1.

X \ Y	-1	0	1	
0	.1	.1	0	.2
1	.2	.2	.1	.5
2	.1	.1	.1	.3
	.4	.4	.2	

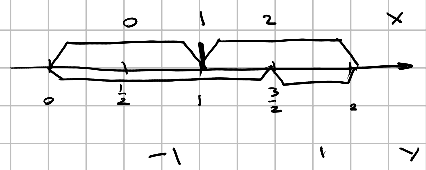
$F(1,0) = P(X \le 1, Y \le 0) = .1 + .1 + .2 + .2 = 0.6$

$F(2,-1) = P(X \le 2, Y \le -1) = .1 + .2 + .1 = 0.4$

$P(X > 2Y) = .1 + .2 + .2 + .1 + .1 = 0.7$

2.

$\Omega = [0,2]$ P – geometryczne $|\Omega| = 2$



$$X(\omega) = \begin{cases} 0 & \omega \in [0,1) \\ 1 & \omega = 1 \\ 2 & \omega \in (1,2] \end{cases} \qquad Y(\omega) = \begin{cases} -1 & \omega \in [0, 1.5] \\ 1 & \omega \in (1.5, 2] \end{cases}$$

$S_X \subseteq \{0,1,2\}$ $S_Y \subseteq \{-1,1\}$

$X \backslash Y$	-1	1
0	.5	0
1	0	0
2	.25	.25

$p(0,-1) = \frac{|[0,1)|}{2} = .5$

$p(0,1) = \frac{1}{2}|[0,1) \cap (1.5,2]| = 0$

$p(1,-1) = \frac{1}{2}|\{1\} \cap [0,1.5]| = 0$

$p(1,1) = \frac{1}{2}|\{1\} \cap (1.5,2]| = 0$

$p(2,-1) = \frac{1}{2}|(1,2] \cap [0,1.5]| = \frac{1}{2} \cdot \frac{1}{2} = 0.25$

$p(2,1) = \frac{1}{2}|(1,2] \cap (1.5,2]| = \frac{1}{2} \cdot \frac{1}{2} = 0.25$

$P(X=2) \cdot P(Y=1) = \frac{1}{2}|(1,2]| \cdot \frac{1}{2}|(1.5,2]| = \frac{1}{2} \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq P(X=2, Y=1) = \frac{1}{4}$

$\Longrightarrow$ X i Y nie są niezależne

3.

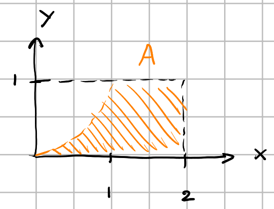
X i Y są niezależne

$X \sim U([0,2]) \qquad Y \sim U([0,1])$

$(X,Y) = U(D) \qquad D = [0,2] \times [0,1]$

$f_{XY}(x,y) = \frac{1}{2} \mathbb{1}_D(x,y) = f_X \cdot f_Y$

$P(Y < X^2) = \iint_A f_{XY}(x,y)\, dx\, dy = |A| \cdot \frac{1}{2} = \frac{2}{3}$



$|A| = \int_0^1 x^2 dx + 1 = \frac{1}{3} x^3 \Big|_0^1 + 1 = \frac{4}{3}$

4.

1 rzut sześcienną kostką

$\Omega = \{1, 2, 3, 4, 5, 6\}$ P – klasyczne $\forall_{\omega \in \Omega} P(\{\omega\}) = \frac{1}{6}$

$$X(\omega) = \begin{cases} 0 & \omega \in \{2, 4, 6\} \\ 1 & \omega \in \{1, 3, 5\} \end{cases}$$

$S_X \in \{0, 1\}$

$S_Y \in \{1, 2\}$

$$Y(\omega) = \begin{cases} 1 & \omega \in \{3, 6\} \\ 2 & \omega \in \{1, 2, 4, 5\} \end{cases}$$

$X \backslash Y$	1	2
0	1/6	2/6
1	1/6	2/6

	$\frac{1}{3}$	$\frac{2}{3}$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{6}$
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{6}$

$p(0,1) = P(\{2,4,6\} \cap \{6\}) = P(\{6\}) = \frac{1}{6} = P(X=0)P(Y=1) = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6}$

$p(0,2) = P(\{2,4\}) = 2/6 = P(X=0)P(X=2) = \frac{3}{6} \cdot \frac{4}{6} = \frac{2}{6}$

$p(1,1) = P(\{3\}) = 1/6 = P(X=1)P(Y=1) = \frac{3}{6} \cdot \frac{2}{6} = \frac{1}{6}$

$p(1,2) = P(\{1,5\}) = 2/6 = P(X=1)P(Y=2) = \frac{3}{6} \cdot \frac{4}{6} = \frac{2}{6}$

Zmienne są niezależne

5.

$$f(x,y) = \begin{cases} e^{-x-y} & x>0,\ y>0 \\ 0 & \text{w p.p.} \end{cases}$$

a)

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_0^{+\infty} e^{-x} \cdot e^{-y}\,dy \ \mathbb{1}_{[0,\infty)}(x) = e^{-x} \cdot \left[-e^{-y}\right]\Big|_0^{+\infty} \cdot \mathbb{1}_{[0,+\infty)}(x) = e^{-x} \cdot \mathbb{1}_{[0,+\infty)}(x)$$

$$f_Y(y) = e^{-y} \cdot \mathbb{1}_{[0,+\infty)}(y) \quad \text{(symetrycznie)}$$

$$f(x,y) = f_X(x) \cdot f_Y(y) \implies \text{są niezależne}$$

b)

$$P(1<X\leq 3,\ 1\leq Y<2) = P(1<X\leq 3)P(1\leq Y<2) = \int_1^3 e^{-x}\,dx \cdot \int_1^2 e^{-y}\,dy = \left[-e^{-x}\right]_1^3 \cdot \left[-e^{-y}\right]_1^2 = (-e^{-3}+e^{-1})(-e^{-2}+e^{-1}) \approx 0.07$$

$$P(Y>1 \mid X\leq 2) = \underbrace{P(Y>1)}_{\text{bo są niezależne}} = \int_1^{+\infty} e^{-y}\,dy = \left[-e^{-y}\right]\Big|_1^{+\infty} = e^{-1}$$

c)

$$P(X+Y>100) = 1 - P(X+Y\leq 100) = 1 - \int_0^{100}\int_0^{100-y} e^{-x-y}\,dx\,dy$$

$$= 1 + \int_0^{100} e^{-x-y}\Big|_0^{100-y} dy$$

$$= 1 + \int_0^{100} e^{-100} - e^{-y}\,dy = 1 + e^{-100} \cdot (e^{-y})\Big|_0^{100} = 101\,e^{-100}$$

6.

$$f_{XY}(x,y) = \frac{1}{6\pi} \exp\left(-\frac{1}{36}\left[10x^2 + 4x(y+1) + 4(y+1)^2\right]\right)$$

$$\frac{1}{6\pi} = \frac{1}{2\pi\sqrt{\det C}} \Rightarrow \sqrt{\det C} = 3 \Rightarrow \det C = 9$$

$$-\frac{1}{2\det C} = -\frac{1}{18}$$

$$f_{XY}(x,y) = \frac{1}{2\pi \cdot 3} \exp\left(-\frac{1}{18}\left[5(x-0)^2 + 2(x-0)(y+1) + 2(y+1)^2\right]\right)$$

$$m = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \quad (X,Y) \sim N(m,C) \Rightarrow X \sim N(0,2) \wedge Y \sim N(-1,5)$$

$\operatorname{cov}(X,Y) \neq 0 \Rightarrow$ nie są niezależne

$$f_X(x) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right) \qquad f_Y(y) = \frac{1}{\sqrt{10\pi}} \exp\left(-\frac{(y+1)^2}{10}\right)$$

7.

$$f_{XY}(x,y) = \frac{1}{36\pi} \exp\left(-\frac{1}{162}\left[9(x-153)^2 - 12(x-153)(y-48) + \frac{25}{4}(y-48)^2\right]\right)$$

$$\frac{1}{36\pi} = \frac{1}{2\pi\sqrt{\det C}} \Rightarrow \sqrt{\det C} = 18 \Rightarrow \det C = 324$$

$$-\frac{1}{2\det C} = -\frac{1}{648}$$

$$-\frac{1}{162}\left[9(x-153)^2 - 12(x-153)(y-48) + \frac{25}{4}(y-48)^2\right]$$

$$= -\frac{1}{648}\left[36(x-153)^2 - 48(x-153)(y-48) + 25(y-48)^2\right]$$

$$m = \begin{bmatrix} 153 \\ 48 \end{bmatrix} \quad C = \begin{bmatrix} 36 & 24 \\ 24 & 25 \end{bmatrix}$$

a) $(X,Y) \sim N(m,C) \Rightarrow X \sim N(153, 36)$

$$P(X < 160) = F(160) = \Phi\left(\frac{160-153}{6}\right) = \Phi\left(\frac{7}{6}\right)$$

$\Phi(1.16) \approx 0.8770$

87.7% ma wzrost poniżej 160 cm

b) $(X,Y) \sim N(m,C) \Rightarrow Y \sim N(48, 25)$

$$P(Y > 36) = 1 - P(Y \le 36) = 1 - \Phi\left(\frac{36-48}{5}\right) = 1 - \Phi(-2.4) = 1 - (1 - \Phi(2.4)) = \Phi(2.4) \approx 99.18\%$$