1.

płaszczyzna ACD $\quad n_1 = \vec{AC} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{i}\cdot(-1) - \hat{j}(-1) + \hat{k}\cdot 1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

a) krawędź nieleżąca na ACD $\rightarrow \vec{AB} = (0,1,1)$

$\alpha = \sphericalangle(\Delta ACD, \vec{AB}) = \frac{\pi}{2} - \sphericalangle(n_1, \vec{AB})$

$\cos(\alpha) = \cos\left(\frac{\pi}{2} - \sphericalangle(n_1, \vec{AB})\right) = \sin\sphericalangle(n_1, \vec{AB}) = \frac{|n_1 \times \vec{AB}|}{|n_1|\cdot|\vec{AB}|}$

$$\cos(\alpha) = \frac{\left\| \begin{vmatrix} \hat{i} & \hat{j} & k \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \right\|}{\sqrt{3}\cdot\sqrt{2}} = \frac{|0\cdot\hat{i} - (-1)\hat{j} + (-1)\hat{k}|}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$$

b) ściana ACB

$$n_2 = \vec{AC} \times \vec{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{i} - \hat{j} + \hat{k} = (-1,-1,1)$$

kąt między ścianami $\quad \beta = \sphericalangle(\Delta ACD, \Delta ACB) = \sphericalangle(n_1, n_2)$

$$\cos(\beta) = \frac{n_1 \circ n_2}{|n_1|\cdot|n_2|} = \frac{1 - 1 + 1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3}$$

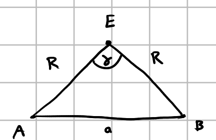
c) środek $\quad E = \frac{1}{4}(A + B + C + D) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

kąt wierzchołek - środek - wierzchołek $\quad \gamma = \sphericalangle(\vec{EA}, \vec{EB})$

$$\cos(\gamma) = \frac{\vec{EA}\circ\vec{EB}}{|\vec{EA}|\cdot|\vec{EB}|} = \frac{\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)\circ\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\left(\sqrt{3\cdot\frac{1}{4}}\right)^2} = \frac{\frac{1}{4} - \frac{1}{4} - \frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3}$$

d) $a = |\vec{AB}| = |\vec{BC}| = \ldots$

$R = |\vec{EA}| = |\vec{EB}| = \ldots$



$a^2 = R^2 + R^2 - 2R\cdot R\cos(\gamma)$

$a^2 = 2R^2(1 - \cos(\gamma))$

$R = \sqrt{\frac{a^2}{2(1 - (-\frac{1}{3}))}} = a\frac{\sqrt{6}}{4}$

e) $n_{ACD} = (-1, 1, 1)$

$(x - A)\circ n_{ACD} = -x + y + z = 0$

$$r = d(E, \Delta ACD) = \frac{|E\circ n_{ACD}|}{|E|} = \frac{\left|-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right|}{\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}}$$

$a = \sqrt{2} \qquad \frac{r}{a} = \frac{1}{\sqrt{6}} \qquad r = \frac{a}{\sqrt{6}} = \frac{a\sqrt{6}}{6}$

f)

$AB \rightarrow A + \alpha B$

$$\tilde{r} = d(E, AB) = \frac{|(E - A)\times B|}{|B|} = \frac{\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{vmatrix} \right\|}{\sqrt{2}} = \frac{|0\cdot\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}|}{\sqrt{2}} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{2}} = \frac{1}{2} = \frac{a}{2\sqrt{2}} = \frac{a\sqrt{2}}{4}$$

g) $V = \frac{1}{6}\left|(\vec{AB}\times\vec{AC})\circ\vec{AD}\right| = \frac{1}{6}|(B\times C)\circ D| = \frac{1}{6}|(1,-1,-1)\circ(0,1,1)| = \frac{1}{6}|0 - 1 - 1| = \frac{2}{6} = \frac{1}{3} = \frac{1}{3}\cdot\left(\frac{a}{\sqrt{2}}\right)^3 = \frac{\sqrt{2}}{12}a^3$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k} = (1,-1,-1)$$

3.

$A = (1,0,0)$ $\qquad \sphericalangle APB = \sphericalangle BPC = \sphericalangle CPA = \frac{\pi}{2}$

$B = (0,2,0)$

$C = (0,0,3)$

$$\vec{PA} \cdot \vec{PB} = 0$$

$$(1-p_x, -p_y, -p_z) \cdot (-p_x, 2-p_y, -p_z) = 0$$

$$-p_x + p_x^2 - 2p_y + p_y^2 + p_z^2 = 0$$

$$p_x^2 + p_y^2 + p_z^2 = p_x + 2p_y \qquad \text{itd analogicznie}$$

$$|P|^2 = p_x + 2p_y$$
$$|P|^2 = p_x + 3p_z$$
$$|P|^2 = 2p_y + 3p_z$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & \alpha \\ 1 & 0 & 3 & \alpha \\ 0 & 2 & 3 & \alpha \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & \alpha \\ 0 & -2 & 3 & 0 \\ 0 & 2 & 3 & \alpha \end{array}\right] \to \left[\begin{array}{cccc} 1 & 0 & 3 & \alpha \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 6 & \alpha \end{array}\right] \quad \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{\alpha}{2} \\ 0 & -2 & 0 & -\frac{\alpha}{2} \\ 0 & 0 & 6 & \alpha \end{array}\right] \quad \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{\alpha}{2} \\ 0 & 1 & 0 & \frac{\alpha}{4} \\ 0 & 0 & 1 & \frac{\alpha}{6} \end{array}\right]$$

$$P = \left(\frac{\alpha}{2}, \frac{\alpha}{4}, \frac{\alpha}{6}\right)$$

$$|P|^2 = \alpha$$

$$\left(\frac{\alpha}{2}\right)^2 + \left(\frac{\alpha}{4}\right)^2 + \left(\frac{\alpha}{6}\right)^2 = \alpha$$

$$\alpha^2\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36}\right) - \alpha = 0 \qquad \frac{49}{144}\alpha\left(\alpha - \frac{144}{49}\right) = 0$$

$$\frac{49}{144}\alpha^2 - \alpha = 0 \qquad \alpha = 0 \ \vee \ \alpha = \frac{144}{49}$$

$P\left(\frac{144}{49}\right)$ jest lustrzanym odbiciem $P(0)$ względem płaszczyzny ABC

metoda geometryczna

$\vec{AB} = (1, -2, 0)$

$\vec{AC} = (1, 0, -3)$

$$n = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 0 & -3 \end{vmatrix} = \hat{\imath} \cdot 6 - \hat{\jmath} \cdot (-3) + \hat{k} \cdot 2 = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

$P = O + \alpha n = \alpha n$ $\qquad$ leży na płaszczyźnie ABC

ABC: $\quad 6(x-1) + 3y + 2z = 0 \qquad 6 \cdot 6\alpha + 3 \cdot 3\alpha + 2 \cdot 2\alpha - 6 = 0$

$\qquad 6x + 3y + 2z - 6 = 0 \qquad \alpha = \frac{6}{49}$

odbicie $\quad O' = O + 2 \cdot \frac{6}{49} \cdot (6,3,2) = O + 2\alpha n$

2.

1) ściana - krawędź $\quad \alpha$

$$n_{ABC} = \vec{AB} \times \vec{AC} \qquad \vec{AB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{AC} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$n_{ABC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \hat{\imath} \cdot 1 - \hat{\jmath} \cdot (-1) + \hat{k} \cdot 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

krawędź $\vec{FE} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$$\alpha = \frac{\pi}{2} - \sphericalangle(n_{ABC}, \vec{FE})$$

$$\cos(\alpha) = \sin(\sphericalangle n_{ABC}, \vec{FE}) = \frac{|n_{ABC} \times \vec{FE}|}{|n_{ABC}| \cdot |\vec{FE}|} = \frac{\left\| \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{matrix} \right\|}{\sqrt{3} \cdot \sqrt{2}} = \frac{|\hat{\imath} \cdot 2 - \hat{\jmath} \cdot 1 + \hat{k} \cdot (-1)|}{\sqrt{6}} = \frac{\sqrt{6}}{\sqrt{6}} = 1$$

są równoległe

2) kąt ściana – ściana

$n_{ABC} = (1,1,1)$

$$\vec{BD} = \begin{bmatrix}-1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\\0\end{bmatrix} \qquad \vec{BC} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$n_{BDC} = \vec{BD} \times \vec{BC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -1 & 0\\ 0 & -1 & 1\end{vmatrix} = \hat{i}\cdot(-1) - \hat{j}\cdot(-1) + \hat{k}\cdot(1) = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\beta = \sphericalangle(ABC, BDC) = \sphericalangle(n_{ABC}, n_{BDC})$$

$$\cos(\beta) = \frac{n_{ABC} \circ n_{BDC}}{|n_{ABC}| \cdot |n_{BDC}|} = \frac{-1+1+1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3}$$ albo $-\frac{1}{3}$

kąt dwuścienny oznacza ten, który jest w środku

3) kąt wierzchołek – środek – wierzchołek

$$S = \frac{1}{6}(A+B+C+D+E+F) = (0,0,0)$$

1° sąsiednie wierzchołki $A - S - B$

$$\gamma = \sphericalangle(\vec{SA}, \vec{SB}) \qquad \cos(\gamma) = \frac{\vec{SA} \circ \vec{SB}}{|\vec{SA}| \cdot |\vec{SB}|} = \frac{0}{1\cdot 1} = 0$$

2° przeciwległe wierzchołki $A - S - D$

$$\delta = \sphericalangle(\vec{SA}, \vec{SD}) \qquad \cos(\delta) = \frac{\vec{SA} \circ \vec{SD}}{|\vec{SA}| \cdot |\vec{SD}|} = \frac{-1}{1} = -1$$

4) promień sfery opisanej

krawędź $a = |\vec{AB}| = \sqrt{2}$ $\qquad \frac{a}{\sqrt{2}} = 1$

$$R = |\vec{SA}| = |\vec{SB}| = \dots = 1$$

ogólnie $R = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}$

5) promień sfery wpisanej

ABC: $(x - A) \circ (\vec{AB} \times \vec{AC}) = 0$

$(x - A) \circ n_{ABC} = 0$

$(x-1, y, z) \circ (1,1,1) = 0$

$x - 1 + y + z = 0$

$x + y + z - 1 = 0$

$$r = \text{dist}(S, ABC) = \frac{|0+0+0-1|}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

ogólnie $r = \frac{\sqrt{3}}{3} \cdot \frac{a}{\sqrt{2}} = \frac{\sqrt{6}}{6} a = \frac{a}{\sqrt{6}}$

6) promień sfery półwpisanej

$$AB: \quad x = A + \alpha \vec{AB} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$S - A = -A = (-1,0,0)$
$\vec{AB} = (-1,1,0)$

$$r_h = dist(S, AB) = \frac{|(S-A)\times \vec{AB}|}{|\vec{AB}|} = \frac{\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 0 & 0\\ -1 & 1 & 0\end{matrix}\right\|}{\sqrt{2}} = \frac{|\hat{i}\cdot 0 - \hat{j}\cdot 0 + \hat{k}\cdot(-1)|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

ogólnie $r_h = \frac{1}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \frac{a}{2}$

$AD = (-2,0,0)$

7) $V = 2V_{ABCDE} = 2 \cdot 2 \cdot V_{ABCD} = 4 \cdot \frac{1}{6} |(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = \frac{2}{3} | n_{ABC} \cdot \vec{AD} | = \frac{2}{3} |-2| = \frac{4}{3}$

ogólnie $V = \frac{4}{3} \cdot \left(\frac{a}{\sqrt{2}}\right)^3 = \frac{4}{3} \cdot \frac{a^3}{2\sqrt{2}} = \frac{4\sqrt{2}\, a^3}{3 \cdot 4} = \frac{\sqrt{2}}{3} a^3$

4.

$A = (2,-3,1)$
$B = (-2,1,-3)$
$C = (-3,3,-1)$
$D = (1,-3,1)$

$\frac{\text{Objętość równoległościanu}}{\text{pole równoległoboku}}$ = wysokość równoległościanu

$$dist(AB, CD) = \left|\frac{|\vec{AC} \circ (\vec{AB} \times \vec{CD})|}{|\vec{AB} \times \vec{CD}|}\right|$$