

1.

O - krasen arbor

$$O \sim B(n, p = \frac{1}{3})$$

K - krasen arbor

k	1	2	3
P(K=k)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\begin{aligned} a) P(O=2) &= P(O=2|K=1)P(K=1) + P(O=2|K=2)P(K=2) + P(O=2|K=3)P(K=3) \\ &= 0 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right) \cdot \frac{1}{3} \\ &= \frac{1}{9} \cdot \frac{1}{3} + \frac{2}{9} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27} + \frac{2}{27} = \frac{3}{27} = \frac{1}{9} \end{aligned}$$

$$b) P(K=3|O=2) = \frac{P(O=2|K=3)P(K=3)}{P(O=2)} = \frac{\frac{2}{27}}{\frac{3}{27}} = \frac{2}{3}$$

2.

$$f_{X,Y}(x,y) = \frac{1}{4\pi} \exp\left(-\frac{1}{2} [2(x-1)^2 + 2(x-1)(y+1) + (y+1)^2]\right)$$

$$\sqrt{\det C} = 2 \Rightarrow \det C = 4$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 2} \exp\left(-\frac{1}{2 \cdot 4} [4(x-1)^2 - 2 \cdot (-2)(x-1)(y+1) + 2(y+1)^2]\right)$$

$$(X,Y) \sim N\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}_m, \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}_C\right)$$

$$a) E(X+2Y) = E(X^2 + 2XY) = EX^2 + 2EXY = 3 + 2 \cdot (-3) = -3$$

$$VX = EX^2 - (EX)^2 \Rightarrow EX^2 = VX + (EX)^2 = 2 + 1^2 = 3$$

$$\text{cov}(X,Y) = EXY - EXEY \Rightarrow EXY = \text{cov}(X,Y) + EXEY = -2 + (1)(-1) = -3$$

$$b) \begin{bmatrix} Z \\ T \end{bmatrix} = \begin{bmatrix} X-Y+1 \\ X+3Y+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}}_A \cdot \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_b + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$m^* = A m + b = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C^* = A C A^T = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -14 \\ -14 & 26 \end{bmatrix}$$

$$(Z,T) \sim N(m^*, C^*)$$

$$c) Y = \sum_{k=1}^{200} X_k \approx N(200 EX, 200 VX) = N(200, 400)$$

$$\begin{aligned} P(|Y-180| > 20) &= P(Y-180 > 20 \vee Y-180 < -20) \\ &= P(Y > 200 \vee Y < 160) = P(Y > 200) + P(Y < 160) \\ &= 1 - F(200) + F(160) = 1 - \Phi\left(\frac{200-200}{20}\right) + \Phi\left(\frac{160-200}{20}\right) \\ &= 1 - \Phi(0) + \Phi(-2) = 1 - \Phi(0) + 1 - \Phi(2) \\ &\approx 0.5223 \end{aligned}$$

3.

$X \backslash Y$	-1	0	1
-2	.3	.2	0
0	0	.3	.2
	.3	.5	.2

$$P(X^2 + Y^2 \leq 1) = 0 + 0.3 + 0.2 = 0.5$$

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{V_X V_Y}} = \frac{0.5}{\sqrt{0.25}} = \frac{5}{4}$$

$$EX = -2 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = -1$$

$$EX^2 = 4 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 2$$

$$VX = 2 - (-1)^2 = 1$$

$$EY = -1 \cdot \frac{2}{5} + 0 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = -\frac{1}{5}$$

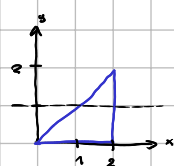
$$EY^2 = 1 \cdot \frac{2}{5} + 0 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{1}{2}$$

$$VY = \frac{1}{2} - \left(-\frac{1}{5}\right)^2 = 0.45$$

$$EXY = 2 \cdot \frac{2}{5} = \frac{4}{5}$$

$$\text{cov}(X, Y) = \frac{4}{5} - (-1)\left(-\frac{1}{5}\right) = \frac{1}{2}$$

4.



$$(X, Y) \sim U(D) \quad |D| = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

$$f_{X,Y}(x, y) = \frac{1}{2} 1_D(x, y)$$

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq x\}$$

$$= \{(x, y) : 0 \leq y \leq 2, y \leq x \leq 2\}$$

a) dann $1 \leq b \leq 2$

$$P(X \leq b | Y < 1)$$

$$= \frac{P(X \leq b, Y < 1)}{P(Y < 1)} = \frac{\frac{1}{2} \left(\frac{1}{2} \cdot 1 \cdot 1 + 1 \cdot (b-1) \right)}{\frac{3}{2}} = \frac{\frac{1}{2} (b-1)}{\frac{3}{2}}$$

$$= \frac{1}{3} \cdot \frac{1}{2} (b-1) = \frac{1}{6} (b-1)$$

$$b) f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \begin{cases} \int_y^2 \frac{1}{2} dx = \frac{1}{2}(2-y) & y \in [0, 2] \\ 0 & \text{sonst.} \end{cases}$$

$$f(x|y) = \begin{cases} \frac{\frac{1}{2}}{\frac{1}{2}(2-y)} = \frac{1}{2-y} & x \in [0, 2] \text{ d.h. } y \in [0, 2] \\ 0 & \text{sonst.} \end{cases}$$

$$E(X|Y=1) = \int_1^2 x \cdot \frac{1}{2-1} dx = \frac{1}{2} x^2 \Big|_1^2 = \frac{3}{2}$$

5.

$$f(x, a) = \begin{cases} (2a+1)x^{2a} & x \in [0,1] \\ 0 & \text{w.p.} \end{cases}$$

$$L = \prod f(x_i, a)$$

$$\begin{aligned} \ln L &= \ln \left(\prod f(x_i, a) \right) = \sum \ln(f(x_i, a)) \\ &= \sum \ln((2a+1)x_i^{2a}) \\ &= \sum \ln(2a+1) + 2a \sum \ln(x_i) \\ &= \sum \ln(2a+1) + \sum 2a \ln(x_i) \\ &= n \ln(2a+1) + 2a \sum \ln(x_i) \\ &= n \ln(2a+1) + 2a \ln \left(\prod x_i \right) \end{aligned}$$

$$\frac{\partial \ln L}{\partial a} = \frac{2n}{2a+1} + 2 \ln \left(\prod x_i \right)$$

$$\frac{\partial^2 \ln L}{\partial a^2} = -\frac{2n}{(2a+1)^2} < 0$$

$$\frac{2n}{2a+1} + 2 \ln \left(\prod x_i \right) = 0$$

$$\frac{n}{2a+1} = -\ln \left(\prod x_i \right)$$

$$n = -2a \ln \left(\prod x_i \right) - \ln \left(\prod x_i \right)$$

$$a = -\frac{n + \ln \left(\prod x_i \right)}{2 \ln \left(\prod x_i \right)}$$

Dla podanej próby

$$n=10$$

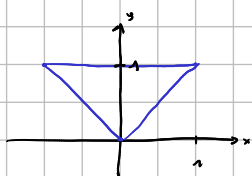
$$\prod x_i = 0.073$$

$$\ln \left(\prod x_i \right) \approx -2.62$$

$$a \approx 1.4$$

6.

$$f(x, y) = \frac{15}{2} x^2 y \cdot 1_D(x, y)$$



$$D = \{(x, y) : -1 \leq x \leq 1, |x| \leq y \leq 1\}$$

$$1_D(x, y) = \begin{cases} 1 & 0 \leq y \leq 1, -y \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \begin{cases} \int_{-y}^y \frac{15}{2} x^2 y dx = \frac{5}{2} y \cdot x^3 \Big|_{-y}^y = \frac{5}{2} y (y^3 - (-y)^3) = 5y^4 & y \in [0, 1] \\ 0 & \text{w.p.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \int_{|x|}^1 \frac{15}{2} x^2 y dy = \frac{15}{4} x^2 \cdot y^2 \Big|_{|x|}^1 = \frac{15}{4} x^2 (1 - x^2) & x \in [-1, 1] \\ 0 & \text{w.p.} \end{cases}$$

$$F_Y(y) = \int_{-\infty}^y f_Y(y) dy = \begin{cases} 0 & y < 0 \\ \int_0^y 5y^4 dy = y^5 & y \in [0, 1] \\ 1 & y > 1 \end{cases}$$

$$EX = \int_{-1}^1 \frac{15}{4} x^3 (1 - x^2) dx = \frac{15}{4} \int_{-1}^1 (x^3 - x^5) dx = 0$$

$$EY = \int_0^1 5y^5 dy = \frac{5}{6} y^6 \Big|_0^1 = \frac{5}{6}$$

$$EXY = \int_{\mathbb{R}^2} xy f(x, y) dx dy = \int_D \frac{15}{2} x^3 y^2 dx dy = \frac{15}{2} \int_0^1 \int_{-y}^y x^3 y^2 dx dy$$

$$= \frac{15}{2} \int_0^1 \frac{1}{4} y^4 x^4 \Big|_{-y}^y dy = \frac{15}{2} \int_0^1 \frac{1}{2} y^4 (y^4 - (-y)^4) dy = \frac{15}{2} \int_0^1 0 dy = 0$$

$$\text{cov}(X, Y) = 0 - 0 \cdot \frac{5}{6} = 0$$

7.

$$X \sim N(0, 4)$$

$$Y \sim N(-1, 9)$$

niezależne

$$\begin{aligned} a) \quad P(X > 0, |Y| < 5) &= P(X > 0) P(|Y| < 5) \\ &= (1 - P(X \leq 0)) P(-5 < Y < 5) \\ &= [1 - \Phi(0)] [\Phi(\frac{5-(-1)}{3}) - \Phi(\frac{-5-(-1)}{3})] \\ &= \frac{1}{2} (\Phi(2) - \Phi(-\frac{4}{3})) \\ &= \frac{1}{2} (\Phi(2) - 1 + \Phi(\frac{4}{3})) \\ &\approx 0,4427 \end{aligned}$$

$$b) \quad (x, y) \sim N \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \right)$$

$$\begin{bmatrix} z \\ t \end{bmatrix} = \begin{bmatrix} x-y+4 \\ 2x+y-5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$m^* = A m + b = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

$$C^* = A C A^T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ -1 & 25 \end{bmatrix}$$

$$(z, t) \sim N(m^*, C^*)$$

$$\det C^* = 324$$

$$f_{z,t}(z, t) = \frac{1}{2\pi\sqrt{324}} \exp\left(-\frac{1}{2 \cdot 324} [25(z-5)^2 + 2(z-5)(t+6) + 13(t+6)^2]\right)$$

8.

$$S_x = \{0, 1, 2\} \quad S_y = \{-1, 1\}$$

$x \backslash y$	-1	1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{4}$	0
	$\frac{1}{2}$	$\frac{1}{2}$

$$P(Y = -1) = P(Y = 1) = \frac{1}{2}$$

$$\begin{aligned} P(X=0, Y=-1) &= P(X=0|Y=-1)P(Y=-1) = \frac{1}{2} \cdot \frac{1}{2} \\ &= P(X=2, Y=-1) = P(X=1|Y=1) \end{aligned}$$

$$S_{x|y=1} = \{0, 1\} \Rightarrow P(X=2, Y=1) = 0$$

$$P(X=1, Y=-1) = \frac{1}{2} - \frac{1}{2} - \frac{1}{8} = \frac{1}{8}$$

$$P(X=0, Y=1) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$A = |X| + |Y| \geq 2$$

$$P(A) = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$P(X=1|A) = \frac{\frac{1}{8} + \frac{1}{8}}{\frac{1}{2}} = \frac{2}{4} = \frac{1}{2}$$

$$P(X=2|A) = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$E(X|A) = 1 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} = \frac{5}{4}$$

$$E(X^2|A) = 1 \cdot \frac{3}{4} + 4 \cdot \frac{1}{4} = \frac{7}{4}$$

$$V(X|A) = \frac{7}{4} - \frac{25}{16} = \frac{3}{16}$$

$$F_{X|A}(x) = \begin{cases} 0 & x < 1 \\ \frac{3}{4} & x \in [1, 2) \\ 1 & x \geq 2 \end{cases}$$

9.

 B_1 - biała u I losowaniu B_2 - biała u II losowaniu X - liczba białych u obu losowaniach

$$S_x = \{0, 1, 2\}$$

$$\begin{aligned} P(B_2) &= P(B_2|B_1)P(B_1) + P(B_2|B_1')P(B_1') \\ &= \frac{5}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{2}{6} = \frac{14}{21} \end{aligned}$$

$$\begin{aligned} P(X=0) &= P(X=0|B_1)P(B_1) + P(X=0|B_1')P(B_1') \\ &= 0 + P(B_2|B_1')P(B_1') = \frac{2}{7} \cdot \frac{2}{6} = \frac{1}{7} \end{aligned}$$

$$\begin{aligned} P(X=1) &= P(X=1|B_1)P(B_1) + P(X=1|B_1')P(B_1') \\ &= P(B_2|B_1)P(B_1) + P(B_2|B_1')P(B_1') \\ &= \frac{2}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{2}{6} = \frac{4}{21} + \frac{4}{21} = \frac{8}{21} \end{aligned}$$

$$\begin{aligned} P(X=2) &= P(X=2|B_1)P(B_1) + P(X=2|B_1')P(B_1') \\ &= P(B_2|B_1)P(B_1) + 0 \\ &= \frac{5}{7} \cdot \frac{4}{6} = \frac{10}{21} \end{aligned}$$

$$P(B_1|B_2) = \frac{P(B_2|B_1)P(B_1)}{P(B_2)} = \frac{\frac{10}{21}}{\frac{14}{21}} = \frac{5}{7}$$

$$E(X^2 - X + 1) = EX^2 - EX + 1 = \frac{48 - 28 + 21}{21} = \frac{41}{21}$$

$$EX = 0 \cdot \frac{1}{7} + 1 \cdot \frac{8}{21} + 2 \cdot \frac{10}{21} = \frac{28}{21}$$

$$EX^2 = 0 \cdot \frac{1}{7} + 1 \cdot \frac{8}{21} + 4 \cdot \frac{10}{21} = \frac{48}{21}$$

10.

$$Z = |X| - |Y|$$

$X \backslash Y$	-1	0	3
1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

$X \backslash Y$	-1	0	3
1	0	1	-2
2	1	2	-1

$$P(Z=2) = \begin{vmatrix} 2 & -2 & 0 & 1 & 2 \\ \hline \frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{vmatrix}$$

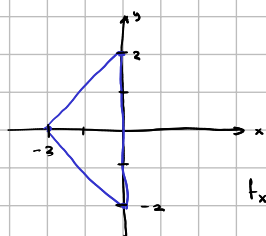
$$F_Z(z) = \begin{cases} 0 & z < -2 \\ \frac{1}{6} & z \in [-2, 0) \\ \frac{2}{6} & z \in [0, 1) \\ \frac{5}{6} & z \in [1, 2) \\ 1 & z \geq 2 \end{cases}$$

$$\begin{aligned} a) \quad V(3X - 6Y - 10) &= V(3X - 6Y) = V(3X) - 2\text{cov}(3X, 6Y) + V(6Y) \\ &= 9VX - 36\text{cov}(X, Y) + 36VY = 9 \cdot \frac{2}{9} - 36 \cdot \frac{2}{9} + 36 \cdot \frac{65}{36} \\ &= 2 - 8 + 65 = 59 \end{aligned}$$

$$\begin{aligned} EX &= 1 \cdot \frac{2}{9} + 2 \cdot \frac{1}{3} = \frac{4}{3} & EY &= -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6} = \frac{1}{2} \\ EX^2 &= 1 \cdot \frac{2}{9} + 4 \cdot \frac{1}{3} = 2 & EY^2 &= 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} + 9 \cdot \frac{1}{6} = \frac{11}{6} \\ VX &= 2 - \frac{16}{9} = \frac{2}{9} & VY &= \frac{11}{6} - \frac{1}{4} = \frac{45}{24} \\ EXY &= -1 \cdot \frac{1}{3} + 3 \cdot \frac{1}{6} - 2 \cdot \frac{1}{6} = 0 \\ \text{cov}(X, Y) &= 0 - \frac{4}{3} \cdot \frac{1}{2} = -\frac{2}{3} \end{aligned}$$

11.

$$f(x, y) = \frac{3}{16}(x+2) \cdot 1_D(x, y)$$



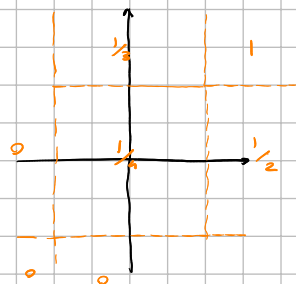
$$\begin{aligned} D &= \{(x, y) : -2 \leq x \leq 0, -x-2 \leq y \leq x+2\} \\ &= \{(x, y) : -2 \leq y \leq 2, |y|-2 \leq x \leq 0\} \end{aligned}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \int_{-x-2}^{x+2} \frac{3}{16}(x+2) dy = \frac{3}{8}(x+2)^2 & x \in [-2, 0] \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-2}^{0} \frac{3}{16}(x+2) dx = \frac{1}{2x+4} & y \in [-2, 2] \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E(Y|X=-1) &= \int_{-1}^1 y f(y|-1) dy = \int_{-1}^1 y \cdot \frac{1}{2} dy = \frac{1}{2} y^2 \Big|_{-1}^1 = 0 \\ E(Y^2|X=-1) &= \int_{-1}^1 y^2 f(y|-1) dy = \int_{-1}^1 y^2 \cdot \frac{1}{2} dy = \frac{1}{6} y^3 \Big|_{-1}^1 = \frac{1}{3} \\ V(Y|X=-1) &= \frac{1}{3} - 0^2 = \frac{1}{3} \end{aligned}$$

12.



$$\begin{aligned} P(X=-1, Y=-1) &= \frac{1}{4} \cdot 0 \cdot 0 = \frac{1}{4} \\ P(X=1, Y=-1) &= \frac{1}{2} - \frac{1}{4} \cdot 0 \cdot 0 = \frac{1}{4} \\ P(X=-1, Y=1) &= \frac{1}{2} - 0 - \frac{1}{4} \cdot 0 = \frac{1}{4} \\ P(X=1, Y=1) &= 1 - \frac{1}{2} - \frac{1}{2} + \frac{5}{12} = \frac{5}{12} \end{aligned}$$

$X \backslash Y$	-1	1
-1	$\frac{1}{4}$	$\frac{1}{12}$
1	$\frac{1}{4}$	$\frac{5}{12}$
	$\frac{1}{2}$	$\frac{1}{2}$

$$\begin{aligned} EX &= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} & EY &= \frac{1}{2} - \frac{1}{2} = 0 \\ EX^2 &= \frac{2}{3} + \frac{1}{3} = 1 & EY^2 &= \frac{1}{2} + \frac{1}{2} = 1 \\ VX &= 1 - \frac{1}{9} = \frac{8}{9} & VY &= 1 - 0^2 = 1 \\ EXY &= \frac{1}{4} \cdot \frac{1}{4} - \frac{1}{12} + \frac{5}{12} = \frac{1}{3} \\ \text{cov}(X, Y) &= \frac{1}{3} - \frac{1}{3} \cdot 0 = \frac{1}{3} \\ \rho(X, Y) &= \frac{\frac{1}{3}}{\sqrt{\frac{8}{9} \cdot 1}} = \frac{1}{\sqrt{6}} \end{aligned}$$

$$X \cdot Y = 0$$

$$P(X+Y=0) = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$$

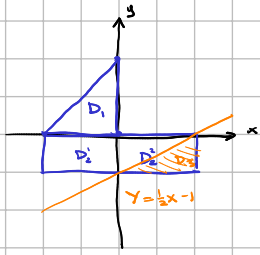
$$P(Y=-1|X+Y=0) = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

$$P(Y=1|X+Y=0) = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

$$E(Y|X+Y=0) = -1 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = -\frac{1}{2}$$

13.

$$f(x,y) = \begin{cases} ax^2 & (x,y) \in D_1 \\ b & (x,y) \in D_2 \\ 0 & \text{u.p.p.} \end{cases} \quad P(Y < 0) = \frac{1}{2}$$



$$D_1 = \{(x,y): -2 \leq x \leq 0, 0 \leq y \leq x+2\}$$

$$= \{(x,y): 0 \leq y \leq 2, y-2 \leq x \leq 0\}$$

$$P(Y < 0) = \iint_{Y < 0} f(x,y) dx dy = \iint_{D_2} b dx dy = |D_2| \cdot b = 4b$$

$$\frac{1}{2} = 4b \Rightarrow b = \frac{1}{8}$$

$$\iint_{\mathbb{R}^2} f(x,y) dx dy = 1 \Rightarrow \iint_{D_1} f(x,y) dx dy = \frac{1}{2}$$

$$\iint_{D_1} f(x,y) dx dy = \int_{-2}^0 \int_0^{x+2} ax^2 dy dx = \int_{-2}^0 ax^2(x+2) dx = a \int_{-2}^0 x^3 + 2x^2 dx$$

$$= a \left[\frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_{-2}^0 = a \left(\frac{1}{4} \cdot 0 + \frac{2}{3} \cdot 0 - \frac{1}{4}(-2)^4 - \frac{2}{3}(-2)^3 \right)$$

$$= a \left(-\frac{16}{4} + \frac{16}{3} \right) = a \frac{64-48}{12} = \frac{16}{12}a = \frac{4}{3}a$$

$$\frac{1}{2} = \frac{4}{3}a \Rightarrow a = \frac{3}{8}$$

$$f_x(x) = \int_{-\infty}^{+\infty} f(x,y) dy = \begin{cases} \int_0^{x+2} \frac{3}{8}x^2 dy + \int_{-1}^0 \frac{1}{8} dy = \frac{3}{8}x^2(x+2) + \frac{1}{8} & x \in [-2, 0] \\ \int_{-1}^0 \frac{1}{8} dy = \frac{1}{8} & x \in (0, 2] \\ 0 & \text{u.p.p.} \end{cases}$$

$$f_y(y) = \int_{-\infty}^{+\infty} f(x,y) dx = \begin{cases} \int_{y-2}^0 \frac{3}{8}x^2 dx = \frac{1}{8}x^3 \Big|_{y-2}^0 = -\frac{1}{8}(y-2)^3 & y \in [0, 2] \\ \int_{-2}^0 \frac{1}{8} dx = \frac{1}{2} & y \in [-1, 0) \\ 0 & \text{u.p.p.} \end{cases}$$

$$P(Y > \frac{1}{2}X - 1) = 1 - P(Y < \frac{1}{2}X - 1) = 1 - \frac{1}{8}|D_3| = 1 - \frac{1}{8} \cdot \frac{1}{2} \cdot 2 \cdot 1 = \frac{7}{8}$$

14.

$$S_{100} \sim B(n=100, p=0.1) \quad \text{binomial}$$

$$S_{100} \approx P(\lambda=10) \quad \lambda=np$$

$$S_{100} \approx N(10, 9)$$

$$\begin{aligned} P(7 < S_{100} < 20) &= P(8 \leq S_{100} \leq 19) = \Phi\left(\frac{19-10}{3}\right) - \Phi\left(\frac{8-10}{3}\right) \\ &= \Phi(3) - \Phi\left(-\frac{2}{3}\right) = \Phi(3) + \Phi\left(\frac{2}{3}\right) - 1 \\ &\approx 0.7441 \end{aligned}$$

$$\begin{aligned} P(S_{100} > 11) &= 1 - P(S_{100} \leq 11) = 1 - \Phi\left(\frac{11-10}{3}\right) = \Phi\left(\frac{10-11}{3}\right) = 0.2099 \\ \frac{10-M}{3} &= 1.34 \Rightarrow M = 5.98 \end{aligned}$$

15.

$$a) X \sim B(n=800, p=0.4) \approx N(320, 192)$$

$$\begin{aligned} P(300 < X < 400) &= \Phi(5.81) - \Phi(-1.45) = \Phi(5.81) + \Phi(1.44) - 1 \\ &\approx \Phi(1.44) \approx 0.9251 \end{aligned}$$

$$b) P(X > N) = 1 - P(X \leq N) = 1 - \Phi\left(\frac{N-320}{\sqrt{192}}\right) = 0.0228$$

$$\Phi\left(\frac{N-320}{\sqrt{192}}\right) = 0.9772 \approx \Phi(2)$$

$$\frac{N-320}{\sqrt{192}} = 2 \Rightarrow N \approx 347$$

16.

Z - zlotane myszu

z	1	2	3
P(Z=z)	1/6	1/2	1/3

$$U - \text{linowa umygu, ktore umygu} \quad U \sim B(2, \frac{1}{2})$$

$$\begin{aligned} P(X=0) &= P(U=1|Z=1)P(Z=1) + P(U=2|Z=2)P(Z=2) + P(U=3|Z=3)P(Z=3) \\ &= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{12} + \frac{1}{4} + \frac{1}{24} = \frac{6}{24} \end{aligned}$$

$$\begin{aligned} P(X=1) &= P(U=0|Z=1)P(Z=1) + P(U=1|Z=2)P(Z=2) + P(U=2|Z=3)P(Z=3) \\ &= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2}\right) \cdot \frac{1}{2} + \frac{1}{3} \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} = \frac{11}{24} \end{aligned}$$

$$\begin{aligned} P(X=2) &= P(U=0|Z=2)P(Z=2) + P(U=1|Z=3)P(Z=3) \\ &= \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{3} \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{6} = \frac{5}{24} \end{aligned}$$

$$\begin{aligned} P(X=3) &= P(U=0|Z=3)P(Z=3) \\ &= \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{24} \end{aligned}$$

$$P(Z=3|X=1) = \frac{P(X=1|Z=3)P(Z=3)}{P(X=1)} = \frac{P(U=2|Z=3)P(Z=3)}{P(X=1)} = \frac{\frac{1}{6}}{\frac{11}{24}} = \frac{3}{11}$$

17.

$$P(X=x) = \begin{cases} 2p & x=1 \\ 1-5p & x=2 \\ 3p & x=3 \end{cases}$$

$$p \in (0, \frac{1}{5})$$

$$L = \prod_{i=1}^n P(X=x_i) = (2p)^{n_1} \cdot (1-5p)^{n_2} \cdot (3p)^{n_3}$$

$$\ln L = n_1 \ln(2p) + n_2 \ln(1-5p) + n_3 \ln(3p)$$

$$\frac{\partial \ln L}{\partial p} = \frac{n_1}{p} - \frac{5n_2}{1-5p} + \frac{n_3}{p}$$

$$\frac{\partial^2 \ln L}{\partial p^2} = -\frac{n_1}{p^2} - \frac{25n_2}{(1-5p)^2} - \frac{n_3}{p^2} < 0$$

$$\frac{n_1}{p} - \frac{5n_2}{1-5p} + \frac{n_3}{p} = 0 \Rightarrow \frac{n_1 + n_3}{p} = \frac{5n_2}{1-5p}$$

$$n_1 - 5pn_1 + n_3 - 5pn_3 = 5n_2p$$

$$n_1 + n_3 = 5pn_1 + 5pn_2 + 5pn_3$$

$$p = \frac{n_1 + n_3}{5(n_1 + n_2 + n_3)}$$

Dla podanej próbki

$$n_1 = 5 \quad n_2 = 3 \quad n_3 = 12$$

$$p = \frac{5+12}{5 \cdot 20} = 0.17$$