Matematika pro fyziky 1: The X-Files 👽

Michal Grňo

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$$\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^2 \qquad H = \|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|^2$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} - \zeta\left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2}\right)$$

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2.1 $\boldsymbol{p}, \boldsymbol{q} = ?$

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \qquad \dot{\boldsymbol{v}} = \begin{pmatrix} -\frac{\partial H}{\partial q_1} - \zeta\left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2}\right) \\ -\frac{\partial H}{\partial q_2} - \zeta\left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2}\right) \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = 2\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\zeta & \zeta \\ 0 & 1 & -\zeta & \zeta \end{pmatrix}}_{M} \boldsymbol{v}$$

$$\boxed{\dot{\boldsymbol{v}} = M\boldsymbol{v} \implies \boldsymbol{v} = \exp(tM)\boldsymbol{v_0}, \quad \boldsymbol{v_0} \in \mathbb{R}^2}$$

2.2 $\exp(tM) = ?$

$$2t\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\zeta & \zeta \\ 0 & 1 & -\zeta & \zeta \end{pmatrix}}_{tM} = \underbrace{\begin{pmatrix} t\zeta & \frac{\zeta}{2}-1 & -t\zeta & \frac{\zeta}{2}+1 \\ t\zeta & \frac{\zeta}{2} & -t\zeta & \frac{\zeta}{2} \\ -t\zeta & 1 & -t\zeta & 1 \\ -t\zeta & 0 & -t\zeta & 0 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -2t & 1 & 0 & 0 \\ 0 & -2t & 0 & 0 \\ 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 2t \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 0 & \frac{1}{2t\zeta} & -\frac{1}{4t} & \frac{\zeta-2}{4t\zeta} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2t\zeta} & \frac{1}{4t} & -\frac{\zeta+2}{4t\zeta} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{P}$$

$$\exp(tM) = \exp(P^{-1}JP) = P^{-1}\exp(J)P$$

$$\exp(tM) = P^{-1}\begin{pmatrix} e^{-2t} & e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{2t} & e^{2t} \\ 0 & 0 & 0 & e^{2t} \end{pmatrix}P$$

$$\exp(tM) = \begin{pmatrix} \frac{\left(-2t\zeta-\zeta+\left(-2t\zeta+\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} & \frac{\zeta\left(2t+(2t-1)e^{4t}+1\right)e^{-2t}}{4} & \frac{\left(t\zeta+(-t\zeta+1)e^{4t}-1\right)e^{-2t}}{2} & t\zeta\sinh(2t) \\ \frac{\zeta\left(-2t+(1-2t)e^{4t}-1\right)e^{-2t}}{4} & \frac{\left(2t\zeta+\zeta+\left(2t\zeta-\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} & -t\zeta\sinh(2t) & \frac{\left(-t\zeta+(t\zeta+1)e^{4t}-1\right)e^{-2t}}{2} \\ \frac{\left(t\zeta+(-t\zeta+1)e^{4t}-1\right)e^{-2t}}{2} & t\zeta\sinh(2t) & \frac{\left(-2t\zeta+\zeta+\left(-2t\zeta-\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} & \frac{\zeta\left(2t+(2t+1)e^{4t}-1\right)e^{-2t}}{4} \\ -t\zeta\sinh(2t) & \frac{\left(-t\zeta+(t\zeta+1)e^{4t}-1\right)e^{-2t}}{2} & \frac{\zeta\left(-2t-(2t+1)e^{4t}+1\right)e^{-2t}}{4} & \frac{\left(2t\zeta-\zeta+\left(2t\zeta+\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} \end{pmatrix}$$

$$\exp(tM)|_{\zeta=0} = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) & 0 \\ 0 & \cosh(2t) & 0 & \sinh(2t) \\ \sinh(2t) & 0 & \cosh(2t) & 0 \\ 0 & \sinh(2t) & 0 & \cosh(2t) \end{pmatrix}$$

2.3 $\det\exp(tM) = ?$

$$\det\exp(tM) = \exp\operatorname{Tr}(tM) = \exp\left(t\operatorname{Tr}(M)\right) = \exp\left(2t\left(\zeta+(-\zeta)\right)\right) = \exp(0) = 1$$

$$\implies \forall\zeta:\quad \frac{\mathrm{d}}{\mathrm{d}t}\iiiint_{\Omega(t)} \mathrm{d}q_1\ \mathrm{d}q_2\ \mathrm{d}p_1\ \mathrm{d}p_2 = 0$$

2.4 $p(t)|_{\zeta=0}$, $q(t)|_{\zeta=0} = ?$

$$\begin{pmatrix} \boldsymbol{q}|_{\zeta=0} \\ \boldsymbol{p}|_{\zeta=0} \end{pmatrix} = \exp(tM)\begin{pmatrix} \boldsymbol{q_0} \\ \boldsymbol{p_0} \end{pmatrix}$$

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix}\Bigg|_{\zeta=0} = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) & 0 \\ 0 & \cosh(2t) & 0 & \sinh(2t) \\ \sinh(2t) & 0 & \cosh(2t) & 0 \\ 0 & \sinh(2t) & 0 & \cosh(2t) \end{pmatrix}\begin{pmatrix} q_{10} \\ q_{20} \\ p_{10} \\ p_{20} \end{pmatrix}$$

$$q_1(t)|_{\zeta=0} = p_{10}\sinh(2t) + q_{10}\cosh(2t)$$

$$q_2(t)|_{\zeta=0} = p_{20}\sinh(2t) + q_{20}\cosh(2t)$$

$$p_1(t)|_{\zeta=0} = p_{10}\cosh(2t) + q_{10}\sinh(2t)$$

$$p_2(t)|_{\zeta=0} = p_{20}\cosh(2t) + q_{20}\sinh(2t)$$