

1.

$x \backslash y$	-1	0	1	
0	.1	.1	0	.2
1	.2	.2	.1	.5
2	.1	.1	.1	.3
	.4	.4	.2	

$$F(1, 0) = P(X \leq 1, Y \leq 0) = .1 + .1 + .2 + .2 = 0.6$$

$$F(2, -1) = P(X \leq 2, Y \leq -1) = .1 + .2 + .1 = 0.4$$

$$P(X > 2Y) = .1 + .2 + .2 + .1 + .1 = 0.7$$

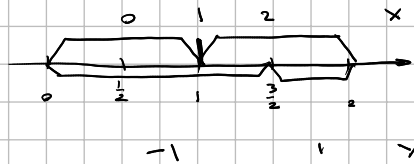
2.

 $\Omega = [0, 2]$ P-geometrische

$| \Omega | = 2$

$$X(\omega) = \begin{cases} 0 & \omega \in [0, 1) \\ 1 & \omega = 1 \\ 2 & \omega \in (1, 2] \end{cases}$$

$$Y(\omega) = \begin{cases} -1 & \omega \in [0, 1.5] \\ 1 & \omega \in (1.5, 2] \end{cases}$$



$$S_X \subseteq \{0, 1, 2\} \quad S_Y \subseteq \{-1, 1\}$$

$X \backslash Y$	-1	1
0	.5	0
1	0	0
2	.25	.25

$$P(0, -1) = \frac{|[0, 1]|}{2} = .5$$

$$P(0, 1) = \frac{1}{2} |[0, 1] \cap (1.5, 2]| = 0$$

$$P(1, -1) = \frac{1}{2} |\{1\} \cap [0, 1.5]| = 0$$

$$P(1, 1) = \frac{1}{2} |\{1\} \cap (1.5, 2]| = 0$$

$$P(2, -1) = \frac{1}{2} |(1, 2] \cap [0, 1.5]| = \frac{1}{2} \cdot \frac{1}{2} = 0.25$$

$$P(2, 1) = \frac{1}{2} |(1, 2] \cap (1.5, 2]| = \frac{1}{2} \cdot \frac{1}{2} = 0.25$$

$$P(X=2) \cdot P(Y=-1) = \frac{1}{2} |(1, 2]| \cdot \frac{1}{2} |(1.5, 2]| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq P(X=2, Y=-1) = \frac{1}{4}$$

$\Rightarrow X$ i Y nie są niezależne

3.

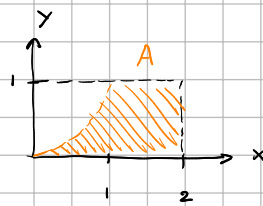
 X i Y są niezależne

$$X \sim U([0, 2]) \quad Y \sim U([0, 1])$$

$$(X, Y) = U(D) \quad D = [0, 2] \times [0, 1]$$

$$f_{X,Y}(x, y) = \frac{1}{2} \mathbb{1}_D(x, y) = f_X \cdot f_Y$$

$$P(Y \leq X^2) = \iint_A f_{X,Y}(x, y) dx dy = |A| \cdot \frac{1}{2} = \frac{2}{3}$$



$$|A| = \int_0^1 x^2 dx + 1 = \frac{1}{3} x^3 \Big|_0^1 + 1 = \frac{4}{3}$$

4.

1 nut oznaczenia kostki

$$\Omega = \{1, 2, 3, 4, 5, 6\} \quad P\text{-klopytane} \quad \forall_{\omega \in \Omega} P(\{\omega\}) = \frac{1}{6}$$

$$X(\omega) = \begin{cases} 0 & \omega \in \{2, 4, 6\} \\ 1 & \omega \in \{1, 3, 5\} \end{cases} \quad \begin{matrix} S_X \in \{0, 1\} \\ S_Y \in \{1, 2\} \end{matrix}$$

$$Y(\omega) = \begin{cases} 1 & \omega \in \{3, 6\} \\ 2 & \omega \in \{1, 2, 4, 5\} \end{cases}$$

$X \backslash Y$	1	2
0	$\frac{1}{6}$	$\frac{2}{6}$
1	$\frac{1}{6}$	$\frac{2}{6}$

$X \backslash Y$	1	2
0	$\frac{1}{6}$	$\frac{2}{6}$
1	$\frac{1}{6}$	$\frac{2}{6}$

$$P(0,1) = P(\{2,4,6\} \cap \{3,6\}) = P(\{6\}) = \frac{1}{6} = P(X=0)P(Y=1) = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6}$$

$$P(0,2) = P(\{2,4\}) = \frac{2}{6} = P(X=0)P(Y=2) = \frac{3}{6} \cdot \frac{4}{6} = \frac{2}{6}$$

$$P(1,1) = P(\{3\}) = \frac{1}{6} = P(X=1)P(Y=1) = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6}$$

$$P(1,2) = P(\{1,5\}) = \frac{2}{6} = P(X=1)P(Y=2) = \frac{3}{6} \cdot \frac{4}{6} = \frac{2}{6}$$

Zmienna s_y niezależna

5.

$$f(x, y) = \begin{cases} e^{-x-y} & x > 0, y > 0 \\ 0 & \text{v p.p.} \end{cases}$$

a)

$$f_x(x) = \int_{-\infty}^{+\infty} f(x, y) dy = \int_0^{+\infty} e^{-x} \cdot e^{-y} dy \quad \mathbb{1}_{\{0, \infty\}}(x) = e^{-x} \cdot [-e^{-y}]_0^{+\infty} \quad \mathbb{1}_{\{0, \infty\}}(x) = e^{-x} \cdot \mathbb{1}_{\{0, \infty\}}(x)$$

$$f_y(y) = e^{-y} \cdot \mathbb{1}_{\{0, \infty\}}(y) \quad (\text{symetryczne})$$

$$f(x, y) = f_x(x) \cdot f_y(y) \rightarrow \text{sq niezależne}$$

$$b) \quad P(1 < X \leq 3, 1 < Y < 2) = P(1 < X \leq 3) P(1 < Y < 2) = \int_1^3 e^{-x} dx \cdot \int_1^2 e^{-y} dy = [-e^{-x}]_1^3 \cdot [-e^{-y}]_1^2 = (-e^{-3} + e^{-1})(-e^{-2} + e^{-1}) \approx 0.07$$

$$P(Y > 1 | X \leq 2) = P(Y > 1) = \int_1^{+\infty} e^{-y} dy = [-e^{-y}]_1^{+\infty} = e^{-1}$$

bo sq niezależne

$$c) \quad P(X + Y > 100) = 1 - P(X + Y \leq 100) = 1 - \int_0^{100} \int_0^{100-y} e^{-x-y} dx dy$$

$$= 1 + \int_0^{100} e^{-x-y} \Big|_0^{100-y} dy$$

$$= 1 + \int_0^{100} e^{-100} - e^{-y} dy = 1 + e^{-100} \cdot (e^{-y}) \Big|_0^{100} = 101 e^{-100}$$



6.

$$f_{x,y}(x,y) = \frac{1}{6\pi} \exp\left(-\frac{1}{26} [10x^2 + 4x(y+1) + 4(y+1)^2]\right)$$

$$\frac{1}{6\pi} = \frac{1}{2\pi \sqrt{\det C}} \Rightarrow \sqrt{\det C} = 3 \Rightarrow \det C = 9$$

$$-\frac{1}{2 \det C} = -\frac{1}{18}$$

$$f_{x,y}(x,y) = \frac{1}{2\pi \cdot 3} \exp\left(-\frac{1}{18} [5(x-0)^2 + 2(x-0)(y+1) + 2(y+1)^2]\right)$$

$$m = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \quad (X,Y) \sim N(m,C) \Rightarrow X \sim N(0,2) \wedge Y \sim N(-1,5)$$

$$\text{cov}(X,Y) \neq 0 \Rightarrow \text{nie są niezależne}$$

$$f_x(x) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right) \quad f_y(y) = \frac{1}{\sqrt{10\pi}} \exp\left(-\frac{(y+1)^2}{10}\right)$$

7.

$$f_{xy}(x, y) = \frac{1}{36\pi} \exp\left(-\frac{1}{162} \left[9(x-153)^2 - 12(x-153)(y-48) + \frac{25}{9}(y-48)^2 \right]\right)$$

$$\frac{1}{36\pi} = \frac{1}{2\pi\sqrt{\det C}} \Rightarrow \sqrt{\det C} = 18 \Rightarrow \det C = 324$$

$$-\frac{1}{2\det C} = -\frac{1}{648}$$

$$-\frac{1}{162} \left[9(x-153)^2 - 12(x-153)(y-48) + \frac{25}{9}(y-48)^2 \right]$$

$$= -\frac{1}{648} \left[36(x-153)^2 - 48(x-153)(y-48) + 25(y-48)^2 \right]$$

$$m = \begin{bmatrix} 153 \\ 48 \end{bmatrix} \quad C = \begin{bmatrix} 36 & 24 \\ 24 & 25 \end{bmatrix}$$

$$a) (X, Y) \sim N(m, C) \Rightarrow X \sim N(153, 36)$$

$$P(X < 160) = F(160) = \Phi\left(\frac{160-153}{6}\right) = \Phi\left(\frac{7}{6}\right)$$

$$\Phi(1.16) = 0.8770$$

87.7% ma wzrost poniżej 160 cm

$$b) (X, Y) \sim N(m, C) \Rightarrow Y \sim N(48, 25)$$

$$P(Y > 36) = 1 - P(Y \leq 36) = 1 - \Phi\left(\frac{36-48}{5}\right) = 1 - \Phi(-2.4) = 1 - (1 - \Phi(2.4)) = \Phi(2.4) \approx 99.18\%$$