Analysis 1

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1

Theorem 1. *Show that Set has a subobject classifier and determine if Grp has a subobject classifier.*

Proof. Let $\omega = \{0, 1\}$, we shall show that the diagram
$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "j"] & 1 \arrow[d, "\psi"] \\ A \arrow[r, "\eta"] & \omega \end{tikzcd}$$
commutes for all monomorphism $j : X \to A$ where 1 is a terminal object of Set. Now, ϕ is unique by virtue of 1 being terminal. If $\eta(x) = 1$ if $x \in X$, and 0 otherwise, and $\psi(1) = 1$. Then necessarily, we see that $\eta(j(x)) = \psi(\phi)$. So, the diagram commutes. Suppose that $\tilde{\eta}$ is another morphism that make the diagram commutes. Then $\eta(j(x)) = \tilde{\eta}(j(x)) \implies \eta(x) = \tilde{\eta}(x)$ whenever $x \in X$. This also implies that $\eta(x) = \tilde{\eta}(x)$ for all x. The two morphisms coincide and are one and the same. □

Theorem 2. *Show that in every category with products/coproducts that*

$$A\pi B \cong B\pi A$$

and

$$A \amalg B \cong B \amalg A$$

Proof. $A\pi B$, and $B\pi A$ are both final objects in the category $C_{A,B}$. $A \amalg B$, and $B \amalg A$ are both initial objects in the category $C^{A,B}$. Initial and final objects are isomorphic when they exists in the same category. □

Theorem 3. *Let X be a set, A an abelian group and $\phi : X \to A$. Prove that there exists a unique abelian group homomorphism $\tilde{\phi} : G/H \to A$ such that*
$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\phi"'] & G/H \arrow[d, "\tilde{\phi}"] \\ & A \end{tikzcd}$$
commutes where $G = F(x)$ and $H = [G, G]$

Proof. We shall show that the following diagram commutes.
$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\phi"'] & G \arrow[r, "\psi"] \arrow[d, "\tilde{\psi}"] & G/H \arrow[dl, "\tilde{\phi}"] \\ & A & \end{tikzcd}$$

G being a free group forces $\tilde{\psi}$ to be unique and the first triangle commutes. The first isomorphism theorem makes the second triangle commutes. Therefore, the outer triangle also commute because $\tilde{\psi}(\psi(j(x))) = \phi(x)$. The uniqueness of $\tilde{\phi}$ implies that G/H is an initial object in the category F^A. $F^{Ab}(X)$ is also an initial object. Therefore $G/H \cong F^{Ab}(X)$.

□

Theorem 4. *Let $\phi : G \to H$, and $\psi : H \to K$ are morphisms in a category with products. Prove that*

$$(\psi \circ \phi)\pi(\psi \circ \phi) = (\psi\pi\psi) \circ (\phi\pi\phi)$$

Proof. ∎

Theorem 5. *Let $H \leq G$ be a subgroup. Prove that*

$$G \times G/H \longrightarrow G/H$$

and

$$G \times G/(gHg^{-1}) \longrightarrow G/(gHg^{-1})$$

are 3isomorphic

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Proof. H is normal. $H = gHg^{-1}$ for all $g \in G$. ∎

Theorem 6. *Let G_1, G_2 be two groups and $\phi : F(G_1) \to G_1$, and $\psi : F(G_2) \to G_2$ be natural epimorphism. Prove that $G = F(G_1 \bigcup G_2)/ < Ker\phi_1, Ker\phi_2 >$ is a coproduct in Grp.*

Proof. First, we see that G come with pair of morphism. $j_{G_1} : g_1 \to [g_1]$ and $j_{G_2} : g_2 \to [g_2]$ by sending each element of the group to their equivalence classes in the quotient group. If we let Z be a group endowed with pair of morphism f_1, f_2, then we have to show that there exists a unique morphism σ such that the diagram

$$\begin{tikzcd}
G_1 \arrow[d, "j_{G_1}"] \arrow[dr, "f_1"] & \\
G \arrow[r, "\sigma"] & Z \\
& G_2 \arrow[u, "f_2"] \arrow[ul, "j_{G_2}"]
\end{tikzcd}$$

. The map given by $\sigma([g_1]) = f_1(g_1)$ and $\sigma([g_2]) = f_2(g_2)$ where $g_1 \in G_1$ and $g_2 \in G_2$ ∎

Theorem 7. *Prove that Grp has cokernels. Determine if Grp has coequalizer.*

Proof. Let $\phi : G \to G'$, we proceed by proving that $coker\phi = G'/N$ where N is the smallest normal subgroup that contains $im\phi$. Intersection of normal subgroups is again normal.

$$\begin{tikzcd}
G \arrow[r, "\phi"] & G' \arrow[r, "\alpha"] \arrow[d, "\phi"] & L \\
& G'/N \arrow[ur, "\psi"] &
\end{tikzcd}$$

The diagram commute with $\psi(gN) = \alpha(g)$ and $\phi(g) = gN$. The function is well-defined because N is normal. $gN = hN \implies gh^{-1} \in N \implies \psi(gN) = \psi(hN)$ because $\psi(gh^{-1}N) = \psi(N)$. The uniqueness of ψ is because it is completely determined by α. ∎