1.

$$z^5\,(\sqrt{3}+j)^3 = z^2\,(1+j)^8$$

a) dla $z=0$

$0=0$ ✓

b) dla $z\neq 0$

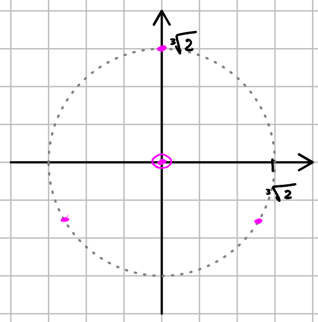
$$r^5e^{5\varphi j}\cdot\left(2e^{\frac{\pi}{6}j}\right)^3 = r^2e^{2\varphi j}\cdot\left(\sqrt{2}\,e^{\frac{\pi}{4}j}\right)^8$$

$$\frac{r^5e^{5\varphi j}}{r^2e^{2\varphi j}} = \frac{16\,e^{2\pi j}}{8\,e^{\frac{\pi}{2}j}} \qquad e^{2\pi j} = e^{0j}$$

$$r^3e^{3\varphi j} = 2e^{-\frac{\pi}{2}j}$$

$r^3=2$ $\qquad 3\varphi = -\frac{\pi}{2}+2k\pi$

$r=\sqrt[3]{2}$ $\qquad \varphi = \dfrac{-\frac{\pi}{2}+2k\pi}{3}$

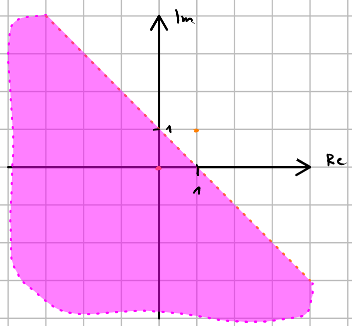


$$\left\{0,\ \sqrt[3]{2}\,e^{-\frac{\pi}{6}j},\ \sqrt[3]{2}\,e^{\frac{\pi}{2}j},\ \sqrt[3]{2}\,e^{-\frac{5\pi}{6}j}\right\}$$

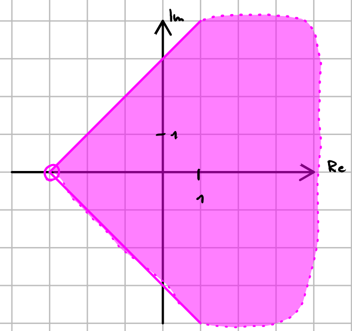
2.

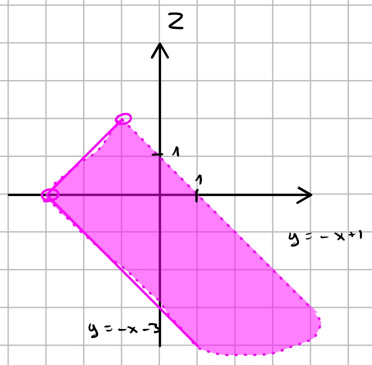
$$\left\{z\in\mathbb{C}:\ |z|<|z-1-j|\ \wedge\ \arg(z+3)\in\left[-\tfrac{\pi}{4},\tfrac{\pi}{4}\right]\right\} = Z$$

$|z|<|z-(1+j)|$



$\arg(z+3)\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$





1) $51-53j$ $\qquad -51+1=-50$

$-51-3=-54$

$-54\le -53 < -50$ ✓

$51-53j\in Z$

2) $67-66j$ $\qquad -67+1=-66$

$-67-3=-70$

$-70\le -66 \cancel{<} -66$

$67-66j\notin Z$

3) $77+76j\notin Z$ na podstawie rysunku (I ćwiartka)

4) $88-85j$ $\qquad -88+1=-87$

$-88-3=-91$

$-91\le -85 \cancel{<} -87$

$88-85j\notin Z$

3.

$$f(x) = \frac{-4x^3}{7(x+4)^2(x^4+16)} = \frac{-4x^3}{7(x+4)^2(x^2-2\sqrt{2}x+4)(x^2+2\sqrt{2}x+4)}$$

$$x^4+8x^2+16-8x^2 = (x^2+4)^2-8x^2 = (x^2-2\sqrt{2}x+4)(x^2+2\sqrt{2}x+4)$$

$\Delta = -8 < 0$ $\quad \Delta = -8 < 0$

nad $\mathbb{R}$

$$f(x) = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{Cx+D}{x^2-2\sqrt{2}x+4} + \frac{Ex+F}{x^2+2\sqrt{2}x+4}$$

$$\sqrt[4]{-16} = \sqrt[4]{16e^{\pi j}} = \left\{ 2e^{\frac{\pi+2k\pi}{4}j} : k=0,1,2,3 \right\} = \left\{ 2e^{\frac{\pi}{4}j}, 2e^{\frac{3\pi}{4}j}, 2e^{-\frac{\pi}{4}j}, 2e^{-\frac{3\pi}{4}j} \right\}$$

$$= \left\{ \sqrt{2}+\sqrt{2}j, -\sqrt{2}+\sqrt{2}j, \sqrt{2}-\sqrt{2}j, -\sqrt{2}-\sqrt{2}j \right\}$$

nad $\mathbb{C}$

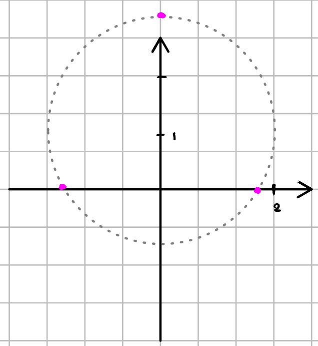
$$f(x) = \frac{-4x^3}{7(x+4)^2(x-(\sqrt{2}+\sqrt{2}j))(x-(-\sqrt{2}+\sqrt{2}j))(x-(\sqrt{2}-\sqrt{2}j))(x-(-\sqrt{2}-\sqrt{2}j))}$$

$$= \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{x-(\sqrt{2}+\sqrt{2}j)} + \frac{D}{x-(-\sqrt{2}+\sqrt{2}j)} + \frac{E}{x-(\sqrt{2}-\sqrt{2}j)} + \frac{F}{x-(-\sqrt{2}-\sqrt{2}j)}$$

4.

$$(z-j)^3 = (2-2j)^2$$
$$(z-j)^3 = (2\sqrt{2}e^{-\frac{\pi}{4}j})^2$$
$$(z-j)^3 = 8e^{-\frac{\pi}{2}j}$$
$$(z-j)^3 = (2e^{-\frac{\pi}{6}j})^3$$

$$z-j \in \sqrt[3]{(2e^{-\frac{\pi}{6}j})^3} = \left\{ 2e^{\frac{-\frac{\pi}{2}+2k\pi}{3}j} : k=0,1,2 \right\} = \left\{ 2e^{-\frac{\pi}{6}j}, 2e^{\frac{\pi}{2}j}, 2e^{-\frac{5\pi}{6}j} \right\}$$
$$z-j \in \{ \sqrt{3}-j, 2j, -\sqrt{3}-j \}$$
$$z \in \{ \sqrt{3}, 3j, -\sqrt{3} \}$$



5. ?

$$\left\{ z \in \mathbb{C} : \operatorname{Im}(z(1-j)) < \operatorname{Re}(\overline{2-7j}) \wedge \arg\left(z - \frac{1}{j^5}\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \right\} = Z$$

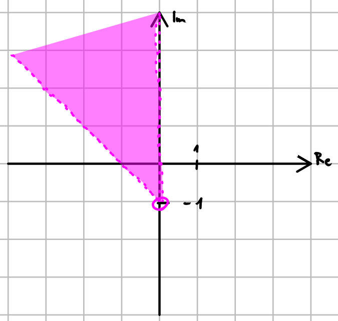
$$\operatorname{Im}(z(1-j)) = \operatorname{Im}(re^{\varphi j} \cdot \sqrt{2}e^{-\frac{\pi}{4}j}) = \operatorname{Im}(\sqrt{2}re^{(\varphi-\frac{\pi}{4})j})$$
$$= \operatorname{Im}(\sqrt{2}r(\cos(\varphi-\tfrac{\pi}{4}) + j\sin(\varphi-\tfrac{\pi}{4}))) = \sqrt{2}r\sin(\varphi-\tfrac{\pi}{4})$$

$$\operatorname{Re}(\overline{2-7j}) = \operatorname{Re}(2+7j) = 2$$

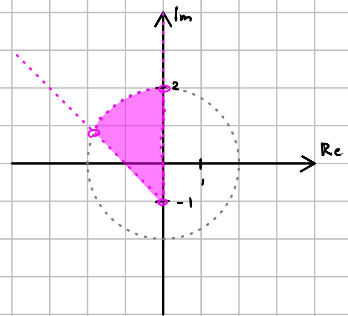
$$\sqrt{2}r\sin(\varphi-\tfrac{\pi}{4}) < 2$$
$r\sin(\varphi-\frac{\pi}{4}) < \sqrt{2}$ $\rightarrow$ dla każdego φ kiedy $r<\sqrt{2}$ bo $-1<\sin(\theta)<1$

$$\frac{1}{j^5} = \frac{1}{j^4 j} = \frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

$$\arg\left(z - \frac{1}{j^5}\right) = \arg(z+j)$$



$\arg(z+j) \in (\frac{\pi}{2}, \frac{3\pi}{4})$



Z

6. ?

$$f(x) = \frac{x^2+2x}{\underset{\Delta<0}{(x^2+4)}(x-2)}$$

nad $\mathbb{R}$

$$f(x) = \frac{x^2+2x}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2} = \frac{Ax(x-2)+B(x-2)+C(x^2+4)}{(x^2+4)(x-2)}$$

$$x^2+2x = Ax(x-2)+B(x-2)+C(x^2+4)$$

$$x=2 \longrightarrow 8 = 8C \iff C=1$$

$$x^2+2x = Ax^2-2Ax+Bx-2B+x^2+4$$

$$x^2+2x = (A+1)x^2+(B-2A)x+(4-2B)$$

$$\begin{cases} A+1=1 \\ B-2A=2 \\ 4-2B=0 \end{cases} \quad \begin{cases} A=0 \\ B=2 \\ B=2 \end{cases}$$

$$f(x) = \frac{2}{x^2+4} + \frac{1}{x-2}$$

nad $\mathbb{C}$

$$f(x) = \frac{x^2+2x}{(x^2+4)(x-2)} = \frac{x^2+2x}{(x+2j)(x-2j)(x-2)} = \frac{A}{x+2j} + \frac{B}{x-2j} + \frac{C}{x-2} = \frac{A(x-2j)(x-2)+B(x+2j)(x-2)+C(x+2j)(x-2j)}{(x+2j)(x-2j)(x-2)}$$

$$x^2+4 = (x+2j)(x-2j)$$

$$x^2+2x = A(x-2j)(x-2)+B(x+2j)(x-2)+C(x+2j)(x-2j)$$

$$x=-2j \longrightarrow -4-4j = A(-4j)(-2-2j)$$

$$-4-4j = A(8+8j)$$

$$A = \frac{-4-4j}{8+8j} \cdot \frac{8-8j}{8-8j} = \frac{-32+32j-32j-32}{64+64} = \frac{-64-64j}{128} = -\frac{1}{2}-\frac{1}{2}j$$

$$x=2j \longrightarrow -4+4j = B(4j)(-2+4j)$$

$$-1+j = B(-4-2j)$$

$$B = \frac{-1+j}{-4-2j} \cdot \frac{-4+2j}{-4+2j} = \frac{4-2j-4j-2}{16+4} = \frac{2-6j}{20} = \frac{1}{10}-\frac{3}{10}j$$

$$x=2 \longrightarrow 8 = C(2+2j)(2-2j)$$

$$8 = C(4+4)$$

$$C=1$$

$$f(x) = \frac{-\frac{1}{2}-\frac{1}{2}j}{x+2j} + \frac{\frac{1}{10}-\frac{3}{10}j}{x-2j} + \frac{1}{x-2}$$

7

$[\exists_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ (n-k<1)] \Rightarrow [\forall_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ n\neq k]$

$n<k+1$ $\quad v(RHS)=0$ bo dla $n=k$ nieprawda że $n\neq k$

$v(LHS)=1$

prawda dla $n=1$

$[\sim(p\Rightarrow q)] \Leftrightarrow [p \wedge \sim q]$

$v(1\Rightarrow 0)=0$

$\sim\Big[[\exists_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ (n-k<1)] \Rightarrow [\forall_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ n\neq k]\Big]$

$[\exists_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ (n-k<1)] \wedge \sim[\forall_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ n\neq k]$

$[\exists_{n\in\mathbb{N}}\ \forall_{k\in\mathbb{N}}\ (n-k<1)] \wedge [\exists_{n\in\mathbb{N}}\ \exists_{k\in\mathbb{N}}\ n=k]$

8.

$\forall_{k\in\mathbb{N}}\ \exists_{n\in\mathbb{N}}\ (k=3n \Rightarrow n=3k)$ $\qquad (p\Rightarrow q) \Leftrightarrow (\sim p \vee q)$

$\forall_{k\in\mathbb{N}}\ \exists_{n\in\mathbb{N}}\ (k\neq 3n \ \vee\ n=3k)$

prawda, bo dla każdej liczby naturalnej można znaleźć $3\cdot k$

$\sim[\forall_{k\in\mathbb{N}}\ \exists_{n\in\mathbb{N}}\ (k=3n \Rightarrow n=3k)]$ $\qquad \sim(p\Rightarrow q) \Leftrightarrow p \wedge \sim q$

$\exists_{k\in\mathbb{N}}\ \forall_{n\in\mathbb{N}}\ (k=3n \ \wedge\ n\neq 3k)$

9.

$\forall_{x\in\mathbb{R}}\ [(x>5 \Rightarrow x<2) \vee x>1]$

$\forall_{x\in\mathbb{R}}\ [(x\leq 5 \ \vee\ x<2) \vee x>1]$

$\forall_{x\in\mathbb{R}}\ [x\in(-\infty,2) \vee x\in(1,+\infty)]$

$\forall_{x\in\mathbb{R}}\ [x\in(-\infty,2)\cup(1,+\infty)]$

$\forall_{x\in\mathbb{R}}\ x\in\mathbb{R}$ tautologia

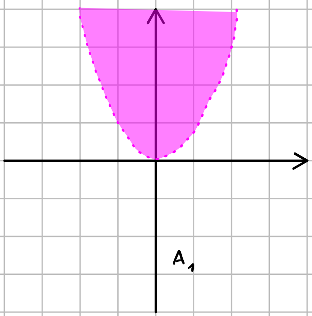
$\sim[\forall_{x\in\mathbb{R}}\ [(x>5 \Rightarrow x<2) \vee x>1]]$

$\exists_{x\in\mathbb{R}}\ [\sim(x>5 \Rightarrow x<2) \wedge \sim(x>1)]$

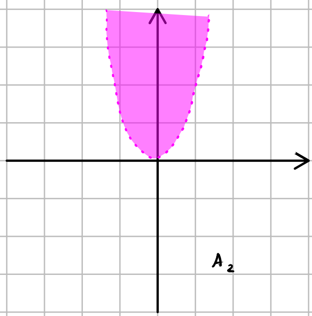
$\exists_{x\in\mathbb{R}}\ [(x>5 \ \wedge\ x\geq 2) \wedge\ x\leq 1]$

10.

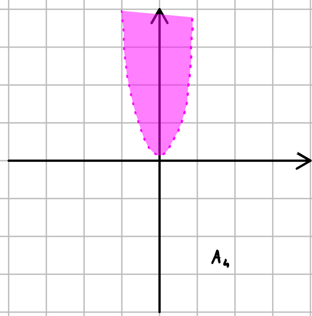
$A_n = \{ (x,y) \in \mathbb{R}^2 : y > nx^2 \}, \quad n \in \mathbb{N}$



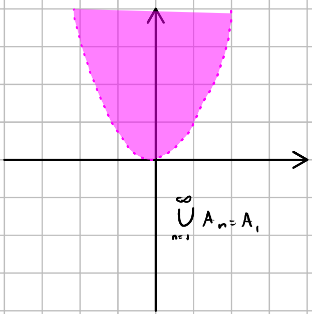
A_1



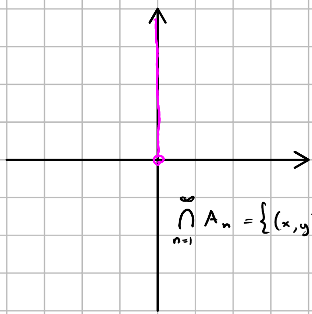
A_2



A_4



$\bigcup_{n=1}^{\infty} A_n = A_1$

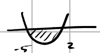


$\bigcap_{n=1}^{\infty} A_n = \{ (x,y) \in \mathbb{R} : x = 0 \wedge y > 0 \}$

11.

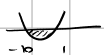
$A_n = \{ x \in \mathbb{R} : (x - \frac{2}{n})(x + 5n) < 0 \}, \quad n \in \mathbb{N}$

A_1 : $(x-2)(x+5) < 0$ $\quad (-5, 2)$



$\bigcup_{n=1}^{\infty} A_n = (-\infty, 2)$

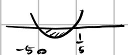
A_2 : $(x-1)(x+10) < 0$

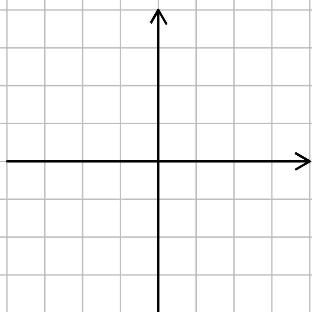


$(-10, 1)$

$\bigcap_{n=1}^{\infty} A_n = (-5, 0]$

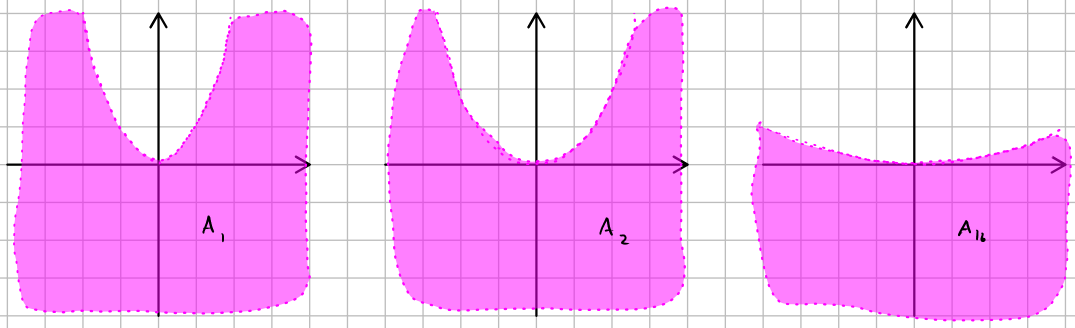
A_{10} : $(x - \frac{1}{5})(x+50) < 0$ $\quad (-50, \frac{1}{5})$

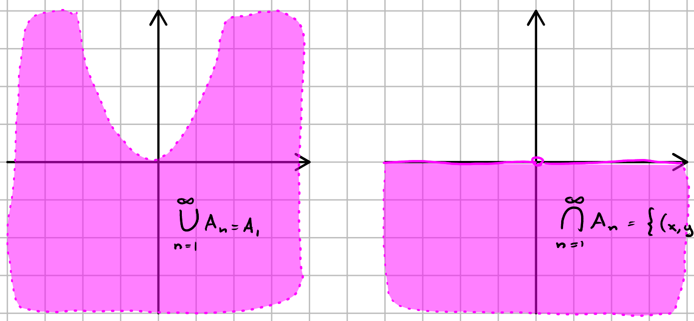




12.

$$A_n = \{(x,y) \in \mathbb{R}^2 : y < \frac{1}{n} x^2\}, \quad n \in \mathbb{N}$$





$$\bigcup_{n=1}^{\infty} A_n = A_1$$

$$\bigcap_{n=1}^{\infty} A_n = \{(x,y) \in \mathbb{R}^2 : y \leq 0\} \setminus \{(0,0)\}$$