1b

$$xy' = \tan(y) \qquad y\left(\tfrac{1}{2}\right) = \tfrac{5\pi}{6}$$

$$\frac{dy}{dx} = \frac{\tan(y)}{x} \to \int \frac{dy}{\tan(y)} = \int \frac{dx}{x}$$

$$\int \frac{dy}{\tan(y)} = \int \frac{\cos(y)}{\sin(y)}\,dy = \left| \begin{matrix} t = \sin(y) \\ dt = \cos(y)dy \end{matrix} \right| = \int \frac{dt}{t}$$

$$\ln|\sin(y)| = \ln|x| + C$$

$$\sin(y) = Cx$$

$$y \in \left[\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right]$$

$$\sin\left(\tfrac{5\pi}{6}\right) = C \cdot \tfrac{1}{2} \qquad y - \pi \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

$$\tfrac{1}{2} = \tfrac{1}{2}C \to C = 1 \qquad y = \pi - \arcsin(x)$$

$$\sin(y) = x$$

2a

$$y' - \frac{2x}{1+x^2}y = 1+x^2$$

$$\frac{dy}{dx} = \frac{2x}{1+x^2}y \to \int \frac{dy}{y} = \int \frac{2x}{x^2+1}dx$$

$$\int \frac{2x}{x^2+1}dx = \left| \begin{matrix} t = x^2+1 \\ dt = 2xdx \end{matrix} \right| = \int \frac{dt}{t} = \ln|x^2+1|$$

$$\ln|y| = \ln|x^2+1| + C$$

$$y = C(x^2+1)$$

$$y' = C'(x^2+1) + 2xC$$

$$C'(x^2+1) + 2xC - \frac{2x}{x^2+1} \cdot C(x^2+1) = x^2+1$$

$$C'(x^2+1) + 2xC - 2xC = x^2+1$$

$$C' = 1 \qquad C = \int 1dx = x + C$$

$$y(x) = (x+C)(x^2+1)$$

2c

$$y' + 4y = 5\sin(3x)$$

1° $y' + 4y = 0$

$$\frac{dy}{dx} = -4y \qquad \int \frac{dy}{y} = \int -4dx$$

$$\ln|y| = -4x + C$$

$$y = e^{-4x+C} = Ce^{-4x} \qquad \text{CORJ}$$

2° $0 + 3j \neq -4 \to y_1 = A\cos(3x) + B\sin(3x)$

$$y_1' = -3A\sin(3x) + 3B\cos(3x)$$

$$-3A\sin(3x) + 3B\cos(3x) + 4A\cos(3x) + 4B\sin(3x) = 5\sin(3x)$$

$$\begin{cases} 4A + 3B = 0 \\ -3A + 4B = 5 \end{cases} \qquad \begin{matrix} B = -\tfrac{4}{3}A \\ -3A - \tfrac{16}{3}A = 5 \\ -\tfrac{25}{3}A = 5 \quad A = -\tfrac{3}{5} \quad B = \tfrac{4}{5} \end{matrix}$$

$$y_1 = -\tfrac{3}{5}\cos(3x) + \tfrac{4}{5}\sin(3x) \qquad \text{CSRN}$$

$$y(x) = -\tfrac{3}{5}\cos(3x) + \tfrac{4}{5}\sin(3x) + Ce^{-4x} \qquad \text{CORN}$$

3b

$$y' - \frac{xy}{2(x^2-1)} = \frac{x}{2y} \qquad y(2) = \sqrt{3}$$

$$y' + p(x)y = q(x)\, y^{-1} \quad \rightarrow v = y^2$$

$$y = \sqrt{v}$$

$$y' = \frac{1}{2\sqrt{v}} v'$$

$$\frac{v'}{2\sqrt{v}} - \frac{x\sqrt{v}}{2(x^2-1)} = \frac{x}{2\sqrt{v}} \quad / \cdot 2\sqrt{v}$$

$$\frac{dv}{dx} - \frac{x}{x^2-1} v = x$$

1°

$$\frac{dv}{dx} = \frac{x}{x^2-1} v \quad \rightarrow \quad \int \frac{dv}{v} = \int \frac{x}{x^2-1} dx$$

$$\ln|v| = \frac{1}{2}\ln|x^2-1| + C$$

$$v = C\sqrt{x^2-1}$$

$$v' = C'\sqrt{x^2-1} + \frac{C}{2\sqrt{x^2-1}} \cdot 2x = C'\sqrt{x^2-1} + \frac{Cx}{\sqrt{x^2-1}}$$

$$C'\sqrt{x^2-1} + \cancel{\frac{Cx}{\sqrt{x^2-1}}} - \cancel{\frac{x}{x^2-1} \cdot C\sqrt{x^2-1}} = x$$

$$C' = \frac{x}{\sqrt{x^2-1}} \qquad C = \int \frac{x}{\sqrt{x^2-1}} dx = \left| \begin{matrix} t = x^2-1 \\ dt = 2x\,dx \end{matrix} \right| = \int \frac{1}{2\sqrt{t}} dt = \sqrt{t} + C = \sqrt{x^2-1} + C$$

$$y^2 = v(x) = (\sqrt{x^2-1} + C)\sqrt{x^2-1} = C\sqrt{x^2-1} + x^2 - 1$$

$$y(2) = \sqrt{3}$$

$$3 = C\sqrt{3} + 3 \quad \rightarrow \quad C = 0$$

$$y = \sqrt{x^2-1}$$

4a

$$y'' - 4y' = 8x \qquad y(0) = 1 \quad y'(0) = -1$$

1°

$$y'' - 4y' = 0$$

$$r^2 - 4r = 0 \qquad r(r-4) = 0$$

$$r_1 = 0 \quad r_2 = 4$$

$$y_0 = C_1 e^{0x} + C_2 e^{4x} = C_1 + C_2 e^{4x} \qquad \text{CORJ}$$

2°

$$y'' - 4y' = 8x$$

$0 + 0j$ jest 1 krotnym pierwiastkiem

$$y_1 = x(Ax + B) = Ax^2 + Bx$$

$$y_1' = 2Ax + B$$

$$y_1'' = 2A$$

3°

$$2A - 4(2Ax + B) = 8x$$

$$2A - 8Ax - 4B = 8x$$

$$\begin{cases} 8 = -8A & A = -1 \\ 0 = 2A - 4B & B = -\frac{1}{2} \end{cases}$$

$$y_1 = -x^2 - \frac{1}{2}x \qquad \text{CSRN}$$

4°

$$y = y_1 + y_0 = C_1 + C_2 e^{4x} - x^2 - \frac{1}{2}x \qquad \text{CORJ}$$

5°

$$y(0) = 1 \quad y'(0) = -1 \qquad y' = 4C_2 e^{4x} - 2x - \frac{1}{2}$$

$$\begin{cases} 1 = C_1 + C_2 & C_2 = -\frac{1}{8} \\ -1 = 4C_2 - \frac{1}{2} & C_1 = \frac{9}{8} \end{cases}$$

CSRN $y = \frac{9}{8} - \frac{1}{8}e^{4x} - x^2 - \frac{1}{2}x$

4b

$$y'' - 2y' + y = 4\sin^2\left(\frac{x}{2}\right) \qquad y(0)=2 \quad y'(0)=1$$

1°

$$y'' - 2y' + y = 0$$
$$r^2 - 2r + 1 = (r-1)^2 = 0$$
$$r_0 = 1 \rightarrow y_0 = C_1 e^x + C_2 x e^x$$

2°

$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$$
$$4\sin^2\left(\frac{x}{2}\right) = 4 \cdot \frac{1}{2}(1-\cos(x)) = 2 - 2\cos(x)$$

$\alpha + \beta j = 0 + j$ nie jest pierwiastkiem

$$y_1 = A\cos(x) + B\sin(x) + C$$
$$y_1' = B\cos(x) - A\sin(x)$$
$$y_1'' = -A\cos(x) - B\sin(x)$$

3°

$$\cancel{-A\cos(x)} - \cancel{B\sin(x)} - 2B\cos(x) + 2A\sin(x) + \cancel{A\cos(x)} + \cancel{B\sin(x)} + C = 2 - 2\cos(x)$$

$$\begin{cases} 0 = 2A & A = 0 \\ -2 = -2B & B = 1 \\ 2 = C & C = 2 \end{cases}$$

$y_1 = \sin(x) + 2$ CSRN

4°

$y = y_0 + y_1 = C_1 e^x + C_2 x e^x + \sin(x) + 2$ CORN

5°

$$y' = C_1 e^x + C_2 x e^x + C_2 e^x + \cos(x)$$

$$y(0) = 2 = C_1 + 2 \rightarrow C_1 = 0$$
$$y'(0) = 1 = C_1 + C_2 + 1 \rightarrow C_2 = 0$$

$y(x) = \sin(x) + 2$ CSRN

5a

$$y'' + y = \tan(x)$$

1°

$$y'' + y = 0$$
$$r^2 + 1 = 0$$
$$r_1 = j \quad r_2 = -j$$
$$\alpha = 0 \quad \beta = 1$$
$$y_0 = C_1 e^0 \cos(x) + C_2 e^0 \sin(x) = C_1\cos(x) + C_2\sin(x)$$

2°

$$\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(x) \end{bmatrix}$$

$$C_1' = \frac{-\sin(x)\tan(x)}{\cos^2(x) + \sin^2(x)} = \frac{-\sin^2(x)}{\cos(x)}$$

$$C_2' = \frac{\cos(x)\tan(x)}{1} = \sin(x)$$

$$C_2 = \int \sin(x)\,dx = -\cos(x) + C_2$$

$$\frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t) + B(1-t)}{1-t^2}$$

$$\int \frac{dt}{1-t} = \left| \begin{matrix} u = 1-t \\ du = -dt \end{matrix} \right| = -\int \frac{du}{u} = -\ln|u|$$

$t = 1 \rightarrow 1 = 2A \quad A = \frac{1}{2}$

$t = -1 \quad 1 = 2B \quad B = \frac{1}{2}$

$$\int \frac{dt}{1-t^2} = \frac{1}{2}\int \frac{dt}{1-t} + \frac{1}{2}\int \frac{dt}{1+t} = -\frac{\ln|1-t|}{2} + \frac{\ln|1+t|}{2} = \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right|$$

$$C_1 = -\int \frac{\sin^2(x)}{\cos(x)}\,dx = -\int \frac{\sin^2(x)}{\cos^2(x)}\cos(x)\,dx = -\int \frac{\sin^2(x)}{1-\sin^2(x)}\cos(x)\,dx = \left| \begin{matrix} t = \sin(x) \\ dt = \cos(x)\,dx \end{matrix} \right| = -\int \frac{t^2}{1-t^2}\,dt = \int \frac{1-t^2-1}{1-t^2}\,dt = \int 1 - \frac{1}{1-t^2}\,dt$$

$$= \int 1\,dt - \int \frac{dt}{1-t^2} = \sin(x) - \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + C_1$$

$$y(x) = \left(\sin(x) - \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + C_1\right)\cos(x) + \left(-\cos(x) + C_2\right)\sin(x)$$

$$y(x) = C_1\cos(x) + C_2\sin(x) - \frac{1}{2}\cos(x)\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + \cancel{\sin(x)\cos(x)} - \cancel{\sin(x)}\cos(x)$$

5b $y'' - 2y' + y = \frac{e^x}{x^2+1}$

1° $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = (r-1)^2 = 0$

$r_0 = 1 \rightarrow y_0 = C_1 e^x + C_2 x e^x$

2° $\begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x^2+1} \end{bmatrix}$

$W = e^x(xe^x + e^x) - xe^x \cdot e^x = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$

$C_1' = \frac{-xe^x \cdot \frac{e^x}{x^2+1}}{e^{2x}} = \frac{-x}{x^2+1}$

$C_2' = \frac{e^x \cdot \frac{e^x}{x^2+1}}{e^{2x}} = \frac{1}{x^2+1}$

$C_1 = -\int \frac{x}{x^2+1}\,dx = -\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \left| \begin{matrix} t = x^2+1 \\ dt = 2x\,dx \end{matrix} \right| = -\frac{1}{2}\int \frac{dt}{t} = -\frac{1}{2}\ln|x^2+1| + C_1$

$C_2 = \int \frac{dx}{x^2+1} = \arctan(x) + C_2$

3° $y(x) = \left(-\frac{1}{2}\ln|x^2+1| + C_1\right)e^x + (\arctan(x) + C_2)\,xe^x$

$= C_1 e^x + C_2 x e^x + x\arctan(x)\,e^x - \frac{1}{2}\ln|x^2+1|\,e^x$

$y'' - 3y' + 2y = 3e^x$

1° $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = r^2 - 2r - r + 2 = (r-2)(r-1) = 0$

$\rightarrow \ y_0 = C_1 e^x + C_2 e^{2x}$

2° $\alpha = 1 \quad \beta = 0 \qquad 1 \rightarrow$ 1-krotny pierwiastek

$y_1 = Axe^x$

$y_1' = Axe^x + Ae^x$

$y_1'' = Axe^x + 2Ae^x$

3° $\cancel{Axe^x} + 2Ae^x - 3\cancel{Axe^x} - 3Ae^x + 2\cancel{Axe^x} = 3e^x$

$3 = -A$

$A = -3 \quad \rightarrow \quad y_1 = -3xe^x$

4° $y = y_0 + y_1 = C_1 e^x + C_2 e^{2x} - 3xe^x$

• $xy' = y \ln\left(\frac{y^2}{x^2}\right)$

$y' = \frac{y}{x} \ln\left(\left(\frac{y}{x}\right)^2\right) \qquad u(x) = \frac{y}{x}$

$y' = \frac{d}{dx} ux = u'x + u$

$u'x + u = u \ln(u^2) = 2u\ln(u)$

$u'x = 2u\ln(u) - u$

$\frac{du}{dx} = \frac{2u\ln(u) - u}{x} \qquad \int \frac{du}{2u\ln(u) - u} = \int \frac{dx}{x}$

$$\int \frac{du}{2u\ln(u) - u} = \int \frac{1}{2\ln(u) - 1} \cdot \frac{du}{u} = \left| \begin{array}{l} t = 2\ln(u) - 1 \\ dt = 2 \cdot \frac{1}{u} du \\ \frac{du}{u} = \frac{1}{2} dt \end{array} \right| = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + C = \frac{1}{2} \ln|2\ln(u) - 1| + C$$

$\frac{1}{2} \ln|2\ln(u) - 1| = \ln|x| + C$

$\sqrt{2\ln(u) - 1} = Cx$

$2\ln(u) - 1 = Cx^2 \qquad C \geq 0$

$\ln(u) = \frac{Cx^2 + 1}{2}$

$u = e^{\frac{Cx^2+1}{2}} \qquad y = xe^{\frac{Cx^2+1}{2}}$

- $y'' - 4y' + 4y = \frac{e^{2x}}{x^2} + 2$

1° $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = (r-2)^2 = 0$

$r_0 = 2 \rightarrow y_0 = C_1 e^{2x} + C_2 x e^{2x}$

2° $\begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{2x}}{x^2} + 2 \end{bmatrix}$

$W = e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x} \cdot xe^{2x} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$

$C_1' = \frac{-xe^{2x} \cdot \left(\frac{e^{2x}}{x^2} + 2\right)}{e^{4x}} = \frac{\frac{-xe^{4x}}{x^2} - 2xe^{2x}}{e^{4x}} = -\frac{1}{x} - 2xe^{-2x}$

$C_2' = \frac{e^{2x}\left(\frac{e^{2x}}{x^2} + 2\right)}{e^{4x}} = \frac{1}{x^2} + 2e^{-2x}$

$\int \frac{1}{x} dx = \ln|x| + C \qquad \int \frac{1}{x^2} dx = -\frac{1}{x} + C$

$\int e^{-2x} dx = -\frac{1}{2} e^{-2x} + C \qquad \int x e^{-2x} dx = \left| \begin{matrix} f = x & g' = e^{-2x} \\ f' = 1 & g = -\frac{1}{2} e^{-2x} \end{matrix} \right| = -\frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx = -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C$

$C_1 = -\int \frac{1}{x} dx - 2 \int x e^{-2x} dx = -\ln|x| + xe^{-2x} + \frac{1}{2} e^{-2x} + C_1$

$C_2 = \int \frac{1}{x^2} dx + 2 \int e^{-2x} = -\frac{1}{x} - e^{-2x} + C_2$

$y = \left(-\ln|x| + xe^{-2x} + \frac{1}{2} e^{-2x} + C_1\right) e^{2x} + \left(-\frac{1}{x} - e^{-2x} + C_2\right) xe^{2x}$

$= C_1 e^{2x} + C_2 xe^{2x} + x + \frac{1}{2} - e^{2x} \ln|x| - e^{2x} - x$

$= C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{2} - e^{2x} \ln|x| - e^{2x}$

$= C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{2} - e^{2x} \ln|x|$

- $xy' = xe^{\frac{y}{x}} + y$

$y' = e^{\frac{y}{x}} + \frac{y}{x}$

$v = \frac{y}{x} \qquad y' = v'x + v$

$v'x + v = e^v + v$

$v'x = e^v$

$\frac{dv}{dx} = \frac{e^v}{x} \qquad \int \frac{dv}{e^v} = \int \frac{dx}{x}$

$\int \frac{dv}{e^v} = \int e^{-v} dv = -e^{-v} + C$

$-e^{-v} = \ln|x| + C$

$e^{-v} = \ln\left|\frac{1}{x}\right| + C$

$\ln|e^{-v}| = \ln\left|\ln\left|\frac{1}{x}\right| + C\right|$

$-v = \ln\left|\ln\left|\frac{C}{x}\right|\right|$

$y = -x \ln\left|\ln\left|\frac{C}{x}\right|\right|$