Chapter 2

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Theorem 1. *Prove that the empty set is a subset of every set.*

Proof. □

Theorem 2. *A complex number is said to be algebraic if there are integers $a_0, ..., a_n$ not all zero such that*

$$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0$$

. Prove that the set of all algebraic numbers is countable.

Proof. For each $N \in \mathbb{N}$, the number of polynomials such that $n + |a_0| + |a_1| + ... + |a_n| = N$ is finite. Hence the number of roots of these polynomials is also finite. Let $U_{n,N}$ be the set of roots of polynomials such that

$$n + |a_0| + |a_1| + ... + |a_n| = N$$

and let $U_n = \bigcup_{i=1}^{\infty} U_{n,i}$. U_n and $\bigcup_{n=1}^{\infty} U_n$ is at most countable. □

Theorem 3. *Prove that there exists real numbers which are not algebraic.*

Proof. Let A be the set of all algebraic numbers. A is countable. As $\mathbb{R}$ is uncountable, $\mathbb{R}$ cannot be a subset of A. Hence, there exists a $x \in \mathbb{R}$ with $x \in A^c$. □

Theorem 4. *Prove that the set of irrational numbers is uncountable.*

Proof. $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{I}$ implies that $\mathbb{I}$ contains an uncountable set because $\mathbb{Q}$ is countable. □

Theorem 5. *Construct a bounded set of real numbers with exactly 3 limit points.*

Proof. $\{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{1 + \frac{1}{n} : n \in \mathbb{N}\} \bigcup \{2 + \frac{1}{n} : n \in \mathbb{N}\}$ with limit points $\{0, 1, 2\}$ □

Theorem 6. *Let K be a subset of real numbers with 0 and $1/n$. Prove that K is compact directly from definition.*

Proof. Let $\bigcup_{i=1}^{\infty} U_i$ be an open cover of K. There exists $i \in \mathbb{N}$ such that $0 \in U_i$. This interval will contain infinitely many elements of K. Thus, there are only finitely many elements not in U_i. The subcover $U_i \bigcup (\bigcup_{n=1}^{m} U_n)$ is a finite subcover of K. □

Theorem 7. *Construct a compact set of real numbers whose limit points form a countable set.*

Proof. Let $U_i = \{\frac{1}{i} + \frac{1}{n} : n \in \mathbb{N}\}$, then $\{0\} \bigcup (\bigcup_{n=1}^{\infty} U_n)$ □

Theorem 8. *Give an example of an open cover of $(0, 1)$ which has no finite subcover.*

Proof. $\bigcup_{n=1}^{\infty} (0, 1 + \frac{1}{n})$ □

Theorem 9. *(a) If A and B are disjoint closed(open) sets in some metric space X, prove that they are seperated.*

(b) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $|x - y| < \delta$, define B similarly, with $<$. Prove that A and B are seperated.

(c) Prove that every connected metric space with at least two points is uncountable.

Proof. (a) Closed sets contains their limit points and $A\bigcap\overline{B}=\overline{A}\bigcap B=\emptyset$.

We exclude clopen sets as we proved the result for closed sets. Let A and B be open sets that are not closed and disjoint Let $p\in X$ be a limit point of A and B, then every neighborhood of p intersects A at a point other than p. Let $U=(p-\epsilon,p+\epsilon)\bigcap A$ and $V=(p-\epsilon,p+\epsilon)\bigcap B$. $U\bigcap V\neq\emptyset$ because we can find an open set contained in $U\bigcup\{p\}$. This would give rise to an open set containing p which only intersect B at p, a contradiction.

(b) $A=\{x\in X:|x-p|<\delta\}$ and $B=\{x\in X:|x-p|>\delta\}$ are disjoint open sets and hence are seperated.

(c) Let $c\in\mathbb{R}$ such that there is no pair of points in A such that their distance is c. By previous excersies, we can seperate X, which is a contradiction. Therefore, for all real numbers, there exists a $y\in A$ such that $|x-y|=c$ which implies it contains an uncountable set. □

Theorem 10. *A metric space is seperable if it contains a countable dense subset. Show that $\mathbb{R}^k$ is seperable.*

Proof. Let $x\in\mathbb{Q}^k$, then $x=(x_1,....,x_k)$. For each x_i, there exists a sequence $u_{i,n}\in\mathbb{Q}$ that converges to x_i. Therefore the sequence $u_n=(u_{1,n},...,u_{k,n})$ converges to x.

$\mathbb{Q}^k$ is a countable dense subset of $\mathbb{R}^k$. □

Theorem 11. *Prove that every seperable metric space has a countable base.*

Proof. The seperable metric space X contains a dense countable subset E. Let $(x-\epsilon,x+\epsilon)$ be a neighborhood of x. Then the neighborhood $(y-r,y+r)\subset(x-\epsilon,x+\epsilon)$ whenever $r<\epsilon$ and $|x-y|<r$. We choose r to be a rational number and $y\in E$. Hence the set $\bigcup_{y,r}N_r(y)$ is a countable base. □

Theorem 12. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is seperable.*

Proof. Let $\delta>0$, then we pick $x_1\in X$, and having chosen $x_1,...,x_j$, we pick x_{j+1} such that $|x_j-x_{j+1}|\geq\delta$. The sequence we chose is an infinite subset of X and has limit point. The infinite subset has a sequence that converges to the limit points and the sequence is cauchy. Therefore, the sequence must have finite range. $U_{i,n}=(x_i-\delta,x_i+\delta)$ are finite open sets that cover X. Hence, every point in X belongs to at least one $U_{i,n}$. Let U_n be the set of all $\{x_i\}$ for $\delta=1/n$, then $U=\bigcup_{n=1}^{\infty}U_n$ is at most countable. Let $x\in X$, then we will construct a sequence of of points as followed. $|x_1-x|<1$, and $|x-x_n|<\frac{1}{n}$. For each n, $x\in\{y:|x-y|<\frac{1}{n}\}$ and we can find $x_n\in U$ to be the point in our sequence. Hence, U is dense in X and it is countable. U is not finite as that would imply X is finite. □

Theorem 13. *Prove that every compact metric space K has a countable base, and that K is therefore seperable.*

Proof. The set $A_x=(x-\frac{1}{n},x+\frac{1}{n})$ form an open cover of K and hence it contains a finite subcover. So, for each n, there exists a finite set of points such that neighborhood around those points with radius 1/n cover K. By previous exercise, it is seperable and it has a countable base. □

Theorem 14. *Prove that the set of condensation points of $E\subset\mathbb{R}^k$ is perfect and that at most countably points in E is not in P.*

Proof. Let $\{V_i\}$ be a countable base of $\mathbb{R}^k$, and define $W=\bigcup_{k=1}^{\infty}V_k$ where $E\bigcap V_k$ is at most countable. Let P be the set of condensation points of E. If $p\in P$ then the intersection of every neighborhood of p with E is uncountable. If U is a neighborhood of p, then there exists V_k such that $V_k\subset U$ which implies that $E\bigcap V_k$ is uncountable and $p\in W^c$. $P\subset W^c$.

If $p\in W^c$, then $E\bigcap V_i$ is uncountable when $p\in V_i$. If U is any neighborhood of p, then we can find V_i that is contained in U. This implies that every neighborhood of p intersect E at an uncountable set and $p\in P$. We have shown that $P=W^c$.

We look at two sets $P\bigcap E$ and $P^c\bigcap E$. $P^c\bigcap E$ is at most countable because for every $p\in P^c$ we can find a neighborhood U such that $E\bigcap U$ is countable. As E is uncountable, $P\bigcap E$ is uncountable.

Suppose that P is not perfect, then we can find a point p in P such that it is not a limit point of P. Let $N/\{p\} \subset P^c$ intersect E at a countable set. $N \bigcup \{p\}$ is countable which is a contradiction. Hence, P is perfect. $\square$