1.

płaszczyzna ACD $\quad n_1 = \vec{AC} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}\cdot(-1) - \hat{j}(-1) + \hat{k}\cdot 1 = \begin{bmatrix}-1\\1\\1\end{bmatrix}$

a) krawędź nieleżąca na ACD $\to \vec{AB} = (0,1,1)$

$$\alpha = \sphericalangle(\triangle ACD, \vec{AB}) = \frac{\pi}{2} - \sphericalangle(n_1, \vec{AB})$$
$$\cos(\alpha) = \cos\left(\frac{\pi}{2} - \sphericalangle(n_1, \vec{AB})\right) = \sin\sphericalangle(n_1, \vec{AB}) = \frac{|n_1 \times \vec{AB}|}{|n_1|\cdot|\vec{AB}|}$$

$$\cos(\alpha) = \frac{\left\| \begin{vmatrix} \hat{i} & \hat{j} & k \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \right\|}{\sqrt{3}\cdot\sqrt{2}} = \frac{|0\cdot\hat{i} - (-1)\hat{j} + (-1)\hat{k}|}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$$

b) ściana ACB

$$n_2 = \vec{AC} \times \vec{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{i} - \hat{j} + \hat{k} = (-1,-1,1)$$

kąt między ścianami $\quad \beta = \sphericalangle(\triangle ACD, \triangle ACB) = \sphericalangle(n_1, n_2)$

$$\cos(\beta) = \frac{n_1 \cdot n_2}{|n_1|\cdot|n_2|} = \frac{1-1+1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3}$$

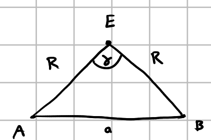
c) środek $\quad E = \frac{1}{4}(A+B+C+D) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

kąt wierzchołek - środek - wierzchołek $\quad \gamma = \sphericalangle(\vec{EA}, \vec{EB})$

$$\cos(\gamma) = \frac{\vec{EA}\cdot\vec{EB}}{|\vec{EA}|\cdot|\vec{EB}|} = \frac{\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)\cdot\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\left(\sqrt{3\cdot\frac{1}{4}}\right)^2} = \frac{\frac{1}{4} - \frac{1}{4} - \frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3}$$

d) $a = |\vec{AB}| = |\vec{BC}| = \dots$

$R = |\vec{EA}| = |\vec{EB}| = \dots$



$$a^2 = R^2 + R^2 - 2R\cdot R\cos(\gamma)$$
$$a^2 = 2R^2(1-\cos(\gamma))$$
$$R = \sqrt{\frac{a^2}{2(1-(-\frac{1}{3}))}} = a\frac{\sqrt{6}}{4}$$

e) $n_{ACD} = (-1,1,1)$

$(X - A)\cdot n_{ACD} = -x + y + z = 0$

$$r = d(E, \triangle ACD) = \frac{|E\cdot n_{ACD}|}{|E|} = \frac{\left|-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right|}{\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}}$$

$a = \sqrt{2} \qquad \frac{r}{a} = \frac{1}{\sqrt{6}} \qquad r = \frac{a}{\sqrt{6}} = \frac{a\sqrt{6}}{6}$

f)

$AB \to A + \alpha B$

$$\tilde{r} = d(E, AB) = \frac{|(E-A)\times B|}{|B|} = \frac{\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{vmatrix} \right\|}{\sqrt{2}} = \frac{|0\cdot\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}|}{\sqrt{2}} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{2}} = \frac{1}{2} = \frac{a}{2\sqrt{2}} = \frac{a\sqrt{2}}{4}$$

g) $V = \frac{1}{6}\left|(\vec{AB}\times\vec{AC})\cdot\vec{AB}\right| = \frac{1}{6}|(B\times C)\cdot D| = \frac{1}{6}|(1,-1,-1)\cdot(0,1,1)| = \frac{1}{6}|0-1-1| = \frac{2}{6} = \frac{1}{3} = \frac{1}{3}\cdot\left(\frac{a}{\sqrt{2}}\right)^3 = \frac{\sqrt{2}}{12}a^3$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k} = (1,-1,-1)$$

3.

$A=(1,0,0)$ $\quad$ $\sphericalangle APB = \sphericalangle BPC = \sphericalangle CPA = \frac{\pi}{2}$

$B=(0,2,0)$

$C=(0,0,3)$

$$\overrightarrow{PA}\cdot\overrightarrow{PB}=0$$

$$(1-p_x, -p_y, -p_z)\cdot(-p_x, 2-p_y, -p_z)=0$$

$$-p_x+p_x^2-2p_y+p_y^2+p_z^2=0$$

$$p_x^2+p_y^2+p_z^2=p_x+2p_y$$ itd analogicznie

$$|P|^2=p_x+2p_y$$
$$|P|^2=p_x+3p_z$$
$$|P|^2=2p_y+3p_z$$

$$\left[\begin{array}{ccc|c}1&2&0&\alpha\\1&0&3&\alpha\\0&2&3&\alpha\end{array}\right]\to\left[\begin{array}{ccc|c}1&2&0&\alpha\\0&-2&3&0\\0&2&3&\alpha\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&3&\alpha\\0&-2&3&0\\0&0&6&\alpha\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&0&\frac{\alpha}{2}\\0&-2&0&-\frac{\alpha}{2}\\0&0&6&\alpha\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&0&\frac{\alpha}{2}\\0&1&0&\frac{\alpha}{4}\\0&0&1&\frac{\alpha}{6}\end{array}\right]$$

$$P=\left(\frac{\alpha}{2},\frac{\alpha}{4},\frac{\alpha}{6}\right)$$
$$|P|^2=\alpha$$
$$\left(\frac{\alpha}{2}\right)^2+\left(\frac{\alpha}{4}\right)^2+\left(\frac{\alpha}{6}\right)^2=\alpha$$
$$\alpha^2\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{36}\right)-\alpha=0 \qquad \frac{49}{144}\alpha\left(\alpha-\frac{144}{49}\right)=0$$
$$\frac{49}{144}\alpha^2-\alpha=0 \qquad \alpha=0 \lor \alpha=\frac{144}{49}$$

$P\left(\frac{144}{49}\right)$ jest lustrzanym odbiciem $P(0)$ względem płaszczyzny ABC

metoda geometryczna

$\overrightarrow{AB}=(1,-2,0)$

$\overrightarrow{AC}=(1,0,-3)$

$$n=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&-2&0\\1&0&-3\end{vmatrix}=\hat{\imath}\cdot6-\hat{\jmath}\cdot(-3)+\hat{k}\cdot2=\begin{bmatrix}6\\3\\2\end{bmatrix}$$

$P=O+\alpha n=\alpha n$ $\quad$ leży na płaszczyźnie ABC

ABC: $6(x-1)+3y+2z=0$ $\qquad$ $6\cdot6\alpha+3\cdot3\alpha+2\cdot2\alpha-6=0$

$6x+3y+2z-6=0$ $\qquad$ $\alpha=\frac{6}{49}$

odbicie $O'=O+2\cdot\frac{6}{49}\cdot(6,3,2)=O+2\alpha n$

2.

1) ściana - krawędź α

$$n_{ABC}=\overrightarrow{AB}\times\overrightarrow{AC} \qquad \overrightarrow{AB}=\begin{bmatrix}0\\1\\0\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\1\\0\end{bmatrix} \qquad \overrightarrow{AC}=\begin{bmatrix}0\\0\\1\end{bmatrix}-\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$n_{ABC}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\-1&1&0\\-1&0&1\end{vmatrix}=\hat{\imath}\cdot1-\hat{\jmath}\cdot(-1)+\hat{k}\cdot1=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\text{krawędź } \overrightarrow{FE}=\begin{bmatrix}0\\-1\\0\end{bmatrix}-\begin{bmatrix}0\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\alpha=\frac{\pi}{2}-\sphericalangle(n_{ABC},\overrightarrow{FE})$$

$$\cos(\alpha)=\sin(\sphericalangle n_{ABC},\overrightarrow{FE})=\frac{|n_{ABC}\times\overrightarrow{FE}|}{|n_{ABC}|\cdot|\overrightarrow{FE}|}=\frac{\left\|\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&1&1\\0&-1&1\end{vmatrix}\right\|}{\sqrt{3}\cdot\sqrt{2}}=\frac{|\hat{\imath}\cdot2-\hat{\jmath}\cdot1+\hat{k}\cdot(-1)|}{\sqrt{6}}=\frac{\sqrt{6}}{\sqrt{6}}=1$$

są równoległe

2) kąt ściana - ściana

$n_{ABC} = (1,1,1)$

$$\vec{BD} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{BC} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$n_{BDC} = \vec{BD} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \hat{i} \cdot (-1) - \hat{j} \cdot (-1) + \hat{k} \cdot (1) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\beta = \sphericalangle(ABC, BDC) = \sphericalangle(n_{ABC}, n_{BDC})$$

$$\cos(\beta) = \frac{n_{ABC} \circ n_{BDC}}{|n_{ABC}| \cdot |n_{BDC}|} = \frac{-1+1+1}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}$$ albo $-\frac{1}{3}$

kąt dwuścienny oznacza ten, który jest w środku

3) kąt wierzchołek - środek - wierzchołek

$$S = \frac{1}{6}(A+B+C+D+E+F) = (0,0,0)$$

1° sąsiednie wierzchołki $A - S - B$

$$\gamma = \sphericalangle(\vec{SA}, \vec{SB}) \qquad \cos(\gamma) = \frac{\vec{SA} \circ \vec{SB}}{|\vec{SA}| \cdot |\vec{SB}|} = \frac{0}{1 \cdot 1} = 0$$

2° przeciwległe wierzchołki $A - S - D$

$$\delta = \sphericalangle(\vec{SA}, \vec{SD}) \qquad \cos(\delta) = \frac{\vec{SA} \circ \vec{SD}}{|\vec{SA}| \cdot |\vec{SD}|} = \frac{-1}{1} = -1$$

4) promień sfery opisanej

krawędź $a = |\vec{AB}| = \sqrt{2}$ $\quad \frac{a}{\sqrt{2}} = 1$

$$R = |\vec{SA}| = |\vec{SB}| = \dots = 1$$

ogólnie $R = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}$

5) promień sfery wpisanej

ABC: $(x - A) \circ (\vec{AB} \times \vec{AC}) = 0$

$$(x - A) \circ n_{ABC} = 0$$
$$(x-1, y, z) \circ (1,1,1) = 0$$
$$x - 1 + y + z = 0$$
$$x + y + z - 1 = 0$$

$$r = \operatorname{dist}(S, ABC) = \frac{|0+0+0-1|}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

ogólnie $r = \frac{\sqrt{3}}{3} \cdot \frac{a}{\sqrt{2}} = \frac{\sqrt{6}}{6} a = \frac{a}{\sqrt{6}}$

6) promień sfery połupisanej

$$AB: \quad x = A + \alpha \vec{AB} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$S - A = -A = (-1,0,0)$
$\vec{AB} = (-1,1,0)$

$$r_h = dist(S, AB) = \frac{|(S-A)\times \vec{AB}|}{|\vec{AB}|} = \frac{\left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{matrix} \right\|}{\sqrt{2}} = \frac{|\hat{i}\cdot 0 - \hat{j}\cdot 0 + \hat{k}\cdot(-1)|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

ogólnie $\quad r_h = \frac{1}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \frac{a}{2}$

$AD = (-2,0,0)$

7) $$V = 2V_{ABCDE} = 2\cdot 2\cdot V_{ABCD} = 4\cdot\frac{1}{6}|(\vec{AB}\times\vec{AC})\cdot\vec{AD}| = \frac{2}{3}|n_{ABC}\circ\vec{AD}| = \frac{2}{3}|-2| = \frac{4}{3}$$

ogólnie $\quad V = \frac{4}{3}\cdot\left(\frac{a}{\sqrt{2}}\right)^3 = \frac{4}{3}\cdot\frac{a^3}{2\sqrt{2}} = \frac{4\sqrt{2}a^3}{3\cdot 4} = \frac{\sqrt{2}}{3}a^3$

4.

$A = (2,-3,1)$
$B = (-2,1,-3)$
$C = (-3,3,-1)$
$D = (1,-3,1)$

$\frac{\text{Objętość równoległościanu}}{\text{pole równoległoboku}}$ = wysokość równoległościanu

$$dist(AB,CD) = \left|\frac{|\vec{AC}\circ(\vec{AB}\times\vec{CD})|}{|\vec{AB}\times\vec{CD}|}\right|$$