Analysis 1

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1

Theorem 1. *Prove that the set* $A = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ *is open in* $\mathbb{R}^2$.

Proof. The set $(0,1)$ is open in $\mathbb{R}$ because the set $B(x) = \{y \in \mathbb{R} : |x-y| < 1-x\}$ is contained in $(0,1)$. So the neighborhood $B(x,y) = \{(x^{'}, y^{'}) : x^{'} \in B(x), y^{'} \in B(y)\}$ is contained in A for $(x,y) \in A$. Thus A is open. □

Theorem 2. *Prove that the set*

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

is closed in $\mathbb{R}^2$

Proof. We defined the set A in terms of the metric on $\mathbb{R}^2$.

$$A = \{z \in \mathbb{R}^2 : |z| \leq 1\}$$

We will show that the compliment of the set is open in $\mathbb{R}^2$

$$A^c = \{z \in \mathbb{R}^2 : |z| > 1\}$$

. We defined the neighborhood

$$B(z) = \{w \in \mathbb{R}^2, z \in A^c : |z-w| < \epsilon\}$$

where $\epsilon = |z| - 1$. We note that if $w \in B$, then $||z| - |w|| \leq |z-w| < |z| - 1$. This implies that $|z| - |w| < |z| - 1 \implies |w| > 1$ or $1 - |z| < |w| - |z| \implies |w| > 1$ and $w \in A^c$. So, $B \subset A^c$. □

Theorem 3. *Let* (X,d) *be a metric space. Define* $\rho : X \; x \; X \to \mathbb{R}$,

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that ρ *is a metric.*

Proof. Since $d(x,y) = d(y,x)$, then $\rho(x,y) = \rho(y,x)$ for all x,y and $\rho(x,y) = 0 \Leftrightarrow x = y$. To verify the triangle inequality, we see that

$$\rho(x,y) \leq \frac{d(x,z + d(z,y)}{1 + d(x,z) + d(y,z)} = \frac{d(x,z)}{1 + d(x,z) + d(y,z)} + \frac{d(y,z)}{1 + d(x,z) + d(y,z)} \leq \frac{d(x,z)}{1 + d(x,z)} + \frac{d(x,z)}{1 + d(y,z)} = \rho(x,z) + \rho(y,z)$$

for all $z \in X$.

Thus, ρ is a metric. □

Theorem 4. *Let* A *be a bounded closed set of* $\mathbb{R}$*, prove that* $inf A$ *and* $sup A$ *are in* A.

Proof. Since A is closed and bounded, it is compact. It implies that its compliment is open. Suppose that $\inf A$ and $\sup A$ are not in A. They must be interior points of the compliment of A. The neighborhoods $(\inf A - \epsilon_1, \inf A + \epsilon_1)$ and $(\sup A - \epsilon_2, \sup A + \epsilon_2)$ will be contained A^c for some choice of $\epsilon_1, \epsilon_2 > 0$. This would imply that there exists a lower bound of A in $(\inf A, \inf A + \epsilon_1)$ and an upper bound of A in $(\sup A - \epsilon_2, \sup A)$ which would give rise to a contradiction. So, either $\sup A, \inf A \in A$ or they are limit points of A. Compact sets contain their limit points. □

Theorem 5. *Let $A_1, ..., A_n$ be subsets of a metric space and let $B = \bigcup_{i=1}^n A_i$. Prove that $\bar{B} = \bigcup_{i=1}^n \bar{A}_i$*

Proof. Let $B^{'}$ be the set of limit points of B.

$$\bar{B} = B \bigcup B^{'} = \bigcup_{i=1}^{n} A_i \bigcup \bigcup_{i=1}^{n} A_i^{'} = \bigcup_{i=1}^{n} (A_i \bigcup A_i^{'}) = \bigcup_{i=1}^{n} \bar{A}_i.$$

It remains to show that $B^{'} = \bigcup_{i=1}^n A_i^{'}$. Let x be a limit points of A_i for some i, then any neighborhood around x must intersect A_i at a point different than x. So, it also intersects B and it is also a limit point of B. So, $\bigcup_{i=1}^n A_i^{'} \subset B^{'}$. If x is a limit point of B, then every neighborhood centered at x must intersect B at some point different than x. This imply that it also intersects A_i for some i and $B^{'} \subset \bigcup_{i=1} A_i^{'}$ □

Theorem 6. *Let $E^{'}$ be the set of limit points of E. Prove that $E^{'}$ is closed.*

Proof. We will show that $(E^{'})^c$ is open. If x is not a limit point of E, then $x \in (E^{'})^c$. This means that there is a neighborhood around x which does not intersect E. The neighborhood in question is a subset of $(E^{'})^c$. so, $(E^{'})^c$ is open. □

Theorem 7. *Let A be the set of interior points of E. Prove the following.*
(a) A is open.
(b) If $G \subset E$ and G is open, prove that $G \subset A$.
(c) Prove that the compliment of A is the closure of the compliment of E.
(d) Does E and $E^{'}$ always have the same interior?

Proof. (a) By the definition of open set, A is open because all of its points are interior.

(b) Since G is open, all its elements are interior point of E. Hence, the elements belong to A, the set of all interior points of E. Therefore, $G \subset A$.

(c) Let $x \in A^c$, then it implies that x is not an interior point of E. So all neighborhoods of x are not contained in E. This can happen because x is not in E or every neighborhood of x intersect E^c. Hence, $x \in E^c \bigcup (E^c)^{'} = \bar{E}^c$. So, $A^c \subset \bar{E}^c$. If $x \in \bar{E}^c$, then either x is a limit point of E^c or $x \in E^c$, either way no neighborhood of x will be contained in E. So $\bar{E}^c \subset A^c$.

(d) No, because interior points are not necessarily limit points. So, if $x \in E^\circ$ and $x \notin E^{'}$, then E° and $E^{'\circ}$ will differ. □

Theorem 8. *Give an example of an open cover of $(0, 1)$ which has no finite subcover.*

Proof. $\bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{2^n})$ □

Theorem 9. *Which of the following sets are compact? Justify.*
(a) $[0, 1] \bigcup [3, 4]$*- Yes, it is closed and bounded.*
(b) $[0, \infty)$ *- It is not bounded. It is not compact.*
(c) $A = \{x \in \mathbb{R} : 0 \leq x \leq 1, x \text{ is irrational } \}$. *- No, it is not closed. The compliment of this set is not open. Pick a rational number in* $(0, 1)$*, then no neighborhood around that rational number will be contained in the compliment of A.*
(d) $\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\} \bigcup \{0\}$ *- Yes, it is closed and bounded. The compliment of this set is open. For any* $n \in \mathbb{N}$*, pick any real number in* $(\frac{1}{n+1}, \frac{1}{n})$*, then we can find a neighborhood around that real number that is contained in* $(\frac{1}{n+1}, \frac{1}{n})$*, and it is open. The compliment is* $\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \bigcup (-\infty, 0) \bigcup (1, \infty)$ *which is open.*

Theorem 10. *Let A and B be compact subsets of a metric space (X, d). Prove that $A \bigcup B$ is compact.*

Proof. Let $\bigcup_{i=1}^{\infty} A_i$ be an open cover of A, and $\bigcup_{j=1}^{\infty} B_j$ be an open cover of B. Since A and B are compact, it implies that we can find a finite subcover of A and B from their respective open cover. So, $A \bigcup B \subset \bigcup_{k=1}^{n} A_i \bigcup \bigcup_{l=1}^{m} B_l$[1] which is a finite subcover of $\bigcup_{i=1}^{\infty} A_i \bigcup \bigcup_{j=1}^{\infty} B_j$. Thus, $A \bigcup B$ is compact. □

Theorem 11. *Let (F_α) be a family of connected sets in a metric space (X, d). Assume that $\bigcap_\alpha F_\alpha \neq \emptyset$. Prove that $\bigcup_\alpha F_\alpha$ is connected.*

Proof. We will prove it by contradiction. Suppose that $\bigcup_\alpha F_\alpha$ is disconnected with the assumption of the theorem. So, $\bigcup_\alpha F_\alpha = A \bigcup B$ where A and B are nonempty seperated sets. If there exists a set F_α such that $F_\alpha \subset A$ or $F_\alpha \subset B$ then it contradicts the assumption that $\bigcap_\alpha F_\alpha$ because seperated sets are disjoint. So, for all α, $F_\alpha = C \bigcup D$ where $C \subset A$ and $D \subset B$. Since $\bar{C} \subset \bar{A}$ and $\bar{D} \subset \bar{B}$, this imply that C and D do not contain the other's limit points and F_α is thus disconnected which is a contradiction. $\bigcup_\alpha F_\alpha$ is connected. □

[1] I hope this doesn't cause any confusion. We picked a finite subcover of A which has n elements. I don't mean to say we chose $A_1, A_2, ...A_n$.