(R_j)$. Let $d = (x_1, y_1)$. Since $d \in E(D_j[V_{R_j} \cup \{u, v\}])$, it follows that an edge incident to x_1 and an edge incident to y_1 both have non-trivial segments on $\mathcal{B}(R_j)$. Without loss of generality, let $e = (x_1, x_2)$ and $f = (y_1, y_2)$ be such edges.

Define e_i to be the segment of e that starts at i and intersects $\mathcal{B}(R_j)$ only at its end and let f_j be the segment f that starts at j and intersects $\mathcal{B}(R_j)$ only at its end, for $i \in \{x_1, x_2\}$ and $j \in \{y_1, y_2\}$. Note that Lemma 3.5 implies that every non-trivial segment of e or f that is not contained in $\mathcal{B}(R_j)$ does not have both ends as crossing points. There are two cases to consider, whether e_{x_1} and f_{y_1} cross or not.

Case 1. *e_{x_1} and f_{y_1} do not cross.*

Note that Figure 10 outlines the edges that will be of importance in this case.

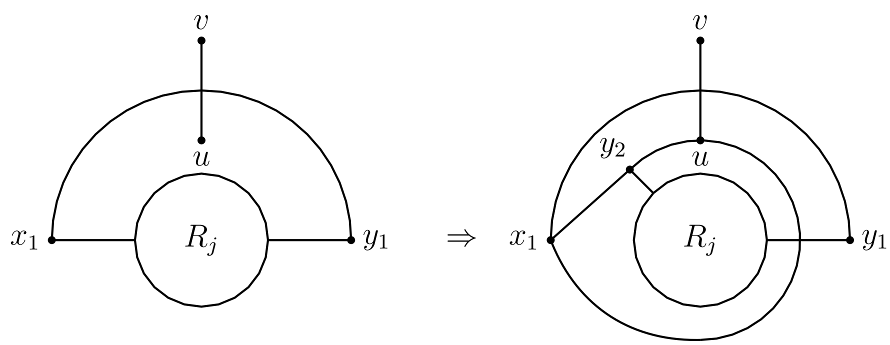


Figure 10: Case 1 of Subclaim 3.1

In $D_j[\{x_1, y_1, u, v\}]$, d crosses c and R_j is in some face F. It follows that the only faces in $D_j[\{x_1, y_1, u, v\}]$ that have a non-trivial segments of c on their boundaries are the faces that have the crossing of c and d on their boundary. By (2), F must be one of these faces.

For this to happen, one of $(u, x_1), (u, y_1), (v, x_1), (v, y_1)$ must cross one of e_{x_1} or f_{y_1}. Since these potential crossings are the same up to relabelling of u, v, e_{x_1} and f_{y_1}, assume without loss of generality that (u, x_1) crosses f_{y_1} as in Figure 10.

Let S be the side of $\gamma_{(u,x_1),(x_1,y_1),f_{y_1}}$ that contains R_j. Note by the drawing of the edge (u, x_1), u is in S. Since f_{y_2} can not cross S and starts on $\mathcal{B}(R_j)$, f_{y_2} is also contained in S.

Since u and y_2 are in S and (u, y_2) can cross $\gamma_{(u,x_1),(x_1,y_1),f_{y_1}}$ at most once (at (x_1, y_1)), it follows that (u, y_2) is drawn inside S. Since y_2 is inside S and (x_1, y_2) can not cross S, (x_1, y_2) is also contained in S. All of these edges have been outlined in Figure 10. The only simple drawing of these edges in D_j has the 3-cycle (u, x_1, y_2) separating c from R_j, a contradiction with (2).

Case 2. *e_{x_1} and f_{y_1} cross.*

Note that it does not matter which direction e_{x_1} and f_{y_1} cross as those cases are symmetric to each other in the sphere. Since c crosses (x_1, y_1) and not f_{y_1} or e_{x_1}, the ends of c are on opposite side of the simple closed curve $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$.

Without loss of generality, assume u is on the side of $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$ that does not contain R_j. If all the edges in $D_j[\{x_1, y_1, u, v\}]$ not in $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$, do not cross $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$, then $D_j[\{x_1, y_1, u, v\}]$ has a K_4 separating c from

R, a contradiction with (2).

Therefore one of these edges crosses $\gamma_{f_{y_1},(x_1,y_1),e_{x_1}}$. Up to symmetry, the two cases are (u, y_1) crosses e_{x_1} or (v, y_1) crosses e_{x_1}.

Case 2.1. (u, y_1) *crosses* e_{x_1}.

Since f_{y_1} crosses e_{x_1} and (u, y_1) crosses e_{x_1}, the region containing f_{y_2} is determined. Furthermore, (u, y_2) must cross e_{x_1} and the 3-cycle (u, y_1, y_2) separates c from R_j, a contradiction with (2).

Case 2.2. (v, y_1) *crosses* e_{x_1}.

e_{x_1} and f_{y_1} partition $\mathcal{B}(R_j)$ as they each intersect $\mathcal{B}(R_j)$ at one point. Let C_{R_j} be a simple closed curve that starts on the crossing of e_{x_1} and f_{y_1}, takes the edge segment f_{y_1} to $\mathcal{B}(R_j)$, walks along $\mathcal{B}(R_j)$ to e_{x_1}, then takes the edge segment e_{x_1} back to the crossing of e_{x_1} and f_{x_1}.

Both e_{x_2} and f_{y_2} must be on the same side of this curve as they can no cross it. If f_{y_2} and (v, y_1) are on the opposite sides of C_{R_j}, then the 3-cycle (v, y_1, y_2) separates c from R_j, a contradiction with (2).

Therefore, f_{y_2} and (v, y_1) are on the same side of C_{R_j}. It follows that e_{x_2} and (v, y_1) are on the same side of C_{R_j}. The 3-cycle (v, y_1, x_2) separates c from R_j, a contradiction with (2). ■

Subclaim 3.2. *d has two endpoints on $\mathcal{B}(R_j)$.*

Proof. By Subclaim 3.1, d has at least one endpoint on $\mathcal{B}(R_j)$. By way of contradiction, assume d has at exactly one endpoint on $\mathcal{B}(R_j)$. Let $d = (x_1, x_2)$ such that x_1 is on $\mathcal{B}(R_j)$. Since $d \in E(D_j[V_{R_j} \cup \{u, v\}])$, it follows that an edge incident to x_2 has a non-trivial segment on $\mathcal{B}(R_j)$. Without loss of generality, let $e = (x_2, x_3)$ be such an edge.

Let γ_{d,e,R_j} be one of two unique simple closed curves on edges d, e and the simple closed curve $\mathcal{B}(R_j)$. Since no edge in $E(\mathcal{B}(R_j))$ crosses c, e does not cross c and c does not cross $\mathcal{B}(R_j)$. It follows that γ_{d,e,R_j} has u and v on opposite sides. Without loss of generality, let u be on the opposite side of x_3. Observe that the edge (x_2, u) is uniquely determined relative to $\mathcal{B}(R_j), d, e, c$.

Let d_{x_i} be the segment of d from x_i to the crossing with c. If (x_3, u) crosses d_{x_1}, then the 3-cycle (x_2, x_3, u) separates c from R_j, a contradiction with (2).

Therefore, (x_3, u) is drawn crossing d_{x_2}. Since (x_1, x_3) can cross $\gamma_{d,e,(e,x_3)}$ or R_j, it follow that (x_1, x_3) is uniquely determined in the drawing involving $d, e, c, (x_1, u), (x_2, u)$ and (x_3, u). In particular, the 3-cycle (x_1, x_3, u) separates c from R_j, a contradiction with (2). ∎

By Subclaim 3.2, w has two endpoints on $\mathcal{B}(R_j)$. Let $w = (x_1, x_2)$. Noting that the edges in the K_4 involving x_1, x_2, u, v are determined relative to R_j, it follows that $D_j[x_1, x_2, u, v]$ has c separated from R_j, a contradiction with (2). ∎

By Claim 3, there exists an edge e^j that crosses c in $E(\mathcal{B}(R_j))$. Without loss of generality, let $e^j = (x_1, x_2)$.

Since e^j has a non-trivial segment on the boundary of R_j, it follows that some segment of e^j, up to relabelling, starts at x_2 crosses c and intersects $\mathcal{B}(R_j)$ only at its end, call this segment $e^j_{x_2}$. Define $e^j_{x_1}$ to be the segment of e^j that starts at x_1 and intersects $\mathcal{B}(R_j)$ only at its end. Note, it is possible that $e^j_{x_1}$ is only the vertex x_1. Note that by Lemma 3.5, every non-trivial segment of e^j that is not contained in $\mathcal{B}(R_j)$ does not have both ends as crossing points on $\mathcal{B}(R_j)$.

Without loss of generality, starting from x_2, $e^j_{x_2}$ crosses $\overrightarrow{(u, v)}$ from right to left as the opposing cases is completely analogous. The end of $e^j_{x_2}$ that is not x_2 is a crossing with an edge $f^j = (y_1, y_2)$ at the intersection of e^j with $\mathcal{B}(R_j)$. Since f^j has a crossing on $\mathcal{B}(R_j)$, it follows by Lemma 3.5 that f^j has segments f^j_i for $i \in \{y_1, y_2\}$ that starts at i and only intersect $\mathcal{B}(R_j)$ at its end.

Without loss of generality, we can assume walking clockwise around $\mathcal{B}(R_j)$, we cross $e^j_{x_2}, f^j_{y_2}, e^j_{x_1}, f^j_{y_1}$, if not we can relabel the ends of f^j.

Claim 4. *$\triangle_{c,e^j,f^j}$ exists.*

Proof. By way of contradiction, assume $f^j_{y_2}$ does not cross c between u and the crossing of c with e^j and $f^j_{y_1}$ does not cross c between v and the crossing of c with e^j.

Subclaim 4.1. *f^j crosses c.*

Proof. By way of contradiction, suppose f^J does not cross c. For the readers convenience, we offer a diagram depicting this case in Figure 11.

It is either (x_1, u) crosses $f^j_{y_1}$, $f^j_{y_2}$, or it does not cross f^j. All cases follow a similar argument with the goal to show that one of the 3-cycles (y_1, y_2, u),

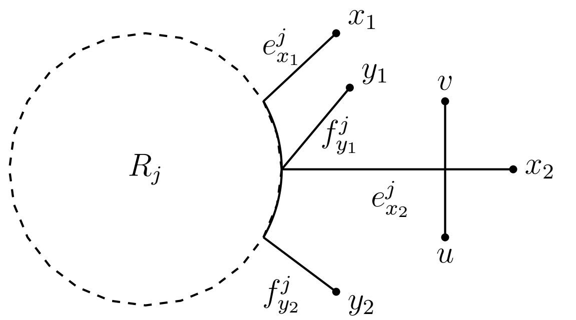


Figure 11: Claim 4 Case 1.

(x_1, y_1, u), or (x_1, y_2, u) contradict (2).

Case 1. (x_1, u) *does not cross* f^j.

In this case, if the edge (y_2, u) crosses e^j, then y_2 and R_j are contained on the same side of $\gamma_{(x_1,u),(u,y_2),e^j}$. In particular, the edge (x_1, y_2) is drawn on the same side of $\gamma_{(x_1,u),(u,y_2),e^j}$ as R_j and the 3-cycle (x_1, y_2, u) separates c from R_j, a contradiction with (2).

It follows that the edge (y_2, u) does not cross e^j. If $\pi_{y_1,y_2,v}(u) = [v, y_1, y_2]$, then the 3-cycle (y_1, y_2, u) separates c from R_j as c does not cross f, a contradiction with (2).

Since $\pi_{\{y_1,y_2,v\}}(u) = [v, y_2, y_1]$, the edge (y_1, u) crosses e^j. If (x_1, y_1) does not cross c, then the 3-cycle (x_1, y_1, u) separates c from R_j, a contradiction with (2).

Therefore, (x_1, y_1) crosses c. In particular, (y_1, v) does not cross (x_1, u) and (x_1, v) does not cross (y_1, u). It follows that the 4-cycle (y_1, v, x_1, u) separates c from R_j, a contradiction with (2).

Case 2. (x_1, u) *crosses* $f^j_{y_2}$.

This case is the exact same as the previous, except both y_2 and (x_1, y_2) will be on the opposite side of $\gamma_{(x_1,u),(u,y_2),e^j}$ when compared to R_j, and c and R_j switch sides in the cycle (x_1, y_2, u).

Case 3. (x_1, u) *crosses* $f^j_{y_1}$.

x_1 is contained on the left side of the closed curved defined by starting at u, taking the c to the crossing of c and e^j, then taking the edge e^j to the crossing of e^j with f^j, then taking the edge f^j to the crossing of f^j with (x_1, u), then taking the edge (x_1, u) to u. In particular, the edge (x_1, y_2) is contained on the left side of this curve.

If (y_2, u) crosses e^j, then the 3-cycle (x_1, y_2, u) separates c from R_j, a contradiction with (2). Therefore, (y_2, u) does not cross e^j and the 3-cycle (y_1, y_2, u) separates c from R_j, a contradiction with (2). ■

By Subclaim 4.1, f^j crosses c. If e^j and f^j cross c in opposite directions on segments e^j_i and f^j_ℓ for $i \in \{x_1, x_2\}$ and $\ell \in \{y_1, y_2\}$, then $D_j[\{i, \ell, u, v\}]$ has the 4-cycle (i, u, ℓ, v) separating c from R_j, a contradiction with (2).

Therefore, e^j and f^j cross c in the same direction, since $\overrightarrow{(x_1, x_2)}$ crosses $\overrightarrow{(u, v)}$ from left to right, so does $\overrightarrow{(y_1, y_2)}$. Without loss of generality, assume $f^j_{y_2}$ crosses c as $f^j_{y_1}$ crossing c is symmetric.

Since $f^j_{y_2}$ does not cross c between u and the crossing of c with e^j, it follows that $f^j_{y_2}$ crosses c between v and the crossing of c with e^j.

Consider the simple drawing $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, f^j, c\}]$. By our choice of oriented crossing on c, $(x_1, y_1), (x_1, v)$ and (y_1, v) are uniquely drawn into $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, f^j, c\}]$ to keep the drawing $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, f^j, c, (x_1, y_1), (x_1, v), (y_1, v)\}]$ simple. In particular, the 3-cycle (x_1, y_1, v) will separate c from R_j, a contradiction with (2).

Therefore, $f^j_{y_2}$ crosses c between u and the crossing of c with e^j. Note by our choice of oriented crossings, that the ends of e^j, f^j and c are on the same side of the simple closed curve $\gamma_{e^j, f^j, c}$. Therefore, $\triangle_{e^j, f^j, c}$ exists. ■

Let $T_j = \triangle_{e^j, f^j, c}$ in D_j in Claim 4. Note that T_j and R_j have empty intersection. It follows that T_j does not contain P.

Claim 5. *T_j does not contain any vertices.*

Proof. By way of contradiction, assume there is some vertex z in T_j as depicted in Figure 12. If (z, v) crosses e^j and (z, u) crosses f^j, then one of the drawings $D_j[\{z, u, v, x_2\}]$ or $D_j[\{z, u, v, y_2\}]$ has c separated from R_j. Note that no simple drawing has (z, v) crossing f^j and (z, u) crossing e^j. Since $\mathcal{B}(T_j)$ separates z from u and v, it follows that (z, u) and (z, v) cross $\mathcal{B}(T_j)$ both on e^j or both on f^j. By symmetry, without loss of generality assumed (z, u) and (z, v) both cross e^j.

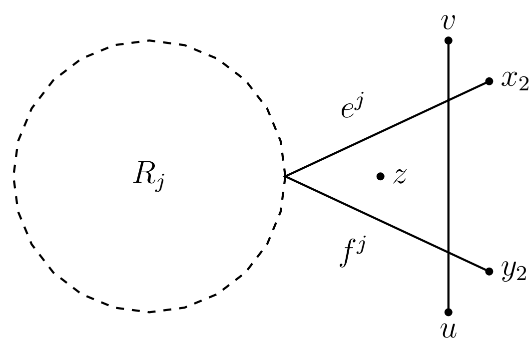


Figure 12: Claim 5.

Since (z, u) and (z, v) both cross e^j, the edges (z, x_1) and (x_1, u) are uniquely drawn in $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, c, (z, u), (z, v)\}]$ to produce a simple drawing $D_j[E(\mathcal{B}(R_j)) \cup \{e^j, c, (z, u), (z, v), (z, x_1), (x_1, u)\}]$. In particular, the 3-cycle (z, x_1, u) separates c from R_j, a contradiction with (2). ■

By Claim 5 there are no vertices in T_j. Since T_j contains no vertices, if an edge in D_j crosses $\mathcal{B}(T_j)$, it does so exactly twice.

Define $N_{D_j}(T_j)$ to be the number of edges that cross $\mathcal{B}(T_j)$ not at c in D_j. Inductively, we will show the following claim.

Claim 6. There is a sequence of simple drawings $D_1, \ldots, D_k$ such that:

1) For $i \in [k]$, each D_i has an e^i, f^i and T_i;
2) For $i \in [k-1]$, $N_{D_{i+1}}(T_{i+1}) < N_{D_i}(T_i)$;
3) $N_{D_k}(T_k) = 0$;
4) For $i \in [k-1]$, there are sets of three edges X^i_ℓ and a sequence of Reidemeister III moves $\{\rho_{X^i_\ell}\}_{\ell=1}^{j_i}$ such that D_{i+1} is $\{\rho_{X^i_\ell}\}_{\ell=1}^{n_i}$ on D_i;
5) For $i \in [k-1]$ and $\ell \in [n_i - 1]$, $\triangle_{X^i_\ell} \subset T_i$ in D_i; and
6) For $i \in [k-1]$, $P \notin T_i$ in D_i.

Proof. If $N_{D_1}(T_1) = 0$, then we are done as P is in R_1 and not T_1.

Therefore, $N_{D_1}(T_1) > 0$. For our induction step, we can assume $N_{D_i}(T_i) > 0$ for some $i \geq 1$. It follows that there exists an edge g^i such that any edge that crosses $\triangle_{e^i,f^i,g^i}$ crosses g^i and $\triangle_{e^i,f^i,g^i} \subset T_i$.

Applying Corollary 2.18 implies that there exists a sequence of Reidemeister III moves $\{\rho_{X^i_\ell}\}_{\ell=1}^{n_i}$ (None of which cross P by careful selection) from D_i to a simple drawing D_{i+1} for some integer n_i such that:

- for $\ell \in [n_i]$, $\triangle_{X^i_\ell} \subset T_i$ in D_i;
- $X^i_{n_i} = \{e^i, f^i, g^i\}$; and
- $g^i \in X^i_\ell$ for all $\ell \in [n_i]$.

In D_{i+1}, a non-trivial segment of g^i is on the boundary of the face containing P, also known as R_i. The result is that e^i, g^i, f^i have consecutive non-trivial segments on R_{i+1}. If g^i does not cross c, then setting $g^i = e^{i+1}$ and $f^i = f^{i+1}$ gives a contradiction to Claim 4.

Therefore, g^i does cross c, and does so outside of T_i by definition of g^i. By setting e^i, f^i and g^i to be the appropriate variables (e^{i+1} or f^{i+1}), it follows by Claims 4 and 5 that $\triangle^{D_i}_{e^i,g^i,c}$ and $\triangle^{D_i}_{f^i,g^i,c}$ exist and are empty of vertices. Since they are both empty of vertices, one of them must contain T_i in D_{i+1}. Without loss of generality, let $T_i \subset \triangle^{D_{i+1}}_{e^i,g^i,c}$ in D_{i+1}.

Set $e^{i+1} = e^i$, $f^{i+1} = g^i$ and $T_{i+1} = \triangle^{D_{i+1}}_{e^{i+1},f^{i+1},c}$. Setting $j = i+1$, we see that e^{i+1}, f^{i+1} and T_{i+1} satisfy Claims 1 - 5.

Note that, since $g^i \in X^i_\ell$ for all $\ell \in [n_i]$, we can choose our Reidemeister III moves so that only the edge g^i is changing. It follows that an edge not g^i crosses $\mathcal{B}(T_i)$ not at c in D_i if and only if it crosses $\mathcal{B}(T_i)$ not at c in D_{i+1}. Since $\rho_{X^i_{n_i}}$ was the last Reidemeister III move, every edge that crosses $\mathcal{B}(T_{i+1})$ not at c in D_{i+1} also crosses $\mathcal{B}(T_i)$ not at c in D_{i+1}. Since g^i does not cross $\mathcal{B}(T_{i+1})$ in D_{i+1}, it follows that $N_{D_{i+1}}(T_{i+1}) < N_{D_i}(T_i)$.

By applying induction, it is clear that 1) - 3) are satisfied. By our use of Claim 4, 4) and 5) are satisfied. Since g^i is the only edge moving by the Reidemeister moves $\{\rho_{X^i_\ell}\}_{\ell=1}^{n_i}$, we choose the Reidemeister III move carefully so that P is not in the disc that contains the Reidemeister III move. By this choice, $P \notin T_{i+1}$ in D_{i+1}, satisfying 6). ■

To complete our proof, we apply Claim 6 to find a sequence of simple drawing $D_1, \ldots, D_k$, a sequence of Reidemeister III moves $\{\rho_{X^i_\ell}\}_{\ell=1}^{n_i}$, and T_i satisfying 1) through 6). Finally we apply Corollary 2.18 to find a sequence of Reidemeister III moves $\{\rho_{X^k_\ell}\}_{\ell=1}^{n_k}$ from D_k to D_{k+1} such that $c \in X^k_\ell$ for all $\ell \in [n_k]$ and $\triangle_{X^k_\ell} \subset \triangle_{X^k_{n_k}}$ in D_k. Since $T_k = \triangle_{X^k_{n_k}}$ in D_k, every Reidemeister III move occurs in some T_i. Since every T_i in D_i does not contain P, it follows no Reidemeister III move contains P, satisfying ii.

Again, since every Reidemeister III move between D_i and D_{i+1} occurs in T_i, all we need to show to satisfy iii. is that a Reidemeister move over $\triangle_{e^i,f^i,c}$ occurs after T_i is emptied.

In D_{k+1}, c is on the boundary of the face containing P, and so is also on the boundary of the face containing P in the simple subdrawing of the K_6 induced by $\triangle_{e^i,f^i,c}$ in D_{k+1}.

However, c is not on the boundary of the face containing P in the simple subdrawing of the K_6 induced by $\triangle_{e^i,f^i,c}$ in $\rho_{X^i_{n_i}}(D_{i+1})$. This implies a Reidemeister move over $\triangle_{e^i,f^i,c}$ occurs after T_i is emptied, as desired.

$\square$

Obviously no simple drawing of K_3 containing c can separate a point P from c, therefore the 4 in Theorem 3.8 is least possible. Moreover, Lemma 3.6 shows that no 3-cycle separates c from P, else there is a K_4 in D containing c that separates c from P.

4 $(6,5)$-Rotation Systems

The goal of this section is to prove the case $n = 6$ of Theorem 1.1. Such a case is interesting as it can be combined with results in [**1**] to prove Theorem 1.1. Although this section contains some flavor of the arguments required to prove Theorem 1.1, it also contains some of the most technical arguments found in this writing. We introduce the notion that orderings of edge crossings on a fixed edge under certain realizability constraints.

Notation 4.1. Let $n \geq 5$, H_n be an n-vertex rotation system, e be a directed edge of H_n, and f and g edges of H_n such that e crosses f and g. Define:

- $f <^e_\wedge g$, if H_n is an $(n,5)$-rotation system, f and g share an endpoint, and e crosses f, then g in the drawing of K_5 induced by e, f, g;
- $f <^e_\| g$, if H_n is an $(n,6)$-rotation system, f and g do not cross in H_n, there is no ordering of f and g of $<^e_\wedge$ relations in the rotation system induced by e, f, g from H_n, and e crosses f, then g in the drawing of the K_6 induced by e, f, g; and
- $f <^e_\triangle g$, if H_n is an $(n,7)$-rotation system, $\triangle_{\{e,f,g\}}$ exists containing a vertex v, and e crosses f, then g in the drawings of the K_7 induced by e, f, g, v.

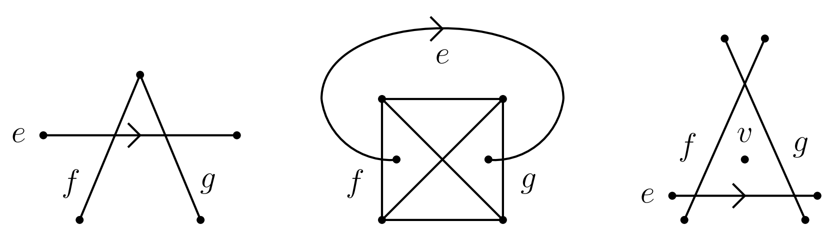


Figure 13: The three relations $f <^e_\wedge g$, $f <^e_\| g$ and $f <^e_\triangle g$.

The three relations are outline in Figure 13. In a simple drawing of a K_5 containing a directed edge e and two edge f and g, $f <^e_\wedge g$ is well defined in that drawing, and by Lemma 2.8, is determined by H. Looking at $f <^e_\| g$ in an associated simple drawing of a K_6 shows that the segments of e from the crossing with f to the crossing with g is uniquely determined by the oriented crossings of e with f and e with g. By Observation 2.7, this ordering is uniquely determined by H. Infrequently, we use $f <_{K_6}^e g$ to say that two edges f and g are ordered on e with either a $<_{\parallel}^e$ relation or a chain of $<_{\wedge}^e$ relations.

Finally, if $f <_{\triangle}^e g$ occurs in some simple drawing of an associated K_7, then we will see in Lemma 5.7 that e must cross f then g in every drawing realizing that associated rotation system by some $<_{\wedge}^e$ and $<_{\parallel}^e$ relations. In particular, $f <_{\triangle}^e g$ is in reality a chain of $<_{\wedge}^e$ and $<_{\parallel}^e$ relations each determined by H.

Our first goal is for a fixed edge e, find a total ordering on the edges it crosses associated with the rotation system we are given. To that end, we want to make sure the relation $<_{\wedge}^e$ on the edges that e crosses induces an acyclic directed graph (As described in Lemma 4.6). If such a graph contained a directed cycle, then there exists (up to relabelling) a directed cycle described in Lemmas 4.4 and 4.5. We first make an observation on rotation system information that is provided from the $<_{\wedge}^e$ relation.

Definition 4.2. Let $n \geq 5$, D be a simple drawing of K_n, and f and g be two edges both incident to the same vertex that both cross a third edge e. Define f and g to *agree* if from their common endpoint, they cross e in the same direction, otherwise the two edges *disagree*.

Observation 4.3. *Let D be a simple drawing of K_5, and $f = (v, f_1)$ and $g = (v, g_1)$ be two edges incident to vertex v that both cross a third edge e. Suppose $f <_{\wedge}^e g$. If f and g agree, then the rotation at v in D is determined by the oriented crossings of e with f or e with g. If f and g disagree, then D is Harborth and the rotation at every vertex is determined by the oriented crossings of e with f or e with g.*

Suppose f and g agree and you are given the oriented crossing of e with f. Since f and g agree, we also have the oriented crossing of e with g. Since $f <_{\wedge}^e g$, there is a unique way to draw the star at v with the edge e in the simple drawing D and this drawing produces the desired rotation at v. In particular, if $e = \overrightarrow{(u,v)}$ is a directed edge that crosses $f = \overrightarrow{(v, f_1)}$, then crosses $g = \overrightarrow{(v, g_1)}$, both from right to left, then $\pi_{u,v,f_1,g_1}(v) = [v, g_1, f_1, u]$.

Similarly, if f and g disagree, then by Observation 2.10, D is Harborth and the drawing is determined by the drawing of e, f and g. It is clear that the rotations at each vertex is determined by the oriented crossings of e with f or e with g given $f <_{\wedge}^e g$.

Lemma 4.4. *Let H be a $(6,5)$-rotation system and e be a directed edge of H. If e_1, e_2, e_3 are three edges sharing an endpoint and all cross e, then $e_1 <^e_\wedge e_2 <^e_\wedge e_3 <^e_\wedge e_1$ does not occur.*

Proof. Let H be a $(6,5)$-rotation system and $e = \overrightarrow{(u,v)}$ be a directed edge of H. Define the simple drawings D_i to be a realization of $H_{-}\{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u,v\}$. Suppose e_1, e_2, e_3 are three edges sharing an endpoint and all cross e.

By way of contradiction, assume such a cycle $e_1 <^e_\wedge e_2 <^e_\wedge e_3 <^e_\wedge e_1$ exists. Without loss of generality this cycle is $(1,2) <^e_\wedge (1,3) <^e_\wedge (1,4) <^e_\wedge (1,2)$. Two of three edges must cross e in the same direction when starting at the vertex 1. Without loss of generality, these two edges are $(1,2)$ and $(1,3)$, and up to relabelling of the ends of e, they cross e from left to right starting at the vertex 1. Since $(1,2) <^e_\wedge (1,3)$, it follows that in D_4, $\pi_{u,v,2,3}(1) = [v,3,2,u]$.

Assume by way of contradiction that $(1,4)$ crosses e in the same direction as $(1,2)$ and $(1,3)$ starting at 1. Then from a similar argument $\pi_{u,v,3,4}(1) = [v,4,3,u]$ and $\pi_{u,v,2,4}(1) = [v,2,4,u]$. Therefore, $\pi(1)$ contains the three incompatible cyclic subrotations $[v,3,2]$, $[v,4,3]$ and $[v,2,4]$, a contradiction.

Therefore, $(1,4)$ crosses e from the opposite side as $(1,2)$ and $(1,3)$ starting at 1. By Obeservation 4.3, both D_3 and D_2 are Harborth drawings of K_5 with e being crossed three times in each of the drawings. In particular, by Observation 4.3, we learn:

- $\pi_{2,4,v}(1) = [2,4,v]$ in D_3;
- $(1,4)$ and $(2,v)$ do not cross in D_3;
- $\pi_{v,2,1}(4) = [v,2,1]$ in D_3;
- $(2,4)$ does not cross $(1,v)$ in D_3; and
- $\overrightarrow{(1,4)}$ crosses $\overrightarrow{(3,v)}$ from right to left in D_2.

We break this into two cases depending on whether $(2,v)$ crosses $(1,3)$ or not.

Case 1. $(2,v)$ *crosses* $(1,3)$.

In D_4, there is a unique direction $(2,v)$ can cross $(1,3)$, in particular, $(2,v)$ crosses $(1,3)$ from right to left. In D_u, this determines the drawing of the K_4 on $1,2,3$ and v. Let S_1 be the side of $\gamma_{(1,3),(2,v),(3,v)}$ into which $\overrightarrow{(1,4)}$ crosses into as it crosses $\overrightarrow{(3,v)}$ from right to left. Since $(1,4)$ does not cross $(2,v)$ or $(1,3)$, the vertex 4 is contained in S_1.

It follows that there is a unique way to draw $(4,v)$ and it determines $\pi_{2,3,4}(v)$ to be $\pi_{2,3,4}(v) = [3,4,2]$. Since $\pi_{1,2,v}(4) = [v,2,1]$, the start of the edge $(2,4)$ at 4 and the vertex 2 are on opposite sides of the 3-cycle $(1,4,v)$. This is not possible as $(2,4)$ does not cross $(1,v)$.

Case 2. $(2,v)$ *does not cross* $(1,3)$.

By the symmetry of $(2,v)$ and $(3,u)$, $(3,u)$ does not cross $(1,2)$. From D_2 and D_3, respectively, $\pi_{1,3,v}(u) = [1,v,3]$ and $\pi_{1,2,u}(v) = [2,u,1]$. Since $(2,v)$ does not cross $(1,3)$ and $(3,u)$ does not cross $(1,2)$, it follows that in D_4, $\overrightarrow{(2,v)}$ crosses $\overrightarrow{(3,u)}$ from right to left.

From these crossings, the edges $(2,u)$ and $(3,v)$ are uniquely determined in D_4 to keep the drawing simple. It follows that the 3-cycles $(1,2,u)$ and $(1,3,v)$ are uniquely determined in the drawing. In particular, $\overrightarrow{(1,v,3)}_R \subset \overrightarrow{(1,v,2)}_R$.

In D_2, 4 is in $\overrightarrow{(1,v,3)}_R$. Deleting u from D_4 and adding 4, we find that 4 is in $\overrightarrow{(1,v,3)}_R \subset \overrightarrow{(1,v,2)}_R$. In D_3, 4 is in $\overrightarrow{(1,v,2)}_L$, a contradiction with 4 in $\overrightarrow{(1,v,2)}_R$. □

Lemma 4.5. *If H is a $(6,5)$-rotation system and $e = \overrightarrow{(u,v)}$ is a directed edge of H, then the cycle $(1,2) <^e_\wedge (2,3) <^e_\wedge (3,4) <^e_\wedge (1,4) <^e_\wedge (1,2)$ does not occur.*

Proof. Define the simple drawings D_i to be a realization of $H - \{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u,v\}$.

By way of contradiction, assume the cycle $\mathcal{C}$ of relations defined by $(1,2) <^e_\wedge (2,3) <^e_\wedge (3,4) <^e_\wedge (1,4) <^e_\wedge (1,2)$ does occur. Observation 4.3 tell us that two edges agreeing or disagreeing provides rotation system information given we know an oriented crossing with e.

Note that the number of relations in $\mathcal{C}$ that agree must be even, else the direction edges cross e is not well defined. It follows that the number of relations in $\mathcal{C}$ that disagree is also even since there is an even number of relations in $\mathcal{C}$.

We will break this into three cases. Either two consecutive relations disagree, two non-consecutive relations disagree, or all relations agree.

Case 1. *Two consecutive relations in $\mathcal{C}$ disagree.*

Assume without loss of generality that $(1,2) <^e_\wedge (2,3)$ and $(2,3) <^e_\wedge (3,4)$ disagree, and $\overrightarrow{(1,2)}$ crosses $\overrightarrow{(u,v)}$ from left to right. Since $(1,2) <^e_\wedge (2,3)$ and the relation disagrees, by Observation 4.3, D_4 is a unique labelled Harborth drawing. Similarly, D_1 is a unique labelled Harborth drawing. In particular, the drawings have:

1. $\pi_{3,4,u,v}(2) = [v,4,3,u]$ in D_1;

2. $\pi_{2,4,u,v}(3) = [v,4,u,2]$ in D_1;

3. $\pi_{2,3,4,v}(u) = [v,4,3,2]$ in D_1;

4. $\pi_{2,3,u,v}(4) = [u,2,3,v]$ in D_1;

5. $\pi_{1,3,u,v}(2) = [u,1,v,3]$ in D_4;

6. $\pi_{1,2,u,v}(3) = [v,u,1,2]$ in D_4; and

7. $\pi_{1,2,3,v}(u) = [1,v,3,2]$ in D_4.

Combining the rotations at $2,3$ and u we get $\pi_{1,3,4,u,v}(2) = [u,1,v,4,3]$, $\pi_{1,2,4,u,v}(3) = [v,4,u,1,2]$, and $\pi_{1,2,3,4,v}(u) = [v,4,3,2,1]$. Observe that in D_4, $\overrightarrow{(3,u)}$ crosses $\overrightarrow{(1,2)}$ from left to right. This determines the drawing of the K_4 in D_v. The rotations of 2 and 3 imply that 4 is in $\overrightarrow{(1,3,2)}_R \cap \overrightarrow{(3,2,u)}_R$.

The rotation at u and the location of 4 in D_v implies that $\overrightarrow{(u,4)}$ crosses $(1,2)$, then $(1,3)$, then $(2,3)$. By Observation 2.9, D_v is Harborth and the rotation of the vertices are determined. In particular, $\pi_{1,2,3,u}(4) = [3,u,1,2]$.

Note that in D_1, $\overrightarrow{(u,v)}$ crosses $\overrightarrow{(2,4)}$ from right to left. In D_4, $\overrightarrow{(u,v)}$ crosses $\overrightarrow{(1,2)}$ from right to left. By definition of $\mathcal{C}$, $\overrightarrow{(u,v)}$ crosses $\overrightarrow{(4,1)}$, therefore, it does so from left to right (else the 3-cycle $(1,2,4)$ has e crossing it from the same side three times, a contradiction in a drawing realizing its 5-vertex rotation system).

Furthermore, we have $(1,4) <^e_\wedge (1,2)$, therefore it follows $(1,4) <^e_\wedge (1,2) <^e_\wedge (2,4)$ and that D_3 is Harborth by Observation 2.9. In particular, $\pi_{1,2,u,v,}(4) = [1,v,u,2]$. Therefore, H contains the three incompatible rotations at 4 $[u,2,3,v], [3,u,1,2]$ and $[1,v,u,2]$, a contradiction.

Case 2. *Two non-consecutive relation in $\mathcal{C}$ disagree.*

Assume without loss of generality that $(1,2) <^e_\wedge (2,3)$ and $(3,4) <^e_\wedge (1,4)$ both disagree. By Observation 4.3, it follows that D_2 and D_4 are both uniquely labelled Harborth drawings up to deciding the oriented crossings of $(1,2)$ and $(3,4)$ with e.

It follows in D_4, independent of the oriented crossing of $(1,2)$ with e, $(1,2) <^e_\wedge (1,3)$. Similarly in D_2, independent of the oriented crossing of $(3,4)$ with e, $(1,3) <^e_\wedge (1,4)$. From $\mathcal{C}$, $(1,4) <^e_\wedge (1,2)$, and so $(1,2) <^e_\wedge (1,3) <^e_\wedge (1,4) <^e_\wedge (1,2)$, a contradiction with Lemma 4.4.

Case 3. *All relations in $\mathcal{C}$ agree.*

Without loss of generality, let $\overrightarrow{(1,2)}$ cross e from left to right. Since all the relations in $\mathcal{C}$ agree, it follows that every edge in $\mathcal{C}$ has its oriented crossing with e determined from the oriented crossing of $(1,2)$ with e. In particular:

- $\pi_{2,4,u,v}(3) = [v,4,2,u]$ from D_1;
- $\pi_{1,3,u,v}(4) = [v,u,3,1]$ from D_2;
- $\pi_{2,4,u,v}(1) = [v,2,4,u]$ from D_3; and
- $\pi_{1,3,u,v}(2) = [v,u,1,3]$ from D_4.

At this point we do analysis on how the edges of D_4 cross each other. These claims will have two crossings that do not occur, however this will require one proof per claim as the edges chosen have a symmetry in H.

Claim 1. $(1,u)$ *crosses* $(2,3)$ *or* $(3,v)$ *crosses* $(1,2)$.

Proof. The two cases are symmetric, so by way of contradiction, we may assume $(1,u)$ crosses $(2,3)$. In D_4, there is a unique direction $(1,u)$ can cross $(2,3)$. Such a crossing determines the simple drawing D_4. In particular, $\pi_{1,3,u}(v) = [u,1,3]$ and $\pi_{1,2,u,v}(3) = [v,2,1,u]$. Combining the rotation at 3 in D_1 and D_4, we get $\pi_{1,2,4,u,v}(3) = [v,4,2,1,u]$.

Observe that $\overrightarrow{(3,4)}$ crosses e from left to right. It follows that the drawing of the K_4 induced on $3,4$ and e is determined, in particular $\overrightarrow{(3,v,u)}_R \cap \overrightarrow{(3,v,4)}_R$ exists. By the rotations at $3,4$ and v, it follows that $v \in \overrightarrow{(3,v,u)}_R \cap$

$\overrightarrow{(3,v,4)}_R$. However, as $\pi_{1,4,u}(3) = [4,1,u]$, the edge $(3,4)$ can not reach the vertex 4 in D_2, a contradiction with the existence of D_2. ∎

Claim 2. $(1,3)$ *crosses* $(2,u)$ *or* $(1,3)$ *crosses* $(2,v)$.

Proof. As per Claim 1, $(1,u)$ does not cross $(2,3)$ and $(3,v)$ does not cross $(1,2)$. It follows that $D_4 - \{(1,3),(1,v),(3,u)\}$ is uniquely determined.

As one would expect, the two cases are symmetric, so by way of contradiction, we may assume $(1,3)$ crosses $(2,v)$. In D_4, there is a unique direction $(1,3)$ crosses $(2,v)$. Again, such a crossing determines the simple drawing D_4. It follows from D_4 that: $\pi_{3,u,v}(1) = [u,3,v]$; $\pi_{1,3,v}(u) = [1,3,v]$; $\pi_{1,3,u}(v) = [u,3,1]$; and $(1,2) <^e_\wedge (1,3)$.

Note that the oriented crossing of $(1,4)$ with e is determined, and so the K_4 induced by $1,4$ and e is uniquely drawn in D_2. By the rotations at $1,4,u$ and v ($\pi_{1,3,u}(4) = [u,3,1]$ determined by D_2), determine the drawing of D_2. In particular, $(1,3) <^e_\wedge (1,4)$.

From $\mathcal{C}$ we have $(1,4) <^e_\wedge (1,2)$, and thus we have $(1,2) <^e_\wedge (1,3) <^e_\wedge (1,4) <^e_\wedge (1,2)$, a contradiction with Lemma 4.4. ∎

Since the crossing in Claims 1 and 2 do not occur, the drawing of $D_4 - \{(1,v),(3,u)\}$ is uniquely determined. Through the symmetry of this case, these arguments extend to D_i for $i \in [4]$. D_1, D_2, and D_4 each give $\pi_{2,4,u,v}(3) = [v,4,2,u]$, $\pi_{1,4,u}(3) = [1,4,u]$, and $\pi_{1,2,v}(3) = [v,2,1]$, respectively. Combining these rotations results in $\pi_{1,2,4,u,v}(3) = [v,4,2,u,1]$. It follows that $D_4 - \{(1,v)\}$ is uniquely determined, which implies D_4 is uniquely determined by symmetry, and D_i for $i \in [4]$ is uniquely determined by symmetry.

D_1 and D_4 give $\pi_{2,3,4,v}(u) = [v,4,2,3]$ and $\pi_{1,2,3,v}(u) = [v,2,1,3]$, respectively. Combining these rotation results in $\pi_{1,2,3,4,v}(u) = [v,4,2,1,3]$, a contradiction with $\pi_{1,3,4,v}(u) = [3,1,v,4]$ in D_2. □

Lemma 4.6. *If H is a $(6,5)$-rotation system, and e is a directed edge of H, then there are no cycles comprised of $<^e_\wedge$ relations in H.*

Proof. Let H be a $(6,5)$-rotation system, e be a directed edge of H. By way of contradiction, assume $\mathcal{C} = (a_0,\dots,a_{k-1},a_0)$ is a shortest cycle of $<^e_\wedge$ relations in H. Without loss of generality, $a_0 = (1,2)$ and $a_1 = (2,3)$. If there exists an i such that $V(\{a_i,a_{i+1},a_{i+2}\}) \subseteq [4]\backslash j$ for some $j \in [4]$, then $(a_0,\dots,a_i,a_{i+2},\dots a_{k-1},a_0)$ is a shorter cycle of $<^e_\wedge$ relations, a contradiction with $\mathcal{C}$.

It follows by Lemma 4.4 and the previous argument that $a_2 = (3,4)$, $a_3 = (4,1)$, and $a_5 = (1,2)$, a contradiction with Lemma 4.5. $\square$

Note that the $<^e_{\parallel}$ relation does not exists in $(6,5)$-rotation systems. Even though the $<^e_{\wedge}$ relation induces an acyclic graph, it is not known in which order two uncrossed disjoint edges f and g cross e. If the induced rotation system of f and g has a planar representation, then for each face, exactly one of f or g bounds that face. If the induced rotation system of f and g has a realization that is a crossing K_4, then in Lemma 4.8, we show that the rotation system implies that the oriented crossings of e with f and e with g are from opposite sides of the uncrossed 4-cycle.

When drawing a realization of a $(6,5)$ rotation system, we choose to draw a specific edge e and depending on the current non-vertex end of our partially drawn edge, it follows that at most one of f and g can be crossed at this time. We proceed by proving Lemma 4.7 and using it as a tool to prove Lemma 4.8.

Lemma 4.7. *Let $n \geq 6$, H_n be an $(n,5)$-rotation system, and $\{x,a,b,c\} \subset V(H_n)$. If $\pi_{a,b,c}(x) = [a,b,c]$ and e is a directed edge of H_n that crosses $\overrightarrow{(x,a)}$, $\overrightarrow{(x,b)}$, and $\overrightarrow{(x,c)}$ from left to right, then the order of these three crossing on e is a cyclic permutation of $[(x,a),(x,b),(x,c)]$.*

Proof. Let $\pi_{a,b,c}(x) = [a,b,c]$ and suppose $e = \overrightarrow{(u,v)}$ is a directed edge that crosses all (x,i) from left to right for $i \in \{a,b,c\}$. Define D_i for $i \in \{a,b,c\}$ to be the realization of the 5-vertex rotation system defined on $(\{a,b,c\}\backslash i) \cup \{x,u,v\}$.

By the symmetry of a,b and c, suppose e crosses (x,a) first. By way of contradiction, assume $(x,a) <^e_{\wedge} (x,c) <^e_{\wedge} (x,b)$. Since the oriented crossing of e and (x,i) are determined, and the order e crosses the edges (x,i) is determined, Observation 4.3 tells us the rotations at x in each D_i are determined. In particular, $\pi_{a,b,u,v}(x) = [u,a,b,v]$, $\pi_{a,c,u,v}(x) = [u,a,c,v]$, and $\pi_{b,c,u,v}(x) = [u,c,b,v]$. Combining these rotations gives $\pi_{a,b,c,u,v}(x) = [u,a,c,b,v]$, a contradiction with $\pi_{a,b,c}(x) = [a,b,c]$. $\square$

Lemma 4.8. *Let H be a $(6,5)$-rotation system, $e = \overrightarrow{(u,v)}$ be a directed edge of H, and f and g two uncrossed edges from a crossing 4-vertex rotation system H_4 in H such that (u,v) crosses both f and g. In H, if there is no chain of $<^e_{\wedge}$ relations ordering f and g, then e crosses f and g from different sides of the uncrossed 4-cycle in H_4.*

Proof. Let H be a $(6,5)$-rotation system, $e = \overrightarrow{(u,v)}$ be a directed edge of H, f and g two uncrossed edges from a crossing 4-vertex rotation system H_4 in H such that e crosses both f and g such that there is not chain of $<^e_\wedge$ relations ordering f and g, and set $V(H_4) = [4]$. Suppose in H that there is no chain of $<^e_\wedge$ relations ordering f and g.

Without loss of generality, assume $e = \overrightarrow{(u,v)}$ and $V(H_4) = [4]$. For $i \in [4] \cup \{u,v\}$, define D_i to be the realization of H_i and D_e to be the realization of $H - \{u,v\}$.

Some edge in D_e crosses another and so without loss of generality, let $\overrightarrow{(1,4)}$ crosses $\overrightarrow{(2,3)}$ from left to right in D_e. Notice that this prescribes the oriented drawing D_e. Without loss of generality, we set $f = (1,2)$ and $g = (3,4)$. We make use of the following observation.

Observation 4.9. *Let u be in side S of a 3-cycle. If v is in S, then e crosses out of S the same number of times it crosses into S. Similarly, if v is not in S, then e crosses out of S exactly once more than it crosses into S.*

For each uncrosses edge (i,j) in D_e, define the face that has both the crossing of $(1,4)$ with $(2,3)$ and the edge (i,j) on the boundary to be $F_{(i,j)}$. Let the fifth and final face of D_e be F_4. To be clear, F_4 is the face that does not have the crossing on the boundary (the side of the uncrossed 4-cycle that does not contain the crossing).

We note the following two facts that will be used extensively in the proofs of the upcoming cases.

Fact 1: Let $y \in \{(1,2),(3,4)\}$, $x \in \{(1,3),(1,4)(2,3)(2,4)\}$ and $z \in \{(1,2),(3,4)\}\backslash\{y\}$. If $x <^e_\wedge y$ ($y <^e_\wedge x$), then $x <^e_\wedge z$ ($z <^e_\wedge x$).

Fact 2: Let C be a 3-cycle. Given the oriented crossings of C with e, the locations of u and v relative to C, and a pair of edges of the three cycle the relation $<^e_\wedge$ is known, then the order the edges of the 3-cycle are crossed is determined.

Fact 1 is an immediate consequence of there not being a chain of $<^e_\wedge$ relations ordering y and z. Fact 2 follows from the consecutive crossings of a C come from opposite sides of C.

To prove the lemma, there are 8 cases up to symmetry of which faces in D_e contain u and v from D_v and D_u, respectively. The cases are as follows with the first five cases having neither u nor v in F_4, while the last three have at least one in F_4:

1. u and v are both in $F_{(1,2)}$;

2. u is in $F_{(1,2)}$ and v is in $F_{(3,4)}$;

3. u is in $F_{(1,2)}$ and v is in $F_{(1,3)}$;

4. u and v are both in $F_{(1,3)}$;

5. u is in $F_{(1,3)}$ and v is in $F_{(2,4)}$;

6. u is in $F_{(1,3)}$ and v is in F_4;

7. u is in $F_{(1,2)}$ and v is in F_4; and

8. u and v are in F_4.

Note that, by way of contradiction, in all cases e crosses $(1,2)$ and $(3,4)$ into F_4 or out of F_4, respectively. This forms 16 cases in total. Case 1.1 will give a detailed explanation how Observation 4.9 determines the oriented crossings of e with edges of D_e. In the cases following Case 1.1, we will apply Observation 4.9 to determine the oriented crossings without explanation.

For each case, we offer Figures 14-29 that on the left describe the faces containing the vertices u and v and the direction e crosses $(1,2)$ and $(3,4)$ and on the right the implied crossings of e with the remaining edges.

Case 1.1. *u and v are both in $F_{(1,2)}$, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

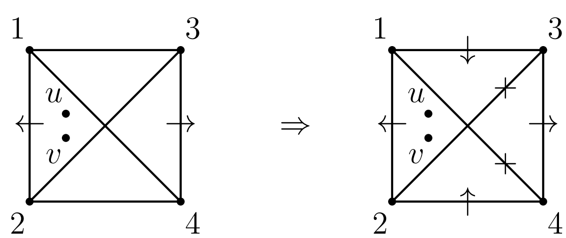


Figure 14: Lemma 4.8 Case 1.1

Because u and v are both in $F_{(1,2)}$, e crosses each 3-cycle in D_e an even number of times and the orientations of these crossings are known. By Observation 4.9, since e crosses out of $\overrightarrow{(1,3,2)}_R$ at $(1,2)$, Observation 4.9 shows e does not cross out of $\overrightarrow{(1,3,2)}_R$ at $(2,3)$.

By Observation 4.9, since e crosses out of $\overrightarrow{(2,3,4)}_R$ at $(3,4)$ and u and v are not inside $\overrightarrow{(2,3,4)}_R$, e does not cross $(2,3)$ out of $\overrightarrow{(2,3,4)}_R$. Combined with the preceding paragraph, e does not cross $(2,3)$. Therefore, e crosses into $\overrightarrow{(1,3,2)}_R$ at $(1,3)$.

By the same arguments, e does not cross $(1,4)$, and e crosses $(1,3)$ and $(2,4)$ into $F_{(1,3)}$ and $F_{(2,4)}$, respectively. Note that each of the 3-cycles on vertices $[4]\backslash\{j\}$, for $j \in [4]$, are crossed exactly twice and the oriented crossings are known.

Since each 3-cycle is crossed twice and we know the oriented crossings and the order in which e crosses the edges of each 3-cycle, Observation 4.3 shows $\pi_{2,3,u,v}(1) = [u,2,3,v]$, $\pi_{1,4,u,v}(2) = [v,4,1,u]$, $\pi_{1,4,u,v}(3) = [v,4,1,u]$, and $\pi_{2,3,u,v}(4) = [u,2,3,v]$. From these rotations and the oriented crossing of $(1,4)$ with $(2,3)$, it follows that D_u is uniquely determined with $\pi_{1,2,3,4}(v) = [1,2,4,3]$.

Since e crosses $\overrightarrow{(1,2)}$ from left to right and $\overrightarrow{(3,4)}$ from right to left, it follows that $\pi_{1,2,u}(v) = [1,u,2]$ and $\pi_{3,4,u}(v) = [4,u,3]$. By the existence of the rotation at v, since we can not combine these rotations at v, we have a contradiction.

Case 1.2. *u and v are both in $F_{(1,2)}$, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F_{(3,4)}$, respectively.*

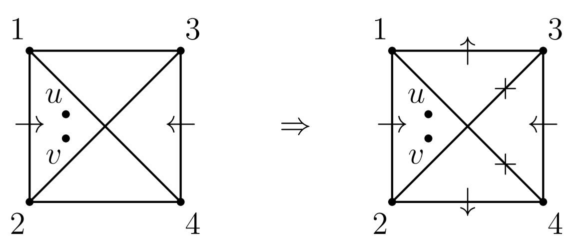


Figure 15: Lemma 4.8 Case 1.2

Redirecting e to go from v to u converts this to Case 1.1, and resolves the case.

Case 2.1. *u in in $F_{(1,2)}$ and v is in $F_{(3,4)}$, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

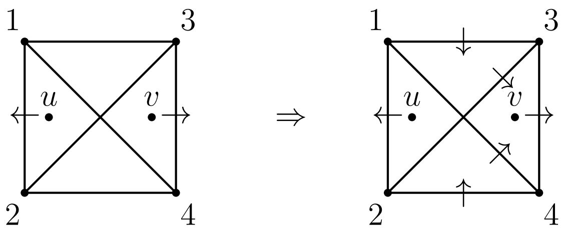


Figure 16: Lemma 4.8 Case 2.1

By Observation 4.9, e crosses $(1,3)$, $(2,4)$, $(1,4)$, and $(2,3)$ into $F_{(1,3)}$, $F_{(2,4)}$, $\overrightarrow{(1,3,4)}_R$, and $\overrightarrow{(2,3,4)}_R$, respectively, and none of the other remaining edges.

If $(2,3) <^e_\wedge (3,4)$, then $(2,3) <^e_\wedge (1,2)$, by Fact 1. By Fact 2 on $(1,2,3)$, $(2,3) <^e_\wedge (1,3) <^e_\wedge (1,2)$. By Fact 1, $(1,3) <^e_\wedge (3,4)$. It follows that $(2,3) <^e_\wedge (1,3) <^e_\wedge (3,4)$, a contradiction with $x = 3$ in Lemma 4.7.

If $(3,4) <^e_\wedge (2,3)$, then by Fact 2 on $(2,3,4)$, $(2,4) <^e_\wedge (3,4) <^e_\wedge (2,3)$. By Fact 1, $(2,4) <^e_\wedge (1,2)$. By Fact 2 on $(1,2,4)$, $(1,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$. It follows that $(1,4) <^e_\wedge (2,4) <^e_\wedge (3,4)$, a contradiction with $x = 4$ in Lemma 4.7.

Case 2.2. *u in in $F_{(1,2)}$ and v is in $F_{(3,4)}$, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F_{(3,4)}$, respectively.*

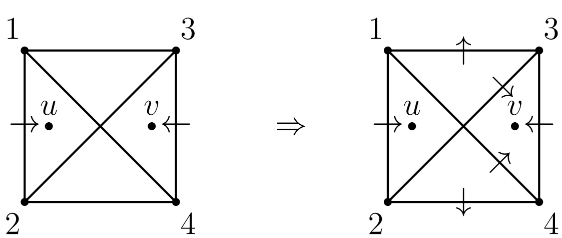


Figure 17: Lemma 4.8 Case 2.2

Redirecting e to go from v to u in Case 2.1 resolves this case.

Case 3.1. *u is in $F_{(1,2)}$ and v is in $F_{(1,3)}$, and e crosses* $(1,2)$ *and* $(3,4)$ *into F_4.*

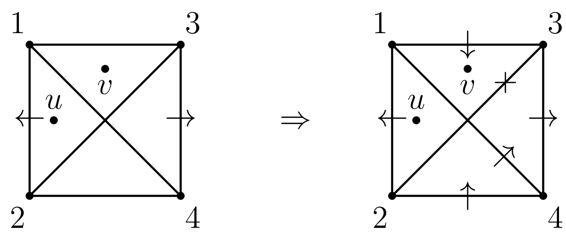


Figure 18: Lemma 4.8 Case 3.1

By Observation 4.9, e crosses $(1,3)$, $(2,4)$, and $(1,4)$ into $F_{(1,3)}$, $F_{(2,4)}$, and $\overrightarrow{(1,3,4)}_R$, respectively, and none of the other remaining edges. By Fact 2 on $(1,2,3)$, $(1,2) <^e_\wedge (1,3)$.

By Fact 1, $(3,4) <^e_\wedge (1,3)$.By Fact 2, $(1,4) <^e_\wedge (3,4) <^e_\wedge (1,3)$. By Fact 1, $(1,4) <^e_\wedge (1,2)$. By Fact 2 on $(1,2,4)$, $(1,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$. By Fact 1, $(2,4) <^e_\wedge (3,4)$. It follows that $(1,4) <^e_\wedge (2,4) <^e_\wedge (3,4)$, a contradiction with $x = 4$ in Lemma 4.7.

Case 3.2. *u is in $F_{(1,2)}$ and v is in $F_{(1,3)}$, and e crosses* $(1,2)$ *and* $(3,4)$ *into $F_{(1,2)}$ and $F_{(3,4)}$, respectively.*

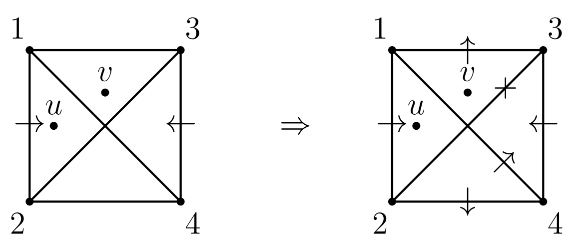


Figure 19: Lemma 4.8 Case 3.2

By Observation 4.9, e crosses $(1,3)$, $(2,4)$, and $(1,4)$ into F_4, F_4, and $\overrightarrow{(1,3,4)}_R$, respectively, and none of the other remaining edges. Fact 2 on $(2,3,4)$ shows $(3,4) <^e_\wedge (2,4)$. By Fact 1, $(1,2) <^e_\wedge (2,4)$. Fact 2 on $(1,2,4)$ shows $(1,4) <^e_\wedge (1,2) <^e_\wedge (2,4)$.

From the oriented crossings, and $(1,4) <^e_\wedge (1,2) <^e_\wedge (2,4)$, by Observation 2.9, D_3 is determined and is Harborth. In particular, $\pi_{2,4,v}(1) = [2,4,v]$.

It follows that in D_u, the edge $(1, v)$ starts in $F_{(1,2)}$ and ends in $F_{(1,3)}$, a contradiction with D_u being simple.

Case 4.1. *u and v are both in $F_{(1,3)}$, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

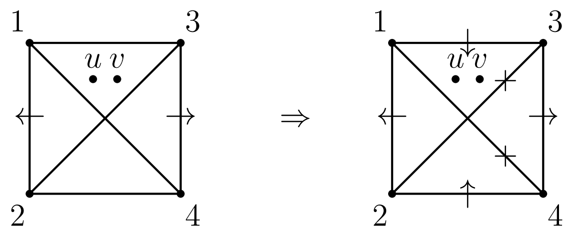


Figure 20: Lemma 4.8 Case 4.1

By Observation 4.9, e crosses $(1,3)$ and $(2,4)$ into $F_{(1,3)}$ and $F_{(2,4)}$, respectively, and none of the other remaining edges. By Fact 2 on $(1,2,3)$, $(1,2) <^e_\wedge (1,3)$.

Given the ordering e crosses $(1,2)$ and $(1,3)$ along with the oriented crossings of e with $(1,2)$ and $(1,3)$, the rotation at 1 is determined in D_4. In particular, $\pi_{2,3,u,v}(1) = [u, 2, 3, v]$. The same arguments apply to every 3-cycle with e. It follows that $\pi_{1,4,u,v}(2) = [v, 1, 4, u]$, $\pi_{1,4,u,v}(3) = [v, 1, 4, u]$, and $\pi_{2,3,u,v}(4) = [u, 2, 3, v]$.

Given the location of v in D_e along with these rotations determines D_u, in particular $\pi_{1,2,3,4}(v) = [1, 2, 4, 3]$. The oriented crossings of e with $(1,2)$ and $(3,4)$ determine that $\pi_{1,2,u}(v) = [1, u, 2]$ and $\pi_{3,4,u}(v) = [4, u, 3]$, respectively. Since the three rotations at v can not be combined, it follows that $\pi_{1,2,3,4,u}(v)$ is not well defined, a contradiction.

Case 4.2. *u and v are both in $F_{(1,3)}$, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F_{(3,4)}$, respectively, and none of the other remaining edges.*

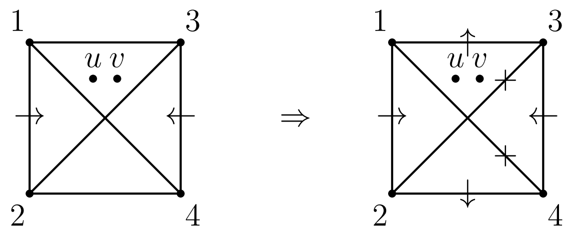


Figure 21: Lemma 4.8 Case 4.2

Redirecting the edge e from v to u and applying the same arguments as Case 4.1 will result in the same conclusion on $\pi_{1,2,3,4,v}(u)$.

Case 5.1. *u is in $F_{(1,3)}$ and v is in $F_{(2,4)}$, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

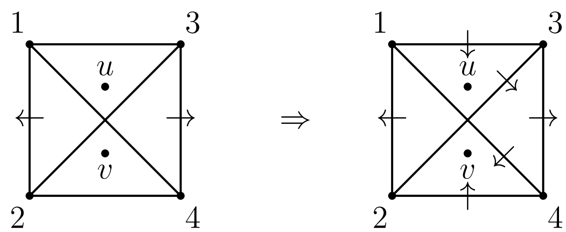


Figure 22: Lemma 4.8 Case 5.1

By Observation 4.9, e crosses $(1,3)$, $(2,4)$, $(1,4)$, and $(2,3)$ into $F_{(1,3)}$, $F_{(2,4)}$, $\overrightarrow{(1,4,2)}_R$ and $\overrightarrow{(2,3,4)}_R$, respectively.

If $(2,3) <^e_\wedge (3,4)$, then $(2,3) <^e_\wedge (1,2)$ by Fact 1.Fact 2 on $(1,2,3)$ shows $(2,3) <^e_\wedge (1,3) <^e_\wedge (1,2)$. Fact 1 then implies $(1,3) <^e_\wedge (3,4)$. Finally, it follows $(2,3) <^e_\wedge (1,3) <^e_\wedge (3,4)$, a contradiction with $x = 3$ in Lemma 4.7.

It follows that $(3,4) <^e_\wedge (2,3)$. This along with the oriented crossings of e with $(3,4)$ and $(2,3)$ imply the rotation at 3 is determined in D_1, in particular $\pi_{2,4,u}(3) = [u,2,4]$. u in $F_{(1,3)}$ in D_v and $(3,u)$ starting in $F_{(3,4)}$ contradicts the fact that D_v is a simple drawing.

Case 5.2. *u is in $F_{(1,3)}$ and v is in $F_{(2,4)}$, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F_{(3,4)}$, respectively.*

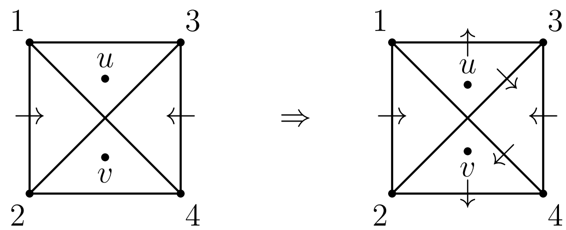


Figure 23: Lemma 4.8 Case 5.2

Redirecting the edge e from v to u and applying the same arguments as Case 5.1 will result in the same conclusion on v in D_u.

Case 6.1. *u is in $F_{(1,3)}$ and v is in F_4, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

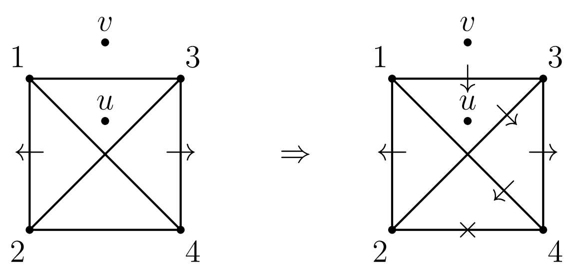


Figure 24: Lemma 4.8 Case 6.1

By Observation 4.9, e crosses $(1,3)$, $(1,4)$, and $(2,3)$ into $F_{(1,3)}$, $\overrightarrow{(1,4,2)}_R$ and $\overrightarrow{(2,3,4)}_R$, respectively, and none of the other remaining edges.

Applying Fact 2 to $(1,2,4)$ gives $(1,4) <^e_\wedge (1,2)$. Fact 1 implies $(1,4) <^e_\wedge (3,4)$. Applying Fact 2 to $(1,3,4)$ gives $(1,4) <^e_\wedge (1,3) <^e_\wedge (3,4)$. By Fact 1, $(1,3) <^e_\wedge (1,2)$. Fact 2 on $(1,2,3)$ gives $(2,3) <^e_\wedge (1,3) <^e_\wedge (1,2)$. It follows that $(2,3) <^e_\wedge (1,3) <^e_\wedge (3,4)$, a contradiction with $x = 3$ in Lemma 4.7.

Case 6.2. *u is in $F_{(1,3)}$ and v is in F_4, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F(3,4)$, respectively.*

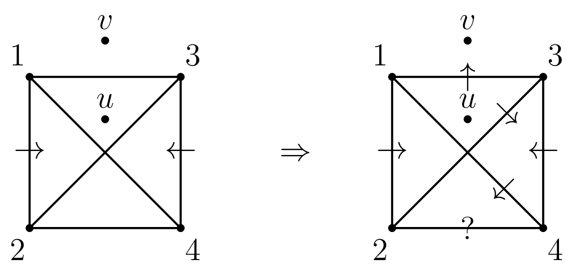


Figure 25: Lemma 4.8 Case 6.2

By Observation 4.9, e crosses $(1,3)$, $(1,4)$ and $(2,3)$) into F_4, $\overrightarrow{(1,4,2)}_R$ and $\overrightarrow{(2,3,4)}_R$, respectively. Observe that $\overrightarrow{(2,3,4)}_R$ has e crossing into it twice and has both u and v on the same side, a contradiction with Observation 4.9.

Case 7.1. *u is in $F_{(1,2)}$ and v is in F_4, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

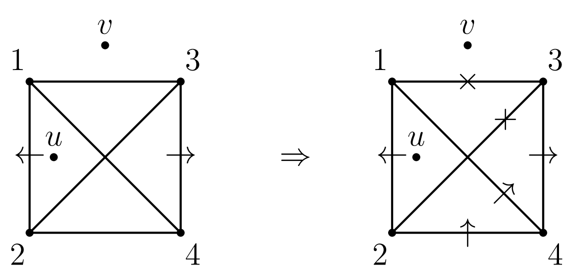


Figure 26: Lemma 4.8 Case 7.1

By Observation 4.9, either e crosses $(1,4)$ and $(2,4)$ into $\overrightarrow{(1,3,4)}_R$ and $F_{(2,4)}$, respectively, or e crosses $(2,3)$ and $(1,3)$ into $\overrightarrow{(2,3,4)}_R$ and $F_{(1,3)}$, respectively. A change of labelling of 3 maps to 4 and 1 maps to 2 implies that both cases are the same. Therefore, without loss of generality e crosses $(1,4)$ and $(2,4)$ into$\overrightarrow{(1,3,4)}_R$ and $F_{(2,4)}$, respectively.

Applying Fact 2 to $(2,3,4)$ gives $(2,4) <^e_\wedge (3,4)$. By Fact 1, $(2,4) <^e_\wedge (1,2)$. Fact 2 on $(1,2,4)$ gives $(1,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$. It follows that $(1,4) <^e_\wedge (2,4) <^e_\wedge (3,4)$, a contradiction with $x = 4$ in Lemma 4.7.

Case 7.2. *u is in $F_{(1,2)}$ and v is in F_4, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F_{(3,4)}$, respectively.*

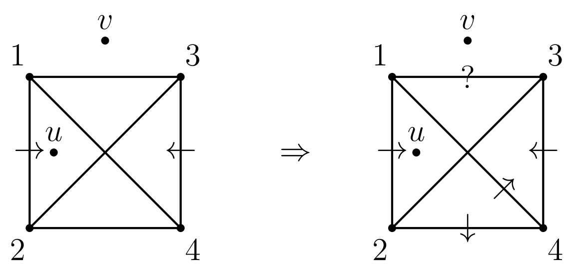


Figure 27: Lemma 4.8 Case 7.2

Applying Observation 4.9 on $(1,2,4)$, gives e crosses out of $\overrightarrow{(1,4,2)}_R$ at both $(1,4)$ and $(2,4)$. It follows that e crosses into $\overrightarrow{(1,3,4)}_R$ twice, once at $(3,4)$ and once at $(1,4)$ implying $v \in \overrightarrow{(1,3,4)}_R$, a contradiction with v in F_4.

Case 8.1. *u and v are in F_4, and e crosses $(1,2)$ and $(3,4)$ into F_4.*

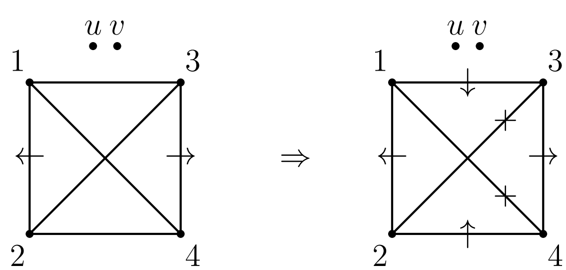


Figure 28: Lemma 4.8 Case 8.1

By Observation 4.9, e crosses $(1,3)$ and $(2,4)$ into $F_{(1,3)}$ and $F_{(2,4)}$, respectively. Applying Fact 2 to $(1,2,3)$ gives, $(1,3) <^e_\wedge (1,2)$.

Since the oriented crossings of e with $(1,3)$ and $(1,2)$ are known and the order e crosses them is known, the rotation at 1 is determined in D_4, in particular $\pi_{2,3,v}(1) = [3,2,v]$. By similar arguments $\pi_{1,4,v}(2) = [v,1,4]$, $\pi_{1,4,v}(3) = [v,4,1]$, and $\pi_{2,3,v}(4) = [2,3,v]$.

Applying these rotations to D_e determines the simple drawing D_u, in particular $\pi_{1,2,3,4}(v) = [1,2,4,3]$. From the oriented crossings of e with $(1,2)$ and $(3,4)$, it follows that $\pi_{1,2,u}(v) = [1,u,2]$ and $\pi_{3,4,u}(v) = [4,u,3]$. The three rotations at v can not be combined, a contradiction with $\pi_{1,2,3,4,u}(v)$ being well defined.

Case 8.2. *u and v are in F_4, and e crosses $(1,2)$ and $(3,4)$ into $F_{(1,2)}$ and $F_{(3,4)}$, respectively.*

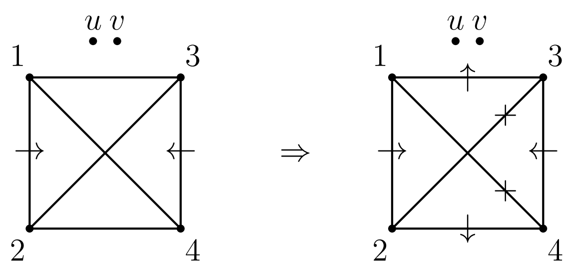


Figure 29: Lemma 4.8 Case 8.2

By Observation 4.9, e crosses $(1,3)$ and $(2,4)$ into F_4 and F_4, respectively. Since the proof of Case 8.1 did not use the fact that $(1,2)$ and $(3,4)$ are not ordered by $<^e_\wedge$ relations, this case follows by symmetry to the previous case. $\square$

We use Lemmas 4.6 and 4.8 to create partial realizations of a (6,5)-rotation system H. We start with Lemma 4.12 that finds a K_4 with an edge e whose endpoints are not in the K_4 that is a partial realization of H. In Lemma 4.13, we extend this result to find a drawing of K_5 with an edge e with exactly one endpoint in the underlying K_5 that is a partial realization of H. Before we do so, we must extend the notion of a simple drawing.

Definition 4.10. Let H_n be an n-vertex rotation system, D be a simple drawing realizing some k-vertex rotation system H_k of H_n for $k < n$, and $e = (u, v)$ be an edge of H_n having u in D and v not in D. If e_a is a partial arc of e starting at u in the sphere, then $D + e_a$ is *simple* if e_a has at most one intersection point with any edge in D.

Definition 4.11. If D is a simple drawing of a graph G such that $u, v \in V(G)$, $(u, v) \notin E(G)$, and e_a is partial arc of e in the sphere starting at u, then $D + e_a$ is *simple* if e_a has at most one intersection point with every edge not incident to v and e_a has no intersection point with any edge incident to v.

Luckily, the definition of a simple drawing of a graph extends nicely to a simple drawing of a graph with a partial arc. We proceed by showing how to find a partial realization of a $(6,5)$-rotation system that is a K_4 with an edge.

Lemma 4.12. *Let H be a $(6,5)$-rotation system on the vertices $[4] \cup \{u, v\}$, $e = \overrightarrow{(u, v)}$, and E be the set of edges that e crosses as determined by H. If D_e is a realization of $H - \{u, v\}$, then there exists a simple drawing $D_e + \{e\}$ that has u and v in their respective faces of D_e as determined by H, the order e crosses the edges of E is consistent with $<^e_\wedge$, and the oriented crossings on e are in the prescribed orientations determined by H.*

Proof. Let D_j be a realization of $H - \{j\}$ for $j \in [4] \cup \{u, v\}$, and D_e be a realization of $H - \{u, v\}$. Let e_i be a segment of e starting at u, crossings exactly i edges satisfying the partial ordering $<^e_\wedge$ in the correct orientation as determined by H, and that ends in a face of D_e for $i \in \{0\} \cup [cr_H(e)]$.

Let D_{e_i} be a drawing of $D_e + e_i$, and E_i be the set of edge e_i crosses in D_{e_i}. It is enough to prove that D_{e_i} exists for all $i \in \{0\} \cup [cr_H(e)]$ inductively on i.

Once this is done, we draw v on the non-vertex end of $e_{cr_H(e)}$ and call this drawing $D_e + \{e\}$. If v is not in its respective face in D_e determined by H, then by Corollary 3.3, there exists some 3-cycle T that has v on opposite sides in H compared to $D_e + \{e\}$.

There is a unique way for e to cross T as defined by the partial ordering $<^e_\wedge$ and the oriented crossing of e with each edge of the 3-cycle. Therefore, any realization of the associated 5-vertex rotation system on e and T has e and T crossing as in $D_e + e$, a contradiction with v being on opposite sides of the 3-cycle in H compared to $D_e + \{e\}$. We continue by proving the inductive statement.

Note that the drawing of D_v that has all the edges sharing u as an endpoint deleted (keeping u in the drawing) satisfies the definition of D_{e_0}.

Assume D_{e_i} exists for some $i \in \{0\} \cup [cr_H(e) - 1]$. By way of contradiction, assume $D_{e_{i+1}}$ does not exist. From the partial ordering $<^e_\wedge$ there is a set C_i of minimal elements in $E \backslash E_i$. It follows that no two edges in C_i share a common endpoint. As C_i is a set of edges in D_e, $|C_i| \leq 2$. Let v_i be the non-vertex end of e_i and R_{v_i} be the face that contains v_i in $D_{e_{i-1}}$ (if $i = 0$, then we let $D_{e_{i-1}} = D_e$).

Extending the drawing D_{e_i} having e_i cross one of the elements in C_i satisfies $<^e_\wedge$ and so we will choose to do so.

For every edge $c \in C_i$, and 3-cycle T containing c in D_e, there are two sides of T. Define the side S_1 to be the side of T bounded by the side of c that e crosses, and the other side of T to be S_2.

If v_i is in S_2, then the induced drawing of T with e_i in D_{e_i} can be extended to a realization of its associated k-vertex rotation system for $k \leq 5$. Since v_i is in S_2, this drawing has e_i not crossing c next, a contradiction with the definition of c being in C_i. Therefore, for every edge $c \in C_i$ and every triangle T containing c, e_i is in the correct side of T to cross c.

If D_e is planar, then every edge in C_i is on R_{v_i} and $|C_i| = 1$ by the $<^e_\wedge$ relation. In this case, we would cross the one edge in C_i to find a drawing of $D_{e_{i+1}}$.

Therefore, D_e is a crossing K_4. If v_i is on the uncrossed side of the uncrossed 4-cycle in D_e, then the $<^e_\wedge$ relation along with Lemma 4.8 imply that there is a unique edge in C_i that e_i can cross on R_{v_i}. Again we cross this edge, to find a drawing of $D_{e_{i+1}}$.

Therefore, v_i is on the crossing side of the uncrossed 4-cycle in D_e. Again if there is a unique edge in C_i to cross, we do so.

If there are two edges in C_i to cross on R_{v_i}, then by the $<^e_\wedge$ relation these edges must be crossing in D_e. In this case, we choose either edge to cross. If there are no such edges to cross, then for any edge $c \in C_i$, e_i is on the correct side of any triangle T containing c.

It follows that $|C_i| = 1$, $c_i \in C_i$ is the unique edge in D_e that has empty intersection with R_{v_i}, the oriented crossing of e with c_i is from the uncrossed side of the uncrossed 4-cycle in D_e to the crosses side.

Without loss of generality, let $c_i = (3, 4)$ and let the clockwise labelling of the uncrossed 4-cycle in D_e from the crossed side be $\overrightarrow{(1, 3, 4, 2)}$. It follows that $\overrightarrow{(1, 4)}$ crosses $\overrightarrow{(2, 3)}$ from left to right and e_i is in the unique face $F_{(1,2)}$ that has $(1, 2)$ and the crossing of $(1, 4)$ with $(2, 3)$ on its boundary in D_e.

For edges (j, k) in the uncrossed 4-cycle, define the other 3 symmetric faces in D_e analogously, and define the face that does not have the crossing of $(1, 4)$ with $(2, 3)$ on the boundary as F_4. Partition the proof into three cases depending on the oriented crossing of e with $(1, 2)$ (e not crossing $(1, 2)$ being one such case).

Case 1. *e crosses $\overrightarrow{(1, 2)}$ from left to right.*

There are two cases to consider, whether $(1, 2)$ has been crossed by e_i or not.

Case 1.1. $(1,2)$ *has been crossed by* e_i.

Since e_i crosses out of $F_{(1,2)}$ at $(1,2)$, it follows that the last edge e_i crossed was not $(1,2)$, in particular $(1,2) <^e_\wedge \cdots <^e_\wedge (3,4)$. After e_i crossed $(1,2)$ it must get back into $F_{(1,2)}$. Without loss of generality, to do so it crosses $(1,2)$ into F_4, then $(2,4)$ into $F_{(2,4)}$, then crosses $(2,3)$ into $F_{(1,2)}$. In particular, $(1,2) <^e_\wedge (2,4) <^e_\wedge (2,3)$

Note by the oriented crossing on $(2,3,4)$, v is in $\overrightarrow{(2,3,4)}_R$. Since $(2,4) <^e_\wedge (2,3)$ and the fact the edges are already crossed, it follows that $(2,4) <^e_\wedge (2,3) <^e_\wedge (3,4)$. Since the oriented crossing are known, by Observation 2.9, the rotations in D_1 are known, and it is Harborth. In particular, $\pi_{2,4,u,v}(3) = [4,v,u,2]$ and v is in $\overrightarrow{(2,3,4)}_R$. Combining this with the rotation at 3 in D_e gives $\pi_{1,2,4,u,v}(3) = [4,v,u,2,1]$.

Since $(1,2) <^e_\wedge (2,3)$ and the oriented crossings are known, it follows in D_4, $\pi_{1,3,u,v}(2) = [v,3,1,u]$. Combining this with the rotation at 2 in D_e gives $\pi_{1,3,4,u,v}(2) = [v,3,4,1,u]$. By the rotation at 2 and the fact $v \in \overrightarrow{(2,3,4)}_R$, in D_u, v is in $F_{(3,4)}$. In D_3, e crosses $(1,2)$, then $(2,4)$ into $\overrightarrow{(1,4,2)}_R$ and must end at v which is outside of $\overrightarrow{(1,4,2)}_R$, therefore e crosses $\overrightarrow{(1,4)}$ from right to left and e_i does not cross $(1,4)$.

Since e_i does not cross $(1,4)$, it follows that $(3,4) <^e_\wedge (1,4)$. By the same analysis on $(1,3,4)$, e crosses $\overrightarrow{(1,3)}$ from right to left and $(3,4) <^e_\wedge (1,3) <^e_\wedge (1,4)$. The rotations in D_2 are determined, in particular, $\pi_{1,4,u,v}(3) = [u,4,1,v]$, a contradiction with $\pi_{1,2,4,u,v}(3) = [4,v,u,2,1]$.

Case 1.2. $(1,2)$ *has not been crossed by* e_i.

Note that $(3,4)$ is the next edge crossed and $(1,2)$ has yet to be crossed. It is clear that there is a chain of $<^e_\wedge$ relations from $(3,4)$ to $(1,2)$, as if not, $(1,2)$ would be an element of C_i, a contradiction with $C_i = \{(3,4)\}$. By symmetry, without loss of generality, assume e crosses one of the edges on $(1,2,4)$ between $(1,2)$ and $(3,4)$.

Again we can assume without loss of generality that it crosses this edge into $\overrightarrow{(1,4,2)}_R$. To justify this, as an example, if the edge crossed was edge $(1,4)$ out of $\overrightarrow{(1,4,2)}_R$ and $(3,4) <^e_\wedge (1,4) <^e_\wedge (1,2)$, then the oriented crossings on $(1,2,4)$ would imply that e crosses $(2,4)$ into $\overrightarrow{(1,4,2)}_R$ and

$(1,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$, in particular $(3,4) <^e_\wedge (1,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$. By Lemma 4.4, it would follow that $(3,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$.

Therefore, either e crosses $y \in \{(2,4),(1,4)\}$ into $\overrightarrow{(1,4,2)}_R$ such that $(3,4) <^e_\wedge y <^e_\wedge (1,2)$. We partition this into the two cases for $y \in \{(2,4),(1,4)\}$.

Case 1.2.1. $y = (2,4)$.

To be clear, in this case e crosses $\overrightarrow{(2,4)}$ from right to left into $F_{(2,4)}$ and $(3,4) <^e_\wedge (2,4) <^e_\wedge (1,2)$. By the oriented crossings of the edges of $(2,3,4)$ and the fact $(3,4) <^e_\wedge (2,4)$, e crosses $\overrightarrow{(2,3)}$ from right to left and $(3,4) <^e_\wedge (2,3) <^e_\wedge (2,4)$. By Observation 2.9, D_1 is Harborth and every rotation in D_1 is determined. In particular, $\pi_{3,4,u,v}(2) = [3,u,v,4]$.

Since $(2,4) <^e_\wedge (1,2)$ and the oriented crossings of e with $(2,4)$ and $(1,2)$ are known, it follows that $\pi_{1,4,u,v}(2) = [v,1,4,u]$. Finally $\pi_{1,3,4}(2) = [1,3,4]$ from D_e. All three of the rotations of 2 can not be combined, therefore we have a contradiction with the existence of $\pi_{1,2,4,u,v}(2)$.

Case 1.2.2. $y = (1,4)$.

In D_v, either $u \in \overrightarrow{(1,4,2,)}_R$ or not.

If $u \notin \overrightarrow{(1,4,2)}_R$, then e_i must have crossed $(2,4)$, then $(2,3)$ to end in $F_{(1,2)}$. In particular, e_i must cross $(2,4)$ into $\overrightarrow{(1,4,2)}_R$. It follows that $(2,4) <^e_\wedge (3,4) <^e_\wedge (1,4) <^e_\wedge (1,2)$, in particular $(2,4) <^e_\wedge (1,4) <^e_\wedge (1,2)$. In D_3, by the oriented crossings e would cross into $\overrightarrow{(1,4,2)}$ at both $(2,4)$ and $(1,4)$ consecutively, a contradiction.

Therefore, $u \in \overrightarrow{(1,4,2)}_R$. By the oriented crossings of the edges of $(1,2,4)$ and the fact $(1,4) <^e_\wedge (1,2)$, it would follows that e crosses $\overrightarrow{(2,4)}$ from left to right and $(2,4) <^e_\wedge (1,4) <^e_\wedge (1,2)$. By the oriented crossings of the edges at 4, the fact that $(2,4) <^e_\wedge (1,4)$, and $(3,4) <^e_\wedge (1,4)$, it follows by Lemma 4.7 that $(2,4) <^e_\wedge (3,4) <^e_\wedge (1,4)$.

Since $(3,4)$ is the next edge e_i must cross, it follows that e_i has already crossed $(2,4)$ and v_i is outside $\overrightarrow{(1,4,2)}_R$. Since $F_{(1,2)}$ is contained in $\overrightarrow{(1,4,2)}_R$, v_i is also inside $\overrightarrow{(1,4,2)}_R$, a contradiction.

Case 2. *e crosses $\overrightarrow{(1,2)}$ from right to left.*

Since e crosses both $(1,2)$ and $(3,4)$ towards the crossing in D_e, it follows by Lemma 4.8 that $(1,2)$ and $(3,4)$ are ordered by $<^e_\wedge$ relations. If $(1,2)$ was the last edge crossed by e_i, then $(3,4)$ would not be the next edge crossed by e_i as $(1,2)$ and $(3,4)$ are ordered by $<^e_\wedge$ relations.

It follows that $(1,2)$ is not the last edge crossed by e_i. If $(1,2)$ has been crossed by e_i, then there is a unique drawing up to symmetry of $D_e + e_i$. Without loss of generality, this drawings has e_i crossing $\overrightarrow{(1,4)}$, $\overrightarrow{(1,3)}$, $\overrightarrow{(2,4)}$, and $\overrightarrow{(2,3)}$ from right to left. The oriented crossings on the edges of $\overrightarrow{(1,4,2)}_R$ and $\overrightarrow{(1,3,4)}_R$ imply that v is in both regions, however, these two regions have empty intersection, a contradiction with D_u.

Therefore, $(1,2)$ has not been crossed by e_i. It follows that $(3,4) <^e_\wedge \cdots <^e_\wedge (1,2)$ as $(3,4)$ is the next edge that e_i must cross. By symmetry, without loss of generality, assume e crosses one of the edges on $(1,2,4)$ between $(1,2)$ and $(3,4)$.

Again we can assume without loss of generality that it crosses this edge into $\overrightarrow{(1,4,2)}_R$. The justification for this is the same as in Case 1.2 with oriented crossings reversed on $(1,2,4)$.

Therefore, either e crosses $y \in \{(1,4),(2,4)\}$ into $\overrightarrow{(1,4,2)}_R$ such that $(3,4) <^e_\wedge y <^e_\wedge (1,2)$. We partition this into the two cases for $y \in \{(1,4),(2,4)\}$.

Case 2.1. $y = (1,4)$.

By the oriented crossings on $(1,3,4)$ and $(3,4) <^e_\wedge (1,4)$ it follows that e crosses $\overrightarrow{(1,3)}$ from right to left and $(3,4) <^e_\wedge (1,3) <^e_\wedge (1,4)$. In particular, $(1,3) <^e_\wedge (1,4) <^e_\wedge (1,2)$, a contradiction with $x = 1$ in Lemma 4.7.

Case 2.2. $y = (2,4)$.

Note that if $(1,4)$ is not crossed, then by the oriented crossings of the edges of $(1,2,4)$, v is in $\overrightarrow{(1,4,2)}_R$. If $(1,4)$ is crossed, then by the oriented crossings of the edges of $(1,2,4)$ or $(1,3,4)$, v is in exactly one of $\overrightarrow{(1,4,2)}_R$ or $\overrightarrow{(1,3,4)}_R$. Therefore, v is in either $\overrightarrow{(1,3,4)}_R$ or $\overrightarrow{(1,4,2)}_R$. We consider these two cases seperately.

Case 2.2.1. $v \in \overrightarrow{(1,3,4)}_R$.

Since v is not in $\overrightarrow{(1,4,2)}_R$ and u is, it follows by the oriented crossings of the edges in $(1,2,4)$ and the fact that $(2,4) <^e_\wedge (1,2)$ that e crosses $\overrightarrow{(1,4)}$ from left to right and $(2,4) <^e_\wedge (1,2) <^e_\wedge (1,4)$.

Since $(3,4) <^e_\wedge (2,4) <^e_\wedge (1,4)$, it follows by Lemma 4.4 that $(3,4) <^e_\wedge (1,4)$. These relations along with the oriented crossings of $(1,3,4)$ implies that $\overrightarrow{(1,3)}$ is crossed from right to left and that $(3,4) <^e_\wedge (1,3) <^e_\wedge (1,4)$.

By Observation 2.9, D_2 is Harborth and the rotations are determined. In particular, $\pi_{3,4,u,v}(1) = [3,u,v,4]$. If u was in $F_{(2,4)}$, then e_i crosses $\overrightarrow{(2,3)}$ from right to left and $(2,3) <^e_\wedge (3,4) <^e_\wedge (1,3)$, a contradiction with $x = 3$ in Lemma 4.7.

Therefore, u in in $F_{(1,2)}$. Knowing the rotation at 1 along with the location of u in D_v, implies that D_v is not simple, a contradiction,

Case 2.2.2. $v \in \overrightarrow{(1,4,2)}_R$.

Since both u and v are in $\overrightarrow{(1,4,2)}_R$ and not $\overrightarrow{(1,3,4)}_R$, it follows that e does not cross $(1,4)$ and e crosses $\overrightarrow{(1,3)}$ form right to left. Note that $(3,4) <^e_\wedge (1,3)$ as $\overrightarrow{(1,3,4)}_R$ does not contain u or v and so e must cross into the $\overrightarrow{(1,3,4)}_R$ at $(3,4)$, then out at $(1,3)$ in D_2.

By Observation 4.3, it follows that $\pi_{1,4,u,v}(3) = [u,4,1,v]$. Combining this with the rotation at 3 in D_e gives $\pi_{1,2,4,u,v}(3) = [u,4,2,1,v]$ Applying the same arguments to $(1,2,4)$ and $(2,3,4)$, and combining the rotations in D_e gives $\pi_{1,3,4,u,v}(2) = [u,4,3,1,v]$ and $\pi_{1,2,3,u,v}(4) = [v,2,1,3,u]$, respectively.

If e crosses $\overrightarrow{(2,3)}$, then it does so from left to right as $u \in F_{(1,2)} \subset \overrightarrow{(1,3,2)}_R$. Since $(3,4) <^e_\wedge (2,4)$, it would follow by the oriented crossings of the edges of $(2,3,4)$ that $(3,4) <^e_\wedge (2,4) <^e_\wedge (2,3)$, in particular $(2,4) <^e_\wedge (2,3)$. Setting $x = 2$, Lemma 4.7 implies that $(2,4) <^e_\wedge (1,2) <^e_\wedge (2,3)$. Since $(1,2) <^e_\wedge (2,3)$, it follows by the oriented crossings of the edges of $(1,2,3)$ that $(1,3) <^e_\wedge (1,2) <^e_\wedge (2,3)$. Since order of the edges crossed in $(1,2,3)$ and e are known, and the oriented crossings of e with these edges are known, it follows that the rotations of the vertices in D_4 are known. In particular, by Observation 2.9, D_4 is Harborth and $\pi_{1,3,u,v}(2) = [3,v,u,1]$, a contradiction with $\pi_{1,3,4,u,v}(2) = [u,4,3,1,v]$.

Therefore, e does not cross $(2,3)$. It follows by the oriented crossings that both u and v are in $F_{(1,2)}$. The oriented crossings on $(1,2,4)$ and the positions of u and v imply that in D_3, $\pi_{1,4,u,v}(2) = [u,4,1,v]$. Applying the same arguments to $(1,2,3)$, $(1,3,4)$, and $(2,3,4)$ gives $\pi_{2,3,u,v}(1) = [v,2,3,u]$, $\pi 1,4,u,v(3) = [u,4,1,v]$, and $\pi_{2,3,u,v}(4) = [v,2,3,u]$, respectively. In D_u, $v \in F_{(1,2)}$, the partial rotation at the vertices $[4]$, and the fact that D_u is simple, implies the rotations at the vertices in D_u are determined, in particular $\pi_{1,2,3,4}(v) = [1,2,4,3]$.

The oriented crossings of e with $(2,4)$ and $(1,3)$ give $\pi_{2,4,u}(v) = [2,u,4]$ and $\pi_{1,3,u}(v) = [3,u,1]$, respectively. The three rotations at v can not be combined, a contradiction with the existence of $\pi_{1,2,3,4,u}(v)$.

Case 3. *e does not cross $(1,2)$.*

Up to symmetry, partition this cases into 4 separate cases depending on the number i of edges e_i crosses. Note that the Figures from this point on are representative of the information at hand, specifically with the edge $(3,4)$ being crosses next by $<^e_\wedge$.

Partition this cases into 5 separate cases up to symmetry on the edges e_i crosses. If $i = 0$, then e_i is determined in D_{e_i}. If $i \geq 1$, then without loss of generality e_i crosses $\overrightarrow{(1,4)}$ from left to right to enter $F_{(1,2)}$. If $i = 1$, then u is in $F_{(1,3)}$. If $i = 2$, then u starts in $F_{(3,4)}$ and e_i crosses $(2,3)$ then $(1,4)$ to end in $F_{(1,2)}$, or $u \in F_4$ and e_i crosses $(1,3)$ then $(1,4)$ to end in $F_{(1,2)}$. If $i \geq 3$, then e_i crosses $(2,4)$ then $(1,3)$ then $(1,4)$ to end in $F_{(1,2)}$.

Case 3.1. $i = 0$.

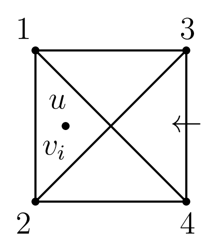


Figure 30: $i = 0$

This cases gives the least amount of crossing information. At this point we know that $(3,4)$ is the next edge e crosses by $<^e_\wedge$ and that $v_i = v_0 = u$ in this instance is inside $F_{(1,2)}$ as in Figure 30.

Either e does not cross $(1,4)$ or $(2,3)$, or e does cross $(1,4)$ or $(2,3)$. If e does cross $(1,4)$ or $(2,3)$, then we can choose the labelling so that it crosses $(1,4)$. In this instance, we split into two cases depending on the oriented crossing and therefore the 3 cases are e does not cross $(1,4)$ or $(2,3)$, e crosses $\overrightarrow{(1,4)}$ from right to left, or e crosses $\overrightarrow{(1,4)}$ from left to right.

Case 3.1.1. *e does not cross $(1,4)$ or $(2,3)$.*

At this point note that e must cross one of $(1,3)$ or $(2,4)$, as if it did not, then the oriented crossings (or lack there of) would imply that v is in both $F_{(3,4)}$ and $F_{(1,2)}$, a contradiction. Since $(1,3)$ and $(2,4)$ are symmetric up to relabelling, we assume without loss of generality e crosses $\overrightarrow{(2,4)}$ from left to right (note the direction e crosses $(2,4)$ is determined). Partition this into two final cases depending on e crossing $(1,3)$ or not.

Case 3.1.1.1. *e crosses $(1,3)$.*

Since u is in $\overrightarrow{(1,3,2)}_R$, e must cross $\overrightarrow{(1,3)}$ from right to left since it is the only edge crossed on $(1,2,3)$. Since $(3,4)$ is the next edge e_i must cross, it follows that $(3,4) <^e_\wedge (1,3)$ and $(3,4) <^e_\wedge (2,4)$. We can also note that by the oriented crossings that $v \in F_4$.

By the oriented crossings of the edges in $(1,3,4)$ and $(2,3,4)$ and the order these edges are crossed by e, it follows that in D_2 and D_1 that $\pi_{1,4,u,v}(3) = [u,4,1,v]$ and $\pi_{2,3,u,v}(4) = [v,2,3,u]$, respectively. Combining these with their respective rotations in D_e gives $\pi_{1,2,3,u,v}(4) = [v,2,1,3,u]$ and $\pi_{1,2,4,u,v}(3) = [u,4,2,1,v]$.

Note that all the oriented crossings of the edges in $(1,2,4)$ with e are known, along with the rotation at 4. It follows that $D_3 - \{(1,v),(2,v)\}$ is determined. In particular, $\pi_{2,4,u}(1) = [4,u,2]$ and $\pi_{1,2,4,v}(u) = [1,v,2,4]$. Similarly on $(1,2,3)$ in $D_4 - \{(1,v),(2,v)\}$, $\pi_{1,3,u}(2) = [1,u,3]$ and $\pi_{1,2,3,v}(u) = [2,3,1,v]$.

The oriented crossing of e with $(3,4)$ gives $\pi_{3,4,v}(u) = [4,v,3]$. Combining the rotations at u gives $\pi_{1,2,3,4,v}(u) = [1,v,2,3,4]$. The rotations at 1,2,3 and 4 along with $u \in F_{(1,2)}$ implies that D_v is determined. In particular, $\pi_{1,2,3,4}(u) = [1,2,4,3]$, a contradiction with $\pi_{1,2,3,4,v}(u) = [1,v,2,3,4]$.

Case 3.1.1.2. *e does not cross $(1,3)$.*

Since $(3,4)$ is the next edge that e_i crosses, it is clear that $(3,4) <^e_\wedge (2,4)$.

By the oriented crossings of the edges on $(2,3,4)$ and the order the edges are crossed, it follows that in D_1, $\pi_{2,3,u,v}(4) = [v,2,3,u]$. Combining the rotation of 4 from D_e results in $\pi_{1,2,3,u,v}(4) = [v,2,1,3,u]$. In D_3, knowing the locations of u and v relative to $(1,2,4)$, e only crosses $(2,4)$ and the rotation at 4, implies $D_3 - \{(1,v),(2,v)\}$ is uniquely determined, in particular, $\pi_{2,4,u}(1) = [2,4,u]$ and $\pi_{1,2,4,v}(u) = [1,v,2,4]$.

Apply the same argument to $(1,3,4)$ and $D_2 - \{(1,u)(3,u)\}$ gives $\pi(1) = [3,v,4]$. From D_e, $\pi_{2,3,4}(1) = [3,4,2]$. Combining the rotations at 1 results in $\pi_{2,3,4,u,v}(1) = [2,3,v,4,u]$. The rotation at 1 implies that $D_2 - \{(3,u)\}$ is uniquely determined from $D_2 - \{(1,u),(3,u)\}$, in particular $\pi_{1,4,v}(u) = [1,4,v]$, a contradiction with $\pi_{1,2,4,v}(u) = [1,v,2,4]$.

Case 3.1.2. *e crosses $\overrightarrow{(1,4)}$ from right to left.*

Since $(3,4)$ is the next edge e_i crosses, it follows that $(3,4) <^e_\wedge (1,4)$. Since e crosses both $(1,4)$ and $(3,4)$ into $\overrightarrow{(1,3,4)}_R$, it follows that e crosses $\overrightarrow{(1,3)}$ from right to left and $(3,4) <^e_\wedge (1,3) <^e_\wedge (1,4)$. Since the order that e crosses the edges on $(1,3,4)$ is known and the oriented crossings are known, it follows that D_2 is uniquely drawn. In particular, by Observation 2.9, D_2 is Harborth and $\pi_{3,4,u,v}(1) = [3,u,v,4]$. Given the rotation at 1 and the fact u is in $F_{(1,2)}$ in D_e, it follows that D_v is not simple, a contradiction.

Case 3.1.3. *e crosses $\overrightarrow{(1,4)}$ from left to right.*

Since u is in $\overrightarrow{(1,4,2)}_R$, it follows that e crosses $\overrightarrow{(2,4)}$ from left to right and $(2,4) <^e_\wedge (1,4)$. Since $i = 0$ and $(3,4)$ is the next edge e_i crosses, it follows that $(3,4) <^e_\wedge (2,4) <^e_\wedge (1,4)$, a contradiction with $x = 4$ in Lemma 4.7.

Case 3.2. *$i = 1$.*

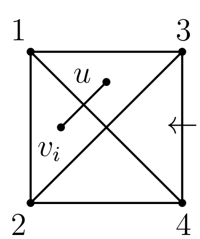


Figure 31: $i = 1$

In this instance, e_i has already crossed one of $(1,4)$ or $(2,3)$ into $F_{(1,2)}$. Up to relabelling, these are the same case, therefore we assume without loss of generality that and e_i crosses $\overrightarrow{(1,4)}$ from left to right into $F_{(1,2)}$ as in Figure 31.

By the oriented crossings on the edges of $(1,2,4)$, either $(2,4)$ is not crossed by e, or e crosses $\overrightarrow{(2,4)}$ from left to right.

Case 3.2.1. *e does not cross* $(2,4)$.

Since $(1,4)$ is crossed by e_i, it follows that $(1,4) <_{\wedge}^{e} (3,4)$. Since the order e crosses $(1,4)$ and $(3,4)$ is known along with the oriented crossings of these edges with e, it follows that in D_2, $\pi_{1,3,u,v}(4) = [v,3,1,u]$. Combining this with the rotation of 4 in D_e gives $\pi_{1,2,3,u,v}(4) = [v,3,2,1,u]$.

The oriented crossings of $(1,2,4)$ imply that $v \in \overrightarrow{(1,4,2)}_R$. This along with the rotation at 4 imply that $v \in F_{(1,2)}$ as D_u is a simple drawing. This implies that v is not in $\overrightarrow{(2,3,4)}_R$ and by the oriented crossing of the edges in $(2,3,4)$ that e crosses $\overrightarrow{(2,3)}$ from right to left. Applying the same arguments to $(1,2,3)$ shows that e crosses $\overrightarrow{(1,3)}$ from right to left. Since u and v are both in $\overrightarrow{(1,3,2)}_R$, it follows by the oriented crossings of the edges that $(1,3) <_{\wedge}^{e} (2,3)$. Since e_i does not cross $(1,3)$ and $(3,4)$ is the next edge that e_i crosses, we have $(3,4) <_{\wedge}^{e} (1,3)$. This implies $(3,4) <_{\wedge}^{e} (1,3) <_{\wedge}^{e} (2,3)$, a contradiction with $x = 3$ in Lemma 4.7.

Case 3.2.2. *e crosses $\overrightarrow{(2,4)}$ from left to right.*

Since e_i crosses $(1,4)$, e_i does not cross $(2,4)$ and the next edge that e_i crosses is $(3,4)$, it follows that $(1,4) <_{\wedge}^{e} (3,4) <_{\wedge}^{e} (2,4)$, a contradiction with $x = 4$ in Lemma 4.7.

Case 3.3. $i = 2$.

Without loss of generality e_i crosses $\overrightarrow{(1,4)}$ from left to right into $F_{1,2}$ as in Case 3.2. Before this crossing occurs, there are two options for how e_i enters $F_{(1,3)}$, either e_i crosses $\overrightarrow{(2,3)}$ from right to left, and u is in $F_{(3,4)}$ or e_i crosses $\overrightarrow{(1,3)}$ from left to right, and u is in F_4.

Case 3.3.1. *e_i crosses $\overrightarrow{(2,3)}$ from right to left, and u is in $F_{(3,4)}$.*

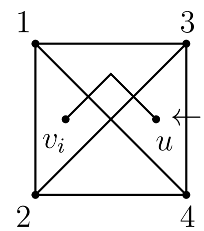


Figure 32: $i = 2$ and e_i crosses $(2,3)$.

If e crosses $\overrightarrow{(2,4)}$, then it does so from left to right as the oriented crossings of the remaining edges of $(2,3,4)$ are determined and $(2,3)$ is the first of the three edges e crosses. Since e_i crosses $(1,4)$, e_i does not cross $(2,4)$ and the next edge e_i crosses is $(3,4)$, it would follow that $(1,4) <^e_\wedge (3,4) <^e_\wedge (2,4)$, a contradiction with $x = 4$ with Lemma 4.7.

Therefore, e does not cross $(2,4)$. By the oriented crossings of $(2,3,4)$, $v \in \overrightarrow{(2,3,4)}_R$ and is not in $\overrightarrow{(1,3,2)}_R$. From the oriented crossings of $(1,2,3)$ and locations of u and v, it follows that e crosses $\overrightarrow{(1,3)}$ from right to left. Since e_i crosses $(2,3)$, e_i does not cross $(1,3)$ and the next edge e_i crosses is $(3,4)$, it follows that $(2,3) <^e_\wedge (3,4) <^e_\wedge (1,3)$, a contradiction with $x = 3$ in Lemma 4.7.

Case 3.3.2. *e_i crosses $\overrightarrow{(1,4)}$ from left to right, e_i crosses $\overrightarrow{(1,3)}$ from left to right, and u is in F_4.*

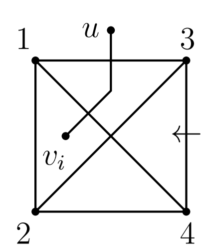


Figure 33: $i = 2$ and e_i crosses $(1,3)$.

By the oriented crossings of $\overrightarrow{(1,3,4)}$, v is in $\overrightarrow{(1,3,4)}_R$. Since $\overrightarrow{(1,3,4)}_R$ and $\overrightarrow{(1,4,2)}_R$ have empty intersection, it follows that v is not in $\overrightarrow{(1,4,2)}_R$. Since

v is not in $\overrightarrow{(1,4,2)}_R$, e_i crosses into $\overrightarrow{(1,4,2)}_R$, and e does not cross $(1,2)$, it follows that e crosses $\overrightarrow{(2,4)}$ from left to right.

Since e_i crosses $(1,4)$, e_i does not cross $(2,4)$, and the next edge e_i crosses is $(3,4)$, it follows that $(1,4) <^e_\wedge (3,4) <^e_\wedge (2,4)$, a contradiction with $x = 4$ in Lemma 4.7.

Case 3.4. $i \geq 3$.

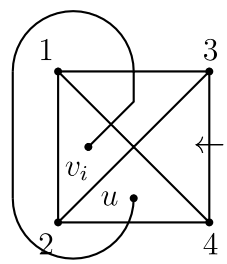


Figure 34: $i = 3$.

Since $i = 3$, the only possibility for the edges that e_i crosses is e_i crossing $(2,4)$ then $(1,3)$ then $(1,4)$ and ends in $F_{(1,2)}$ as in Figure 34. e_i crosses $(2,4)$, then $(1,4)$ and does not cross $(3,4)$. It follows that $(2,4) <^e_\wedge (1,4) <^e_\wedge (3,4)$, a contradiction with $x = 4$ in Lemma 4.7. □

Lemma 4.13. *Let H be a $(6,5)$ rotation system on the vertices $[4] \cup \{u,v\}$, $e = \overrightarrow{(u,v)}$, and E be the set of edges that e crosses determined by H. If D_v is a realization of $H - \{v\}$, then there exists a simple drawing $D_v + \{e\}$ that has v in its respective face in D_v determined by H, e crosses exactly the edges of E satisfying $<^e_\wedge$, the oriented crossings involving e are in the prescribed orientations determined by H, and the rotation at u is the same in both H and $D_v + \{e\}$.*

Proof. Let H be a $(6,5)$-rotation system on the vertices $[4] \cup \{u,v\}$, $e = \overrightarrow{(u,v)}$, D_i be a realization of $H - \{i\}$ for $i \in ([4] \cup \{u,v\}$, D_e be a realization of $H - \{u,v\}$, E be the set of edges that e crosses determined by H, and $\{c_j\}_{j=1}^{cr_H(e)}$ be the sequence in which e crosses the edges of E in Lemma 4.12.

Let e_i be the segment of e starting at u that has exactly i crossings for $i \in [cr_H(e)]_0$, and D_{e_i} be the drawing of $D_v + e_i$. We will prove D_{e_i} exists for all $i \in [cr_H(e)]_0$ inductively on i.

Note that there is no need to talk about the order e crosses edge related to $<_{\wedge}^{e}$ as this is satisfied by the definition of $\{c_j\}_{j=1}^{cr_H(e)}$. By adding a small segment at the correct rotation at u in D_v, we find a simple drawing of D_{e_0}.

Assume D_{e_i} exists for some $i \in [cr_H(e) - 1] \cup \{0\}$. By way of contradiction, assume $D_{e_{i+1}}$ does not exist. Let R_{v_i} be the face in D_v containing the non-vertex end of e_i. There are two cases depending upon the existence of c_{i+1}.

Case 1. c_{i+1} *does not exist.*

If c_{i+1} does not exist, then the edge e_i has crossed all edges of the sequence $\{c_j\}_{j=1}^{cr_H(e)}$ in D_{e_i}. Let T be the intersection of sides of triangles in D_v containing v determined by H. If R_{v_i} does not correspond to T, then by Corollary 3.3, there is some 3-cycle having e_i and v on separate sides, a contradiction with the induced 5-vertex rotation system in H being realizable.

It follows that R_{v_i} correspond to T. Drawing the vertex v at the end of e_i in D_{e_i} produces the desired simple drawing $D_v + \{e\}$.

Case 2. c_{i+1} *exists.*

Let (z, u_{i+1}, v_{i+1}) be some 3-cycle containing $c_{i+1} = (u_{i+1}, v_{i+1})$ in D_{e_i}, $y \in V(H)\backslash\{z, u_{i+1}, v_{i+1}, u, v\}$, and let F_y be the induced drawing of D_{e_i} without y. Since F_y is defined on five vertices, it is clear F_y extends to a realization of H_y. Since c_{i+1} is the next edge crossed in F_y, it follows that e_i is on the correct side S of (z, u_{i+1}, v_{i+1}) to cross c_{i+1} in the correct orientation to extend the drawing. Note that R_{v_i} is contained in S since c_{i+1} is the next edge crossed in F_y and S is defined in D_{e_i}.

By Lemma 3.4, the only segment of e_i in R_{v_i} is the segment formed by e_{i-1} crossing into R_{v_i}. It follows that if c_{i+1} is on the boundary of R_{v_i}, then e_i can be extended in D_{e_i} to cross c_{i+1} in the correct orientation to create $D_{e_{i+1}}$, a contradiction.

Therefore, c_{i+1} is not on the boundary of R_{v_i}. By Theorem 3.8, since there are no Reidemeister III moves in D_v, there is some drawing $\mathcal{D}$ of D_v on a K_4 containing c_{i+1} such that no face of $\mathcal{D}$ contains R_{v_i} and has c_{i+1} on its boundary.

Consider the drawing F of D_{e_i} containing $\mathcal{D}$ and e_i. If both u and v are not in $\mathcal{D}$, then by definition of $\{c_j\}_{j=1}^{cr_H(e)}$ and Lemma 4.12, F can be extended with e_i crossing c_{i+1}, a contradiction with no face of $\mathcal{D}$ containing R_{v_i} and having c_{i+1} on its boundary, and $\mathcal{D}$ being contained in F.

Therefore, at least one of u or v is in $\mathcal{D}$. If follows that $\mathcal{F}$ is a partial drawing of some realizable 5-vertex rotation system Z of H containing u, v, u_{i+1}, v_{i+1}. Thus $\mathcal{F}$ extends to a realization of Z, with e_i crossing c_{i+1} next, a contradiction with no face of $\mathcal{D}$ containing R_{v_i} and having c_{i+1} on its boundary. $\square$

To find a realization of a $(6, 5)$-rotation system, we would like to apply Lemma 4.13 to all possible edges incident to one vertex v, then glue the drawings along their common K_5. However, such a strategy could create a drawing that is not simple since two edges having v as an endpoint could cross multiple times. Therefore, we first create a lemma that finds structure in two tangled edges incident to the same vertex.

At this point, I would like to thank Bruce Richter for offering a beautiful proof of the following lemma about structure in tangled edges.

Lemma 4.14. *Let D be a drawing of a path of length 2. If the two edges have a finite positive number of crossings, then there exists a face of D whose boundary is exactly a non-trivial segment of each edge.*

Proof. Let D be a drawing of a path of length 2 with $e = (v, u_1)$ and $f = (v, u_2)$ being the two crossing edges. Traversing f from v to u_2, let the crossings of e and f be labelled $x_1 = v, x_2, \ldots, x_k$ for some $k \in \mathcal{Z}$. Define B_i to be the bigon on $e[x_i, x_{i+1}] \cup f[x_i, x_{i+1}]$ and S_i to be the closed side of B_i that does not contain u_1.

Claim 1. *There exists an i such that u_1, u_2, v are all in S_i.*

Proof. Either every crossing of e and f is in the same direction, or for some i, x_i and x_{i+1} are crossings of e and f in opposing directions.

Case 1. *Every crossing of e and f is in the same direction.*

Consider B_1. e only crosses $f[x_1, x_2]$ at its ends. For f to cross $e[x_1, x_2]$ it would have to do it at least once in the opposite direction of the crossing at x_2. Since this is not the case, f does not cross $e[x_1, x_2]$ and B_1 is our desired bigon.

Case 2. *There exists an i such that x_i and x_{i+1} are crossings of e and*

f in opposing directions.

Topologically, there is one way to draw $f[x_i, x_{i+1}]$ relative to e, and that is to have v not in S_i. Direct e from v to u_1 and let e_L and e_R be the sides of e (L and R represent left and right, respectively, or right and left respectively). Without loss of generality e_R bounds B_i and there is no path from any point in S_i to e_L (if we consider edges to have width ϵ, then this is natural). By way of contradiction, u_2 is in S_i, else B_i is our desired bigon..

For $i+1 < j$, let x_j be the next time f crosses e in the same direction as the crossing of x_1. We know such a j exists since f must cross e_R into B_i to end at u_2. It follows that B_{j-1} is a bigon with S_{j-1} not containing v. By the orientation of the crossings at x_{j-1} and x_j is follows that S_{j-1} is bounded by e_L and there is no path from a point in S_{j-1} to e_R.

The only way for B_i to intersect B_j would be for $f[x_{j-1}, x_j]$ to intersect B_i. It is clear this does not happen on f unless $x_{i+1} = x_{j-1}$, in which case the intersection is at x_{i+1}. Since $f[x_{j-1}, x_j]$ contains only intersection of e on e_L, it follows that B_i intersects B_{j-1} on at most x_{i+1} (Again, we distinguish between e_L and e_R).

If u_s is not in S_{j-1}, then B_{j-1} is our desired bigon. Therefore u_2 is in S_{j-1}. It follows that one u_1-empty side of a bigon is contained in the u_1-empty side of the other bigon. Without loss of generality, assume $S_{j-1} \subset S_i$, as the opposing case is completely analogues. It follows that u_2 has a path from itself to both e_L and e_R in S_i, a contradiction. ■

Let N_j be the number of bigons in S_j that are described in Claim 1. Let B_i be a bigon as described in Claim 1 such that N_i is minimum. If $e[x_i, x_i + 1]$ has a crossing with f other than its ends, then it is at some x_j. Since u_2 is not in S_i, it follows that x_{j+1} is determined and B_j is a bigon as in Claim 1 with $S_j \subset S_i$ and $N_j < N_i$, a contradiction.

Therefore, $e[x_i, x_i+1]$ is uncrossed, $f[x_i, x_i+1]$ is uncrossed by definition, and S_i contains none of u_1, u_2, v. It follows that S_i is a face with B_i bounding it, as desired. □

To finish this section and prove Lemma 4.15, we use Lemma 4.13 to form a not necessarily simple drawing D of H that has the rotations at each vertex the same as in H, then use Lemma 4.14 to find the associated simple drawing that realizes H.

Lemma 4.15. *If H_6 is a $(6,5)$-rotation system, then H_6 is realizable.*

Proof. Let H_6 be a $(6,5)$-rotation system, v be a vertex of $V(H_6)$, $H_v = H_6 - v$, $e_i = (v, u_i)$ be the edges in H_6 having v as an endpoint, and D_v be a realization of H_v. Since H_v is a 5-vertex rotation system, D_v is the unique simple drawing realizing H_v.

By Lemma 4.13, there are simple drawings $D_v + e_i$ such that have v in its respective face in D_v determined by H, e_i crosses exactly the edges determined by H_6 in their prescribed orientations in $D_v + e_i$, and the rotations at u_i in each of the respective drawings is the same as in H_6. Since there is a unique way to draw D_v, it follows we can glue all the $D_v + e_i$ along D_v. Call this new drawing, D. If D is simple, then we are done.

Therefore, D is not simple. Since $D_v + e_i$ is simple for every i, it follows that the reason D is not simple is because two edges e_i and e_j cross. By Lemma 4.14, there exists a closed curve δ comprised of a non-trivial segment e_{i_c} of e_i and a non-trivial segments e_{j_c} of e_j such that a side S of δ has empty intersection with e_i and e_j. In D, S could contain a vertex.

Case 1. *S contains a vertex z.*

Since the intersection of $S \cap e_i$ and $S \cap e_j$ are empty, it follows that u_i and u_j are not in S. In D, it must be the case that (z, u_i) crosses δ. Since D contains the drawing of $D_v + e_i$, it follows that (z, u_i) crosses e_j in D, and thus in $D_v + e_j$. Similarly, (z, u_j) crosses e_i in D, and thus in $D_v + e_i$. By definition of $D_v + e_i$ and $D_v + e_j$, H_6 has (z, u_i) crossing e_j and (z, u_j) crossing e_i. The rotation on v, z, u_i and u_j is realizable, a contradiction with simple drawings of K_4 having at most one crossing.

Case 2. *S contains no vertices.*

Let f be an edge that crosses δ. Since S contains no vertices, it follows that f crosses δ twice, once at e_{i_c} and once at e_{j_c}. It follows that an edge crosses e_{i_c} if and only if it crosses e_{j_c}.

Let $\bar{e}_i$ be the edge e_i rerouted to take e_{j_c} instead of e_{i_c}. Similarly, let $\bar{e}_j$ to be the edge e_j rerouted to take e_{i_c} instead of e_{j_c}. Since e_{i_c} and e_{j_c} cross the same edges, it follows that $\bar{e}_i$ and $\bar{e}_j$ cross the same edges as e_i and e_j, respectively.

Consider the new drawing of $\bar{D}$ that has $\bar{e}_i$ replacing e_i and $\bar{e}_j$ replacing e_j from D with the crossings on δ uncrossed.

If δ has v on the boundary, then the number of crossings between (v, u_i)

and (v, u_j) has reduced by 1. If δ does not have v on the boundary, then the number of crossings between (v, u_i) and (v, u_j) has reduced by 2.

Comparing the rotations of the vertices, only the rotation at v could have possible changed. Repeatedly applying this procedure results in a drawing $\mathcal{D}$ that is simple, and has all the rotations at every vertex the same as H other than v. By Observation 2.6, the rotation at v at a triple of vertices is the same in both H and $\mathcal{D}$. It follows that the rotation at v is the same in both H and $\mathcal{D}$. Therefore, $\mathcal{D}$ is a realization of H, as desired. $\square$

5 Acyclic Orderings from Rotation Systems

The beginning of Section 4 showed that the $<^e_\wedge$ relation on a fixed edge e induces am acyclic graph for $(6,5)$-rotation systems. The purpose of this section is to show that the $<^e_\wedge$ relation and $<^e_\parallel$ relation on a fixed edge e induces am acyclic graph on $(n, n-1)$-rotation systems. We partition this into sections depending on the value of n. In Section 5.1 we show this for $(7,6)$-rotation systems, in Section 1 we show this for $(8,7)$-rotation system, and in Section 5.3 we show this on a $(n,8)$-rotation systems for $n \geq 9$, which is a stronger conclusion than required.

5.1 7-vertex Rotation Systems

The main goal of this section is to prove that $<^e_\wedge$ and $<^e_\parallel$ induce an acyclic directed graph for $(7,6)$ rotation systems. To that end, we want to show the following theorem:

Theorem 5.1. *If H is a $(7,6)$-rotation system, and e is a directed edge of H, then there are no cycles comprised of $<^e_\wedge$ and $<^e_\parallel$ relations in H.*

We prove this result by showing if such a cycle existed, then Lemma 5.3 shows no two consecutive relations can be a $<^e_\parallel$ relation. From this, we show that up to relabelling one of the cycles in Lemma 5.4 and 5.5 occurs in the graph for a contradiction. Before we prove these lemmas, we require a small observation to shorten the arguments.

Observation 5.2. *Let H be a $(7,6)$-rotation system, $e = \overrightarrow{(u,v)}$ be a directed edge of H, $V(H) = [5] \cup \{u,v\}$, and $\mathcal{C} = (a_0, \ldots, a_{k-1}, a_0)$ be a cycle of $<^e_\wedge$ and $<^e_\parallel$ relations such that $a_i <^e a_{i+1}$, for all $i \in \mathbb{Z}_k$.*

For all $i \in \mathbb{Z}_k$, if a_i and a_{i+2} are adjacent, then either:

- *$V(\{a_i, a_{i+1}, a_{i+2}\}) = [5]$; or*
- *$(a_0, \ldots, a_i, a_{i+2}, \ldots, a_{k-1}, a_0)$ is a cycle of $<^e_\wedge$ and $<^e_\parallel$ relations.*

Proof. Suppose there exists some i such that a_i and a_{i+2} are adjacent, and $V(\{a_i, a_{i+1}, a_{i+2}\}) \subseteq [5] \backslash \{j\}$, for some $j \in [5]$. Let the simple drawing D_j be a realization of $H - \{j\}$ for $j \in V(H)$. It follows that $a_i <^e a_{i+1} <^e a_{i+2}$ in D_j, and that $a_i <^e_\wedge a_{i+2}$. Therefore, there exists some cycle of relations $\bar{\mathcal{C}} = (a_0, \ldots, a_i, a_{i+2}, \ldots, a_{k-1}, a_0)$, as desired. $\square$

Lemma 5.3. *Let H be a $(7,6)$-rotation system, $e = \overrightarrow{(u,v)}$ be a directed edge of H. If $\mathcal{C} = (a_0, \ldots, a_{k-1}, a_0)$ is a minimal cycle of $<^e_\wedge$ and $<^e_\parallel$ relations such that $a_i <^e a_{i+1}$, for all $i \in \mathbb{Z}_k$, then for all $j \in \mathbb{Z}_k$, one of $a_j <^e a_{j+1}$ or $a_{j+1} <^e a_{j+2}$ is a $<^e_\wedge$ relation.*

Proof. Define the simple drawings D_i to be a realization of $H - \{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u, v\}$.

By way of contradiction, suppose both $a_i <^e_\parallel a_{i+1}$ and $a_{i+1} <^e_\parallel a_{i+2}$. Without loss of generality, $a_i = (1,2)$ and $a_{i+1} = (3,4)$. By Observation 5.2, it follows that $a_{i+2} \in \{(1,5),(2,5)\}$. Since vertices 1 and 5 are symmetric, without loss of generality pick $a_{i+2} = (1,5)$. By minimality of $\mathcal{C}$, $a_{i+2} <^e_\wedge a_i$. To prove this claim, we must find a contradiction to the relations $(1,2) <^e_\parallel (3,4) <^e_\parallel (1,5) <^e_\wedge (1,2)$. Note that vertex 3 and 4 are symmetric in these relations, therefore there are at most two cases to draw the partial K_6 illustrating the relation $(1,2) <^e_\parallel (3,4)$ determined by the direction e crosses $\overrightarrow{(1,2)}$. In actuality, there is one case as the inverse rotation system flips the oriented crossings of edges and preserves the relation of edge crossings over any fixed edge. By applying the analysis of the first case to the inverse rotation system, the result would be the second case. Without loss of generality assume e crosses $\overrightarrow{(1,2)}$ from right to left.

D_5 is determined by the relation $(1,2) <^e_\parallel (3,4)$ with e crossing $\overrightarrow{(1,2)}$ from right to left. By the symmetry of vertices 3 and 4 we can assume without loss of generality that e crosses $\overrightarrow{(3,4)}$ from left to right. It follows that $H - \{u, v, 5\}$ is determined and that $\overrightarrow{(1,3)}$ crosses $\overrightarrow{(2,4)}$ from right to left.

In D_2, consider when e crosses the edges of $\overrightarrow{(1,3,4)}$ and $(1,5)$ after crossing the edge $(3,4)$. For $x \in \{(1,3),(1,4)\}$, if $(3,4) \prec^e_{D_2} x \prec^e_{D_2} (1,5)$, then $(1,2) <^e_\parallel (3,4) <^e_\wedge x <^e_\wedge (1,5) <^e_\wedge (1,2)$. By Observation 5.2, this can be reduced to $(1,2) <^e_\parallel (3,4) <^e_\wedge x <^e_\wedge (1,2)$, a contradiction with the existence of D_5.

It follows that the crossing of e and $(1,5)$ is in $\overrightarrow{(1,3,4)}_R$ as e crosses $(3,4)$ into $\overrightarrow{(1,3,4)}_R$ then crosses $(1,5)$. Since $(3,4)$ and $(1,5)$ are $<^e_\parallel$ related, they do not cross, and the edge $(1,5)$ is contained in $\overrightarrow{(1,3,4)}_R$, in particular, the vertex 5 is contained in $\overrightarrow{(1,3,4)}_R$.

In D_e, the vertex 5 is not in $\overrightarrow{(1,2,3)}_R$ as $\overrightarrow{(1,2,3)}_R \cap \overrightarrow{(1,3,4)}_R = \emptyset$. From D_5, $\pi_{2,3,4}(1) = [2,3,4]$ and from D_2, $\pi_{3,4,5}(1) = [3,5,4]$. It follows that

$\pi_{2,3,4,5}(1) = [2,3,5,4]$. In D_4, from the rotation at 1 and the directed crossing of e with $\overrightarrow{(1,2)}$, $(2,3)$ crossing $(1,5)$ would imply that $5 \in \overrightarrow{(1,2,3)}_R$. Since we know that $5 \notin \overrightarrow{(1,2,3)}_R$, $(2,3)$ does not cross $(1,5)$. This implies that $(2,3)$ must cross e, in particular, $(2,3) <^e_\wedge (1,2)$ and e crosses $(2,3)$ from left to right.

In D_1, e crosses into $\overrightarrow{(2,3,4)}_R$ at $(3,4)$ and $(2,3)$. It follows that e crosses out of $\overrightarrow{(2,3,4)}_R$ at $(2,4)$ and that $(3,4) <^e_\wedge (2,4) <^e_\wedge (2,3)$. We now have that $(1,2) <^e_\| (3,4) <^e_\wedge (2,4) <^e_\wedge (2,3) <^e_\wedge (1,2)$, a contradiction with H containing realizable 6-vertex rotation systems. $\square$

Lemma 5.4. *If H is a $(7,6)$-rotation system and $e = \overrightarrow{(u,v)}$ is a directed edge of H, then the cycle $(1,2) <^e_\wedge (2,3) <^e_\wedge (3,4) <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,2)$ does not occur.*

Proof. Define the simple drawings D_i to be a realization of $H - \{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u,v\}$.

By way of contradiction, assume such a cycle exists. Three consecutive edges in this cycle form a 6-vertex rotation system with e. We will show that the second of any three consecutive edges can not be replaced in this cycle with some other edge in the induced 6-vertex rotation system.

By symmetry, it is enough to consider the order $(1,2) <^e_\wedge (2,3) <^e_\wedge (3,4)$ and check if $(2,3)$ can be replaced. Since D_5 is simple, it is clear that no edge incident to e crosses e. Therefore, suppose some edge $x \in \{(1,3),(1,4),(2,4)\}$ has the property that $(1,2) <^e_\wedge x <^e_\wedge (3,4)$. If this were the case, then the cycle $(1,2) <^e_\wedge x <^e_\wedge (3,4) <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,2)$ exists.

If $x \in \{(1,3),(1,4)\}$, then this cycle reduces to $(1,5) <^e_\wedge x <^e_\wedge (3,4) <^e_\wedge (4,5) <^e_\wedge (1,5)$, a contradiction with 6-vertex rotation systems being realizable. Similarly, if $x = (2,4)$, then the cycle reduces to $(1,2) <^e_\wedge x <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,2)$, again a contradiction with realizable 6-vertex rotation systems. Therefore, no such edge x can replace $(2,3)$.

In D_1, if $(1,2)$ and $(3,4)$ cross, then e can not cross $(1,2),(2,3)$, and $(3,4)$ consecutively as after crossing $(1,2)$, e would cross $(2,3)$ from the crossing side of the uncrossed 4-cycle to the non-crossing side of the uncrossed 4-cycle and would not be able to cross $(3,4)$.

If $(1,2)$ and $(3,4)$ are not crossing and are in a crossing K_4, then after e has crossed $(1,2)$, then $(2,3)$, it would be contained in a face containing crossing edges and an edge that shares an endpoint with $(1,2)$. Thus, e would not be able to cross $(3,4)$ after crossing $(2,3)$. It follows that a $H - \{u, v, 5\}$ can only be realized by a planar K_4. By symmetry, $H - \{u, v, 1\}, H - \{u, v, 2\}, H - \{u, v, 3\}$, and $H - \{u, v, 4\}$ are also all realized by planar drawings. However, every realization of the 5-vertex rotation system $H - \{u, v\}$ contains at least one crossing, a contradiction. $\square$

Lemma 5.5. *If H is a $(7, 6)$-rotation system and $e = \overrightarrow{(u, v)}$ is a directed edge of H, then the cycle $(1, 2) <^e_{\parallel} (3, 4) <^e_{\wedge} (4, 5) <^e_{\wedge} (1, 5) <^e_{\wedge} (1, 2)$ does not occur.*

Proof. Define the simple drawings D_i to be a realization of $H - \{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u, v\}$.

By way of contradiction, assume such a cycle exists. Notice since $(1, 2)$ and $(3, 4)$ are in a crossing K_4 where they do not cross, that one of the crossing diagonals is $y \in \{(1, 3), (1, 4)\}$.

Let C_i be the oriented cycle defined by the vertices $[4]\backslash\{i\}$ such that in D_i, e crosses into $C_{i,R}$ at $(3, 4)$ or e crosses out of $C_{i,R}$ at $(1, 2)$.

Consider the order e crosses the edges of $(1, 3, 4)$ and $(1, 5)$ after crossing into $C_{2,R}$ at $(3, 4)$. For $x \in \{(1, 3), (1, 4)\}$, if $(3, 4) \prec^e_{D_2} x \prec^e_{D_2} (1, 5)$, then $(1, 2) <^e_{\parallel} (3, 4) <^e_{\wedge} x <^e_{\wedge} (1, 5) <^e_{\wedge} (1, 2)$. By Observation 5.2, this can be reduced to $(1, 2) <^e_{\parallel} (3, 4) <^e_{\wedge} x <^e_{\wedge} (1, 2)$, a contradiction with the existence of D_5.

It follows that the crossing of e and $(1, 5)$ is inside $C_{2,R}$ as e crosses $(3, 4)$ into $C_{2,R}$ then crosses $(1, 5)$. Since the crossing of e and $(4, 5)$ happens between the crossing of e with $(3, 4)$ and the crossing of e with $(1, 5)$ on e, the crossing of e with $(4, 5)$ must also be in $C_{2,R}$.

Since a K_4 contains at most one crossing, one of the edges $(1, 5)$ or $(4, 5)$ is contained in $C_{2,R}$. It follows that 5 is contained in $C_{2,R}$.

By considering the order in which e crosses the edges of $C_{3,R}$ and $(1, 5)$, the same argument shows that 5 is contained in $C_{3,R}$.

If $y = (1, 4)$, then $C_{2,R}$ and $C_{3,R}$ have empty intersection in D_5, and thus have empty intersection in D_e. By the existence of D_e and $5 \in C_{2,R} \cap C_{3,R}$, $y \neq (1, 4)$. Therefore, $y = (1, 3)$.

There are two cases to finish this claim depending on whether $(2, 3)$ is crossed by e or not.

Case 1. *$(2, 3)$ is not crossed by e.*

Consider the order the edges of $C_{4,R}$ and $(1, 5)$ are crossed ny e. If $(1, 5) <^e_{\wedge}$

$(1,3) <^e_\wedge (1,2)$, then $(1,2) <^e_\| (3,4) <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,3) <^e_\wedge (1,2)$. By Observation 5.2, this cycle can be reduced to $(3,4) <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,3) <^e_\wedge (3,4)$, a contradiction with the existence of D_2.

Since $(2,3)$ does not cross e, $C_{4,R}$ contains the crossing of $(1,5)$ with e. Note that $C_{2,R}$ contains the vertex 5 and the intersection of $C_{2,R}$ and $C_{4,R}$ is empty in D_e. It follows that $C_{4,R}$ does not contain the vertex 5 and that $(1,5)$ starts inside $C_{4,R}$ since the intersection of $(1,5)$ and e is contained in $C_{4,R}$. This is a contradiction as D_6 can not be a simple drawing if the edge $(1,5)$ starts in $C_{4,R}$ and $5 \in C_{2,R} \cap C_{3,R}$.

Case 2. *e crosses* $(2,3)$.

Each of C_1 and C_4 have e crossing from one side of the cycle to the other at $(3,4)$ and $(1,2)$, respectively. By our choice of C_1 and C_4, either e crosses $(2,3)$ from the same side as $(1,2)$ in C_4, or e crosses $(2,3)$ from the same side as $(3,4)$ in C_1 (as observed in D_1). Without loss of generality, we will assume that e crosses $(2,3)$ from the same side as $(1,2)$ in C_4 as the argument for the opposing assumption is completely analogous.

Working in D_4, we note that e also crosses $(1,3)$ into $C_{4,R}$ as both $(1,2)$ and $(2,3)$ are crossed outward by e. In particular, $(1,3)$ is the second edge crossed by e of the edges in C_4.

If $(1,5) \prec^e_{D_4} (2,3) \prec^e_{D_4} (1,2)$, then $(1,5) <^e_\wedge (1,3) <^e_\wedge (1,2)$ as $(1,3)$ is the second edge crossed in C_4 by e. It follows that $(1,2) <^e_\| (3,4) <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,3) <^e_\wedge (1,2)$. By Observation 5.2, these relations can be reduced to $(1,3) <^e_\wedge (3,4) <^e_\wedge (4,5) <^e_\wedge (1,5) <^e_\wedge (1,3)$, a contradiction with the existence of D_2.

It follows that the crossing of $(1,5)$ and e is contained in $C_{4,R}$. Since the vertex 5 is contained in the exterior of $C_{4,R}$, it follows that the edge $(1,5)$ starts inside $C_{4,R}$ at 1.

This is a contradiction, as the drawing D_e is simple, therefore, it can not have $(1,5)$ start in $C_{4,R}$ and end at $5 \in C_{2,R} \cap C_{3,R}$. □

We end this section by applying Observation 5.2 and Lemmas 5.3, 5.4 and 5.5 to prove Theorem 5.1.

Proof of Theorem 5.1. Let H be a $(7,6)$-rotation system and $e = \overrightarrow{(u,v)}$ be a directed edge of H. By way of contradiction, let $\mathcal{C} = (a_0, a_1, ..., a_{k-1})$ be a minimal cycle of $<^e_\wedge$ and $<^e_\|$ relations such that $a_i <^e a_{i+1}$ (mod k) for $i \in [k]$.

If $\mathcal{C}$ does not contain a $<^e_{\parallel}$ relation, then repeated applications of Observation 5.2 would give us that up to relabelling $\mathcal{C} = ((1,2),(2,3),(3,4),(4,5),(5,1),(1,2))$, a contradiction with Lemma 5.4.

Therefore, $\mathcal{C}$ contains a $<^e_{\parallel}$ relation $a_i < a_{i+1}$, for some $i \in \mathbb{Z}_k$. Without loss of generality, let $a_i = (1,2)$ and $a_{i+1} = (3,4)$. By Observation 5.2, Lemma 5.3, and by symmetry of $(3,4)$, it must be the case that $a_{i+1} <^e_{\wedge} a_{i+2}$ and that $a_{i+2} = (4,5)$. Similarly, $a_{i-1} <^e_{\wedge} a_i$ and $a_{i-1} = (1,5)$.

If $(1,5) <^e_{\wedge} (4,5)$, then the cycle $(a_0,\ldots,a_i,a_{i-1},a_{i+2},\ldots,a_{k-1},a_0)$ exists and $\mathcal{C}$ is not minimal, a contradiction. Therefore, $(4,5) <^e_{\wedge} (1,5)$ and $(1,2) <^e_{\parallel} (3,4) <^e_{\wedge} (4,5) <^e_{\wedge} (1,5) <^e_{\wedge} (1,2)$ exists, a contradiction with Lemma 5.5, as desired. □

5.2 8-Vertex Rotation Systems

Similar to the previous section, this section proves that $<^e_{\wedge}$ and $<^e_{\parallel}$ induce an acyclic directed graph for $(8,7)$ rotation systems. To that end, we want to show the following theorem:

Theorem 5.6. *If H is a $(8,7)$-rotation system and e is a directed edge of H, then there are no cycles comprised of $<^e_{\wedge}$ and $<^e_{\parallel}$ relations in H.*

Similar to the proof of Theorem 5.1, we restrict the structure of a minimal cycle of relations using Observation 5.9 and show that up to relabelling one of the cycles in Lemmas 5.10 or 5.11 exists for a contradiction. Before we can prove these statements we must first learn more about the $<^e_{\triangle}$ relation. In particular, the $<^e_{\triangle}$ relation can always be represented as a chain of $<^e_{\wedge}$ and $<^e_{\parallel}$ relations. Many thanks go to Bruce Richter for taking a very crude statement and argument of the following lemma and turning it into what it is today.

Lemma 5.7. *Let H be an $(n,7)$-rotation system, a,b be edges of H and e a directed edges of H such that $a <^e_{\triangle} b$, as certified by the vertex v. Let u_a and v_a be the ends of a such that traversing a from u_a to v_a the crossings of e precedes the crossings of b. Likewise, changing the roles of a and b, we get u_b and v_b. Then either:*

- *There exists an $i \in \{a,b\}$ such that (u_i,v) crosses e, and does not cross a or b; or*
- *$a <^e_{\parallel} (v_b,v) <^e_{\wedge} (u_a,v) <^e_{\wedge} (u_b,v) <^e_{\wedge} (v_a,v) <^e_{\parallel} b$.*

Proof. The proof involves drawing the edge (u_a, v). It starts at v on the side $\triangle$ of $\gamma_{a,b,e}$ not incident to any vertex in $V(\{a, b, e\})$ and finishes at u_a on the other side of $\gamma_{a,b,e}$. Evidently, it crosses $\triangle$ an odd number of times and does not cross a, therefore it crosses $\triangle$ exactly once.

As we start from v, we leave $\triangle$ by crossing one of e or b.

Case 1. *The $\triangle$-leaving crossing of (u_a, v) is with b.*

One side of $\gamma_{a,b,(u_a,v)}$ contains exactly v, a non-trivial segment of b starting at u_b, and a non-trivial segment of e. It follows that (u_b, v) crosses $\gamma_{a,b,(u_a,v)}$ and even number of times, but can only cross a. It follows that (u_b, v) does not cross $\gamma_{a,b,(u_a,v)}$. The only thing (u_b, v) can cross is the segment of e on the same side of $\gamma_{a,b,(u_a,v)}$. This implies that (u_b, v) leave $\triangle$ from e and does not cross a or b, as desired.

Case 2. *The $\triangle$-leaving crossing of (u_a, v) is with e.*

Case 1 with the roles of a and b interchanged implies that (u_b, v) also leaves $\triangle$ by crossing e. We may assume that (u_i, v) crosses j for $\{i, j\} = \{a, b\}$, else we are done. In this case, $\gamma_{e,b,(u_a,v)}$ exists.

Trivially, as (u_a, v) does not cross b at $\gamma_{a,b,e}$, it crosses b at one of the two components in $b\backslash\gamma_{a,b,e}$.

Case 2.1. *(u_a, v) crosses b on the component containing u_b in $b\backslash\gamma_{a,b,e}$.*

In this case, one side of $\gamma_{a,b,(u_a,v)}$ contains exactly v, u_b and a segment of e. Evidently, (u_b, v) crosses $\gamma_{a,b,(u_a,v)}$ an even number of times, but can only cross a. It follows that (u_b, v) does not cross $\gamma_{a,b,(u_a,v)}$. Since the only edge segment on the same side of $\gamma_{a,b,(u_a,v)}$ containing v and u_b is e, it follows that (u_b, v) crosses e and not a or b, as desired.

Case 2.2. *(u_a, v) crosses b on the component containing v_b in $b\backslash\gamma_{a,b,e}$.*

Similarly, (u_b, v) crosses a on the component containing v_a in $b\backslash\gamma_{a,b,e}$. These crossings uniquely determine the drawing D of $a, b, e, (u_a, v)$ and (u_b, v). Since every 7-vertex rotation system in H is realizable, it follows that the drawing D can be extended to a drawing $\bar{D}$ containing the induced K_4s on v, v_a, v_b, u_a and v, v_a, v_b, u_b.

There is a unique way to extend D to $\bar{D}$ to keep $\bar{D}$ simple. In $\bar{D}$, $a <^e_{\parallel} (v_b, v) <^e_{\wedge} (u_a, v)) <^e_{\wedge} (u_b, v) <^e_{\wedge} (v_a, v) <^e_{\parallel} b$, as desired. ☐

Corollary 5.8. *Let H be an $(n, 7)$-rotation system, a, b be edges of H and e a directed edges of H such that $a <^e_{\triangle} b$, as certified by the vertex v. Let u_a and v_a be the ends of a such that traversing a from u_a to v_a the crossings of e proceeds the crossings of b. Likewise, changing the roles of a and b, we get u_b and v_b. Then either:*

- *$a <^e_{\wedge} (u_a, v) <^e_{\parallel} b$;*
- *$a <^e_{\parallel} (u_b, v) <^e_{\wedge} b$; or*
- *$a <^e_{\wedge} (u_a, v)) <^e_{\wedge} (u_b, v) <^e_{\wedge} b$.*

Lemma 5.7 is used primarily in the proof of Lemma 5.10 to find structure in realizations of 7-vertex rotation system that share a common 6-vertex rotation system. As for Corollary 5.8, it is used to simplify the proof of Theorem 5.12. Let us make an observation on the structure of cycles of $<^e_{\wedge}$ and $<^e_{\parallel}$ relations in $(8, 7)$-rotation systems.

Observation 5.9. *Let H be an $(8, 7)$-rotation system, $e = \overrightarrow{(u, v)}$ be a directed edge of H, $\mathcal{C} = (a_0, \ldots, a_{k-1}, a_0)$ be a cycle of $<^e_{\wedge}$ and $<^e_{\parallel}$ relations (i.e. for all $i \in \mathbb{Z}_k$, $a_i <^e_{\wedge} a_{i+1}$ or $a_i <^e_{\parallel} a_{i+1}$).*

For all $i, j \in \mathbb{Z}_k$, if a_i and a_j are related by an $<^e_{\wedge}$ or $<^e_{\parallel}$ relation and $j \notin \{i-1, i+1\}$, then either:

- *$V(\{a_i, \ldots, a_j\}) = [6]$; or*
- *$(a_0, \ldots, a_i, a_j, \ldots, a_{k-1}, a_0)$ is a cycle of $<^e_{\wedge}$ and $<^e_{\parallel}$ relations.*

Proof. Suppose there exists some i and j such that a_i and a_j are related by an $<^e_{\wedge}$ or $<^e_{\parallel}$ relation, and $V(\{a_i, \ldots, a_j\}) \subseteq [6] \backslash \{k\}$, for some $k \in [6]$. Let the simple drawing D_k be a realization of $H - \{k\}$. It follows that $a_i < \cdots <^e a_j$ in D_k, and that $a_i <^e a_j$ is a $<^e_{\wedge}$ or $<^e_{\parallel}$ relation. Therefore, there exists some cycle of relations $\bar{\mathcal{C}} = (a_0, \ldots, a_i, a_j, \ldots, a_{k-1}, a_0)$, as desired. ☐

Since the $<^e_{\triangle}$ relation is always a chain of $<^e_{\wedge}$ and $<^e_{\parallel}$ relations, Lemma 5.10 has more substance than one would expect. In particular, it is used in every case of the proof of Theorem 5.6.

Lemma 5.10. *If H is an $(8,7)$-rotation system, and e is a directed edge of H, then the cycle order $(1,2) <^e_\triangle< (3,4) <^e_\triangle< (1,2)$ does not occur.*

Proof. Define the simple drawings D_i to be a realization of $H - \{i\}$ and D_e to be a realization of $H - \{u, v\}$.

Suppose such a cycle relation exists with vertex 5 being the certificate for $(3,4) <^e_\triangle (1,2)$ and vertex 6 being the certificate for $(1,2) <^e_\triangle (3,4)$. Up to relabelling, without loss of generality it can be assumed that e crosses $\overrightarrow{(1,2)}$ from right to left, e crosses $\overrightarrow{(3,4)}$ from right to left and $\overrightarrow{(1,2)}$ crosses $\overrightarrow{(3,4)}$ from right to left. We will break this proof into cases depending on how Lemma 5.7 relates to D_6.

Case 1. $(3,4) <^e_\parallel (1,5) <^e_\wedge (2,5) <^e_\wedge (4,5) <^e_\wedge (3,5) <^e_\parallel (1,2)$.

It follow that $(3,4) <^e_\wedge (3,5) <^e_\parallel (1,2)$ and $(3,4) <^e_\parallel (1,5) <^e_\wedge (1,2)$.

If $(1,2) <^e_{K_6} (i,6) <^e_{K_6} (3,4)$ where $i \in \{1,3\}$, then in D_2, $(i,6) <^e_{K_6} (3,4) <^e_{K_6} (i,5)$, in particular $(i,6) <^e_\wedge (i,5)$. It would follow that $(1,2) <^e_{K_6} (i,6) <^e_\wedge (i,5) <^e_{K_6} (1,2)$, a contradiction with the existence of D_3.

Therefore, by Lemma 5.7, $(1,2) <^e_\parallel (4,6) <^e_\wedge (1,6) <^e_\wedge (3,6) <^e_\wedge (2,6) <^e_\parallel (3,4)$, in particular, $(1,2) <^e_\wedge (1,6) <^e_\wedge (3,6) <^e_\wedge (3,4)$. Since $(1,5) <^e_\parallel (1,2) <^e_\wedge (1,6)$, it follows that $(1,5) <^e_\wedge (1,6)$ and $(3,4) <^e_\parallel (1,5) <^e_\wedge (1,6) <^e_\wedge (3,6) <^e_\wedge (3,4)$, a contradiction with the existence of D_2.

Case 2. *There exists an $i \in \{2,4\}$ such that $(i,5)$ crosses e, and does not cross $(1,2)$ or $(3,4)$.*

Since the edges $(2,5)$ and $(4,5)$ are symmetric at this point, without loss of generality the edge $(2,5)$ crosses e and not $(1,2)$ and $(3,4)$. Let $\bar{e}$ be the segment of e that starts at the crossing of e and $(3,4)$ and ends at the crossing of e and $(1,2)$. Observe that the edges $(1,2), (3,4), \bar{e}, (2,5), (2,3)$ and $(1,3)$ are uniquely drawn in D_6. In particular, e crosses $(1,2)$ into $\overrightarrow{(1,3,2)}_R$, and $(2,5)$ is contained in $\overrightarrow{(1,2,3)}_R$.

The same argument as Case 1 on D_5 shows $(1,2) <^e_\parallel (4,6) <^e_\wedge (1,6) <^e_\wedge (2,6) <^e_\wedge (3,6) <^e_\wedge (3,4)$ does not occur. By Lemma 5.7, there exists a $j \in \{1,3\}$ such that $(j,6)$ crosses e, and does not cross $(1,2)$ or $(3,4)$.

Case 2.1. $(3,6)$ *crosses e and not* $(1,2)$ *or* $(3,4)$.

In D_5, the edges $(1,2), \bar{e}, (3,4), (3,6), (1,3)$ and $(2,3)$ are uniquely drawn. In particular, $\pi_{1,2,6}(3) = [2,6,1]$. In D_4, since the edge $(3,6)$ starts inside $\overrightarrow{(1,3,2)}_R$ and does not cross $(1,2)$, it follows that $(3,6)$ is contained in $\overrightarrow{(1,3,2)}_R$. Since $(3,6)$ and $(2,5)$ are contained in opposite sides of the 3-cycle $(1,2,3)$, it follows that $(2,5)$ and $(3,6)$ do not cross.

In D_5, $(1,2) <^e_{K_6} (3,6)$ as they do no cross and e crosses $(1,2)$ first.

Similarly, in D_6, $(3,4) <^e_{K_6} (2,5)$. Since $(2,5) <^e_{\wedge} (1,2) <^e_{K_6} (3,6)$ and $(2,5)$ does not cross $(3,6)$, it follows that in D_1, $(2,5) <^e_{K_6} (3,6)$. This results in the cycle $(3,4) <^e_{K_6} (2,5) <^e_{K_6} (3,6) <^e_{\wedge} (3,4)$, a contradiction with the existence of D_1.

Case 2.2. $(1,6)$ *crosses* e *and not* $(1,2)$ *or* $(3,4)$.

We break this into two smaller cases depending on if $(3,5)$ crosses $\bar{e}$ in D_6 or not.

Case 2.2.1. $(3,5)$ *does not cross* $\bar{e}$ *in* D_6.

By the symmetry of Case 2.1, we also have that $(4,5)$ does not both cross e and not $(1,2)$ and $(3,4)$.

Let $E_{1,2,3,4}$ be the edge set of the K_4 involving $(1,2)$ and $(3,4)$. Observe that the edges of $E_{1,2,3,4}, (2,5), (3,5), (4,5)$ and $\bar{e}$ are uniquely drawn in D_6, in particular that $\pi_{1,4,5}(3) = [5,4,1]$ and $\pi_{1,3,5}(4) = [1,3,5]$.

Similarly, the edges of $E_{1,2,3,4}, (1,6)$ and $\bar{e}$ are uniquely drawn in D_5, in particular that $\pi_{2,3,6}(1) = [3,6,2]$. Let C_1 be the closed curve in D_4 defined by starting at the vertex 1, taking the edge $(1,3)$ to the vertex 3, then taking the edge $(3,5)$ to the crossing of $(3,5)$ with $(1,2)$, and taking the edge $(1,2)$ back to the vertex 1. Let $C_{1,R}$ be the side of C_1 that is bounded by the right side of $\overrightarrow{(1,3)}$. Since $\pi_{2,3,6}(1) = [3,6,2]$, the edge $(1,6)$ starts inside $C_{1,R}$.

Let us now go to D_4. Note that the K_4 on $(3,5)$ and $(1,2)$ is determined as $\overrightarrow{(1,2)}$ crosses $\overrightarrow{(3,5)}$ from right to left in D_6. Since $\pi_{2,3,6}(1) = [3,6,2]$, $(1,6)$ is either contained on one side of $\gamma_{(1,3),(3,5),(1,2)}$ or $\overrightarrow{(1,6)}$ crosses $\overrightarrow{(3,5)}$ from right to left in D_4.

Case 2.2.1.1. $(1,6)$ *is contained on one side of* $\gamma_{(1,3),(3,5),(1,2)}$ *in* D_4

It follows that $(1,6)$ does not cross $(3,5)$. Observe that the drawing of the induced K_4 on $1,2,3,5$ along with the edge $(1,6)$ is determined. In any realization of these 5-vertices along with e, e crosses $\overrightarrow{(2,5)}$ from left to right, then crosses $(1,2)$, then crosses $(1,6)$. In particular, e crosses into $\overrightarrow{(3,2,5)}_R$ at $(2,5)$ and this region does not contain (1.6) as it does not contain $\gamma_{(1,3),(3,5),(1,2)}$.

It follows that e crosses $x \in \{(2,3),(3,5)\}$ after crossing $(2,5)$ and before crossing $(1,6)$. Note that both these edge share an endpoint with $(2,5)$ and neither of these edge crosses $(1,6)$ since they are contained on the wrong side of $\gamma_{(1,3),(3,5),(1,2)}$. It follows that $(2,5) <^e_{K_6} x <^e_{K_6} (1,6)$.

Therefore, we have $(3,4) <^e_{\|} (2,5) <^e_{K_6} x <^e_{K_6} (1,6) <^e_{\|} (3,4)$. Note that D_1, it follows that $(3,4) <^e_{\wedge} x <^e_{K_6} (1,6) <^e_{\|} (3,4)$, a contradiction in one of D_2 or D_5.

Case 2.2.1.2. $\overrightarrow{(1,6)}$ *crosses* $\overrightarrow{(3,5)}$ *from right to left in* D_4.

Now we make some observations on D_2. Noe that the K_4 involving $(1,6)$ and $(3,5)$ is uniquely determined by their oriented crossings. Since $\pi_{1,4,5}(3) = [5,4,1]$ and $(3,4)$ does not cross $(1,6)$ in D_5, it follows that $(3,4)$ is one side of $\gamma_{(1,3),(1,6),(3,5)}$, in particular on the side S that is a face in the induced K_4 on $1,3,5,6$.

Suppose $(1,4)$ is drawn inside S. It would follow that $\overrightarrow{(1,3,4)}_L$ contains $(1,6)$. After e crosses $(1,6)$ in D_2, it would have to cross $(1,3,4)$ at $(1,3)$ or $(1,4)$ to cross $(3,4)$ in the correct orientation. It would follows that $(1,6) <^e_{\wedge} x <^e_{\wedge} (3,4)$ for $x \in \{(1,3),(1,4)\}$. Expanding our cycle, we would have $(3,4) <^e_{\|} (2,5) <^e_{\wedge} (1,2) <^e_{\wedge} (1,6) <^e_{\wedge} x <^e_{\wedge} (3,4)$. Reducing over D_5 would give, $(3,4) <^e_{\|} (2,5) <^e_{\wedge} (1,2) <^e_{\wedge} x <^e_{\wedge} (3,4)$, a contradiction with D_6.

Therefore, $(1,4)$ is not drawn inside S. There is a unique was to draw $(1,4)$, starting at 4 it crosses $(3,5)$, then $(3,6)$, then ends at 1. In particular, there is a closed curve $\gamma_{(3,4),(1,4),(3,5)}$ that has 5 on one side and the starting of the edge $(4,5)$ at 4 on the other (as per $\pi_{1,3,5}(4) = [1,3,5]$), a contradiction.

Case 2.2.2. $(3,5)$ *crosses* $\bar{e}$ *in* D_6.

In D_6, the order e crosses edges is $(3,5)$, then $(1,2)$. Without loss of generality, by deleting the vertex 4 and adding the vertex 6 and its edges, we obtain D_4 from D_5. In particular, the order e crosses edge in D_4 is $(3,5)$, then $(1,2)$, then $(1,6)$. From D_4, we obtain D_2 in a similar manner with e crossing $(3,5)$ then $(1,6)$. However, $(3,4)$ must be crossed after $(1,6)$ and before $(3,5)$ in D_2, a contradiction. $\square$

In the same vein as Lemma 5.10, we show in Lemma 5.11 that there are no short cycles of $<_{\parallel}^{e}$ relations.

Lemma 5.11. *If H is an $(8,7)$-rotation system, and e is a directed edge of H, then the cycle order $(1,2) <_{\parallel}^{e} (3,4) <_{\parallel}^{e} (5,6) <_{\parallel}^{e} (1,2)$ does not occur.*

Proof. Let $e- = \overrightarrow{(u,v)}$ be a directed edge of H. Define the simple drawings D_i to be a realization of $H - \{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u,v\}$. For all i, let $\bar{e}_i$ be the segment of e in D_i between the two edges e crosses in $\{(1,2),(3,4),(4,5)\}$. In the drawing of D_i, define $\mathcal{D}_i$ to be the unique drawing of $\bar{e}_i$ with the induced K_4 on the two edges in $\{(1,2),(3,4),(5,6)\}$ that cross e.

By way of contradiction, assume the cycle order $(1,2) <_{\parallel}^{e} (3,4) <_{\parallel}^{e} (5,6) <_{\parallel}^{e} (1,2)$ does occur. Up to relabelling, we may assume without loss of generality that e crosses $\overrightarrow{(1,2)}$, $\overrightarrow{(3,4)}$, and $\overrightarrow{(5,6)}$ from right to left.

Claim 1. *Let $W = \{1,2,3,4\}$. If $w \in W$, then $(5,w)$ does not cross $\bar{e}_6$ in D_6.*

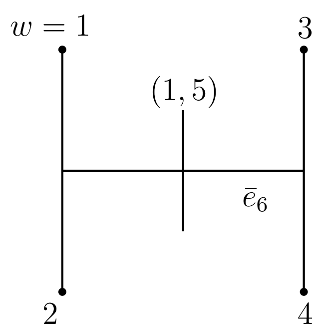


Figure 35: Lemma 5.11 Claim 1.

Proof. Without loss of generality, we can assume that $w = 1$. By way of contradiction, assume $(1,5)$ crosses $\bar{e}_6$, in particular, e crosses $(1,5)$ before $(3,4)$ and $(1,2) <_{\wedge}^{e} (1,5)$ as in Figure 35.

By deleting the vertex 2 in D_6 and adding the vertex 6 along with all of its incident edges, we create a realization Z of $H - \{2\}$. In Z, e crosses $(1,5)$, then $(3,4)$, then $(5,6)$. In particular, $(1,5) <^e_\wedge (5,6)$. It follows that $(1,2) <^e_\wedge (1,5) <^e_\wedge (5,6) <^e_\parallel (1,2)$, a contradiction with the existence of D_3. ■

One remark about Claim 1 is that there is a symmetry. In particular, we can replace W with any four vertices that are the ends of two of $(1,2), (3,4)$ and $(5,6)$. Furthermore, we can replace 5 with 6, or alternatives with any end vertex $[6]$ that is not in W and consider the drawing D_i, where i is not used.

Claim 2. *Let* $W = \{1,2,3,4\}$. *If* $w \in W$, *then* $(5,w)$ *does not cross any edge in* $\mathcal{D}_6$.

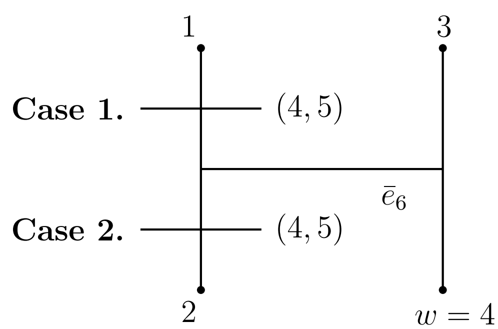


Figure 36: Lemma 5.11 Claim 2.

Proof. Without loss of generality, we may assume that $w = 4$. Since D_6 is a simple drawing, $(4,5)$ and $(3,4)$ do not cross. By Claim 1, we need only prove that $(4,5)$ does not cross the edge $(1,2)$. By way of contradiction, suppose $(4,5)$ does cross $(1,2)$. We will break this into cases depending on the location that $(4,5)$ crosses $(1,2)$ on $(1,2)$ as depicted in Figure 36.

Case 1. *Assume* $(4,5)$ *crosses* $(1,2)$ *between the crossing of* e *with* $(1,2)$ *and the vertex* 1 *in* D_6.

Consider how the edge $(3,5)$ and $(2,3)$ are drawn with the partial edge $\bar{e}_6$, $(3,4)$, $(4,5)$, and the segment of $(1,2)$ starting at 2 and ending at the crossing of $\bar{e}_6$. Since D_6 is a simple drawing, the drawing of these edges is unique, in particular, $\overrightarrow{(4,5)}$ crosses $\overrightarrow{(2,3)}$ from left to right and $\pi_{2,4,5}(3) = [2,4,5]$.

In $\mathcal{D}_1 + (2,3)$, by Claim 1, $(2,3)$ does not cross $\bar{e}_1$. Let C_1 be the unique close curve defined on $\bar{e}_1$ and segments of $(3,4), (3,5)$ and $(5,6)$ and let $C_{1,R}$ be the side of C_1 that is bounded by the right side of $\overrightarrow{(3,5)}$. In particular, the edge $(4,5)$ is outside $C_{1,R}$.

Since $\pi_{2,4,5}(3) = [2,4,5]$, $(2,3)$ starts at 3 in $C_{1,R}$. Since $(2,3)$ crosses $(4,5)$ and does not cross $\bar{e}_1$, it must cross C_1 at $(5,6)$, then cross $(4,5)$.

Since there is a unique way to do this, 2 must be in the unique region bounded by the edge $(3,5)$ and the crossing of $(3,6)$ with $(4,5)$. It follows that $(2,6)$ is uniquely determined in $\mathcal{D}_1 + (2,3) + (2,6)$, in particular, $\overrightarrow{(4,5)}$ crosses $\overrightarrow{(2,6)}$ from left to right and $\pi_{2,4,6}(5) = [2,4,6]$.

In $\mathcal{D}_3 + (4,5)$, let the $C_2 = \overrightarrow{(1,2,6,5)}$. Since $\overrightarrow{(4,5)}$ crosses $\overrightarrow{(2,6)}$ from left to right, if $(4,5)$ starts inside $C_{2,R}$, then of the edges of C_2 it would cross $(2,6)$ first. It follows that if $(4,5)$ starts inside $C_{2,R}$, then it would cross $\bar{e}_3$, a contradiction with Claim 1.

Therefore, $(4,5)$ starts outside of $C_{2,R}$. In particular, $\pi_{1,4,6}(5) = [6,4,1]$. $\pi_{2,4,6}(5) = [2,4,6]$ implies that $\pi_{1,2,4,6}(5) = [2,4,1,6]$. Given this rotation at 5 and $\overrightarrow{(4,5)}$ crosses $\overrightarrow{(2,6)}$ from left to right, there is a unique drawing of $(4,5)$ in $\mathcal{D}_3 + (4,5)$. In particular, 4 is contained in the region bounded by $(2,6)$ and segments of $(2,5)$ and $(1,6)$. Extending this drawing to a drawing of $\mathcal{D}_3 + (4,5) + (1,4)$ shows that $(1,4)$ must cross $\bar{e}_3$, a contradiction with Claim 1.

Case 2. *Assume $(4,5)$ crosses $(1,2)$ between the crossing of e with $(1,2)$ and the vertex 2 in D_6.*

In D_6, by deleting 3 and adding the vertex 6 and its incident edges we get a realization $\bar{D}_3$ of $H - \{3\}$ with $(4,5)$ crossing $(1,2)$ between 2 and the crossing of e. Setting $\bar{x} = (5,6), \bar{y} = (1,2), \bar{j} = 4$ and $\bar{z} = 5$, we have a contradiction with Case 1. ■

One remark about Claim 2 is that there is a symmetry. In particular, we can replace W with any four vertices that are the ends of two of $(1,2), (3,4)$ and $(5,6)$. Furthermore, we can replace 5 with 6, or alternatives with any end vertex $[6]$ that is not in W and consider the drawing D_i, where i is not used.

Now we finish the proof of Lemma 5.11. Since D_5 is a simple drawing, the drawing of $\mathcal{D}_5 + (1,3) + (1,4)$ in D_5 is uniquely determined. In particular,

$2 \in \overrightarrow{(1,3,4)}_R$.

By Claims 1 and 2, the simple drawings of $\mathcal{D}_4 + (1,5) + (1,6)$ in D_4 is uniquely determined. In particular, $2 \in \overrightarrow{(1,5,6)}_L$.

By Claims 1 and 2, the simple drawings of $\mathcal{D}_2+(1,3)+(1,4)+(1,5)+(1,6)$ in D_2 is uniquely determined. In particular, $\overrightarrow{(1,3,4)}_R \subset \overrightarrow{(1,5,6)}_R$. In D_e, this must also be the case.

In D_e, since $2 \in \overrightarrow{(1,5,6)}_L$, $2 \notin \overrightarrow{(1,5,6)}_R$. Since $\overrightarrow{(1,3,4)}_R \subset \overrightarrow{(1,5,6)}_R$, $2 \notin \overrightarrow{(1,3,4)}_R$, a contradiction with $2 \in \overrightarrow{(1,3,4)}_R$.

□

We conclude this section with the proof of Theorem 5.6.

Proof of Theorem 5.6. Let H be an $(8,7)$-rotation system and $e = \overrightarrow{(u,v)}$ be a directed edge of H. By way of contradiction, let $\mathcal{C} = (a_0, a_1, ..., a_{k-1}, a_0)$ be a smallest cycle of $<^e_\wedge$ and $<^e_\parallel$ relations such that $a_i <^e a_{i+1}$ for $i \in \mathbb{Z}_k$.

Define the simple drawings D_i to be realizations of $H - \{i\}$ for $i \in V(H)$, and D_e to be a realization of $H - \{u,v\}$.

Let $i \in [k]$. Without loss of generality, assume $a_i = (1,2)$. Let us partition this proof into three cases depending on if $\mathcal{C}$ contains two consecutive $<^e_\parallel$ relations, if $\mathcal{C}$ contains a $<^e_\parallel$ relations and does not contain two consecutive $<^e_\parallel$ relations, or if $\mathcal{C}$ contains no $<^e_\parallel$ relations.

Case 1. *$\mathcal{C}$ contains no $<^e_\parallel$ relation.*

Without loss of generality, set $a_{i+1} = (2,3)$. If an end of a_{i+2} is the vertex 2, then by Observation 5.9, $\mathcal{C}$ is not a smallest cycle of $<^e_\wedge$ and $<^e_\parallel$ relations.

This shows that any three consecutive edges in $\mathcal{C}$ do not contain the same vertex. Without loss of generality, it follows that $a_{i+2} = (3,4)$. By repeated use of Observation 5.9, we find that $\mathcal{C} = (1,2) <^e_\wedge (2,3) <^e_\wedge (3,4) <^e_\wedge (4,5) <^e_\wedge (5,6) <^e_\wedge (6,1) <^e_\wedge (1,2)$. $(1,2)$ and $(4,5)$ can not be ordered in their 6-vertex rotation system with e as they are crossed in different orders in D_6 and D_3. Since they are ordered in each of D_6 and D_3, it follows that $(1,2) <^e_\triangle (4,5) <^e_\triangle (1,2)$, a contradiction with of Lemma 5.10 up to relabelling.

Case 2. *$\mathcal{C}$ contains two consecutive $<^e_\parallel$ relations.*

Without loss of generality, assume $a_i <^e_\parallel a_{i+1} <^e_\parallel a_{i+2}$ and let $a_{i+1} = (3,4)$.

By Observation 5.9, it follows that $a_{i+2} = (5,6)$. If $a_{i+2} <^e_{\parallel} a_{i+3}$, then Observation 5.9 implies that $a_{i+3} = (1,2)$, a contradiction with Lemma 5.11. Therefore, $a_{i+2} <^e_{\wedge} a_{i+3}$ and by Observation 5.9, without loss of generality $a_{i+3} = (1,5)$. If $(1,2) <^e_{\wedge} (1,5)$, then the cycle $(a_0, \ldots, a_i, a_{i+3}, \ldots, a_{k-1})$ is shorter than $\mathcal{C}$, a contradiction.

It follows that $(1,2) <^e_{\parallel} (3,4) <^e_{\parallel} (5,6) <^e_{\wedge} (1,5) <^e_{\wedge} (1,2)$. $(3,4)$ and $(1,5)$ can not be ordered in their 6-vertex rotation system with e as they are in different orders in D_6 and D_2. Since they are ordered in D_6 and D_2, it follows that $(3,4) <^e_{\triangle} (1,5) <^e_{\triangle} (3,4)$, a contradiction with Lemma 5.10 up to relabelling.

Case 3. *$\mathcal{C}$ contains a $<^e_{\parallel}$ relation and does not contain two consecutive $<^e_{\parallel}$ relations.*

Without loss of generality, assume $a_i <^e_{\parallel} a_{i+1}$ and $a_{i+1} = (3,4)$. By Case 2, $a_{i-1} <^e_{\wedge} a_i$ and $a_{i+1} <^e_{\wedge} a_{i+2}$. By Observation 5.9, without loss of generality $a_{i-1} = (1,5)$ and $a_{i+2} = (3,6)$. By Observation 5.9, it is clear that $a_{i+3} = (x,5)$ where $x \in [6]\backslash\{5\}$. If $x \in \{1,3,4\}$, then any outcome of Observation 5.9 would produce a smaller cycle than $\mathcal{C}$, a contradiction. Therefore, $x \in \{2,6\}$.

Case 3.1. *$x = 2$.*

Note that $(2,5) <^e_{\wedge} (1,2) <^e_{\parallel} (3,4) <^e_{\wedge} (3,6) <^e_{\parallel} (2,5)$. $(2,5)$ and $(3,4)$ can not be ordered in their 6-vertex rotation system with e as they are in different orders in D_6 and D_1. Since they are ordered in D_6 and D_1, it follows that $(2,5) <^e_{\triangle} (3,4) <^e_{\triangle} (2,5)$, a contradiction with Lemma 5.10 up to relabelling.

Case 3.2. *$x = 6$.*

Note that $(2,5) <^e_{\wedge} (1,2) <^e_{\parallel} (3,4) <^e_{\wedge} (3,6) <^e_{\wedge} (5,6)$. If $(2,5) <^e_{\wedge} (5,6)$, then the cycle $(a_0, \ldots, a_{i-1}, a_{i+3}, \ldots, a_{k-i})$ is shorter than $\mathcal{C}$, a contradiction. Therefore, $(5,6) <^e_{\wedge} (2,5)$ and $(2,5) <^e_{\wedge} (1,2) <^e_{\parallel} (3,4) <^e_{\wedge} (3,6) <^e_{\wedge} (5,6) <^e_{\wedge} (2,5)$.

$(2,5)$ and $(3,4)$ can not be ordered in their 6-vertex rotation system with e as they are in different orders in D_6 and D_1. Since they are ordered in D_6 and D_1, it follows that $(2,5) <^e_{\triangle} (3,4) <^e_{\triangle} (2,5)$, a contradiction with

Lemma 5.10 up to relabelling. □

5.3 n-Vertex Rotation Systems

We finish Section 5 by proving that $<^e_\wedge$ and $<^e_\|$ induce an acyclic directed graph for $(n, 8)$ rotation systems. To that end, this short section starts with a statement of Theorem 5.12 and is followed by a direct proof.

Theorem 5.12. *If $n \geq 9$, H is an $(n, 8)$-rotation system and e is a directed edge of H, then there are no cycles comprised of $<^e_\wedge$ and $<^e_\|$ relations in H.*

We obtain a short proof by building a weight function and using Claim 1 to show that a minimum weight cycle $\mathcal{C}$ does not exist.

Proof. Let $n \geq 9$, H be an $(n, 8)$-rotation system and $e = \overrightarrow{(u, v)}$ be a directed edge of H. For an arbitrary cycle (string) $\mathcal{A}$ of relations, define $\mathcal{A}_\ell$ to be length $\mathcal{A}$, $\mathcal{A}_\|$ to be the number of $<^e_\|$ relations in $\mathcal{A}$ and $\mathcal{A}_w = \mathcal{A}_\ell + \mathcal{A}_\|$ to be the weight of $\mathcal{A}$.

By way of contradiction, let $\mathcal{C} = (a_0, a_1, ..., a_{k-1}, a_0)$ be a cycle of $<^e_\wedge$ and $<^e_\|$ relations with minimum cycle weight such that $a_i <^e a_{i+1}$ for $i \in \mathbb{Z}_k$.

Claim 1. *There is no string of relations $\mathcal{A}$ in $\mathcal{C}$ defined on at most 8 vertices and having $\mathcal{A}_w \geq 4$.*

Proof. By way of contradiction, assume $\mathcal{C} = (a_i, \ldots, a_j, \ldots, a_i)$ and $\mathcal{A} = (a_i, \ldots, a_j)$ such that $|V(a_i, \ldots, a_j, e)| \leq 8$ and $\mathcal{A}_w \geq 4$. $V(a_i, \ldots, a_j, e)$ induces a realizable 8-vertex rotation system H_1 in H. Let D be a realization of H_1. In D, $a_i <^e a_j$ for some $<^e_\wedge, <^e_\|$ or $<^e_\triangle$ relation.

If this relation is $<^e_\wedge$ or $<^e_\|$, then the cycle $(a_i, a_j, \ldots, a_{i-1}, a_i)$ exists in H and has smaller weight than $\mathcal{C}$, a contradiction.

It follows that $a_i <^e_\triangle a_j$. By Corollary 5.8, there is a string of $<^e_\wedge$ and $<^e_\|$ relations $a_i <^e \cdots <^e a_j$ in H_1 with at most 7-vertices that contributes 3 to the weight of any cycle containing it. Replacing $a_i <^e \cdots <^e a_j$ with this string of relations reduces the weight of $\mathcal{C}$, a contradiction. ■

Let $i \in [k]$. Let us partition this proof into three cases depending on if $\mathcal{C}$ contains two consecutive $<^e_\|$ relations, if $\mathcal{C}$ contains a $<^e_\|$ relation and does not contain two consecutive $<^e_\|$ relations, or if $\mathcal{C}$ contains no $<^e_\|$ relations.

Case 1. *$\mathcal{C}$ contains two consecutive $<^e_\|$ relations.*

Without loss of generality $a_i <^e_{\parallel} a_{i+1} <^e_{\parallel} a_{i+2}$. By Claim 1, this does not occur.

Case 2. *$\mathcal{C}$ contains no $<^e_{\parallel}$ relations.*

In this case, $\mathcal{C}_\ell > 4$ as a cycle of $<^e_{\wedge}$ relations of length at most 4 is defined on some realizable 8-vertex rotation system in H, a contradiction. It follows that $a_i <^e_{\wedge} a_{i+1} <^e_{\wedge} a_{i+2} <^e_{\wedge} a_{i+3} <^e_{\wedge} a_{i+4}$ is a string of relations in $\mathcal{C}$ with distinct edges defined on at most 8 vertices, a contradiction with Claim 1.

Case 3. *$\mathcal{C}$ contains a $<^e_{\parallel}$ relation, but does not contain two consecutive $<^e_{\parallel}$ relations.*

Without loss of generality, assume $a_i <^e_{\parallel} a_{i+1}$. By Case 1, without loss of generality $a_{i-1} <^e_{\wedge} a_i <^e_{\parallel} a_{i+1} <^e_{\wedge} a_{i+2}$. It is clear that a_{i-1} and a_{i+2} are distinct edges as they share an end with a_i and a_{i+1}, respectively. Observe that $a_{i-1} <^e_{\wedge} a_i <^e_{\parallel} a_{i+1} <^e_{\wedge} a_{i+2}$ is defined on at most 8 vertices, a contradiction with Claim 1, as desired. $\square$

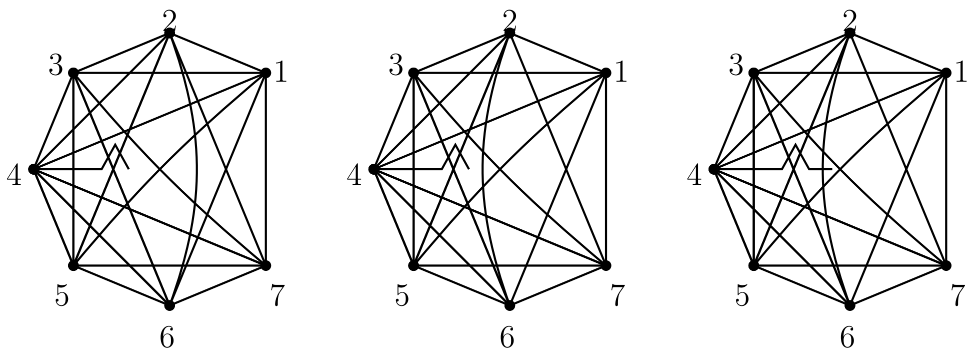


Figure 37: Drawing a partial edge $(4, 8_4)$ into a drawing of K_7.

6 Simple Drawings of $(n, 6)$-Rotation Systems

It is important to keep in mind the process in which we will find a realization of an $(n, n-1)$ rotation system H_n. Given such a rotation system, we will draw the smallest known realizable rotation system contained in H_n, which in this section will be a simple drawing of K_6. From there, we find a new drawing that contains a single new edge to a new vertex v. For each edge incident to v after the first, we will draw that edge from v sequentially by its crossing segments, possibly choosing to cross other edges incident to v and possibly choosing to change the underlying drawing without v. The resulting drawing is not guaranteed to be simple, however, it will have the rotation at every vertex other than v coincide with its rotation in H_n. Finally, untangling the edges will result in a simple drawing, and a short argument will show that the simple drawing is a realization of H_n.

We start this section by showing in Lemma 6.1 that there is a simple drawing of $K_{n-1} + e$ for some $(n, n-1)$-rotation system. The statement and proof of this is similar to Lemma 4.13, however, Lemma 6.1 will only be used to form a base case for the induction of $(n, n-1)$-rotation systems. After the base case has been established, we have all the tools to make an inductive proof that $(n, n-1)$-rotation systems are realizable for $n \geq 6$, an equivalent statement to Theorem 1.1.

The proof of Lemma 6.1 is straight forward. We start with an $(n, n-1)$-rotation system H and a realization D of K_{n-1}. Commence by starting to draw an edge e from a vertex u in K_{n-1} to an nth vertex v. Since we know that

$<^e$ is a partial ordering, we can always find an edge c that must be crossed next. Since no drawing of K_4 has c separated from the face containing the non-vertex end of our partial edge we are drawing, by Theorem 3.8, there is a set of Reidemeister III moves that results in a drawing of c on the boundary of the face that our partial edge has crossed into. Crossing c and applying a simple induction gives us the result.

As an example, we see in Figure 37 that the edge $(4, 8)$ is being drawn into a simple drawing of K_7. In the left simple drawing, $(4, 8)$ needs to cross $(2, 6)$, however $(2, 6)$ does not bound the correct face. The center drawing is obtained from the left by performing a Reidemeister III move on edges $\triangle_{(2,6),(3,7),(1,5)}$ to move a non-trivial segment of $(2, 6)$ to the appropriate face. Finally, the right drawing is obtained from the center drawing by having $(4, 8)$ cross $(2, 6)$.

Lemma 6.1. *Let $n \geq 6$, H be an $(n, n-1)$-rotation system, and $e = \overrightarrow{(u, v)}$ a directed edge of H. If D_v is a realization of $H - \{v\}$, then there exists a simple drawing $\bar{D}_v + e$ that has v in its respective region determined by H, the rotation at u is the same as H_n, and the rotation at every vertex in $V(H_n - \{u, v\})$ is the same in both $\bar{D}_v + e$ and $H_n - v$.*

Proof. By Lemma 4.15, we let $n \geq 7$. Let H be an $(n, n-1)$-rotation system on the vertices $[n-2] \cup \{u, v\}$, $e = \overrightarrow{(u, v)}$ be a directed edge of H, D_i be a realization of $H - \{i\}$ for $i \in V(H)$, D_e be a realization of $H - \{u, v\}$, and E be the set of edges that e crosses determined by H.

Let G be the directed graph where $V(G) = E$ and the arc $\overrightarrow{(f, g)}$ exists if $f <^e_\wedge g$ or $f <^e_\parallel g$. By Theorems 5.1, 5.6, and 5.12, G contains no directed cycles.

By greedily picking and deleting source vertices of G, it follows that there exists a sequence of vertices $\{c_i\}_{i=1}^{cr_H(e)}$ such that for $i_1 > i_2$, then there is no directed path from c_{i_1} to c_{i_2} in G. In particular, no chain of $<^e_\wedge$ or $<^e_\parallel$ relations has $c_{i_1} <^e \cdots <^e c_{i_2}$.

Let e_i be the segment of e starting at u in the correct rotation that has crossed c_1 to c_i and ends at a point v_i in some face of D_v, and D_i be the drawing of $D_v + e_i$ with the rotation at u the same as H_n. We will prove D_i exists for all $i \in \{0\} \cup [cr_H(e)]$ inductively on i.

For the sake of the reader, we illustrate an example of $D_v + e_i$ in Figure 38 for $i = 3$ with $e_i = (4, 8_3)$ represented by the dashed line in the figure and

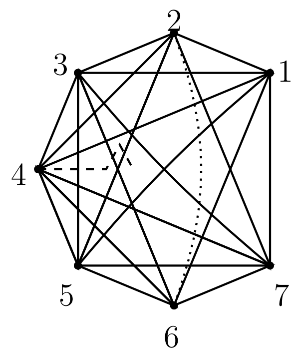


Figure 38: $D_8 + (4, 8_3)$.

$c_{i+1} = (2, 6)$ dotted line in the figure representing the next edge in $\{c_i\}_{i=1}^{cr_H(e)}$ that e_i crosses.

Note that the drawing of D_v satisfies the definition of D_0 by starting e_0 in the correct rotation at u.

Assume D_i exists for some $i \in \{0\} \cup [cr_H(e) - 1]$. By way of contradiction, assume D_{i+1} does not exist. Let R_{v_i} be the face in D_{i-1} containing v_i (if i=0, then let $D_{i-1} = D_v$). There are two cases dependent upon $i = cr_H(e)$ or $i < cr_H(e)$.

Case 1. $i = cr_H(e)$.

In this case, e_i has crossed all edges of the sequence $\{c_i\}_{i=1}^{cr_H(e)}$ in D_i. Let T be the intersection of sides of triangles in D_v containing v determined by H. If R_{v_i} does not correspond to T, then by Corollary 3.3, there is some 3-cycle having e_i and v on separate sides.

Let D_{T+e} be a realization of the 5-vertex rotation system induced on T and e in H. Since e crosses edges of D_v in the order of the sequence $\{c_i\}_{i=1}^{cr_H(e)}$, and does so in the correct orientation, it follows that the crossings of e with T are the same in both D_{e_i} and D_{T+e}, a contradiction with v and e_i being on opposite sides of T in D_{e_i}.

It follows that R_{v_i} correspond to T. Drawing the vertex v at the end of e_i in D_i produces the desired simple drawing $\bar{D}_v + e$.

Case 2. $i < cr_H(e)$.

By Lemma 3.4, the only segment of e_i in R_{v_i} is the segment formed by e_{i-1} crossing into R_{v_i}. It follows that if c_{i+1} is on the boundary of R_{v_i}, then e_i can be extended in D_i to cross c_{i+1} in the correct orientation to create D_{i+1}. If the orientation of this crossing does not follow H, then for any 3-cycle T containing c_{i+1}, e_i is on the wrong side of T to cross c_{i+1}.

Let D_{T+e} be a realization of the 5-vertex rotation system induced on T and e in H. Since e crosses edges of D_v in the order of the sequence $\{c_i\}_{i=1}^{cr_H(e)}$, and does so in the correct orientation, it follows that the crossings of e with T are the same in both D_{e_i} and D_{T+e}. By definition of $\{c_i\}_{i=1}^{cr_H(e)}$, c_{i+1} is the next edge e_i crosses in D_{T+e}, a contradiction with e_i being on wrong sides of T in D_{e_i} to cross c_{i+1}.

It follows that c_{i+1} is not on the boundary of R_{v_i}, else we would cross it. Theorem 3.8 give two possibilities for c_{i+1}

Case 2.1. *There is some drawing $\mathcal{D}$ of D_v on a K_4 containing c_{i+1} such that no face of $\mathcal{D}$ contains R_{v_i} and has c_{i+1} on its boundary.*

By Lemma 4.12, there is some 3-cycle T in $\mathcal{D}$ that has e_i on the wrong side of T to cross c_{i+1}. Let D_{T+e} be a realization of the 5-vertex rotation system induced on T and e in H. Since e crosses edges of D_v in the order of the sequence $\{c_i\}_{i=1}^{cr_H(e)}$, and does so in the correct orientation, it follows that the crossings of e with T are the same in both D_{e_i} and D_{T+e}. By definition of $\{c_i\}_{i=1}^{cr_H(e)}$, c_{i+1} is the next edge e_i crosses in D_{T+e}, a contradiction with e_i being on wrong sides of T in D_{e_i} to cross c_{i+1}.

Case 2.2. *There are sets of edges X_i and a sequence of Reidemeister III moves $\{\rho_{X_i}\}_{i=1}^{k}$ in D_v that places c_{i+1} on the boundary of the component containing v_i such that the intersection of each $\triangle_{X_i}$ with v_i is empty, and for each Reidemeister III move ρ_{X_i} not on the edge c, there exists a j such that ρ_{X_j} is on the edge c and $\triangle_{X_i}$ is contained in $\triangle_{X_j}$.*

If the non-vertex end of e_i was inside any $\triangle_{X_i}$, then R_{v_i} is in $\triangle_{X_i}$, and thus, $P \in \triangle_{X_i}$, a contradiction. Therefore, the non-vertex end of v_i is not in $\triangle_{X_i}$ for any i. By Lemma 3.4, it follows that there is at most one non-trivial segment of e_i in any $\triangle_{X_i}$ that crosses in and out of the region. Applying a Reidemeister III move over this segment, allows us to perform next apply

ρ_{X_i} and continue.

It follows that c_{i+1} can be placed on the boundary of R_{v_i}. Having e_i cross c_{i+1} produces the drawing D_{i+1}, a contradiction with D_{i+} not existing. $\square$

We come to the first of two main theorems in this thesis, the proof of which is similar to the proof of Lemma 6.1. To finish this section and prove Theorem 6.2, we use Lemma 6.1 to form a base case for induction. In the inductive step, we draw a new edge $f = (u_f, v)$ sequentially by its crossing from v to u_f and use the partial ordering $<^e$ found in Section 5 at each iteration to determine which edge c is crossed next.

To make sure we can produce a drawing in which c is crossed next, we use Theorem 3.8 from Section 3 to show we can change the underlying drawing without v (by using Reidemeister III moves) so that our partial edge can cross c (possibly having to cross other edges incident to v). It is important that we check that for each Reidemeister III move on $\triangle_{x,y,z}$ for arbitrary edges x, y, z, that $v \notin \triangle_{x,y,z}$. To prove this, we use the special conditions ii. and iii. in Theorem 3.8

Once all the crossings have been performed, we argue we can connect this partial edge at u_f to form f. The only problem with the resulting drawing is that the edge (u_f, v) may have crosses other edges incident to v. We finish by turning the associated drawing into a simple drawing by untangling the edges and argue that the rotation at each vertex is the same as the rotations in H_n, as desired.

Theorem 6.2. *Let $n \geq 6$. If H_n is an $(n, n-1)$-rotation system, then H_n is realizable.*

Proof. Let $n \geq 6$, H_n be an $(n, n-1)$-rotation system and v be a vertex in $V(H_n)$. By Lemma 4.15, without loss of generality $n \geq 7$.

Let E_v be the set of edges having v as an endpoint. Let us build a simple drawing D of K_n with vertex set $V(H_n)$ that has the rotation at every vertex other than v the same as in H_n. By Observation 2.6 and the fact that H_n is an $(n, 4)$-rotation system, it would follow that the rotation at v is correct and D is a simple drawing realizing H_n

To show such a drawing D exists, we will show there is a simple drawing $D_i + F_i$ such that $F_i \subseteq E(v)$, $1 \leq i \leq n-1$, $|F_i| = i$, $D_i + F_i$ has the same rotations at each vertex as $H_n - v$, v is in the correct region in D_i determined by H_n, and for every edge (u_j, v) in F_i, the rotation at u_j in $D_i + F_i$ is the same as the rotation at u_j in H_n.

Let us prove this by induction on i. Let $e = (u_e, v)$ be an edge having v as an endpoint and $F_1 = \{e\}$. By Lemma 6.1, there is a simple drawings $D_1 + F_1$ that has v in its respective region determined by H_n, the rotation at u_e is the same as in H_n, and the rotation at every vertex in $V(H_n - \{u_1, v\})$ is the same in both $D_1 + F_1$ and $H_n - v$.

Assume for some i, $D_i + F_i$ exists. All we need to show it $D_{i+1} + F_{i+1}$ exists. Let $f = (u_f, v)$ be an edge in $E(v)\backslash F_i$ and E_f be the set of edges f crosses in H_n.

Let f be directed from v to u_f and G be the directed graph where $V(G) = E_f$ and the arc $\overrightarrow{(g,h)}$ exists if $g <^f_\wedge h$ or $g <^f_\| h$. By Theorems 5.1, 5.6, and 5.12, G contains no directed cycles.

By greedily picking and deleting source vertices of G, it follows that there exists a sequence of vertices $\{c_k\}_{k=1}^{cr_{H_n}(f)}$ such that for $k_1 > k_2$ there is no directed path from c_{k_1} to c_{k_2} in G. In particular, no chain of $<^f_\wedge$ or $<^f_\|$ relations has $c_{k_1} <^f < \cdots <^f c_{k_2}$.

Similar to the proof of Lemma 6.1, we show by induction there is a simple drawing $D_i^j + F_i + f_j$ for $0 \leq j \leq cr_{H_n}(f)$ such that f_j is an edge starting at v crossing $\{c_k\}_{k=1}^{j}$ in their respective orders and oriented crossings, $D_i^j + F_i + f_j$ has the same rotations at each vertex as $H_n - v$, and for every edge (u_j, v) in $F_i \cup f$, the rotation at u_j in $D_i^j + F_i + f_j$ is the same as the rotation at u_j in H_n.

For the sake of the reader, we illustrate an example of $D_i^j + F_i + f_j$ in Figure 39 for $i = 3$ $j = 2$ with $F_2 = \{(1, 8), (7, 8),$ $f_j = (4, 8_3)$ represented by the dashed line in the figure and $c_{j+1} = (2, 6)$ dotted line in the figure representing the next edge in $\{c_k\}_{k=1}^{cr_{H_n}(f)}$ that f_j crosses.

By drawing a small segment of an edge at v in $D_i + F_i$, it is clear $D_i^0 + F_i + f_0$ exists.

By induction, assume $D_i^j + F_i + f_j$ exists. Let v_j be the non-vertex end of f_j and let R_{v_j} be the face in $D_i^j + F_i + f_j$ that contains v_j. There are two cases depending on if $j = cr_H(f)$ or $j < cr_f(H)$.

Case 1. $j = cr_H(f)$.

In this case, the edge f_j has crossed all edges of $\{c_k\}_{k=1}^{cr_{H_n}(f)}$ in $D_i^j + F_i + f_j$. Let T be the intersection of sides of triangles in D_j^i containing u_f determined by H_n. If R_{v_j} does not correspond to T, then by Corollary 3.3, there is some 3-cycle T in $\mathcal{D}$ that has f_j and u_f on opposite sides of T.

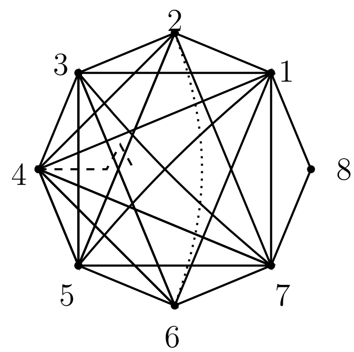


Figure 39: $D_2^3 + \{(1,8),(7,8)\} + (4,8_3)$.

Let D_{T+f} be a realization of the 5-vertex rotation system induced on T and f in H_n. Since f crosses edges of $D_i^j + F_i + f_j$ in the order of the sequence $\{c_k\}_{k=1}^{cr_{H_n}(f)}$, and does so in the correct orientation, it follows that the crossings of f with T are the same in both $D_i^j + F_i + f_j$ and D_{T+f}, a contradiction with u_f and f_j on opposite sides of T.

Therefore, R_{v_j} correspond to T. It follows that u_f is in R_{v_j}. Since D_j^i is simple, f_j can be extended to connect v_j to u_f (possibly crossing edges of F_i) in R_{v_j} producing a drawing $D_{i+1} + F_{i+1}$ with $D_{i+1} + f_j$ being simple. Since R_{v_j} corresponds to T, observation 3.7 applied to the drawing of the K_4 involving u_f, v and the former and ladder consecutive vertices around v in the rotation at u_f in H_n implies that the rotation at u_f in the drawings corresponds to the rotation at u_f in H_n.

This drawing may not be simple, to be exact, f may cross the edges in F_i at least once, but finitely many times. Assume this is the case, as if not we are done. We will show there is a simple drawing $D_{i+1} + \bar{F}_{i+1}$ that preserves the rotation of the vertices in $D_{i+1} + F_{i+1}$ other than v such that $D_{i+1} + \bar{F}_i$ is simple and corresponds to H_n.

Let $g \in F_i$ be an edge that crosses f in $D_{i+1} + F_{i+1}$ such that $g = (u_g, v)$. By Lemma 4.14, there exists a closed curve δ comprised of a non-trivial segment f^c of f and a non-trivial segments g^c of g such that a side S of δ does not contain v, u_g and u_f. In $D_{i+1} + F_{i+1}$, S could contain a vertex.

Case 1.1. *S contains a vertex z.*

Let $x \in \{u_g, u_f\}$. In $D_{i+1} + F_{i+1}$, since u_x is not in S and z is in S, it must be the case that (z, u_x) crosses δ. Since $D_{i+1} + (u_x, v)$ is simple, it follows that (z, u_x) crosses (u_y, v) for $y \in \{u_f, u_g\} \backslash \{x\}$ in $D_{i+1} + F_{i+1}$, and thus in $D_{i+1} + (u_x, v)$.

The rotation system on v, u_f, u_g and z in H_n is realizable, and contains at most one crossing. It follows that at least one of the crossings (z, u_x) with (u_y, v) does not occur. By Observation 2.6, one of the rotations at u_g, u_f or z in $D_{i+1} + f$ or $D_{i+1} + g$ does not match its rotation in H_n, a contradiction.

Case 1.2. *S contains no vertices.*

Let h be an edge that crosses δ. Since S contains no vertices, it follows that h crosses δ twice, once at f_c and once at g_c. It follows that an edge crosses f_c if and only if it crosses g_c.

Let $\bar{f}$ be the edge f rerouted to take g_c instead of f_c. Similarly, let $\bar{g}$ to be the edge g rerouted to take f_c instead of g_c. Since f_c and g_c cross the same edges, it follows that $\bar{f}$ and $\bar{g}$ cross the same edges as f and g, respectively.

Consider the new drawing of $D_{i+1} + \bar{F}_{i+1}$ that has $\bar{f}$ replacing f and $\bar{g}$ replacing g in $D_{i+1} + F_{i+1}$ with the crossings on δ uncrossed.

If δ has v on the boundary, then the number of crossings between f and g has reduced by 1. If δ does not have v on the boundary, then the number of crossings between f and g has reduced by 2.

Comparing the rotations of the vertices, only the rotation at v could have possible changed. Repeatedly applying this procedure results in a drawing that is simple, and has all the rotations at every vertex the same as in H_n other than v, as desired.

Case 2. *$j < cr_f(H)$.*

If c_{j+1} is on the boundary of R_{v_j}, then we cross edges of F_i and cross c_{j+1} to form a drawing $D_i^{j+1} + F_i + f_{j+1}$. Such a drawing may not be simple, however, a similar argument as in Case 1.2. using Lemma 4.14, produces a desired simple drawing. Therefore, without loss of generality, c_{j+1} is not on the boundary of R_{v_j}.

Suppose there is a drawing D_c on some K_4 containing c_{j+1} in D_i^j that separates c_{j+1} from R_{v_j}. The vertices of such a K_4 along with the ends of the edge f are defined on a 6-vertex rotation system in H_n.

If both u_f and v are not in D_c, then by definition of $\{c_j\}_{j=1}^{cr_H(e)}$ and Lemma 4.12, D_c can be extended with f_j crossing c_{j+1}, a contradiction with no face of D_c containing R_{v_i} and having c_{j+1} on its boundary.

Therefore, at least one of u_f or v is in D_c. If follows that $D_c + f_j$ is a partial drawing of some realizable 5-vertex rotation system of H. A realization os such a 5-vertex rotation system must contain $D_c + f_j$ as D_c is unique to the rotation system and f_j follows $\{c_j\}_{j=1}^{cr_H(e)}$. Therefore, f_j crosses c_{j+1} next in the realization and can not do so in $D_c + f_j$, a contradiction.

1. There is a sequence (possibly empty) of Reidemeister III moves $\{\rho_{X_i}\}_{i=1}^{k}$ with sets of edges X_i such that $D_{i+1} = \rho_{X_i}(D_i)$ with:

   i. A non-trivial segment of c is on the boundary of the face of D_{k+1} containing P;

   ii. $P \notin \triangle_{X_i}$, $\forall i \in [k]$; and

   iii. For $i \in [k]$, if $c \notin X_i$, then there exists $j > i$ in $[k]$ such that $c \in X_j$ and $\triangle_{X_i} \subset \triangle_{X_j}$ in D_i; or

It follows by Theorem 3.8, that there are sets of edges sets of edges X_ℓ and a sequence of Reidemeister III moves $\{\rho_{X_\ell}\}$ in D_i^j that places c on the boundary of the component containing v_j such that the intersection of each $\triangle_{X_\ell}$ with v_j is empty, and for each Reidemeister III move $\rho_{X_{\ell_1}}$ not on the edge c, there exists an $\ell_2 > \ell_1$ such that $\rho_{X_{\ell_2}}$ is on the edge c and $\triangle_{X_{\ell_1}}$ is contained in $\triangle_{X_{\ell_2}}$.

There are two cases, either for every ρ_{X_ℓ}, $\triangle_{X_\ell}$ does not contain v, or there exists a ρ_{X_ℓ} such that $\triangle_{X_\ell}$ contains v.

Case 2.1. *For every ρ_{X_ℓ}, $\triangle_{X_\ell}$ does not contain v.*

Apply the Reidemeister III moves ρ_{X_ℓ} to $D_i^j + F_i + f_j$ until we find an ℓ_1 such that $\triangle_{X_{\ell_1}}$ is not empty. If ℓ_1 does not exist, then we can extend f_j (crossing edges in F_i) to cross c_{i+1} after applying the Reidemeiter III moves to form a drawing $D_i^{j+1} + F_i + f_{j+1}$. Such a drawing again can be reduced to be simple, similar to Case 1.2. using Lemma 4.14.

Therefore, such an ℓ_1 exists. Let $\tilde{D}_i^j + F_i + f_j$ be the simple drawings after applying the Reidemeister III moves $\{\rho_{X_\ell}\}_{\ell=1}^{\ell_1-1}$. Note P, v, and nothing in $\tilde{D}_i^j$ is in $\triangle_{X_{\ell_1}}$. It follows that only segments of edges in F_i and segments of f_j can be in $\triangle_{X_{\ell_1}}$.

Since $\tilde{D}_i^j + F_i + f_j$ is simple, it follows that these segments are not crossing. Since these segments are not crossings, they each cross the same pair of edges g_1 and g_2 on the boundary of $\triangle_{X_{\ell_1}}$.

It follows that there exists an edge segment h that crosses both g_1 and g_2 such that one side of $\gamma_{g_1,g_2,h}$ is empty and contained in $\triangle_{X_{\ell_1}}$. Applying a Reidemeister III move to this triple of edges over $\gamma_{g_1,g_2,h}$ reduces the number of edge segments crossing $\triangle_{X_{\ell_1}}$.

Repeatedly applying this process results in a drawing with $\triangle_{X_{\ell_1}}$ empty, and thus $\rho_{X_{\ell_1}}$ can be applied.

Repeatedly applying the rest of the Reidemeister III moves in $\{\rho_{X_\ell}\}$ in a similar fashion will produce a simple drawing $D_v^\rho + F_i + f_j$ with c_{i+1} on the boundary of the region containing v_j in D_v^ρ. Extend f_j at v_j in this drawing to cross c_{j+1} (possibly crossing edges of F_i) to form a drawing $D_i^{j+1} + F_i + f_{j+1}$. Such a drawing may not be simple, however, a similar argument as in Case 1.2. using Lemma 4.14, produces a desired simple drawing. Therefore, without loss of generality, c_{i+1} is not on the boundary of R_{v_j}.

Case 2.2. *There exists a ρ_{X_ℓ} such that $\triangle_{X_\ell}$ contains v.*

By Theorem 3.8, there exists X_c such that $v \in \triangle_{X_c}$ for $\ell < c$. We will show the existence of such a drawing contradicts the fact that c_{i+1} is the next edge to cross as per the sequence $\{c_j\}_{j=1}^{cr_H(e)}$. Let $(v_1, w_1), (v_2, w_2)$ and $c_{i+1} = (v_c, w_c)$ be the three edges that bound $\triangle_{X_c}$.

Consider the simple subdrawing $D_7 + f_j$ in $D_i^j + F_i + f_j$ induced on f_j and the vertices v_x, w_x and u_f, for $x \in \{1, 2, c\}$ and let D_6 be the induced drawing on the vertices v_x, w_x.

By our assumption, $v \in \triangle_{X_c}$. Without loss of generality, label the ends of (v_x, w_x) in such a way that when traversing (v_x, w_x), starting at v_x the crossing of c_{i+1} precedes the crossings of (v_y, w_y), for $y \in \{1, 2\}\backslash\{x\}$.

By Theorem 3.8, R_{v_j} is not in $\triangle_{X_c}$. Since there is exactly one Reidemeister III move in D_6 on c, it follows that R_{v_j} is contained on the side S_w of $\gamma_{(v_1,w_1),(v_2,w_2),(w_1,w_2)}$ not containing v, and the face containing R_{v_j} in $D_7 + f_j$ has the crossing of (v_1, w_1) and (v_2, w_2) on the boundary.

Claim 1. $|V(D_7 + f_j)| < 8$.

Proof. Since $|V(D_7 + f_j| \geq 8$, $n \geq 8$. Consider the simple subdrawing D_{u_f} on edges $(v_1.w_1), (v_2, w_2), c, (w_1, w_1)$ and vertex v in $D_7 + f_j$. Such a drawing is contained in every realization of its associated 7-vertex rotation system. Let $\bar{D}_f$ be such a realization.

By the position of v, there is no sequence of Reidemeister III moves that brings c_{i+1} to the boundary of R_{v_j} in $\bar{D}_f$. By Theorem 3.8, that there is a K_4 in $\bar{D}_f$ that separates c_{i+1} from R_{v_j}. Since there exists a sequence of Reidemeister III moves in D_i^j that bring c_{i+1} to R_{v_j}, it follows that this K_4 contains v.

Since the drawing of this K_4 is uniquely determined by H_n and f_j follows $\{c_k\}_{k=1}^{cr_{H_n}(f)}$, it follows that the drawing of this K_4 along with f_j is contained in any realization of its induced 5-vertex rotation system, a contradiction with c_{i+1} being the next edge crosses in this K_4 and the K_4 separating R_{v_j} from c_{i+1}. ■

Let $D_{v_x} + f_j$ be the drawing of $D_7 + f_j$ without v_x, for $x \in \{1, 2\}$. Since v_j is the non-vertex end of f_j and f_j does not cross c_{i+1}, it follows up to symmetry on (v_1, w_1) and (v_2, w_2) that starting at v, f_j crosses (v_1, w_1) then crosses (w_1, w_2) into S_w, or f_j crosses (v_1, w_1) then crosses (v_2, w_2) into S_w.

Case 2.2.1. *f_j crosses (v_1, w_1) then crosses (w_1, w_2).*

It is clear that the edges (v_x, w_x) all have distinct endpoints for $x \in \{1, 2, c\}$, as each pair of edges crosses in a simple drawing.

By Claim 1, $|V(D_7 + f_j)| < 8$. By the crossings of f_j, $u_f = v_2$. Let D_f be a realization of the induced 5-vertex rotation system on v, u_f, w_1, w_2, v_1 in H_n. The drawing of the K_4 on w_1, w_2, v_1, v_2 is the same in both D_f and $D_7 + f_j$. Since f_j crosses edges in the order $\{c_k\}_{k=1}^{cr_{H_n}(f)}$, it follows that the drawing of w_1, w_2, v_1, v_2 and f_j is the same in both D_f and $D_7 + f_j$.

Note that f_j can not cross out $\gamma_{(v_1,w_1),(v_2,w_2),(w_1,w_2)}$ as it has crossed two of the three edges already, and is adjacent to the third at u_f in D_f. However, f ends at u_f and so it does cross this curve, a contradiction.

Case 2.2.2. *f_j crosses (v_1, w_1) then crosses (v_2, w_2).*

Each pair of edge of $c, f, (v_1, w_1), (v_2, w_2)$ cross in H_n, therefore $|V(D_7 + f_j)| = 8$, a contradiction with Claim 1. □

A simple induction using Theorem 6.2 gives way to a proof of Theorem 1.1.

7 Gioan's Theorem

This section is dedicated to Theorem 1.3 which states that two simple drawings D and D' of the same rotation system are the same up to a set of Reidemeister III moves applied to D. Together with Theorem 1.1, we have a complete characterization for when two simple drawings of K_n are different. In particular, two simple drawings of K_n either differ on their associated rotation systems or there is a sequence of Reidemeister III moves from one to the other. We restate Theorem 1.3 for the readers convenience.

Theorem 1.3. *Let n be a positive integer, H_n be a realizable complete n-vertex rotation system. If D and D' are two simple drawings realizing H_n, then one can be obtained from the other through a series of Reidemeister III moves.*

To prove Theorem 1.3, we start by comparing two simple drawings D and D' with the same rotation system. Suppose D and D' agree on $K[\{v_1, \ldots, v_r\}]$ plus some edges at v_{r+1} and a partial edge e_i starting at v_{r+1}, but they do not agree on e_{i+1} the extension of e_i in both drawings. In particular, c the last edge e_{i+1} crosses in D is not the same edge as c' the last edge e_{i+1} crosses in D'. The idea behind the proof of Theorem 7 is that e_i can cross an edge h that is one step closer to c' than c resulting in a realization $\bar{D}$ that has the same rotation system as D and D' as seen in Figure 40. Furthermore, Lemma 7.2 shows there is a sequence of Reidemeister III moves from D to $\bar{D}$ if $\triangle_{c,e,h}$ exists (where e is the edge we are drawing). Lemma 7.4 shows that consecutive edges like c and h always have the property that $\triangle_{c,e,h}$ exists. Applying a simple induction proves Theorem 1.3.

Observation 7.1. *Let D be a simple drawing of three directed edges e, f and g such that $\triangle_{\{e,f,g\}}$ exists. If the order that e crosses f and g is known and pairwise two oriented crossings of e, f and g are known, then topologically there are exactly two drawings that realize the given information each having different oriented crossings.*

Suppose D is a simple drawing of K_n, L is a subdrawing of D with a partial edge e_i and c is the next edge e_i crosses in D. We proceed by characterizing when a neighbour of c can be crossed in L to extend to a simple drawing of K_n with the same associated rotation system as D. Furthermore, we connect this new drawing and D by a series of Reidemeister III moves.

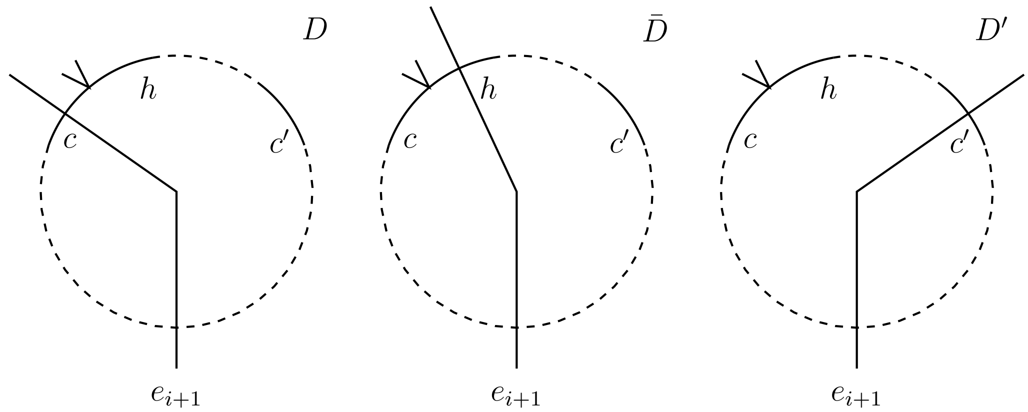


Figure 40: From D to $\bar{D}$ and eventually to D'.

Lemma 7.2. *Let $n \geq 6$, H_n be a realizable n-vertex rotation system, D be a simple drawing realizing H_n, $e = (u, v)$ be an edge of H_n and F_v be a subset of edges having v as an endpoint in D. Let $r \geq 5$, L be a subdrawing of D on some complete graph K_r along with the edges F_v containing a segment of e labelled e_i starting at v and has i crossings in L, and R_L be the region containing the non-vertex end of e_i in L.*

If e_i crosses f next in D, f and g are consecutive edges on the boundary of R_L, $\triangle_{\{e,f,g\}}$ exists in D and e_i has not crossed g in L, then there is a simple drawing $\bar{D}$ containing L with e_i crossing g next that realizes H_n and there is a sequence of Reidemeister III moves $\{\rho_j\}_{j=1}^{k}$ from D to $\bar{D}$.

Proof. The goal of this proof is to find a series of Reidemeister III moves from D to a drawing $\bar{D}$ such that each move is on a triangle that contains an edge segment not in L and the last Reidemeister III move is on $\triangle_{\{e,f,g\}}$. Such a drawing $\bar{D}$ keeps the rotations at the vertices the same, contains the subdrawing L and has e_i crossing g next.

Let H_n be a realizable n-vertex rotation system and D be a simple drawing realizing H_n, and $e = (u, v)$ be an edge of H_n.

Let $n \geq 6$, H_n be a realizable n-vertex rotation system, D be a simple drawing realizing H_n, $e = (u, v)$ be an edge of H_n, E_v be the set of edges having v as an endpoint in D, and $F_v \subset E_v$. Let $r \geq 5$, L be a subdrawing of D on some complete graph K_r along with the edges F_v containing a segment of e labelled e_i starting at v and has i crossings in L, and R_L be the region containing the non-vertex end of e_i in L. Assume e_i crosses f next in D, f

and g are consecutive edges on the boundary of R_L, $\triangle_{\{e,f,g\}}$ exists in D and e_i has not crossed g in L.

Define e', f' and g' to be the segments of e, f and g respectively, that form the boundary of $\triangle aef$ in D and let $\bar{f}$ and $\bar{g}$ be the consecutive segments of f and g respectively, that are on the boundary of R_L.

Observe that f' is the segment of $\bar{f}$ from the crossings of f and g to the crossing of e and f in D. It follows that, other than g, no edge in L crosses f'. In D, since $\triangle aef$ contains no vertices, their are three types of segments that could cross $\triangle aef$:

- X the set of segments that cross f' and g';
- Y the set of segments that cross f' and e'; and
- Z the set of segments that cross e' and g'.

At this point, note that no edge segment in L is in X, and the only edge segment in L that could be in Y is a segment of g. Let a_1 be the first edge segment from the f, g-crossing on f' that is in X, if no such edge segment exists, then we have successfully emptied X. We will empty $\triangle_{a_1,e,f}$ of crossings by using Reidemeister III moves on edges that are not in L, showing we can empty $\triangle_{\{e,f,g\}}$ of a_1 and thus empty $\triangle_{\{e,f,g\}}$ of segments in X.

Note that the segment of f on $\triangle_{a_1,f,g}$ has no crossings other than at its ends. By Lemma 2.17, there is a triangle $\triangle_{f,g_1,h_1}$ contained in $\triangle_{a_1,f,g}$ that is empty of edges and contains a segment of f. Since $\triangle_{f,g_1,h_1}$ is contained in $\triangle_{a_1,f,g}$, it follows for both g_1 and h_1 that some segment of g_1 and h_1 is contained $X \cup Y$. Since one of these edges is not g, it follows that one of the segments on the boundary of $\triangle_{f,g_1,h_1}$ is not in L.

Applying a Reidemeister III move to $\triangle_{f,g_1,h_1}$ reduces the number of crossings in $\triangle_{a_1,f,g}$, as desired. Using Reidemeister III moves to repeat this process shows we can empty $\triangle_{a_1,f,g}$. It follows that there is a series of Reidemeister III moves from D to a drawing D_1 each having an edge segment not in L that empties $\triangle_{\{e,f,g\}}$ of segments of type X. This implies D_1 contains the subdrawing L.

Applying the same argument to segments in Y results in a series of Reidemeister III moves each containing an edge segment not in L from D_1 to a drawing D_2 that contains the subdrawing L which has $\triangle_{\{e,f,g\}}$ empty of segments from $X \cup Y$.

Let a_2 be the first edge segment from the e, g-crossing on e' that is in Z in D_2, if no such edge segment exists, then we have successfully emptied Z.

Note that $\triangle_{a_2,e,g}$ is contained in $\triangle_{\{e,f,g\}}$. We will empty $\triangle_{a_2,e,g}$ of crossings by using Reidemeister III moves on edges that are not in L, showing we can empty $\triangle_{\{e,f,g\}}$ of a_2 and thus empty $\triangle_{\{e,f,g\}}$ of segments in Z.

By definition of a_2, Lemma 2.17 applies and implies that there is a triangle $\triangle_{e,g_2,h_2}$ that is contained in $\triangle_{a_2,e,g}$ that is empty of edges and has a segment of e on the boundary. Not that the segment of e on the boundary of $\triangle_{e,g_2,h_2}$ is contained on the segment of e on the boundary of $\triangle_{a_2,e,g}$ which is contained on the segment of e on the boundary of $\triangle_{\{e,f,g\}}$. It follows that the segment of e on the boundary of $\triangle_{e,g_2,h_2}$ is not in L.

Applying a Reidemeister III move to $\triangle_{e,g_2,h_2}$ reduces the number of crossings in $\triangle_{a_2,e,g}$ and preserves the drawing of L. Using Reidemeister III moves to repeat this process shows we can empty $\triangle_{a_2,e,g}$ and preserve the drawing of L. Noting again that the segment of e in $\triangle_{a_2,e,g}$ is not in L, we can apply a Reidemeister III move over $\triangle_{a_2,e,g}$ to reduce the number of segments in Z and preserve the drawing L.

It follows that there is a series of Reidemeister III moves from D_2 to a drawing D_3 that empties $\triangle_{\{e,f,g\}}$ of segments of type Z and D_3 contains L. Noting that the segment of e bounding $\triangle_{\{e,f,g\}}$ is not in L, we finally apply a Reidemeister III move to $\triangle_{\{e,f,g\}}$ resulting in a drawing $\bar{D}$ obtained from a series of Reidemeister III moves from D such that $\bar{D}$ contains L and has e crossing g as the next step, as desired. □

Now that we can describe when neighbouring edges have extensions to simple drawing of K_n with the same rotation system. We look at the set of all such edge on the boundary of L and show that they appear consecutively. To that end, let us define the set of choice edges.

Notation 7.3. Let $n \geq 6$, H_n be a realizable n-vertex rotation system, and D a simple drawing that realizes H_n. Let $e = (u, v)$ be an edge in H_n, L be a subdrawing of D containing a segment of e labelled e_i, starting at v and has i crossings in L, and R_L be the region containing the non-vertex end of e_i in L. Define $C_L(e_i)$ to be the set of edges that e_i can cross on the boundary of R_L such that the drawing after the crossing of e_i with the boundary can be extended to a simple drawing that realizes H_n.

We proceed by showing that the edges in $C_L(e_i)$ appear consecutively along the boundary of the face in L containing the non-vertex end of e_i.

Lemma 7.4. *Let $n \geq 6$, H_n be a realizable n-vertex rotation system, D be a simple drawing realizing H_n, $e = (u, v)$ be an edge of H_n, and F_v be a subset* *of edge incident to v in D not containing e. For $r \geq 5$, let L be a subdrawing of D on some K_r not containing v, along with the edges F_v, and a segment of e starting at v that has i crossings in L labelled e_i. If R_L is the region containing the non-vertex end of e_i in L, then the elements of $C_L(e_i)$ appear consecutively on the boundary of R_L.*

Proof. Let $n \geq 6$, H_n be a realizable n-vertex rotation system, D be a simple drawing realizing H_n, $e = (u, v)$ be an edge of H_n, F_v be a subset of edge incident to v in D that does not contain e, L be a subdrawing of D containing a segment of e labelled e_i, and starting at u and has i crossings in L. Suppose R_L is the region containing the non-vertex end of e_i in L and let z be the point on $\mathcal{B}(R_L)$ that e_i crosses (z could be the vertex v if $i = 0$). By way of contradiction, assume $C_L(e_i)$ contains two edges f and g such that $\bar{f}$ and $\bar{g}$ are their segments on $\mathcal{B}(R_L)$ respectively, and there is no path of edges in $C_L(e_i)$ on R_L that connects the segments $\bar{f}$ and $\bar{g}$.

Claim 1. *Let d be an edge in L. If d has at least two non-trivial segments on $\mathcal{B}(R_L)$, then there exists a simple closed curve ϕ on $d \cup \mathcal{B}(R_L)$ containing v such that there is a side S_ϕ of ϕ does not contain any vertex and $\phi \cap R_L = \{\bar{d_1}, \bar{d_2}\}$, where $\bar{d_1}$ and $\bar{d_2}$ are segments of edges d_1 and d_2 that start at vertex v and end at their respective crossings with d.*

Proof. Let $d = (u_d, v_d)$. Since d has at least two non-trivial segments on $\mathcal{B}(R_L)$, it follows that there is a non-trivial segment of d, call it s_d that intersects $\mathcal{B}(R_L)$ only at its ends and these intersections are at crossing edges with d.

Let ϕ be any simple closed curved defined on $\mathcal{B}(R_L)$ and s_d that uses s_d. If both sides of ϕ contain an end of d, then without loss of generality there exists a vertex $w \neq v$ in L such that w and u_d are on opposite sides of ϕ. (w, u_d) is in L by definition of L, and so (w, u_d) must cross ϕ. Since L is simple this is not possible, and so some side of ϕ, labelled S_ϕ, contains no end of d.

Similarly, S_ϕ contains no vertex $w \neq v$. The crossings of s_d with $\mathcal{B}(R_L)$ on ϕ are at edges d_1 and d_2. For each of these edges, there is an end that is inside S_ϕ and an end that is outside S_ϕ since L is simple. It follows that each of d_1 and d_2 have v as an endpoint.

If S_ϕ contained a segment of an edge not having v as an end, then one of the ends of that edge would be contained in S_ϕ, a contradiction. This shows

$\bar{d_1}$ and $\bar{d_2}$ the segments of d_1 and d_2 respectively, in S_ϕ cross no edges. Waling along the boundary of R_L from the (d_1, d) crossing to the (d_2, d) crossing shows that we only walk along $\bar{d_1}$ and $\bar{d_2}$. Therefore, $\gamma_{d_1,d_2,d}$ is a closed curve defined on $\mathcal{R}_\mathcal{L}$ and s_d that uses s_d. Setting $\phi = \gamma_{d_1,d_2,d}$ completes the proof. ■

By repeated use of Lemma 7.2, there is a maximal set of consecutive edge segments B that are choices on $\mathcal{B}(R_L)$ containing $\bar{f}$ such that crossing any of these segments can complete into a drawing. Without loss of generality, let the clockwise boundary walk of $\mathcal{B}(R_L)$ be $(\bar{b_1}, \ldots, \bar{b_k}, \ldots, \bar{g}, \ldots, z, \ldots)$ with $B = \{\bar{b_1}, \ldots, \bar{b_k}\}$ and let b_i be the respective edge of $\bar{b_i}$. Since g has a crossing with each b_i or they are segments of the same edge, we can orient each $\bar{b_i}$ towards the crossing of b_i and g, or towards the segment $\bar{g}$.

Claim 2. *For all i, $\bar{b_i}$ is oriented clockwise along $\mathcal{B}(R_L)$.*

Proof. By way of contradiction, assume $\bar{b_i}$ is oriented counter clockwise for some i. Without loss of generality, let $\bar{b_i}$ be the edge segment in B that is closest in B to $\bar{b_1}$ that is oriented counterclockwise.

Case 1. *$b_i = g$.*

It follows by taking a edge walk on g that from $\bar{b_i}$ to $\bar{g}$, we see $\bar{b_i}$, then the crossing of g with some edge g_2 on the boundary of R_L, then $\bar{g}$. Claim 1 implies that g_2 has v as an endpoint.

Let D_1 be a realization of H_n with e_i crossing $\bar{b_i}$ in L and D_2 be a realization of H_n with e_1 crossings $\bar{g}$ in L. In D_1, $e <^g_\wedge g_2$ and in D_2 $g_2 <^g_\wedge e$, a contradiction with both drawings being a realization of H_n.

Case 2. *$b_i \neq g$.*

Since $\bar{b_i}$ and $\bar{g}$ are separated by z, it must be the case that the head of $\bar{b_i}$ crosses some segment $\bar{h}_1$ on $\mathcal{B}(R_L)$. Let h_1 be the respective edge for $\bar{h}_1$.

Case 2.1. *$h_1 = g$.*

Orient $\bar{g}$ towards the (b_i, h_1) crossing. By Claim 1, a simple closed curve ϕ on $g \cup \mathcal{B}(R_L)$ containing v such that there is a side S_ϕ of ϕ does not contain any vertex and $\phi \cap R_L = \{\bar{d_1}, \bar{d_2}\}$, where $\bar{d_1}$ and $\bar{d_2}$ are segments of edges d_1 and d_2 that start at vertex v and end at their respective crossings with d.

Since b_i is not incident to v, it follows that both $\bar{g}$ and $\bar{h_1}$ are oriented clockwise on $\mathcal{B}(R_L)$. As per Claim 1, up to relabelling, starting at $\bar{g}$ and following its orientation, g crosses d_1, then d_2, then b_i.

Let D_1 be a realization of H_n with e_i crossing $\bar{b}_i$ in L and D_2 be a realization of H_n with e_1 crossings $\bar{g}$ in L. From D_2 we get $e \prec^g_{D_2} d_1 \prec^g_{D_2} b_i$. It follows from D_2, that $e <^g_\wedge d_1$ in H_n.

Since e crosses b_i then g in D_1 and all three edges are in a R3-triangle, it follows that the order g crosses e and b_i is reversed in D_1. In particular, g crosses $d_1 \prec^g_{D_1} b_i \prec^g_{D_1} e$. It follows from D_1, that $d_1 <^g_\wedge e$, a contradiction with both drawings being a realization of H_n.

Case 2.2. $h_1 \neq g$.

If e and h_1 are ordered on b_i or e has already crossed h_1 in L, then in any drawing realizing H_n and containing L, $e <_{b_i} h_1 <_{b_i} g$. It would follow by Observation 7.1 that the drawing of $\gamma_{e,b_i,g}$ is determined and one of $\bar{g}$ or $\bar{b_i}$ is not a choice. Therefore, $\triangle_{e,b_i,h_1}$ exists and h_1 has not been crossed by e in L.

By Lemma 7.2, $\bar{h}_1 \in B$ and $\bar{h}_1$ is oriented clockwise on $\mathcal{B}(R_L)$ by minimality of $\bar{b_i}$. It follows that b_i crosses h_1 from right to left and e crosses h_i from right to left from H_n.

We will prove that at this moment g is not an option for a contradiction by looking at the drawing of the $K_6 + e_i$ in L induced on the ends of h_1, b_i and g. Let $D_4 + e_i$ be the drawing of the K_4 induced on the ends of h_1 and b_i. Since $\mathcal{B}(R_L)$ has the crossing of h_1 and b_i, the region R_4 containing R_L in D_4 is determined. Let u_h and u_b be the ends of h_1 and b_i respectively, that are on the boundary of R_4.

Now extend D_4+e_i to include the drawing of g from L. By the orientation of b_i and h_1 in L, and the definition of $\triangle_{b_i,h_1,g}$, it follows that g does not cross R_4 on b_i or h_1. Since g is on the boundary of R_L, some segment of g is in R_4. It follows that g crosses R_4 once at (u_b, u_h). Let u_g be the end of g that is in R_4 (Such an end exists as g crosses the boundary of R_4 once).

Again extend this drawing to be $D_4 + e_i + \{(u_g, u_h), (u_g, u_b)\}$, the original drawing extended to include the drawing of the edges (u_g, u_h) and (u_g, u_b) from L. Since this drawing is simple, $\gamma_{b_i,g,(u_b,u_h)}$ implies that (u_g, u_b) is contained in R_4. Similarly, $\gamma_{h_1,g,(u_b,u_h)}$ implies that (u_g, u_h) is contained in R_4. It follows that there is no facial region in R_4 that both contain a segment of g the crossing of b_i and h_1, a contradiction with the existence of R_L. ∎

By Claim 2, every edge segment in B is directed clockwise (in particular $\bar{b_k}$). Let g_k be the edge that the head of b_k crosses on $\mathcal{B}(R_L)$.

If $g = g_k$, we orient $\bar{g}$ towards the (b_k, g_k) crossing. Since b_k is not incident to v, it follows that both $\bar{g}$ and $\bar{g_k}$ are oriented counterclockwise on $\mathcal{B}(R_L)$. As per Claim 1, up to relabelling, starting at $\bar{g}$ and following its orientation, g crosses d_3, then d_4, then b_i, where d_3 and d_4 are edges incident to v

Let D_1 be a realization of H_n with e_i crossing $\bar{b}_k$ in L and D_2 be a realization of H_n with e_1 crossings $\bar{g}$ in L. From D_2 we get $e \prec^g_{D_2} d_3 \prec^g_{D_2} b_k$. It follows from D_2, that $e <^g_{\wedge} d_3$ in H_n.

Since e crosses b_k then g in D_1 and all three edges are in a R3-triangle, it follows that the order g crosses e and b_k is reversed in D_1. In particular, g crosses $d_3 \prec^g_{D_1} b_k \prec^g_{D_1} e$. It follows from D_1, that $d_3 <^g_{\wedge} e$, a contradiction with both drawings being a realization of H_n.

Therefore, $g \neq g_k$. If e and g_k are ordered on b_k or e has already crossed g_k in L, then in any drawing realizing H_n and containing L, $e <_{b_k} g_k <_{b_k} g$. It would follow by Observation 7.1 that the drawing of $\gamma_{e,b_k,g}$ is determined and one of $\bar{g}$ or $\bar{b}_k$ is not a choice. Therefore, $\triangle_{e,b_k,g_k}$ exists and g_k has not been crossed by e in L. By Lemma 7.2, g_k is a choice, a contradiction with the definition of b_k. $\square$

Finally, we end this section with a proof of Theorem 1.3 using Lemma 7.2 and Lemma 7.4.

Proof of Gioan's Theorem Let n be a positive integer, H_n be a realizable n-vertex rotation system, and D and D' be two simple drawings realizing H_n. If $n \leq 5$, then H_n uniquely determines its associated realizable drawings and $D = D'$. Therefore, $n \geq 6$.

Let $r \geq 5$ be the largest integer such that there exists a common drawing L_r of K_r in D and D'. Since every common K_5 in D and D' is uniquely drawn the same, it follows that such an L_r exists. Since $D \neq D'$, there is a vertex v in D not in L_r.

Let E^v_r be the edges having an endpoint in L_r and the other endpoint being v, and $F^v_r \subset E^v_r$ such that $L_r + F^v_r$ is a common drawing in both of D and D'. $L_r + F^v_r$ exists because L_r exists and F^v_r can be empty.

Since L_{r+1} does not exists, it follows that $E^v_r \neq F^v_r$. Let $e = (u, v)$ be an edge in $E^v_r \backslash F^v_r$, and i be the largest integer such that there exists a segment of e, labelled e_i, starting at v having i crossings such that $L_r + F^v_r + e_i$ is a common drawing in both D and D'. Such an $L_r + F^v_r + e_i$ exists since it is possible that $i = 0$ and e_0 is a small segment of e starting at the correct spot in the rotation at v.

If we can show that there is a sequence of Reidemeister III moves from D to some drawing D_1 such that the common drawing of D_1 and D' is $L_r + F_r^v + e_{i+1}$, then induction will imply that there there is a sequence of Reidemeister III moves from D to D'.

Let R_L be the region in $L_r + F_r^v + e_i$ that contains the non-vertex end of e_i. Let f be the next edge that e_i crosses in D and f' be the next edge that e_i crosses in D'. By Lemma 7.4, the elements of $C_{L_r+F_r^v+e_i}(e_i)$ appear consecutively on the boundary of R_L.

Let $(b_1 = f, \ldots, b_k = f')$ be such a path of segments. Starting from $D = D_1$, define D_j for $2 \leq j \leq k$ to be the simple drawing $\bar{D}$ produced by Lemma 7.2 with e_i crossing b_j. Lemma 7.2 implies there is a sequence of Reidemeister III moves $\{\rho^j_{\ell_j}\}$ that takes D_j to D_{j+1}.

Note that D_k and D' have $L_r + F_r^v + e_{i+1}$ in common (if e_{i+1} has crossed all its edges, then this is $L_r + (F_r^v \cup \{e\})$, and if $F_r^v \cup \{e\} = E_r^v$, then this is L_{r+1}).

Taking the sequence of Reidemeister III moves $\{\rho^j_{\ell_j}\}_{j=1}^{k-1}$ from D to D_k gives us the desired inductive result. $\square$

8 Concluding Remarks

The proof of Theorem 1.1, provide insight into how to draw a realizable rotation system using a combinatorial algorithm. Restricting this proof to the case $n = 6$, offers the first non-computational proof that $(6, 5)$-rotation systems are realizable. Taking a combinatorial approach to the proof of Theorem 1.1 resulted in Theorem 3.8, which characterizes edges and faces in simple drawings of K_n independent of rotation systems. Stand alone such a result is interesting as the Harary-Hill conjecture remains open and the characterization applies directly to every simple drawing of K_n. Finally, we provide a simplified proof of Theorem 1.3 and provide intuition as to how graphs with the same rotation system are drawn.

We end this thesis with three separate open questions. The first two questions are motivated by Theorems 1.1 and 1.3 respectively. The last questions is the open question posed by Dan Archdeacon cited in [**4**].

Let us extend the definitions of realizable rotation systems and (n, k)-rotation systems. Define a complete n-vertex rotation system H to be g-*realizable* if there is a simple drawing D of K_n in a surface of Euler genus g having the rotations at the vertices the same as H. Let a rotation system H be an (n, k, g)-*rotation system* if H is an n-vertex rotation system and every rotation system H_k that is obtained from H restricted to a set of k vertices is g-realizable.

Open Question 1. *Does there exist an integer function $k(g)$ such that for every $n \geq k(g)$, every $(n, k(g), g)$-rotation system is g-realizable?*

Open Question 2. *If D and D' are two simple drawings of K_n in a surface of Euler genus g have the same rotation system, then does there exist a sequence of Reidemeister III moves applied to D that transforms D into D' (or possibly a more generalized set of operations)?*

Open Question 3 (Archdeacon, [**4**])**.** *Is the number of non-planar K_4's in any n-vertex complete rotation system at least $H(n)$?*

References

[1] J. Kynčl. “Simple realizability of complete abstract topological graphs in P.” Discrete and Computational Geometry 45.3 (2011): 383-399.

[2] J. Kynčl. “Simple realizability of complete abstract topological graphs simplified.” International Symposium on Graph Drawing and Network Visualization. Springer, Cham, (2015): 309-320.

[3] E. Gioan. “Complete graph drawings up to triangle mutations” WG 2005. Lecture Notes in Computer Science, vol 3787. Springer, Berlin, Heidelberg, (2005): 139-150.

[4] A. Arroyo, D. McQuillan, R. B. Richter, and G. Salazar. “Drawings of Kn with the same rotation scheme are the same up to triangle-flips (Gioan’s Theorem)”. Australasian Journal of Combinatorics 67.2 (2017): 131–144

[5] J. Kynčl,“Improved enumeration of simple topological graphs. “ Discrete & Computational Geometry 50.3 (2013): 727-770.

[6] J. Pach, J. Solymosi, and G. Tóth, “Unavoidable Configurations in Complete Topological Graphs” Discrete Comput. Geom. (2003) 30: 311.

[7] Ábrego, Bernardo M., et al. “All good drawings of small complete graphs.” Proc. 31st European Workshop on Computational Geometry (EuroCG). (2015): 57-60.

[8] O.Aichholzer, Private Communications, 2016.

[9] J. Kynčl.“Enumeration of simple complete topological graphs.” European Journal of Combinatorics 30.7 (2009): 1676-1685.

[10] P. Shengjun, and R. B. Richter. “The crossing number of K_{11} is 100.” Journal of Graph Theory 56.2 (2007): 128-134.

[11] M. Balko, R. Fulek, and J. Kynčl.“Crossing Numbers and Combinatorial Characterization of Monotone Drawings of K_n.” Discrete & Computational Geometry 53.1 (2015): 107-143.

[12] J. Blažek and M. Koman. “A minimal problem concerning complete plane graphs.” Theory of graphs and its applications, Czech. Acad. of Sci., (1964): 113–117.

[13] J. Kynčl and V. Pavel, “On edges crossing few other edges in simple topological complete graphs.” Discrete mathematics 309.7 (2009): 1917-1923.

[14] J. Pach and G. Tóth. “How many ways can one draw a graph?.” Combinatorica 26.5 (2006): 559-576.

[15] J. Pach, J. Solymosi, and G. Tóth. “Unavoidable configurations in complete topological graphs.” Discrete Computational Geometry 30.2 (2003): 311-320.

[16] J. Pach, N. Rubin, and G. Tardos. “Planar point sets determine many pairwise crossing segments.” Advances in Mathematics 386 (2021): 107779.

Appendix A

We present a rudimentary drawing algorithm for complete rotation systems that takes as input a complete rotation system H, and outputs either a simple drawing D that realizes H or a 5-vertex non-realizable rotation system H_5.

Edge Drawing Subroutine $\mathcal{E}(H, D, (u, v))$

- Input: A complete rotation system H, a desired edge (u, v) to be drawn, and a partial drawing D on a complete graph containing u along with a set of edges incident to v from H.
- Output: A drawing $\bar{D}$ that is a partial drawing of H and contains the edges D and (u, v), or a non-realizable at most 8-vertex rotation system in H

1) Set $i = 0$ and start the edge (u, v) at u in the appropriate position according to H in D and call this new drawing D_0.
2) Define $(u, v)_i$ to be the partial edge of (u, v) in D_i, v_i to be the non-vertex end of $(u, v)_i$ and the new drawing D_i. Define E_i to be a set of minimal edges that $(u, v)_i$ has yet to cross defined by the $<^e_\wedge$ and $<^e_\parallel$ relations. If such a set does not exist, define $E_i = \infty$.
   a) If $E_i = \infty$, then Section 5 implies we can find a non-realizable 8-vertex rotation system H_8 in H. Output H_8 and stop the subroutine
   b) If $E_i = \emptyset$ and v is currently on the boundary of the region containing v_i, then extend $(u, v)_i$ by connecting it to v, set the new drawing to be $\bar{D}$. Output $\bar{D}$ and stop the subroutine.
   c) If $E_i = \emptyset$ and v is not currently on the boundary of the region containing v_i, then v_i and v are in different faces of $D_i - v$, in particular, there is a 3-cycle T separating v_i from v. The induced at most 5-vertex rotation system H_5 on T and (u, v) is non-realizable. Output H_5 and stop the subroutine.
   d) If $E_i \neq \emptyset$ and some edge of $c_i \in E_i$ is on the boundary of the region containing v_i do:

i) if crossing c_i forms the correct oriented crossing with (u, v), then extend $(u, v)_i$ in D_i to cross e, set the new drawing to be D_{i+1}, $i := i + 1$, and start at step 2).

ii) if crossing c_i does not form the correct oriented crossing with (u, v), then define H_4 to a 4-vertex rotation system in H induced on the ends of c_i and (u, v). H_4 is not realizable, therefore output H_4 and stop the subroutine.

e) If $E_i \neq \emptyset$ and no edge of E_i is on the boundary of the region containing v_i, then let c_i be an edge in E_i and do:

i) If Theorem 3.8 finds a K_4 separating v_i from c_i in $D_i - v$, then the at most 6-vertex rotation system H_6 induced on this K_4 and (u, v) is non-realizable. Output H_6 and stop the subroutine.

ii) If Theorem 3.8 finds a sequence of Reidemeister III moves $\{\rho_{X_j}\}$ that places c_i on the boundary of the region containing v_i in $D_{-}v$, then do:

1) For some j, if a vertex v is in $\triangle_{X_j}$, then there is some K_4 involving v and c_i separating c_i from the region containing v_i. Such a K_4 along with (u, v) form a non-realizable H_5. Output H_5 and stop the subroutine.

2) For all j, if every $\triangle_{X_j}$ contains no vertices, then perform the sequence of Reidemeister III moves $\{\rho_{X_j}\}$ on D_i and go to step 2.d.i).

Basic Algorithm $\mathcal{A}(H)$

- Input: A complete rotation system H on the vertices $[n]$.
- Output: A drawing D that is a realization of H, or a non-realizable 5-vertex rotation system H_5 in H..

1) Start by drawing the 3-cycle $(1, 2, 3)$, set $i = 3$ and call the drawing D.

2) For i from 4 to n do: For j from 1 to $i - 1$ do:

i) Process $\mathcal{E}(H, D, (i, j))$.

ii) If the output of the subroutine is some non-realizable H_k for $k \leq 8$, then check every 5-vertex rotation system in H_8 till a non-realizable 5-vertex rotation system H_5 is found. Output H_5 and stop the algorithm.

iii) Otherwise, the output of the subroutine is $\bar{D}$. Set $D := \bar{D}$. If $j = n - 1$, then output D and stop the algorithm.