Empirical exam
Econometrics of Financial Markets

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Appendix

Exercise 1

Firstly, we visually inspect all the monthly series by plotting it. All the three series seems to hover around a constant 0 mean without exhibiting volatility clustering, they all may suggest stationarity (see p. 6). However, for a more statistically robust assessment, we adopt the **Augmented Dickey-Fuller test (ADF)**, the latter checks for the presence of a unit root, which is indicative of non-stationarity. As we can see from the p-values of the three tests (section 1.2), we have evidence to reject the null, i.e. the series are stationary.
Before estimating the VAR model, it's essential to determine the optimal lag length. We proceed by minimizing information criteria such as AIC, BIC, HQC and FPE. In our case, a single lag was found to be optimal as it minimized all these criteria (section 1.3).
To further validate the VAR(1) model, we do a residual diagnosis (Not autocorrelated, homoskedastic, Normal distributed) (see p. 7). The residual autocorrelation was tested with the **Ljung-Box Test** and found to be non-significant, with all p-values exceeding 50%, indicating a well-specified model. Additionally, an **ARCH Test** was conducted to assess heteroskedasticity, resulting in a failure to reject the null hypothesis, suggesting homoskedasticity. Finally, we take the **Anderson-Darling Test** to verify that the singular residuals are normally distributed, confirming that the residual vector has multivariate white noise characteristics, indicating a well-specified VAR model. We're now ready to move on to evaluating the structural impulse response function (IRF) (Section 1.4) in order to evaluate the dynamic impact of oil price shocks on the inflation rate and the growth rate . Unlike the VAR model, which primarily reveals time-correlated relationships, structural VAR is more effective for structural analysis. They integrate economic theories and assumptions about the interactions between variables. This approach helps in identifying and interpreting the causal structure within the system, offering a deeper understanding of the underlying economic dynamics.
In order to discuss the propagation mechanism of oil price shocks on the inflation rate and the growth rate, we focus on the first column of the IRFs' plots (see p. 9).
The first plot (**oil** $\rightarrow$ **oil**) refers to the impact of Oil price Shock on the Oil price. The plot suggests that the response is positive and significant at the onset, as expected, since a shock in oil prices directly affects oil prices. The effect starts with the peak and then decays, which indicates that the market for oil prices adjusts relatively quickly to shocks (after 6-8 months the shock is "digested"). The economic motivation could be due to various market mechanisms such as supply-demand adjustments, speculation or other market participants' reactions.
The second plot (**oil** $\rightarrow$ **inf**) refers to the impact of Oil price Shock on the inflation rate. The plot suggests that the response is positive and an increase in oil prices can lead to higher inflation, consistent with economic theory that oil price increases can be passed through to consumers in the form of higher prices for goods and services. The impact peaks in the second month (with magnitude 0.17575) from the shock, and then gradually diminishes(after 7-10 months the shock is "digested"), indicating that the effect of oil price shocks on inflation may be temporary or that monetary policy may respond to stabilize inflation.
The third plot (**oil** $\rightarrow$ **growth**) refers to the impact of Oil price Shock on the growth rate. The graph shows a sharp initial increase in the response of growth rate to a positive oil price shock. This is somewhat atypical because empirical evidence [1] often suggest that positive oil price shocks (increases in oil prices) tend to have a negative impact on economic growth, particularly for oil-importing countries. This is due to higher production costs and potential reductions in consumer spending power. The response quickly drops below zero within 4 months, indicating a negative impact on growth which aligns more closely with standard economic theory.
Finally, after 15-17 months, the response hovers around zero, implying that any negative effects

[1] Hamilton, J.D. (1983) 'Oil and the Macroeconomy Since World War II', *Journal of Political Economy*, 91(2), pp. 228-248.

of the oil price shock on growth are short-lived or are counterbalanced by other factors in the economy. In conclusion, it appears that the system absorbs shocks to oil prices with significant short-term effects on inflation and growth, but these effects tend to diminish over time.
The last task of this exercise is to estimate an appropriate ARMA model for the preferred variable (Oil) (section 1.5). The general form of an ARMA(p, q) model for the variable Oil_t is given by:

$$\text{Oil}_t = c + \sum_{i=1}^{p} \phi_i \text{Oil}_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t \tag{1}$$

Having confirmed the stationarity of oil price series with the **ADF test** (section 1.2), we proceeded to evaluate heteroskedasticity using the **ARCH test** (section 1.5). The test yielded a significant p-value of 0.00106, leading us to reject the null hypothesis and have evidence of homoskedasticity. The next step is determine the appropriate 'p' and 'q' values for the ARMA model, that represent the number of lags necessary to eliminate serial autocorrelation for an accurate estimation of the model. For this purpose we first take a look on the correlograms **ACF** and **PACF**, which suggest potential models such as ARMA(1,0) or ARMA(1,1). To refine our selection, we then focused on minimizing information criteria like AIC and BIC (p. 10). This Method leads on the ARMA(1,0) i.e. AR(1) as optimal. We proceed to estimate the AR(1) ARMA(1,1) and ARMA(2,1), we found that AR(1) without constant was the most parsimonious. Moreover we conducted an analysis of the residuals of the same model (the simpler, and for now the best). The results of the test in the summary (p. 12) are:

- **Ljung-Box (Q)**: The p-value for the Ljung-Box test is 0.57, suggesting that there is no significant autocorrelation in the residuals, not rejecting the null hypothesis of no autocorrelation.
- **Jarque-Bera (JB)**: The Jarque-Bera test p-value is 1.00, indicating that the residuals are normally distributed, not rejecting the null hypothesis of normality.
- **Heteroskedasticity (H)**: The p-value is 0.87, suggesting no heteroskedasticity in the residuals.

The model appears to fit the data well, with no significant autocorrelation in residuals. The latters appear normally distributed. The non-significance of the constant term implies that it might be unnecessary in this model. Moreover the F-test (p. 12) shows that the ARMA(1,0) model is significantly better than a mean-only model, it suggests that the autoregressive component of lag 1 is essential in modeling the Oil time series and cannot be omitted.
Finally we can compare the equation of Oil in The VAR model with the AR(1) process:
In the AR(1) (p. 12) the coefficient for oil is significant (p ¡ 0.001), indicating a strong autoregressive component. The model fits the data well as evidenced by the low AIC and BIC values. The Ljung-Box test suggests no autocorrelation in the residuals, indicating a good fit. On the other side, in the VAR Model, oil is also influenced by past values of growth and inflation rate, (p. 8) but the coefficients for inf (p-value = 14.8%) and growth (p-value = 7.5%) are not significant at traditional levels (of 5%). The oil equation within the VAR model shows a significant autoregressive component similar to the AR(1) model. We can say that this model captures the interdependencies among oil, growth, and inflation. However, the lack of significant coefficients for inf and growth in the oil equation suggests that oil might be predominantly driven by its own past values, similar to the findings in the AR(1) model.
In conclusion, while the AR(1) model provides a parsimonious fit for the *oil* series, the VAR model offers a broader perspective by including growth and inflation. The choice between the two would depend on the specific research question and whether the additional complexity of the VAR model provides valuable insights for growth and inflation.

Exercise 2

From an empirical point of view, if the series are non-stationary, one possibility to test for the condition of PPP is to refer to the notion of cointegration. We starts importing the data of pa (prices in countr A), pb (prices in countr B) and e (exchange rate) already in logarithmic form, then we plot the graphs and e can observe the presence of persistence and a trend, suggesting that the series are likely non-stationary. (p. 13). As before, we adopt the **ADF test** in order to confirm or discard the hypothesis of non-stationarity. The results (section 2.2) indicates a very high p-value for each series so we can NOT reject H_0 : presence of unit root, i.e. we have evidence of non-stationarity.
We can now proceed to evaluate if there are some cointegration relation between the three time series. For this purpose we utilize the Johansen test (section 2.3), the latter checks for the presence of multiple cointegrating vectors, and the null hypothesis for each number of cointegrating vectors is tested in sequence, starting from 0. The results (p. 14) shows that there is at least one cointegrating relationship among the *pa*, *pb*, and *e* series. This implies a long-term equilibrium relationship among these variables, which is consistent with the Purchasing Power Parity (PPP) hypothesis. Finally, the Max-Eigen test confirms that there is one strong cointegrating relationship and no additional relationships.
The next step is to ,through an error correction model, verify the reaction of the three variables to disequilibria from the cointegration relation. For this purpose we adopt the vector error correction model (VECM)[2].
To estimate the VECM model (section 2.4), it remains to find the optimal lag of differences. We use the same method as before, i.e. minimizing the information criterions, the suggested one is the 1 lag order (p. 15).
We are ready now to estimate the VECM model, the summary (p. 17) shows that the model provides a wealth of information regarding the dynamic relationships and long-term equilibrium among the variables *pa*, *pb*, and *e*.
We have:

1. **Loading Coefficients (Alpha)**:
   These coefficients are associated with the error correction term ($ec1$) and indicate how each variable adjusts in the short run to restore the long-term equilibrium following a shock. The coefficient for $ec1$ in the equation for *pa* is **-0.3052** and is highly significant (p-value of 0.000). This suggests that *pa* adjusts to correct for any disequilibrium. For *pb* and *e*, the coefficients of $ec1$ are **0.0120** and **0.0102**, respectively, but are not statistically significant.

2. **Cointegration Relations (Beta)**:
   These coefficients represent the long-term equilibrium relationships among the variables. The coefficients $beta.1$, $beta.2$, and $beta.3$ correspond to *pa*, *pb*, and *e*, respectively. The coefficients for *pb* and *e* are **-1.0940** and **-0.9952**, respectively, and are both highly significant. This suggests a strong long-term relationship among these variables.

In conclusion the results (p. 17), especially the cointegration relations, provide support for the PPP hypothesis, indicating that the exchange rate (*e*) and the price levels of the two countries (*pa* and *pb*) are in a long-term equilibrium relationship. This supports the idea that over the long term, the exchange rate adjusts to equalize the price levels between the two countries. This is consistent with the PPP hypothesis, which posits a long-term equilibrium between exchange rates and price levels.

[2]Engle, R.F. and Granger, C.W.J., 1987. Co-integration and Error Correction: Representation, Estimation, and Testing. Econometrica, 55(2), pp.251-276.

1 Exercise Appendix

1.1 Importing the Data and Plot

I have utilized Python as econometrics software, the code and the results follow:

```python
import pandas as pd
import statsmodels.api as sm
from statsmodels.tsa.api import VAR
import matplotlib.pyplot as plt

# Import the Dataset
file_path = 'data/Contucci_Marco.csv'
data = pd.read_csv(file_path)
# Visualize the head of the dataset
data.head()

# Selection of the variables for the VAR model
var_data = data[["oil", "inf", "growth"]]
# Convert time format
var_data.index = pd.to_datetime(data['obs'], format='%YM%m')
# Control if there are missing values
missing_values = var_data.isnull().sum()
# No Missing Values :)

# Plot of the 3 time series
fig, axes = plt.subplots(nrows=3, ncols=1, dpi=120, figsize=(10,6))
for i, ax in enumerate(axes.flatten()):
    data = var_data.iloc[:, i]
    ax.plot(data, color='blue', linewidth=1)
    # Better Visualization
    ax.set_title(var_data.columns[i])
    ax.xaxis.set_ticks_position('none')
    ax.yaxis.set_ticks_position('none')
    ax.spines["top"].set_alpha(0)
    ax.tick_params(labelsize=6)
plt.tight_layout()
```

The plots (Figure 1) suggests Stationarity, let's test it in the following section.

1.2 Check for Stationarity

I have applied the Augmented Dickey-Fuller(ADF) in order to check if the series are Stationary:

```python
# Check for Stationarity
import statsmodels.tsa.stattools as ts

# Test Augmented Dickey-Fuller(ADF) for Stationarity
adf_results = {}
for col in var_data.columns:
    result = ts.adfuller(var_data[col], autolag='AIC')
    adf_results[col] = result[1]  # p-value
print("ADF p-values:", adf_results)
```

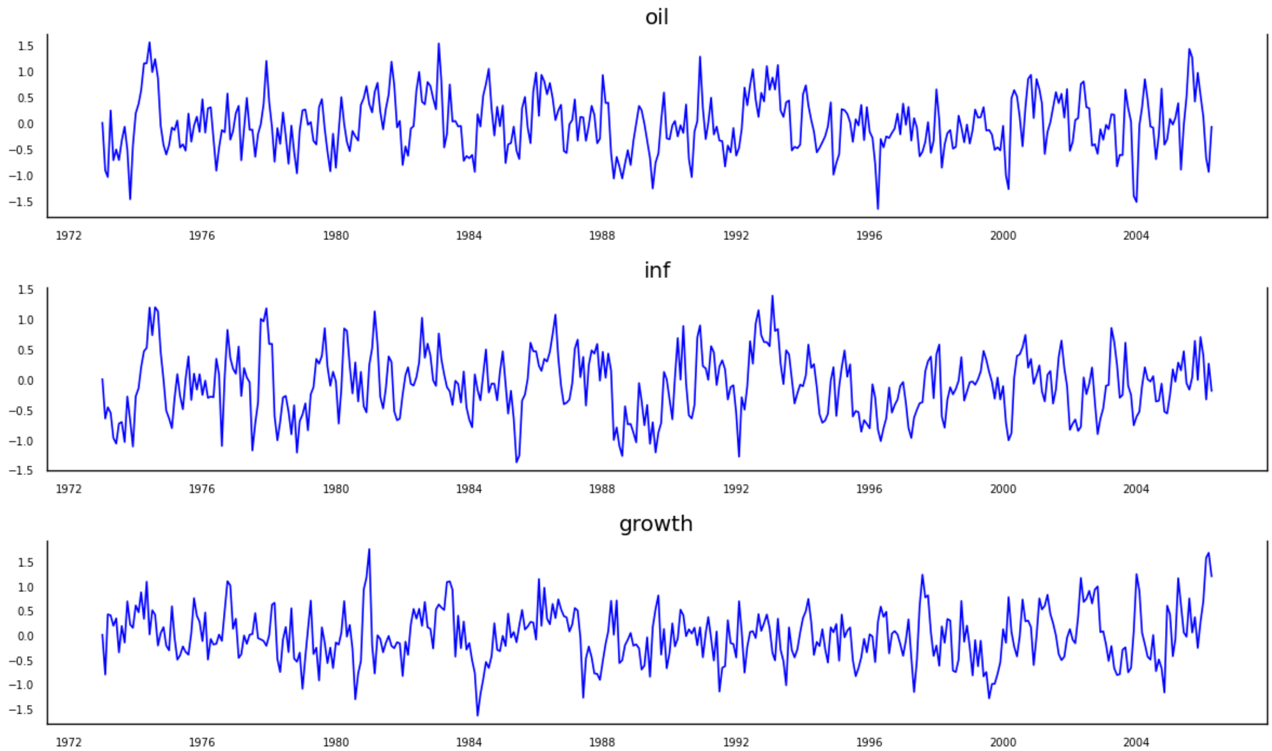

Figure 1: Plot of the three time series

The output is the following:
`ADF p-values: {'oil': 5.060e-20, 'inf': 4.380e-16, 'growth': 1.269e-20}`
In conclusion we have evidence of Stationarity since the Null Hypothesis of precence of unit root (non Stationary) is rejected for every time series.

1.3 Optimal Order VAR estimation

First of all we check which order of the VAR model is the optimal minimizing information criterion such as AIC, BIC and so on. We chose the smallest one. Moreover we take a look to the behaviour(Homoskedasticity, Autocorrelation,Distribution) of the residuals of the optimal VAR model, assumed to be a White-Noise distributed.

```python
# Determining optimal order for the VAR
model = VAR(var_data)
order_selection = model.select_order(maxlags = 15)
selected_order = order_selection.selected_orders
# Print the selected order
print(selected_order)

# Fit the model VAR(1) and take the residuals
var_model = model.fit(1)
residuals = var_model.resid

# Tests on the separated residuals vector
from statsmodels.stats.diagnostic import acorr_ljungbox, het_arch, normal_ad
residual_tests = {}

for col in var_data.columns:
```

```python
    res = residuals[col]
    # Ljung-Box test
    lb_pvalue = acorr_ljungbox(res, lags=[10],
    ↪  return_df=True)['lb_pvalue'].iloc[0]
    # ARCH test
    arch_stat, arch_pvalue, _, _ = het_arch(res)
    # Normality test (Anderson-Darling)
    ad_stat, ad_pvalue = normal_ad(res)
    # Update the results
    residual_tests[col] = {
        "Ljung-Box Test p-value": lb_pvalue,
        "ARCH Test Statistic": arch_stat,
        "ARCH Test p-value": arch_pvalue,
        "Normality Test (Anderson-Darling) Statistic": ad_stat,
        "Normality Test (Anderson-Darling) p-value": ad_pvalue
    }
print(residual_tests)
```

Out: {'aic': 1, 'bic': 1, 'hqic': 1, 'fpe': 1}
The Information criterions suggest that the optimal order is 1.
The results for the residuals of the VAR(1) model follow:
Oil Residuals:

- Ljung-Box Test p-value: 0.558 (suggests no autocorrelation)
- ARCH Test Statistic: 13.37, p-value: 0.204 (no significant ARCH effects)
- Normality Test (Anderson-Darling) Statistic: 0.208, p-value: 0.864 (residuals appear normally distributed)

Inflation Residuals:

- Ljung-Box Test p-value: 0.909 (suggests no autocorrelation)
- ARCH Test Statistic: 13.70, p-value: 0.187 (no significant ARCH effects)
- Normality Test (Anderson-Darling) Statistic: 0.298, p-value: 0.587 (residuals appear normally distributed)

Growth Residuals:

- Ljung-Box Test p-value: 0.717 (suggests no autocorrelation)
- ARCH Test Statistic: 8.07, p-value: 0.622 (no significant ARCH effects)
- Normality Test (Anderson-Darling) Statistic: 0.355, p-value: 0.458 (residuals appear normally distributed)

In summary, the residuals of the VAR(1) model for *oil*, *inf*, and *growth* exhibit no significant autocorrelation, no ARCH effects, and they appear to be normally distributed. This suggests that the residuals can be considered as multivariate white noise, which is a good indication for the adequacy of the VAR(1) model.

1.4 Structural impulse response function(IRF)

```python
# Summary of the model VAR(1)
print(var_model.summary())

# calculating structural impulse response functions
irf = var_model.irf(15)  # 15 periods

# Plot the structural impulse response functions
irf.plot(orth = True)
plt.show()
```

```
Summary of Regression Results
==================================================================
No. of Equations:         3.00000    BIC:                  -4.95241
Nobs:                     399.000    HQIC:                 -5.02487
Log likelihood:          -674.529    FPE:                0.00626749
AIC:                     -5.07238    Det(Omega_mle):     0.00608271
------------------------------------------------------------------
Results for equation oil
==========================================================================
            coefficient       std. error           t-stat            prob
--------------------------------------------------------------------------
const         -0.004659         0.023365           -0.199           0.842
L1.oil         0.486214         0.049620            9.799           0.000
L1.inf         0.074181         0.051280            1.447           0.148
L1.growth      0.077511         0.043489            1.782           0.075
==========================================================================

Results for equation inf
==========================================================================
            coefficient       std. error           t-stat            prob
--------------------------------------------------------------------------
const         -0.038568         0.019901           -1.938           0.053
L1.oil         0.234544         0.042263            5.550           0.000
L1.inf         0.505048         0.043676           11.563           0.000
L1.growth      0.046508         0.037041            1.256           0.209
==========================================================================

Results for equation growth
==========================================================================
            coefficient       std. error           t-stat            prob
--------------------------------------------------------------------------
const         -0.009416         0.023567           -0.400           0.689
L1.oil         0.033708         0.050048            0.674           0.501
L1.inf        -0.051221         0.051722           -0.990           0.322
L1.growth      0.505437         0.043865           11.523           0.000
==========================================================================
```

```
Correlation matrix of residuals
              oil       inf     growth
oil       1.000000  0.345930  0.028666
inf       0.345930  1.000000  0.011494
growth    0.028666  0.011494  1.000000
```

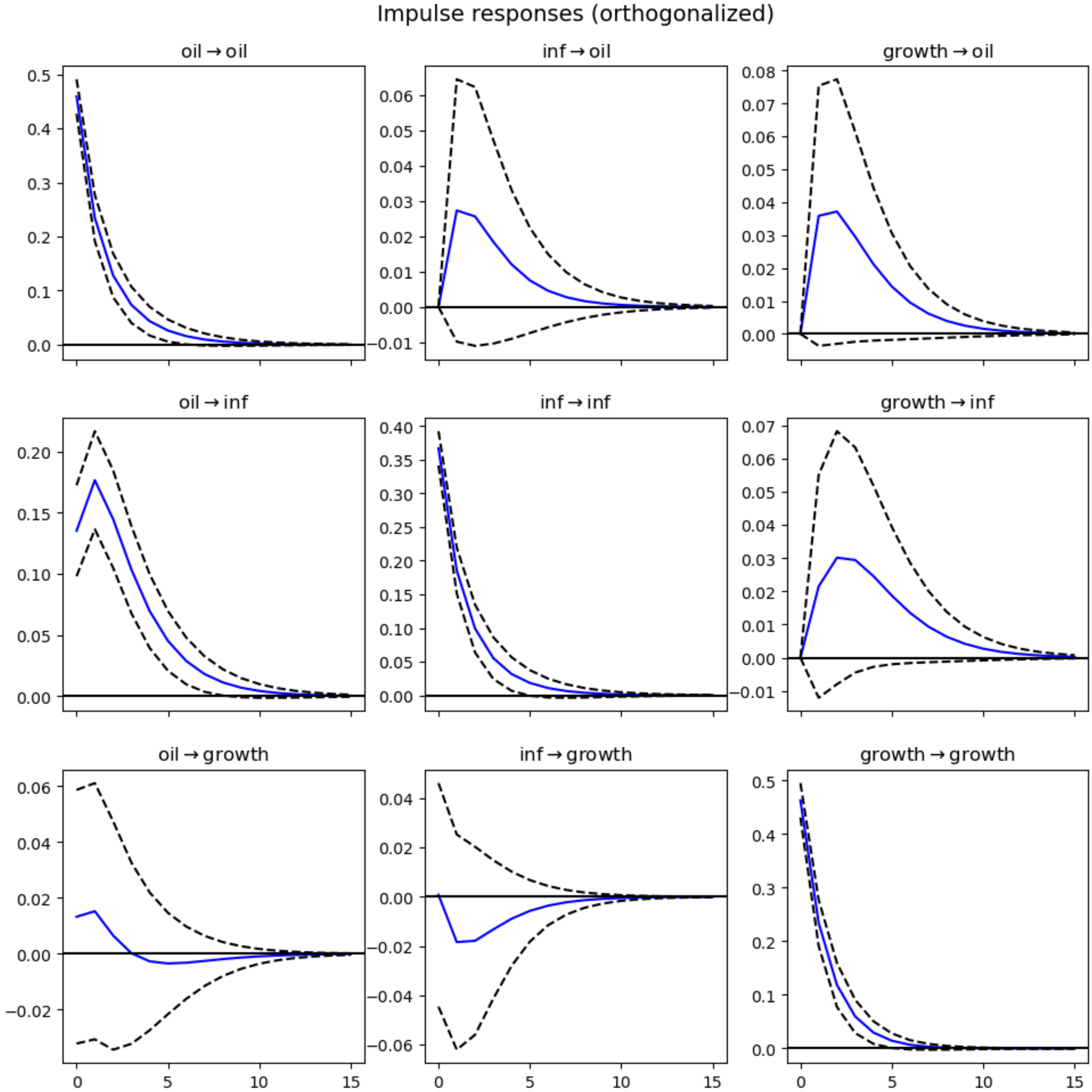

Figure 2: structural IRFs (orthogonalized)

1.5 ARMA model

Now we proceed to estimate the ARMA model for the Oil time series.

```python
# ARMA of oil
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
from statsmodels.tsa.stattools import arma_order_select_ic
```

```python
import matplotlib.pyplot as plt
oil_series = data['oil']

# Check for Homoskedasticity
arch_testo = het_arch(oil_series)
print("ARCH Test Statistic:", arch_testo[0])
print("p-value:", arch_testo[1])

# ACF and PACF plots
fig, axes = plt.subplots(1, 2, figsize = (15, 4))
plot_acf(oil_series, lags = 20, ax = axes[0])
plot_pacf(oil_series, lags = 20, ax = axes[1])
plt.show()
```

```
ARCH Test Statistic: 29.43105213476845
p-value:  0.0010607001507988829
```

The ARCH test, rejecting the null hypothesis of presence of ARCH effects, confirms that the series is **Homoskedastic**.

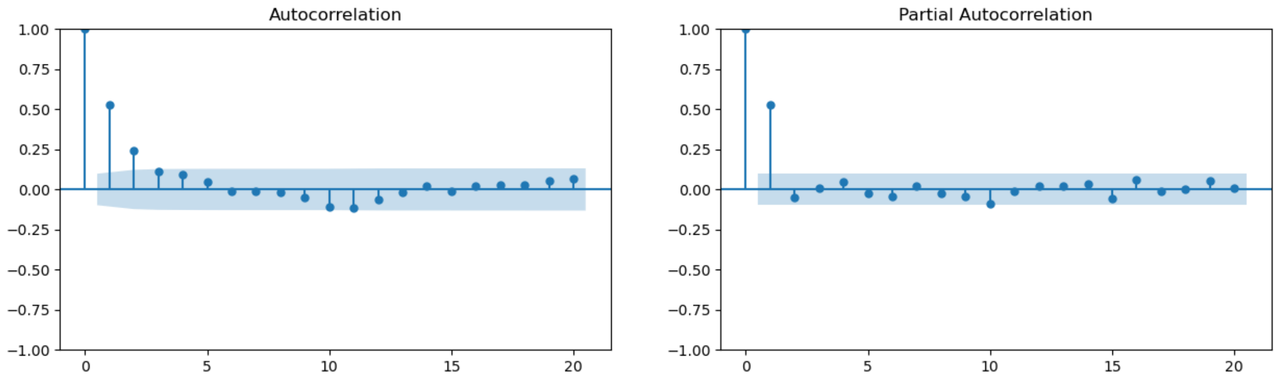

Figure 3: Plot of the ACF and PACF

The ACF and PACF suggest an ARMA model with orders (1,1) or (1,0).
We proceed finding the optimal orders minimizing the information criterion such as AIC and BIC:

```
{'aic':      0           1           2           3           4
0  648.263774  540.736475  521.600802  522.755645  523.844082
1  519.021287  519.908598  521.877342  523.098110  524.643899
2  519.914060  521.902617  523.603611  524.903501  517.451246
3  521.879641  523.836896  524.182290  525.076956  526.325354
4  522.819171  524.710992  524.550165  526.994423  528.376228,

'bic':      0           1           2           3           4
0  656.246703  552.710869  537.566660  542.712968  547.792870
1  530.995681  535.874456  541.834665  547.046897  552.584150
2  535.879919  541.859940  547.552398  552.843753  549.382962
3  541.836964  547.785684  552.122542  557.008672  562.248535
4  546.767958  552.651243  556.481881  562.917604  568.290873,
```

```
'aic_min_order': (1, 0),
'bic_min_order': (1, 0)}
```

Both the information criterions confirm ARMA(1,0) model to estimate the Oil time series. In the next section we provide the estimation of the fit of this model for our data and analyse its residuals.

Fit and residual analysis of ARMA(1,0)=AR(1)

We tested in the section 1.2 that the Oil time series is Stationary so we can proceed to estimate the coefficients with OLS estimation.

```python
from statsmodels.tsa.arima.model import ARIMA
from scipy import stats
import numpy as np

# Estimating ARMA(1,0) = ARIMA(1,0,0)
arma_model = ARIMA(oil_series, order = (1, 0, 0), trend = 'n') # No const
arma_result = arma_model.fit()

# Summary of the model
print(arma_result.summary())


# F-test
rss_full = np.sum(arma_result.resid**2)
# Fitting a null model (mean only, no AR or MA components)
null_model = ARIMA(oil_series, order=(0, 0, 0)).fit()
rss_null = np.sum(null_model.resid**2)

# Calculating the F-statistic
n = len(oil_series)
p = 1
f_stat = ((rss_null - rss_full) / p) / (rss_full / (n - p - 1))
f_p_value = 1 - stats.f.cdf(f_stat, p, n - p - 1)

print("F-test: ",f_stat,"\np-value: ", f_p_value)
```

```
F-test:  154.65909965180455
p-value:  1.1102230246251565e-16
```

```
                               SARIMAX Results
==============================================================================
Dep. Variable:                    oil   No. Observations:                  400
Model:                 ARIMA(1, 0, 0)   Log Likelihood                -256.638
Date:                Thu, 04 Jan 2024   AIC                            517.276
Time:                        12:53:00   BIC                            525.259
Sample:                             0   HQIC                           520.438
                                - 400
Covariance Type:                  opg
==============================================================================
```

```
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ar.L1          0.5291      0.042     12.506      0.000       0.446       0.612
sigma2         0.2111      0.015     14.201      0.000       0.182       0.240
==============================================================================
Ljung-Box (L1) (Q):                   0.32   Jarque-Bera (JB):              0.01
Prob(Q):                              0.57   Prob(JB):                      1.00
Heteroskedasticity (H):               1.03   Skew:                         -0.00
Prob(H) (two-sided):                  0.87   Kurtosis:                      3.02
==============================================================================
```

The estimated ARIMA(1, 0, 0) model for the ‘oil‘ series is given by:

$$oil_t = 0.5281 \cdot oil_{t-1} + \varepsilon_t \tag{2}$$

where:

- oil_t is the current value of the ‘oil‘ series at time t.
- ϕ_1 is the coefficient of the first lag of the series, estimated as 0.5281.
- oil_{t-1} is the observed value of the ‘oil‘ series at time $t-1$.
- ε_t is the error term at time t, assumed to be white.

We can take a look to the Diagnostic Tests on the residuals:

- **Ljung-Box (Q)**: The p-value for the Ljung-Box test is 0.57, suggesting that there is no significant autocorrelation in the residuals, not rejecting the null hypothesis of no autocorrelation.
- **Jarque-Bera (JB)**: The Jarque-Bera test p-value is 1.00, indicating that the residuals are normally distributed, not rejecting the null hypothesis of normality.
- **Heteroskedasticity (H)**: The p-value is 0.87, suggesting no heteroskedasticity in the residuals.

The model appears to fit the data well, with no significant autocorrelation in residuals. The latters appear normally distributed. The non-significance of the constant term implies that it might be unnecessary in this model. Moreover the F-test shows that the ARMA(1,0) model is significantly better than a mean-only model, it suggests that the autoregressive component of lag 1 is essential in modeling the Oil time series and cannot be omitted.

2 Exercise Appendix

2.1 Plot of the time series

We take from initial data the data of the two price indices related to Country A and Country B and the nominal exchange rate. We can see that the data provided are already in logarithmic form.

```python
# Plot of the 3 series
plt.figure(figsize=(12, 6))
plt.plot(data['pa'], label='pa', color='darkviolet')
plt.plot(data['pb'], label='pb', color='deeppink')
plt.plot(data['e'], label='e', color='navy',linestyle='--')
plt.title('Comparison of pa and pb')
plt.xlabel('Time')
plt.ylabel('Values')
plt.legend()
plt.show()
```

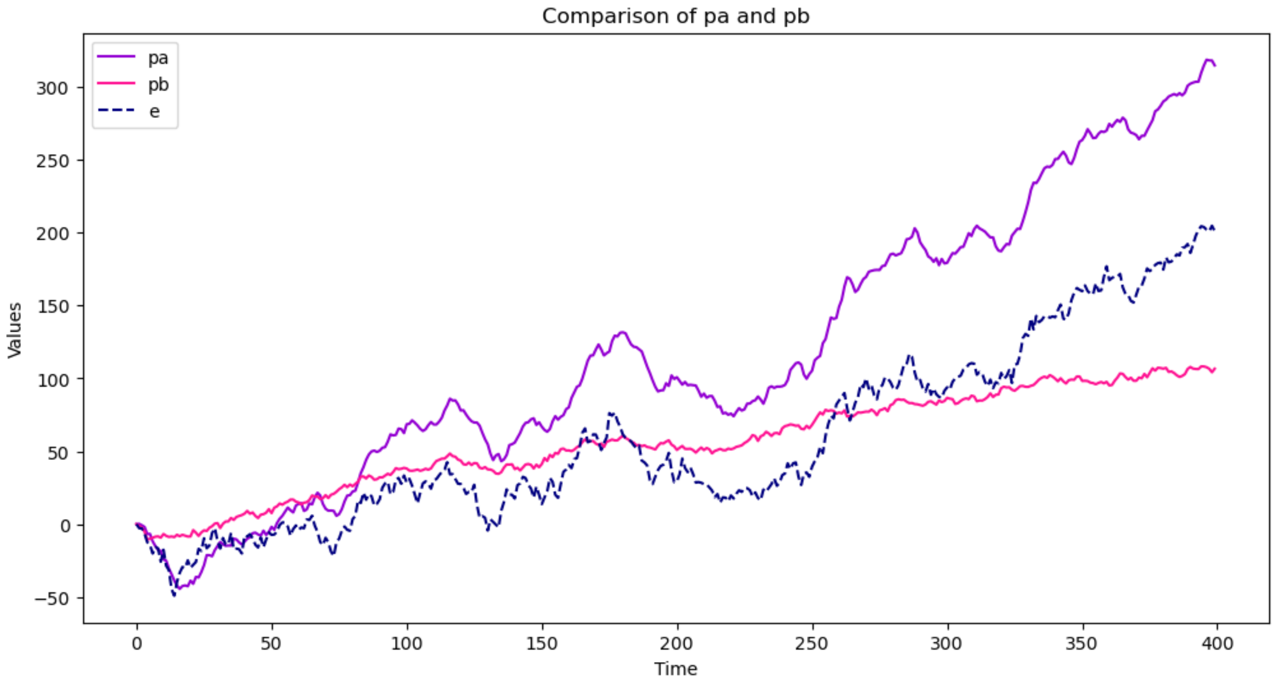

Figure 4: Plot of the logarithm of the two price indices related to Country A and Country B and the nominal exchange rate

In the above plot (Figure 4) we can observe the presence of persistence and a trend, suggesting that the 3 series are likely non-stationary. We will now conduct the Augmented Dickey-Fuller(ADF) test in the following section to verify this suggestion.

2.2 Check for Stationarity

```python
from statsmodels.tsa.stattools import adfuller

adf_pa = adfuller(data['pa'])
adf_pb = adfuller(data['pb'])
```

```python
adf_e = adfuller(data['e'])

print('ADF Statistic for pA:', adf_pa[0], 'p-value:', adf_pa[1])
print('ADF Statistic for pB:', adf_pb[0], 'p-value:', adf_pb[1])
print('ADF Statistic for eAB:', adf_e[0], 'p-value:', adf_e[1])
```

```
ADF Statistic for pA:  0.5550961879781255  p-value: 0.986454911861702
ADF Statistic for pB:  -1.804511687626314  p-value: 0.3782240777455919
ADF Statistic for eAB: 0.4920015128893162  p-value: 0.9846279504810067
```

We have a confirm of the previous suggestion of the Plot in Figure 4. For the three time series we have evidence to not reject the null hypothesis of unit root, i.e. the time series are non stationary.

2.3 Cointegration Test

We utilize Johansen test in order to see if there are some cointegration relation between the three time series. The test checks for the presence of multiple cointegrating vectors, and the null hypothesis for each number of cointegrating vectors is tested in sequence, starting from 0.

```python
from statsmodels.tsa.vector_ar.vecm import coint_johansen
coint_data = data[['pa', 'pb', 'e']]

# Johansen cointegration test
johansen_test = coint_johansen(coint_data, det_order = 0, k_ar_diff = 1)

# Trace statistics
trace_stats = johansen_test.lr1
crit_values = johansen_test.cvt
max_eig_stats = johansen_test.lr2
critical_values_max_eig = johansen_test.cvm

# Print the results
print("Eigenvalues:\n", johansen_test.eig)
print("Trace Statistics:\n", trace_stats)
print("Critical Values Trace (90%, 95%, 99%):\n", crit_values)
print("Max-Eigen Statistic:\n", max_eig_stats)
print("Critical Values Max-Eig (90%, 95%, 99%):\n", critical_values_max_eig)
```

```
Eigenvalues:
 [0.41849657 0.016192   0.0042997 ]
Trace Statistics:
 [223.98321671   8.2121284    1.71496913]
Critical Values Trace (90%, 95%, 99%):
 [[27.0669 29.7961 35.4628]
 [13.4294 15.4943 19.9349]
 [ 2.7055  3.8415  6.6349]]
Max-Eigen Statistic:
 [215.77108831   6.49715927   1.71496913]
Critical Values Max-Eig (90%, 95%, 99%):
 [[18.8928 21.1314 25.865]
 [12.2971 14.2639 18.52]
 [ 2.7055  3.8415  6.6349]]
```

The next step is the interpretation of the above test (Johansen):

- **For 0 cointegrating relations**:
  The trace statistic (223.98) is much higher than the 99% critical value (35.46). This strongly suggests rejecting the null hypothesis of no cointegrating vector.
- **For 1 cointegrating relation**:
  The second trace statistic (8.21) is lower than the 95% critical value (15.49) but higher than the 90% critical value (13.43). This suggests that the null hypothesis of at most one cointegrating vector can be rejected at the 90% confidence level but not at the 95% level.
- **For 2 cointegrating relations**:
  The third trace statistic (1.71) is lower than all the critical values. This suggests accepting the null hypothesis of at most two cointegrating vectors.

Based on this test, it can be inferred that there is at least one cointegrating relationship among the *pa*, *pb*, and *e* series. This implies a long-term equilibrium relationship among these variables, which is consistent with the Purchasing Power Parity (PPP) hypothesis. The Max-Eigen test confirms that there is one strong cointegrating relationship and no additional relationships (*coint.rank = 1*).

2.4 VECM model

We start finding the optimal lag of differences, starting from minimize the information criterions such as AIC, BIC, HQIC and so on.

```python
from statsmodels.tsa.vector_ar.vecm import select_order

# Order selecting
maxlags = 10
order_results = select_order(coint_data, maxlags = maxlags, deterministic =
    "ci")
print(order_results.summary())
```

```
VECM Order Selection (* highlights the minimums)
==================================================
       AIC         BIC         FPE         HQIC
--------------------------------------------------
1       5.338*      5.583*      208.1*      5.435*
2        5.359       5.696       212.6       5.493
3        5.373       5.801       215.5       5.542
4        5.389       5.909       219.1       5.595
5        5.413       6.024       224.3       5.655
6        5.443       6.146       231.3       5.722
7        5.463       6.258       235.9       5.778
8        5.485       6.371       241.2       5.836
9        5.502       6.480       245.4       5.890
10       5.505       6.575       246.3       5.929
```

We observe that the information criterions suggest the order 1 as the optimal one.
(*k.ar.diff = 1*)
Now we proceed to estimate the VECM model with 1 cointegrating vector and 1 lag differences.

```python
from statsmodels.tsa.vector_ar.vecm import VECM

# fitting the VECM model
vecm_model = VECM(coint_data, coint_rank = 1, k_ar_diff = 1)  # coint_rank = 1
vecm_fit = vecm_model.fit()
print(vecm_fit.summary())

# Getting the coefficients of the error correction term
ecm_coefficients = vecm_fit.alpha

# Summary of the ECM coefficients
print("ECM coefficients:"
    "\nPA (pa):", ecm_coefficients[0][0],
    "\nEAB (e):", ecm_coefficients[1][0],
    "\nPB (pb):", ecm_coefficients[2][0])
```

```
Det. terms outside the coint. relation & lagged endog. parameters for pa
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
L1.pa          0.0197      0.036      0.540      0.589      -0.052       0.091
L1.pb         -0.0457      0.061     -0.750      0.453      -0.165       0.074
L1.e          -0.0147      0.026     -0.574      0.566      -0.065       0.036
Det. terms outside the coint. relation & lagged endog. parameters for pb
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
L1.pa          0.0510      0.034      1.485      0.137      -0.016       0.118
L1.pb         -0.0695      0.057     -1.211      0.226      -0.182       0.043
L1.e           0.0484      0.024      2.003      0.045       0.001       0.096
Det. terms outside the coint. relation & lagged endog. parameters for e
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
L1.pa          0.0797      0.108      0.740      0.459      -0.131       0.291
L1.pb          0.0737      0.180      0.409      0.682      -0.279       0.426
L1.e           0.0011      0.076      0.015      0.988      -0.147       0.150
                 Loading coefficients (alpha) for equation pa
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ec1           -0.3052      0.018    -17.071      0.000      -0.340      -0.270
                 Loading coefficients (alpha) for equation pb
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
ec1            0.0120      0.017      0.714      0.475      -0.021       0.045
                 Loading coefficients (alpha) for equation e
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
```

```
ec1             0.0102      0.053      0.194      0.846     -0.093       0.114
          Cointegration relations for loading-coefficients-column 1
==============================================================================
                  coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
beta.1          1.0000          0          0      0.000       1.000       1.000
beta.2         -1.0940      0.013    -87.060      0.000      -1.119      -1.069
beta.3         -0.9952      0.010   -103.663      0.000      -1.014      -0.976
==============================================================================
ECM coefficients:
PA (pa): -0.3051926285335777
EAB (e): 0.012016903715797432
PB (pb): 0.010245630012984997
```

The Vector Error Correction Model (VECM) provides a wealth of information regarding the dynamic relationships and long-term equilibrium among the variables pa, pb, and e.
The interpretation of the above output is the following:

1. **Loading Coefficients (Alpha)**:
   These coefficients are associated with the error correction term ($ec1$) and indicate how each variable adjusts in the short run to restore the long-term equilibrium following a shock. The coefficient for $ec1$ in the equation for pa is **-0.3052** and is highly significant (p-value of 0.000). This suggests that pa adjusts to correct for any disequilibrium. For pb and e, the coefficients of $ec1$ are **0.0120** and **0.0102**, respectively, but are not statistically significant.
2. **Cointegration Relations (Beta)**:
   These coefficients represent the long-term equilibrium relationships among the variables. The coefficients $beta.1$, $beta.2$, and $beta.3$ correspond to pa, pb, and e, respectively. The coefficients for pb and e are **-1.0940** and **-0.9952**, respectively, and are both highly significant. This suggests a strong long-term relationship among these variables.

In conclusion the results, especially the cointegration relations, provide support for the PPP hypothesis, indicating that the exchange rate (e) and the price levels of the two countries (pa and pb) are in a long-term equilibrium relationship. This supports the idea that over the long term, the exchange rate adjusts to equalize the price levels between the two countries.