

1.

b) f ist parzyst

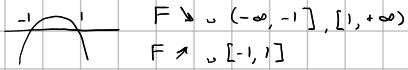
$$\int_{-a}^a f(x) dx = \int_0^a [f(x) + f(-x)] dx = 2 \int_0^a f(x) dx$$

2.

$$a) F(x) = \int_x^{2x} \frac{dt}{t^2+2t+2} \quad D_F = \mathbb{R}$$

$$F(x) = G(2x) - G(x) \quad G(x) = \int \frac{dt}{t^2+2t+2}$$

$$\begin{aligned} F'(x) &= 2 \frac{1}{t^2+2t+2} - \frac{1}{x^2+2x+2} = \frac{1}{2x^2+2x+1} - \frac{1}{x^2+2x+2} \\ &= \frac{x^2+2x+2 - 2x^2 - 2x - 1}{(2x^2+2x+1)(x^2+2x+2)} = \frac{(1-x)(1+x)}{(2x^2+2x+1)(x^2+2x+2)} \end{aligned}$$

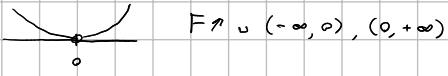


$$b) F(x) = \int_x^{2x} \frac{e^t}{t} dt \quad G(x) = \int \frac{e^x}{x} dx \quad x \neq 0$$

$$F(x) = G(2x) - G(x)$$

$$F'(x) = \frac{d}{dx} G(2x) - \frac{d}{dx} G(x) = \frac{e^{2x}}{2x} \cdot 2 - \frac{e^x}{x} = \frac{e^{2x} - e^x}{x} = e^x \frac{e^x - 1}{x}$$

$$F'(x) > 0 \iff e^x(e^x - 1)x > 0$$



3.

a)

$$\lim_{x \rightarrow 0^+} \frac{\int_0^{\sin(x)} \sqrt{\tan(t)} dt}{\int_0^{\tan(x)} \sqrt{\sin(t)} dt} \stackrel{0}{=} \lim_{x \rightarrow 0^+} \frac{\sqrt{\tan(\sin(x))} \cos(x)}{\sqrt{\sin(\tan(x))} \frac{1}{\cos^2(x)}} = \lim_{x \rightarrow 0^+} \frac{\sqrt{\tan(\sin(x))}}{\sqrt{\sin(\tan(x))}} \cos^2(x) = 1$$

$$\frac{\tan(\sin(x))}{\sin(\tan(x))} = \frac{\frac{\tan(\sin(x))}{\sin(x)}}{\frac{\sin(\tan(x))}{\sin(x)}} = \frac{\frac{\tan(\sin(x))}{\sin(x)} \rightarrow 1}{\frac{\sin(\tan(x))}{\sin(x)} \cdot \frac{1}{\cos^2(x)} \rightarrow 1}$$

$$G(x) = \int \sqrt{\tan(x)} dx$$

$$F(x) = G(\sin(x)) - G(0)$$

$$\frac{d}{dx} F(x) = G'(\sin(x)) \cos(x) = \sqrt{\tan(\sin(x))} \cos(x)$$

b)

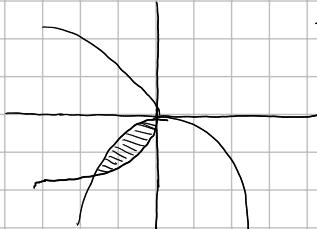
$$\lim_{x \rightarrow \infty} \frac{\int_{x^2}^{x^2+1} (\ln(t+1) dt)}{x^2} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{\ln(x^2+2) \cdot 2x - \ln(x^2+1) \cdot 2x}{2x} = \lim_{x \rightarrow \infty} \ln\left(\frac{x^2+2}{x^2+1}\right) = \ln(1) = 0$$

$$\begin{aligned} c) \lim_{x \rightarrow \infty} \frac{\int_{e^x}^{e^{3x}} \frac{dt}{\ln^3(t)}}{e^{3x}} &\stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{3} \cdot e^{3x} \cdot 3 - \frac{1}{\ln^3(e^x)} \cdot e^x}{3e^{3x}} = \lim_{x \rightarrow \infty} \frac{\frac{3e^{3x}}{(3x)^3} - \frac{e^x}{x^3}}{3e^{3x}} = \lim_{x \rightarrow \infty} \frac{3e^{3x} - 27e^x}{81e^{3x}x^3} \\ &= \lim_{x \rightarrow \infty} \frac{e^{2x} - 9}{27x^3e^{3x}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{2e^{2x}}{27 \cdot 3x^2e^{3x} + 27 \cdot 3e^{3x}x^3} = \frac{2}{81} \lim_{x \rightarrow \infty} \frac{e^{2x}}{e^{3x}(x^3+x^2)} = \frac{2}{81} \lim_{x \rightarrow \infty} \frac{1}{e^x(x^3+x^2)} = 0 \end{aligned}$$

4.

$$\text{a)} \quad 2y = -x^2 \rightarrow y = -\frac{1}{2}x^2$$

$$2x = -y^2 \rightarrow y = \pm \sqrt{-2x} \quad x = -\frac{1}{2}y^2$$



$$-\sqrt{-2x} = -\frac{1}{2}x^2$$

$$-2x = \frac{1}{2}x^4$$

$$-8 = x^3 \quad x = -2$$

$$|D| = \int_{-2}^0 \left( -\frac{1}{2}x^2 - (-\sqrt{-2x}) \right) dx$$

$$= \int_{-2}^0 \left( -\frac{1}{2}x^2 + \sqrt{-2x} \right) dx = -\frac{1}{2} \int_{-2}^0 x^2 dx + \int_{-2}^0 \sqrt{-2x} dx$$

$$\int_{-2}^0 x^2 dx = \frac{1}{3}x^3 \Big|_{-2}^0 = +\frac{8}{3}$$

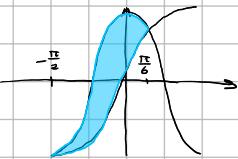
$$\int_{-2}^0 \sqrt{-2x} dx = \left| \frac{t = -2x}{dt = -2dx} \right| = -\frac{1}{2} \int_4^0 \sqrt{t} dt = \frac{1}{2} \int_0^4 \sqrt{t} dt = \frac{1}{2} \cdot \left( \frac{2}{3}t^{1/2} \right) \Big|_0^4 = \frac{8}{3}$$

$$|D| = \left( -\frac{1}{2} \right) \left( \frac{8}{3} \right) + \frac{8}{3} = \frac{4}{3}$$

$$\text{b)} \quad y = \sin(x)$$

$$y = \cos(2x)$$

$$(0, \frac{1}{2})$$



$$\sin(x) = \cos(2x)$$

$$\sin(x) = 2\cos^2(x) + 1 = 1 - 2\sin^2(x)$$

$$2\sin^2(x) + \sin(x) - 1 = 0$$

$$2\sin^2(x) + 2\sin(x) - \sin(x) - 1 = 0$$

$$2\sin(x)(\sin(x) + 1) - (\sin(x) + 1) = 0$$

$$(2\sin(x) - 1)(\sin(x) + 1) = 0$$

$$\sin(x) = \frac{1}{2} \quad \vee \quad \sin(x) = -1$$

$$x = \frac{\pi}{6} \quad \vee \quad x = -\frac{\pi}{2}$$

$$|D| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} [\cos(2x) - \sin(x)] dx = \frac{1}{2} \sin(2x) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{6}} + \cos(x) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2} \left( \frac{\sqrt{3}}{2} - 0 \right) + \frac{\sqrt{3}}{2} - 0 = \frac{3\sqrt{3}}{4}$$

5.

$$\text{a)} \quad 0 \leq x \leq \frac{\pi}{2}$$

$$0 \leq y \leq \sin(x) + \cos(x)$$

$$|V| = \pi \int_0^{\frac{\pi}{2}} [\sin(x) + \cos(x)]^2 dx = \pi \int_0^{\frac{\pi}{2}} [1 + \sin(2x)] dx = \pi \left[ x \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \cos(2x) \Big|_0^{\frac{\pi}{2}} \right] = \pi \left( \frac{\pi}{2} - \frac{1}{2} (-1 - 1) \right) = \pi \left( \frac{\pi}{2} + 1 \right)$$

$$\text{b)} \quad 0 \leq x \leq \pi$$

$$\sin(x) \leq y \leq 2\sin(x)$$

$$|V| = |V_1| - |V_2| = \pi \int_0^{\pi} [2\sin(x)]^2 dx - \pi \int_0^{\pi} [\sin(x)]^2 dx = 4\pi \int_0^{\pi} \sin^2(x) dx - 2\pi \int_0^{\pi} \sin^2(x) dx$$

$$= 3\pi \int_0^{\pi} \frac{1 - \cos(2x)}{2} dx = \frac{3\pi}{2} \left( x - \frac{1}{2} \sin(2x) \right) \Big|_0^{\pi} = \frac{3\pi^2}{2}$$

Kołos  $\rightarrow$  zadania 3/25  $\rightarrow$  4/28-2

6.

$$a) \quad y = x\sqrt{x} \quad x \in [0, 1]$$

$$|L| = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$\frac{d}{dx} x\sqrt{x} = \frac{d}{dx} x^{\frac{3}{2}} = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$|L| = \int_0^1 \sqrt{1 + (\frac{3}{2}\sqrt{x})^2} dx = \int_0^1 \sqrt{1 + \frac{9}{4}x} dx = \left| t = \frac{9}{4}x + 1 \right|_{dt = \frac{9}{4}dx}^{\frac{13}{4}} \sqrt{t} dt = \left[ \frac{2}{3}t^{\frac{1}{2}} \right]_1^{\frac{13}{4}} = \frac{2}{3} \cdot \frac{13\sqrt{13} - 8}{8} = \frac{13\sqrt{13} - 8}{24}$$

$$|L| = \frac{4}{3} \cdot \frac{2}{3} \left( \frac{13}{4} \cdot \frac{\sqrt{13}}{2} - 1 \right) = \frac{8}{27} \cdot \frac{13\sqrt{13} - 8}{8} = \frac{13\sqrt{13} - 8}{27}$$

$$b) \quad y = \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh^2(x) - \sinh^2(x) = 1$$

$$x \in [0, 1]$$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$|L| = \int_0^1 \sqrt{1 + \sinh^2(x)} dx = \int_0^1 \cosh(x) dx = \left[ \sinh(x) \right]_0^1 = \frac{e - e^{-1}}{2}$$

7.

$$a) \quad \int_0^\infty x e^{-x^2} dx = \lim_{T \rightarrow \infty} \int_0^T x e^{-x^2} dx \left| \begin{array}{l} t = x^2 \\ dt = 2x dx \end{array} \right| = \lim_{T \rightarrow \infty} \int_0^{T^2} e^{-t} \cdot \frac{1}{2} dt = \frac{1}{2} \lim_{T \rightarrow \infty} (-e^{-t}) \Big|_0^{T^2} = \frac{1}{2} \lim_{T \rightarrow \infty} (-e^{-T^2} + 1) = \frac{1}{2}$$

$$b) \quad \int e^x \cos(x) dx = \left| \begin{array}{l} f = \cos(x) \\ f' = -\sin(x) \end{array} \right. \left| \begin{array}{l} g = e^x \\ g' = e^x \end{array} \right. = e^x \cos(x) + \int e^x \sin(x) dx = \left| \begin{array}{l} f = \sin(x) \\ f' = \cos(x) \end{array} \right. \left| \begin{array}{l} g = e^x \\ g' = e^x \end{array} \right. = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) dx$$

$$2 \int e^x \cos(x) dx = e^x \cos(x) + e^x \sin(x)$$

$$\int e^x \cos(x) dx = \frac{e^x}{2} (\cos(x) + \sin(x))$$

$$\int_{-\infty}^0 e^x \cos(x) dx = \lim_{s \rightarrow -\infty} \int_s^0 e^x \cos(x) dx = \frac{1}{2} \lim_{s \rightarrow -\infty} e^x [\cos(x) + \sin(x)] \Big|_s^0 = \frac{1}{2} \lim_{s \rightarrow -\infty} [1 - e^s (\cos(s) + \sin(s))] = \frac{1}{2}$$

↓ ograniczona

$$c) \quad \int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5} = \int_{-\infty}^0 \frac{dx}{x^2 + 4x + 5} + \int_0^{\infty} \frac{dx}{x^2 + 4x + 5} = \lim_{s \rightarrow -\infty} \int_s^0 \frac{dx}{x^2 + 4x + 5} + \lim_{T \rightarrow \infty} \int_0^T \frac{dx}{x^2 + 4x + 5}$$

$$= \lim_{s \rightarrow -\infty} \arctan(x+2) \Big|_s^0 + \lim_{T \rightarrow \infty} \arctan(x+2) \Big|_0^T$$

$$= \lim_{s \rightarrow -\infty} [\arctan(2) - \arctan(s+2)] + \lim_{T \rightarrow \infty} [\arctan(T) - \arctan(2)]$$

$$= \arctan(2) - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \arctan(2) = \pi$$

8.

$$a) \int_1^e \frac{dx}{x \sqrt{\ln(x)}} = \lim_{a \rightarrow 1^+} \int_a^e \frac{dx}{x \sqrt{\ln(x)}} = \lim_{a \rightarrow 1^+} [2\sqrt{\ln(e)} - 2\sqrt{\ln(a)}] = 2$$

$$\int \frac{dx}{x \sqrt{\ln(x)}} = \left| \begin{array}{l} t = \ln(x) \\ dt = \frac{1}{x} dx \end{array} \right| = \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{\ln(x)}$$

$$b) \int_1^2 \frac{x}{\sqrt{x-1}} dx = \lim_{a \rightarrow 1^+} \int_a^2 \frac{x}{\sqrt{x-1}} dx = \dots$$

$$\int \frac{x}{\sqrt{x-1}} dx = \left| \begin{array}{l} t = x-1 \\ dt = dx \end{array} \right| = \int \frac{t+1}{\sqrt{t}} dt = \int \left[ \sqrt{t} + \frac{1}{\sqrt{t}} \right] dt = \int \sqrt{t} dt + \int \frac{1}{\sqrt{t}} dt = \frac{3}{2} t^{\frac{1}{2}} + 2\sqrt{t} + C = \frac{3}{2}(x+1)\sqrt{x+1} + 2\sqrt{x+1} + C$$

$$c) f(x) = \frac{1}{x^2 - 4x + 3} = \frac{1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{A(x-3) + B(x-1)}{(x-1)(x-3)} = \frac{-\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x-3}$$

$$1 = -2A \quad A = -\frac{1}{2}$$

$$1 = 2B \quad B = \frac{1}{2}$$

$$\int \frac{dx}{x^2 - 4x + 3} = -\frac{1}{2} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x-3} dx = \frac{1}{2} \ln|x-3| - \frac{1}{2} \ln|x-1| + C$$

$$\int_0^2 \frac{dx}{x^2 - 4x + 3} = \int_0^1 \frac{dx}{x^2 - 4x + 3} + \int_1^2 \frac{dx}{x^2 - 4x + 3} = \lim_{a \rightarrow 1^-} \int_0^a \frac{dx}{x^2 - 4x + 3} + \lim_{b \rightarrow 1^+} \int_b^2 \frac{dx}{x^2 - 4x + 3}$$

$$= \lim_{a \rightarrow 1^-} \left[ \frac{1}{2} \ln|x-3| - \frac{1}{2} \ln|x-1| \right] \Big|_0^a + \lim_{b \rightarrow 1^+} \left[ \frac{1}{2} \ln|x-3| - \frac{1}{2} \ln|x-1| \right] \Big|_b^2$$

$$= \frac{1}{2} \lim_{a \rightarrow 1^-} \left[ \ln|a-3| - \ln|a-1| - \ln|-3| + \ln|-1| \right] \Big|_{+\infty}^0 + \frac{1}{2} \lim_{b \rightarrow 1^+} \left[ \ln|b-3| - \ln|b-1| - \ln|b-3| + \ln|b-1| \right] \Big|_{-\infty}^2$$

całka nie istnieje