1.

a) $\varphi: \mathbb{R}^2 \to \mathbb{R}^3 \quad \varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ xy \\ x-y \end{bmatrix}$

$$\varphi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \varphi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \varphi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

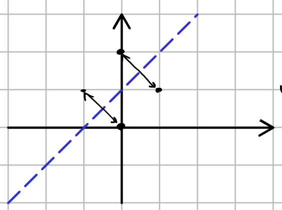
nie jest przekształceniem liniowym

b) $\varphi: \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-3 \\ 0 \\ y-z \end{bmatrix}$

$$\varphi\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \neq \varphi\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) + \varphi\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

nie jest przekształceniem liniowym

c) $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ symetria względem $y = x+1$



$$\varphi\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + \varphi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\neq$$

$$\varphi\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

nie jest przekształceniem liniowym

2.

a) $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2z \\ 3x \\ x-2z \end{bmatrix} = x\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} 2z = 0 \\ 3x = 0 \\ x-2z = 0 \end{cases} \to \begin{cases} x = 0 \\ y \in \mathbb{R} \\ z = 0 \end{cases} \quad \operatorname{Ker}\varphi = \left\{\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} : y \in \mathbb{R}\right\} = \underbrace{\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}}_{\text{baza Ker } \varphi} \qquad \operatorname{Im}\varphi = \operatorname{span}\underbrace{\left\{\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}\right\}}_{\text{baza Im } \varphi}$$

b) $\varphi: \mathbb{R}^2 \to \mathbb{R}^4$

$$\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ 0 \\ 4y-2x \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -2 & 4 \\ 0 & 0 \end{bmatrix}}_{M^{\mathcal{E}_2}_{\mathcal{E}_4}(\varphi)} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \end{array}\right] \to \begin{array}{l} x - 2y = 0 \\ x = 2y \end{array}$$

$\operatorname{rank}(\varphi) = 1 \quad \operatorname{Im}\varphi = \operatorname{span}\underbrace{\{(1,0,-2,0)\}}_{\text{baza}} \qquad \operatorname{Ker}\varphi = \left\{\begin{bmatrix} 2y \\ y \end{bmatrix} : y \in \mathbb{R}\right\} = \operatorname{span}\underbrace{\{(2,1)\}}_{\text{baza}}$

c) $\varphi: \mathbb{R}^4 \to \mathbb{R}^3$

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ x+2z \\ x-y+z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$k_3 = 2k_1 + k_2$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[w_3 - w_1]{w_2 - w_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ \cancel{0} & \cancel{2} & \cancel{2} & \cancel{0} & \cancel{0} \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{w_1 + 2w_3} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x & y & \uparrow & \uparrow \end{array} \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = z \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad z, t \in \mathbb{R}$$

$$\operatorname{Ker} \varphi = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

bazy

$$\operatorname{rank}\left(M_{\mathcal{E}_4}^{\mathcal{E}_3}(\varphi)\right) = 2 \qquad \operatorname{Im} \varphi = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

d) $\varphi: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$

$$(\varphi(v))(x) = (x-1)\, v'(x) - 2v(x) \qquad \mathcal{E}_3 = (x^2, x, 1)$$

$$\varphi(ax^2 + bx + c) = (x-1)(2ax + b) - 2(ax^2 + bx + c) = 2ax^2 + bx - 2ax - b - 2ax^2 - 2bx - 2c$$
$$= (-2a - b)x + (-b - 2c)1$$

$$\varphi\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -2a-b \\ -b-2c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$k_2 = \frac{1}{2}k_1 + \frac{1}{2}k_3$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \; c \in \mathbb{R}$$

$\operatorname{Ker} \varphi = \operatorname{span}\{(1, -2, 1)\} = \operatorname{span}\{x^2 - 2x + 1\}$ $\qquad \operatorname{rank}\left(M_{\mathcal{E}_3}^{\mathcal{E}_3}(\varphi)\right) = 2$

bazy

$\operatorname{Im} \varphi = \operatorname{span}\{(0,1,0), (0,0,1)\} = \operatorname{span}\{x, 1\}$

3.

$\varphi: \mathbb{R}^3 \to \mathbb{R}^2$

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-3y \\ x-y+2z \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathcal{A} = \left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right) \qquad B = \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)$$

$$M^{\mathcal{E}_3}_{\mathcal{E}_2}(\varphi) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

$$M^{\mathcal{A}}_{\mathcal{E}_2}(\varphi) = M^{\mathcal{E}_3}_{\mathcal{E}_2}(\varphi) \cdot M^{\mathcal{A}}_{\mathcal{E}_3}(id) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -3 \\ -5 & -5 & 5 \end{bmatrix}$$

$$M^{\mathcal{E}_3}_{B}(\varphi) = M^{\mathcal{E}_2}_{B}(id) \cdot M^{\mathcal{E}_3}_{\mathcal{E}_2}(\varphi) = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 10 \\ 1 & -3 & -6 \end{bmatrix}$$

$$M^{\mathcal{A}}_{B}(\varphi) = M^{\mathcal{E}_2}_{B}(id) \cdot M^{\mathcal{E}_3}_{\mathcal{E}_2}(\varphi) \cdot M^{\mathcal{A}}_{\mathcal{E}_3}(id) = M^{\mathcal{E}_2}_{B}(id) \cdot M^{\mathcal{A}}_{\mathcal{E}_2}(\varphi) = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & -3 \\ -5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -31 & -16 & 34 \\ 19 & 9 & -21 \end{bmatrix}$$

4. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$

$$\varphi\left(\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix} \qquad \varphi\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \varphi\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$M_{\mathcal{E}_3}(\mathcal{A}) = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ det $A = 4 \neq 0$ kolumny są bazą $\mathbb{R}^3$

więc istnieje dokładnie 1 przekształcenie spełniające warunki

$$M^{\mathcal{A}}_{\mathcal{E}_3}(\varphi) = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 0 \\ -8 & 0 & -2 \end{bmatrix} \qquad M^{\mathcal{E}_3}_{\mathcal{E}_3}(\varphi) = M^{\mathcal{A}}_{\mathcal{E}_3}(\varphi) \cdot M_{\mathcal{A}}(\mathcal{E}_3) = M^{\mathcal{A}}_{\mathcal{E}_3}(\varphi) \cdot \left[M^{\mathcal{A}}_{\mathcal{E}_3}(id)\right]^{-1}$$

→ można policzyć rank(φ) od razu z $M^{A}_{\mathcal{E}_3}(\varphi)$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 2 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_2+w_1-3w_3]{} \left[\begin{array}{ccc|ccc} 0 & -1 & 2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \frac{1}{4}w_2 \\ \\ w_1-2w_3 \end{matrix} \longrightarrow \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & -3/4 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$M^{\mathcal{E}_3}_{\mathcal{E}_3}(\varphi) = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 0 \\ -8 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1/4 & 1/4 & -3/4 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ -2 & -2 & 4 \end{bmatrix}$$

rank (φ) = 1 $\quad$ $\operatorname{Im}\varphi = \operatorname{span}\{(1,0,-2)\}$

baza Im φ

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \end{array}\right] \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad y,z \in \mathbb{R}$$

x ↑ ↑

$$\operatorname{Ker}\varphi = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

baza Ker φ

5.

$$M_B^A(\varphi) = \begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} \qquad \varphi: \mathbb{R}^2 \to \mathbb{R}^3$$

$$M_{\mathcal{E}_2}(A) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \qquad M_A^{\mathcal{E}_2}(id) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = (M_{\mathcal{E}_2}(A))^{-1}$$

$$M_{\mathcal{E}_3}^B(id) = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & -3 & 0 \end{bmatrix} \qquad M_{\mathcal{E}_3}^{\mathcal{E}_2}(\varphi) = M_{\mathcal{E}_3}^B(id) \cdot M_B^A(\varphi) \cdot M_A^{\mathcal{E}_2}(id) =$$

$$M_{\mathcal{E}_3}^{\mathcal{E}_2}(\varphi) = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 28 & 22 \\ -8 & -6 \\ -6 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 50 & 72 \\ -14 & -20 \\ -10 & -14 \end{bmatrix}$$

$$\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 50 & 72 \\ -14 & -20 \\ -10 & -14 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 50x + 72y \\ -14x - 20y \\ -10x - 14y \end{bmatrix}$$

$$\varphi\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 194 \\ -54 \\ -38 \end{bmatrix} \neq 0 \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \notin \operatorname{Ker} \varphi$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in \operatorname{Im}\varphi \iff \exists \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \quad \varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 50 & 72 & 1 \\ -14 & -20 & 0 \\ -10 & -14 & -1 \end{array}\right] \xrightarrow[w_2 - \frac{7}{5}w_3]{w_1 + 5w_3} \left[\begin{array}{cc|c} 0 & 2 & -4 \\ 0 & -\frac{2}{5} & \frac{7}{5} \\ -10 & -14 & -1 \end{array}\right] \xrightarrow{-5 \cdot w_2} \left[\begin{array}{cc|c} 0 & 2 & -4 \\ 0 & 2 & -7 \\ -10 & -14 & -1 \end{array}\right]$$

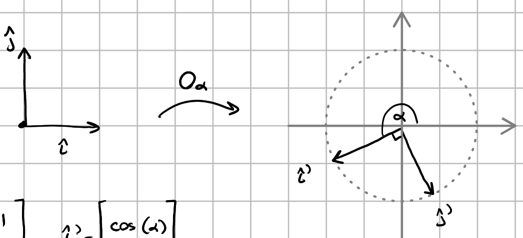
układ sprzeczny

więc $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \notin \operatorname{Im}\varphi$

$$\frac{14 \cdot 7}{5} - 20 = \frac{98 - 100}{5}$$

6.

$O_\alpha: \mathbb{R}^2 \to \mathbb{R}^2$ obrót o α wokół $(0,0)$ przeciwnie do wskazówek zegara



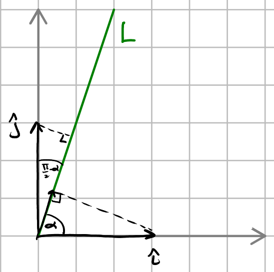
$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{\imath}' = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$$

$$\hat{\jmath} = \begin{bmatrix} \cos(\alpha + \frac{\pi}{2}) \\ \sin(\alpha + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{bmatrix} \qquad O_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$O_\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\alpha) - y\sin(\alpha) \\ x\sin(\alpha) + y\cos(\alpha) \end{bmatrix}$$

7.

$\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ rzut prostokątny na prostą $y = 3x$ (L)



$\tan(\alpha) = 3 \qquad \alpha = \arctan(3)$

$|\varphi(\hat{\imath})| = |\hat{\imath}| \cos(\alpha) = \cos(\alpha)$

$|\varphi(\hat{\jmath})| = |\hat{\jmath}| \cos(\frac{\pi}{2} - \alpha) = \sin(\alpha)$

$\varphi(\hat{\imath}) = \begin{bmatrix} \cos(\alpha) \cdot \cos(\alpha) \\ \cos(\alpha) \cdot \sin(\alpha) \end{bmatrix}$

$\varphi(\hat{\jmath}) = \begin{bmatrix} \sin(\alpha) \cos(\alpha) \\ \sin(\alpha) \cdot \sin(\alpha) \end{bmatrix}$

$M_{\mathcal{E}_2}^{\mathcal{E}_2}(\varphi) = \begin{bmatrix} \cos^2(\alpha) & \sin(\alpha)\cos(\alpha) \\ \sin(\alpha)\cos(\alpha) & \sin^2(\alpha) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix} \qquad \varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.1x + 0.3y \\ 0.3x + 0.9y \end{bmatrix}$

$\mathrm{Im}\,\varphi = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$ – prosta $y = 3x$

$\mathrm{Ker}\,\varphi = \mathrm{span}\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$ – prosta $y = -\frac{1}{3}x$ (prostopadła)

$\left[\begin{array}{cc|c} 0.1 & 0.3 & 0 \\ 0.3 & 0.9 & 0 \end{array}\right] \qquad 0.1x + 0.3y = 0 \qquad y = -\frac{1}{3}x$

$B = (v_1, v_2) \quad v_1 \in L \quad v_2 \perp L$

$v_1 = a\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad v_2 = b\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\varphi(v_1) = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} a \\ 3a \end{bmatrix} = \begin{bmatrix} a \\ 3a \end{bmatrix} = v_1$

$\varphi(v_2) = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} 3a \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

$M_B^B(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$M_B^B(\varphi) = M_B^{\mathcal{E}_2}(id) M_{\mathcal{E}_2}^{\mathcal{E}_2}(\varphi) M_{\mathcal{E}_2}^{B}(id)$

$= \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$

$= \frac{-1}{10}\begin{bmatrix} -1 & -3 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & -0.1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$