Inverse Problems for Biomedical Systems

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Model : RTE

Evaluation of the integral

$$\frac{1}{c}\frac{\partial L_\nu(s,\boldsymbol{u},t)}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} L_\nu(s,\boldsymbol{u},t) = -(\mu_a + \mu_s)L_\nu(s,\boldsymbol{u},t) + \frac{\mu_s}{4\pi}\int_{4\pi} ph_\nu(s,\boldsymbol{u}',\boldsymbol{u})\, L_\nu(s,\boldsymbol{u}',t)d\Omega' ds + Q(\boldsymbol{r},\boldsymbol{u},t)$$

Developments:
PN Approximations
P1: Diffusing wave theory
P3:…
Diffusion Approximation

Stochastics
Monte Carlo simulations
Random walk

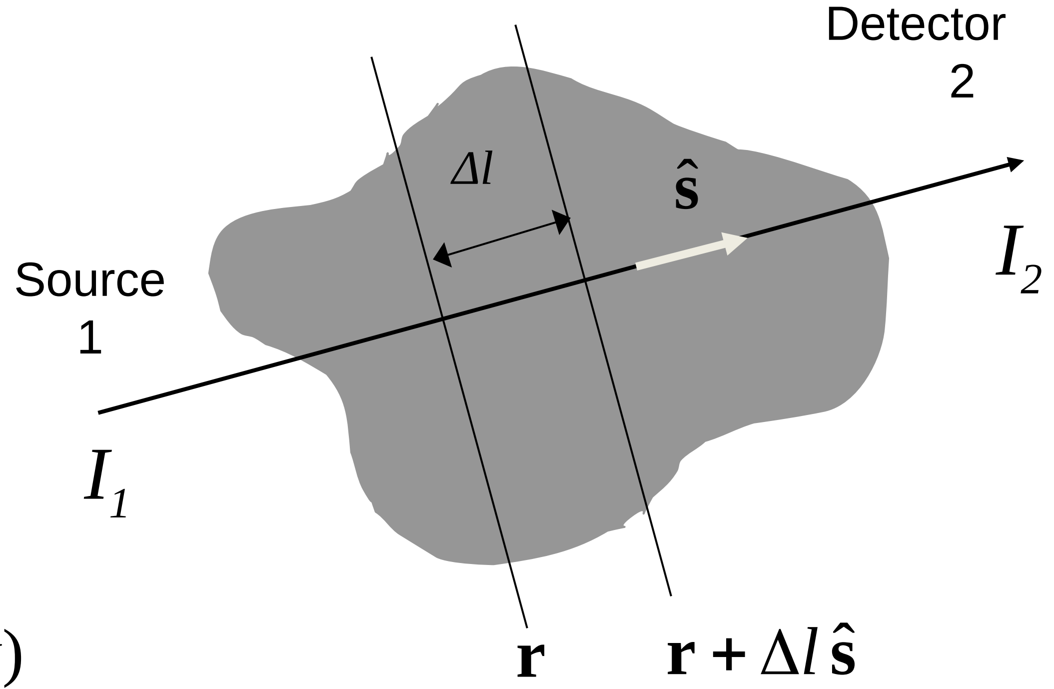
Discretizations (space, angular)
Discrete Ordinates Method(DOM=SN)
Kubelka-Munk (2Flux)
Adding-doubling
FDM,FEM

+ Hybrid Methods

Case 1: Non-scattering media

X-ray propagation

- X-rays propagate along straight lines
- Suffer absorption along their paths



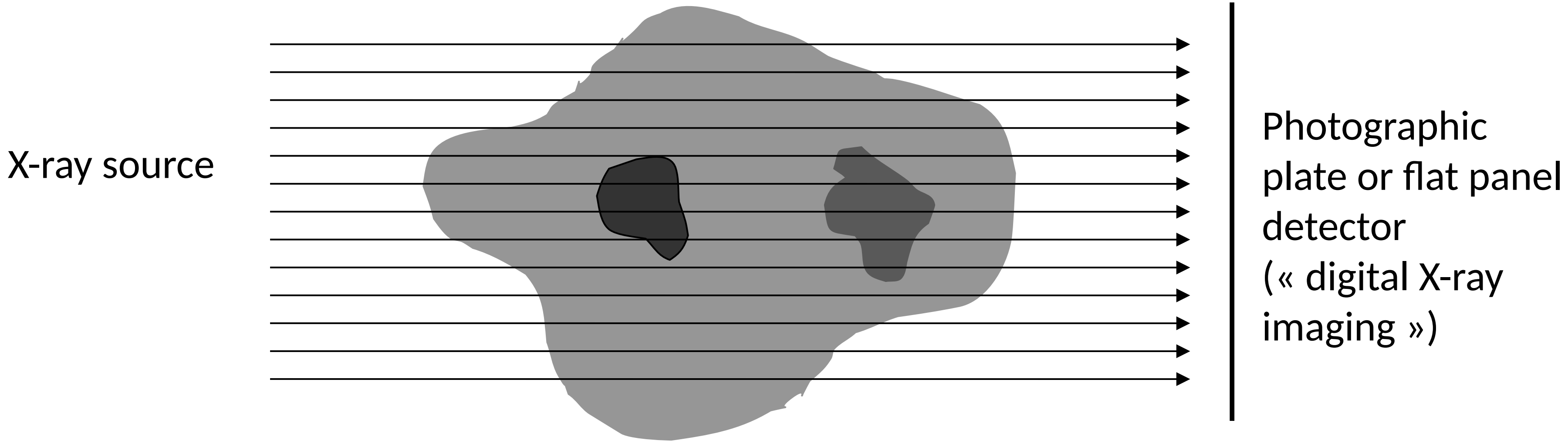
$$I(\mathbf{r} + \Delta l\,\hat{\mathbf{s}}) = I(\mathbf{r})e^{-\alpha(\mathbf{r})\Delta l} \quad \text{(Beer' law)}$$

$$\frac{dI(\mathbf{r})}{ds} = -\alpha(\mathbf{r})I(\mathbf{r}) \Leftrightarrow \hat{\mathbf{s}} \cdot \vec{\nabla} I + \alpha(\mathbf{r})I(\mathbf{r}) = 0$$

$$\Rightarrow -\ln\left(\frac{I_2}{I_1}\right) = \int_1^2 \alpha(\mathbf{r})dl \quad \text{(line integral)}$$

$\hat{\mathbf{s}}$ = unit vector along the straight line

Traditional X-ray imaging



- Problem: All the information acquired along a ray is projected on a single point on the photographic plate (or flat panel electronic detector nowadays)

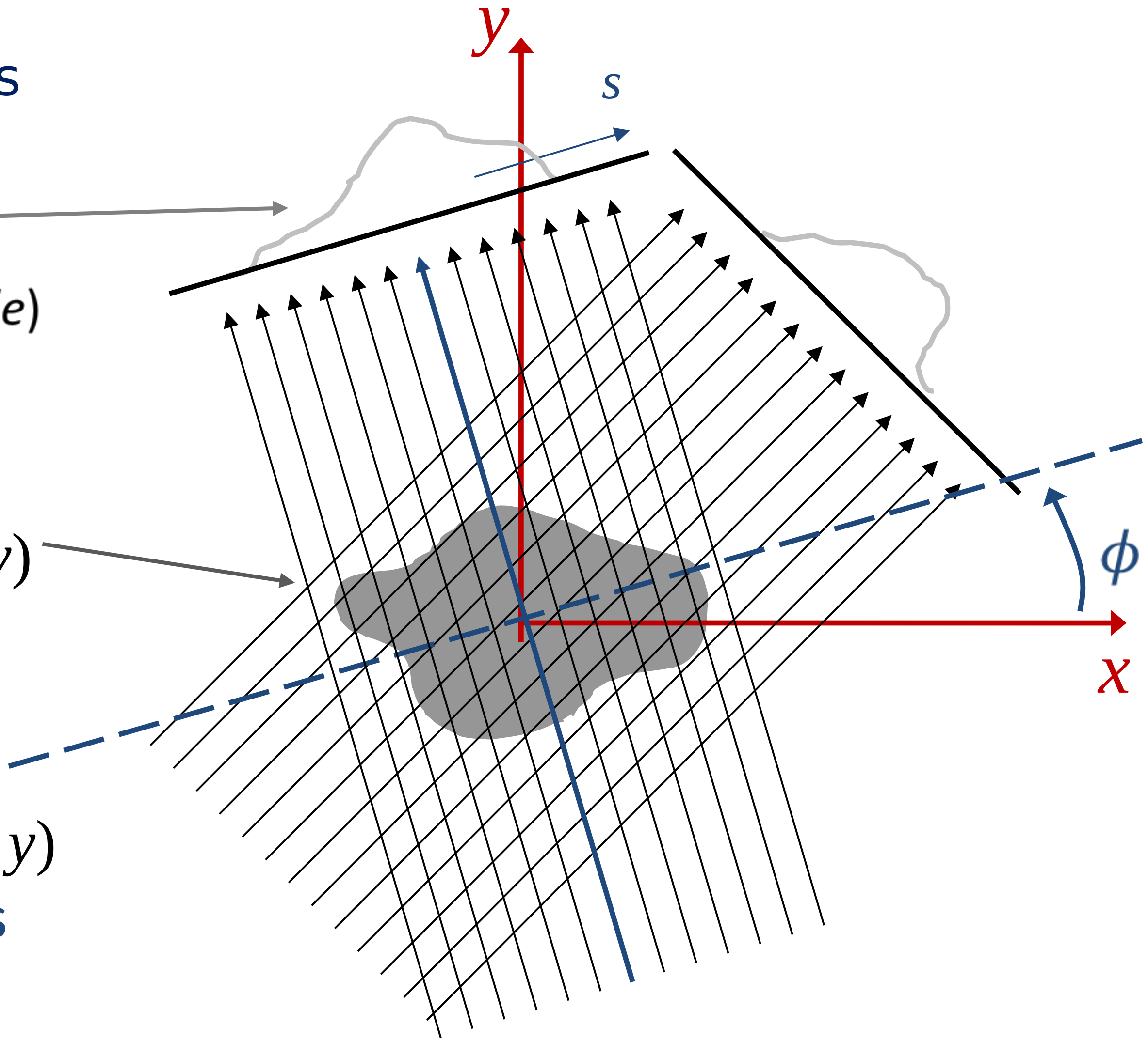
X-ray computed tomography (CT)

- Parallel projections

Projection $p(s, \phi)$

(~*transmitted intensity profile*)

Object $\alpha(x, y)$

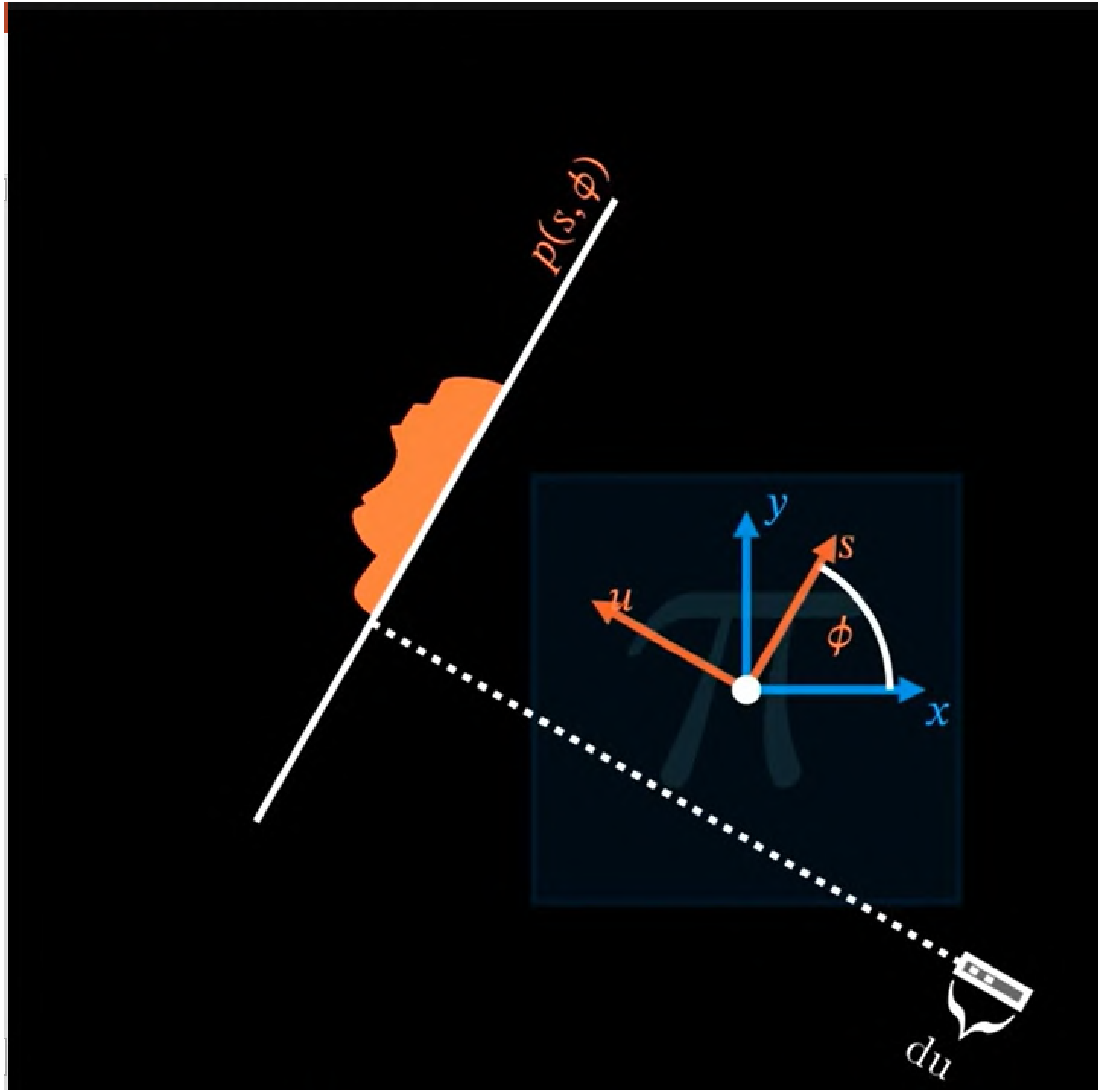


Goal of CT: recover $\alpha(x, y)$ from several projections

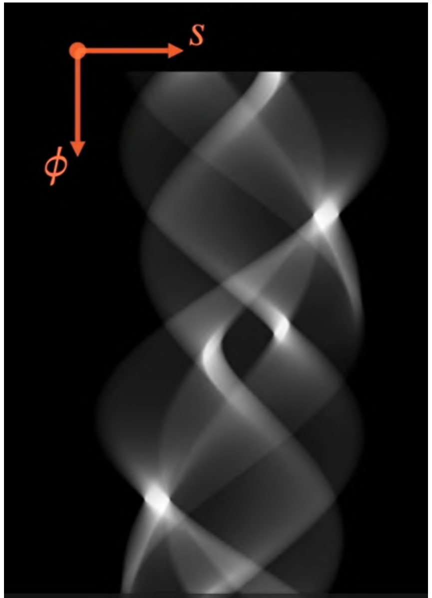
Radon Transform : F= $\mathcal{R}$

Johann Radon (1887-1956)

1917

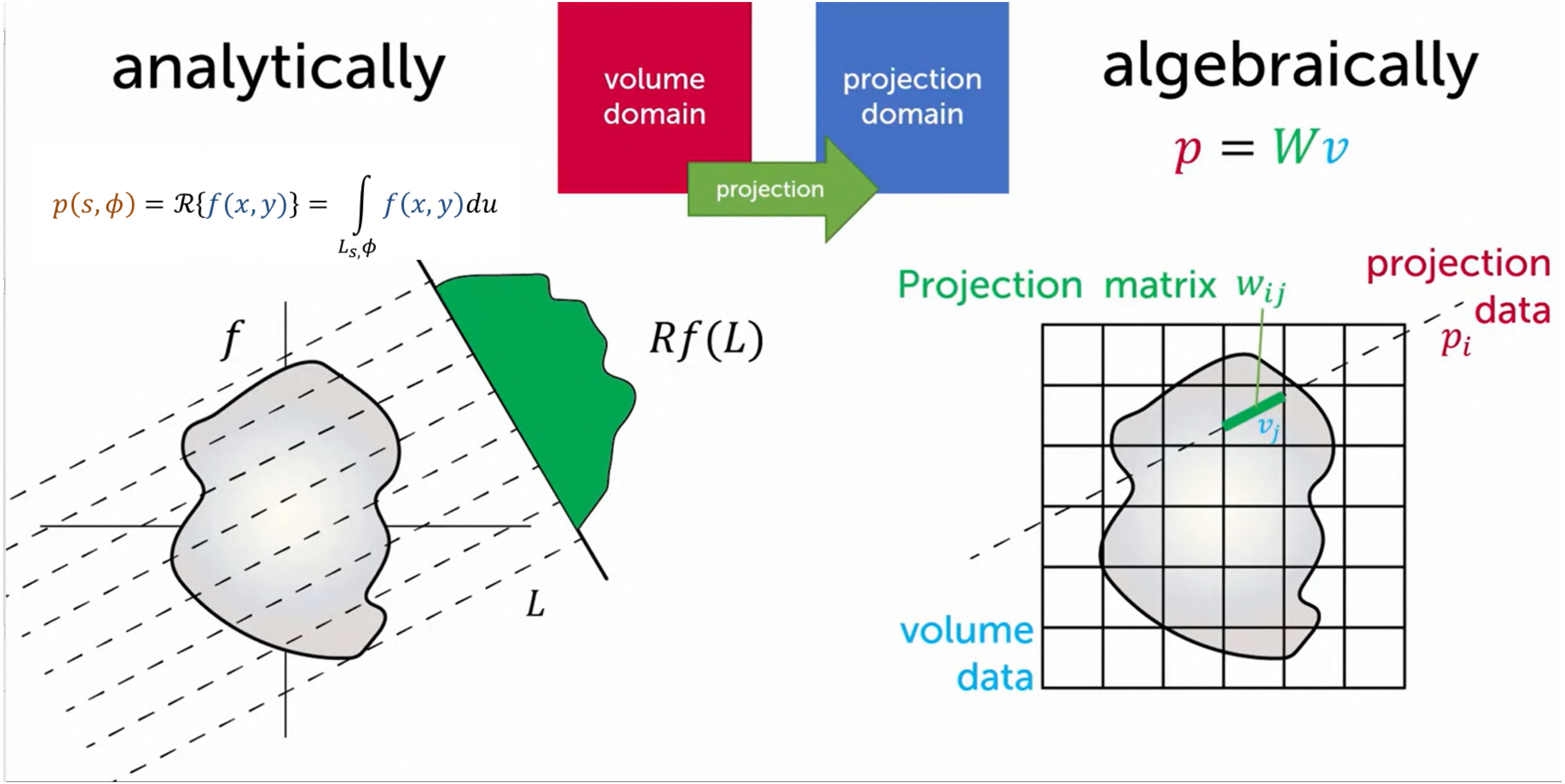


Sinogram



Solving an inverse problem

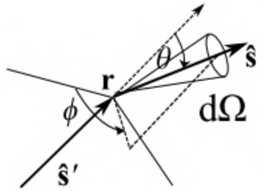
Analytical vs Algebraic



Case 2: Highly-scattering media

Diffusion Approximation

1-Model: From RTE to DE



Expansion of the Radiance in Spherical Harmonics $Y_{n,m}(\theta,\phi)$

$$L(\mathbf{r},\hat{\mathbf{s}},t)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}L_{n,m}(\mathbf{r},t)\,Y_{n,m}(\hat{\mathbf{s}})$$

Ex.

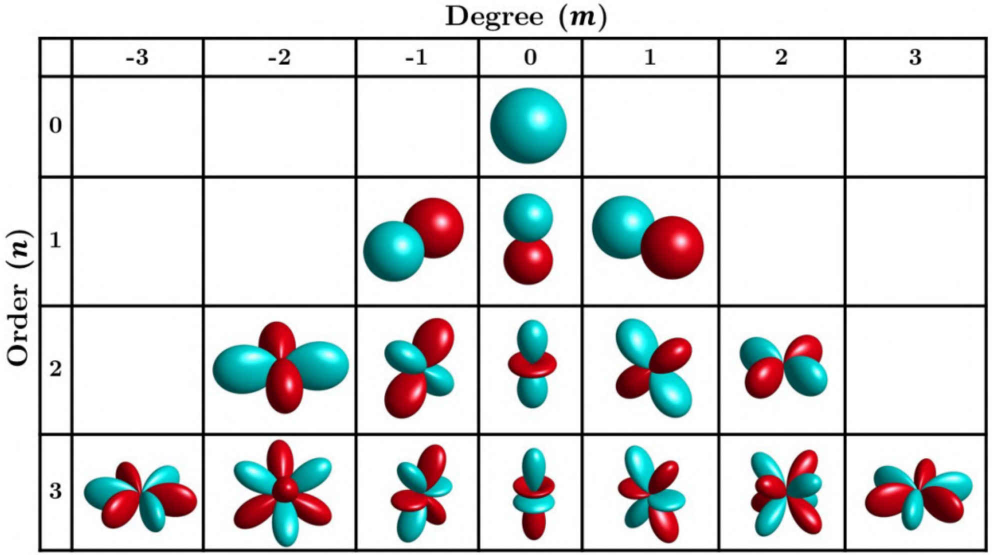
$$Y_{0,0}(\theta,\phi)=\frac{1}{\sqrt{4\pi}},$$

$$Y_{1,-1}(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi},$$

$$Y_{1,0}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta,$$

$$Y_{1,1}(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}.$$

Truncation at order N: « **PN approximation** »



SH orthogonality property:

$$\int_{4\pi}Y_{n,m}(\hat{\mathbf{s}})\,Y^{*}_{n',m'}(\hat{\mathbf{s}})\mathrm{d}\Omega=\int_0^{2\pi}\int_0^{\pi}Y_{n,m}(\theta,\phi)\,Y^{*}_{n',m'}(\theta,\phi)\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi=\delta_{n,n'}\delta_{m,m'}$$

Diffusion Approximation

$$L_{n,m}(\mathbf{r},t) = \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t)\, Y^*_{n,m}(\hat{\mathbf{s}})\, \mathrm{d}\Omega$$

P1 approximation

$$L(\mathbf{r},\hat{\mathbf{s}},t) = \underbrace{\boxed{L_{0,0}(\mathbf{r},t)Y_{0,0}(\hat{\mathbf{s}})}}_{(1)} + \underbrace{\boxed{L_{1,1}(\mathbf{r},t)Y_{1,1}(\hat{\mathbf{s}}) + L_{1,-1}(\mathbf{r},t)Y_{1,-1}(\hat{\mathbf{s}})}}_{(2)}$$

(1) $\Rightarrow$

$$\begin{aligned} L_{0,0}(\mathbf{r},t) &= \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t)\, Y^*_{0,0}(\hat{\mathbf{s}})\, \mathrm{d}\Omega \\ &= \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t) \frac{1}{\sqrt{4\pi}}\, \mathrm{d}\Omega \\ &= \frac{1}{\sqrt{4\pi}} \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t)\, \mathrm{d}\Omega = \frac{\phi(\mathbf{r},t)}{\sqrt{4\pi}} \end{aligned}$$

Fluence Rate [$W.m^{-2}$]

(2) $\hat{\mathbf{s}} = (s_x, s_y, s_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

Current Density Vector [$W.m^{-2}$]: $\boldsymbol{J}(\mathbf{r},t) = \int_{4\pi} L(\mathbf{r},\hat{\boldsymbol{s}},t)\hat{\boldsymbol{s}}\,\mathrm{d}\Omega$

$$Y_{1,-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}(s_x - i s_y), \qquad L_{1,-1} = \sqrt{\frac{3}{8\pi}} \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t)(s_x + i s_y)\mathrm{d}\Omega$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}} s_z, \qquad L_{1,0} = \sqrt{\frac{3}{4\pi}} \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t) s_z \mathrm{d}\Omega$$

$$Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}(s_x + i s_y), \qquad L_{1,1} = -\sqrt{\frac{3}{8\pi}} \int_{4\pi} L(\mathbf{r},\hat{\mathbf{s}},t)(s_x - i s_y)\mathrm{d}\Omega$$

$\Rightarrow$

$$\mathbf{J} = \sqrt{\frac{2\pi}{3}} \left(L_{1,-1} - L_{1,1}, -i(L_{1,-1} + L_{1,1}), \sqrt{2}L_{1,0}\right)$$

$$J_x = \sqrt{\frac{2\pi}{3}}(L_{1,-1} - L_{1,1}),$$

$$J_y = -i\sqrt{\frac{2\pi}{3}}(L_{1,-1} + L_{1,1}),$$

$$J_z = \sqrt{\frac{4\pi}{3}} L_{1,0},$$

$\Rightarrow$

$$\mathbf{J} \cdot \hat{\mathbf{s}} = \sqrt{\frac{2\pi}{3}} \sum_{m=-1}^{1} L_{1,m}(\mathbf{r},t) Y_{1,m}(\hat{\mathbf{s}})$$

P1 approximation: $L(\mathbf{r},\hat{\mathbf{s}},t) = \frac{\phi(\mathbf{r},t)}{\sqrt{4\pi}} + \frac{3}{4\pi}\mathbf{J}.\hat{\mathbf{s}}$

Diffusion Equation

Combination of

$$\left\{ \begin{array}{l} \int_{4\pi}(RTE)ds \\ \int_{4\pi}(RTE).\mathbf{s}ds \end{array} \right.$$

$$\frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r},t) + \nabla.\mathbf{J} + \mu_a(\mathbf{r})\phi(\mathbf{r},t) = Q(\mathbf{r},t)$$

+ additional assumptions

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t) + \frac{1}{3}\nabla\phi(\mathbf{r},t) + [\mu_a(\mathbf{r}) + \mu_s'(\mathbf{r})]\mathbf{J}(\mathbf{r},t) = \mathbf{0}$$

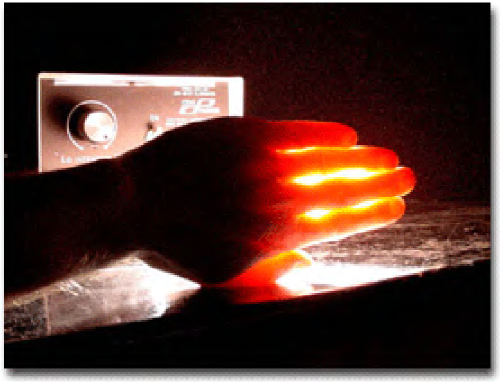
$\mathbf{J}(\mathbf{r},t) = -D\nabla\phi(\mathbf{r},t)$ Fick's law

+ additional assumptions= *Diffusion Approximation (DA)* $\left|\frac{1}{c}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t)\right| \ll (\mu_a + \mu_s')|\mathbf{J}(\mathbf{r},t)|$ ← Slowly varying phenomena

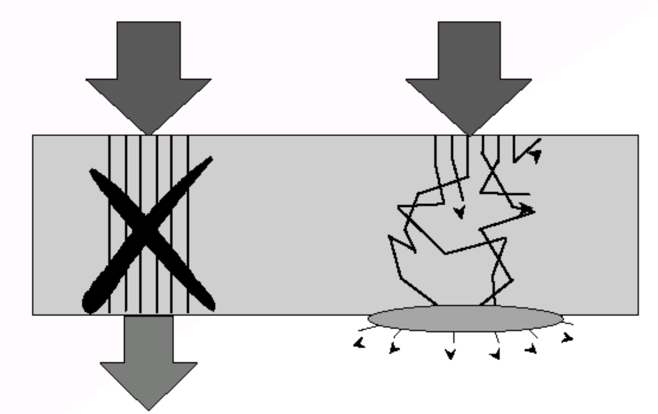
$\mu_s' \equiv \mu_s(1-g)$ *Reduced scattering coefficient*

$D = \frac{1}{3(\mu_s' + \mu_a)}$ *Diffusion coefficient*

$$\frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r},t) - \nabla.(D(\mathbf{r})\nabla\phi(\mathbf{r},t)) + \mu_a(\mathbf{r})\phi(\mathbf{r},t) = Q(\mathbf{r},t)$$



Absorption (μ_a)



Diffusion (μ_s')

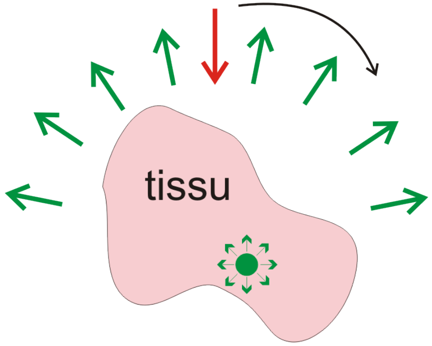
$$-\nabla.\left(\frac{1}{3\mu_s^{'}(\mathrm{r})}\nabla\phi(\mathrm{r},t)\right)+\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathrm{r},t)+\mu_a(\mathrm{r})\phi(\mathrm{r},t)=S(\mathrm{r},\mathrm{r}_0,t)$$

Diffusion Propagation Absorption Source

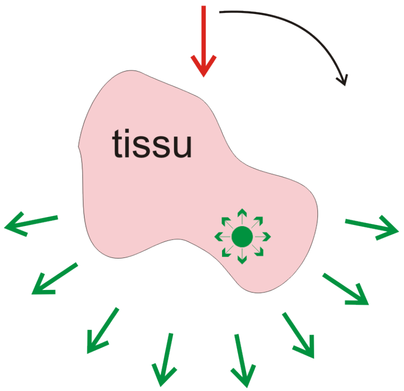
Several configurations are possible

Pogue, *et al, Comparison of imaging geometries for DOT of tissue* Opt. Expr., 1999, 4, 270-86

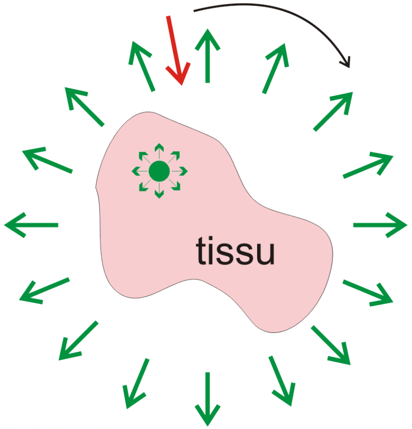
Epi-illumination



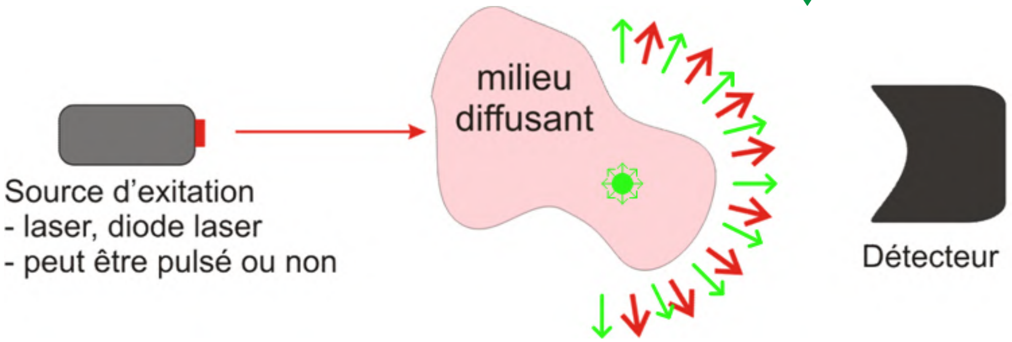
Trans-illumination



Epi-/Trans-illumination

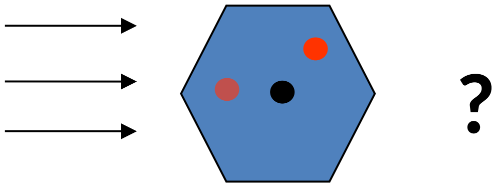


Fluorescence

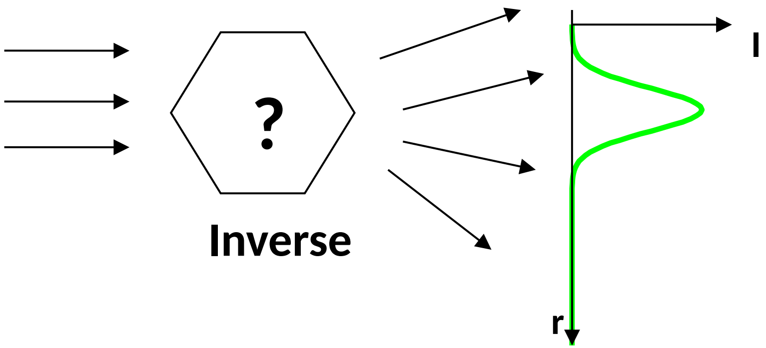


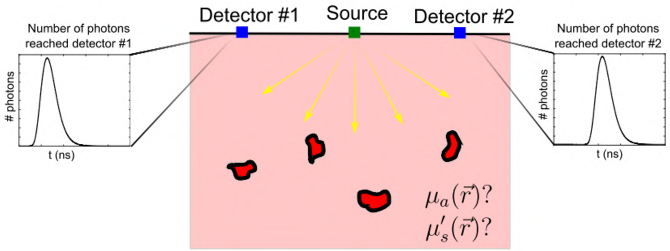
The inverse problem:

Forward problem ➔ Inverse problem



forward



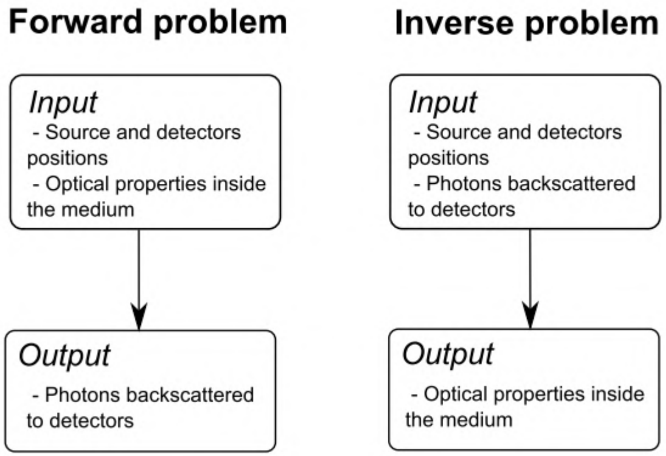


Forward problem:

Knowing the **volume** distribution of the unknowns (absorption coefficient, diffusion coefficient, distribution of the markers) ➔ determine profil of **surface** measurements

Inverse problem:

Knowing the profil of **surface** measurements ➔ determine **volume** distribution of the unknowns (absorption coefficient, diffusion coefficient, distribution of the markers)



Principal problem: # of unknowns > # of measurements

- ➔Ill-posed problem
- ➔Difficulties in determining a unique solution