1.

$$\lim_{n\to\infty}\left(1+3n^2\sin\left(\frac{1}{n^3}\right)\right)^n$$

$$\infty\cdot 0$$

$$3n^2\sin\left(\frac{1}{n^3}\right)=\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}\cdot\frac{1}{n^3}\cdot 3n^2=\frac{3}{n}\cdot\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}=\frac{1}{\frac{n}{3}\cdot\frac{\frac{1}{n^3}}{\sin\left(\frac{1}{n^3}\right)}}$$

$$n=\frac{n}{3}\cdot\frac{\frac{1}{n^3}}{\sin\left(\frac{1}{n^3}\right)}\cdot 3\cdot\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}$$

$$\lim_{n\to\infty}\left(1+3n^2\sin\left(\tfrac{1}{n^3}\right)\right)^n=\lim_{n\to\infty}\left(\left[1+\frac{1}{\frac{n}{3}\cdot\frac{\frac{1}{n^3}}{\sin\left(\frac{1}{n^3}\right)}}\right]^{\frac{n}{3}\cdot\frac{\frac{1}{n^3}}{\sin\left(\frac{1}{n^3}\right)}}\right)^{3\cdot\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}}=e^3$$

$$\lim_{n\to\infty}\frac{n}{3}\cdot\frac{\frac{1}{n^3}}{\sin\left(\frac{1}{n^3}\right)}=+\infty\qquad\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}=1\qquad\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

2.

$$\lim_{x\to\frac{\pi}{2}^-}(\cot(x))^{\cot(x)}=\lim_{x\to\frac{\pi}{2}^-}e^{\ln\left((\cot(x))^{\cot(x)}\right)}=\lim_{x\to\frac{\pi}{2}^-}e^{\cot(x)\cdot\ln(\cot(x))}=e^0=1$$

$$\lim_{x\to\frac{\pi}{2}^-}\cot(x)=\lim_{x\to\frac{\pi}{2}^-}\frac{\cos(x)}{\sin(x)}=\left|\frac{0^+}{1}\right|=0$$

$$\frac{d}{dx}\cot(x)=\frac{d}{dx}\frac{\cos(x)}{\sin(x)}=\frac{-\sin^2(x)-\cos^2(x)}{\sin^2(x)}=\frac{-1}{\sin^2(x)}$$

$$\frac{d}{dx}x^{-1}=-x^{-2}$$

$$\lim_{x\to\frac{\pi}{2}^-}\cot(x)\ln(\cot(x))=\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\cot(x))}{\frac{1}{\cot(x)}}\overset{\frac{\infty}{\infty}}{\underset{H}{=}}\lim_{x\to\frac{\pi}{2}^-}\frac{\frac{1}{\cot(x)}\cdot\frac{-1}{\sin^2(x)}}{-\frac{1}{\cot^2(x)}\cdot\frac{-1}{\sin^2(x)}}=\lim_{x\to\frac{\pi}{2}^-}-\cot(x)=0$$

3.

$$\int e^{\cos^2(x)}\sin(x)\cos(x)\,dx=\begin{vmatrix}t=\cos(x)\\ dt=-\sin(x)dx\end{vmatrix}=-\int t e^{t^2}dt=\begin{vmatrix}u=t^2\\ du=2t\,dt\end{vmatrix}=-\frac{1}{2}\int e^u du=-\frac{1}{2}e^u+C=-\frac{1}{2}e^{t^2}+c=-\frac{1}{2}e^{\cos^2(x)}+C$$

$$\int_0^\infty e^{\cos^2(x)}\sin(x)\cos(x)\,dx=\lim_{T\to+\infty}\left[-\frac{1}{2}e^{\cos^2(T)}+\frac{1}{2}e^{\cos^2(0)}\right]=\frac{e}{2}-\frac{1}{2}\lim_{T\to+\infty}e^{\cos^2(T)}$$

$\lim_{T\to+\infty}\cos(T)$ nie istnieje więc całka jest rozbieżna

$$T_n'=2\pi n\to\lim_{n\to\infty}\cos(T_n')=1$$
$$T_n''=2\pi n+\frac{\pi}{2}\to\lim_{n\to\infty}\cos(T_n'')=0$$

?

5. $xy'=y\ln\left(\frac{y^2}{x^2}\right)$

$$y'=\frac{y}{x}\ln\left(\frac{y^2}{x^2}\right)$$

$$u=\frac{y}{x}\qquad y'=\frac{d}{dx}ux=u'x+u$$

$$u'x+u=u\ln(u^2)=2u\ln(u)$$

$$\frac{du}{dx}=\frac{2u\ln(u)-u}{x}\qquad\int\frac{du}{2u\ln(u)-u}=\int\frac{dx}{x}$$

$$\int\frac{du}{2u\ln(u)-u}=\int\frac{1}{2\ln(u)-1}\cdot\frac{du}{u}=\begin{vmatrix}t=2\ln(u)-1\\ dt=2\frac{1}{u}du\end{vmatrix}=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln|2\ln(u)-1|+c$$

$$\frac{1}{2}\ln|2\ln(u)-1|=\ln|x|+C$$
$$\ln|2\ln(u)-1|=2\ln|x|+C$$
$$\ln|2\ln(u)-1|-\ln|x^2|=C$$
$$\ln\left|\frac{2\ln(u)-1}{x^2}\right|=C$$
$$\frac{2\ln(u)-1}{x^2}=C$$
$$2\ln(u)-1=Cx^2$$
$$\ln(u)=\frac{Cx^2+1}{2}=Cx^2+\frac{1}{2}$$
$$u=e^{Cx^2+\frac{1}{2}}\qquad y=xe^{Cx^2+\frac{1}{2}}$$

sprawdzenie $C=1$

$$y=xe^{x^2+\frac{1}{2}}$$
$$y'=x\cdot 2xe^{x^2+\frac{1}{2}}=2x^2e^{x^2+\frac{1}{2}}\qquad y'x=2x^3e^{x^2+\frac{1}{2}}$$

$$y\ln\left(\frac{y^2}{x^2}\right)=xe^{x^2+\frac{1}{2}}\cdot\ln\left(\frac{x^2e^{2x^2+1}}{x^2}\right)=xe^{x^2+\frac{1}{2}}\ln\left(e^{2x^2+1}\right)$$
$$=e^{x^2+\frac{1}{2}}\cdot x(2x^2+1)=2x^3e^{x^2+\frac{1}{2}}+xe^{x^2+\frac{1}{2}}$$

6. $y'' - 4y' + 4y = \frac{e^{2x}}{x^2} + 2$

$r^2 - 4r + 4 = (r-2)^2 = 0 \quad r_0 = 2$

$y(x) = C_1 e^{2x} + C_2 x e^{2x}$ $\qquad \frac{d}{dx} x e^{2x} = x \cdot 2e^{2x} + e^{2x} = e^{2x}(2x+1)$

$$\begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{2x}}{x^2} + 2 \end{bmatrix}$$

$$C_1' = \frac{-xe^{2x} \cdot \left(\frac{e^{2x}}{x^2} + 2\right)}{(2x+1)e^{2x}e^{2x} - xe^{2x} \cdot 2e^{2x}} = \frac{-\frac{e^{4x}}{x} - 2xe^{2x}}{2xe^{4x} + e^{4x} - 2xe^{4x}} = \frac{e^{4x}\left(-\frac{1}{x} - 2xe^{-2x}\right)}{e^{4x}} = -\frac{1}{x} - 2xe^{-2x}$$

$$C_2' = \frac{e^{2x}\left(\frac{e^{2x}}{x^2} + 2\right)}{e^{4x}} = \frac{1}{x^2} + 2e^{-2x}$$

$$C_1 = \int -\frac{1}{x} - 2xe^{-2x}\, dx = -\int \frac{dx}{x} - 2\int xe^{-2x}\, dx$$

$$\int xe^{-2x}\, dx = \left| \begin{matrix} f = x & g' = e^{-2x} \\ f' = 1 & g = -\frac{1}{2}e^{-2x} \end{matrix} \right| = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

$$C_1 = -\ln|x| + xe^{-2x} + \frac{1}{2}e^{-2x} + C_1$$

$$C_2 = \int \frac{1}{x^2}dx + 2\int e^{-2x}\, dx = -\frac{1}{x} - e^{-2x} + C_2$$

$$\begin{aligned} y(x) &= \left(-\ln|x| + xe^{-2x} + \frac{1}{2}e^{-2x} + C_1\right)e^{2x} + \left(-\frac{1}{x} - e^{-2x} + C_2\right)xe^{2x} \\ &= -e^{2x}\ln|x| + x + \frac{1}{2} - e^{2x} - x + C_1e^{2x} + C_2xe^{2x} \\ &= C_1e^{2x} + C_2xe^{2x} - e^{2x}\ln|x| - e^{2x} + \frac{1}{2} \qquad C_1, C_2 \in \mathbb{R} \end{aligned}$$