

Matematika pro fyziky 1: The X-Files 🧐

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1 ???

$$\mathbf{p}, \mathbf{q} \in \mathbb{R}^2 \quad H = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right)$$

2 !!!

2.1 $\mathbf{p}, \mathbf{q} = ?$

$$\mathbf{v} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \quad \dot{\mathbf{v}} = \begin{pmatrix} -\frac{\partial H}{\partial q_1} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \\ -\frac{\partial H}{\partial q_2} - \zeta \left(\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_2} \right) \\ \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \end{pmatrix} = 2 \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\zeta & \zeta \\ 0 & 1 & -\zeta & \zeta \end{pmatrix}}_M \mathbf{v}$$

$$\dot{\mathbf{v}} = M\mathbf{v} \implies \mathbf{v} = \exp(tM)\mathbf{v}_0, \quad \mathbf{v}_0 \in \mathbb{R}^2$$

2.2 $\exp(tM) = ?$

$$\underbrace{2t \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\zeta & \zeta \\ 0 & 1 & -\zeta & \zeta \end{pmatrix}}_{tM} = \underbrace{\begin{pmatrix} t\zeta & \frac{\zeta}{2} - 1 & -t\zeta & \frac{\zeta}{2} + 1 \\ t\zeta & \frac{\zeta}{2} & -t\zeta & \frac{\zeta}{2} \\ -t\zeta & 1 & -t\zeta & 1 \\ -t\zeta & 0 & -t\zeta & 0 \end{pmatrix}}_{P^{-1}} \underbrace{\begin{pmatrix} -2t & 1 & 0 & 0 \\ 0 & -2t & 0 & 0 \\ 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 2t \end{pmatrix}}_J \underbrace{\begin{pmatrix} 0 & \frac{1}{2t\zeta} & -\frac{1}{4t} & \frac{\zeta-2}{4t\zeta} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2t\zeta} & \frac{1}{4t} & -\frac{\zeta+2}{4t\zeta} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_P$$

$$\exp(tM) = \exp(P^{-1}JP) = P^{-1}\exp(J)P$$

$$\exp(tM) = P^{-1} \begin{pmatrix} e^{-2t} & e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{2t} & e^{2t} \\ 0 & 0 & 0 & e^{2t} \end{pmatrix} P$$

$$\exp(tM) = \begin{pmatrix} \frac{(-2t\zeta - \zeta + (-2t\zeta + \zeta + 2)e^{4t} + 2)e^{-2t}}{4} & \frac{\zeta(2t + (2t-1)e^{4t} + 1)e^{-2t}}{4} & \frac{(t\zeta + (-t\zeta + 1)e^{4t} - 1)e^{-2t}}{2} & \frac{t\zeta \sinh(2t)}{2} \\ \frac{\zeta(-2t + (1-2t)e^{4t} - 1)e^{-2t}}{4} & \frac{(2t\zeta + \zeta + (2t\zeta - \zeta + 2)e^{4t} + 2)e^{-2t}}{4} & -t\zeta \sinh(2t) & \frac{(-t\zeta + (t\zeta + 1)e^{4t} - 1)e^{-2t}}{2} \\ \frac{(t\zeta + (-t\zeta + 1)e^{4t} - 1)e^{-2t}}{2} & \frac{t\zeta \sinh(2t)}{2} & \frac{(-2t\zeta + \zeta + (-2t\zeta - \zeta + 2)e^{4t} + 2)e^{-2t}}{4} & \frac{\zeta(2t + (2t-1)e^{4t} - 1)e^{-2t}}{4} \\ -t\zeta \sinh(2t) & \frac{(-t\zeta + (t\zeta + 1)e^{4t} - 1)e^{-2t}}{2} & \frac{\zeta(-2t - (2t-1)e^{4t} + 1)e^{-2t}}{4} & \frac{(2t\zeta - \zeta + (2t\zeta + \zeta + 2)e^{4t} + 2)e^{-2t}}{4} \end{pmatrix}$$

$$\exp(tM)|_{\zeta=0} = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) & 0 \\ 0 & \cosh(2t) & 0 & \sinh(2t) \\ \sinh(2t) & 0 & \cosh(2t) & 0 \\ 0 & \sinh(2t) & 0 & \cosh(2t) \end{pmatrix}$$

2.3 $\det \exp(tM) = ?$

$$\det \exp(tM) = \exp \operatorname{Tr}(tM) = \exp(t \operatorname{Tr}(M)) = \exp(2t(\zeta + (-\zeta))) = \exp(0) = 1$$

$$\Rightarrow \forall \zeta : \frac{d}{dt} \iiint \int_{\Omega(t)} dq_1 dq_2 dp_1 dp_2 = 0$$

2.4 $p(t)|_{\zeta=0}, q(t)|_{\zeta=0} = ?$

$$\begin{pmatrix} q|_{\zeta=0} \\ p|_{\zeta=0} \end{pmatrix} = \exp(tM) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$$

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} \bigg|_{\zeta=0} = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) & 0 \\ 0 & \cosh(2t) & 0 & \sinh(2t) \\ \sinh(2t) & 0 & \cosh(2t) & 0 \\ 0 & \sinh(2t) & 0 & \cosh(2t) \end{pmatrix} \begin{pmatrix} q_{10} \\ q_{20} \\ p_{10} \\ p_{20} \end{pmatrix}$$

$$q_1(t)|_{\zeta=0} = p_{10} \sinh(2t) + q_{10} \cosh(2t)$$

$$q_2(t)|_{\zeta=0} = p_{20} \sinh(2t) + q_{20} \cosh(2t)$$

$$p_1(t)|_{\zeta=0} = p_{10} \cosh(2t) + q_{10} \sinh(2t)$$

$$p_2(t)|_{\zeta=0} = p_{20} \cosh(2t) + q_{20} \sinh(2t)$$

