1.

rozkład dyskretny

$S_X = \{-1, 0, k\}$

$P(X=-1) = \frac{1}{3}$
$P(X=0) = \frac{1}{3k}$
$P(X=k) = \frac{1}{k}$

$1 = \sum_{x_i \in S_X} P(X=x_i) = \frac{1}{3} + \frac{1}{3k} + \frac{1}{k}$

$\frac{2}{3} = \frac{4}{3k} \Rightarrow k = 2$

$EX = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} = \frac{2}{3}$

$E(X^2) = (-1)^2 \frac{1}{3} + 0^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{2} = \frac{7}{3}$

$VX = E(X - EX)^2 = E(X^2) - (EX)^2 = \frac{7}{3} - (\frac{2}{3})^2 = \frac{21}{9} - \frac{4}{9} = \frac{17}{9}$

$V(3X+1) = 3^2\, VX = 9 \cdot \frac{17}{9} = 17$

2.

$$f_X(x) = cx^2 \cdot \mathbb{1}_{[0,6]}(x)$$

a)

$$1 = \int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^6 cx^2\,dx = \frac{1}{3}c \cdot x^3\Big|_0^6 = 72c \implies c = \frac{1}{72}$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx = \begin{cases} 0 & t < 0 \\ \int_0^t \frac{1}{72}x^2\,dx = \frac{1}{216}t^3 & 0 \le t \le 6 \\ 1 & t > 6 \end{cases}$$

b)

$$P(X < 2) = \int_{-\infty}^{2} f_X(x)\,dx = \int_0^2 f_X(x)\,dx = F_X(2) - F_X(0) = \frac{1}{27}$$

c)

$$EX = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_0^6 x \cdot \frac{1}{72}x^2\,dx = \frac{1}{288} \cdot x^4\Big|_0^6 = 4.5$$

$$VX = E(X - EX)^2 = E(X^2) - (EX)^2$$

$$EX^2 = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_0^6 \frac{1}{72}x^4 = \frac{1}{360} \cdot x^5\Big|_0^6 = 21.6$$

$$\sigma = \sqrt{21.6 - (4.5)^2} \simeq 1.16$$

3.

rozkład dyskretny

$S_x = \{0, 1, 2\}$

$EX = 0.9 \qquad VX = 0.69$

$VX = EX^2 - (EX)^2 \implies EX^2 = 0.69 + 0.9^2 = 1.5$

$$\begin{cases} 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 = 0.9 \\ 0 \cdot p_0 + 1 \cdot p_1 + 4 \cdot p_2 = 1.5 \end{cases}$$

i	0	1	2
p_i	0.4	0.3	0.3

$2 p_2 = 0.6 \implies p_2 = 0.3$

$p_1 = 1.5 - 4 \cdot 0.3 = 0.3$

$p_0 = 1 - p_1 - p_2 = 0.4$

$E|X - EX| = 0.4 \cdot |0 - 0.9| + 0.3\,|1 - 0.9| + 0.3\,|2 - 0.9| = 0.72$

4.

a) Rozkład Bernoulliego $X \sim B(2000, 0.005)$

b) awaria : $X \geq 4$

$$P(X \geq 4) = 1 - P(X < 4) = 1 - \sum_{k=0}^{3} \binom{2000}{k} \cdot (0.005)^k \cdot (0.995)^{2000-k} = 0.989796$$

Przybliżenie Poissona $\lambda = 2000 \cdot 0.005 = 10$

$$P(X \geq 4) \simeq 1 - \sum_{k=0}^{3} e^{-10} \cdot \frac{10^k}{k!} \simeq 0.989664$$

5.

rozkład ciągły

$f_X(x) = 12x^2(1-x) \cdot \mathbb{1}_{[0,1]}(x)$ 4 niezależne próbki

a) Dokładnie 1 próbka zawiera ponad połowę zanieczyszczeń

X – zawartość zanieczyszczeń w próbce

$$\int 12x^2(1-x)dx = \int -12x^3 + 12x^2 dx = -3x^4 + 4x^3$$

$$P(X > \tfrac{1}{2}) = \int_{\frac{1}{2}}^{+\infty} f_X(x)dx = \int_{\frac{1}{2}}^{1} 12x^2(1-x)dx = (-3x^4 + 4x^3)\Big|_{\frac{1}{2}}^{1} = \frac{11}{16}$$

$$P(X \le \tfrac{1}{2}) = 1 - \frac{11}{16} = \frac{5}{16}$$

$$Y \sim B(4, \tfrac{11}{16})$$

$$P(Y=1) = \binom{4}{1} \cdot \left(\frac{11}{16}\right)^1 \cdot \left(\frac{5}{16}\right)^3 \simeq 0.08$$

b) Co najmniej 1 próbka zawiera ponad połowę zanieczyszczeń

$$P(Y \geq 1) = 1 - P(Y<1) = 1 - P(Y=0) = 1 - \binom{4}{0} \cdot \left(\frac{11}{16}\right)^0 \cdot \left(\frac{5}{16}\right)^4 \simeq 0.99$$

6.

$$f_X(x) = 5 \cdot \mathbb{1}_{[0.4, 0.6]}(x)$$

norma : $x \in [0.41, 0.59]$

a) $P(0.41 \le X \le 0.59) = \int_{0.41}^{0.59} 5 \, dx = 5 \cdot (0.59 - 0.41) = 0.9$

b) Rozkład dwumianowy $Y \sim B(999, 0.9)$

$$(n+1) \cdot p - 1 \le k_0 \le (n+1) \cdot p$$

$$899 \le k_0 \le 900$$

Najbardziej prawdopodobna liczba kulek w normie spośród 999 losowych to 899 lub 900