1.

a) $\lim_{n\to\infty} \frac{4^n+5^n}{2^{2n+1}+5^{n+1}} = \lim_{n\to\infty} \frac{5^n\left[\left(\frac{4}{5}\right)^n+1\right]}{5^n\left[\frac{2\cdot 2^{2n}}{5^n}+5\right]} = \lim_{n\to\infty} \frac{\left(\frac{4}{5}\right)^n+1}{2\cdot\left(\frac{4}{5}\right)^n+5} = \frac{0+1}{0+5} = \frac{1}{5}$

b) $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n}} = ?$ $\quad x = \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n} \quad q=\frac{1}{2},\ |q|<1$

$x = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

$y = \lim_{n\to\infty} 1+\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n} \quad q_y = \frac{1}{3},\ |q_y|<1$

$y = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{2^n}}{1+\frac{1}{3}+\dots+\frac{1}{3^n}} = \frac{x}{y} = \frac{2}{\frac{3}{2}} = 2\cdot\frac{2}{3} = \frac{4}{3}$

c) $\lim_{n\to\infty} \sqrt[n]{\frac{3^n+4^n}{4^n+5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{3^n+4^n}}{\sqrt[n]{4^n+5^n}}$

$x = \lim_{n\to\infty} \sqrt[n]{3^n+4^n} \qquad \forall_{n>1}\ \sqrt[n]{x} \nearrow$ dla $x>0$

$\lim_{n\to\infty} \sqrt[n]{4^n} \le \lim_{n\to\infty} \sqrt[n]{3^n+4^n} \le \lim_{n\to\infty} \sqrt[n]{4^n+4^n} = \lim_{n\to\infty} \sqrt[n]{2\cdot 4^n} = \lim_{n\to\infty} 4\cdot\sqrt[n]{2}$

$4 \le x \le 4 \implies x=4$

$y = \lim_{n\to\infty} \sqrt[n]{4^n+5^n}$

$\lim_{n\to\infty} \sqrt[n]{5^n} \le \lim_{n\to\infty} \sqrt[n]{4^n+5^n} \le \lim_{n\to\infty} \sqrt[n]{5^n+5^n}$

$5 \le y \le 5 \implies y=5$

$\lim_{n\to\infty} \sqrt[n]{\frac{3^n+4^n}{4^n+5^n}} = \frac{x}{y} = \frac{4}{5}$

d) $\lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{2}n\right)}{n+1} \qquad \cos\left(\frac{\pi}{2}\right)=0 \quad \cos\left(2\cdot\frac{\pi}{2}\right)=-1 \quad \cos\left(3\cdot\frac{\pi}{2}\right)=0 \quad \cos$

1) $\lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{2}\cdot(2n+1)\right)}{(2n+1)+1} = \lim_{n\to\infty} \frac{0}{2n+2} = 0$

2) $\lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{2}(4n)\right)}{4n+1} = \lim_{n\to\infty} \frac{1}{4n+1} = 0$

3) $\lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{2}(4n+2)\right)}{(4n+2)+1} = \lim_{n\to\infty} \frac{-1}{4n+3} = 0$

Więc $\lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{2}n\right)}{n+1} = 0$

albo łatwiej:

$\lim_{n\to\infty} \underbrace{\cos\left(n\frac{\pi}{2}\right)}_{\text{ograniczony}} \cdot \underbrace{\frac{1}{n+1}}_{0} = 0$

e) $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}\cdot 4^n + n\cdot 3^n + 5n^3} = 4$ z twierdzenia o 3 ciągach

$$\sqrt[n]{\frac{1}{n^2}4^n} \le \sqrt[n]{\frac{1}{n^2}4^n + n\cdot 3^n + 5n^3} \le \sqrt[n]{n^3 4^n + n^3 4^n + n^3 4^n}$$

$$\sqrt[n]{\frac{1}{n^2}4^n} = 4\cdot\frac{1}{(\sqrt[n]{n})^2} \to 4$$

$$\sqrt[n]{n^3 4^n + n^3 4^n + n^3 4^n} = 4\cdot\sqrt[n]{3n^3} \to 4$$

f) $\lim_{n\to\infty} \frac{\binom{n+2}{n}}{1+2+3+\dots+n}$

$$a_n = \binom{n+2}{n} = \frac{(n+2)(n+1)\,n!}{2\,n!} = \frac{(n+2)(n+1)}{2}$$

$$b_n = 1+2+3+\dots+n = \frac{1+n}{2}\cdot n = \frac{(n+1)\cdot n}{2}$$

$$\lim_{n\to\infty} \frac{\frac{(n+2)(n+1)}{2}}{\frac{(n+1)\,n}{2}} = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{1} = 1$$

g) $\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{\frac{n}{\sqrt{n}}} = [e^\infty] = \infty$

$$\lim_{n\to\infty}\sqrt{n} = +\infty \implies \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = e$$

h) $\lim_{n\to\infty}\left(\frac{2n^2+2n+1}{2n^2+2}\right)^{n+1} = \lim_{n\to\infty}\left(\frac{2n^2+2+2n+1-2}{2n^2+2}\right)^{n+1}$

$$\lim_{n\to\infty}\left(1+\frac{2n-1}{2n^2+2}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{2n^2+2}{2n-1}}\right)^{n+1}$$

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{2n^2+2}{2n-1}}\right)^{\frac{2n^2+2}{2n-1}}\right]^{\frac{2n-1}{2n^2+2}\cdot(n+1)} = e^1 = e$$

$$(2n-1)(n+1) = 2n^2+2n-n-1$$

$$\lim_{n\to\infty}\frac{2n^2+n-1}{2n^2+2} = 1$$

$$\lim_{n\to\infty}\frac{2n^2+2}{2n-1} = \infty$$

i) $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2019} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2019} = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]^{2019} = 1^{2019} = 1$

j) $\lim_{n\to\infty} n\cdot[\ln(n+3)-\ln(n)] = \lim_{n\to\infty} n\ln\left(\frac{n+3}{n}\right) = \lim_{n\to\infty}\ln\left(\left(1+\frac{3}{n}\right)^n\right)$

$$= \ln\left(\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}\cdot 3}\right) = \ln(e^3) = 3$$

2.

a) $\lim\limits_{n\to\infty} \frac{\cos(n\pi)+\sqrt{3}}{2\cos(n\pi)+\sqrt{2}}$

1) $\lim\limits_{n\to\infty} \frac{\cos((2n+1)\pi)+\sqrt{3}}{2\cos((2n+1)\pi)+\sqrt{2}} = \frac{-1+\sqrt{3}}{-2+\sqrt{2}}$

2) $\lim\limits_{n\to\infty} \frac{\cos(2n\pi)+\sqrt{3}}{2\cos(2n\pi)+\sqrt{2}} = \frac{1+\sqrt{3}}{2+\sqrt{2}}$

podciągi mają różne granice więc granica ciągu nie istnieje

b) $\lim\limits_{n\to\infty} a^n$, gdzie $a \in \mathbb{R}$ i $a < -1$

$$\left.\begin{array}{l} 1)\ a<-1 \Rightarrow \lim\limits_{n\to\infty} a^{2n} = +\infty \\ 2)\ a<-1 \Rightarrow \lim\limits_{n\to\infty} a^{2n+1} = -\infty \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty} a^{2n} \neq \lim\limits_{n\to\infty} a^{2n+1}$$

granice dwóch podciągów są różne więc granica ciągu nie istnieje

3.

a) $\lim_{x\to 0} \frac{1-\cos(4x)}{x^2} = \lim_{x\to 0} \frac{1-(1-2\sin^2(2x))}{x^2} = \lim_{x\to 0} \frac{2\sin^2(2x)}{x^2}$

$\lim_{x\to 0} \frac{8\sin^2(2x)}{(2x)^2} = \lim_{x\to 0} 8\cdot\left[\frac{\sin(2x)}{2x}\right]^2 = 8\cdot 1^2 = 8$

b) $\lim_{x\to 0} \frac{\arctan(x)}{3x}$ $\quad \lim_{y\to 0} \frac{y}{3\tan(y)} = \lim_{y\to 0} \frac{y}{\sin(y)}\cdot\cos(y)\cdot\frac{1}{3}$

$y = \arctan(x)$
$x = \tan(y)$

$= 1\cdot 1\cdot\frac{1}{3} = \frac{1}{3}$

$x\to 0 \Rightarrow \arctan(x)\to 0$

c) $\lim_{x\to 0} (1-\sin(x))^{\frac{1}{x}} = \lim_{x\to 0}\left(\left(1+\frac{1}{-\frac{1}{\sin(x)}}\right)^{-\frac{1}{\sin(x)}}\right)^{-\frac{\sin(x)}{x}} = e^{-1}$

d) $\lim_{x\to 0} (\cos(x))^{\cot^2(x)} = \lim_{x\to 0} \sqrt{1-\sin^2(x)}^{\cot^2(x)} = \lim_{x\to 0} (1-\sin^2(x))^{\frac{1}{2}\cot^2(x)}$

$\lim_{x\to 0}\left[\left(1+\frac{1}{-\frac{1}{\sin^2(x)}}\right)^{-\frac{1}{\sin^2(x)}}\right]^{(-\sin^2(x))\cdot\frac{1}{2}\frac{\cos^2(x)}{\sin^2(x)}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

$\lim_{x\to 0} -\frac{1}{2}\cos^2(x) = -\frac{1}{2}$

e) $\lim_{x\to 0} \frac{\sin(4x)}{4-\sqrt{5x+16}} = \lim_{x\to 0} \frac{\sin(4x)\,[4+\sqrt{5x+16}]}{(4-\sqrt{5x+16})(4+\sqrt{5x+16})} = \lim_{x\to 0} \frac{\sin(4x)\,[4+\sqrt{5x+16}]}{16-5x-16}$

$\lim_{x\to 0} \frac{\sin(4x)\,(4+\sqrt{5x+16})}{-5x} = \lim_{x\to 0} \frac{\sin(4x)}{4x}\cdot\left(-\frac{4}{5}\right)\cdot(4+\sqrt{5x+16})$

$1\cdot\left(-\frac{4}{5}\right)\cdot(4+4) = -\frac{32}{5}$

f) $\lim_{x\to -1} \frac{\sin(x^2-1)}{x+1} = \lim_{x\to -1} \frac{\sin(x^2-1)}{x+1}\cdot\frac{x-1}{x-1} = \lim_{x\to -1} \frac{\sin(x^2-1)}{x^2-1}\cdot(x-1) = 1\cdot(-2) = -2$

g) $\lim_{x\to -\infty} \frac{\sqrt{2x^2+x+1}}{x} = \lim_{x\to -\infty} \frac{\sqrt{2x^2+x+1}}{-|x|}\quad \lim_{x\to -\infty} \frac{\sqrt{2x^2+x+1}}{-\sqrt{x^2}} = \lim_{x\to -\infty} \sqrt{\frac{2x^2+x+1}{-x^2}} = \sqrt{\lim_{x\to -\infty} -\frac{2x^2+x+1}{x^2}}$

$\sqrt{\lim_{x\to -\infty} -\frac{x^2\left(2+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\cdot 1}} = -\sqrt{2}$

4.

a) $\lim_{x \to -\infty} \sin(3x)$

$x_n' = -\frac{\pi}{2} \cdot (4n+1)$ $\lim_{n\to\infty} \sin\left(-3 \cdot \frac{\pi}{2}(4n+1)\right) = \lim_{n\to\infty} \sin\left(-6\pi(n+1) - \frac{\pi}{2}\right) = -1$

$x_n'' = -\pi n$

$\lim_{n\to\infty} \sin(-3\pi n) = 0$

$\lim_{n\to\infty} \sin(3x_n') \neq \lim_{n\to\infty} \sin(3x_n'') \Longrightarrow$ granica $\lim_{x\to-\infty} \sin(3x)$ nie istnieje

b) $\lim_{x\to 0} (1+|x|)^{\frac{1}{x}}$

$\lim_{x\to 0^+} (1+|x|)^{\frac{1}{x}} = \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e$

$\lim_{x\to 0^-} (1+|x|)^{\frac{1}{x}} = \lim_{x\to 0^-} (1-x)^{\frac{1}{x}} = \lim_{x\to 0^-} \left(1+(-x)\right)^{\frac{1}{(-x)}\cdot(-1)} = e^{-1} = \frac{1}{e}$

ponieważ $\lim_{x\to 0^-} (1+|x|)^{\frac{1}{x}} \neq \lim_{x\to 0^+} (1+|x|)^{\frac{1}{x}}$ to $\lim_{x\to 0} (1+|x|)^{\frac{1}{x}}$ nie istnieje

c) $\lim_{x\to 0} \sin\left(\arctan\left(\frac{1}{x}\right)\right)$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\lim_{t\to -\infty} \arctan(t) = -\frac{\pi}{2}$

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$ $\lim_{t\to +\infty} \arctan(t) = \frac{\pi}{2}$

$\lim_{x\to 0^+} \sin\left(\arctan\left(\frac{1}{x}\right)\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$\neq$ więc $\lim_{x\to 0} \sin\left(\arctan\left(\frac{1}{x}\right)\right)$ nie istnieje

$\lim_{x\to 0^-} \sin\left(\arctan\left(\frac{1}{x}\right)\right) = \sin\left(-\frac{\pi}{2}\right) = -1$