Analysis 1

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Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$, and suppose that*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is consant.

Proof. As $-(x-y)^2 \leq f(x) - f(y) \leq (x-y)^2$, we see that

$$0 = \lim_{x \to y} \frac{-(x-y)^2}{x-y} \leq f^{'}(x) = \lim_{x \to y} \frac{f(x) - f(y)}{x-y} \leq \lim_{x \to y} \frac{(x-y)^2}{x-y} = 0$$

. Hence $f^{'}(x) = 0 \implies$ f is constant. □

Theorem 2. *Suppose g is real function on $\mathbb{R}$, with bounded derivative $|g^{'}| \leq M$. Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one, if ϵ is small enough.*

Proof. $f(x)$ is differentiable with derivative $f^{'}(x) = 1 + \epsilon g^{'}(x)$. The Mean value Theorem guarantee the existence of a real number c such that $f^{'}(c) = \frac{f(x)-f(y)}{x-y}$. Hence if $0 = |f(x) - f(y)| = |f^{'}(c)(x-y)|$, tt remains to show that $f^{'}(c) \neq 0$ which is given when $\epsilon M < 1$.

$$-\epsilon M \leq \epsilon g^{'}(x) \leq \epsilon M \implies 1 - \epsilon M \leq 1 + \epsilon g^{'}(x) = f^{'}(x) \leq 1 + \epsilon M$$

with $1 - \epsilon M > 0$. The claims follows because when $\epsilon < \frac{1}{M}$, $|f(x) - f(y)| = 0 \implies |x-y| = 0$. □

Theorem 3. *If*

$$C_0 + \frac{C_1}{2} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

, where $C_0, ..., C_n$ are real constants, prove that the equation

$$f(x) = C_0 + C_1 x + ... + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Let $g(x) = C_0 x + \frac{C_1 x^2}{2} + ... + \frac{C_{n-1} x^n}{n} + \frac{C_n x^{n+1}}{n+1}$, then $g(x)$ is continuous and differentiable. Its derivative is given by $f(x)$. Since $g(0) = g(1) = 0$, there exists a $c \in (0,1)$ such that $g^{'}(c) = f(c) = 0$. This is because of the Mean Value Theorem. □

Theorem 4. *Suppose f is defined and differentiable for every $x > 0$, and $f^{'}(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.*

Proof. As $g(x)$ is differentiable, the mean value theorem allow us to find c such that $g(x) = f(x+1) - f(x) = (x+1-x)f^{'}(c) = f^{'}(c)$ where $x < c < x+1$ and hence $g(x) \to 0$ as x goes to infinity. This is because as $x \to \infty$, $c \to \infty$. □

Theorem 5. *Suppose, for a fixed x, $f^{'}(x)$ and $g^{'}(x)$ exist, $g^{'}(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that*

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f^{'}(x)}{g^{'}(x)}$$

.

Proof. As $t \to x$, $f(t) = f(x) + [f^{'}(x) + v(t)](t - x)$ and $g(t) = g(x) + [g^{'}(x) + u(t)](t - x)$ where $u(t) \to 0$, and $v(t) \to 0$. Hence $\frac{f(t)}{g(t)} = \frac{f(x)+f^{'}(x)(t-x)}{g(x)+g^{'}(x)(t-x)} = \frac{f^{'}(x)}{g^{'}(x)}$ as $t \to x$. □

Theorem 6. *Suppose $f^{'}$ is continous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that*

$$|\frac{f(t) - f(x)}{t - x} - f^{'}(x)| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

Proof. Since $f^{'}$ is defined on $[a, b]$, let $x, t \in [a, b]$ and $\phi(t) = \frac{f(t)-f(x)}{t-x}$ then $f^{'}(x) = \lim_{t \to x} \phi(t)$ for all $x \in [a, b]$ and for all $\epsilon > 0$, there exists $\delta(x, \epsilon) > 0$ such that $|t - x| < \delta(x, \epsilon) \implies |\phi(t) - f^{'}(x)| < \epsilon$. Hence for all $x, t \in [a, b]$, $\epsilon > 0$, there exists $\delta > 0$ such that $|t - x| < \delta \implies |\phi(t) - f^{'}(x)| < \epsilon$ by setting $\delta = \delta(x, \epsilon)$. □

Theorem 7. *Suppose f is defined in a neighborhood of x and suppose $f^{''}(x)$ exists. Show that*

$$\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f^{''}(x)$$

Proof. Let $\epsilon > 0$, then we can find δ_0 such that

$$|\frac{f(x + h) - f(x)}{h} - f^{'}(x)| < \epsilon$$

and

$$|\frac{f(x) - f(x - h)}{h} - f^{'}(x - h)| < \epsilon$$

whenever $|h| < \delta_0$. Hence ,

$$-h(f^{'}(x) - f^{'}(x - h) + 2\epsilon) \leq f(x + h) + f(x - h) - 2f(x) \leq h(f^{'}(x) - f^{'}(x - h) + 2\epsilon)$$

and

$$|\frac{f(x + h) + f(x - h) - 2f(x)}{h^2}| \leq |\frac{f^{'}(x) - f^{'}(x - h)}{h}| + 2\epsilon$$

and

$$f^{''}(x) - \epsilon \leq \frac{f^{'}(x) - f^{'}(x - h)}{h} \leq f^{''}(x) + \epsilon$$

whenever $|h| < \delta_1$. Set $\delta = \min\{\delta_0, \delta_1\}$ and the claim follows. □