Analysis 1

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Question 1. *Suppose $f^{'}(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that*

$$g^{'}(f(x)) = \frac{1}{f^{'}(x)}(a < x < b)$$

.

Proof. Let $x, y \in (a, b)$ with $y - x > 0$. By the mean value theorem, there exists a $c \in (x, y)$ such that

$$f^{'}(c) = \frac{f(y) - f(x)}{y - x} > 0 \implies f(y) - f(x) > 0$$

.

$g(f(x)) = x$ which is differentiable. Its derivative is $g^{'}(f(x))f^{'}(x) = 1$ by the chain rule. Hence $g^{'}(f(x)) = \frac{1}{f^{'}(x)}$. g is differentiable on (a, b) because $f^{'}(x) > 0$. □

Question 2. *Suppose*
(a) f is continuous on $[0, \infty)$
(b) f is differentiable on $(0, \infty)$
(c) $f(0) = 0$
(d) $f^{'}$ is monotonically increasing.
Prove that

$$g(x) = \frac{f(x)}{x}(x > 0)$$

is monotonically increasing.

Proof. This is a consequence of the Mean Value Theorem which guarantee the existence of $c > 0$ such that $f^{'}(c) = \frac{f(x)-f(y)}{x-y}$ with $y = 0$. We get $g(x) = f^{'}(c)$ which is monotonically increasing. □

Question 3. *Let f be a continuous real function on R, of which it is known that $f^{'}(x)$ exists for all $x \neq 0$ and that $f^{'}(x) \to 3$ as $x \to 0$. Does it follow that $f^{'}(0)$ exists?.*

Proof. Yes, it exists. To see this, let $\{y_n\}$ be a sequence that converges to 0. The Mean Value Theorem guarantee the existence of a sequence of c_n such that $f^{'}(c_n) = \frac{f(y_n)-f(0)}{y_n}$ where $c_n \to 0$. Thus, $f^{'}(0) = \lim_{n\to\infty} f^{'}(c_n) = \lim_{n\to\infty} \frac{f(y_n)-f(0)}{y_n}$.

$$\lim_{y_n \to 0} \frac{f^{'}(y_n)}{1} = 3$$

.

By L'Hospital rule, and continuity of f, $f^{'}(0) = 3$. □

Question 4. *Suppose $a \in \mathbb{R}$, f is twice differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f^{'}(x)|, |f^{''}(x)|$ respectively on (a, ∞). Prove that $M_1^2 \leq 4M_0M_2$*

Proof. By Taylor's theorem,

$$f(x) = P(x) + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n$$

. With $n = 2, \alpha = x + 2h$, We have

$$f(x) = f(x+2h) + f^{'}(x+2h)(-2h) + f^{''}(c)(2h^2)$$

. We will show that

$$f^{''}(c) = \frac{f^{'}(x+2h) - f^{'}(x)}{2h}$$

Set $g(t) = f(t) - f(\alpha) - f^{'}(\alpha)(t-\alpha) - \frac{f^{''}(c)}{4}$ with second derivative

$$g^{'}(t) = f^{'}(t) - f^{'}(\alpha) - \frac{f^{''}(c)(t-\alpha)}{2}$$

. Set $t = x, \alpha = x + 2h$, we get the result.

$$f(x) = f(x+2h) - 4h^2 f^{''}(c) - 2hf^{'}(x) + 2h^2 f^{''}(c)$$

.

$$f^{'}(x) = \frac{f(x+2h) - f(x) - 2h^2 f^{''}(c)}{2h}$$

.

$$|f^{'}(x)| \le M_1 \le \frac{M_0 + h^2 M_2}{h} \implies M_1^2 \le 1 + 3M_0M_1 \le 4M_0M_1$$

when $h = M_0$. □

Question 5. *Suppose f is a real, three times differentiable function on* $[-1, 1]$, *such that*

$$f(-1) = 0, f(0) = 0, f(1) = 1, f^{'}(0) = 0$$

.

Prove that $f^{'''}(x) \ge 3$ *for some* $x \in (-1, 1)$.

Proof. Taylor's theorem gives us

$$f(1) = f(0) + f^{'}(0) + \frac{f^{''}(0)}{2} + \frac{f^{'''}(c)}{6}$$

and

$$f(-1) = f(0) - f^{'}(0) + \frac{f^{''}(0)}{2} - \frac{f^{'''}(s)}{6}$$

and

$$f^{'''}(c) = f^{''}(1) - f^{''}(0)$$

and

$$f^{'''}(s) = f^{''}(-1) - f^{''}(0)$$

which gives us

$$f^{'''}(s) + f^{'''}(c) = 6$$

with $c \in (0, 1)$ and $(-1, 0)$. One of the two is at least 3. The result follows. □