Zestaw 11

1.

a) $y' = y - y^2 \qquad y(0) = 0.5$

$\frac{dy}{dx} = y - y^2 \rightarrow \int \frac{dy}{y-y^2} = \int 1 dx$

$\frac{1}{y-y^2} = \frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{A(1-y)+By}{y(1-y)}$

$y=1 \rightarrow 1 = B \qquad y = 0 \rightarrow 1 = A$

$\int \frac{1}{y-y^2} dy = \int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln|y| - \ln|y-1| + C$

$\ln|y| - \ln|y-1| = x + C$

$\ln\left|\frac{y}{y-1}\right| = x + C \qquad \frac{y}{y-1} = e^{x+C} = Ce^x$

$y = Ce^x y - Ce^x$

$Ce^x y - y = Ce^x$

$y = \frac{Ce^x}{Ce^x - 1}$

$\frac{1}{2} = \frac{C \cdot e^0}{C \cdot e^0 - 1} = \frac{C}{C-1}$

$C - 1 = 2C \qquad C = -1$

$y = \frac{-e^x}{-e^x - 1} = \frac{e^x}{e^x+1}$

b) $x \cdot y' = \tan(y) \qquad y(\frac{1}{2}) = \frac{5\pi}{6}$

$\frac{dy}{dx} = \frac{\tan(y)}{x} \qquad \int \frac{dy}{\tan(y)} = \int \frac{dx}{x}$

$\int \frac{dy}{\tan(y)} = \int \frac{\cos(y)}{\sin(y)} dy = \left| \begin{matrix} t = \sin(y) \\ dt = \cos(y) dy \end{matrix} \right| = \int \frac{dt}{t}$

$\ln|\sin(y)| = \ln|x| + C$

$\sin(y) = Cx$

$y = \arcsin(Cx)$

$y = \arcsin(x)$

$\frac{5\pi}{6} = \arcsin(C \cdot \frac{1}{2})$

$\sin(\frac{5\pi}{6}) = \frac{C}{2}$

$C = 2 \cdot \frac{1}{2} = 1$

c) $y(1) = -2$

$y' = \frac{y-x}{x} = \frac{y}{x} - \frac{x}{x} = \frac{y}{x} - 1$

$u = \frac{y}{x} \qquad y = ux \qquad y' = u'x + u$

$u'x + u = u - 1$

$u'x = -1$

$\frac{du}{dx} = \frac{-1}{x} \qquad \int 1 du = -\int \frac{dx}{x}$

$u = -\ln|x| + C$

$y = -x\ln|x| + Cx$

$-2 = -\ln|1| + C$

$-2 = C$

$y = -x\ln|x| - 2x$

$y' = -x \cdot \frac{1}{x} - \ln|x| - 2 = -\ln|x| - 3$

$\frac{y-x}{x} = \frac{-x\ln|x| - 3x}{x} = -\ln|x| - 3$

2.

a) $y' - \frac{2x}{1+x^2} y = 1+x^2$

1° $y' - \frac{2x}{1+x^2} y = 0$

$\frac{dy}{dx} = \frac{2x}{1+x^2} y \qquad \int \frac{dy}{y} = \int \frac{2x}{1+x^2} dx = \left| \begin{matrix} t = 1+x^2 \\ dt = 2x dx \end{matrix} \right| = \int \frac{dt}{t}$

$\ln|y| = \ln|1+x^2| + C$

$y = C(1+x^2)$

2° $y' = C(x) \cdot 2x + C'(x)(1+x^2)$

$2C(x)x + C'(x)(1+x^2) - \frac{2x}{1+x^2} \cdot C(x)(1+x^2) = 1+x^2$

$2xC(x) + C'(x)(1+x^2) - 2xC(x) = 1+x^2$

$C'(x) = 1 \qquad C(x) = \int 1 dx = x + C$

$y = (x+C)(1+x^2) = x^3 + Cx^2 + x + C$

Sprawdzenie

$y' - \frac{2x}{1+x^2} y = 1+x^2$

$y = x^3 + Cx^2 + x + C$

$y' = 3x^2 + 2Cx + 1$

$3x^2 + 2Cx + 1 - \frac{2x}{1+x^2}(x^3 + Cx^2 + x + C)$

$3x^2 + 2Cx + 1 - \frac{2x}{1+x^2}\left[x^2(x+C) + (x+C)\right]$

$3x^2 + 2Cx + 1 - 2x(x+C)$

$3x^2 + 2Cx + 1 - 2x^2 - 2Cx = x^2+1$

b) $y' + 2xy = xe^{-x^2}$

1° $y' + 2xy = 0$

$\frac{dy}{dx} = -2xy \qquad \int \frac{dy}{y} = \int -2x dx$

$\ln|y| = -2 \cdot \frac{1}{2}x^2 + C = -x^2 + C$

$y = e^{-x^2+C} = Ce^{-x^2}$

2° $y' = \frac{d}{dx}\left[C(x)e^{-x^2}\right] = C'(x)e^{-x^2} + C(x)e^{-x^2} \cdot (-2x)$

$C'(x)e^{-x^2} - 2xC(x)e^{-x^2} + 2xC(x)e^{-x^2} = xe^{-x^2}$

$C'(x) = x \qquad C(x) = \int x dx = \frac{1}{2}x^2 + C$

$y(x) = (\frac{1}{2}x^2 + C)e^{-x^2}$

$y(x) = (\frac{1}{2}x^2 + C)e^{-x^2}$

$y' = xe^{-x^2} + (\frac{1}{2}x^2 + C)e^{-x^2} \cdot (-2x)$

$xe^{-x^2} + (\frac{1}{2}x^2 + C)e^{-x^2} \cdot (-2x) + 2x(\frac{1}{2}x^2 + C)e^{-x^2} = xe^{-x^2}$

c) $y' + 4y = 5\sin(3x)$

1° $y' + 4y = 0$

$\frac{dy}{dx} = -4y \qquad \int \frac{dy}{y} = -4\int 1 dx$

$\ln|y| = -4x + C$

$y = e^{-4x+C} = Ce^{-4x}$

2° $y_1 = A\sin(3x) + B\cos(3x)$

$y_1' = 3A\cos(3x) - 3B\sin(3x)$

$3A\cos(3x) - 3B\sin(3x) + 4A\sin(3x) + 4B\cos(3x) = 5\sin(3x)$

$(3A+4B)\cos(3x) + (4A-3B)\sin(3x) = 5\sin(3x)$

$\begin{cases} 0 = 3A + 4B \\ 5 = 4A - 3B \end{cases} \qquad \begin{matrix} B = -\frac{3}{4}A \\ 5 = 4A + \frac{3}{4}A = \frac{25}{4}A \end{matrix} \qquad \begin{cases} A = \frac{4}{5} \\ B = -\frac{3}{5} \end{cases}$

3° $y(x) = \frac{4}{5}\sin(3x) - \frac{3}{5}\cos(3x) + Ce^{-4x}$

d) $y' - 2y = \cos(x) - x\sin(x)$

1° $\frac{dy}{dx} = 2y \qquad \int \frac{dy}{y} = 2\int 1 dx$

$\ln|y| = 2x + C$

$y = e^{2x+C} = Ce^{2x}$

2° $y_1 = (Ax+B)\cos(x) + (Cx+D)\sin(x)$

$y_1' = -(Ax+B)\sin(x) + A\cos(x) + (Cx+D)\cos(x) + C\sin(x)$

$-Ax\sin(x) - B\sin(x) + A\cos(x) + Cx\cos(x) + D\cos(x) + C\sin(x) - 2Ax\cos(x) - 2B\cos(x) - 2Cx\sin(x) - 2D\sin(x) = \cos(x) - x\sin(x)$

$(-Ax - B + C - 2Cx - 2D)\sin(x) + (A + Cx + D - 2Ax - 2B) = \cos(x) - x\sin(x)$

$\begin{cases} 0 = -A - 2C \\ 1 = -B + C - 2D \\ -1 = C - 2A \\ 0 = A + D - 2B \end{cases} \qquad \left[\begin{array}{cccc|c} -1 & 0 & -2 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 \\ -2 & 0 & 1 & 0 & -1 \\ 1 & -2 & 0 & 1 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 0.4 \\ -0.08 \\ -0.2 \\ -0.56 \end{bmatrix}$

$y_1 = (0.4x - 0.08)\cos(x) + (-0.2x - 0.56)\sin(x) + Ce^{2x}$

3.

a) $y' + 2y = 5\cos(x) \qquad y\left(\frac{\pi}{2}\right) = 1$

1° $\frac{dy}{dx} = -2y \qquad \int \frac{dy}{y} = -2\int 1 dx$

$\ln|y| = -2x + C$

$y = Ce^{-2x}$

2° $y_1 = A\cos(x) + B\sin(x)$

$y_1' = -A\sin(x) + B\cos(x)$

$-A\sin(x) + B\cos(x) + 2A\cos(x) + 2B\sin(x) = 5\cos(x)$

$(B+2A)\cos(x) + (2B-A)\sin(x) = 5\cos(x)$

$\begin{cases} 5 = B + 2A \\ 0 = 2B - A \end{cases} \quad \begin{cases} A = 2B \\ 5 = 5B \end{cases} \quad \begin{cases} A = 2 \\ B = 1 \end{cases}$

3° $y(x) = 2\cos(x) + \sin(x) + Ce^{-2x}$

$1 = 2\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) + Ce^{-\pi}$

$1 = 1 + Ce^{-\pi}$

$0 = Ce^{-\pi} \qquad C = 0$

$y(x) = 2\cos(x) + \sin(x)$

b) ?

4.

a) $y'' - 4y' = 8x \qquad y(0) = 1 \quad y'(0) = -1$

1° $r^2 - 4r = 0$

$r(r-4) = 0$

$y(x) = C_1 e^{0\cdot x} + C_2 e^{4x} = C_1 + C_2 e^{4x}$

2° $\begin{bmatrix} 1 & e^{4x} \\ 0 & 4e^{4x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 8x \end{bmatrix}$

$C_1'(x) = \frac{-8xe^{4x}}{4e^{4x}} = -2x \qquad C_1(x) = -2\int x dx = -x^2 + C_1$

$C_2'(x) = \frac{8x}{4e^{4x}} = 2xe^{-4x} \qquad C_2 = 2\int xe^{-4x} dx = \begin{vmatrix} x & -\frac{1}{4}e^{-4x} \\ 1 & e^{-4x} \end{vmatrix} = 2\cdot\left(-\frac{1}{4}xe^{-4x}\right) + 2\cdot\frac{1}{4}\int e^{-4x} = -\frac{1}{2}xe^{-4x} + \frac{1}{2}\cdot\left(-\frac{1}{4}\right)e^{-4x} + C_2$

$= -\frac{1}{2}xe^{-4x} - \frac{1}{8}e^{-4x} + C_2$

$= -\frac{1}{8}e^{-4x}(4x+1) + C_2$

$y = -x^2 + C_1 + \left(-\frac{1}{8}e^{-4x}(4x+1) + C_2\right)\cdot e^{4x}$

$= -x^2 + C_1 - \frac{1}{8}(4x+1) + C_2 e^{4x}$

$= -x^2 - \frac{1}{2}x - \frac{1}{8} + C_1 + C_2 e^{4x}$

$y' = -2x - \frac{1}{2} + 4C_2 e^{4x}$

$\begin{cases} 2 = -\frac{1}{8} + C_1 + C_2 \\ 1 = -\frac{1}{2} + 4C_2 \end{cases} \qquad C_2 = \frac{3}{8} \quad C_1 = \frac{7}{4}$

$y(x) = -x^2 - \frac{1}{2}x - \frac{1}{8} + \frac{7}{4} + \frac{3}{8}e^{4x}$

$y(x) = \frac{3}{8}e^{4x} - x^2 - \frac{1}{2}x + \frac{13}{8}$

sprawdzenie

$y' = \frac{3}{2}e^{4x} - 2x - \frac{1}{2}$

$y'' = 6e^{4x} - 2$

$y'' - 4y' = 6e^{4x} - 2 - 6e^{4x} + 8x + 2 = 8x$

b) $y'' - 2y' + y = 4\sin^2(\frac{x}{2})$ $y(0) = 2$ $y'(0) = 1$ $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

1° $r^2 - 2r + 1 = 0$ $4\sin^2(\frac{x}{2}) = 4 \cdot \frac{1}{2}(1 - \cos(x)) = 2 - 2\cos(x)$

$(r-1)^2 = 0$

$\rightarrow y(x) = C_1 e^x + C_2 x e^x$

2° $\begin{bmatrix} e^x & xe^x \\ e^x & e^x(x+1) \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 4\sin^2(\frac{x}{2}) \end{bmatrix}$

$$C_1'(x) = \frac{-4xe^x \sin^2(\frac{x}{2})}{\cancel{xe^{2x}} + e^{2x} - \cancel{xe^{2x}}} = -4xe^{-x}\sin^2(\frac{x}{2})$$

$$C_1(x) = -4\int x e^{-x}\sin^2(\frac{x}{2})\,dx = \dots = e^{-x}\left((x+1)(\sin(x)+2) - x\cos(x)\right) + C_1$$

$$C_2' = \frac{4e^x\sin^2(\frac{x}{2})}{e^{2x}} = 4e^{-x}\sin^2(\frac{x}{2})$$

$$C_2(x) = 4\int e^{-x}\sin^2(\frac{x}{2})\,dx = \dots = e^{-x}\left(-\sin(x) + \cos(x) - 2\right) + C_2$$

$$y(x) = C_1 e^x + \cancel{x\sin(x)} + \cancel{2x} + \sin(x) + 2 - \cancel{x\cos(x)} + C_2 x e^x - \cancel{x\sin(x)} + \cancel{x\cos(x)} - \cancel{2x}$$

$$y(x) = C_1 e^x + C_2 x e^x + \sin(x) + 2$$

3° $y' = C_1 e^x + C_2 e^x(x+1) + \cos(x)$

$\begin{cases} 1 = C_1 + C_2 + 1 \\ 2 = C_1 + 2 \end{cases}$ $C_1 = 0$, $C_2 = 0$

$y(x) = \sin(x) + 2$

$$\int e^{-x}\, 4\sin^2\left(\tfrac{x}{2}\right)dx = \int e^{-x}(2-2\cos(x))\,dx = \int\left[2e^{-x} - 2e^{-x}\cos(x)\right]dx$$
$$= 2\left[\int e^{-x}dx - \int e^{-x}\cos(x)\,dx\right]$$

$$\int e^{-x}dx = -e^{-x} + C$$
$$\int e^{-x}\cos(x)\,dx = \begin{vmatrix} \cos(x) & -e^{-x} \\ -\sin(x) & e^{-x} \end{vmatrix} = -e^{-x}\cos(x) - \int e^{-x}\sin(x) = \begin{vmatrix} \sin(x) & -e^{-x} \\ \cos(x) & e^{-x} \end{vmatrix} = -e^{-x}\cos(x) - \left(-e^{-x}\sin(x) + \int e^{-x}\cos(x)\,dx\right)$$

$$\int e^{-x}\cos(x)\,dx = -e^{-x}\cos(x) + e^{-x}\sin(x) - \int e^{-x}\cos(x)\,dx$$
$$2\int e^{-x}\cos(x)\,dx = e^{-x}(\sin(x) - \cos(x))$$
$$\int e^{-x}\cos(x)\,dx = \frac{e^{-x}}{2}\left[\sin(x) - \cos(x)\right] + C$$

$$4\int e^{-x}\sin^2\left(\tfrac{x}{2}\right)dx = -2e^{-x} - e^{-x}\left[\sin(x) - \cos(x)\right] + C = e^{-x}\left[-\sin(x) + \cos(x) - 2\right] + C$$

$$-4\int x e^{-x}\sin^2\left(\tfrac{x}{2}\right)dx =$$
$$-\int x e^{-x}\, 4\sin^2\left(\tfrac{x}{2}\right)dx = -\int x e^{-x}(2-2\cos(x))\,dx$$
$$= -\int\left[2xe^{-x} - 2xe^{-x}\cos(x)\right]dx$$
$$= -\int 2xe^{-x}\,dx \quad - \int -2xe^{-x}\cos(x)\,dx$$
$$= -2\int xe^{-x}\,dx \quad + 2\int xe^{-x}\cos(x)\,dx$$

$$\int xe^{-x}dx = \begin{vmatrix} x & -e^{-x} \\ 1 & e^{-x} \end{vmatrix} = -xe^{-x} - \int e^{-x}dx = -xe^{-x} + e^{-x} + C = e^{-x}(1-x) + C$$
$$\int x e^{-x}\cos(x)\,dx = \begin{vmatrix} f = x & g = \frac{1}{2}e^{-x}\left[\sin(x) - \cos(x)\right] \\ f' = 1 & g' = e^{-x}\cos(x) \end{vmatrix} = x\cdot\frac{1}{2}e^{-x}\left[\sin(x) - \cos(x)\right] - \frac{1}{2}\int e^{-x}\left[\sin(x) - \cos(x)\right]dx$$
$$=$$

$$\int e^{-x}\sin(x)\,dx = \begin{vmatrix} \sin(x) & -e^{-x} \\ \cos(x) & e^{-x} \end{vmatrix} = -e^{-x}\sin(x) + \int e^{-x}\cos(x)$$

$$\frac{1}{2}xe^{-x}\left[\sin(x) - \cos(x)\right] - \frac{1}{2}\int e^{-x}\sin(x)\,dx + \frac{1}{2}\int e^{-x}\cos(x)\,dx$$
$$= \frac{1}{2}xe^{-x}\left[\sin(x) - \cos(x)\right] - \frac{1}{2}\left[-e^{-x}\sin(x) + \int e^{-x}\cos(x)\,dx\right] + \frac{1}{2}\int e^{-x}\cos(x)\,dx$$
$$= \frac{1}{2}xe^{-x}\left[\sin(x) - \cos(x)\right] + \frac{1}{2}e^{-x}\sin(x) \cancel{- \frac{1}{2}\int e^{-x}\cos(x)\,dx} + \cancel{\frac{1}{2}\int e^{-x}\cos(x)\,dx}$$
$$= \frac{1}{2}e^{-x}\left[x\sin(x) - x\cos(x) + \sin(x)\right] + C$$

$$-4\int xe^{-x}\sin^2\left(\tfrac{x}{2}\right)dx = -2e^{-x}(1-x) + e^{-x}\left[(x+1)\sin(x) - x\cos(x)\right] + C$$
$$= e^{-x}\left[(x+1)\sin(x) - x\cos(x) + 2(x-1)\right] + C$$
$$= e^{-x}\left[(x\right.$$

5. ?

a) $y'' + y = \tan(x)$

1° $r^2 + 1 = 0$

$r \in \sqrt{-1} = \sqrt{e^{\pi j}} = \{ e^{\frac{\pi}{2} j}, e^{-\frac{\pi}{2} j} \} = \{ 0 + j, 0 - j \}$

$\alpha = 0 \quad \beta = 1$

$\rightarrow y(x) = C_1 \cos(x) + C_2 \sin(x)$

2° $$\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(x) \end{bmatrix}$$

$$C_1' = \frac{-\sin(x)\tan(x)}{\cos^2(x) + \sin^2(x)} = -\frac{\sin^2(x)}{\cos(x)}$$

$$C_1(x) = -\int \frac{\sin^2(x)}{\cos(x)} dx = \int \frac{\cos^2(x) - 1}{\cos(x)} dx = \int \cos(x) dx - \int \frac{dx}{\cos(x)}$$

$$\int \frac{dx}{\cos(x)} = \left| \begin{array}{l} t = \tan(\frac{x}{2}) \\ dx = \frac{2dt}{1+t^2} \\ \cos(x) = \frac{1-t^2}{1+t^2} \end{array} \right| = \int \frac{\frac{2}{1+t^2}}{\frac{1-t^2}{1+t^2}} dt = 2 \int \frac{1}{1-t^2} dt = 2 \int \frac{dt}{(1-t)(1+t)} = \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = \ln|1 + \tan(\tfrac{x}{2})| + \ln|1 - \tan(\tfrac{x}{2})| + C$$

$$\frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t) + B(1-t)}{1-t^2}$$

$t = 1 \rightarrow 1 = 2A \quad A = \frac{1}{2}$

$t = -1 \rightarrow 1 = 2B \quad B = \frac{1}{2}$

$$C_1(x) = \sin(x) - \ln|1 - \tan^2(\tfrac{x}{2})| + C_1$$

$C_2'(x) = \cos(x)\tan(x) = \sin(x) \qquad C_2(x) = \int \sin(x) dx = -\cos(x) + C_2$

$$y(x) = C_1 \cos(x) + C_2 \sin(x) + \sin(x)\cos(x) - \ln|1 - \tan^2(\tfrac{x}{2})| - \sin(x)\cos(x)$$
$$y(x) = C_1 \cos(x) + C_2 \sin(x) - \ln|1 - \tan^2(\tfrac{x}{2})|$$

b) $y'' - 2y' + y = \dfrac{e^x}{x^2+1}$

1° $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$\rightarrow y(x) = C_1 e^x + C_2 x e^x$

2° $$\begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x^2+1} \end{bmatrix}$$

$$C_1' = \frac{-xe^x \frac{e^x}{x^2+1}}{xe^{2x} + e^{2x} - xe^{2x}} = \frac{\frac{-x}{x^2+1} e^{2x}}{e^{2x}} = -\frac{x}{x^2+1} \qquad C_1(x) = -\int \frac{x}{x^2+1} dx = -\frac{1}{2} \int \frac{2x}{x^2+1} dx = \left| \begin{array}{l} t = x^2+1 \\ dt = 2x dx \end{array} \right| = -\frac{1}{2} \int \frac{dt}{t} = -\frac{1}{2} \ln|x^2+1| + C_1$$

$$C_2' = \frac{\frac{e^{2x}}{x^2+1}}{e^{2x}} = \frac{1}{x^2+1} \qquad C_2(x) = \int \frac{dx}{x^2+1} = \arctan(x) + C_2$$

$$y(x) = C_1 e^x + C_2 x e^x - \frac{1}{2} e^x \ln|x^2+1| + x \arctan(x) e^x$$

1 b

$$xy' = \tan(y) \qquad y\left(\tfrac{1}{2}\right) = \tfrac{5\pi}{6}$$

$\tan(y) = 0 \iff y = k\pi,\ k \in \mathbb{Z}$

$y(x) = k\pi,\ k\in\mathbb{Z}$ są rozwiązaniami

$$\frac{dy}{dx}\cdot x = \tan(y) \longrightarrow \int \frac{dy}{\tan(y)} = \int \frac{dx}{x}$$

$$\int \frac{\cos(y)}{\sin(y)}\,dy = \left| \begin{array}{l} t = \sin(y) \\ dt = \cos(y)\,dy \end{array} \right| = \int \frac{dt}{t} = \ln|\sin(y)| + C$$

$\ln|\sin(y)| = \ln|x| + C$

$|\sin(y)| = e^{\ln|x| + C} = e^C|x| = \tilde{C}|x| \qquad \tilde{C} > 0$

$\sin(y) = \pm\tilde{C}x$

$\sin(y) = Cx \qquad C \neq 0$

$y = \arcsin(Cx) \qquad C \in \mathbb{R}$

$\frac{5\pi}{6} = \arcsin\left(\frac{1}{2}C\right)$

$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2} = \frac{1}{2}C \longrightarrow C = 1$

$y(x) = \arcsin(x)$

1 c

$$y' = \frac{y}{x} - 1 \qquad y(1) = -2$$

$$\left| \begin{array}{l} u = \frac{y}{x} \\ y' = u'x + u \end{array} \right|$$

$u'x + u = u - 1$

$\frac{du}{dx} = \frac{-1}{x} \longrightarrow \int 1\,du = -\int \frac{dx}{x}$

$u = -\ln|x| + C$

$y = -x\ln|x| + Cx \qquad C \in \mathbb{R}$

$-2 = -\ln(1) + C$

$-2 = C$

$y(x) = -x\ln|x| - 2x$

2.

a) $y' - \frac{2x}{1+x^2}y = 1 + x^2$

1° $y' - \frac{2x}{1+x^2}y = 0$

$y' = \frac{2x}{1+x^2}y \longrightarrow \int \frac{dy}{y} = 2\int \frac{x}{1+x^2}\,dx$

$\ln|y| = \ln|x^2+1| + C$

$|y| = C(x^2+1)$

$y = C(x^2+1) \qquad C \in \mathbb{R}$

2° $y' = \frac{d}{dx}\left[C(x)(x^2+1)\right] = 2x\,C(x) + (x^2+1)C'(x)$

$2xC + C'(x^2+1) - \frac{2x}{1+x^2}\cdot C(x^2+1) = 1 + x^2$

$2xC + C'(x^2+1) - 2xC = x^2 + 1$

$C'(x^2+1) = x^2 + 1$

$C' = 1 \longrightarrow C = \int 1\,dx = x + C$

$y(x) = (x + C)(x^2+1) = x^3 + x + C(x^2+1)$

b) $y' + 2xy = xe^{-x^2}$

1° $y' = -2xy$

$\frac{dy}{dx} = -2xy \longrightarrow \int \frac{dy}{y} = -2\int x\,dx$

$\ln|y| = -x^2 + C$

$y = Ce^{-x^2} \qquad C \in \mathbb{R}$

2° $y' = -2xCe^{-x^2} + C'e^{-x^2}$

$C'e^{-x^2} \cancel{- 2xCe^{-x^2}} + \cancel{2xCe^{-x^2}} = xe^{-x^2}$

$C'e^{-x^2} = xe^{-x^2}$

$C' = x \qquad C = \int x\,dx = \frac{1}{2}x^2 + C$

$y(x) = \left(\frac{1}{2}x^2 + C\right)e^{-x^2} = Ce^{-x^2} + \frac{1}{2}x^2e^{-x^2}$

4.

a) $y'' - 4y' = 8x \qquad y(0) = 1 \qquad y'(0) = -1$

1° $y'' - 4y' = 0$

$r^2 - 4r = r(r-4) \qquad r_1 = 0 \quad r_2 = 4$

$y_0 = C_1 e^0 + C_2 e^4 = C_1 + C_2 e^{4x} \qquad C_1, C_2 \in \mathbb{R}$

2° $y'' - 4y' = 8x$

$y = (Ax + B) \cdot x = Ax^2 + Bx$

bo $y = C$ jest rozwiązaniem RJ

$y' = 2Ax + B$

$y'' = 2A$

$2A - 4(2Ax + B) = 8x \quad \rightarrow \begin{cases} 8 = -8A \\ 0 = 2A - 4B \end{cases} \quad \begin{cases} A = -1 \\ B = -\frac{1}{2} \end{cases}$

$2A - 8Ax - 4B = 8x$

$y(x) = -x^2 - \frac{1}{2}x \qquad$ CSRN

$y(x) = -x^2 - \frac{1}{2}x + C_1 + C_2 e^{4x} \qquad$ CORN

b) $y'' - 2y' + y = 4\sin^2\left(\frac{x}{2}\right)$

metoda uzmienniania

1° $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = (r-1)^2 = 0 \qquad r_0 = 1$

$y(x) = C_1 e^{1x} + C_2 x e^{1x} = C_1 x e^x + C_2 e^x$

metoda przewidywań

$4\sin^2\left(\frac{x}{2}\right) = 4 \cdot \frac{1}{2}\left[1 - \cos(x)\right] = 2 - 2\cos(x)$

$\rightarrow y = A + B\cos(x) + C\sin(x)$

...

z egzaminu

$y'' - 5y' + 6y = x e^{2x}$

$r^2 - 5r + 6y = r^2 - 2r - 3r + 6y = r(r-2) - 3(r-2) = (r-3)(r-2)$

$y = C_1 e^{3x} + C_2 e^{2x}$

$y = e^{2x}(Ax + B) \cdot x = e^{2x}(Ax^2 + Bx)$

$y' = e^{2x}(2Ax + B) + 2e^{2x}(Ax^2 + Bx)$

$y'' = e^{2x}(2A) + 2e^{2x}(2Ax + B) + 2e^{2x}(2Ax + B) + 4e^{2x}(Ax^2 + Bx)$

... podstawić