1.

X i Y są niezależne

$X, Y \sim g(p)$

$\Rightarrow P(Y=k) = P(X=k) = (1-p)^{k-1} p$ dla $k \in \mathbb{N}$

$\Rightarrow S_x = S_y = \mathbb{N}$

$Z = X + Y$ $\quad S_z = \{2, 3, 4, \dots\} = \mathbb{N} \setminus \{1\}$

dla $k \in S_z$

$$P(Z=k) = P(X+Y=k) = P(X=a, Y=k-a, a \in \{1, \dots k-1\})$$

$$= \sum_{a=1}^{k-1} P(X=a, Y=k-a)$$

$$= \sum_{a=1}^{k-1} P(X=a) P(Y=k-a)$$

$$= \sum_{a=1}^{k-1} p(1-p)^{a-1} \cdot p(1-p)^{k-a-1}$$

$$= \sum_{a=1}^{k-1} p^2 (1-p)^{k-2}$$

$$= (k-1) p^2 (1-p)^{k-2}$$

Z nie ma rozkładu geometrycznego

2.

$X \backslash Y$	-1	0	1
0	0.1	0.1	0
1	0.2	0.2	0.1
2	0.1	0.1	0.1

$Z = \max(X,Y) - \min(X,Y) = |X - Y|$

wartości Z

$X \backslash Y$	-1	0	1
0	1	0	1
1	2	1	0
2	3	2	1

$S_Z = \{0, 1, 2, 3\}$

$P(Z=0) = p(0,0) + p(1,1) = 0.1 + 0.1 = 0.2$

$P(Z=1) = p(0,-1) + p(1,0) + p(0,1) + p(2,1) = 0.1 + 0.2 + 0.1 + 0 = 0.4$

$P(Z=2) = p(1,-1) + p(2,0) = 0.2 + 0.1 = 0.3$

$P(Z=3) = p(2,-1) = 0.1$

$(U, V) = (|X - Y|, X^2 + Y^2)$

wartości (U,V)

$X \backslash Y$	-1	0	1
0	$(0,1)$	$(0,0)$	$(0,1)$
1	$(1,2)$	$(0,1)$	$(1,2)$
2	$(2,5)$	$(0,4)$	$(2,5)$

$S_{UV} = \{(0,0), (0,1), (1,2), (0,4), (2,5)\}$

$P(U=0, V=0) = p(1,0) = 0.1$

$P(U=0, V=1) = p(0,-1) + p(0,1) + p(1,0) = 0.1 + 0.2 + 0 = 0.3$

$P(U=1, V=2) = p(1,-1) + p(1,1) = 0.2 + 0.1 = 0.3$

$P(U=0, V=4) = p(2,0) = 0.1$

$P(U=2, V=5) = p(2,-1) + p(2,1) = 0.1 + 0.1 = 0.2$

$U \backslash V$	0	1	2	4	5
0	0.1	0.3	0	0.1	0
1	0	0	0.3	0	0
2	0	0	0	0	0.2

3.

$D = [0,2]\times[0,2] \qquad |D| = 4$

$(X,Y) \sim U(D) \qquad$ P geometryczne

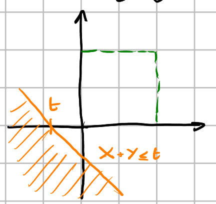
$f_{XY}(x,y) = \frac{1}{4}\cdot \mathbb{1}_D(x,y)$

$Z = X+Y \qquad S_Z = [0,4]$

$F_Z(t) = P(Z \le t) = P(X+Y \le t) = \iint\limits_{x+y\le t} f(x,y)\,dx\,dy$

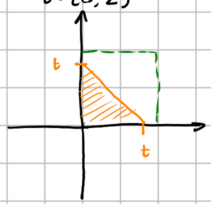
$1 - \frac{1}{8}(16 - 8t + t^2) = 1 - 2 + t - \frac{1}{8}t^2 \to 1 - \frac{1}{4}t$

$t<0$



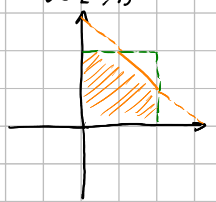
$F_Z(t) = 0$

$t\in[0,2)$



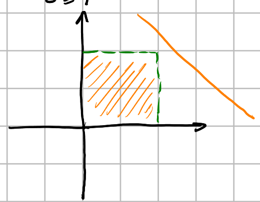
$F_Z(t) = \frac{\frac{1}{2}t^2}{4} = \frac{1}{8}t^2$

$t\in[2,4)$



$F_Z(t) = \frac{4 - \frac{1}{2}(4-t)^2}{4}$

$t \ge 4$



$F_Z(t) = 1$

$$F_Z(t) = \begin{cases} 0 & t<0 \\ \frac{1}{8}t^2 & t\in[0,2) \\ 1 - \frac{1}{8}(4-t)^2 & t\in[2,4) \\ 1 & t\ge 4 \end{cases}$$

$$f_Z(t) = \frac{d}{dt}F_Z(t) = \begin{cases} \frac{1}{4}t & t\in[0,2) \\ 1 - \frac{1}{4}t & t\in[2,4] \\ 0 & \text{w p.p.} \end{cases}$$

4.

X i Y niezależne

$X, Y \sim \exp(\lambda = 1)$

$$f_X(t) = f_Y(t) = e^{-t} \cdot \mathbb{1}_{[0,+\infty)}(t)$$

$$F(t) = F_X(t) = F_Y(t) = (1 - e^{-t}) \cdot \mathbb{1}_{[0,+\infty)}(t)$$

$$Z = \min(X, Y)$$

$$F_Z(t) = P(Z \leq t) = P(\min(X, Y) \leq t)$$

$$= 1 - P(\min(X, Y) > t)$$

$$= 1 - P(X > t, Y > t)$$

$$= 1 - P(X > t)P(Y > t)$$

$$= 1 - (1 - F(t))(1 - F(t))$$

$$= (1 - e^{-2t}) \cdot \mathbb{1}_{[0,+\infty)}(t)$$

5.

X i Y niezależne

$X \sim N(-2, 3) \qquad Y \sim N(2, 4)$

$Z = X - Y$

$F_Z(t) = P(X - Y \leq t)$

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 3}} \exp\left(-\frac{(x+2)^2}{6}\right) \cdot \frac{1}{\sqrt{2\pi \cdot 4}} \exp\left(-\frac{(y-2)^2}{8}\right)$$

$$= \frac{1}{2\pi\sqrt{12}} \exp\left(-\frac{1}{6}(x+2)^2 - \frac{1}{8}(y-2)^2\right)$$

$$(X, Y) \sim N\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\right)$$

Z to przekształcenie liniowe $(X, Y) \Rightarrow Z \sim N(m_Z, \sigma_Z^2)$

$m_Z = EZ = EX - EY = -4$

$\sigma_Z^2 = VZ = V(X - Y) = VX + VY - 2\operatorname{cov}(X, Y) = VX + VY = 7$

$Z \sim N(-4, 7)$

Alternatywnie

$$Z = X - Y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$Z \sim N(Am, ACA^T)$

$$Am = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -4$$

$$ACA^T = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 7$$

6.

$$f_{xy}(x,y) = \frac{1}{2\sqrt{2}\pi} \exp\left(-\frac{1}{2}\left(x^2 + 2x(y+1) + \frac{3}{2}(y+1)^2\right)\right) \qquad \det C = 2$$

$$= \frac{1}{2\pi\sqrt{2}} \exp\left(-\frac{1}{2\cdot 2}\left(2x^2 - 2\cdot(-2)\, x(y+1) + 3(y+1)^2\right)\right)$$

$$m = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$(Z, T) = (2X + Y + 1,\ 2X - Y - 1)$$

$$\begin{bmatrix} 2X + Y + 1 \\ 2X - Y - 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b}$$

$$m^* = Am + b = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C^* = ACA^T = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 10 & 22 \end{bmatrix}$$

$$(Z, T) \sim N(m^*, C^*) \qquad \det C^* = 32$$

$$f_{ZT}(z,t) = \frac{1}{2\pi\sqrt{32}} \exp\left(-\frac{1}{2\sqrt{32}}\left(22z^2 - 20zt + 6t^2\right)\right)$$