Inverse Problems for Biomedical Systems

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What is an inverse problem?

From a mathematical point of view

- We consider the indirect measurement of an unknown physical quantity $p \in X$. The measurement $m \in Y$ is related to the unknown by a **physical or mathematical model**

$$m = F(p)$$

where $F: X \to Y$ is called the **forward mapping**.

- Computing m for a given p is called the **forward problem**.

- Finding p for a given measurement m (the data) is called the **inverse problem**.

$$p = F^{-1}(m)$$

- The inverse problem is often **ill-posed**, making it more difficult than the corresponding direct problem.

Well/Ill-posedness

- A problem is called **well-posed** (in the sense of Jacques Hadamard, 1923), if
  - (a) Existence a solution exists, at least one
  - (b) Uniqueness the solution is unique, at most one
  - (c) Stability the solution depends continuously on the data.
- If one or more of these conditions are violated, the problem is called **ill-posed**.

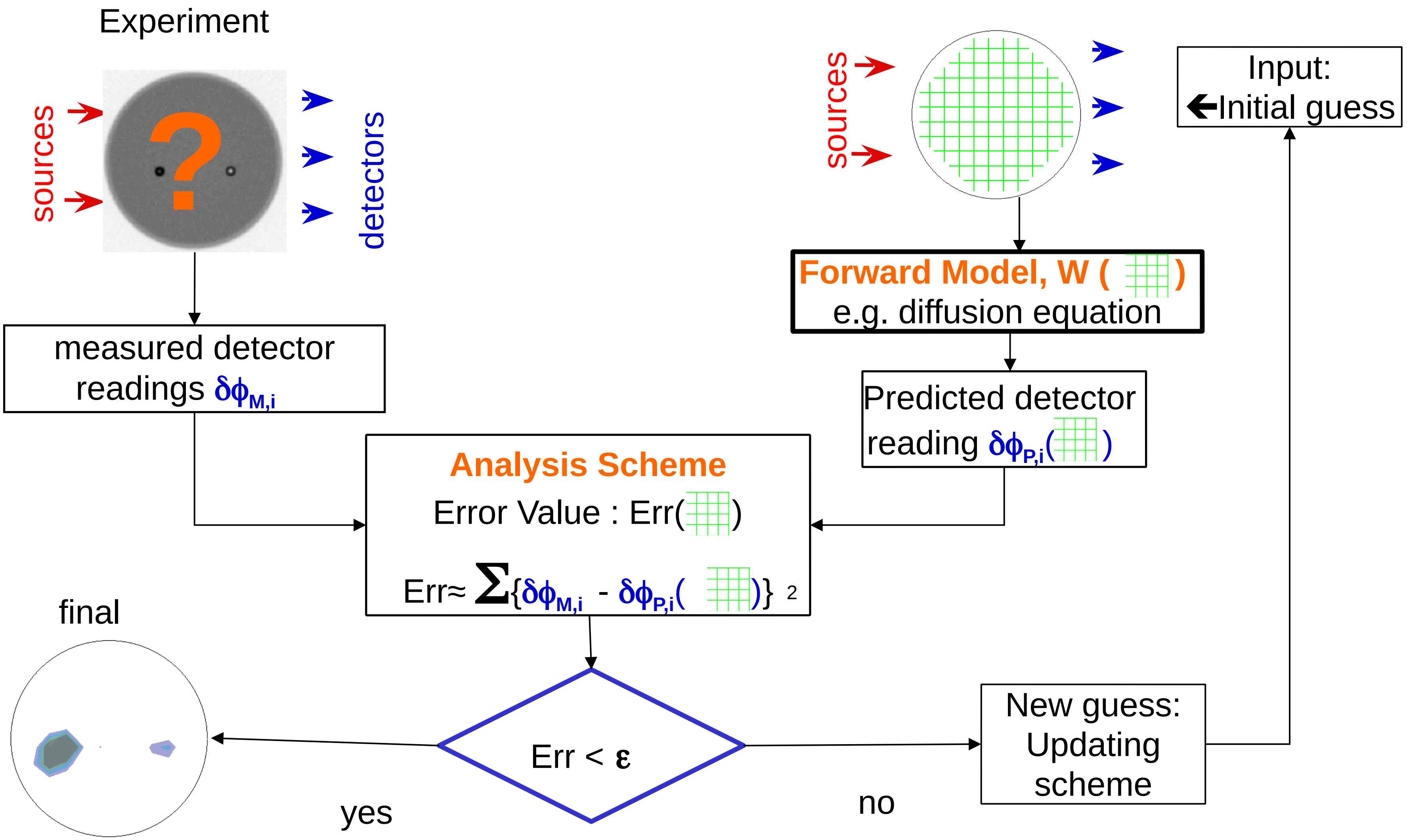
- Some examples of ill-posed inverse problems are X-ray tomography, image deblurring, the inverse heat equation, and electrical impedance tomography…

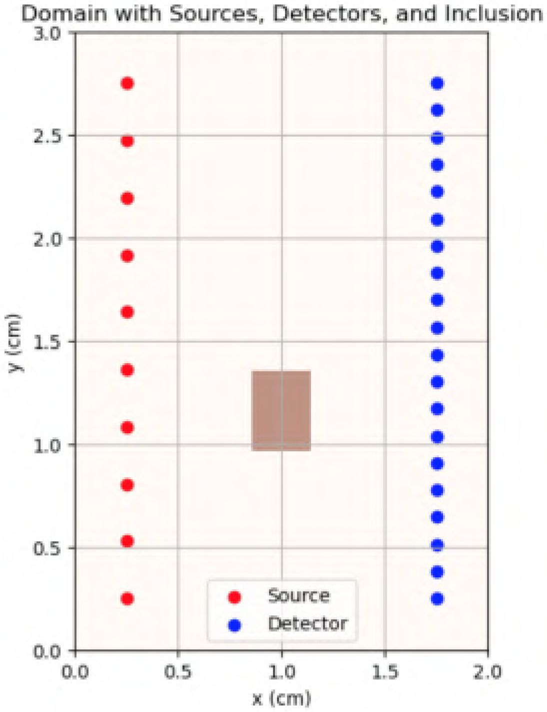
- The ill-posedness of an inverse problem poses a challenge because usually **errors are present in the measurements**.

- Incorporating these into model in the form of additive noise η leads to **a more realistic model**

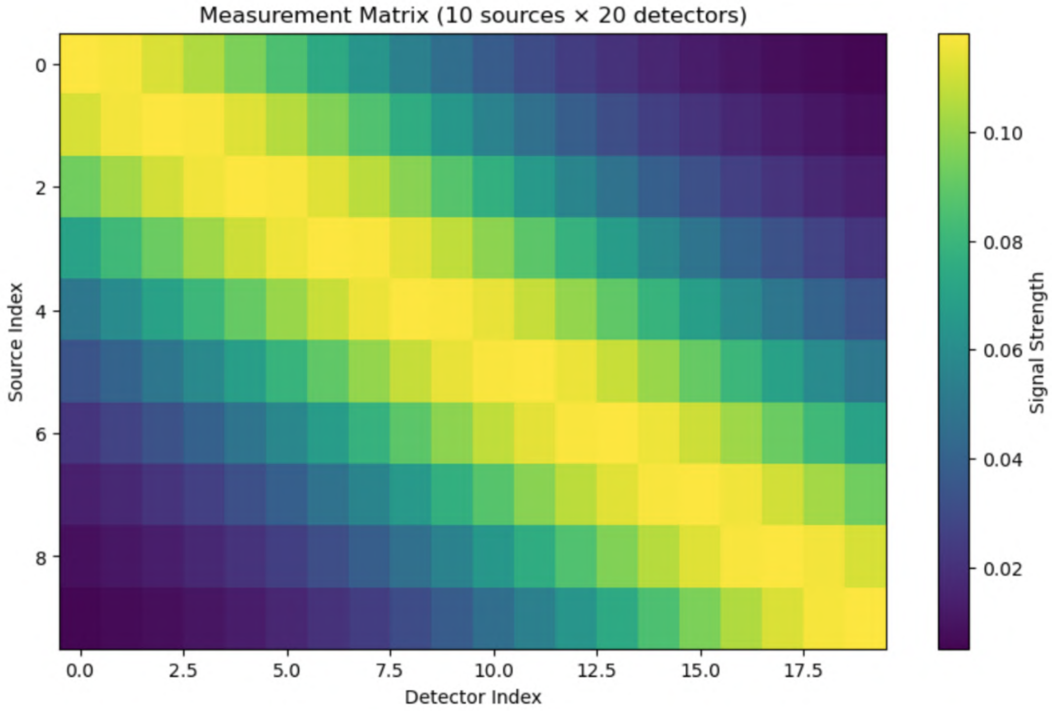
$$m = F(p) + \eta$$

Reconstruction=inverse problem





Domain with Sources, Detectors, and Inclusion
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y (cm)
0.0
0.5
1.0
1.5
2.0
x (cm)
Source
Detector



Measurement Matrix (10 sources × 20 detectors)
0
2
4
6
8
Source Index
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
Detector Index
0.10
0.08
0.06
0.04
0.02
Signal Strength

The inverse problem

Pseudo-code

$$\begin{bmatrix} \delta\phi_{11} \\ \delta\phi_{12} \\ \vdots \\ \delta\phi_{N_s N_d} \end{bmatrix} \approx \begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} (\delta\mu_a)_1 \\ (\delta\mu_a)_2 \\ \vdots \\ (\delta\mu_a)_M \end{bmatrix} + \begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} (\delta D)_1 \\ (\delta D)_2 \\ \vdots \\ (\delta D)_M \end{bmatrix} \quad \Longleftrightarrow \quad \delta\phi_{sd} = \mathbf{J}\delta\mu_a$$

$N_{\max}$ = maximum number of iterations

Set initial guess of absorption coefficient $\mu_a^0(\mathbf{x})$ (usually uniform)

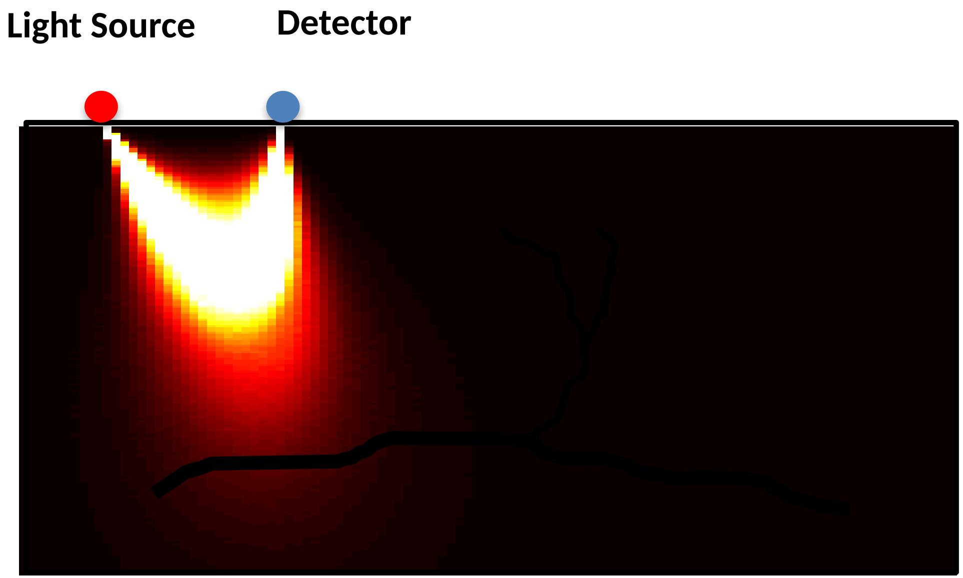
For loop ($k = 0 : N_{\max}$):

1. Solve forward problem with $\mu_a^k(\mathbf{x})$ from iteration k-th.
2. Set $\delta\phi_{sd}$: signal difference between unknown and simulated medium.
3. Compute the jacobian matrix $\mathbf{J}$.
4. Solve linear system $\delta\phi_{sd} = \mathbf{J}\delta\mu_a$ by applying some regularization. $\mathbf{J}^T\mathbf{J}(\delta\mu_a + \lambda I) = \mathbf{J}^T\delta\phi_{sd}$
5. $\mu_a^{k+1}(\mathbf{x}) = \mu_a^k(\mathbf{x}) + \delta\mu_a^k(\mathbf{x})$

If $\|\mu_a^{k+1} - \mu_a^k\|_2 / \|\mu_a^{k+1}\|_2 < \varepsilon$ then convergence has been reached and the algorithm is stopped.

DOT

Forward / Inverse problem



$$S_0 - S = \int \underbrace{K(r)}\, S_{in}\ \underbrace{\delta\mu_a(r)}\ d^3r$$

Change at detector

Sensitivity for given source and detector position

Object with change in absorption