Matrix Theory

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Theorem 1. *Let V be a vector space and $\alpha \in End(V)$.*
(i) Prove that if V is dimensional, then α is injective if and only if α is also surjective.
(ii) Provide an example that shows that this is not true if V had infinite dimension.

Proof. (i) Suppose that α is injective, then it implies that $ker(\alpha) = \{0\} \implies \dim(Ker\alpha) = 0$. The image of α can be written as the direct sum $ker(\alpha) + im(\alpha)$. Thus, $\dim(V) = \dim(ker(\alpha)) + \dim(im(\alpha)) = n \implies \dim(V) = \dim(im(\alpha))$. Therefore, α is surjective.

Suppose that α is surjective, then it implies that $\dim(V) = \dim(im(\alpha)) \implies \dim(ker(\alpha)) = 0$. Therefore, α is injective.

(ii) Let $V = C[a, b]$, and $I(f) = \int_a^b f(x)dx$. This is a surjective linear operator that is not injective. □

Theorem 2. *Let $\alpha, \beta \in Hom(V, W)$ where $dim(V) < \infty$. Prove that*

$$rk(\alpha + \beta) \leq rk(\alpha) + rk(\beta)$$

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Proof. $rk(\alpha + \beta) = rk(\alpha) + rk(\beta) - rk(Im(\alpha) \bigcap Im(\beta)) \implies rk(\alpha) + rk(\beta) + rk(Im(\alpha) \bigcap Im(\beta))$ which implies the result. □

Theorem 3. *Let $\alpha \in End(V)$ for a finite dimensional vector space V. Suppose that α is nilpotent of index k. Prove that $I - A$ is an automorphism of v.*

Theorem 4. *Let V be a finite dimensional vector space over $\mathbb{F}$ and α be a linear map from V to $\mathbb{F}$. Assume that α is not the zero map. Prove that there exists a vector $u \in V$ such that $\alpha(u) = 1$ and V is the direct sum of $Ker\alpha$ and $span\{u\}$.*

Proof. Since α is not the zero map, there exists a vector in V such that $\alpha(v) \neq 0$. $\alpha(v) = u \implies \alpha(v/u) = 1$. We know that $\mathbb{F}$ =null(α) + rk(α) □