Analysis 1

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Theorem 1. *Prove, using the definition, that the function $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{x}$ is continuous.*

Proof. We will show that for any $c \in (0, \infty)$, $\lim_{x \to c} f(x) = f(c)$. Let $\epsilon > 0$, then a choice for $\delta = \epsilon\sqrt{c}$

$$|x - c| = |\sqrt{x} - \sqrt{c}||\sqrt{x} + \sqrt{c}| \leq \delta \implies |\sqrt{x} - \sqrt{c}| \leq \frac{\delta}{|\sqrt{x} + \sqrt{c}|} \leq \frac{\delta}{\sqrt{c}} \leq \epsilon$$

.

For the case when $c = 0$, $\sqrt{0} = 0 \implies \lim_{x \to 0} \sqrt{x} = 0$ because the choice $\delta = \sqrt{\epsilon}$ implies that $|x| = |\sqrt{x}|^2 \leq (\sqrt{\epsilon})^2 \implies |\sqrt{x}| \leq \epsilon$. □

Theorem 2. *if f is a continuous function of a metric space X into a metric space Y, prove that*

$$f(\overline{E}) \subset \overline{f(E)}$$

Proof.
If $x \in E$, then $f(x) \in f(E) \subset \overline{f(E)}$. Let x be a limit of E that is not in E, then $f(x) \in V$, where V is an open set of Y. $f^{-1}(V)$ is an open set that contains x and must intersects E at a point different than x. Let $y \in f^{-1}(V) \bigcap E$. $f(y) \in f(E) \subset \overline{f(E)}$. Thus, $f(y) \in V \bigcap f(E)$. It matters not whether $f(x) = f(y)$ or $f(x) \neq f(y)$ because in the former $f(x) \in \overline{f(E)}$ and in the latter $f(x)$ is a limit point of $f(E)$. Hence, the result is proven. □

Theorem 3. *If f is a continuous function from a metric space X to $\mathbb{R}$, and let $a \in \mathbb{R}$. Prove that the set $\{x \in X : f(x) > a\}$ is open and the sets $\{x \in X : f(x) \geq a\}$, $\{x \in X : f(x) = a\}$ are closed. Use this to show that the set $A_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^3 + z^4 < 1\}$ is open and $A_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + xy + y^2 + yz + z^2 = 1\}$.*

Proof. $\{x \in X : f(x) > a\} = f^{-1}(a, \infty)$ which is an open set because $f(x)$ is continuous. The compliment of $\{x \in X : f(x) \geq a\}$ is $\{x \in X : f(x) < a\}$. The compliment of $\{x \in X : f(x) = a\}$ is $\{x \in X : f(x) > a\} \bigcup \{x \in X : f(x) < a\}$ which are both open. $\{x \in X : f(x) < a\}$ is open in X because it is the preimage of $(-\infty, a)$ under f.

A_1 is open because $f(x, z, z) = x^2 + y^3 + z^4$ is continuous in $\mathbb{R}^3$.

A_2 is closed because $g(x, y, z) = x^2 + xy + y^2 + yz + z^2$. □

Theorem 4. *let f, g be continuous function from a metric space X to $\mathbb{R}$ into a metric space Y. Let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.*

Proof. E is dense in X. Hence $\overline{E} = X \implies f(\overline{E}) = f(X) \subset \overline{f(E)}$. Let $y \in Y$ be a limit point of $f(E)$. Then there exists a sequence $\{y_n\} \in f(E)$ that converges to y. There is also a corresponding sequence in Esuch that $x_n \in \underline{f^{-1}(y_n)}$ where $\{x_n\}$ converges in X. Hence there exists a $x \in X$, such that $f(x) = y \implies y \in f(X) \implies \overline{f(E)} \subset f(X)$. We have shown that $f(X) = \overline{f(E)}$.

$f(E)$ is dense in $f(X)$. For all $x \in X$, we can find a sequence $\{x_n\}$ in E such that $x_n \to x$. Continuity of $f(x)$ implies that the sequence $\{f(x_n)\}$ converges to $f(x)$. Hence, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x_n - x| < \delta \implies |f(x_n) - f(x)| = |g(x_n) - f(x)| < \epsilon$. $\{g(x_n)\} \to f(x)$. The uniqueness of limits give us our results. ☐

Theorem 5. *Define f, g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$ $f(x,y) = \frac{xy^2}{x^2+y^4}$, $g(x,y) = \frac{xy^2}{x^2+y^6}$, if $(x,y) \neq (0,0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; neverhteless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous.*

Proof. $0 \leq (x - y^2)^2 = x^2 + y^4 - 2xy^2$ for all $x, y \in \mathbb{R}$. If $x > 0$, then $\frac{-xy^2}{x^2+y^4} \leq \frac{xy^2}{x^2+y^4} \leq \frac{2xy^2}{x^2+y^4} \leq 1$. Hence $f(x,y)$ is bounded.

$0 \leq (x - y^3)^2 = x^2 + y^6 - 2xy^3$, then

$$g(x,y) = \frac{xy^2}{x^2+y^6} = \frac{1}{y}\frac{xy^3}{x^2+y^6} \leq \frac{1}{y}$$

which goes to infinity or negative infinity as y approaches zero. Hence, $g(x,y)$ is unbounded.

$f(x,kx) = \frac{k^2x^3}{x^2+k^4x^4} = \frac{xk^2}{1+x^2k^4}$, and $g(x,kx) = \frac{xk^2}{1+k^6x^4}$ which are continuous.

Let $(y^(2), y) \to (0,0$, then $g(y^2, y) = \frac{y^4}{2y^4} = \frac{1}{2}$ which not 0. Hence it is discontinuous. ☐

Theorem 6. *Let E be a dense subset of X. Let $f(x)$ be a uniformly continuous real function defined on E. Prove that f has a continuous extension from E to X.*

Proof. Let

$$g(x) = \begin{cases} f(x) & x \in E \\ \lim_{p \to x} f(p) & x \notin E \end{cases}$$

As X is the closure of E, $x \in X/E$ is a limit point of E. To see that it is continuous. Suppose that $\{y_n\}$, $\{x_n\}$ are both sequences in E that converges to y. Then the sequence $b_{2n} = x_n$ and $b_{2n+1} = y_n$ also converges to y. Let N be a large number such that $|y_n - y| < \epsilon$ and $|x_n - y| < \epsilon$. Then

$$|x_n - y| \leq |x_n - y_m| + |y_m - y| \leq |x_n - y_m| + \epsilon \leq 2\epsilon$$

Hence, the sequence $\{b_m\}$ is cauchy. It suffices to show that $f(b_m)$ is also cauchy. Take N large enough, such that $|b_n - b_m| < \delta$. f is uniformly continuous on E. We can find a $\delta > 0$ such that for all points $p, q \in E$ with $|p - q| < \delta \implies |f(p) - f(q)| < \epsilon$. Take $p = b_n$, and $q = b_m$. Hence $\{f(b_m\}$ is cauchy and convergent. Since it converges, all of its subsequence must converge to $f(y)$. ☐

Theorem 7. *let*

$$f(x) = \begin{cases} \frac{1}{n} & \frac{m}{n} \in \mathbb{Q} \\ 0 & x \in \mathbb{R}/\mathbb{Q} \end{cases}$$

Prove that $f(x)$ is continuous on $\mathbb{R}/\mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Proof. $\mathbb{Q}$ is dense in $\mathbb{R}$. So for $r \in \mathbb{R}/\mathbb{Q}$, we can find a sequence of rational numbers that converges to r. For N large enough, we can find rational numbers $\frac{m}{n}$ such that $|\frac{m}{n} - r| \leq \delta \implies |m - nr||\frac{1}{n}| < \delta \implies |\frac{1}{n}| \leq \frac{\delta}{|m-nr|} \leq \frac{\delta}{r}$ by choosing $\delta = r\epsilon$. Hence, $f(x)$ goes to zero on the sequence which is $f(r)$. To see that the function is discontinuous on rational numbers, let $\{\frac{1}{n}\}$ goes to zero but the sequence $\{f(\frac{1}{n}) = \frac{1}{n}\}$ also goes to zero as opposed to $f(0) = 1$.

Let $\frac{p}{q} \in \mathbb{Q}$, then the sequence $a_n = \frac{p}{q} + \frac{1}{n} = \frac{pn+q}{nq}$ converges to $\frac{p}{q}$, but $f(a_n) = \frac{1}{nq}$ converges to zero as opposed to $f(p/q) = \frac{1}{q}$. Thus, f is discontinuous at every rational points. ☐

Theorem 8. *A real valued function f defined in (a,b) is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $a < x < y < b$?. What is the geometrical interpretation of this inequality? Prove that every convex function is continuous. Show that if $a < s < t < u < b$ then

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$$

.

Proof. The geometrical interpretation is that $y - \lambda(x-y)$ is the line segment with x, y as endpoints. The definition of convexity says that and two points in (a, b), the image of the line segment between them will be under the line segment between the points $f(x), f(y)$.

Let $t = s+\lambda(u-s)$ where $\lambda = \frac{t-s}{u-s}$. Then by convexity, $f(t) \leq \lambda f(u)+(1-\lambda)f(s)$, and hence $f(t)-f(s) \leq \lambda(f(u)-f(s) \implies \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$. By the same token, $(u-s)(f(t)-f(s)) \leq (t-s)(f(u)-f(s))$ implies that $-(u-t+t-s)f(s) \leq (t-u+u-s)f(u)-(u-s)f(t) \implies (u-t)(f(u)-f(s)) \leq (u-s)(f(u)-f(t)$which implies the inequality.

Let $x, y, s, t \in (a, b)$, with $a < s < y < x < t < b$ Hence,

$$\frac{f(x)-f(y)}{x-y} \leq \frac{f(t)-f(s)}{t-s} \leq \frac{f(t)-f(x)}{t-x}$$

.

$$\frac{f(y)-f(s)}{y-s} \leq \frac{f(x)-f(s)}{x-s} \leq \frac{f(x)-f(y)}{x-y}$$

.

These inequalities implies that $|f(x)-f(y)| \leq M|x-y| \leq M\epsilon$ where $M = \max \frac{|f(t)-f(x)|}{|t-x|}, \frac{|f(x)-f(s)|}{|x-s|}$. Hence, with $\delta = \frac{\epsilon}{M}$ f is continous on (a, b). $\square$