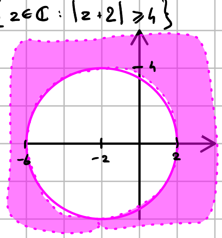
1.

a) $A=\left\{ z\in\mathbb{C}: |z+2| \geqslant \mathrm{Im}\left(8e^{\frac{5\pi j}{6}}\right) \vee \mathrm{Im}(z^2)\leqslant 0\right\}$

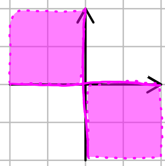
$\mathrm{Im}\left(8e^{\frac{5\pi j}{6}}\right)=\mathrm{Im}\left(8\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}j\right)\right)=\mathrm{Im}(-4\sqrt{3}+4j)=4$

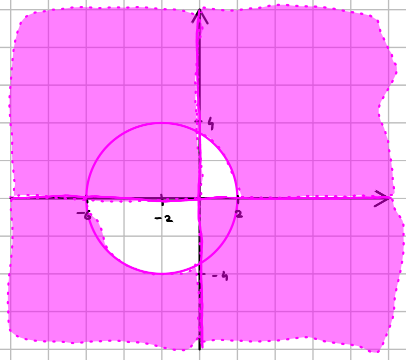
$\{z\in\mathbb{C}: |z+2|\geqslant 4\}$



$\mathrm{Im}(z^2)=\mathrm{Im}((a+bj)^2)=\mathrm{Im}(a^2+2abj-b^2)=2ab$

$2ab\leqslant 0 \iff ab\leqslant 0 \iff (a\geqslant 0 \wedge b\leqslant 0)\vee(a\leqslant 0\wedge b\geqslant 0)$





b) $(z-j)^3(1+j\sqrt{3})^3=j\,|(\sqrt{7}-j)^4|$

$(1+j\sqrt{3})^3=\left[2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)\right]^3=8\cdot\left[e^{\frac{\pi}{3}j}\right]^3=8e^{\pi j}=-8$

$|(\sqrt{7}-j)^4|=|\sqrt{7}-j|^4=(\sqrt{8})^4=64$

$(z-j)^3\cdot(-8)=64j$

$(z-j)^3=-8j$

$z-j\in\sqrt[3]{-8j}=\sqrt[3]{8e^{-\frac{\pi}{2}j}}=\left\{2e^{\frac{-\frac{\pi}{2}+2k\pi}{3}j} : k=0,1,2\right\}=\left\{2e^{-\frac{\pi}{6}j},\ 2e^{\frac{\pi}{2}j},\ 2e^{-\frac{5\pi}{6}j}\right\}=\left\{2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}j\right),\ 2(0+j),\ 2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}j\right)\right\}=\left\{\sqrt{3}-j,\ 2j,\ -\sqrt{3}-j\right\}$

$z\in\{\sqrt{3},\ 3j,\ -\sqrt{3}\}$

2.

$\varphi:\mathbb{R}^4\to\mathbb{R}^3$

$$\varphi\left(\begin{bmatrix}x\\y\\z\\t\end{bmatrix}\right)=\begin{bmatrix}2x+y-3z\\3x+y-2z+t\\2x+2z+2t\end{bmatrix}=x\begin{bmatrix}2\\3\\2\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}-3\\-2\\2\end{bmatrix}+t\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}2&1&-3&0\\3&1&-2&1\\2&0&2&2\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\\t\end{bmatrix}$$

$$M_{E_3}^{E_4}(\varphi)=\begin{bmatrix}2&1&-3&0\\3&1&-2&1\\2&0&2&2\end{bmatrix}\qquad M_{E_4}(A)=M_{E_4}^{A}(id)=\begin{bmatrix}0&1&0&0\\3&-2&0&0\\1&0&2&1\\-1&-1&1&2\end{bmatrix}$$

$$M_{E_3}^{A}(\varphi)=M_{E_3}^{E_4}(\varphi)\cdot M_{E_4}^{A}(id)=\begin{bmatrix}2&1&-3&0\\3&1&-2&1\\2&0&2&2\end{bmatrix}\cdot\begin{bmatrix}0&1&0&0\\3&-2&0&0\\1&0&2&1\\-1&-1&1&2\end{bmatrix}=\begin{bmatrix}0&0&-6&-3\\0&0&-3&0\\0&0&6&6\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}-2&2&-1&1&0&0\\0&1&0&0&1&0\\3&-2&2&0&0&1\end{array}\right]\xrightarrow{v_1+v_3}\left[\begin{array}{ccc|ccc}1&0&1&1&0&1\\0&1&0&0&1&0\\3&-2&2&0&0&1\end{array}\right]\xrightarrow[v_3-3v_1]{}\left[\begin{array}{ccc|ccc}1&0&1&1&0&1\\0&1&0&0&1&0\\0&-2&-1&-3&0&-2\end{array}\right]\xrightarrow[v_3+2v_2]{}\left[\begin{array}{ccc|ccc}1&0&1&1&0&1\\0&1&0&0&1&0\\0&0&-1&-3&2&-2\end{array}\right]\xrightarrow[v_3\cdot(-1)]{v_1+v_3}\left[\begin{array}{ccc|ccc}1&0&0&-2&2&-1\\0&1&0&0&1&0\\0&0&1&3&-2&2\end{array}\right]$$

$$M_{B}^{A}(\varphi)=M_{B}^{E_3}(id)\cdot M_{E_3}^{E_4}(\varphi)\cdot M_{E_4}^{A}(id)=[M_{E_3}(B)]^{-1}\cdot M_{E_3}^{A}(\varphi)=\begin{bmatrix}-2&2&-1\\0&1&0\\3&-2&2\end{bmatrix}\cdot\begin{bmatrix}0&0&-6&-3\\0&0&-3&0\\0&0&6&6\end{bmatrix}=\begin{bmatrix}0&0&0&0\\0&0&-3&0\\0&0&0&3\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}2&1&-3&0&0\\3&1&-2&1&0\\2&0&2&2&0\end{array}\right]\xrightarrow[\substack{v_2-v_1\\v_3-v_1}]{}\left[\begin{array}{cccc|c}2&1&-3&0&0\\1&0&1&1&0\\0&-1&5&2&0\end{array}\right]\xrightarrow{v_1-2v_2}\left[\begin{array}{cccc|c}0&1&-5&-2&0\\1&0&1&1&0\\0&-1&5&2&0\end{array}\right]\to\begin{array}{c}\left[\begin{array}{cccc|c}1&0&1&1&0\\0&1&-5&-2&0\end{array}\right]\\ \begin{array}{cccc}x&y&\uparrow a_1&\uparrow a_2\end{array}\end{array}$$

$$\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=a_1\begin{bmatrix}-1\\5\\1\\0\end{bmatrix}+a_2\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}$$

$\dim \mathrm{Ker}\,\varphi=\dim \mathrm{span}\left\{\begin{bmatrix}-1\\5\\1\\0\end{bmatrix},\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}\right\}=2$ $\dim\mathbb{R}^4=2+2=4$

$\dim \mathrm{Im}\,\varphi=\mathrm{rank}(\varphi)=2$

$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \in \mathrm{Im}\varphi$ bo układ ma rozwiązanie

$$\left[\begin{array}{cccc|c} 2 & 1 & -3 & 0 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ 2 & 0 & 2 & 2 & -2 \end{array}\right] \begin{array}{c} \\ \xrightarrow{} \\ w_2 - w_1 \\ w_3 - w_1 \end{array} \left[\begin{array}{ccccc} 2 & 1 & -3 & 0 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 5 & 2 & -3 \end{array}\right] \xrightarrow{w_1 - 2w_2} \left[\begin{array}{ccccc} 0 & 1 & -5 & -2 & 3 \\ 1 & 0 & 1 & 1 & -1 \\ \cancel{0} & \cancel{-1} & \cancel{5} & \cancel{2} & \cancel{-3} \end{array}\right]$$

3.

$$\left[\begin{array}{cccc|c} 3 & 2 & -1 & 5 & -2 \\ 1 & 0 & -a & 3a & 2a \\ 2 & a & -a & -2a & 3a \end{array}\right] \qquad \begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -a \\ 2 & a & -a \end{vmatrix} \xrightarrow{w_3 - w_2} \begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -a \\ 1 & a & 0 \end{vmatrix} = 0 - 2a - a - 0 - (-3a^2) - 0$$

$$= 3a^2 - 3a = 3a(a-1)$$

dla $a \neq 0 \wedge a \neq 1$

$\mathrm{rank}(A|B) = \mathrm{rank}(A) = 3$

nieskończenie wiele rozwiązań zależnych od 1 parametru

dla $a = 0$

$$\left[\begin{array}{cccc|c} 3 & 2 & -1 & 5 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ \cancel{2} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 2 & -1 & 5 & -2 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathrm{rank}(A|B) = \mathrm{rank}(A) = 2$

nieskończenie wiele rozwiązań zależnych od 2 parametrów

dla $a = 1$

$$\left[\begin{array}{cccc|c} 3 & 2 & -1 & 5 & -2 \\ 1 & 0 & -1 & 3 & 2 \\ 2 & 1 & -1 & -2 & 3 \end{array}\right] \begin{array}{c} w_1 - 3w_2 \\ \longrightarrow \\ w_3 - 2w_2 \end{array} \left[\begin{array}{ccccc} 0 & 2 & 2 & -4 & -8 \\ 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 1 & -8 & -1 \end{array}\right] \xrightarrow{w_1 - 2w_3} \left[\begin{array}{ccccc} 0 & 0 & 0 & 12 & -6 \\ 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 1 & -8 & -1 \end{array}\right] \xrightarrow{\frac{1}{12}w_1} \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 1 & -8 & -1 \end{array}\right] \begin{array}{c} \longrightarrow \\ w_2 - 3w_1 \\ w_3 + 8w_1 \end{array} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 1 & 0 & -1 & 0 & \frac{7}{2} \\ 0 & 1 & 1 & 0 & -5 \end{array}\right]$$

$\mathrm{rank}(A|B) = \mathrm{rank}(A) = 3$

nieskończenie wiele rozwiązań zależnych od 1 parametru

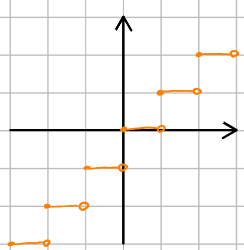
$a = -1$

$$\left[\begin{array}{cccc|c} 3 & 2 & -1 & 5 & -2 \\ 1 & 0 & 1 & -3 & -2 \\ 2 & -1 & 1 & 2 & -3 \end{array}\right] \begin{array}{c} w_1 - 3w_2 \\ \longrightarrow \\ w_3 - 2w_2 \end{array} \left[\begin{array}{ccccc} 0 & 2 & -4 & 14 & 4 \\ 1 & 0 & 1 & -3 & -2 \\ 0 & -1 & -1 & 8 & 1 \end{array}\right] \begin{array}{c} w_1 + 2w_3 \\ \longrightarrow \\ -1 \cdot w_3 \end{array} \left[\begin{array}{ccccc} 0 & 0 & -6 & 30 & 6 \\ 1 & 0 & 1 & -3 & -2 \\ 0 & 1 & 1 & -8 & -1 \end{array}\right] \xrightarrow{-\frac{1}{6}w_1} \left[\begin{array}{ccccc} 0 & 0 & 1 & -5 & -1 \\ 1 & 0 & 1 & -3 & -2 \\ 0 & 1 & 1 & -8 & -1 \end{array}\right] \begin{array}{c} \longrightarrow \\ w_2 - w_1 \\ w_3 - w_1 \end{array} \begin{array}{c} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -5 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 0 \end{array}\right] \\ \begin{array}{ccccc} x & y & z & \uparrow t & \end{array} \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 5 \\ 1 \end{bmatrix} \qquad t \in \mathbb{R}$$

4. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \lfloor x \rfloor$

- $f^{-1}((-2,2)) = f^{-1}(\{-1,0,1\}) = [-1,2)$
- $f(f^{-1}([\frac{1}{2}, e])) = f(f^{-1}(\{1,2\})) = f([1,3)) = \{1,2\}$
- $f^{-1}(f([\frac{1}{2}, e])) = f^{-1}(\{0,1,2\}) = [0,3)$



5.

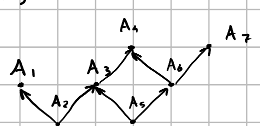
$A_1 = [0,3]$ $\quad A_5 = [2,4]$

$A_2 = [1,3]$ $\quad A_6 = [2,5]$

$A_3 = [1,4]$ $\quad A_7 = [2,6]$

$A_4 = [1,5]$

$(2^{\mathbb{R}}, \subseteq)$



elementy maksymalne A_4, A_7

elementy minimalne A_2, A_5

$\sup(A_t) = [0,6]$

$\inf(A_t) = [2,3]$

6.

$$W = \operatorname{span}\{x^3+x^2-2,\ x^2-2x,\ x^3+2x-2,\ x^3+2x^2-2x-2\} \subseteq \mathbb{R}_3[x]$$

$$\dim W = \operatorname{rank}\left[\begin{matrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & -2 & 2 & -2 \\ -2 & 0 & -2 & -2 \end{matrix}\right]\begin{matrix} \\ v_2-v_1 \\ \frac{1}{2}v_3 \\ v_4+2v_1 \end{matrix} \longrightarrow \left[\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{matrix}\right]\begin{matrix} \\ \\ v_3+v_2 \\ \\ \end{matrix} \longrightarrow \left[\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{matrix}\right] \to \dim W = 2$$

$W = \operatorname{span}\{x^3+x^2-2,\ x^2-2x\}$

baza $(x^3+x^2-2,\ x^2-2x)$

7.

$A = \{x\in\mathbb{R} :\ \forall_{n\in\mathbb{N}}\ \ 2-\frac{1}{n} \le x < 7+\sqrt{n}\} = \bigcap_{i=1}^{\infty} P_i = [2, 8)$

$B = \{x\in\mathbb{R} :\ \exists n\in\mathbb{N}\ \ 2-\frac{1}{n} \le x < 7+\sqrt{n}\} = \bigcup_{i=1}^{\infty} P_i = [1, +\infty)$

$P_i = \{x\in\mathbb{R} :\ 2-\frac{1}{n} \le x < 7+\sqrt{n}\}$

$P_1 = [1, 8)$

$P_4 = [1.75, 9)$

$P_n \to [2, \infty)$