Chapter 2

Mike Desgrottes

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Theorem 1. *Prove that the empty set is a subset of every set.*

Proof. Let W be a set, then $W \bigcup \emptyset = W \implies \emptyset \subset W$ $\square$

Theorem 2. *A complex number is said to be algebraic if there are integers $a_0, ..., a_n$ not all zero such that*

$$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0$$

. Prove that the set of all algebraic numbers is countable.

Proof. For each $N \in \mathbb{N}$, the number of polynomials such that $n + |a_0| + |a_1| + ... + |a_n| = N$ is finite. Hence the number of roots of these polynomial is also finite. Let $U_{n,N}$ be the set of roots of polynomial such that

$$n + |a_0| + |a_1| + ... + |a_n| = N$$

and let $U_n = \bigcup_{i=1}^{\infty} U_{n,i}$. The map

$$n + |a_0| + ... + |a_n| \to (n, a_0, ..., a_n)$$

ensure that U_n is isomorphic to $\mathbb{Z}^{n+1}$. U_n and $\bigcup_{n=1}^{\infty} U_n$ is countable. $\square$

Theorem 3. *Prove that there exists real numbers which are not algebraic.*

Proof. Let A be the set of all algebraic numbers. A is countable. As $\mathbb{R}$ is uncountable, $\mathbb{R}$ cannot be a subset of A. $\square$

Theorem 4. *Prove that the set of irrational numbers is uncountable.*

Proof. $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{I}$ implies that $\mathbb{I}$ contains an uncountable set. $\square$

Theorem 5. *Construct a bounded set of real numbers with exactly 3 limit points.*

Proof. $\{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{1 + \frac{1}{n} : n \in \mathbb{N}\} \bigcup \{2 + \frac{1}{n} : n \in \mathbb{N}\}$ with limit points $\{0, 1, 2\}$ $\square$

Theorem 6. *Let K be a subset of real numbers with 0 and $1/n$. Prove that K is compact directly from definition.*

Proof. Let $\bigcup_{i=1}^{\infty} U_i$ be an open cover of K. There exists i such that $0 \in U_i$. This interval will contain infinitely many element of K. Thus, there are only finitely many number of elements not in U_i. The finite subcover $U_i \bigcup (\bigcup_{n=1}^{m} U_n)$ is finite subcover of K. $\square$

Theorem 7. *Construct a compact set of real numbers whose limit points form a countable set.*

Proof. Let $U_i = \{i + \frac{1}{n} : n \in \mathbb{N}\}$, then $\{0\} \bigcup \mathbb{N} \bigcup (\bigcup_{n=1}^{\infty} U_n)$ $\square$

Theorem 8. *Give an example of an open cover of $(0, 1)$ which has no finite subcover.*

Proof. $\bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$ $\square$

Theorem 9. *(a) If A and B are disjoint closed(open) sets in some metric space X, prove that they are seperated.*

(b) Fix $p \in X$*,* $\delta > 0$*, define A to be the set of all* $q \in X$ *for which* $|x - y| < \delta$*, define B similarly, with* $<$*. Prove that A and B are seperated.*

(c) Prove that every connected metric space with at least two points is uncountable.

Proof. (a) Closed sets contains their limit points and $A \bigcap \overline{B} = \overline{A} \bigcap B = \emptyset$. If A and B are disjoint open set, then their complement is closed and the result follows.

(b) $A = \{x \in X : |x - p| < \delta\}$ and $B = \{x \in X : |x - p| > \delta\}$ are disjoint open sets and hence are seperated.

(c) Let $c \in \mathbb{R}$ such that there is no pair of points in A such that their distance is c. By previous excersies, we can seperate X, which is a contradiction. Therefore, for all real numbers, there exists a $y \in A$ such that $|x - y| = c$ which implies it contains an uncountable set. □

Theorem 10. *A metric space is seperable if it contains a countable dense subset. Show that* $\mathbb{R}^k$ *is seperable.*

Proof. $\mathbb{Q}^k$ is a countable dense subset of $\mathbb{R}^k$. □

Theorem 11. *Prove that every seperable metric space has a countable base.*

Proof. The metric space have a countable dense subset E. Hence between any two elements in X, there exists an element of E. Take $(y - r, r + y)$ with r,y being rational. The set of these interval is countable and contains x. □

Theorem 12. *Let X be a metric space in which every infinite subset has a limit point. Prove that X is seperable.*

Proof. Let $\delta > 0$, then we pick $x_1 \in X$,a nd having chosen $x_1, ..., x_j$, we pick x_{j+1} such that $|x_j - x_{j+1}| \geq \delta$. The sequence we chose is an infinite subset of X and has limit point. Therefore, it must terminate at a finite point. $U_i = (x_i - \delta, x_i + \delta)$ are finite open sets that cover X. The triangle inequality prevent any element of X to be too close to two points of U_i. Hence, every point in X belongs to at least one U_i. $\bigcup_{i=1}^{\infty} U_i$ with $\delta = 1/n$ is a countable dense subset. □

Theorem 13. *Prove that every compact metric space K has a countable base, and that K is therefore seperable.*

Proof. The set $A_x = (x - \frac{1}{n}, x + \frac{1}{n})$ form an open cover of K and hence it contains a finite subcover. So, for each n, there exists a finite set of points such that neighborhood around those points with radius 1/n cover K. By previous exercise, it is seperable. □