

1.

$$\Omega = \{0, \dots, 9\} \quad \forall k \in \Omega \quad P(\{k\}) = 0.1$$

$$X(k) = \cos(k\pi) \quad S_X = \{1, -1\}$$

$$Y(k) = \sin\left(\frac{k\pi}{2}\right) \quad S_Y = \{0, 1, -1\}$$

$x \backslash y$	-1	0	1
-1	.2	0	.3
1	0	.5	0

k	0	1	2	3	4	5	6	7	8	9
X	1	-1	1	-1	1	-1	1	-1	1	-1
Y	0	1	0	-1	0	1	0	-1	0	1

$$P(X=Y) = 0.2 + 0 = 0.2$$

2.

$x \backslash y$	-1	0	1	
-1	$a - \frac{1}{16}$	$\frac{1}{4} - a$	0	$\frac{3}{16}$
0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{7}{16}$
1	$a + \frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4} - a$	$\frac{6}{16}$
	$2a + \frac{1}{8}$	$\frac{1}{2} - a$	$\frac{3}{8} - a$	

$$p(x, y) := P(X=x, Y=y)$$

$$a) \begin{cases} a - \frac{1}{16} \geq 0 \\ a + \frac{1}{16} \geq 0 \\ \frac{1}{4} - a \geq 0 \end{cases} \Rightarrow \begin{cases} a \geq \frac{1}{16} \\ a \geq -\frac{1}{16} \\ a \leq \frac{1}{4} \end{cases} \Rightarrow a \in \left[\frac{1}{16}, \frac{1}{4} \right]$$

$$\sum_{x, y} p(x, y) = 1 \quad (\text{nie zależy od } a)$$

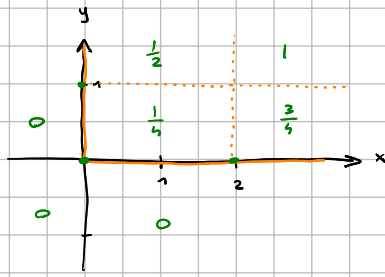
$$\begin{aligned} b) \quad P(X > 2Y) &= \frac{7}{16} \\ &= p(-1, -1) + p(0, -1) + p(1, -1) + p(1, 0) \\ &= a - \frac{1}{16} + \frac{1}{8} + a + \frac{1}{16} + \frac{1}{16} \\ &= 2a + \frac{3}{16} \end{aligned}$$

$$2a + \frac{3}{16} = \frac{7}{16} \Rightarrow a = \frac{1}{8} \in \left[\frac{1}{16}, \frac{1}{4} \right]$$

$$c) \quad F_{X,Y}(0, 1) = P(X \leq 0, Y \leq 1) = \frac{3}{16} + \frac{7}{16} = \frac{10}{16} = \frac{5}{8}$$

$$F_{X,Y}\left(-\frac{1}{2}, \frac{1}{2}\right) = P\left(X \leq -\frac{1}{2}, Y \leq \frac{1}{2}\right) = p(-1, -1) + p(-1, 0) = a - \frac{1}{16} + \frac{1}{4} - a = \frac{3}{16}$$

3.



$$S_{xy} = \{(0,0), (0,1), (2,0)\}$$

x \ y	0	1	
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$
	$\frac{3}{4}$	$\frac{1}{2}$	

$$P(X=\alpha, Y=\beta) = F(\alpha, \beta) - F(\alpha^-, \beta) - F(\alpha, \beta^-) + F(\alpha^-, \beta^-)$$

$$P(X=0, Y=0) = \frac{1}{4} - 0 - 0 + 0 = \frac{1}{4}$$

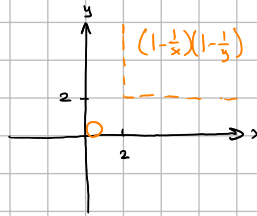
$$P(X=0, Y=1) = \frac{1}{2} - 0 - \frac{1}{4} + 0 = \frac{1}{4}$$

$$P(X=2, Y=0) = \frac{3}{4} - \frac{1}{4} - 0 + 0 = \frac{1}{2}$$

$$P(X=2, Y=1) = 1 - \frac{1}{2} - \frac{3}{4} + \frac{1}{4} = 0$$

4.

$$F(x, y) = \begin{cases} (1 - \frac{1}{x}) \cdot (1 - \frac{1}{y}) & x \geq 2 \wedge y \geq 2 \\ 0 & x < 2 \vee y < 2 \end{cases}$$



$$F_x(x) = \lim_{y \rightarrow \infty} F(x, y) = (1 - \frac{1}{x}) \cdot \mathbb{1}_{[2, \infty)}(x)$$

$$F_y(y) = \lim_{x \rightarrow \infty} F(x, y) = (1 - \frac{1}{y}) \cdot \mathbb{1}_{[2, \infty)}(y)$$

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_x(2) = 1 - (1 - \frac{1}{2}) = \frac{1}{2}$$

$$P(1 < X \leq 3, 1 < Y \leq 4) = F(3, 4) - F(1, 4) - F(3, 1) + F(1, 1) = (1 - \frac{1}{3})(1 - \frac{1}{4}) - 0 - 0 + 0 = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$$

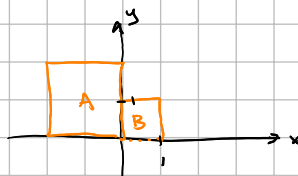
$$P(X=2, Y=2) = F(2, 2) - F(2^-, 2) - F(2, 2^-) + F(2^-, 2^-) = \frac{1}{2} \cdot \frac{1}{2} - 0 - 0 + 0 = \frac{1}{4}$$

5.

$$A = [-2, 0] \times [0, 2]$$

$$B = (0, 1] \times (0, 1]$$

$$f_{xy}(x, y) = \begin{cases} a & -2 \leq x \leq 0 \wedge 0 \leq y \leq 2 \\ \frac{1}{2} & 0 < x \leq 1 \wedge 0 < y \leq 1 \\ 0 & \text{в пр} \end{cases}$$



$$a \in \mathbb{R}$$

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x, y) dx dy = \iint_{\mathbb{R}^2} f_{xy}(x, y) dx dy = \iint_A f_{xy}(x, y) dx dy + \iint_B f_{xy}(x, y) dx dy + 0$$

$$\iint_A f_{xy}(x, y) dx dy = \int_{-2}^0 \left[\int_0^2 a dy \right] dx = \int_{-2}^0 2a dx = 2 \cdot 2a = 4a$$

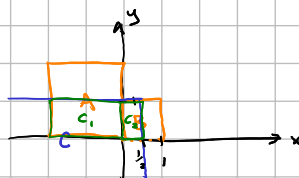
$$\iint_B f_{xy}(x, y) dx dy = \int_0^1 \left[\int_0^1 \frac{1}{2} dy \right] dx = \int_0^1 \frac{1}{2} dx = 1 \cdot \frac{1}{2} = \frac{1}{2} \quad \rightarrow \text{объём прѣдѣлаѣнїя}$$

$$1 = \frac{1}{2} + 4a \Rightarrow a = \frac{1}{8}$$

$$F_{xy}\left(\frac{1}{2}, 1\right) = \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^1 f_{xy}(x, y) dy dx = \iint_C f_{xy}(x, y) dx dy \quad C = (-\infty, \frac{1}{2}] \times (-\infty, 1]$$

$$= \iint_{C_1} f_{xy}(x, y) dx dy + \iint_{C_2} f_{xy}(x, y) dx dy$$

$$= \frac{1}{8} \cdot (2 \cdot 1) + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot 1\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$



6.

$$f_{X,Y}(x,y) = \begin{cases} e^{-y} & x \geq 0 \wedge y \geq x \\ 0 & \text{sonst} \end{cases}$$

$$D = \{(x,y) : 0 \leq x, x \leq y\}$$

$$= \{(x,y) : 0 \leq y, 0 \leq x \leq y\}$$



$$S_X = [0, +\infty) \quad S_Y = [0, +\infty)$$

$$\text{f\u00fcr } x \in S_X: f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy = \int_x^{+\infty} e^{-y} dy = -e^{-y} \Big|_x^{+\infty} = e^{-x}$$

$$\text{f\u00fcr } y \in S_Y: f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx = \int_0^y e^{-y} dx = e^{-y} \int_0^y 1 dx = y e^{-y}$$

$$\begin{aligned} P(X+Y \leq 2) &= \iint_{x+y \leq 2} f_{X,Y}(x,y) dx dy = \int_0^1 \left[\int_x^{2-x} e^{-y} dy \right] dx = \int_0^1 (-e^{-y}) \Big|_x^{2-x} dx = \int_0^1 -e^{x-2} + e^{-x} dx \\ &= -e^{x-2} \Big|_0^1 - e^{-x} \Big|_0^1 = -e^{-1} + e^{-2} + -e^{-1} + e^0 = (1-e^{-1})^2 > 0 \end{aligned}$$

