Matrix Theory

Mike Desgrottes

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Theorem 1. *Let V be a real vector space. Let $f : V \to [-2, 2]$ be a function satisfying*

$$f(au + bv) \geq \min f(u), f(v)$$

for all real numbers a, and b and all $u, v \in V$. Prove that $f(0) \geq f(v)$ for all $v \in V$. Also if $f(0) \geq h \geq 0$, then prove that $W = \{v \in V : f(v) \geq h\}$ is a subspace.

Proof.

$$f(0) = f(v - v) \geq f(v)$$

for all $v \in V$.

We proceed to prove that W is a subspace. First, $0 \in W$ because $f(0) \geq h$. Let a and b be real numbers and $u, v \in V$, then $f(au+bv) \geq \min f(u), f(v) \geq h$. Therefore, $af(u)+bf(v) \in W$. W is closed under linear combination. W contains inverse because $f(-v) \geq f(v) \implies -v \in W$. W is a subspace. □

Theorem 2. *Let W_1, W_2 be subspaces of a vector space V. Prove that $W_1 \bigcup W_2$ is a subspace if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.*

Proof. We note that $W = (W_1/W_2 \bigcup W_2/W_1) \bigcup W_1 \bigcap W_2$. Let $u \in W_1/W_2$ and $v \in W_2/W_1$, we begin by looking at an arbitrary linear combination of u and v.

If $c_1u + c_2v \in W_1 \bigcap W_2$, then it implies that $c_1u + c_2v - c_1u \in W_1$ which is a contradiction since $u \notin W_1$.

Without loss of generality, let $c_1u+c_2v \in W_1/W_2$, then either $c_1u+c_2v-c_1u \in W_1/W_2$ or $c_1u+c_2v-c_1u \in W_1 \bigcap W_2$. The former cannot happen due to the fact that $v \in W_2/W_1$. $c_1u + c_2v - c_1u \in W_1 \implies c_2v \in W_1$ which is a contradiction.

We have shown that if W is a subspace, then $W_2 \subset W_1$ or $W_1 \subset W_2$.

WLOG let $W_1 \subset W_2$, then we have to show that $W_1 \bigcup W_2$ is a subspace. it's trivial due to the fact that $W_1 \bigcup W_2 = W_2$ which is itself a subspace. □

Theorem 3. *Let W be a subspace of a finite dimensional vector space V. Prove that there exists a subspace U such that U is the compliment of W.*

Proof. We have to show that there exists a subspace U such that

(1) $W + U = V$

(2) $W \bigcap U = \emptyset$

Let $B = \{v_1, ..., v_k\}$ be a basis for V. $k < \infty$ since V is finite dimensional. Let $1 \leq n \leq k$, then if $B_1 = \{v_1, ..., v_n\}$ is a basis for W. We choose U to be the subspace generated by the span of $B_2 = \{v_{n+1}, ..., v_k\}$. First, if $x \in V$, then $x = \sum_1^k a_iv_i = \sum_1^n a_iv_i + \sum_{i=n+1}^k a_iv_i$. There can be no vectors in the intersection of the subspaces because it would contradict the fact that both subspaces are linearly independent. □

Theorem 4. *Let W_1 and W_2 be two finite-dimensional subspaces of the vector space V. Prove that W_1 and W_2 are independent if and only if* $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$

Proof. Suppose that W_1 and W_2 are independent subspaces, and $\{v_1, ..., v_k\}$ and $\{w_1, ..., w_m\}$ be their respective bases. then we will look at the subspace $W_1 + W_2$. If we pick a vector $v \in W_1 + W_2$, then $v = \sum_{i=1}^{k} v_i + \sum_{i=1}^{m} w_i$. The set $\{v_i, ..., v_k, w_1, ..., w_m\}$ is a spanning set. Now, it suffices to show that the union of the two bases form a linearly independent set. Independent subspaces are disjoint. the vectors in the basis of W_1 and W_2 are linearly independent. Thus, the set $\{v_i, ..., v_k, w_1, ..., w_m\}$ is a basis for the subspace $W_1 + W_2$.

Suppose that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$. If $\dim W_1 = k$ and $\dim W_2 = m$, then a basis for $W_1 + W_2$ will have $m + k$ linearly independent vectors. To show that W_1 and W_2 are independent, we show that $W_1 \bigcap W_2 = \emptyset$. To see this, suppose WLOG that $v_1 \in W_2$, then this contradict the fact that the dimension of $W_2 + W_1$ is $m + k$ because the set $\{v_2, ..., v_k, w_1, ..., w_m\}$ would also be a basis for $W_1 + W_2$. □

Theorem 5. *Let V be a vector space with basis $\{v_1, ..., v_n\}$. Is $\{w_1, ..., w_n\}$ necessarily a basis for V with $w_i = \sum_{j=1}^{i} v_j$*

Proof. We see that $\sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{i} v_j$ by swapping the order of summation we arrive at $\sum_{j=1}^{n} v_j \sum_{i=j}^{n} a_i = 0$. This implies that $\sum_{i=j}^{n} a_i = 0$ for all $j \in [1, ..n]$. Upon closer inspection, we see this implies that $a_i = 0$ for all i. This is because the sum in question is $a_n v_n + (a_n + a_{n-1})v_{n-1} + ... + (a_n + ... + a_1)v_1$. it implies that all of the $a_i = 0$ □

Theorem 6. *(i) Let $W = \{p \in \mathbb{P}_4 : \int_{-1}^{1} p(t)dt = 0\}$. Find a basis for W.*

(ii)Extend the basis in (i) to a basis of $\mathbb{P}_4$.

(iii) Find a subspace U such that $\mathbb{P}_4 = W + U$.

Proof. Let $p(t) = at^4 + bt^3 + ct^2 + dt + e$, then $\rho(p) = \int_{-1}^{1} p(t)dt$. $\rho(p) = 0 \implies e = \frac{-a}{5} - \frac{c}{3}$. A basis for W is $\{1, x, x^3, ax^4 + cx^2 - (\frac{a}{5} + \frac{c}{3})\}$.

(iii) The subspace $U = \{p \in \mathbb{P}_4 : \rho(p) \neq 0\}$. □

Theorem 7. *Let V be a vector space of dimension n, and W be a subspace of V of dimension n - 1. Prove that if U is a subspace of V not contained in W, then $dimW \bigcap U) = dimU$ - 1.*

Proof. Let $\{w_1, ..., w_{n-1}\}$ be a basis for W, and $\{u_1, ..., u_k\}$ a basis for U. Since U is not contained in W. there must be a vector in the basis of U that is linearly independent from the basis of W. Let us call this vector v_1. Then we extend the basis for W to a basis for V by $\{w_1, ..., w_{n-1}, v_1\}$. We see that there can be at most one vector from the basis of U that is linear independent from the basis of W. $U = U/W + (U \bigcap W) \implies \dim U = \dim U/W + \dim U \bigcap w$ with $\dim U/W = 1$. □

Theorem 8. *Let $f(x)$ be a polynomial of degree n. Prove that $\{f(x), f^{'}(x), ..., f^n(x)\}$.*

Proof. The only function for which $f(x) = cf^n(x)$ is the function $f(x) = e^x$ which is not a polynomial. To see the linear indepence of the higher derivatives, we will use the fundemental theorem of Calculus. $f^n(x) = \int f^{n+1}(x)dx$[1]. So, if $f(x) \neq cf^n(x)$ then $f^i(x) \neq cf^j(x)$ for any i, j. So, the set is linearly indepedent. □

Theorem 9. *Let $a_0, a_1, ..., a_n$ be scalars and*

$$p_i(x)\ \Pi_{k \neq i} \frac{x - a_j}{a_i - a_j}$$

Prove that the set $\{p_1(x), ..., p_n(x)\}$ is a basis for $\mathbb{P}_n$.

Proof. □

[1]As always, this is true up to a constant