1.

a) $w(z) = z^{17} + z^{12} - z^7 - z^2$

$$z^{12}(z^5+1) - z^2(z^5+1)$$

$$(z^5+1)(z^{12}-z^2)$$

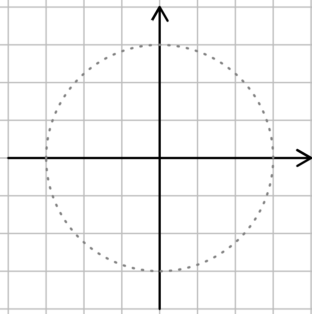
$$z^2(z^5+1)(z^{10}-1)$$

$$z^2(z^5+1)(z^5+1)(z^5-1)$$

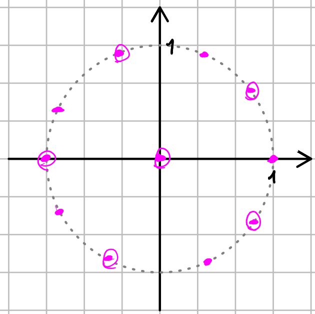
$$z^2(z^5+1)^2(z^5-1)$$

$z^5+1=0$
$z^5=-1$
$r^5 e^{5\varphi j} = e^{\pi j}$
$r^5 = 1 \quad r=1$
$5\varphi = \pi + 2k\pi$
$\varphi = \frac{\pi+2k\pi}{5}$

$z^5-1=0$
$z^5=1$
$r^5 e^{5\varphi j} = e^{0j}$
$r^5=1 \quad r=1$
$5\varphi = 2k\pi$
$\varphi = \frac{2k\pi}{5}$



$$\left\{ \underbrace{0,\ e^{\frac{\pi}{5}j},\ e^{\frac{3\pi}{5}j},\ -1,\ e^{-\frac{\pi}{5}j},\ e^{-\frac{3\pi}{5}j}}_{\text{dwukrotne}},\ \underbrace{1,\ e^{\frac{2\pi}{5}j},\ e^{\frac{4\pi}{5}j},\ e^{-\frac{2\pi}{5}j},\ e^{-\frac{4\pi}{5}j}}_{\text{jednokrotne}} \right\}$$



⊙ pierwiastek 2-krotny

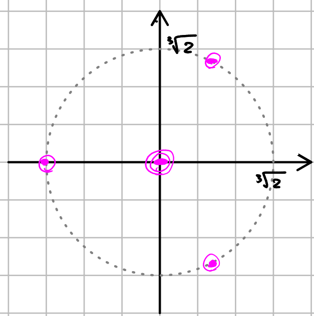
3 różne p. rzeczywiste, 11 różnych p. zespolonych

b) $w(z) = z^9 + 4z^6 + 4z^3 = z^3(z^6+4z^3+4) = z^3(z^3+2)^2$

$$\sqrt[3]{-2} = \left\{ \sqrt[3]{2}\, e^{\frac{\pi+2k\pi}{3}j};\ k=0,1,2 \right\}$$

$$\left\{ 0,\ \sqrt[3]{2}\,e^{\frac{\pi}{3}j},\ -\sqrt[3]{2},\ \sqrt[3]{2}\,e^{-\frac{\pi}{3}j} \right\}$$

$\downarrow$ 3, $\downarrow$ 2, $\downarrow$ 2, $\downarrow$ 2 krotność



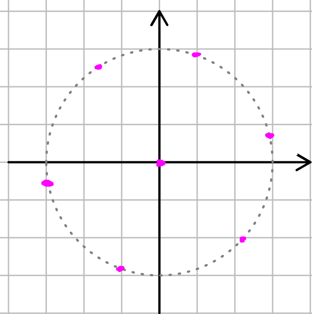
2 różne rzeczywiste
4 różne zespolone

c) $w(z) = z^7 - jz = z(z^6 - j)$

$$z^6 - j = 0 \rightarrow \sqrt[6]{j} = \sqrt[6]{e^{\frac{\pi}{2}j}} = \left\{ e^{\frac{\frac{\pi}{2}+2k\pi}{6}j};\ k=0,1,2,3,4,5 \right\}$$

$$\left\{ 0,\ e^{\frac{\pi}{12}j},\ e^{\frac{5\pi}{12}j},\ e^{\frac{9\pi}{12}j},\ e^{-\frac{3\pi}{12}j},\ e^{-\frac{7\pi}{12}j},\ e^{-\frac{11\pi}{12}j} \right\}$$

$\downarrow$ 1, $\downarrow$ 1, $\downarrow$ 1, $\downarrow$ 1, $\downarrow$ 1, $\downarrow$ 1, $\downarrow$ 1 krotności



1 rzeczywisty, 7 różnych zespolonych

2.

$w(z) = z^4 + 2z^3 + 2z^2 + 10z + 25$

$z_0 = 1 + 2j \qquad w(z_0) = 0 \iff w(\overline{z_0}) = 0$ bo współczynniki są rzeczywiste

$z_1 = 1 - 2j$

$w(z) = (z - 1 - 2j)(z - 1 + 2j)(z^2 + bz + c)$

$= (z^2 - z + \cancel{2j}z - z + 1 - \cancel{2j} - \cancel{2j}z + \cancel{2j} + 4)(z^2 + bz + c)$

$(z^2 - 2z + 5)(z^2 + bz + c)$

$z^4 + bz^3 + cz^2 - 2z^3 - 2bz^2 - 2cz + 5z^2 + 5bz + 5c$

$z^4 + (b-2)z^3 + (c - 2b + 5)z^2 + (5b - 2c)z + 5c$

$2 = b - 2 \longrightarrow b = 4$

$2 = c - 2b + 5$

$10 = 5b - 2c$

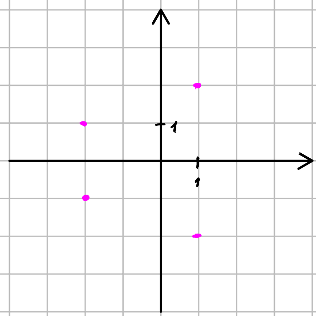
$25 = 5c \longrightarrow c = 5$

$z^2 + 4z + 5 \qquad \Delta = 4^2 - 4 \cdot 5 = -4$

$\delta = 2j \in \sqrt{\Delta}$

$z_2 = \frac{-4 - 2j}{2} = -2 - j$

$z_3 = \frac{-4 + 2j}{2} = -2 + j$



$\{1 + 2j,\ 1 - 2j,\ -2 - j,\ -2 + j\}$

$$\begin{array}{l}
\phantom{z^4 + 2z^3 + {}} z^2 + 4z + 5 \\
\hline
z^4 + 2z^3 + 2z^2 + 10z + 25 \ \big(\ z^2 - 2z + 5 \\
\ominus\ \underline{z^4 - 2z^3 + 5z^2} \\
\phantom{z^4 + {}} 4z^3 - 3z^2 + 10z \\
\phantom{z^4 + {}} \ominus\ \underline{4z^3 - 8z^2 + 20z} \\
\phantom{z^4 + 2z^3 + {}} 5z^2 - 10z + 25 \\
\phantom{z^4 + 2z^3 + {}} \ominus\ \underline{5z^2 - 10z + 25} \\
\phantom{z^4 + 2z^3 + 2z^2 + 10z + {}} 0
\end{array}$$

3.

a) $w(z) = z^4 + 6z^2 + 25$ $\Delta = 36 - 100 = -64$ $\delta = 8j \in \sqrt{\Delta}$

$= (z^2 - (-3-4j))(z^2 - (-3+4j))$

$p_1 = \frac{-6-8j}{2} = -3-4j$

$p_2 = -3+4j$

$z^2 = -3-4j \quad \vee \quad z^2 = -3+4j$

1) $z = x+yj \qquad x^2+y^2 = |z|^2 = |z^2| = 5$

$x^2 - y^2 = -3 \qquad 2x^2 = 2$

$2xyj = -4j \qquad x = 1 \vee x = -1$

$x^2 + y^2 = 5 \qquad y = -2 \quad y = 2$

2) $z = x+yj$

$x^2 - y^2 = -3 \qquad 2x^2 = 2$

$2xy = 4 \qquad x = 1 \quad \vee \quad x = -1$

$x^2 + y^2 = 5 \qquad y = 2 \qquad y = -2$

$z_0 = 1-2j \qquad z_1 = -1+2j \qquad z_2 = 1+2j \qquad z_3 = -1-2j$

$w(z) = (z-1+2j)(z+1-2j)(z-1-2j)(z+1+2j)$

b) $w(z) = z^4 - (3+2j)^4 = [z^2 - (3+2j)^2][z^2 + (3+2j)^2]$

$= (z-3-2j)(z+3+2j)(z-2+3j)(z+2-3j)$ $(a+bj)(a-bj) = a^2 - abj + abj - bj^2 = a^2+b^2$

c) $w(z) = (1+\sqrt{3}j)z^4 + 8 = 2e^{\frac{\pi}{3}j}z^4 + 8$

$z^4 = \frac{-8}{2e^{\frac{\pi}{3}j}} = \frac{8e^{\pi j}}{2e^{\frac{\pi}{3}j}} = 4e^{\frac{2\pi}{3}j}$ $\sqrt[4]{4} = 2^{\frac{2}{4}} = \sqrt{2}$

$\sqrt[4]{4e^{\frac{2\pi}{3}j}} = \{\sqrt[4]{4}\, e^{\frac{\frac{2\pi}{3}+2k\pi}{4}j};\ k=0,1,2,3\}$

$\{\sqrt{2}e^{\frac{\pi}{6}j}, \sqrt{2}e^{\frac{2\pi}{3}j}, \sqrt{2}e^{-\frac{\pi}{3}j}, \sqrt{2}e^{-\frac{5\pi}{6}j}\}$

$\{\sqrt{2}(\frac{\sqrt{3}}{2} + j\frac{1}{2}), \sqrt{2}(-\frac{1}{2} + j\frac{\sqrt{3}}{2}), \sqrt{2}(\frac{1}{2} - j\frac{\sqrt{3}}{2}), \sqrt{2}(-\frac{\sqrt{3}}{2} - j\frac{1}{2})\}$

$\{\frac{\sqrt{6}}{2} + j\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + j\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2} - j\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2} - j\frac{\sqrt{2}}{2}\}$

$w(z) = (z - \frac{\sqrt{6}}{2} - j\frac{\sqrt{2}}{2})(z + \frac{\sqrt{2}}{2} - j\frac{\sqrt{6}}{2})(z - \frac{\sqrt{2}}{2} + j\frac{\sqrt{6}}{2})(z + \frac{\sqrt{6}}{2} + j\frac{\sqrt{2}}{2})$

4.

a) $$\frac{2x^2-7x}{(x-3)^3(x^2+4)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

b) $$\frac{3x^4-2}{x^4-2x^3+2x-1} = \frac{3x^4-2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} + E$$

Rozkład jest tylko dla funkcji wymiernych właściwych

$$x^4-1-2x^3+2x = (x^2+1)(x^2-1) - 2x(x^2-1) = (x^2-1)(x^2-2x+1) = (x+1)(x-1)(x-1)^2 = (x+1)(x-1)^3$$

c) $$\frac{5x^3}{(x^4+4)^2} = \frac{5x^3}{(x^2-2x+2)^2(x^2+2x+2)^2} = \frac{Ax+B}{x^2-2x+2} + \frac{Cx+D}{(x^2-2x+2)^2} + \frac{Ex+F}{x^2+2x+2} + \frac{Gx+H}{(x^2+2x+2)^2}$$

$$x^4+4 = x^4+4x^2+4-4x^2 = (x^2+2)^2-4x^2 = (x^2-2x+2)(x^2+2x+2)$$

$\Delta = -4$ $\quad\Delta = -4$

5.

a) $$\frac{4x-10}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$4x-10 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

dla $x=3$ $\quad 12-10 = C\cdot 2\cdot 1$ $\quad C=1$

dla $x=2$ $\quad 8-10 = B\cdot 1\cdot(-1)$ $\quad B=2$

dla $x=1$ $\quad 4-10 = A(-1)(-2)$ $\quad A=-3$

$$\frac{-3}{x-1} + \frac{2}{x-2} + \frac{1}{x-3}$$

b) $$\frac{8}{x^2-4x+2} = \frac{8}{(x-2+\sqrt{3})(x-2-\sqrt{3})} = \frac{A}{x-2+\sqrt{2}} + \frac{B}{x-2-\sqrt{2}} = \frac{A(x-2-\sqrt{2})+B(x-2+\sqrt{2})}{(x-2+\sqrt{2})(x-2-\sqrt{2})}$$

$\Delta = 16-8 = 8$ $\quad \sqrt{\Delta} = 2\sqrt{2}$ $\qquad 8 = A(x-2-\sqrt{2}) + B(x-2+\sqrt{2})$

$x_1 = \frac{4-2\sqrt{2}}{2} = 2-\sqrt{2}$ $\qquad$ dla $x = 2+\sqrt{2}$ $\quad 8 = B(\cancel{2}+\sqrt{2}-\cancel{2}+\sqrt{2})$ $\quad B = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$

$x_2 = 2+\sqrt{2}$ $\qquad$ dla $x = 2-\sqrt{2}$ $\quad 8 = A(2-\sqrt{2}-2-\sqrt{2})$ $\quad A = \frac{8}{-2\sqrt{2}} = -2\sqrt{2}$

$$\frac{2\sqrt{2}}{x-2-\sqrt{2}} - \frac{2\sqrt{2}}{x-2+\sqrt{2}}$$

c) $$\frac{x^2-3x+6}{x^4-5x^2+4} = \frac{x^2-3x+6}{(x+1)(x-1)(x-2)(x+2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x+2}$$

$$x^4-5x^2+4 = x^4-4x^2-x^2+4 = x^2(x^2-4)-(x^2-4) = (x^2-1)(x^2-4) = (x+1)(x-1)(x+2)(x-2)$$

$$x^2-3x+6 = A(x-1)(x-2)(x+2) + B(x+1)(x-2)(x+2) + C(x+1)(x-1)(x+2) + D(x-1)(x+1)(x-2)$$

dla $x=-1$ $\quad 1+3+6 = A(-2)(-3)(1)$ $\quad A = \frac{10}{6} = \frac{5}{3}$

dla $x=1$ $\quad 1-3+6 = B(2)(-1)(3)$ $\quad B = \frac{4}{-6} = -\frac{2}{3}$

dla $x=2$ $\quad 4-6+6 = C(3)(1)(4)$ $\quad C = \frac{4}{12} = \frac{1}{3}$

dla $x=-2$ $\quad 4+6+6 = D(-3)(-1)(-4)$ $\quad D = \frac{16}{-12} = -\frac{4}{3}$

$$\frac{5/3}{x+1} - \frac{2/3}{x-1} + \frac{1/3}{x-2} - \frac{4/3}{x+2}$$

d) $\dfrac{2x^3+5x^2+3x+1}{x^2(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

$$2x^3+5x^2+3x+1 = Ax(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)x^2(x^2+1) + (Ex+F)x^2$$

dla $x=0 \qquad 1 = B$ $\qquad\qquad (x^2+1)^2 = x^4+2x^2+1$

$$Ax^5 + 2Ax^3 + Ax + Bx^4 + 2Bx^2 + B + Cx^5 + Cx^3 + Dx^4 + Dx^2 + Ex^3 + Fx^2$$

$$(A+C)x^5 + (B+D)x^4 + (2A+C+E)x^3 + (2B+D+F)x^2 + Ax + B$$

$$\begin{cases} 0 = A+C \\ 0 = B+D \\ 2 = 2A+C+E \\ 5 = 2B+D+F \\ 3 = A \\ 1 = B \end{cases} \qquad \begin{aligned} A &= 3 \\ B &= 1 \\ C &= -3 \\ D &= -1 \\ E &= -1 \\ F &= 4 \end{aligned} \qquad \frac{3}{x} + \frac{1}{x^2} - \frac{3x+1}{x^2+1} - \frac{x-4}{(x^2+1)^2}$$

albo można podstawić $x=j$
(pierwiastki zespolone nierozkładalne)

6.

a) $\dfrac{4z+3j}{z^2-2jz+3} = \dfrac{4z+3j}{(z+j)(z-3j)} = \dfrac{A}{z+j} + \dfrac{B}{z-3j}$

$$z^2-3jz+jz+3 = z(z-3j)+j(z-3j) = (z+j)(z-3j)$$

b) $\dfrac{z^4+3z-2}{z^2(z^2+9)(z+3j)^2} = \dfrac{z^4+3z-2}{z^2(z+3j)^3(z-3j)} = \dfrac{A}{z} + \dfrac{B}{z^2} + \dfrac{C}{z-3j} + \dfrac{D}{z+3j} + \dfrac{E}{(z+3j)^2} + \dfrac{F}{(z+3j)^3}$

c) $\dfrac{z^7+z^2}{(z^4+4)^2} = \dfrac{A}{z-1-j} + \dfrac{B}{(z-1-j)^2} + \dfrac{C}{z+1+j} + \dfrac{D}{(z+1+j)^2} + \dfrac{E}{z-1+j} + \dfrac{F}{(z-1+j)^2} + \dfrac{G}{z+1-j} + \dfrac{H}{(z+1-j)^2}$

$$(z^4+4)^2 = \left[(z^2-2j)(z^2+2j)\right]^2 = (z-1-j)^2(z+1+j)^2(z-1+j)^2(z+1-j)^2$$

$$\sqrt{2j} = \sqrt{2e^{\frac{\pi}{2}j}} = \left\{\sqrt{2}e^{\frac{\pi}{4}j}, \sqrt{2}e^{-\frac{3\pi}{4}j}\right\} = \{1+j, -1-j\}$$

$$\sqrt{-2j} = \sqrt{2e^{-\frac{\pi}{2}j}} = \left\{\sqrt{2}e^{-\frac{\pi}{4}j}, \sqrt{2}e^{\frac{3\pi}{4}j}\right\} = \{1-j, -1+j\}$$

7.

a) $\dfrac{4z^2+1}{z^4-1}$ $\quad f(z)=4z^2+1$ $\quad z^4-1=(z^2-1)(z^2+1)=(z-1)(z+1)(z-j)(z+j)$

$g(z)=z^4-1$

$g'(z)=4z^3$ $\quad a(z)=\dfrac{4z^2+1}{4z^3}$ $\quad \dfrac{1}{j}\cdot\dfrac{j}{j}=\dfrac{j}{-1}=-j$

$a(1)=\dfrac{5}{4}$ $\quad a(-1)=-\dfrac{5}{4}$ $\quad a(j)=\dfrac{-3}{-4j}=-\dfrac{3}{4}j$ $\quad a(-j)=\dfrac{-3}{-j\cdot(-1)\cdot 4}=-\dfrac{3}{4j}=\dfrac{3}{4}j$

$$\frac{4z^2+1}{z^4-1}=\frac{5/4}{z-1}-\frac{5/4}{z+1}-\frac{j3/4}{z-j}+\frac{j\,3/4}{z+j}$$

b) $\dfrac{z^{2020}}{z^{2021}+1}=\dfrac{f(z)}{g(z)}$ $\quad g'(z)=2021\,z^{2020}$ $\quad \dfrac{f(z)}{g'(z)}=\dfrac{z^{2020}}{2021\,z^{2020}}=\dfrac{1}{2021}$

$z^{2021}+1=0 \rightarrow \sqrt[2021]{-1}=\sqrt[2021]{e^{\pi j}}=\left\{e^{\frac{\pi+2k\pi}{2021}j} : k\in[0,2020]\cap\mathbb{Z}\right\}$

$$\frac{z^{2020}}{z^{2021}+1}=\sum_{i=0}^{2020}\frac{1/2021}{z-e^{\frac{\pi+2k\pi}{2021}j}}$$