1.

$$X(\omega) = \begin{cases} -2 & \omega \in [0,1) \\ -1 & \omega \in [1,4) \\ 1 & \omega \in [4,6] \end{cases} \qquad \Omega = [0,6]$$

$$Y(\omega) = \begin{cases} -1 & \omega \in [0,2] \cup [4,6] \\ 2 & \omega \in (2,4) \end{cases}$$

a)

$X \backslash Y$	-1	2	
-2	$\frac{1}{6}$	0	$\frac{1}{6}$
-1	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$
1	$\frac{2}{6}$	0	$\frac{2}{6}$
	$\frac{4}{6}$	$\frac{2}{6}$	

b) $(Z, T) = (X - Y, |X+Y|)$

Wartości (Z, T)

$X \backslash Y$	-1	2
-2	$-1, 3$	$-4, 0$
-1	$0, 2$	$-3, 1$
1	$2, 0$	$-1, 3$

$P(Z=-1, T=3) = \frac{1}{6} + 0 = \frac{1}{6}$

$P(Z=0, T=2) = \frac{1}{6}$

$P(Z=-3, T=1) = \frac{2}{6}$

$P(Z=2, T=0) = \frac{2}{6}$

c) $\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{VX \cdot VY}}$

$cov(X,Y) = E(X \cdot Y) - EX \cdot EY$

$EX = -2 \cdot \frac{1}{6} - 1 \cdot \frac{3}{6} + 1 \cdot \frac{2}{6} = \frac{-2-3+2}{6} = -\frac{1}{2}$

$E(X^2) = 4 \cdot \frac{1}{6} + 1 \cdot \frac{3}{6} + 1 \cdot \frac{2}{6} = \frac{4+3+2}{6} = \frac{9}{6} = \frac{3}{2}$

$VX = E(X^2) - (EX)^2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$

$EY = -1 \cdot \frac{4}{6} + 2 \cdot \frac{2}{6} = 0$

$E(Y^2) = 1 \cdot \frac{4}{6} + 4 \cdot \frac{2}{6} = \frac{12}{6} = 2$

$VY = 2 - 0^2 = 2$

$E(XY) = (-2)(-1)\frac{1}{6} + (-1)(-1)\frac{1}{6} + (-1)(2)\frac{2}{6} + (1)(-1)\frac{2}{6} = \frac{2}{6} + \frac{1}{6} - \frac{4}{6} - \frac{2}{6} = -\frac{1}{2}$

$cov(X,Y) = -\frac{1}{2} - (-\frac{1}{2}) \cdot 0 = -\frac{1}{2}$

$\rho(X,Y) = \frac{-\frac{1}{2}}{\sqrt{\frac{5}{4} \cdot 2}} = -\frac{1}{\sqrt{10}}$

2.

$S_x = \{-1, 1, 2\}$ $S_y = \{0, 1\}$

$P(Y=0) = P(Y=1) = P(X=-1) = \frac{1}{2}$ bo $S_y = \{0,1\}$

$P(X=1) = P(X=2) = (1 - P(X=-1))\frac{1}{2} = \frac{1}{4}$

$P(X=-1, Y=0) = P(X=-1, Y=1) = P(X=2, Y=0) = \frac{1}{2} P(X=-1) = \frac{1}{4}$

a)

$X \backslash Y$	0	1	
-1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{4}$	0	$\frac{1}{4}$
	$\frac{1}{2}$	$\frac{1}{2}$	

b) $Z = \sin\left(\frac{\pi}{2}(X+Y)\right) + \cos(\pi(X+Y))$

$Z(-1,0) = \sin(-\frac{\pi}{2}) + \cos(-\pi) = -1 + (-1) = -2$

$Z(-1,1) = \sin(0) + \cos(0) = 0 + 1 = 1$

$Z(1,1) = \sin(\pi) + \cos(2\pi) = 0 + 1 = 1$

$Z(2,0) = \sin(\pi) + \cos(2\pi) = 1$

$S_z = \{1, -2\}$

$P(Z=1) = P(X=-1, Y=1) + P(X=1, Y=1) + P(X=2, Y=0) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

$P(Z=2) = P(X=-1, Y=0) = \frac{1}{4}$

c) $V(4X - 2Y + 3) = V(4X) + V(2Y) - 2\,cov(4X, 2Y)$

$= 16VX + 4VY - 16\,cov(X,Y)$

$= 16 \cdot \frac{27}{16} + 4 \cdot \frac{1}{4} - 16 \cdot (-\frac{1}{8}) = 27 + 1 + 2 = 30$

$EX = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{-2+1+2}{4} = \frac{1}{4}$

$EX^2 = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{2+1+4}{4} = \frac{7}{4}$

$VX = \frac{7}{4} - \frac{1}{16} = \frac{27}{16}$

$EY = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$EY^2 = \frac{1}{2}$

$VY = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$E(XY) = -\frac{1}{4} + \frac{1}{4} = 0$

$cov(X,Y) = 0 - \frac{1}{4} \cdot \frac{1}{2} = -\frac{1}{8}$

3.

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{5}} \exp\left(-\frac{1}{5}\left[(x+1)^2 + (x+1)(y-1) + \frac{3}{2}(y-1)^2\right]\right)$$

$\det C = 5$

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{5}} \exp\left(-\frac{1}{2\cdot 5}\left[2(x+1)^2 - 2\cdot(-1)(x+1)(y-1) + 3(y-1)^2\right]\right)$$

$$m = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

a) $E(X(X+2Y)) = E(X^2 + 2XY) = EX^2 + 2\,EXY = 4 + 2\cdot 5 = 13$

$VX = EX^2 - (EX)^2 \Rightarrow EX^2 = VX + (EX)^2 = 3 + (-1)^2 = 4$

$\mathrm{cov}(X,Y) = EXY - EX\,EY \Rightarrow EXY = \mathrm{cov}(X,Y) + EX\cdot EY = -1 + 3\cdot 2 = 5$

b) $(X,Y) \sim N(m, C) \Rightarrow X \sim N(-1, 3), \quad Y \sim N(1, 2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{3}} \exp\left(-\frac{(x+1)^2}{2\cdot 3}\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{2}} \exp\left(-\frac{(y-1)^2}{2\cdot 2}\right)$$

c) $(Z, T) = (2X - Y + 1, X + Y - 2)$

$$\begin{bmatrix} 2X - Y + 1 \\ X + Y - 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{b}$$

$(Z, T) \sim N(m^*, C^*)$

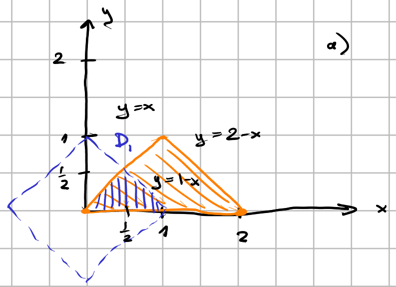
$$m^* = Am + b = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$C^* = ACA^T = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 3 \\ 3 & 3 \end{bmatrix}$$

4.

$f_{XY}(x,y) = x \cdot \mathbb{1}_D(x,y)$ $\qquad D = \{(x,y):\ 0 \le x \le 2,\ \ 0 \le y \le 1 - |x-1|\}$

$\qquad\qquad\qquad\qquad\qquad\quad = \{(x,y):\ 0 \le y \le 1,\ \ y \le x \le 2-y\}$



a)

$$f_X(x) = \begin{cases} \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy = \int_0^{1-|x-1|} x\, dy = x - x|x-1| & x \in [0,2] \\ 0 & \text{υ β.β.} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dx = \int_y^{2-y} x\, dx = \frac{1}{2}x^2 \Big|_y^{2-y} = \frac{1}{2}(2-y)^2 - \frac{1}{2}y^2 = 2-2y & y \in [0,1] \\ 0 & \text{υ β.β} \end{cases}$$

b) $\quad D_1 = \{(x,y):\ 0 \le y \le \frac{1}{2},\ \ y \le x \le 1-y\}$

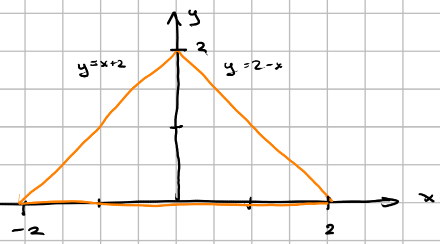
$$P(|X|+|Y| < 1) = \iint_{D_1} f_{XY}(x,y)\,dx\,dy = \int_0^{\frac{1}{2}} \int_y^{1-y} x\, dx\, dy$$

$$= \int_0^{\frac{1}{2}} \frac{1}{2}x^2 \Big|_y^{1-y} dy = \frac{1}{2} \int_0^{\frac{1}{2}} 1 - 2y + y^2 - y^2\, dy = \frac{1}{2} \int_0^{\frac{1}{2}} 1 - 2y\ dy$$

$$= \frac{1}{2}\left(y - y^2\right)\Big|_0^{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{8}$$

5.

$f_{XY}(x,y) = ay \cdot 1\!\!1_D(x,y)$



$D = \{(x,y) : -2 \le x \le 2,\ 0 \le y \le 2-|x|\}$

$= \{(x,y) : 0 \le y \le 2,\ y-2 \le x \le 2-y\}$

a) $1 = \iint_{\mathbb{R}^2} f_{XY}(x,y)\,dx\,dy = \iint_D ay\,dx\,dy = \int_{-2}^{2}\int_{0}^{2-|x|} ay\,dy\,dx$

$= a\int_{-2}^{2} \frac{1}{2}y^2\Big|_0^{2-|x|} dx = a\int_{-2}^{2} \frac{1}{2}(4-4|x|+|x|^2)\,dx$

$= \frac{1}{2}a\int_{-2}^{2} 4-4|x|+x^2\,dx = a\int_0^2 4-4x+x^2\,dx$

$= a\left[4x-2x^2+\frac{1}{3}x^3\right]_0^2 = a\cdot\left(8-8+\frac{8}{3}\right) = \frac{8}{3}a \Rightarrow a = \frac{3}{8}$

$1 = \int_0^2\int_{y-2}^{2-y} ay\,dx\,dy = a\int_0^2 y(2-y-(y-2))\,dy = a\int_0^2 y(4-2y)\,dy = 2a\int_0^2 2y-y^2\,dy$

$= 2a\cdot\left[y^2-\frac{1}{3}y^3\right]_0^2 = 2a\cdot\left(4-\frac{8}{3}\right) = 2a\cdot\frac{4}{3} = \frac{8}{3}a \Rightarrow a = \frac{3}{8}$

$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} \int_{y-2}^{2-y} \frac{3}{8}y\,dx = \frac{3}{8}y\cdot(2-y-y+2) = \frac{3}{8}y(4-2y) = \frac{3}{2}y - \frac{3}{4}y^2 & y\in[0,2] \\ 0 & \text{w p.p.} \end{cases}$

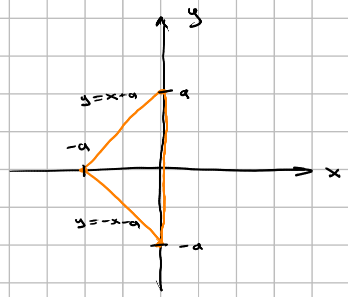
b) $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_0^{2-|x|} \frac{3}{8}y\,dy = \frac{3}{8}\int_0^{2-|x|} y\,dy = \frac{3}{16}y^2\Big|_0^{2-|x|} = \frac{3}{16}(4-4|x|+x^2) & x\in[-2,2] \\ 0 & \text{w p.p.} \end{cases}$

$P(X>0) = \int_0^{+\infty} f_X(x)\,dx = \int_0^2 \frac{3}{16}(4-4|x|+x^2)\,dx = \frac{3}{16}\int_0^2 4-4x+x^2\,dx$

$= \frac{3}{16}\left[4x-2x^2+\frac{1}{3}x^3\right]_0^2 = \frac{3}{16}\left[8-8+\frac{8}{3}\right] = \frac{1}{2}$

(ma sens bo to $\frac{1}{2}$ trójkąta, a wartość f_{XY} nie zależy od x)

6.



$(X,Y) \sim U(D)$

$|D| = \frac{1}{2} \cdot a \cdot 2a = a^2$

$E(3X-Y) = -2 = 3EX - EY$

$f_{XY}(x,y) = \frac{1}{a^2} \mathbb{1}_D(x,y)$ $\qquad D = \{(x,y): -a \le x \le 0,\ -x-a \le y \le x+a\}$

$= \{(x,y): -a \le y \le a,\ |y|-a \le x \le 0\}$

a)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_{-x-a}^{x+a} \frac{1}{a^2}\,dy = 2\frac{x+a}{a^2} & x \in [-a, 0] \\ 0 & \text{u p.p.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} \int_{|y|-a}^{0} \frac{1}{a^2}\,dx = \frac{a-|y|}{a^2} & y \in [-a, a] \\ 0 & \text{u p.p.} \end{cases}$$

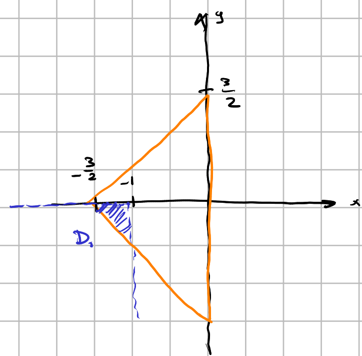
$$EX = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-a}^{0} 2x\frac{x+a}{a^2}\,dx = \frac{2}{a^2}\int_{-a}^{0} x^2+ax\,dx = \frac{2}{a^2}\left[\frac{1}{3}x^3+\frac{a}{2}x^2\right]_{-a}^{0} = \frac{2}{a^2}\left(0-\left(-\frac{a^3}{3}+\frac{a^3}{2}\right)\right) = \frac{2}{a^2}\cdot\frac{-a^3}{6} = -\frac{a}{3}$$

$$EY = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = \int_{-a}^{a} y\frac{a-|y|}{a^2}\,dy = 2\int_0^a y\frac{a-y}{a^2}\,dy = \frac{2}{a^2}\int_0^a ay-y^2\,dy = \frac{2}{a^2}\left[\frac{a}{2}y^2-\frac{1}{3}y^3\right]_0^a = \frac{2}{a^2}\left(\frac{a^3}{2}-\frac{a^3}{3}\right) = \frac{2}{a^2}\cdot\frac{a^3}{6} = \frac{a}{3}$$

$-2 = 3\left(-\frac{a}{3}\right) - \frac{a}{3} \Rightarrow -2 = -\frac{4}{3}a \Rightarrow a = \frac{3}{2}$

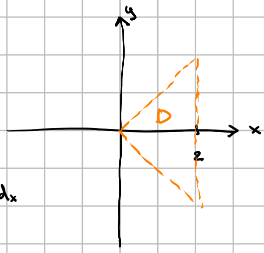
b)

$$F_{XY}(-1,0) = P(X \le -1, Y \le 0) = \iint_{D_1} f_{XY}(x,y)\,dx\,dy = \frac{|D_1|}{|D|} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2}{\left(\frac{3}{2}\right)^2} = \frac{\frac{1}{8}}{\frac{9}{4}} = \frac{2}{9}$$



7.

$$f_{XY}(x,y) = \begin{cases} a(x-y) & 0<x<2, \quad -x<y<x \\ 0 & \text{w p.p.} \end{cases}$$



$$1 = \iint_{\mathbb{R}^2} f_{XY}(x,y)\,dx\,dy = \iint_D a(x-y)\,dx\,dy = a\int_0^2\int_{-x}^{x} x-y \; dy\; dx$$

$$= a\int_0^2 \left[xy - \tfrac{1}{2}y^2\right]_{-x}^{x} dx = a\int_0^2 x^2 - \tfrac{1}{2}x^2 - (-x^2 - \tfrac{1}{2}x^2)\,dx$$

$$= a\int_0^2 2x^2\,dx = \frac{2a}{3}x^3\Big|_0^2 = \frac{16a}{3} \implies a = \frac{3}{16}$$

$$D = \{(x,y): 0<x<2, \; -x<y<x\}$$
$$= \{(x,y): -2<y<2, \; |y|<x<2\}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_{-x}^{x} a(x-y)\,dy = 2a\int_0^x x-y\,dy = 2a\left[xy - \tfrac{1}{2}y^2\right]_0^x = ax^2 = \frac{3}{16}x^2 & \text{dla } x\in(0,2) \\ 0 & \text{w p.p.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} \int_{|y|}^{2} \frac{3}{16}(x-y)\,dx = \frac{3}{16}\left[\tfrac{1}{2}x^2 - yx\right]_{|y|}^{2} = \frac{3}{16}\left[2 - 2y - \tfrac{1}{2}y^2 + y|y|\right] & \text{dla } y\in(-2,2) \\ 0 & \text{w p.p.} \end{cases}$$

$$(f_X \cdot f_Y)(1,0) = \frac{3}{16}\cdot\frac{3}{16}\cdot 2 \neq f_{XY}(1,0) = \frac{3}{16} \implies f_X \cdot f_Y \neq f_{XY}$$

X i Y nie są niezależne

$$P(Y>0) = \int_0^{+\infty} f_Y(y)\,dy = \frac{3}{16}\int_0^2 2 - 2y - \tfrac{1}{2}y^2 + y^2\,dy = \frac{3}{16}\int_0^2 2 - 2y + \tfrac{1}{2}y^2\,dy = \frac{3}{16}\left[2y - y^2 + \tfrac{1}{6}y^3\right]_0^2 = \frac{3}{16}\left(4 - 4 + \frac{8}{6}\right) = \frac{1}{4}$$

8.

$S_x = \{-2, -1, 0\} \quad S_y = \{0, 1\}$

$P(Y=0) + P(Y=1) = 1 \Rightarrow P(Y=0) = P(Y=1) = P(X=-1) = \frac{1}{2}$

$P(X=-2) = 1 - P(X=0) - P(X=-1) = \frac{1}{3}$

$X \backslash Y$	0	1	
-2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
-1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
0	0	$\frac{1}{6}$	$\frac{1}{6}$
	$\frac{1}{2}$	$\frac{1}{2}$	

$P(X=-2, Y=1) + P(X=-1, Y=1) + P(X=0, Y=1) = P(Y=1) = \frac{1}{2}$

$P(X=-2, Y=1) = P(X=-1, Y=1) = P(X=0, Y=1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{VX \cdot VY}}$$

$EX = -2 \cdot \frac{1}{3} + (-1) \cdot \frac{1}{2} + 0 \cdot \frac{1}{6} = -\frac{7}{6}$

$EX^2 = 4 \cdot \frac{1}{3} + 1 \cdot \frac{1}{2} = \frac{11}{6}$

$EY = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$EY^2 = 1 \cdot \frac{1}{2} = \frac{1}{2}$

$EXY = -2 \cdot 1 \cdot \frac{1}{6} + (-1) \cdot 1 \cdot \frac{1}{6} = -\frac{1}{2}$

$VX = \frac{11}{6} - \frac{49}{36} = \frac{17}{36}$

$VY = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$cov(X, Y) = -\frac{1}{2} - (-\frac{7}{6})(\frac{1}{2}) = -\frac{1}{2} + \frac{7}{12} = \frac{1}{12}$

$$\rho(X, Y) = \frac{\frac{1}{12}}{\sqrt{\frac{17}{36} \cdot \frac{1}{4}}} = \frac{1}{\sqrt{17}}$$

9.

$$F_{XY}(x,y) = \begin{cases} (1-\frac{2}{x})(1-e^{-2y}) & x \geq 2 \wedge y \geq 0 \\ 0 & \text{w p.p.} \end{cases}$$

X i Y niezależne $\iff F_{XY} = F_X \cdot F_Y$ $\qquad \lim\limits_{y \to +\infty} e^{-2y} = 0$

$$F_X(x) = \lim_{y \to +\infty} F_{XY}(x,y) = \begin{cases} 1-\frac{2}{x} & x \geq 2 \\ 0 & x < 2 \end{cases} = (1-\frac{2}{x}) \mathbb{1}_{[2,+\infty)}(x)$$

$$F_Y(y) = \lim_{x \to +\infty} F_{XY}(x,y) = \begin{cases} 1-e^{-2y} & y \geq 0 \\ 0 & y < 0 \end{cases} = (1-e^{-2y}) \mathbb{1}_{[0,+\infty)}(y)$$

$$F_X(x) \cdot F_Y(y) = (1-\frac{2}{x})(1-e^{-2y}) \cdot \mathbb{1}_{[2,+\infty)}(x) \mathbb{1}_{[0,+\infty)}(y) = F_{XY}(x,y)$$

X i Y są niezależne

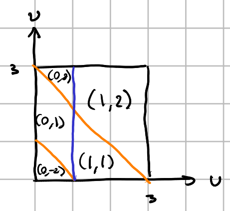
$$f_X(x) = \frac{dF_X}{dx} = \frac{2}{x^2} \cdot \mathbb{1}_{[2,+\infty)}(x)$$

kol_batch2_1

1.

$\Omega = [0,3]^2$ $\quad P$ geometryczne

$$X(u,v) = \begin{cases} 0 & u < 1 \\ 1 & u \geq 1 \end{cases} \qquad Y(u,v) = \begin{cases} -2 & u+v < 1 \\ 1 & 1 \leq u+v \leq 3 \\ 2 & \text{w p.p.} \end{cases}$$



$X \backslash Y$	-2	1	2	
0	$1/18$	$2/9$	$1/18$	$1/3$
1	0	$2/9$	$4/9$	$2/3$
	$1/18$	$8/18$	$9/18$	

$|\Omega| = 9$

$EX = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$

$EX^2 = 1 \cdot \frac{2}{3} = \frac{2}{3}$

$VX = \frac{2}{3} - \frac{4}{9} = \frac{6-4}{9} = \frac{2}{9}$

$EY = -2 \cdot \frac{1}{18} + 1 \cdot \frac{8}{18} + 2 \cdot \frac{9}{18} = \frac{24}{18} = \frac{4}{3}$

$EY^2 = 4 \cdot \frac{1}{18} + 1 \cdot \frac{8}{18} + 4 \cdot \frac{9}{18} = \frac{48}{18} = \frac{24}{9} = \frac{8}{3}$

$VY = \frac{8}{3} - \frac{16}{9} = \frac{24-16}{9} = \frac{8}{9}$

$EXY = 1 \cdot 1 \cdot \frac{2}{9} + 1 \cdot 2 \cdot \frac{4}{9} = \frac{10}{9}$

$\operatorname{cov}(X,Y) = \frac{10}{9} - \frac{2}{3} \cdot \frac{4}{3} = \frac{2}{9}$

$$C = \begin{bmatrix} 2/9 & 2/9 \\ 2/9 & 8/9 \end{bmatrix}$$

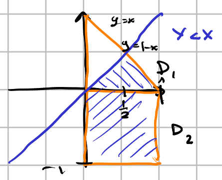
$V(X \mid X > Y) = ?$ nie było

2

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2} & x, y \geq 0, \quad x+y \leq 1 \\ 1+xy & 0<x<1, \quad -1<y<0 \\ 0 & \text{o.p.p.} \end{cases}$$

$$D = \{(x,y) : 0<x<1, \ -1<y<1-x\}$$

$$P(Y<X) = \iint_{Y<X} f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy$$



$$D_1 = \{(x,y) : 0 \leq x \leq 1, \ 0 < y < 1 - |x - \tfrac{1}{2}|\}$$

$$= \{(x,y) : 0 \leq y < \tfrac{1}{2}, \ y \leq x \leq 1-y\}$$

$$D_2 = \{(x,y) : 0 \leq x \leq 1, \ -1 \leq y < 0\}$$

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_0^{\frac{1}{2}} \int_y^{1-y} \tfrac{1}{2}\,dx\,dy = \int_0^{\frac{1}{2}} \tfrac{1}{2}(1-2y)\,dy = \tfrac{1}{2}\left[y - y^2\right]_0^{\frac{1}{2}} = \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{8}$$

$$\iint_{D_2} f(x,y)\,dx\,dy = \int_0^1 \int_{-1}^0 1+xy\,dy\,dx = \int_0^1 \left[y + \tfrac{1}{2}xy^2\right]_{-1}^0 dx = \int_0^1 1 - \tfrac{1}{2}x\,dx = x - \tfrac{1}{4}x^2\Big|_0^1 = \tfrac{3}{4}$$

$$P(Y<X) = \tfrac{1}{8} + \tfrac{3}{4} = \tfrac{7}{8}$$

3.

$$f(x,y) = \frac{1}{4\pi\sqrt{2}} \exp\left(-\frac{3}{4}\left[\frac{1}{4}x^2 + \frac{1}{3}x(y+2) + \frac{1}{3}(y+2)^2\right]\right)$$

$$\sqrt{\det C} = 2\sqrt{2} = \sqrt{8} \Rightarrow \det C = 8$$

$$f(x,y) = \frac{1}{2\pi\sqrt{8}} \exp\left(-\frac{1}{16}\left[3x^2 - 2\cdot(-2)\,x(y+2) + 4(y+2)^2\right]\right)$$

$$m = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \qquad \det C = 12 - 4 = 8$$

$$\begin{bmatrix} Z \\ T \end{bmatrix} = \begin{bmatrix} X - 1 \\ -2X + Y + 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b}$$

$$m^* = Am + b = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$C^* = ACA^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -10 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -10 \\ -10 & 27 \end{bmatrix}$$

$$(Z, T) \sim N(m^*, C^*)$$

$$E(Z(Z-T)) = E(ZZ - ZT) = 2EZ - EZT = 2\cdot(-1) - (-1)(-1) = -3 \quad ?$$

$$U = Z + 2T - 1$$

$$[U] = \underbrace{[1 \;\; 2]}_{A} \begin{bmatrix} Z \\ T \end{bmatrix} + \underbrace{[-1]}_{b}$$

$$EU = [1 \;\; 2]\begin{bmatrix} -1 \\ -1 \end{bmatrix} + [-1] = -4$$

$$VU = [1 \;\; 2]\begin{bmatrix} 4 & -10 \\ -10 & 27 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [-16 \;\; 17]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 18$$

$$U \sim N(-4, 18)$$

2B1. 3

$Z = X - 2Y \qquad S_Z = \{-7, -1, 0, 1\}$

Z

$X \backslash Y$	0	1	
-5	0	.4	.4
0	.1	0	.1
1	.3	.2	.5
	.4	.6	

$X \backslash Y$	0	1
-5	-5	-7
0	0	-2
1	1	-1

z	-7	-1	0	1
$P(Z=z)$	.4	.2	.1	.3

$$EX = -5 \cdot 0.4 + 0 \cdot 0.1 + 1 \cdot 0.5 = -\frac{3}{2}$$

$$EX^2 = 25 \cdot \frac{4}{10} + 1 \cdot \frac{5}{10} = \frac{21}{2}$$

$$VX = \frac{21}{2} - \frac{9}{4} = \frac{42-9}{4} = \frac{33}{4}$$

$$EY = 0 \cdot \frac{4}{10} + 1 \cdot \frac{6}{10} = \frac{6}{10} = \frac{3}{5}$$

$$EXY = 1 \cdot 1 \cdot \frac{2}{10} + (-5) \cdot 1 \cdot \frac{4}{10} = -1.8 = -\frac{9}{5}$$

$$EZ = -7 \cdot \frac{4}{10} - 1 \cdot \frac{2}{10} + 0 \cdot \frac{1}{10} + 1 \cdot \frac{3}{10} = \frac{-28 - 2 + 3}{10} = -\frac{27}{10}$$

$$EZ^2 = 49 \cdot \frac{4}{10} + 1 \cdot \frac{2}{10} + 1 \cdot \frac{3}{10} = \frac{101}{10}$$

$$VZ = \frac{1010}{100} - \frac{729}{100} = \frac{281}{100}$$

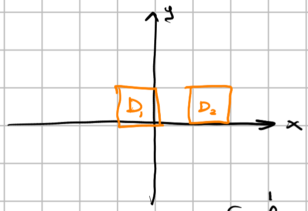
$$\rho(Z, X) = \frac{\operatorname{cov}(Z, X)}{\sqrt{VZ \cdot VX}}$$

$$\operatorname{cov}(Z, X) = \operatorname{cov}(X - 2Y, X) = E(X(X-2Y)) - E(X-2Y)EX$$
$$= E(X^2 - 2XY) - E(X - 2Y) \cdot EX$$
$$= EX^2 - 2EXY - EX(EX - 2EY)$$
$$= \left(\frac{21}{2}\right)^2 - 2 \cdot \left(-\frac{9}{5}\right) - \left(-\frac{3}{2}\right)\left(-\frac{3}{2} - 2 \cdot \frac{3}{5}\right)$$

...

3.

$$f(x,y)=\begin{cases} 2x^2 & -1\le x\le 0, \quad 0\le y\le 1 \\ \frac{1}{3} & 1\le x\le 2, \quad 0\le y\le 1 \\ 0 & \text{u p.p.} \end{cases}$$



$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy=\begin{cases} \int_0^1 2x^2\,dy = 2x^2 & x\in[-1,0] \\ \int_0^1 \frac{1}{3}dy=\frac{1}{3} & x\in[1,2] \\ 0 & \text{u p.p.} \end{cases}$$

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\,dx=\begin{cases} \int_{-1}^{0} 2x^2\,dx+\int_1^2\frac{1}{3}dx=\frac{2}{3}x^3\Big|_{-1}^{0}+\frac{1}{3}=-+\frac{2}{3}+\frac{1}{3}=1 & y\in[0,1] \\ 0 & \text{u p.p.} \end{cases}$$

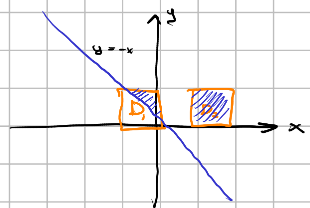
$$EXY=\iint_{\mathbb{R}^2} xy\,f(x,y)\,dx\,dy=\iint_{D_1}2x^3y\,dx\,dy+\iint_{D_2}\frac{1}{3}xy\,dx\,dy$$

$$=\int_{-1}^{0}\int_0^1 2x^3y\,dy\,dx+\int_1^2\int_0^1\frac{1}{3}xy\,dy\,dx$$

$$=\int_{-1}^{0}x^3\cdot y^2\Big|_0^1\,dx+\int_1^2\frac{1}{3}x\cdot\frac{1}{2}y^2\Big|_0^1\,dx$$

$$=\int_{-1}^{0}x^3\,dx+\int_1^2\frac{1}{6}x\,dx=\frac{1}{4}x^4\Big|_{-1}^{0}+\frac{1}{12}x^2\Big|_1^2$$

$$=-\frac{1}{4}+\frac{3}{12}=0$$



$$P(X+Y>0)=\iint_{X+Y>0}f(x,y)\,dx\,dy=\iint_{D_1'}2x^2\,dx\,dy+\iint_{D_2}\frac{1}{3}\,dx\,dy$$

$$D_1'=\{(x,y):\ -1\le x\le 0,\ -x\le y\le 1\}$$

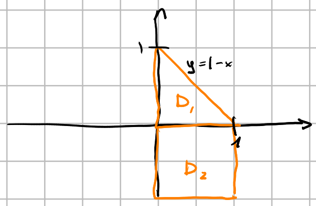
$$=\int_{-1}^{0}\int_{-x}^{1}2x^2\,dy\,dx+\frac{1}{3}\cdot|D_2|$$

$$=\int_{-1}^{0}2x^2(1+x)\,dx+\frac{1}{3}=\int_{-1}^{0}2x^2+2x^3\,dx+\frac{1}{3}$$

$$=\frac{1}{3}+\frac{2}{3}x^3\Big|_{-1}^{0}+\frac{1}{2}x^4\Big|_{-1}^{0}=\frac{1}{3}+\frac{2}{3}-\frac{1}{2}=\frac{1}{2}$$

2.

$$f(x,y) = \begin{cases} \frac{1}{2} & x,y \geq 0 , \quad x+y \leq 1 \\ 1+xy & 0 < x < 1, \quad -1 < y < 0 \\ 0 & \text{w p.p.} \end{cases}$$



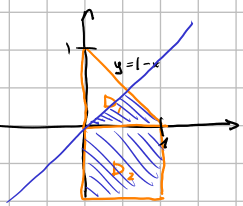
$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} \frac{3}{2} - x & x \in [0,1] \\ 0 & \text{w p.p.} \end{cases}$$

$$D_1 = \{(x,y) : 0 \leq x \leq 1 , 0 \leq y \leq 1-x\}$$
$$= \{(x,y) : 0 \leq y \leq 1 , 0 \leq x \leq 1-y\}$$

$$\int_0^{1-x} \frac{1}{2} dy + \int_{-1}^{0} 1+xy\,dy = \frac{1}{2} - \frac{1}{2}x + y\Big|_{-1}^{0} + x \cdot \frac{1}{2}y^2\Big|_{-1}^{0} = \frac{1}{2} - \frac{1}{2}x + 1 - \frac{1}{2}x = \frac{3}{2} - x$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \begin{cases} \int_0^{1-y} \frac{1}{2} dx = \frac{1}{2} - \frac{1}{2}y & y \in [0,1] \\ \int_0^{1} 1+xy\,dx = x\Big|_0^1 + \frac{1}{2}yx^2\Big|_0^1 = 1+\frac{1}{2}y & y \in [-1,0] \\ 0 & \text{w p.p.} \end{cases}$$

$$P(Y<X) = \iint_{D_1'} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy$$



$$= \frac{1}{2}|D_1'| + \int_0^1 \int_{-1}^{0} 1+xy\,dy\,dx$$

$$= \frac{1}{8} \quad \int_0^1 y\Big|_{-1}^{0} + \frac{1}{2}xy^2\Big|_{-1}^{0}\,dx$$

$$= \frac{1}{8} + \int_0^1 1 - \frac{1}{2}x\,dx = \frac{1}{2} + x\Big|_0^1 - \frac{1}{4}x^2\Big|_0^1 = \frac{1}{8} + 1 - \frac{1}{4} = \frac{7}{8}$$