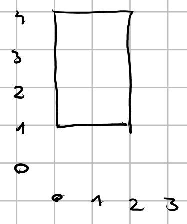
1.1

a) $\iint_D xy\ dx\,dy \qquad D=[0,2]\times[1\times4]$



$\int_0^2 \left(\int_1^4 xy\,dy\right)dx$

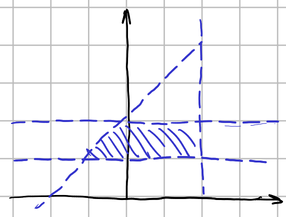
$\int_0^2 x\cdot\left(\int_1^4 y\,dy\right)dx$

$\int_0^2 \frac{15}{2}x\,dx = \frac{15}{2}\cdot\frac{1}{2}x^2\Big|_0^2 = 15$

$f_1(x)=1 \qquad f_2(x)=4$

$a=0 \qquad b=2$

$\frac{1}{2}y^2\Big|_1^4 = 8-\frac{1}{2}=\frac{15}{2}$



b)

D: $x=2 \qquad y=1 \qquad y=2 \qquad x=y-2$

I $\quad \iint_D 3\,dx\,dy = \int_1^2\left(\int_{y-2}^2 3\,dx\right)dy = 3\int_1^2[x]_{y-2}^2\,dy = 3\int_1^2 2-y+2\,dy = 3\int_1^2 4-y\,dy$

$= 3\left[4y-\frac{1}{2}y^2\right]_1^2 = 3\left[8-2-4+\frac{1}{2}\right]=7.5$

II $\quad \iint_D 3\,dx\,dy = \int_{-1}^0\int_1^{x+2} 3\,dy\,dx + \int_0^2\int_1^2 3\,dy\,dx$

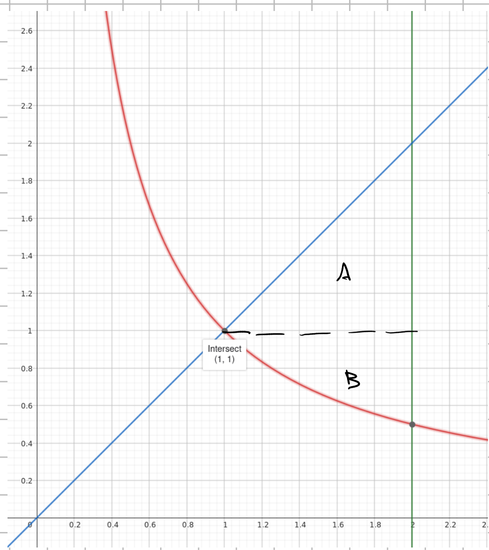
III $\quad \iint_D 3\,dx\,dy = 3\cdot\frac{2+3}{2}\cdot 1 = \frac{15}{2}$

1.2

$f(x,y)=\frac{3}{4}x\cdot 1_D(x,y)$

D: $x=2 \qquad y=x \qquad xy=1$

$1_D(x,y)=\begin{cases}1 & \text{dla } (x,y)\in D\\ 0 & \text{dla } (x,y)\notin D\end{cases}$



a) $\iint_D f(x,y)\,dx\,dy$

I $\quad \int_1^2\int_{\frac{1}{x}}^{x}\frac{3}{4}x\,dy\,dx$

$\int_1^2\left(\frac{3}{4}xy\right)\Big|_{\frac{1}{x}}^{x}dx$

$\int_1^2 \frac{3}{4}x^2-\frac{3}{4}\,dx$

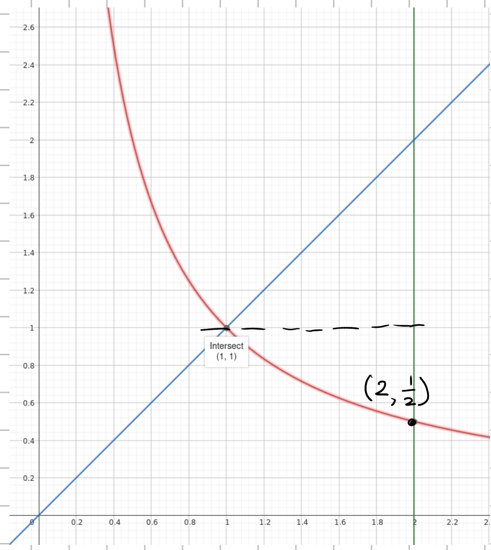
$\frac{1}{4}x^3-\frac{3}{4}x\Big|_1^2 = \frac{8}{4}-\frac{3}{2}-\frac{1}{4}+\frac{3}{4}=1$

II $\quad \iint_D f(x,y)\,dx\,dy = \int_1^2\int_y^2\frac{3}{4}x\,dx\,dy + \int_{\frac{1}{2}}^{1}\int_{\frac{1}{y}}^{2}\frac{3}{4}x\,dx\,dy$

b) $f_1 : \mathbb{R} \to \mathbb{R}$

$$f_1(x) = \int_{-\infty}^{+\infty} f(x,y)\, dy$$

$$\int_{-\infty}^{+\infty} \frac{3}{4} x \cdot \mathbb{1}_D(x,y)\, dy$$



$$f_1(x) = \begin{cases} 0 & \text{dla} \quad x \in (-\infty, 1] \cup (2, +\infty) \\ \int_{\frac{1}{x}}^{x} \frac{3}{4} x \, dy = \frac{3}{4} x \left(x - \frac{1}{x}\right) = \frac{3}{4}(x^2 - 1) & \text{dla} \quad x \in (1, 2] \end{cases}$$

$$f_2(y) = \int_{-\infty}^{+\infty} f(x,y)\, dx = \begin{cases} 0 & \text{dla} \quad y \in (-\infty, \frac{1}{2}] \cup [2, +\infty) \\ \int_{\frac{1}{y}}^{2} \frac{3}{4} x \, dx = \frac{3}{8} x^2 \Big|_{\frac{1}{y}}^{2} = \frac{3}{8}\left(4 - \frac{1}{y^2}\right) & y \in (\frac{1}{2}, 1) \\ \int_{y}^{2} \frac{3}{4} x \, dx = \frac{3}{8}(4 - y^2) & \text{dla} \quad y \in [1, 2) \end{cases}$$

$$f_2(y) = \frac{3}{8}\left(4 - \frac{1}{y^2}\right) \cdot \mathbb{1}_{(\frac{1}{2}, 1)}(y) + \frac{3}{8}(4 - y^2) \cdot \mathbb{1}_{[1,2)}(y)$$

2.1

$$f(x) = x^2$$

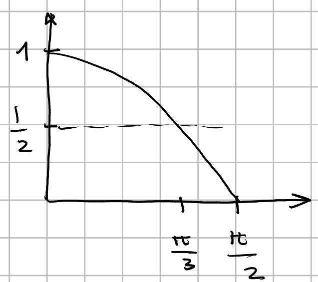
$$f^{-1}([0,4)) = (-2, 2)$$

$$f^{-1}((-2,-1)) = \emptyset$$

$$f^{-1}((0,1]) = [-1, 0) \cup (0, 1]$$

2.2 $\quad g(t) = f^{-1}((-\infty, t])$

a) $f : [0, \frac{\pi}{2}] \to \mathbb{R} \qquad f(x) = \cos(x)$



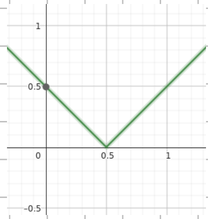
$$f^{-1}((-\infty, 0]) = \{\tfrac{\pi}{2}\}$$

$$f^{-1}((-\infty, 1]) = [0, \tfrac{\pi}{2}]$$

$$f^{-1}((-\infty, \tfrac{1}{2}]) = [\tfrac{\pi}{3}, \tfrac{\pi}{2}]$$

$$g(t) = \begin{cases} \emptyset & \text{dla} \quad t < 0 \\ [0, \frac{\pi}{2}] & \text{dla} \quad t \geq 1 \\ [\arccos(t), \frac{\pi}{2}] & \text{dla} \quad t \in [0, 1) \end{cases}$$

b) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = |x - \frac{1}{2}|$



$f^{-1}((-\infty, -1]) = \emptyset$

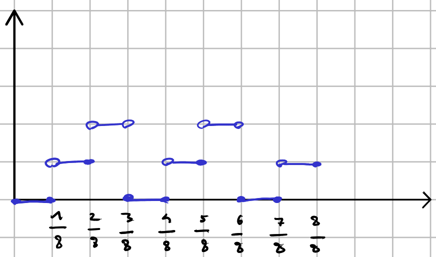
$f^{-1}((-\infty, 0]) = \{\frac{1}{2}\}$

$f^{-1}((-\infty, \frac{1}{2}]) = [0, 1]$

$f^{-1}((-\infty, 1]) = [-0.5, 1.5]$

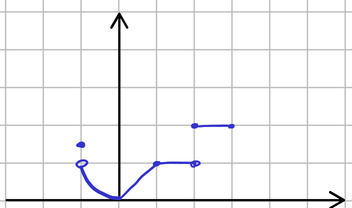
$$g(t) = \begin{cases} \emptyset & \text{dla } t < 0 \\ [\frac{1}{2} - t, \frac{1}{2} + t] & \text{dla } t \geq 0 \end{cases}$$

c) $f: [0,1] \to \mathbb{R}$



$$g(t) = \begin{cases} \emptyset & \text{dla } t \in (-\infty, 0) \\ [0, \frac{1}{8}] \cup [\frac{3}{8}, \frac{4}{8}] \cup [\frac{6}{8}, \frac{7}{8}] & \text{dla } t \in [0, 1) \\ [0, \frac{2}{8}] \cup [\frac{3}{8}, \frac{5}{8}] \cup [\frac{6}{8}, 1] & \text{dla } t \in [1, 2) \\ [0, 1] & \text{dla } t \in [2, +\infty) \end{cases}$$

d) $f: [-1, 3]$

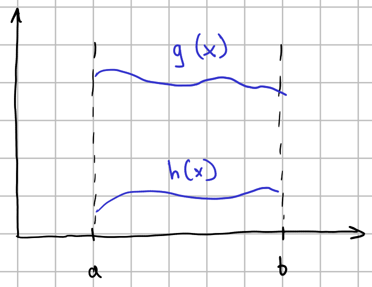


$$g(t) = \begin{cases} \emptyset & \text{dla } t \in (-\infty, 0) \\ [-\sqrt{t}, t] & \text{dla } t \in [0, 1) \\ (-1, 2) & \text{dla } t \in [1, \frac{3}{2}) \\ [-1, 2) & \text{dla } t \in [\frac{3}{2}, 2) \\ [-1, 3] & \text{dla } t \in [2, +\infty) \end{cases}$$

$f^{-1}((-\infty, 0])\ \{0\}$

$f^{-1}((-\infty, \frac{1}{4}]) = [-\frac{1}{2}, \frac{1}{4}]$

$f^{-1}((-\infty, 1]) = (-1, 2)$



$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{h(x)}^{g(x)} f(x,y)\,dy \right) dx$$