Matematika pro fyziky 1: The X-Files 👽

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1 ? ? ?

$$\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^2 \qquad H = \|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|^2$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} + (-1)^j \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right)$$

2 ! ! !

2.1 $\boldsymbol{p}, \boldsymbol{q} = ?$

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \qquad \dot{\boldsymbol{v}} = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ -\frac{\partial H}{\partial q_1} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \\ -\frac{\partial H}{\partial q_2} + \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \end{pmatrix} = 2 \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -\zeta & \zeta \\ 0 & -1 & \zeta & -\zeta \end{pmatrix}}_{M} \boldsymbol{v}$$

$$\boxed{\dot{\boldsymbol{v}} = M\boldsymbol{v} \quad \Longrightarrow \quad \boldsymbol{v} = \exp(tM)\boldsymbol{v_0}, \quad \boldsymbol{v_0} \in \mathbb{R}^2}$$

2.2 $\exp(tM) = ?$

$$tM = P^{-1}\ J\ P$$

$$\underbrace{2t \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -\zeta & \zeta \\ 0 & -1 & \zeta & -\zeta \end{pmatrix}}_{tM} = \underbrace{\begin{pmatrix} \mathrm{i} & -\mathrm{i} & \frac{1}{\zeta+\sqrt{\zeta^2-1}} & \frac{1}{\zeta-\sqrt{\zeta^2-1}} \\ \mathrm{i} & -\mathrm{i} & -\frac{1}{\zeta+\sqrt{\zeta^2-1}} & \frac{1}{-\zeta+\sqrt{\zeta^2-1}} \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{P^{-1}}$$

$$\underbrace{\begin{pmatrix} -2\mathrm{i}t & 0 & 0 & 0 \\ 0 & 2\mathrm{i}t & 0 & 0 \\ 0 & 0 & -2t\left(\zeta + \sqrt{\zeta^2-1}\right) & 0 \\ 0 & 0 & 0 & 2t\left(-\zeta + \sqrt{\zeta^2-1}\right) \end{pmatrix}}_{J}$$

$$\underbrace{\frac{1}{4} \begin{pmatrix} -\mathrm{i} & -\mathrm{i} & 1 & 1 \\ \mathrm{i} & \mathrm{i} & 1 & 1 \\ -\frac{1}{\sqrt{\zeta^2-1}} & \frac{1}{\sqrt{\zeta^2-1}} & -\frac{\zeta}{\sqrt{\zeta^2-1}} - 1 & \frac{\zeta}{\sqrt{\zeta^2-1}} + 1 \\ \frac{1}{\sqrt{\zeta^2-1}} & -\frac{1}{\sqrt{\zeta^2-1}} & \frac{\zeta}{\sqrt{\zeta^2-1}} - 1 & -\frac{\zeta}{\sqrt{\zeta^2-1}} + 1 \end{pmatrix}}_{P}$$

$$\exp(tM) = \exp(P^{-1}JP) = P^{-1}\exp(J)P$$

$$\exp(tM) = P^{-1}\begin{pmatrix} e^{-2it} & 0 & 0 & 0 \\ 0 & e^{2it} & 0 & 0 \\ 0 & 0 & e^{-2t\left(\zeta+\sqrt{\zeta^2-1}\right)} & 0 \\ 0 & 0 & 0 & e^{-2t\left(\zeta-\sqrt{\zeta^2-1}\right)} \end{pmatrix} P$$

$$\exp(tM) = \underbrace{\begin{pmatrix} \frac{e^{2it}}{4}+\frac{e^{-2it}}{4}-\frac{e^{-2t\left(\zeta+\sqrt{\zeta^2-1}\right)}}{4\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}}+\frac{e^{-2t\left(\zeta-\sqrt{\zeta^2-1}\right)}}{4\left(\zeta-\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}} & \frac{e^{2it}}{4}+\frac{e^{-2it}}{4}+\frac{e^{-2t\left(\zeta+\sqrt{\zeta^2-1}\right)}}{4\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}}-\frac{e^{-2t\left(\zeta-\sqrt{\zeta^2-1}\right)}}{4\left(\zeta-\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}} & \frac{\left(i(1-\zeta^2)e^{4t(\zeta+i)}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t\zeta}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(i(1-\zeta^2)e^{4t(\zeta+i)}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t\zeta}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} \\ \frac{e^{2it}}{4}+\frac{e^{-2it}}{4}+\frac{e^{-2t\left(\zeta+\sqrt{\zeta^2-1}\right)}}{4\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}}-\frac{e^{-2t\left(\zeta-\sqrt{\zeta^2-1}\right)}}{4\left(\zeta-\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}} & \frac{e^{2it}}{4}+\frac{e^{-2it}}{4}-\frac{e^{-2t\left(\zeta+\sqrt{\zeta^2-1}\right)}}{4\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}}+\frac{e^{-2t\left(\zeta-\sqrt{\zeta^2-1}\right)}}{4\left(\zeta-\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}} & \frac{\left(i(1-\zeta^2)e^{4t(\zeta+i)}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t\zeta}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(i(1-\zeta^2)e^{4t(\zeta+i)}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t\zeta}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} \\ \frac{\left(i(1-\zeta^2)e^{4t\zeta}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(i(1-\zeta^2)e^{4t\zeta}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(\left(-\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+(\zeta^2-1)e^{4t\zeta}+(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(\left(\zeta-\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}-\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+(\zeta^2-1)e^{4t\zeta}+(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} \\ \frac{\left(i(1-\zeta^2)e^{4t\zeta}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(i(1-\zeta^2)e^{4t\zeta}+\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}-\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+i(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(\left(\zeta-\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}-\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+(\zeta^2-1)e^{4t\zeta}+(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} & \frac{\left(\left(-\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta+\sqrt{\zeta^2-1}+i\right)}+\left(\zeta+\sqrt{\zeta^2-1}\right)\sqrt{\zeta^2-1}\,e^{2t\left(\zeta-\sqrt{\zeta^2-1}+i\right)}+(\zeta^2-1)e^{4t\zeta}+(\zeta^2-1)e^{4t(\zeta+i)}\right)e^{-2t(2\zeta+i)}}{4(\zeta^2-1)} \end{pmatrix}}_{\text{😂👌}}$$

$$\exp(tM)|_{\zeta=0} = \begin{pmatrix} \cos(2t) & 0 & \sin(2t) & 0 \\ 0 & \cos(2t) & 0 & \sin(2t) \\ -\sin(2t) & 0 & \cos(2t) & 0 \\ 0 & -\sin(2t) & 0 & \cos(2t) \end{pmatrix}$$

2.3 $\det\exp(tM) = ?$

$$\det\exp(tM) = \exp\mathrm{Tr}(tM) = \exp\left(t\,\mathrm{Tr}(M)\right) = \exp\left(2t\,(-2\zeta)\right) = \exp(-4t\zeta)$$

$$\zeta = 0 \quad\Longrightarrow\quad \det\exp(tM) = 1 \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\iiiint_{\Omega(t)} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}p_1\,\mathrm{d}p_2 = 0$$

$$\zeta \neq 0 \quad\Longrightarrow\quad \iiiint_{\Omega(t)} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}p_1\,\mathrm{d}p_2 = \mathrm{e}^{-4t\zeta}\iiiint_{\Omega(0)} \mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}p_1\,\mathrm{d}p_2$$

2.4 $\boldsymbol{p(t)|_{\zeta=0},\ q(t)|_{\zeta=0} = ?}$

$$\begin{pmatrix} \boldsymbol{q}|_{\zeta=0} \\ \boldsymbol{p}|_{\zeta=0} \end{pmatrix} = \exp(tM) \begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{p_0} \end{pmatrix}$$

$$\left.\begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix}\right|_{\zeta=0} = \begin{pmatrix} \cos(2t) & 0 & \sin(2t) & 0 \\ 0 & \cos(2t) & 0 & \sin(2t) \\ -\sin(2t) & 0 & \cos(2t) & 0 \\ 0 & -\sin(2t) & 0 & \cos(2t) \end{pmatrix} \begin{pmatrix} q_{10} \\ q_{20} \\ p_{10} \\ p_{20} \end{pmatrix}$$

$$q_1(t)|_{\zeta=0} = p_{10}\sin(2t) + q_{10}\cos(2t)$$

$$q_2(t)|_{\zeta=0} = p_{20}\sin(2t) + q_{20}\cos(2t)$$

$$p_1(t)|_{\zeta=0} = p_{10}\cos(2t) - q_{10}\sin(2t)$$

$$p_2(t)|_{\zeta=0} = p_{20}\cos(2t) - q_{20}\sin(2t)$$

2.5 $\boldsymbol{MichalGr\check{n}o = ?}$