1.

a) $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)}$

$n=0 \rightarrow A=1 \qquad n=-1 \rightarrow -B=1 \quad B=-1$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

b) $\sum_{n=1}^{\infty} \frac{n-1}{n!} = \sum_{n=1}^{\infty} \frac{n}{n!} - \frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - \frac{1}{n!}$

$$S_n = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{(n-2)!} - \frac{1}{(n-1)!} + \frac{1}{(n-1)!} - \frac{1}{n!}$$

$$S_n = 1 - \frac{1}{n!}$$

$$\sum_{n=1}^{\infty} \frac{n-1}{n!} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n!} = 1$$

2.

a) $\sum_{n=3}^{\infty} \frac{\ln(n)}{n} \qquad \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$$\int \frac{\ln(x)}{x} dx = \left| \begin{matrix} f = \ln(x) & g' = \frac{1}{x} \\ f' = \frac{1}{x} & g = \ln(x) \end{matrix} \right| = \ln^2(x) - \int \frac{\ln(x)}{x} dx$$

$$2 \int \frac{\ln(x)}{x} dx = \ln^2(x) + C \qquad \int \frac{\ln(x)}{x} dx = \frac{1}{2} \ln^2(x) + C$$

$$\int_3^{\infty} \frac{\ln(x)}{x} dx = \lim_{T\to+\infty} \frac{1}{2}\ln^2(T) - \frac{1}{2}\ln(3) = +\infty \quad \text{całka rozbieżna}$$

więc szereg rozbieżny

b) $\sum_{n=1}^{\infty} \left[\arccos\left(\frac{n}{2n+1}\right) - \frac{\pi}{4}\right]$

$$\lim_{n\to\infty} \arccos\left(\frac{n}{2n+1}\right) - \frac{\pi}{4} = \arccos\left(\frac{1}{2}\right) - \frac{\pi}{4} = \frac{\pi}{3} - \frac{\pi}{4} \neq 0$$

szereg jest rozbieżny (1 warunek konieczny)

c) $\sum_{n=1}^{\infty} \frac{3^n \cdot n^{n^2}}{(n+1)^{n^2}}$

$$\frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{3^n \cdot n^{n\cdot n}}{(n+1)^{n\cdot n}}} = \lim_{n\to\infty} \frac{3\cdot n^n}{(n+1)^n} = 3 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 3 \lim_{n\to\infty} \left(1 + \frac{1}{-n-1}\right)^{(-n-1)\cdot\frac{n}{-n-1}} = 3\cdot e^{-1} = \frac{3}{e} > 1$$

szereg rozbieżny

d) $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

$$\frac{2}{\pi}\frac{1}{n} \leq \sin\left(\frac{1}{n}\right) \leq \frac{1}{n} \Rightarrow 0 \leq \sin^2\left(\frac{1}{n}\right) \leq \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ zbieżny (szereg Dirichleta)}$$

zbieżny (kryterium porównawcze)

$$\lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n+1}\right)}{\sin^2\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\frac{\sin^2\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n+1}\right)^2}\cdot\left(\frac{1}{n+1}\right)^2}{\frac{\sin^2\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)^2}\cdot\left(\frac{1}{n}\right)^2} = \lim_{n\to\infty} \left(\frac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}}\right)^2 \cdot \frac{1}{(n+1)^2} \cdot \left(\frac{\frac{1}{n}}{\sin\left(\frac{1}{n}\right)}\right)^2 \cdot n^2 = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1$$

e) $\sum_{n=1}^{\infty} \frac{2^n+5^n}{3^n+4^n}$

$$\underset{\downarrow \atop 0}{\frac{1}{2^n}} = \frac{2^n+2^n}{4^n+4^n} \leq \frac{2^n+5^n}{3^n+4^n} \leq \frac{5^n+5^n}{3^n+3^n} = \underset{\downarrow \atop \infty}{\left(\frac{5}{3}\right)^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n+5^n}{3^n+4^n}} = \lim_{n\to\infty} = \lim_{n\to\infty} \frac{\sqrt[n]{2^n+5^n}}{\sqrt[n]{3^n+4^n}} = \frac{\lim_{n\to\infty}\sqrt[n]{2^n+5^n}}{\lim_{n\to\infty}\sqrt[n]{3^n+4^n}} = \frac{5}{4} > 1 \quad \text{więc szereg rozbieżny}$$

$$5 = \sqrt[n]{5^n} \leq \sqrt[n]{2^n+5^n} \leq \sqrt[n]{5^n+5^n} = \sqrt[n]{5}\cdot 5$$

f) $\sum_{n=1}^{\infty} \frac{n+1}{(n+2)\cdot 3^n}$

$$\lim_{n\to\infty} \frac{\frac{n+2}{(n+3)3\cdot 3^n}}{\frac{n+1}{(n+2)\cdot 3^n}} = \lim_{n\to\infty} \frac{(n+2)}{3(n+3)\cdot 3^n} \cdot \frac{(n+2)\cdot 3^n}{(n+1)} = \lim_{n\to\infty} \frac{n^2+4n+4}{3n^2+12n+9} = \frac{1}{3} < 1$$

szereg zbieżny

g) $\sum_{n=1}^{\infty} \frac{\arcsin\left(\frac{n^2}{n^2+1}\right)}{2^n}$

$$0 \le \frac{\arcsin\left(\frac{n^2}{n^2+1}\right)}{2^n} \le \frac{\frac{\pi}{2}}{2^n} \qquad \frac{\pi}{2}\sum \frac{1}{2^n} \quad |q|<1 \quad \text{zbieżny}$$

szereg zbieżny (kryterium porównawcze)

h) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$

$$\int \frac{\ln(x)}{x^2}\,dx = \begin{vmatrix} f=\ln(x) & g'=\frac{1}{x^2} \\ f'=\frac{1}{x} & g=\frac{-1}{x} \end{vmatrix} = -\frac{\ln(x)}{x} + \int \frac{dx}{x^2} = -\frac{\ln(x)}{x} - \frac{1}{x} + C = -\frac{\ln(x)+1}{x} + C$$

$$\int_2^{\infty} \frac{\ln(x)}{x^2}\,dx = \lim_{T\to+\infty} -\frac{\ln(T)+1}{T} + \frac{\ln(2)+1}{2} = \frac{\ln(2)+1}{2} < 1$$

$$\lim_{T\to+\infty} \frac{\ln(T)+1}{T} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{T\to+\infty} \frac{1}{T} = 0$$

szereg zbieżny

i) $\sum_{n=1}^{\infty} \frac{(n+1)!\, n^n}{(2n)!\cdot 3^n}$

$$\lim_{n\to\infty} \frac{\frac{(n+2)!\cdot(n+1)^{n+1}}{(2n+2)!\cdot 3^{n+1}}}{\frac{(n+1)!\cdot n^n}{(2n)!\cdot 3^n}} = \lim_{n\to\infty} \frac{(n+2)\cancel{(n+1)!}\cdot(n+1)\cdot(n+1)^n}{(2n+2)(2n+1)\cancel{(2n)!}\cdot\cancel{3^n}\cdot 3} \cdot \frac{\cancel{(2n)!}\cdot\cancel{3^n}}{\cancel{(n+1)!}\cdot n^n} = \lim_{n\to\infty} \frac{1}{3}\cdot\left(\frac{n+1}{n}\right)^n \cdot \frac{n^2+3n+2}{4n^2+6n+2} = \frac{1}{3}\cdot e\cdot\frac{1}{4} = \frac{e}{12} < 1$$

szereg zbieżny

3.

a) $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{2n+1}{3n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{3n+1}\right)^n} = \lim_{n\to\infty} \frac{2n+1}{3n+1} = \frac{2}{3} < 1 \Rightarrow \sum_{n=1}^{\infty}\left|(-1)^{n+1}\left(\frac{2n+1}{3n+1}\right)^n\right| \text{ jest zbieżny} \Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2n+1}{3n+1}\right)^n \text{ zbieżny}$$

bezwzględnie zbieżny

b) $\sum_{n=1}^{\infty} (-1)^{n+1}\sin\left(\frac{1}{n}\right)$

1° $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0 \Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$ zbieżny (kryterium Leibnitza)

2° $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\sin\left(\frac{1}{n}\right)\right| = \sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$

$\frac{2}{\pi}\cdot\frac{1}{n} \le \sin\left(\frac{1}{n}\right) \le \frac{1}{n}$ $\quad \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}$ rozbieżny (szereg harmoniczny) $\Rightarrow \sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ rozbieżny (kryterium porównawcze)

zbieżny warunkowo