Chapter 1

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Theorem 1. *If r is rational ($r \neq 0$) and x is irrational. Prove that $r + x$ and rx are irrational.*

Proof. Suppose that $r + x$ and rx were rational with $r + x = \frac{p}{q}$ and $rx = \frac{a}{b}$. Then we see that $x = \frac{a}{rb} = \frac{p-qr}{q}$ which is a contradiction. Therefore, $r + x$, and rx are both irrational. $\square$

Theorem 2. *Prove that there is no rational number whose square is 12.*

Proof. Let $12 = \frac{p^2}{q^2}$ with $(p, q) = 1$.

$$12q^2 = p^2 \implies 2\sqrt{3}q = p \implies 3|p$$

and

$$3 = \frac{p^2}{4q^2} = \frac{3k}{4q^2} \implies (p, q) \geq 3$$

which is a contradiction. $\square$

Theorem 3. *Prove the following*
(a) If $x \neq 0$, $xy = xz \implies y = z$
(b) If $x \neq 0$, $xy = x \implies x = 1$
(c) If $x \neq 0$, $xy = 1 \implies y = \frac{1}{x}$
(d) If $x \neq 0$ $\frac{1}{\frac{1}{x}} = x$

Proof. (a) $xy = xz \implies x^{-1}(xz) = x^{-1}(xy) \implies x = y$
(b) $xy = x \implies x^{-1}xy = x^{-1}x \implies y = 1$
(c) $xy = 1 \implies x^{-1}(xy) = x^{-1} \implies y = x^{-1} = \frac{1}{x}$
(d) $(\frac{1}{x})(\frac{1}{\frac{1}{x}}) = \frac{x}{x} = 1$ $\square$

Theorem 4. *Let E be a nonempty ssubset of an ordered set. Suppose α is a lower bound of E, β an upper bound of E. Prove that $\alpha \leq \beta$.*

Proof. $\alpha \leq x \leq \beta$ for all $x \in E$, then it follows that $\alpha \leq \beta$. $\square$

Theorem 5. *Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of $-x$ for $x \in A$. Prove that* $\inf A = -\sup(-A)$.

Proof. Let B be the set of all lower bound of A. For all $\beta \in B$, $\beta \leq x$ for all x in A. Then $-x \leq -\beta$. So, $-B$ is the set of all upper bound of $-A$ and $\inf A \geq \beta$. This implies that $-\beta \leq -\inf A = -\sup(-A)$. As the infinum is the smallest upper bound for $-A$, it implies that $-\sup(-A) \leq \inf A$ and the result follows. $\square$

Theorem 6. *For $b > 1$, prove the following*

(a) *If* m, n, p, q *are integers with* $n > 0$ *and* $q > 0$,

$$r = \frac{m}{n} = \frac{p}{q}$$

. *Prove that* $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$.

(b) *Prove that* $b^{r+s} = b^r b^s$ *if* r, s *are rational.*

(c) *If* $x \in \mathbb{R}$, *define* $B(x) = \{b^t : t \leq x\}$. *Prove that* $b^r = \sup B(r)$ *when* r *is rational.*

(d) *Prove that* $b^{r+s} = b^r b^s$ *for all real number r,s.*

Theorem 7. *Prove that no order can be defined in the complex field that turn it into an ordered field.*

Proof. Let $<$ be an order of $\mathbb{C}$, then either $i > 0 \implies i^2 > 0$ or $-i > 0 \implies (-i)^2 < 0$. In either case, the ordered will violate one of the axiom of an ordered field. □

Theorem 8. *Prove that* $||x| - |y|| \leq |x - y|$

Proof.

$$|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$$

□