Analysis 1

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Theorem 1. *Prove that the supremum of* $\{\frac{m}{m+n}\}$ *is 1 for* $m, n \in \mathbb{N}$

Proof. Since $m + n > m$, $\frac{m}{m+n} < 1$ for all $m, n \in \mathbb{N}$. So, 1 is an upper bound for the sequence. Suppose that $\gamma < 1$, then we will show that γ cannot be an upper bound for the sequence. We first begin by showing that $\gamma = \frac{r}{s} \in \{\frac{m}{m+n}\}$. This is evident by setting $m = r$ and $n = s - r$. Since $\gamma = \frac{cm}{c(m+n)} < \frac{cm}{c(m+n)-1}$ for any natural number $c > 1$.This implies that there is an element of $\{\frac{m}{m+n}\}$ which is greater than γ. Thus, γ cannot be an upper bound. We've shown that 1 is an upper bound and any number less than 1 cannot be an upper bound. We proved that 1 is the supremum of the sequence. $\square$

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Theorem 2. *Let* $\{a_n\}$, *and* $\{b_n\}$ *be two sequences. Prove that* $\inf a_n + \inf b_n \leq inf(a_n + b_n)$.

Proof. Let $\gamma_1 = \inf a_n$, and $\gamma_2 = \inf b_n$, we know that $\gamma_1 \leq a_n$, and $\gamma_2 \leq b_n$ for all $n \in \mathbb{N}$. This implies that $\gamma_1 + \gamma_2 \leq a_n + b_n$ for all $n \in \mathbb{N}$. So, $\gamma_1 + \gamma_2$ is a lower bound. By definition, $\gamma_1 + \gamma_2 \leq \inf(a_n + b_n)$. $\square$

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Theorem 3. *Prove that*

$$\inf\{\frac{1}{x^2+1} : x \in \mathbb{R}\} = 0$$

Proof. First, $\frac{1}{x^2+1} > 0$ for all $x \in \mathbb{R}$. 0 is a lower bound for the sequence. The sequence in question is bounded above by 1. It suffice to show that any real number $\gamma \in (0, 1)$ cannot be a lower bound. We begin by showing that γ is in the sequence. This is evident by setting $x = \sqrt{\frac{1}{\gamma} - 1}$ or $x = -\sqrt{\frac{1}{\gamma} - 1}$. We see that $\frac{1}{(x+1)^2+1} < \gamma$ and γ is not a lower bound. $\square$

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Theorem 4. *Find the supremum and infinum of the sequence*

$$\{\frac{m}{n} + \frac{4n}{m} : m, n \in \mathbb{N}\}$$

Proof. The supremum of the sequence is infinity because the subsequence $\{\frac{1}{n} + 4n : n \in \mathbb{N}\}$ increase without bound. For the infinum, we will show that 4 is a lower bound and. This is evident because

$$\frac{m}{n} + \frac{4n}{m} = \frac{m^2 + 4n^2}{mn} = \frac{(m + 2n)^2}{mn} - 4 \geq 4$$

.

$$(m+2n)^2 - 4mn = m^2 + 4n^2 - 4mn = m^2 + 4n(n-m) \geq 0$$

for all m and n. We proceed by noting that if $n \geq m$, then $m^2 + 4n(n-m) \geq 0$ and if $n < m$ and $m = n + \gamma$,then

$$m^2 + 4n(n-m) = (n+\gamma)^2 - 4n\gamma = (n-\gamma)^2 \geq 0$$

. Thus, 4 is a lower bound for the sequence. We will show that any number greater than four cannot be a lower bound. Suppose $r \in \mathbb{Q}$, the function $f(r) = \frac{r^2+4}{r}$ has derivative $g(r) = \frac{r^2-4}{r^2}$ by setting it equal to zero. we get a minimum value at $r = 2$ and $f(2) = 4$. so, any number bigger than 4 cannot be a lower bound for the sequence. $\square$