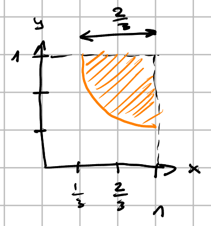
1.

$\Omega = \{(x,y)\in\mathbb{R}^2 : (x,y)\in[0,1]\times[0,1]\}$

P - prawdopodobieństwo geometryczne

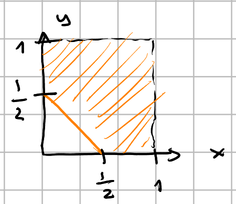
a) $A = \{(x,y)\in\Omega : (x-1)^2+(y-1)^2 \le \frac{2}{3}\}$



$$P(A) = \frac{|A|}{|\Omega|} = \frac{\frac{1}{4}\cdot\pi\cdot\left(\frac{2}{3}\right)^2}{1^2} = \frac{\pi}{9}$$

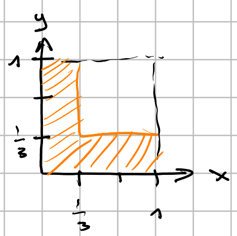
b) $B = \{(x,y)\in\Omega : x+y > \frac{1}{2}\}$

$y > -x + \frac{1}{2}$



$$P(B) = \frac{|B|}{|\Omega|} = \frac{1^2 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{1^2} = \frac{7}{8}$$

c) $C = \{(x,y)\in\Omega : \min(x,y) \le \frac{1}{3}\}$



$$P(C) = \frac{|C|}{|\Omega|} = \frac{1^2 - \left(\frac{2}{3}\right)^2}{1^2} = \frac{5}{9}$$

2.

P – prawdopodobieństwo klasyczne

K – kobieta 2000

K' – mężczyzna 1500

R – ogląda

R' – nie ogląda

$P(R|K) = \frac{8}{10}$ $P(R'|K) = \frac{2}{10}$

$P(R|K') = \frac{45}{100}$ $P(R'|K') = \frac{55}{100}$

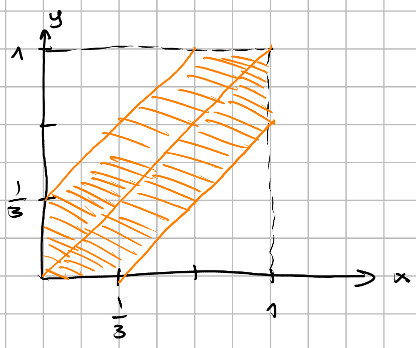
a) $P(R) = P(R|K)P(K) + P(R|K')P(K') = \frac{8}{10} \cdot \frac{2000}{3500} + \frac{45}{100} \cdot \frac{1500}{3500} = \frac{13}{20}$

b) $P(K'|R) = \frac{P(K' \cap R)}{P(R)} = \frac{P(R|K')P(K')}{P(R)} = \frac{\frac{45}{100} \cdot \frac{1500}{3500}}{\frac{13}{20}} = \frac{27}{91}$

3.

$\Omega = \{(x,y) \in [0,1] \times [0,1]\}$

x, y – czas przyjścia do punktu startowego między 7:30 a 7:45



$A = \{(x,y) \in \Omega : y \in [x, x+\frac{1}{3}] \vee x \in [y, y+\frac{1}{3}]\}$

$y \in [x, x+\frac{1}{3}] \vee x \in [y, y+\frac{1}{3}]$

$x \leq y \leq x + \frac{1}{3} \quad \vee \quad y \leq x \leq y + \frac{1}{3}$

$(y \geq x \wedge y \leq x + \frac{1}{3}) \vee (y \leq x \wedge y \geq x - \frac{1}{3})$

P – prawdopodobieństwo geometryczne

$P(A) = \frac{|A|}{|\Omega|} = \frac{1^2 - 2 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3}}{1^2} = \frac{5}{9}$

4.

$\Omega = \{(a, b) \in [-1, 1] \times [-1, 1]\}$

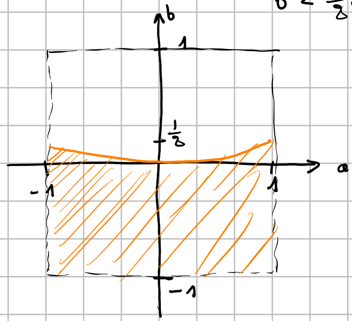
P – p. geometryczne

a) A – $x^2 + ax + 2b$ ma 2 różne pierwiastki rzeczywiste

$\Delta = a^2 - 4 \cdot 1 \cdot 2b = a^2 - 8b$

$A = \{(a,b) \in \Omega : a^2 - 8b > 0\}$

$b < \frac{1}{8}a^2$



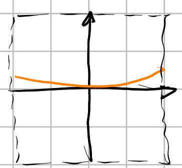
$|A| = 2 \cdot 1 + \int_{-1}^{1} \frac{1}{8}a^2 \, da = 2 + \frac{1}{8} \cdot \left[\frac{1}{3}a^3\right]_{-1}^{1} = 2 + \frac{1}{8}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right)$

$= 2 + \frac{1}{8} \cdot \frac{2}{3} = \frac{25}{12}$

$P(A) = \frac{|A|}{|\Omega|} = \frac{\frac{25}{12}}{4} = \frac{25}{48}$

b) B – $x^2 + ax + 2b$ ma 1 podwójny pierwiastek rzeczywisty

$B = \{(a,b) \in \Omega : a^2 - 8b = 0\}$



$P(B) = \frac{|B|}{|\Omega|} = \frac{0}{4} = 0$

5.

D – urządzenie ma defekt

W – metoda informuje o defekcie

$P(W|D) = 80\% \implies P(W'|D) = 20\%$

$P(W|D') = 1\% \implies P(W'|D') = 99\%$

$P(D) = 1\% \implies P(D') = 99\%$

$$P(D|W) = \frac{P(D \cap W)}{P(W)} = \frac{P(W|D)P(D)}{P(W)} = \frac{P(W|D)P(D)}{P(W|D)P(D) + P(W|D')P(D')}$$

$$= \frac{0.8 \cdot 0.01}{0.8 \cdot 0.01 + 0.01 \cdot 0.99} \simeq 0.45$$

6.

$\Omega = \{O, R\} \times \{1,2,3,4,5,6\}$

Prawdopodobieństwo klasyczne $\forall_{\omega \in \Omega} P(\omega) = \frac{1}{|\Omega|}$ $\quad |\Omega| = 12$

$$X(\omega) = \begin{cases} 3 & \omega = (R, 1) \\ 1 & \omega \in \{O\}\times\{1,2,3,4,5,6\} \cup \{R\}\times\{2,4,6\} \\ -3 & \text{w p.p.} \end{cases}$$

$S_X = \{-3, 1, 3\}$

i	3	1	-3
p_i	$\frac{1}{12}$	$\frac{9}{12}$	$\frac{2}{12}$

$EX = 3 \cdot \frac{1}{12} + 1 \cdot \frac{9}{12} - 3 \cdot \frac{2}{12} = \frac{3+9-6}{12} = \frac{1}{2}$

$EX^2 = 9 \cdot \frac{1}{12} + 1 \cdot \frac{9}{12} + 9 \cdot \frac{2}{12} = \frac{9+9+18}{12} = \frac{36}{12} = 3$

$VX = EX^2 - (EX)^2 = 3 - \frac{1}{4} = \frac{11}{4}$

$V(4X - 10) = 4^2 VX = 16 \cdot \frac{11}{4} = 44$

7.

$$f_X(x) = \frac{1}{3} \cdot \mathbb{1}_{[-1,0]\cup[1,3]}(x)$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx = \begin{cases} 0 & t < -1 \\ \frac{1}{3}(t+1) & t \in [-1,0] \\ \frac{1}{3} & t \in (0,1) \\ \frac{1}{3} + \frac{1}{3}(t-1) & t \in [1,3] \\ 1 & t > 3 \end{cases}$$

$$|x-1| > x$$

$$\underbrace{x < 1}: \quad -x+1 > x, \quad 1 > 2x, \quad x < \frac{1}{2}$$

$$\underbrace{x \geq 1}: \quad x-1 > x, \quad -1 > 0$$

$$P(|X-1| > X) = P((X \geq 1 \wedge X-1 > X) \vee (X < 1 \wedge -X+1 > X))$$

$$= P(X \geq 1 \wedge -1 > 0) + P(X < 1 \wedge X < \tfrac{1}{2})$$

$$= P(X < \tfrac{1}{2})$$

$$= F_X(\tfrac{1}{2}) = \frac{1}{3}$$

8.

$$f_X(x) = a(x-1)\cdot \mathbb{1}_{[0,1]}(x)$$

1. $\int_{-\infty}^{+\infty} f_X(x)dx = 1$

2. $\forall_{x\in\mathbb{R}}\ f_X(x) \geq 0$

$$\int_{-\infty}^{+\infty} f_X(x)dx = \int_0^1 a(x-1)\,dx = a\left[\frac{1}{2}x^2 - x\right]\Big|_0^1 = a\left(\frac{1}{2}-1-0\right) = -\frac{1}{2}a \implies a=-2$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)dx = \begin{cases} 0 & t<0 \\ \int_0^t -2(x-1)dx = -2\left[\frac{1}{2}x^2-x\right]\Big|_0^t = -2\cdot\frac{1}{2}t^2+2t = 2t-t^2 & t\in[0,1] \\ 1 & t>1 \end{cases}$$

$$EX = \int_{-\infty}^{+\infty} x\, f_X(x)dx = \int_0^1 -2\cdot x\cdot(x-1)\,dx = -2\int_0^1 x^2-x\,dx = -2\left[\frac{1}{3}x^3-\frac{1}{2}x^2\right]\Big|_0^1 = -2\left(\frac{1}{3}-\frac{1}{2}\right) = \frac{1}{3}$$

9.

$$F_X(x) = \begin{cases} 0 & x < -2 \\ ax + b & -2 \le x < 0 \\ \frac{1}{3} & 0 \le x < 1 \\ cx + d & 1 \le x < 2 \\ 1 & x \geq 2 \end{cases}$$

F_X jest ciągła

$F_X(-2) = F_X(-2^-) \implies -2a + b = 0$

$F_X(0) = F_X(0^-) \implies \frac{1}{3} = b$

$F_X(1) = F_X(1^-) \implies c + d = \frac{1}{3}$

$F_X(2) = F_X(2^-) \implies 1 = 2c + d$

$a = \frac{1}{6}$
$b = \frac{1}{3}$
$c = \frac{2}{3}$
$d = -\frac{1}{3}$

$$P(|X| > \tfrac{1}{2}) = P(X > \tfrac{1}{2} \vee X < -\tfrac{1}{2}) = P(X > \tfrac{1}{2}) + P(X < -\tfrac{1}{2})$$

$$= 1 - P(X < \tfrac{1}{2}) + P(X < -\tfrac{1}{2})$$

$$= 1 - F_X(\tfrac{1}{2}) + F_X(-\tfrac{1}{2})$$

$$= 1 - \cancel{\tfrac{1}{3}} + \tfrac{1}{6} \cdot (-\tfrac{1}{2}) + \cancel{\tfrac{1}{3}} = \tfrac{11}{12}$$

10.

$\Omega = \{(a,b) : a, b \in [0,6]\}$

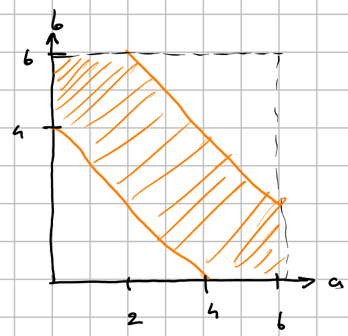
P – p. geometryczne

a) $A = \{(a,b) \in \Omega : \frac{a+b}{2} \in [2,4]\}$

$2 \leq \frac{a+b}{2} \leq 4$

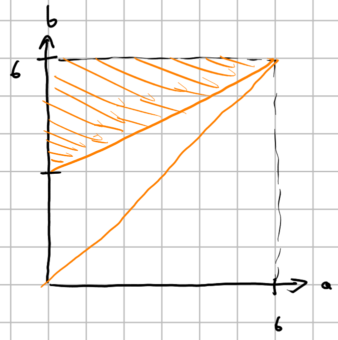
$4 \leq a+b \leq 8$

$-a+4 \leq b \leq -a+8$



$P(A) = \frac{|A|}{|\Omega|} = \frac{6^2 - 2\cdot\frac{1}{2}\cdot 4\cdot 4}{6^2} = \frac{20}{36} = \frac{5}{9}$

b) $B = \{(a,b) \in \Omega : |a-b| > |6-b|\}$



$|a-b| > |6-b|$

$|a-b| > 6-b$

$a \geq b$:

$a-b > 6-b$

$a > 6$

$a < b$:

$-a+b > 6-b$

$2b > a+6$

$b > \frac{1}{2}a + 3$

$P(B) = \frac{|B|}{|\Omega|} = \frac{\frac{1}{2}\cdot 3\cdot 6}{6^2} = \frac{1}{4}$

11.

$$F_X(x) = \begin{cases} 0 & x < -3 \\ a & -3 \le x < -1 \\ bx + c & -1 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$P(X = -3) = P(X = -1) = P(X = 1) = \frac{1}{4}$$

punkty skokowe $-3, -1, 1$

$$P(X=-3) = F_X(-3) - F_X(-3^-) \Longrightarrow \frac{1}{4} = a - 0 = a$$

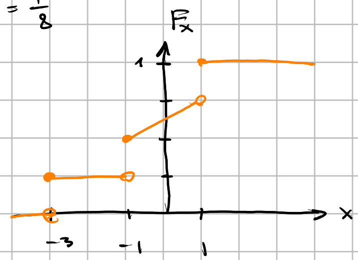
$$P(X=-1) = F_X(-1) - F_X(-1^-) \Longrightarrow \frac{1}{4} = -b + c - \frac{1}{4} \Longrightarrow c = b + \frac{1}{2}$$

$$P(X=1) = F_X(1) - F_X(1^-) \Rightarrow \frac{1}{4} = 1 - b - c \Rightarrow c = \frac{3}{4} - b$$

$$b + \frac{1}{2} = \frac{3}{4} - b \Rightarrow 2b = \frac{1}{4} \Rightarrow b = \frac{1}{8}$$

$$c = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$a = \frac{1}{4}$$



$$P(-2 \le X < 1) = F_X(1^-) - F_X(-2^-) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$1 \times K_1$
$2 \times K_2$
$3 \times K_3$

$P(O) = \frac{1}{3}$

Tyle rzutów monetą ile cyfr na kuli

$(O_k | C_n) \sim B(n, \frac{1}{3})$
$P(O_k | C_n) = \binom{n}{k} p^k (1-p)^{n-k}$

C_i - wylosujemy kulę oznaczoną cyfrą i
A - wypadną co najmniej 2 orły = 2 lub 3

$P(A) = P(O_2) + P(O_3)$

$P(O_2) = P(O_2 | C_1) \cdot P(C_1) + P(O_2 | C_2) P(C_2) + P(O_2 | C_3) P(C_3)$

$P(O_2) = 0 + \binom{2}{2}(\frac{1}{3})^2(\frac{2}{3})^0 \cdot \frac{2}{6} + \binom{3}{2}(\frac{1}{3})^2(\frac{2}{3})^1 \cdot \frac{3}{6} = \frac{1}{9} \cdot \frac{2}{6} + \cancel{3} \cdot \frac{1}{9} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} = \frac{1}{27} + \frac{1}{9} = \frac{4}{27}$

$P(O_3) = P(O_3 | C_1) P(C_1) + P(O_3 | C_2) P(C_2) + P(O_3 | C_3) P(C_3)$

$P(O_3) = 0 + 0 + \binom{3}{3}(\frac{1}{3})^3(\frac{2}{3})^0 \cdot \frac{3}{6} = \frac{1}{27} \cdot \frac{1}{2} = \frac{1}{54}$

$P(A) = \frac{4}{27} + \frac{1}{54} = \frac{9}{54} = \frac{1}{6}$

B - wypadnie dokładnie 1 reszka $= O_2$

$P(C_3 | B) = \frac{P(C_3 \cap B)}{P(B)} = \frac{P(B | C_3) P(C_3)}{P(B)}$

$P(B | C_3) P(C_3) = \binom{3}{1}(\frac{2}{3})^1(\frac{1}{3})^2 \cdot \frac{3}{6} = \cancel{3} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{9} \cdot \frac{1}{\cancel{2}} = \frac{1}{9}$

$P(B) = P(O_2) = \frac{4}{27}$

$P(C_3 | B) = \frac{\frac{1}{9}}{\frac{4}{27}} = \frac{3}{4}$

rozkład ciągły $a, b \in \mathbb{R}$

$$f_X(x) = \begin{cases} a|x| & -1 \le x \le 2 \\ bx & 3 \le x \le 4 \\ 0 & \text{w p.p.} \end{cases} = \begin{cases} -ax & -1 \le x < 0 \\ ax & 0 \le x \le 2 \\ bx & 3 \le x \le 4 \\ 0 & \text{w p.p.} \end{cases}$$

$$P(X>2) = 0.7 \implies 0.7 = \int_2^{+\infty} f_X(x)\,dx = \int_3^4 bx\,dx = b \cdot \left[\tfrac{1}{2}x^2\right]_3^4 = b(8-4.5) = 3.5b \implies b = 0.2$$

$$1 = \int_{-\infty}^{+\infty} f_X(x)\,dx = \int_{-1}^{0} -ax\,dx + \int_0^2 ax\,dx + \int_3^4 0.2x\,dx$$

$$1 = -a \cdot \left[\tfrac{1}{2}x^2\right]_{-1}^{0} + a \cdot \left[\tfrac{1}{2}x^2\right]\Big|_0^2 + 0.7$$

$$0.3 = -a \cdot (0 - \tfrac{1}{2}) + a \cdot 2$$

$$0.3 = 2.5a$$

$$a = 0.12$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx = \begin{cases} 0 & t < -1 \\ \int_{-1}^{t} -0.12x\,dx = -0.12 \cdot \left[\tfrac{1}{2}x^2\right]_{-1}^{t} = -0.12\left(\tfrac{t^2}{2} - \tfrac{1}{2}\right) & -1 \le t < 0 \\ \int_{-1}^{0} -0.12x\,dx + \int_0^t 0.12x\,dx = 0.06 + 0.06t^2 & 0 \le t \le 2 \\ \int_{-1}^{0} -0.12x\,dx + \int_0^2 0.12x\,dx = 0.3 & 2 < t < 3 \\ 0.3 + \int_3^t 0.2x\,dx = 0.3 + 0.2\left(\tfrac{1}{2}t^2 - \tfrac{9}{2}\right) = 0.3 + 0.1(t^2 - 9) & 3 \le t \le 4 \\ 1 & t > 4 \end{cases}$$

$$P(|X-1|>1) = P(X-1>1 \lor X-1<-1) = P(X>2 \lor X<0) = P(X<0) + P(X>2)$$

$$= P(X<0) + 1 - P(X \le 2)$$

$$= F_X(0) + 1 - F_X(2)$$

$$= 0.06 + 1 - 0.3 = 0.76$$

14.

$\Omega = \{-1, 0, 1, 2, 3\}$

ω	-1	0	1	2	3
X	2	1	0	1	2
Y	0	1	2	1	0

$P(\{1\}) = \frac{1}{3}$ $\quad P(\{\omega\}) = \frac{1}{6}$ dla $\omega \in \{-1, 0, 2, 3\}$

$X(\omega) = |\omega - 1|$ $\quad Y(\omega) = 2 - |\omega - 1|$

$S_X = \{2, 1, 0\}$ $\quad S_Y = \{0, 1, 2\}$

$P(X=0) = P(\{1\}) = \frac{1}{3}$

$P(X=1) = P(\{0\}) + P(\{2\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P(X=2) = P(\{-1\}) + P(\{3\}) = \frac{1}{3}$

$P(Y=0) = P(\{-1\}) + P(\{3\}) = \frac{1}{3}$

$P(Y=1) = P(\{0\}) + P(\{2\}) = \frac{1}{3}$

$P(Y=2) = P(\{1\}) = \frac{1}{3}$

Mają taki sam rozkład

$P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}) = P(\{0, 2\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$V(3X + 10) = 9\,VX$

$VX = EX^2 - (EX)^2$

$EX^2 = 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{3} = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$

$EX = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = \frac{1}{3} + \frac{2}{3} = 1$

$VX = \frac{5}{3} - 1^2 = \frac{2}{3}$

$V(3X+10) = 9 \cdot \frac{2}{3} = 6$

15.

$\Omega = \{Z, N, B\}$

$P(\{Z\}) = \frac{2}{4}$

$P(\{N\}) = P(\{B\}) = \frac{1}{4}$

X – liczba niebieskich ścian w 3 rzutach

$X \sim B(n=3, p=\frac{1}{4})$

$S_X = \{0, 1, 2, 3\}$

$P(X=k) = \binom{3}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{3-k}$

$P(X=0) = \frac{27}{64}$

$P(X=1) = 3 \cdot \frac{1}{4} \cdot \frac{9}{16} = \frac{27}{64}$

$P(X=2) = 3 \cdot \frac{1}{16} \cdot \frac{3}{4} = \frac{9}{64}$

$P(X=3) = \frac{1}{64}$

$V(4X-500) = 16\, VX = 16\left(EX^2 - (EX)^2\right)$

$EX = 0 \cdot \frac{27}{64} + 1 \cdot \frac{27}{64} + 2 \cdot \frac{9}{64} + 3 \cdot \frac{1}{64} = \frac{27+18+3}{64} = \frac{48}{64} = \frac{3}{4}$

$EX^2 = 0 \cdot \frac{27}{64} + 1 \cdot \frac{27}{64} + 4 \cdot \frac{9}{64} + 9 \cdot \frac{1}{64} = \frac{27+36+9}{64} = \frac{72}{64} = \frac{9}{8}$

$V(4X-500) = 16 \cdot \left(\frac{9}{8} - \frac{9}{16}\right) = 16 \cdot \frac{9}{16} = 9$

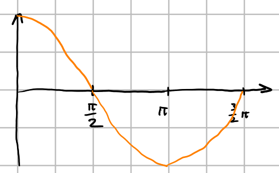
$\sigma_{4X-500} = \sqrt{9} = 3$

*

$[-1,1]$ - wybieramy 2 punkty a, b

X: liczba rozwiązań $\cos t = a+b$ dla $t \in [0, \frac{3}{2}\pi]$

znaleźć rozkład X

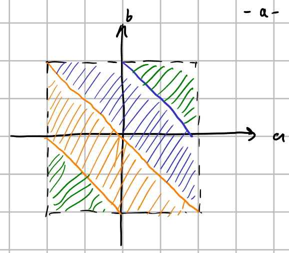


$$X = \begin{cases} 0 & \text{w p.p.} \\ 1 & a+b \in (0,1] \cup \{-1\} \\ 2 & a+b \in (-1, 0) \end{cases}$$

$\Omega = [-1,1]^2 \quad |\Omega| = 4$

$P(X=2) = P(\{(a,b) \in \Omega : \ -1 < a+b < 0\})$

$-a-1 < b < -a$

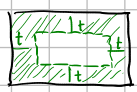


$P(X=1) = P(\{(a,b) \in \Omega : \ 0 < a+b < 1\})$

**

Prostokąt 2×4, losujemy punkt w środku

T - odległość punktu od najbliższego boku



$$P(T \le t) = \frac{8 - (4-2t)(2-2t)}{8}$$

***

Autobusy A: 10 - 11
B: 10 - 12

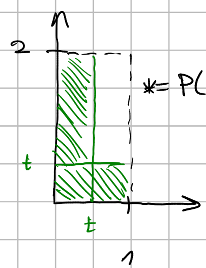
T czas oczekiwania na autobus

$\Omega = \{(a,b) \in [0,1] \times [0,2]\}$

$T(a,b) = \min(a,b) \in [0,1]$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ * & t \in [0,1] \\ 1 & t > 1 \end{cases}$$

$T \le t = \min(a,b) \le t$



$$* = P(T \le t) = \frac{2 - (2-t)(1-t)}{2}$$

Kol 12A - 1

1.

$2\times B,\ 3\times C$

X – liczba czarnych kul spośród 3 wyjętych bez zwracania
P – klasyczne

$S_X=\{1,2,3\}$ $\qquad \binom{5}{3}=\frac{5!}{2!3!}=\frac{5\cdot 4}{2}=10$

$$P(X=1)=\frac{\binom{2}{2}\binom{3}{1}}{\binom{5}{3}}=\frac{3}{10}$$

$$P(X=2)=\frac{\binom{2}{1}\binom{3}{2}}{\binom{5}{3}}=\frac{2\cdot 3}{10}=\frac{6}{10}$$

$$P(X=3)=\frac{\binom{2}{0}\binom{3}{3}}{\binom{5}{3}}=\frac{1}{10}$$

Y – ile razy wypadły 4 oczka w n rzutach kostką czworościenną

$$P(X=1\mid Y=1)=\frac{P(Y=1\mid X=1)\,P(X=1)}{P(Y=1)}$$

$$P(Y=1)=P(Y=1\mid X=1)P(X=1)+P(Y=1\mid X=2)P(X=2)+P(Y=1\mid X=3)P(X=3)$$

$$=\binom{1}{1}\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^0\cdot\frac{3}{10}+\binom{2}{1}\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^1\cdot\frac{6}{10}+\binom{3}{1}\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2\cdot\frac{1}{10}$$

$$=\frac{1}{4}\cdot\frac{3}{10}+2\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{6}{10}+3\cdot\frac{1}{4}\cdot\frac{9}{16}\cdot\frac{1}{10}$$

$$=\frac{3}{40}+\frac{9}{40}+\frac{27}{640}=\frac{48+144+27}{640}=\frac{219}{640}$$

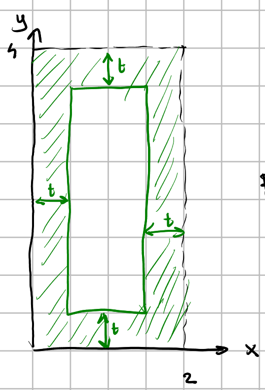
$$P(X=1\mid X=1)=\frac{3}{40}\cdot\frac{640}{219}=\frac{48}{219}$$

2.

$\Omega=\{(x,y)\in[0,2]\times[0,4]\}$

T – odległość punktu od najbliższego boku prostokąta

$$\frac{1}{8}\left(8-(8-4t-8t+4t^2)\right)=\frac{1}{8}(8-8+12t-4t^2)$$
$$=\frac{1}{8}(12t-4t^2)=\frac{1}{2}(3t-t^2)$$
$$\frac{d}{dt}=\frac{1}{2}\cdot(3-2t)=\frac{3}{2}-t$$



$$F_T(t)=P(T\le t)=\begin{cases}0 & t<0\\ \dfrac{8-(2-2t)(4-2t)}{8} & 0\le t\le 1\\ 1 & t>1\end{cases}$$

P – geometryczne

$$f_T(t)=\frac{d}{dt}F_T(t)=\begin{cases}\frac{3}{2}-t & t\in[0,1]\\ 0 & \text{w p.p.}\end{cases}$$

$$ET=\int_{-\infty}^{+\infty}t\,f_T(t)=\int_0^1 t\left(\tfrac{3}{2}-t\right)dt=\int_0^1 -t^2+\tfrac{3}{2}t\,dt=\left[-\tfrac{1}{3}t^3+\tfrac{3}{4}t^2\right]_0^1=-\frac{1}{3}+\frac{3}{4}=\frac{5}{12}$$

3.

T = temperatura

$T \sim N(20, \sigma^2)$

$P(T > 30) = 15\% \implies P(T \leq 30) = F_T(30) = 85\%$

$P(T < 15) = ?$

$F_T(a) = \Phi\left(\frac{a-m}{\sigma}\right)$

$85\% = \Phi\left(\frac{30-20}{\sigma}\right) = \Phi\left(\frac{10}{\sigma}\right)$

$\Phi(1.04) \simeq 0.85 \implies \frac{10}{\sigma} \simeq 1.04 \implies -\frac{5}{\sigma} \simeq -0.52$

$P(T < 15) = F_T(15) = \Phi\left(\frac{15-20}{\sigma}\right) = \Phi\left(-\frac{5}{\sigma}\right) \simeq \Phi(-0.52)$

$= 1 - \Phi(0.52) \simeq 30\%$

4.

$X \sim$ rozkład dyskretny $\implies F_X$ ma postać "schodkową"

$\implies a = c = 0$

$$F_X = \begin{cases} 0 & x < 0 \\ b & x \in [0, 1) \\ \frac{3}{4} & x \in [1, 2) \\ d & x \in [2, 3) \\ 1 & x \geq 3 \end{cases}$$

$EX = 1$

$4\,P(X=0) = P(X<3)$

$S_X = \{0, 1, 2, 3\}$

i	0	1	2	3
p_i	b	$\frac{3}{4} - b$	$4b - \frac{3}{4}$	$1 - 4b$

$P(X=0) = F_X(0) - F_X(0^-) = b$

$P(X<3) = F_X(3^-) = d$

$4b = d$

$1 = EX = 0 \cdot b + 1 \cdot \left(\frac{3}{4} - b\right) + 2 \cdot \left(4b - \frac{3}{4}\right) + 3(1 - 4b)$

$1 = \frac{3}{4} - b + 8b - \frac{3}{2} + 3 - 12b$

$5b = \frac{5}{4} \qquad b = \frac{1}{4}$

$V(3 - 4X) = (-4)^2 VX = 16\, VX$

$EX^2 = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 9 \cdot 0 = \frac{1}{2} + 1 = \frac{3}{2}$

$V(3 - 4X) = 16\left(EX^2 - (EX)^2\right) = 16 \cdot \left(\frac{3}{2} - 1^2\right) = 8$

kol 12 B

1.

Urna 1	Urna 2
$3 \times B$	$2 \times B$
$1 \times C$	$3 \times C$

C_1 – przełożona czarna kula

C_1' – przełożona biała kula

X – liczba czarnych kul wylosowanych z 2 urny

$$P(X=2) = P(X=2 \mid C_1)P(C_1) + P(X=2 \mid C_1')P(C_1')$$

$$= \binom{3}{2}\left(\frac{4}{6}\right)^2\left(\frac{2}{6}\right)^1 \cdot \frac{1}{4} + \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) \cdot \frac{3}{4}$$

$$= \cancel{3} \cdot \frac{4}{9} \cdot \frac{1}{\cancel{3}} \cdot \frac{1}{\cancel{4}} + 3 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{3}{4}$$

$$= \frac{1}{9} + \frac{9}{32}$$

$$P(C_1 \mid X=2) = \frac{P(X=2 \mid C_1)P(C_1)}{P(X=2)} = \frac{\frac{1}{9}}{\frac{1}{9} + \frac{9}{32}} \approx 0.28$$

2.

$$\Omega = \{(a,b) : a, b \in [-1, 1]\}$$

X – liczba dodatnich pierwiastków równania

$$x^2 - (2a+b)x + a(a+b) = 0$$

$$\Delta = (2a+b)^2 - 4a(a+b) = 4a^2 + 4ab + b^2 - 4a^2 - 4ab = b^2$$

$X = 2 \iff \Delta > 0 \wedge x_1 + x_2 > 0 \wedge x_1 x_2 > 0$

$x_1 + x_2 = 2a + b$

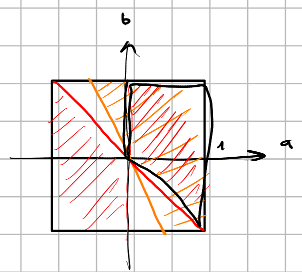
$X = 1 \iff (\Delta > 0 \wedge x_1 x_2 < 0)$
$\vee (\Delta = 0 \wedge x_0 > 0)$

$x_1 x_2 = a(a+b)$

$x_0 = \frac{2a+b}{2}$

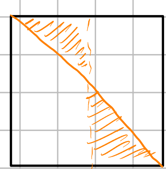
$S_X = \{0, 1, 2\}$

$$X = 2 \quad \begin{cases} b^2 > 0 \iff b \neq 0 \\ 2a + b > 0 \iff b > -2a \\ a(a+b) > 0 \iff (a > 0 \wedge b > -a) \vee (a < 0 \wedge b < -a) \end{cases}$$



$$P(X=2) = \frac{\frac{3}{2}}{4} = \frac{3}{8}$$

$$X = 1 \quad \begin{cases} b^2 > 0 \\ a(a+b) < 0 \end{cases} \iff \begin{cases} b \neq 0 \\ (a < 0 \wedge b > -a) \vee (a > 0 \wedge b < -a) \end{cases}$$



$$\begin{cases} b^2 = 0 \\ a + \frac{1}{2}b < 0 \end{cases} \iff \begin{cases} b = 0 \\ b < -2a \end{cases}$$

$$P(X=1) = \frac{1}{4} + 0 = \frac{1}{4}$$

$$P(X=0) = 1 - \frac{3}{8} - \frac{1}{4} = \frac{3}{8}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{8} & x \in [0, 1) \\ \frac{5}{8} & x \in [1, 2) \\ 1 & x \geq 2 \end{cases}$$

$$EX = 0 \cdot \frac{3}{8} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} = 1$$

3.

T – czas przejazdu

$T \sim N(50, \sigma^2)$

$P(T \le 60) = 0.9 = F_T(60) = \Phi\left(\frac{60-50}{\sigma}\right) = \Phi\left(\frac{10}{\sigma}\right)$

$\Phi(1.28) \simeq 0.9 \implies \frac{10}{\sigma} \simeq 1.28 \implies -\frac{5}{\sigma} \simeq -0.64$

$P(T > 55) = 1 - P(T \le 55) = 1 - F_T(55) = 1 - \Phi\left(\frac{55-60}{\sigma}\right) \simeq 1 - \Phi(-0.64)$
$= 1 - (1 - \Phi(0.64)) \simeq 74\%$

4.

X ~ rozkład ciągły

$$F(x) = \begin{cases} 0 & x < 0 \\ ax^2 + b & x \in [0,1) \\ \frac{3}{4} & x \in [1,2) \\ cx + d & x \in [2,3) \\ 1 & x \ge 3 \end{cases}$$

F jest ciągła

$F(0) = F(0^-) \implies 0 = a \cdot 0 + b \implies b = 0$

$F(1) = F(1^-) \implies \frac{3}{4} = a \cdot 1 + 0 \implies a = \frac{3}{4}$

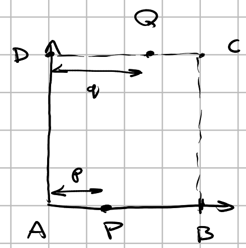
$\left.\begin{array}{l} F(2) = F(2^-) = 2c + d = \frac{3}{4} \\ F(3) = F(3^-) = 1 = 3c + d \end{array}\right\} \quad \begin{array}{l} c = \frac{1}{4} \\ d = \frac{1}{4} \end{array}$

$$F(x) = \begin{cases} 0 & \dots \\ \frac{3}{4}x^2 \\ \frac{3}{4} \\ \frac{1}{4}x + \frac{1}{4} \\ 1 \end{cases} \qquad f(x) = \frac{d}{dx}F = \begin{cases} 0 & \text{w p.p.} \\ \frac{3}{2}x & x \in [0,1) \\ \frac{1}{4} & x \in [2,3) \end{cases}$$

$$EX = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 \frac{3}{2}x^2 dx + \int_2^3 \frac{1}{4}x\,dx = \frac{1}{2}[x^3]_0^1 + \frac{1}{8}[x^2]_2^3 = \frac{1}{2} + 1 - \frac{1}{2} = 1$$

$P(|X - 1.5| > 0.5) = P(X - 1.5 > 0.5 \lor X - 1.5 < -0.5) = P(X > 2 \lor X < 1) = P(X > 2) + P(X < 1) = 1 - F(2) + F(1)$
$= 1 - \frac{3}{4} + \frac{3}{4} = 1$

ProbT 1



$P = (p, 1)$
$Q = (q, 1)$

2.

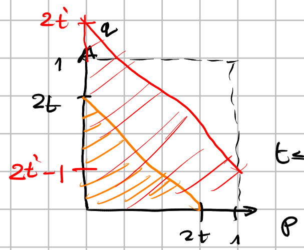
$\Omega = \{(p, q) : p, q \in [0, 1]\} \quad |\Omega| = 1$

P – geometryczne

X – pole $APQD$

$X(p, q) = \frac{1}{2}(p + q) \in [0, 1]$

$\frac{1}{2}p + \frac{1}{2}q \leq t$
$p + q \leq 2t$
$q \leq -p + 2t$



$F_X(t) = \begin{cases} 0 & t < 0 \\ 2t^2 & t \in [0, \frac{1}{2}] \\ -2t^2 + 4t - 1 & t \in (\frac{1}{2}, 1] \\ 1 & t > 1 \end{cases}$

$t \leq \frac{1}{2}$: $\quad P(X \leq t) = \frac{1}{2} \cdot (2t)^2 = 2t^2$

$f_X(t) = \frac{dF_X}{dt}$

$t > \frac{1}{2}$: $P(X \leq t) = 1 - \frac{1}{2}(1 - (2t - 1))^2$
$= 1 - \frac{1}{2}(2 - 2t)^2$
$= 1 - \frac{1}{2}(4 - 8t + 4t^2)$
$= 1 - 2 + 4t - 2t^2$
$= -2t^2 + 4t - 1$

$f_X(t) = \begin{cases} 4t & t \in [0, \frac{1}{2}] \\ -4t + 4 & t \in (\frac{1}{2}, 1] \\ 0 & \text{w p.p.} \end{cases}$

$-2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} - 1 = -\frac{1}{2} + 2 - 1 = \frac{1}{2}$