1.

$$\lim_{n\to+\infty}\left[1+\sin\left(\frac{2n+1}{n^2}\right)\right]^{n+1}=\lim_{n\to\infty}\left[1+\frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}}\cdot\frac{2n+1}{n^2}\right]^{n+1}=\lim_{n\to\infty}\left[1+\frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}}\cdot\frac{1}{\frac{n^2}{2n+1}}\right]^{\frac{n^2}{2n+1}\cdot\frac{2n+1}{n^2}\cdot(n+1)}$$

$$=\lim_{n\to\infty}\left(\left[1+\frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}}\cdot\frac{1}{\frac{n^2}{2n+1}}\right]^{\frac{n^2}{2n+1}}\right)^{\frac{2n^2+3n+1}{n^2}}=e^2$$

$$\left(1+\frac{1}{n}\right)^n\to e \qquad \lim_{n\to\infty}\frac{2n+1}{n^2}=0 \qquad \lim_{n\to\infty}\frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}}=1$$

$$\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2}=2 \qquad \lim_{n\to\infty}\left[1+\frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}}\cdot\frac{1}{\frac{n^2}{2n+1}}\right]^{\frac{n^2}{2n+1}}=e$$

2.

$$\forall_{x>3}\quad \sqrt[4]{x}-\sqrt[4]{3}<\sqrt[4]{x-3}$$

$$\iff \forall_{x>3}\quad \sqrt[4]{x}-\sqrt[4]{3}-\sqrt[4]{x-3}<0$$

$$\iff \forall_{x>3}\quad f(x)<0$$

$$f(x)=\sqrt[4]{x}-\sqrt[4]{3}-\sqrt[4]{x-3}$$

$$f'(x)=\frac{1}{4}x^{-\frac{3}{4}}-\frac{1}{4}(x-3)^{-\frac{3}{4}}\cdot 1-0$$

$$=\frac{1}{4}\left[x^{-\frac{3}{4}}-(x-3)^{-\frac{3}{4}}\right]$$

$$f'(x)\geq 0 \iff x^{-\frac{3}{4}}-(x-3)^{-\frac{3}{4}}\geq 0$$

$$\frac{1}{x^{\frac{3}{4}}}-\frac{1}{(x-3)^{\frac{3}{4}}}\geqslant 0 \qquad \frac{(x-3)^{\frac{3}{4}}-x^{\frac{3}{4}}}{(x(x-3))^{\frac{3}{4}}}\geqslant 0$$

$$x>3 \implies x(x-3)>0 \qquad \left(\sqrt[4]{x-3}-\sqrt[4]{x}\right)\left((x-3)^{\frac{2}{4}}+((x-3)x)^{\frac{1}{4}}+x^{\frac{2}{4}}\right.\geq 0$$

$$\iff \sqrt[4]{x-3}-\sqrt[4]{x}\geq 0$$

$$\sqrt[4]{x-3}\geqslant\sqrt[4]{x}$$

sprzeczność dla $x\in(3,+\infty)$ bo $g(t)=\sqrt[4]{t}$ jest rosnąca w $\mathbb{R}_+$

więc f jest malejąca w $(3,+\infty)$

$$\left.\begin{array}{l}\lim\limits_{x\to3^+}f(x)=\lim\limits_{x\to3^+}\left[\sqrt[4]{x}-\sqrt[4]{3}-\sqrt[4]{x-3}\right]=0\\ \text{i } f \searrow \text{ w } (3,+\infty)\end{array}\right\}\implies \forall_{x>3}\ f(x)<0$$

3.

$$\int\frac{e^{2x}}{1+e^x}\,dx=\left|\begin{matrix}t=e^x\\ dt=e^x dx\end{matrix}\right|=\int\frac{t}{1+t}\,dt=\int\left[1-\frac{1}{1+t}\right]dt=\int 1\,dt-\int\frac{1}{1+t}\,dt=t-\ln|1+t|+C$$

$$=e^x-\ln|1+e^x|+C$$

4. $$\int \frac{\ln(x^4+x^2)}{x^2}dx = \left| \begin{matrix} f=\ln(x^4+x^2) & g'=\frac{1}{x^2} \\ f'=\frac{4x^2+2}{x^3+x} & g=-\frac{1}{x} \end{matrix} \right| = -\frac{\ln(x^4+x^2)}{x} + \int \frac{4x^2+2}{x^4+x^2}dx$$

$$\frac{d}{dx}\ln(x^4+x^2) = \frac{1}{x^4+x^2}\cdot(4x^3+2x) = \frac{4x^3+2x}{x^4+x^2} = \frac{4x^2+2}{x^3+x}$$

$$\int x^{-2}dx = \frac{1}{-2+1}x^{-2+1} = -x^{-1} = \frac{-1}{x}$$

$$\frac{4x^2+2}{x^4+x^2} = \frac{4x^2+2}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1)+B(x^2+1)+Cx^3+Dx^2}{x^4+x^2}$$

$$4x^2+2 = Ax^3+Ax+Bx^2+B+Cx^3+Dx^2$$
$$4x^2+2 = (A+C)x^3+(B+D)x^2+Ax+B$$

$$\begin{cases} 0=A+C \\ 4=B+D \\ 0=A \\ 2=B \end{cases} \qquad \begin{matrix} A=0 \\ B=2 \\ C=0 \\ D=2 \end{matrix}$$

$$\frac{4x^2+2}{x^4+x^2} = \frac{2}{x^2}+\frac{2}{x^2+1}$$

$$\int \frac{\ln(x^4+x^2)}{x^2}dx = -\frac{\ln(x^4+x^2)}{x} + 2\int\frac{dx}{x^2} + 2\int\frac{dx}{x^2+1} = -\frac{\ln(x^4+x^2)}{x} - \frac{2}{x} + 2\arctan(x) + C$$

$$\int_1^\infty \frac{\ln(x^4+x^2)}{x^2}dx = \lim_{T\to+\infty} -\frac{\ln(T^4+T^2)}{T} - \frac{2}{T} + 2\arctan(T) + \ln(2) - 2 - 2\arctan(1)$$

$$\lim_{T\to+\infty}\frac{\ln(T^4+T^2)}{T} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{T\to+\infty}\frac{\frac{4x^2+2}{x^3+x}}{1} = 0$$

$$\int_1^\infty \frac{\ln(x^4+x^2)}{x^2}dx = 0-0+2\cdot\frac{\pi}{2}+\ln(2)-2-2\cdot\frac{\pi}{4} = \frac{\pi}{2}+\ln(2)-2$$

5. $$f(x,y) = \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{\Delta x^2\cdot 0}{\Delta x^4}-0}{\Delta x} = 0$$

$$\frac{\partial}{\partial x}\frac{x^2y^3}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\frac{x^2y^3}{x^4+2y^2x^2+y^4} = \frac{2y^3x(x^2+y^2)^2 - x^2y^3(4x^3+4y^2x)}{(x^2+y^2)^4} = \frac{2y^3x^5+\cancel{4y^5x^3}+2y^7x-4x^5y^3-\cancel{4x^3y^5}}{(x^2+y^2)^4} = \frac{2y^7x-2y^3x^5}{(x^2+y^2)^4}$$

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{2y^7x-2y^3x^5}{(x^2+y^2)^4} & \text{dla } (x,y)\neq(0,0) \\ 0 & \text{dla } (x,y)=(0,0) \end{cases} \qquad 2xy\,\frac{y^6-y^2x^4}{(x^2+y^2)^4}$$

$$(x_n,y_n) = \left(\frac{1}{n},\frac{1}{n}\right)\to(0,0)$$

$$\frac{\partial f}{\partial x}(x_n,y_n) = \frac{2\cdot\frac{1}{n^7}\cdot\frac{1}{n} - 2\cdot\frac{1}{n^3}\cdot\frac{1}{n^5}}{\left(\frac{1}{n^2}+\frac{1}{n^2}\right)^4} = \frac{\frac{2}{n^8}-\frac{2}{n^8}}{\frac{16}{n^8}} = 0$$

$$(\tilde{x}_n,\tilde{y}_n) = \left(\frac{1}{n},\frac{2}{n}\right)$$

$$\frac{\partial f}{\partial x}(\tilde{x}_n,\tilde{y}_n) = \frac{2\cdot\frac{128}{n^7}\cdot\frac{1}{n} - 2\cdot\frac{8}{n^3}\cdot\frac{1}{n^5}}{\left(\frac{1}{n^2}+\frac{4}{n^2}\right)^4} = \frac{\frac{256-16}{n^8}}{\frac{5^4}{n^8}} = \frac{240}{625}\neq 0$$

pochodna w $(0,0)$ istnieje i jest nieciągła

6.

$$y'' - 4y' + 3y = 6e^{3x} + 9x$$

$$r^2 - 4r + 3 = r^2 - 3r - r + 3 = (r-1)(r-3)$$

$$r_1 = 1 \quad r_2 = 3$$

$$\rightarrow y_0(x) = C_1 e^x + C_2 e^{3x}$$

$$\begin{bmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 6e^{3x}+9x \end{bmatrix} \qquad C_1' = \frac{\begin{vmatrix} 0 & e^{3x} \\ 6e^{3x}+9x & 3e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix}} = \frac{-e^{3x}(6e^{3x}+9x)}{3e^{4x} - e^{4x}} = \frac{-6e^{6x} - 9xe^{3x}}{2e^{4x}} = -3e^{2x} - \frac{9}{2}xe^{-x}$$

$$C_1(x) = \int -3e^{2x}\,dx - \frac{9}{2}xe^{-x}\,dx = -3\int e^{2x}\,dx - \frac{9}{2}\int xe^{-x}\,dx = -3\cdot\frac{1}{2}e^{2x} - \frac{9}{2}\cdot(-e^{-x})(x+1) = -\frac{3}{2}e^{2x} + \frac{9}{2}e^{-x}(x+1) + C_1$$

$$C_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 6e^{3x}+9x \end{vmatrix}}{2e^{4x}} = \frac{6e^{4x} + 9xe^x}{2e^{4x}} = 3 + \frac{9}{2}xe^{-3x}$$

$$\int xe^{-3x}\,dx = \begin{vmatrix} f = x & g' = e^{-3x} \\ f' = 1 & g = -\frac{1}{3}e^{-3x} \end{vmatrix} = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}$$

$$C_2(x) = 3\int 1\,dx + \frac{9}{2}\int xe^{-3x}\,dx = 3x - \frac{3}{2}xe^{-3x} - \frac{1}{2}e^{-3x} + C_2$$

$$y(x) = \left[-\frac{3}{2}e^{2x} + \frac{9}{2}e^{-x}(x+1) + C_1\right]e^x + \left[3x - \frac{3}{2}xe^{-3x} - \frac{1}{2}e^{-3x} + C_2\right]e^{3x}$$

$$= C_1e^x + C_2e^{3x} - \frac{3}{2}e^{3x} + \frac{9}{2}x + \frac{9}{2} + 3xe^{3x} - \frac{3}{2}x - \frac{1}{2}$$

$$= C_1e^x + C_2e^{3x} + 3xe^{3x} - \frac{3}{2}e^{3x} + 3x + 4$$