

$$c = 7 \quad d = 5$$

1.

a) $m = 2 \cdot 5 + 3 = 13$

$$\left| (5 + 5j) e^{\frac{j\pi}{4}} \right| \cdot 2^6 = j^{13} \cdot |2^7| \quad z = 0 \quad \checkmark$$

d.h. $z \neq 0 \quad r > 0$ $j^{13} = j^{12} \cdot j = 1 \cdot j = j$

$$\left| 5\sqrt{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \cdot e^{\frac{j\pi}{4}} \right| \cdot r^6 e^{6j\frac{\pi}{4}} = j |r^7 e^{7j\frac{\pi}{4}}|$$

$$\left| 5\sqrt{2} \cdot e^{(\frac{\pi}{4} + \frac{\pi}{4})j} \right| \cdot r^6 e^{6j\frac{\pi}{4}} = r^7 j$$

$$5\sqrt{2} \cdot r^6 e^{6j\frac{\pi}{4}} = r^7 e^{7j\frac{\pi}{4}}$$

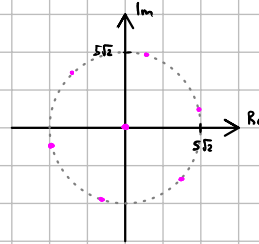
$$5\sqrt{2} e^{6j\frac{\pi}{4}} = r e^{7j\frac{\pi}{4}}$$

$$5\sqrt{2} = r \quad \wedge \quad 6\frac{\pi}{4} = \frac{\pi}{4} + 2k\pi$$

$$\varphi = \frac{\pi}{12} + \frac{k\pi}{3} = \frac{\pi + 4k\pi}{12} \quad k \in \mathbb{Z}$$

$$5\sqrt{2} e^{\frac{2\pi}{12}j} = 5\sqrt{2} e^{\frac{\pi}{6}j} = 5\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j \right) = -5 + 5j$$

$$5\sqrt{2} e^{-\frac{3\pi}{12}j} = 5\sqrt{2} e^{-\frac{\pi}{4}j} = 5\sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j \right) = 5 - 5j$$



b) $k = 7 + 5 + 1 = 13$

$$B = \left\{ z \in \mathbb{C} : \operatorname{Re}(2e^{\frac{7\pi}{4}j}) \leq |z + 2j^{13}| \leq \left| \frac{50j}{(1+2j)^4} \right| \vee \arg(3-3j) < \arg(z + 2j^{13} - 2) < \arg(2+2j) \right\}$$

$$\operatorname{Re}(2e^{\frac{7\pi}{4}j}) = \operatorname{Re}(2e^{\frac{\pi}{4}j}) = \operatorname{Re}(2(\frac{1}{2} + \frac{j}{2})) = \operatorname{Re}(1 + j) = 1$$

$$\left| \frac{50j}{(1+2j)^4} \right| = \frac{|50j|}{|1+2j|^4} = \frac{50}{(\sqrt{5})^4} = \frac{50}{25} = 2$$

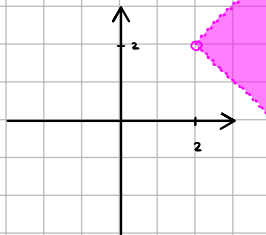
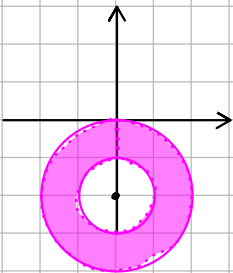
$$2j^{13} = 2j^{12}j = 2(j^4)^3j = 2j$$

$$\arg(3-3j) = \arg(3\sqrt{2}(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}})) = \arg(3\sqrt{2}e^{-\frac{\pi}{4}j}) = -\frac{\pi}{4}$$

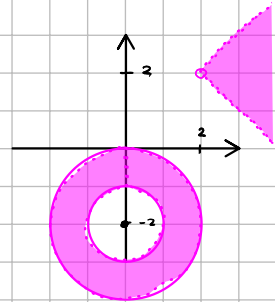
$$\arg(2+2j) = \arg(2\sqrt{2}(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}})) = \arg(2\sqrt{2}e^{\frac{\pi}{4}j}) = \frac{\pi}{4}$$

$$2j^{13} - 2 = -2 + 2j$$

$$B_1 = \{ z \in \mathbb{C} : |z - (-2j)| \leq 2 \} \quad B_2 = \{ z \in \mathbb{C} : -\frac{\pi}{4} < \arg(z - (2-2j)) < \frac{\pi}{4} \}$$



$$B = B_1 \cup B_2$$



2. $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\varphi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10x + 15y - 5z \\ 5y \\ 10x + 30y - 5z \end{pmatrix} = \begin{pmatrix} 10 & 15 & -5 \\ 0 & 5 & 0 \\ 10 & 30 & -5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad M_{E_3}^{E_3}(\varphi) = \begin{pmatrix} 10 & 15 & -5 \\ 0 & 5 & 0 \\ 10 & 30 & -5 \end{pmatrix}$$

$$M_{E_3}^A(\varphi) = M_{E_3}^{E_3}(\varphi) \cdot M_{E_3}^A(id) = \begin{pmatrix} 10 & 15 & -5 \\ 0 & 5 & 0 \\ 10 & 30 & -5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & -5 \\ 0 & 5 & -15 \end{pmatrix}$$

$$\begin{array}{c} M_{E_3}(A) \quad I_3 \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{u_3 - 2u_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array} \right] \xrightarrow{u_1 \leftrightarrow u_3} \left[\begin{array}{ccc|ccc} 0 & -1 & -3 & -2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array} \right] \xrightarrow{u_1 + 3u_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array} \right] \xrightarrow{u_1 + 3u_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & -3 & 1 \end{array} \right] \xrightarrow{u_1 \leftrightarrow u_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & 2 & 3 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array} \right] \end{array}$$

$$M_A^A(\varphi) = M_A^{E_3}(id) \cdot M_{E_3}^{E_3}(\varphi) \cdot M_{E_3}^A(id) = (M_{E_3}(A))^{-1} \cdot M_{E_3}^A(\varphi) = \begin{pmatrix} -1 & -3 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & -5 \\ 0 & 5 & -15 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 10 & 15 & -5 & 0 \\ 0 & 5 & 0 & 0 \\ 10 & 30 & -5 & 0 \end{array} \right] \xrightarrow{u_1 - 10u_2} \left[\begin{array}{ccc|c} 10 & 15 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{array} \right] \xrightarrow{u_1 - 15u_2} \left[\begin{array}{ccc|c} 10 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{u_1 : \cdot \frac{1}{10}} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} \frac{1}{2} \\ 0 \\ 1 \end{matrix} \quad t \in \mathbb{R}$$

$$\dim \ker \varphi = 1$$

$$\dim \operatorname{Im} \varphi = \operatorname{rank}(\varphi) = 2$$

$$\dim \mathbb{R}^3 = 3 = 2 + 1$$

$$M_{E_3}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & -3 \end{bmatrix} \quad \text{Ker } \varphi = \text{span} \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in \text{Ker } \varphi \quad \text{reszta nie daje się wyrazić jako kombinacja } (\frac{1}{2}, 0, 1)$$

$$\begin{bmatrix} 10 & 15 & -5 & 1 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & -1 \\ 10 & 30 & -5 & 2 & 1 & -3 \end{bmatrix} \xrightarrow{\substack{3 \cdot u_2 \\ u_3 - u_1}} \begin{bmatrix} 10 & 15 & -5 & 1 & 1 & 0 \\ 0 & 15 & 0 & 0 & 0 & -3 \\ 0 & 15 & 0 & 1 & 0 & -3 \end{bmatrix} \xrightarrow{\substack{\frac{1}{10} u_1 \\ \frac{1}{15} u_2}} \begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \times & \frac{1}{10} & 0 \\ 0 & 1 & 0 & \times & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & \times & 0 & -\frac{1}{5} \end{bmatrix} \xrightarrow{u_3 - u_2} \begin{bmatrix} 1 & 0 & -0.5 & \times & 0.1 & 0.3 \\ 0 & 1 & 0 & \times & 0 & -0.2 \\ 0 & 1 & 0 & \times & 0 & -0.2 \end{bmatrix}$$

$M_{E_3}(\varphi)$

v_1

v_2

v_3

v_1

$$u'_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix} \quad u'_3 = \begin{bmatrix} 0.3 \\ -0.2 + t \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{rank}(A|u_2) = \text{rank}(A|u_3) = \text{rank}(A) = 2$$

$$v_1 \notin \text{Im } \varphi$$

$$v_2, v_3 \in \text{Im } \varphi$$

$$3. \quad \begin{bmatrix} 3 & a^2 & 4-a^2 & a \\ 1 & 2 & a & a^2+1 \\ 2 & 1 & 2a & 2a^2+1 \end{bmatrix} \xrightarrow{u_3 - 2u_1} \begin{bmatrix} 3 & a^2 & 4-a^2 & a \\ 1 & 2 & a & a^2+1 \\ 0 & -3 & 0 & -3 \end{bmatrix} \xrightarrow{u_1 \leftrightarrow u_3} \begin{bmatrix} 1 & 2 & a & a^2+1 \\ 3 & a^2 & 4-a^2 & a \\ 0 & -3 & 0 & -3 \end{bmatrix} \xrightarrow{u_2 - 3u_1} \begin{bmatrix} 1 & 2 & a & a^2+1 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 0 & -3 \end{bmatrix} \xrightarrow{u_2 - u_3} \begin{bmatrix} 1 & 2 & a & a^2+1 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-3)} \begin{bmatrix} 1 & 2 & a & a^2+1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot (-3) = (3a - 4 + a^2) \cdot 3 = 3(a^2 + 4a - a - 4) = 3(a+4)(a-1)$$

$$1^\circ \quad a \in \mathbb{R} \setminus \{-4, 1\}$$

$$\text{Rank}(A|B) = \text{Rank}(A) = 3$$

doladanie 1 rozwiązanie

$$2^\circ \quad a = -4$$

$$\begin{bmatrix} 3 & 16 & -12 & -4 \\ 1 & 2 & -4 & 17 \\ 2 & 1 & -8 & 31 \end{bmatrix} \xrightarrow{\substack{u_1 - 3u_2 \\ u_3 - 2u_1}} \begin{bmatrix} 0 & 10 & 0 & -55 \\ 1 & 2 & -4 & 17 \\ 0 & -3 & 0 & -3 \end{bmatrix}$$

$$\text{układ sprzeczny} \quad \text{rank}(A) = 2 \quad \text{rank}(A|B) = 3$$

$$3^\circ \quad a = 1$$

$$\begin{bmatrix} 3 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{\substack{u_1 - 3u_2 \\ u_3 - 2u_1}} \begin{bmatrix} 0 & -5 & 0 & -5 \\ 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{5}u_1} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \end{bmatrix} \xrightarrow{\substack{u_1 \leftrightarrow u_2 \\ u_3 + 3u_2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}u_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{u_1 - u_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{u_2 - u_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1+t \\ 0 \end{bmatrix} \quad t \in \mathbb{R}$$

$$\text{rank}(A|B) = \text{rank}(A) = 2$$

niekoniwencyjnie wiele rozwiązań, 1 parametr

4.

$$A = \{x \in \mathbb{R} : \exists n \in \mathbb{N} \quad 2 + \frac{3}{n} \leq x < 5 + \frac{8}{n}\} = \bigcup_{i=1}^{\infty} P_i = (2, 13)$$

$$P_i = \{x \in \mathbb{R} : 2 + \frac{3}{i} \leq x < 5 + \frac{8}{i}\}$$

$$P_1 = [5, 13)$$

$$P_2 = [2.75, 9)$$

$$P_n \rightarrow (2, 5]$$

$$B = \{x \in \mathbb{R} : \forall n \in \mathbb{N} \quad 2 + \frac{3}{n} \leq x < 5 + \frac{8}{n}\} = \bigcap_{i=1}^{\infty} P_i = \{5\}$$

$$C = \{x \in \mathbb{R} : x < 8 \Rightarrow x \in B\} = \{x \in \mathbb{R} : x \geq 8 \vee x \in B\} = \{5\} \cup [8, \infty)$$

5.

$$W = \{(x, y, z, t, u) : |x - 2z + 3t| + |3z - 2t - u| = 0\}$$

$$|x - 2z + 3t| + |3z - 2t - u| = 0 \Leftrightarrow x - 2z + 3t = 0 \wedge 3z - 2t - u = 0 \Leftrightarrow \begin{cases} x = 2z - 3t \\ u = 3z - 2t \end{cases}$$

$$v = (x, y, z, t, u) \quad \alpha v = (\alpha x, \alpha y, \alpha z, \alpha t, \alpha u)$$

$$|\alpha x - 2\alpha z + 3\alpha t| + |\alpha(3z - 2t - u)| = |\alpha| \cdot |x - 2z + 3t| + |\alpha| \cdot |3z - 2t - u| = |\alpha| \cdot [|x - 2z + 3t| + |3z - 2t - u|] = |\alpha| \cdot 0 = 0 \rightarrow \alpha v \in W$$

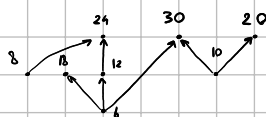
$$v_1, v_2 \rightarrow |x_1 + x_2 - 2z_1 - 2z_2 + 3t_1 + 3t_2| + |3z_1 + 3z_2 - 2t_1 - 2t_2 - u_1 - u_2| = |(x_1 - 2z_1 + 3t_1) + (x_2 - 2z_2 + 3t_2)| + |(3z_1 - 2t_1 - u_1) + (3z_2 - 2t_2 - u_2)| = |0 + 0| + |0 + 0| = 0$$

$$\dim W = \text{rank} \begin{bmatrix} 0 & 2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix} = 3$$

$$\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

6.

$$(\{6, 8, 10, 12, 18, 20, 24, 30\}, |)\}$$



elementy maksymalne 18, 20, 24, 30

elementy minimalne 6, 8, 10

$$\sup(A) = 120 = \text{lcm}(20, 24, 30)$$

$$\inf(A) = 2 = \text{gcd}(6, 8, 10)$$

ciągłańców maksymalnej długości

8, 12, 18, 20, 30

7.

$$A \in M_{3 \times 3}(\mathbb{C}) \quad \det A = 3 \cdot \cos\left(\frac{\pi}{2}\right) = 3 \cdot \cos\left(\frac{\pi}{2}\right) = 3$$

$$\det(A+A) = \det(2A) = \det(2 \cdot I_3 \cdot A) = \det(2I_3) \det A = 2^3 \cdot 3 = 24$$

$$\det(A^3) = (\det A)^3 = 27$$

$$\det(-jA) = \det(-jI_3) \cdot \det A = (-j)^3 \cdot 3 = -1 \cdot j^3 \cdot 3 = 3j$$

$$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{3}$$

$$\text{rank}(4 \cdot A^6) = 3 \quad \text{bo} \quad \det(4A^6) = 4^3 (\det A)^6 = 6^6 \neq 0$$