Radon Transform

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1 Light Propagation model

The Radon Transform lies at the heart of X ray tomography. Consider the propagation of light in tissue where scattering is negligible

$$\frac{1}{c}\frac{\partial I(\mathbf{r},\mathbf{u})}{\partial t} + \mathbf{u}\cdot\nabla I(\mathbf{r},\mathbf{u}) = -\mu_a(\mathbf{r})I(\mathbf{r},\mathbf{u}) + s(\mathbf{r},\mathbf{u}) \tag{1}$$

Suppose that there are no internal sources, $s(\mathbf{r},\mathbf{u}) = 0$, and that we consider the steady-state equilibrium, $\partial I/\partial t = 0$. Note that each for each direction $\mathbf{u}$, the intensity is independent of other directions. We can find a closed form solution

$$I(\mathbf{r}_0 + \ell\mathbf{u}, \mathbf{u}) = I(\mathbf{r}_0, \mathbf{u}) \exp\left(-\int_0^{\ell} \mu_a(\mathbf{r}_0 + s\mathbf{u})\mathrm{d}s\right) \tag{2}$$

where s [m] is a distance away from a boundary point $\mathbf{r}_0$ along direction $\mathbf{u}$.

2 Forward problem

Suppose that we have detectors around the surface of the domain of interest, that the input intensity is known. We can measure then the ratio between the output/input intensities and take the log, such that

$$\log\left(\frac{I(\mathbf{r},\mathbf{u})}{I(\mathbf{r}_0,\mathbf{u})}\right) = -\int_{-\infty}^{\infty} \mu_a(\mathbf{r}_0 + s\mathbf{u})\mathrm{d}s \tag{3}$$

Here, the limits are taken for all space assuming that outside the domain of interest there is no absorption.

To get all the information along direction $\mathbf{u}$, we consider all lines that do not intersect. We can do this by choosing a normal vector perpendicular to $\mathbf{u}$, such that $\mathbf{r}_0 = t\mathbf{u}^{\perp}$ letting $t \in (-\infty, \infty)$.

$$\log\left(\frac{I(\mathbf{r}_0,\mathbf{u})}{I(\mathbf{r},\mathbf{u})}\right) = \int_{-\infty}^{\infty} \mu_a(t\mathbf{u}^{\perp} + s\mathbf{u})\mathrm{d}s \tag{4}$$

We can define the Radon transform as

$$\mathcal{R}(\mathbf{u}, t) \triangleq \int_{-\infty}^{\infty} \mu_a(t\mathbf{u}^{\perp} + s\mathbf{u})\mathrm{d}s \tag{5}$$

If we choose a coordinate system $\mathbf{r} = (x, y)$, then we can parametrize $\mathbf{u}$ using some angle θ with respect to x:

$$\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad , \quad \mathbf{u}^{\perp} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{6}$$

Note that the choice of $\mathbf{u}^{\perp}$ is not unique (can be mirrored). The Radon transform is then

$$\log\left(\frac{I(\mathbf{r}_0,\mathbf{u})}{I(\mathbf{r},\mathbf{u})}\right) = \mathcal{R}(\theta, t) \triangleq \int_{-\infty}^{\infty} \mu_a(s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta)\mathrm{d}s \tag{7}$$

3 Inverse problem

Suppose a sinogram (θ, t) is given. We wish to recover $\mu_a(\mathbf{r})$ that produced the data. Let the 2D Fourier transform of the absorption coefficient be

$$M(\boldsymbol{\omega}) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mu_a(\mathbf{r}) e^{-2\pi i \mathbf{r}\cdot\boldsymbol{\omega}} \mathrm{d}^2\mathbf{r} \tag{8}$$

Similarly, the 1D Fourier transform of the sinogram for a given θ is

$$\Pi(\theta, \omega) = \mathcal{F}\left\{\mathcal{R}(\theta, t)\right\} = \int_{-\infty}^{\infty} \mathcal{R}(\theta, t) e^{-2\pi i \omega t} \mathrm{d}t = M(\omega \mathbf{u}^{\perp}) \tag{9}$$

Where the last equality comes from the Fourier slice theorem. Indeed, let $\mathbf{r} = t\mathbf{u}^{\perp} + s\mathbf{u}$

$$\begin{aligned} M(\omega \mathbf{u}^{\perp}) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mu_a(\mathbf{r}) e^{-2\pi i (t\mathbf{u}^{\perp} + s\mathbf{u})\cdot(\omega \mathbf{u}^{\perp})} \mathrm{d}t\mathrm{d}s \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mu_a(\mathbf{r}) e^{-2\pi i t \omega} \mathrm{d}t\mathrm{d}s \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mu_a(t\mathbf{u}^{\perp} + s\mathbf{u}) \mathrm{d}s \right) e^{-2\pi i t \omega} \mathrm{d}t \\ &= \int_{\mathbb{R}} \mathcal{R}(\mathbf{u}, t) e^{-2\pi i t \omega} \mathrm{d}t \\ &= \Pi(\theta, \omega) \end{aligned} \tag{10}$$

The most common inversion approach is the Back projection formula, which can be derived starting from the inverse Fourier transform of M:

$$\mu_a(\mathbf{r}) = \int_{\mathbb{R}} \int_{\mathbb{R}} M(\boldsymbol{\omega}) e^{2\pi i \boldsymbol{\omega}\cdot\mathbf{r}} \mathrm{d}^2\boldsymbol{\omega} \tag{11}$$

consider a change of variables from cartesian to polar coordinates in the frequency plane, using the angle θ and the 'radius' ω, such that $\boldsymbol{\omega} = (-\omega \sin\theta, \omega\cos\theta) = \omega\mathbf{u}^{\perp}$:

$$\mu_a(\mathbf{r}) = \int_0^{\pi} \int_{\mathbb{R}} M(\omega \mathbf{u}^{\perp}) e^{2\pi i \omega \mathbf{u}^{\perp}\cdot\mathbf{r}} |\omega| \mathrm{d}\omega\mathrm{d}\theta \tag{12}$$

$$= \int_0^{\pi} \int_{\mathbb{R}} \Pi(\theta, \omega) e^{2\pi i \omega t} |\omega| \mathrm{d}\omega\mathrm{d}\theta \tag{13}$$

$$= \int_0^{\pi} \mathcal{F}^{-1}\left\{\Pi(\theta, \omega)|\omega|\right\}(t) \mathrm{d}\theta \tag{14}$$

Note that interior integral is the inverse Fourier transform of a high-pass filtered version of the spectrum of the sinogram.