1.

X i Y są niezależne

$X, Y \sim g(p)$

$\Rightarrow P(Y=k) = P(X=k) = (1-p)^{k-1} p \qquad$ dla $k \in \mathbb{N}$

$\Rightarrow S_x = S_y = \mathbb{N}$

$Z = X+Y \qquad S_z = \{2,3,4,\dots\} = \mathbb{N} \setminus \{1\}$

dla $k \in S_z$

$P(Z=k) = P(X+Y=k) = P(X=a, Y=k-a, a \in \{1,\dots k-1\})$

$= \sum_{a=1}^{k-1} P(X=a, Y=k-a)$

$= \sum_{a=1}^{k-1} P(X=a) P(Y=k-a)$

$= \sum_{a=1}^{k-1} p(1-p)^{a-1} \cdot p(1-p)^{k-a-1}$

$= \sum_{a=1}^{k-1} p^2 (1-p)^{k-2}$

$= (k-1) p^2 (1-p)^{k-2}$

Z nie ma rozkładu geometrycznego

2.

X\Y	-1	0	1
0	0.1	0.1	0
1	0.2	0.2	0.1
2	0.1	0.1	0.1

$Z = \max(X,Y) - \min(X,Y) = |X - Y|$

wartości Z

$S_Z = \{0, 1, 2, 3\}$

X\Y	-1	0	1
0	1	0	1
1	2	1	0
2	3	2	1

$P(Z=0) = \rho(0,0) + \rho(1,1) = 0.1+0.1 = 0.2$

$P(Z=1) = \rho(0,-1) + \rho(1,0) + \rho(0,1) + \rho(2,1) = 0.1 + 0.2 + 0.1 + 0 = 0.4$

$P(Z=2) = \rho(1,-1) + \rho(2,0) = 0.2 + 0.1 = 0.3$

$P(Z=3) = \rho(2,-1) = 0.1$

$(U, V) = (|X \cdot Y|, X^2 - Y^2)$

wartości (U, V)

X\Y	-1	0	1
0	(0, 1)	(0, 0)	(0, 1)
1	(1, 2)	(0, 1)	(1, 2)
2	(2, 5)	(0, 4)	(2, 5)

$S_{UV} = \{(0,0), (0,1), (1,2), (0,4), (2,5)\}$

$P(U=0, V=0) = \rho(1,0) = 0.1$

$P(U=0, V=1) = \rho(0,-1) + \rho(0,1) + \rho(1,0) = 0.1 + 0.2 + 0 = 0.3$

$P(U=1, V=2) = \rho(1,-1) + \rho(1,1) = 0.2 + 0.1 = 0.3$

$P(U=0, V=4) = \rho(2,0) = 0.1$

$P(U=2, V=5) = \rho(2,-1) + \rho(2,1) = 0.1 + 0.1 = 0.2$

U\V	0	1	2	4	5
0	0.1	0.3	0	0.1	0
1	0	0	0.3	0	0
2	0	0	0	0	0.2

3.

$D = [0, 2] \times [0, 2]$ $|D| = 4$
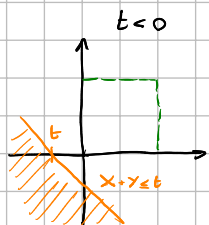
$(X, Y) \sim U(D)$ P geometryczne

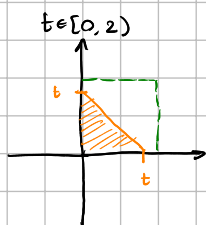$f_{XY}(x, y) = \frac{1}{4} \cdot \mathbb{1}_D(x, y)$

$Z = X + Y$ $S_Z = [0, 4]$

$F_Z(t) = P(Z \leq t) = P(X + Y \leq t) = \iint\limits_{X+Y \leq t} f(x,y)\, dx\,dy$

$1 - \frac{1}{8}(16 - 8t + t^2) = 1 - 2 + t - \frac{1}{8}t^2 \;\to\; 1 - \frac{1}{4}t$

$t < 0$ $t \in [0, 2)$
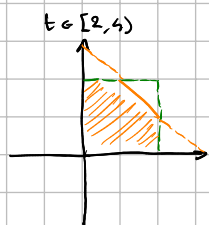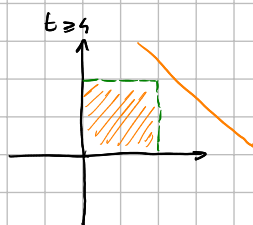
$F_Z(t) = 0$ $F_Z(t) = \frac{\frac{1}{2}t^2}{4} = \frac{1}{8}t^2$

$$F_Z(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{8}t^2 & t \in [0, 2) \\ 1 - \frac{1}{8}(4-t)^2 & t \in [2, 4) \\ 1 & t \geq 4 \end{cases}$$

$$f_Z(t) = \frac{d}{dt}F_Z(t) = \begin{cases} \frac{1}{4}t & t \in [0, 2) \\ 1 - \frac{1}{2}t & t \in [2, 4] \\ 0 & \text{w. pp.} \end{cases}$$

$t \in [2, 4)$ $t \geq 4$

$F_Z(t) = \dfrac{4 - \frac{1}{2}(4-t)^2}{4}$ $F_Z(t) = 1$

4.

X i Y niezależne
X, Y ~ exp($\lambda = 1$)

$f_x(t) = f_y(t) = e^{-t} \cdot \mathbb{1}_{[0,+\infty)}(t)$

$F(t) = F_x(t) = F_y(t) = (1 - e^{-t}) \cdot \mathbb{1}_{[0,+\infty)}(t)$

$Z = \min(X, Y)$

$F_z(t) = P(Z \leq t) = P(\min(X, Y) \leq t)$

$\quad = 1 - P(\min(X, Y) > t)$

$\quad = 1 - P(X > t, Y > t)$

$\quad = 1 - P(X > t)P(Y > t)$

$\quad = 1 - (1 - F(t))(1 - F(t))$

$\quad = (1 - e^{-2t}) \cdot \mathbb{1}_{[0,+\infty)}(t)$

5.

X i Y niezależne

$X \sim N(-2, 3) \qquad Y \sim N(2, 4)$

$Z = X - Y$

$F_Z(t) = P(X - Y \leq t)$

$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 3}} \exp\left(-\frac{(x+2)^2}{6} \right) \cdot \frac{1}{\sqrt{2\pi \cdot 4}} \exp\left(-\frac{(y-2)^2}{8} \right)$

$\qquad = \frac{1}{2\pi \sqrt{12}} \exp\left(-\frac{1}{6}(x+2)^2 - \frac{1}{8}(y-2)^2 \right)$

$(X, Y) \sim N\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \right)$

Z to przekształcenie liniowe $(X,Y) \implies Z \sim N(m_Z, \sigma_Z^2)$

$\quad m_Z = EZ = EX - EY = -4$

$\quad \sigma_Z^2 = VZ = V(X-Y) = VX + VY - 2\,cov(X,Y) = VX + VY = 7$

$Z \sim N(-4, 7)$

Alternatywnie

$\quad Z = X - Y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\quad Z \sim N(Am, \; ACA^T)$

$\quad Am = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -4$

$\quad ACA^T = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 7$

6.

$$f_{xy}(x,y) = \frac{1}{2\sqrt{2}\,\pi}\,\exp\left(-\frac{1}{2}\left(x^2 + 2x(y+1) + \frac{3}{2}(y+1)^2\right)\right) \qquad \det C = 2$$

$$= \frac{1}{2\pi\sqrt{2}}\,\exp\left(-\frac{1}{2\cdot 2}\left(2x^2 - 2\cdot(-2)\,x(y+1) + 3(y+1)^2\right)\right)$$

$$m = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$(Z,T) = (2x + y + 1,\ 2x - y - 1)$$

$$\begin{bmatrix} 2x + y + 1 \\ 2x - y - 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b}$$

$$m^* = Am + b = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C^* = ACA^T = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & -6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 10 & 22 \end{bmatrix}$$

$$(Z,T) \sim N(m^*, C^*) \qquad \det C^* = 32$$

$$f_{ZT}(z,t) = \frac{1}{2\pi\sqrt{32}}\,\exp\left(-\frac{1}{2\sqrt{32}}\left(22z^2 - 20zt + 6t^2\right)\right)$$