Matematika pro fyziky 1: The X-Files 👽

Michal Grňo

4. ledna 2020

1 ? ? ?

$$\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^2 \qquad H = \|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|^2$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right)$$

2 ! ! !

2.1 $\boldsymbol{p}, \boldsymbol{q} = ?$

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \qquad \dot{\boldsymbol{v}} = \begin{pmatrix} -\frac{\partial H}{\partial q_1} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \\ -\frac{\partial H}{\partial q_2} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = 2 \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\zeta & \zeta \\ 0 & 1 & -\zeta & \zeta \end{pmatrix}}_{M} \boldsymbol{v}$$

$$\boxed{\dot{\boldsymbol{v}} = M\boldsymbol{v} \quad \Longrightarrow \quad \boldsymbol{v} = \exp(tM)\boldsymbol{v_0}, \quad \boldsymbol{v_0} \in \mathbb{R}^2}$$

2.2 $\exp(tM) = ?$

$$\underbrace{2t \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\zeta & \zeta \\ 0 & 1 & -\zeta & \zeta \end{pmatrix}}_{tM} = \underbrace{\begin{pmatrix} t\zeta & \frac{\zeta}{2} - 1 & -t\zeta & \frac{\zeta}{2} + 1 \\ t\zeta & \frac{\zeta}{2} & -t\zeta & \frac{\zeta}{2} \\ -t\zeta & 1 & -t\zeta & 1 \\ -t\zeta & 0 & -t\zeta & 0 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -2t & 1 & 0 & 0 \\ 0 & -2t & 0 & 0 \\ 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 2t \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 0 & \frac{1}{2t\zeta} & -\frac{1}{4t} & \frac{\zeta-2}{4t\zeta} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2t\zeta} & \frac{1}{4t} & -\frac{\zeta+2}{4t\zeta} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{P}$$

$$\exp(tM) = \exp(P^{-1}JP) = P^{-1}\exp(J)P$$

$$\exp(tM) = P^{-1} \begin{pmatrix} e^{-2t} & e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{2t} & e^{2t} \\ 0 & 0 & 0 & e^{2t} \end{pmatrix} P$$

$$
\exp(tM) = \begin{pmatrix}
\frac{\left(-2t\zeta-\zeta+\left(-2t\zeta+\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} & \frac{\zeta\left(2t+(2t-1)e^{4t}+1\right)e^{-2t}}{4} & \frac{\left(t\zeta+\left(-t\zeta+1\right)e^{4t}-1\right)e^{-2t}}{2} & t\zeta\sinh\left(2t\right) \\
\frac{\zeta\left(-2t+(1-2t)e^{4t}-1\right)e^{-2t}}{4} & \frac{\left(2t\zeta+\zeta+\left(2t\zeta-\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} & -t\zeta\sinh\left(2t\right) & \frac{\left(-t\zeta+\left(t\zeta+1\right)e^{4t}-1\right)e^{-2t}}{2} \\
\frac{\left(t\zeta+\left(-t\zeta+1\right)e^{4t}-1\right)e^{-2t}}{2} & t\zeta\sinh\left(2t\right) & \frac{\left(-2t\zeta+\zeta+\left(-2t\zeta-\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4} & \frac{\zeta\left(2t+(2t+1)e^{4t}-1\right)e^{-2t}}{4} \\
-t\zeta\sinh\left(2t\right) & \frac{\left(-t\zeta+\left(t\zeta+1\right)e^{4t}-1\right)e^{-2t}}{2} & \frac{\zeta\left(-2t-(2t+1)e^{4t}+1\right)e^{-2t}}{4} & \frac{\left(2t\zeta-\zeta+\left(2t\zeta+\zeta+2\right)e^{4t}+2\right)e^{-2t}}{4}
\end{pmatrix}
$$

$$
\exp(tM)|_{\zeta=0} = \begin{pmatrix}
\cosh\left(2t\right) & 0 & \sinh\left(2t\right) & 0 \\
0 & \cosh\left(2t\right) & 0 & \sinh\left(2t\right) \\
\sinh\left(2t\right) & 0 & \cosh\left(2t\right) & 0 \\
0 & \sinh\left(2t\right) & 0 & \cosh\left(2t\right)
\end{pmatrix}
$$

2.3 $\det\exp(tM) = ?$

$$
\det\exp(tM) = \exp\operatorname{Tr}(tM) = \exp\big(t\operatorname{Tr}(M)\big) = \exp\Big(t\left(\zeta+(-\zeta)\right)\Big) = \exp(0) = 1
$$

$$
\implies \frac{\mathrm{d}}{\mathrm{d}t}\iiiint_{\Omega(t)} \mathrm{d}q_1\ \mathrm{d}q_2\ \mathrm{d}p_1\ \mathrm{d}p_2 = 0 \quad \forall\zeta
$$