

Radon Transform

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1 Light Propagation model

The Radon Transform lies at the heart of X ray tomography. Consider the propagation of light in tissue where scattering is negligible

$$\frac{1}{c} \frac{\partial I(\mathbf{r}, \mathbf{u})}{\partial t} + \mathbf{u} \cdot \nabla I(\mathbf{r}, \mathbf{u}) = -\mu_a(\mathbf{r})I(\mathbf{r}, \mathbf{u}) + s(\mathbf{r}, \mathbf{u}) \quad (1)$$

Suppose that there are no internal sources, $s(\mathbf{r}, \mathbf{u}) = 0$, and that we consider the steady-state equilibrium, $\partial I / \partial t = 0$. Note that each for each direction \mathbf{u} , the intensity is independent of other directions. We can find a closed form solution

$$I(\mathbf{r}_0 + \ell\mathbf{u}, \mathbf{u}) = I(\mathbf{r}_0, \mathbf{u}) \exp \left(- \int_0^\ell \mu_a(\mathbf{r}_0 + s\mathbf{u}) ds \right) \quad (2)$$

where s [m] is a distance away from a boundary point \mathbf{r}_0 along direction \mathbf{u} .

2 Forward problem

Suppose that we have detectors around the surface of the domain of interest, that the input intensity is known. We can measure then the ratio between the output/input intensities and take the log, such that

$$\log \left(\frac{I(\mathbf{r}, \mathbf{u})}{I(\mathbf{r}_0, \mathbf{u})} \right) = - \int_{-\infty}^{\infty} \mu_a(\mathbf{r}_0 + s\mathbf{u}) ds \quad (3)$$

Here, the limits are taken for all space assuming that outside the domain of interest there is no absorption.

To get all the information along direction \mathbf{u} , we consider all lines that do not intersect. We can do this by choosing a normal vector perpendicular to \mathbf{u} , such that $\mathbf{r}_0 = t\mathbf{n}$ letting $t \in (-\infty, \infty)$.

$$\log \left(\frac{I(\mathbf{r}_0, \mathbf{u})}{I(\mathbf{r}, \mathbf{u})} \right) = \int_{-\infty}^{\infty} \mu_a(t\mathbf{n} + s\mathbf{u}) ds \quad (4)$$

We can define the Radon transform as

$$\mathcal{R}(\mathbf{n}, t) \triangleq \int_{-\infty}^{\infty} \mu_a(t\mathbf{n} + s\mathbf{u}) ds \quad (5)$$

If we choose a coordinate system $\mathbf{r} = (x, y)$, then we can parametrize n using some angle θ with respect to x :

$$\mathbf{n} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix} \quad (6)$$

Note that the choice of \mathbf{n} is not unique (can be mirrored). The Radon transform is then

$$\log \left(\frac{I(\mathbf{r}_0, \mathbf{u})}{I(\mathbf{r}, \mathbf{u})} \right) = \mathcal{R}(\theta, t) \triangleq \int_{-\infty}^{\infty} \mu_a(s \sin \theta + t \cos \theta, -s \cos \theta + t \sin \theta) ds \quad (7)$$

3 Inverse problem

Suppose a sinogram (θ, t) is given. We wish to recover $\mu_a(\mathbf{r})$ that produced the data. Let the 2D Fourier transform of the absorption coefficient be

$$M(\boldsymbol{\omega}) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mu_a(\mathbf{r}) e^{-2\pi i \mathbf{r} \cdot \boldsymbol{\omega}} d^2 \mathbf{r} \quad (8)$$

Similarly, the 1D Fourier transform of the sinogram for a given θ is

$$\Pi(\theta, \omega) = \mathcal{F}\{\mathcal{R}(\theta, t)\} = \int_{-\infty}^{\infty} \mathcal{R}(\theta, t) e^{-2\pi i \omega t} dt = M(\omega \mathbf{n}) \quad (9)$$

Where the last equality comes from the Fourier slice theorem. Indeed, let $\mathbf{r} = t\mathbf{n} + s\mathbf{u}$

$$\begin{aligned} M(\omega \mathbf{n}) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mu_a(\mathbf{r}) e^{-2\pi i (t\mathbf{n} + s\mathbf{u}) \cdot (\omega \mathbf{n})} dt ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mu_a(\mathbf{r}) e^{-2\pi i t \omega} dt ds \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mu_a(t\mathbf{n} + s\mathbf{u}) ds \right) e^{-2\pi i t \omega} dt \\ &= \int_{\mathbb{R}} \mathcal{R}(\mathbf{n}, t) e^{-2\pi i t \omega} dt \\ &= \Pi(\theta, \omega) \end{aligned} \quad (10)$$

The most common inversion approach is the Back projection formula, which can be derived starting from the inverse Fourier transform of M :

$$\mu_a(\mathbf{r}) = \int_{\mathbb{R}} \int_{\mathbb{R}} M(\boldsymbol{\omega}) e^{2\pi i \boldsymbol{\omega} \cdot \mathbf{r}} d^2 \boldsymbol{\omega} \quad (11)$$

consider a change of variables from cartesian to polar coordinates in the frequency plane, using the angle θ and the ‘radius’ ω , such that $\boldsymbol{\omega} = (\omega \cos \theta, \omega \sin \theta) = \omega \mathbf{n}$:

$$\mu_a(\mathbf{r}) = \int_0^\pi \int_{\mathbb{R}} M(\omega \mathbf{n}) e^{2\pi i \omega \mathbf{n} \cdot \mathbf{r}} |\omega| d\omega d\theta \quad (12)$$

$$= \int_0^\pi \int_{\mathbb{R}} \Pi(\theta, \omega) e^{2\pi i \omega t} |\omega| d\omega d\theta \quad (13)$$

$$= \int_0^\pi \mathcal{F}^{-1}\{\Pi(\theta, \omega) |\omega|\}(t) d\theta \quad (14)$$

Note that interior integral is the inverse Fourier transform of a high-pass filtered version of the spectrum of the sinogram.