

Matematika pro fyziky 1: The X-Files 🦋

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5. ledna 2020

1 ???

$$\mathbf{p}, \mathbf{q} \in \mathbb{R}^2 \quad H = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} + (-1)^j \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right)$$

2 !!!

2.1 $\mathbf{p}, \mathbf{q} = ?$

$$\mathbf{v} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \quad \dot{\mathbf{v}} = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ -\frac{\partial H}{\partial q_1} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \\ -\frac{\partial H}{\partial q_2} + \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \end{pmatrix} = 2 \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -\zeta & \zeta \\ 0 & -1 & \zeta & -\zeta \end{pmatrix}}_M \mathbf{v}$$

$$\dot{\mathbf{v}} = M\mathbf{v} \implies \mathbf{v} = \exp(tM)\mathbf{v}_0, \quad \mathbf{v}_0 \in \mathbb{R}^2$$

2.2 $\exp(tM) = ?$

$$tM = P^{-1} J P$$

$$\underbrace{2t \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -\zeta & \zeta \\ 0 & -1 & \zeta & -\zeta \end{pmatrix}}_{tM} = \underbrace{\begin{pmatrix} i & -i & \frac{1}{\zeta + \sqrt{\zeta^2 - 1}} & \frac{1}{\zeta - \sqrt{\zeta^2 - 1}} \\ i & -i & -\frac{1}{\zeta + \sqrt{\zeta^2 - 1}} & \frac{1}{-\zeta + \sqrt{\zeta^2 - 1}} \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -2it & 0 & 0 & 0 \\ 0 & 2it & 0 & 0 \\ 0 & 0 & -2t(\zeta + \sqrt{\zeta^2 - 1}) & 0 \\ 0 & 0 & 0 & 2t(-\zeta + \sqrt{\zeta^2 - 1}) \end{pmatrix}}_J \underbrace{\frac{1}{4} \begin{pmatrix} -i & -i & 1 & 1 \\ i & i & 1 & 1 \\ -\frac{1}{\sqrt{\zeta^2 - 1}} & \frac{1}{\sqrt{\zeta^2 - 1}} & -\frac{\zeta}{\sqrt{\zeta^2 - 1}} - 1 & \frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1 \\ \frac{1}{\sqrt{\zeta^2 - 1}} & -\frac{1}{\sqrt{\zeta^2 - 1}} & \frac{\zeta}{\sqrt{\zeta^2 - 1}} - 1 & -\frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1 \end{pmatrix}}_P$$

$$\exp(tM) = \exp(P^{-1}JP) = P^{-1} \exp(J)P$$

$$\exp(tM) = P^{-1} \begin{pmatrix} e^{-2it} & 0 & 0 & 0 \\ 0 & e^{2it} & 0 & 0 \\ 0 & 0 & e^{-2t(\zeta + \sqrt{\zeta^2 - 1})} & 0 \\ 0 & 0 & 0 & e^{-2t(\zeta - \sqrt{\zeta^2 - 1})} \end{pmatrix} P$$

$$\exp(tM) = \underbrace{\begin{pmatrix} \frac{\zeta^2 + \sqrt{\zeta^2 - 1}}{2(\zeta + \sqrt{\zeta^2 - 1})} e^{-2it} & \frac{\zeta^2 + \sqrt{\zeta^2 - 1}}{2(\zeta + \sqrt{\zeta^2 - 1})} e^{2it} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta - \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} \\ \frac{\zeta^2 + \sqrt{\zeta^2 - 1}}{2(\zeta + \sqrt{\zeta^2 - 1})} e^{-2it} & \frac{\zeta^2 + \sqrt{\zeta^2 - 1}}{2(\zeta + \sqrt{\zeta^2 - 1})} e^{2it} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta - \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} \\ \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta - \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} \\ \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta + \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} & \frac{(4\zeta - \zeta^2 + \sqrt{\zeta^2 - 1}) e^{-2t(\zeta - \sqrt{\zeta^2 - 1})}}{4\sqrt{\zeta^2 - 1}} \end{pmatrix}}_{\text{🤔🤔}}$$

$$\exp(tM)|_{\zeta=0} = \begin{pmatrix} \cos(2t) & 0 & \sin(2t) & 0 \\ 0 & \cos(2t) & 0 & \sin(2t) \\ -\sin(2t) & 0 & \cos(2t) & 0 \\ 0 & -\sin(2t) & 0 & \cos(2t) \end{pmatrix}$$

2.3 $\det \exp(tM) = ?$

$$\det \exp(tM) = \exp \text{Tr}(tM) = \exp(t \text{Tr}(M)) = \exp(2t(-2\zeta)) = \exp(-4t\zeta)$$

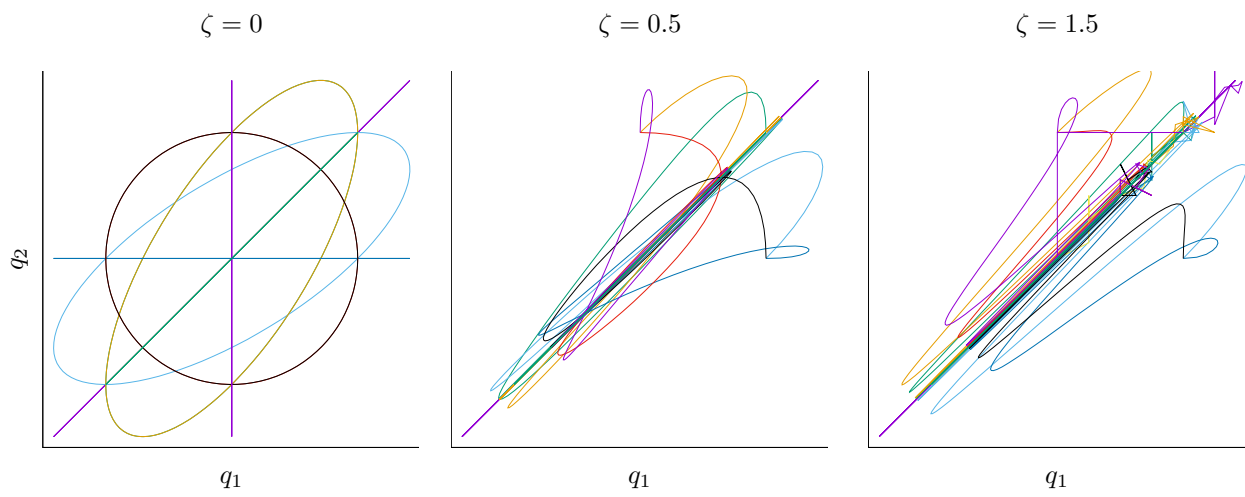
$$\zeta = 0 \implies \det \exp(tM) = 1 \implies \frac{d}{dt} \iiint \int_{\Omega(t)} dq_1 dq_2 dp_1 dp_2 = 0$$

$$\zeta \neq 0 \implies \iiint \int_{\Omega(t)} dq_1 dq_2 dp_1 dp_2 = e^{-4t\zeta} \iiint \int_{\Omega(0)} dq_1 dq_2 dp_1 dp_2$$

2.4 $p(t)|_{\zeta=0}, q(t)|_{\zeta=0} = ?$

$$\begin{pmatrix} q|_{\zeta=0} \\ p|_{\zeta=0} \end{pmatrix} = \exp(tM) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$$

$$\left. \begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} \right|_{\zeta=0} = \begin{pmatrix} \cos(2t) & 0 & \sin(2t) & 0 \\ 0 & \cos(2t) & 0 & \sin(2t) \\ -\sin(2t) & 0 & \cos(2t) & 0 \\ 0 & -\sin(2t) & 0 & \cos(2t) \end{pmatrix} \begin{pmatrix} q_{10} \\ q_{20} \\ p_{10} \\ p_{20} \end{pmatrix}$$



$$q_1(t)|_{\zeta=0} = p_{10} \sin(2t) + q_{10} \cos(2t)$$

$$q_2(t)|_{\zeta=0} = p_{20} \sin(2t) + q_{20} \cos(2t)$$

$$p_1(t)|_{\zeta=0} = p_{10} \cos(2t) - q_{10} \sin(2t)$$

$$p_2(t)|_{\zeta=0} = p_{20} \cos(2t) - q_{20} \sin(2t)$$

2.5 *MichalGrňo* = ?

