1.

$$F_X(x) = \begin{cases} 0 & x < 1 \\ a\sqrt{x} + b & 1 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

rozkład ciągły

$$\frac{d}{dx}\left[a\sqrt{x} + b\right] = a\,\frac{1}{2}x^{-\frac{1}{2}} = \frac{a}{2\sqrt{x}}$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx$$

$$\frac{d}{dx}F_X(x) = f_X(x) = \begin{cases} 0 & x < 1 \\ \frac{a}{2\sqrt{x}} & 1 \leq x < 4 \\ 0 & x \geq 4 \end{cases}$$

$$\int_{-\infty}^{+\infty} f_X(x) = 1 = \int_{1}^{4} \frac{a}{2\sqrt{x}}\,dx = a\sqrt{x}\,\Big|_1^4 = 2a - a = a$$

$$1 = \lim_{x \to 4^-} \sqrt{x} + b \qquad 0 = \lim_{x \to 1^+} \sqrt{x} + b$$

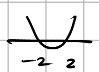
$$1 = 2 + b \qquad 0 = 1 + b$$

$$b = -1 \qquad b = -1$$

$$P(X^2 \leq 4) = P(-2 \leq X \leq 2) = \int_{-2}^{2} f_X(x)\,dx = F_X(2) - F_X(-2) = \sqrt{2} - 1 - 0 = \sqrt{2} - 1$$

$$x^2 - 4 \leq 0$$

$$(X-2)(x+2) \leq 0$$



2.

$$f(x) = (ax-1)\cdot \mathbb{1}_{(0,1)}(x) = \begin{cases} ax-1 & x\in(0,1) \\ 0 & \text{w p.p.} \end{cases}$$

I $\int_{-\infty}^{+\infty} f(x)dx = 1$

$$1 = \int_0^1 (ax-1)\,dx = \left[\tfrac{1}{2}ax^2 - x\right]\Big|_0^1 = \tfrac{1}{2}a - 1 \Rightarrow a = 4$$

II $f(x) \geqslant 0$

$$f(0) = 4\cdot 0 - 1 = -1 < 0$$

nie istnieje takie $a\in\mathbb{R}$

3.

rozkład ciągły

$$f_X(x) = \begin{cases} a & x \in [-1,0) \\ b(x^2+x) & x \in [0,1] \\ 0 & \text{w p.p.} \end{cases}$$

$a, b \in \mathbb{R}$ $\qquad P(X<0) = \frac{1}{6}$

a)

$$\frac{1}{6} = P(X<0) = \int_{-\infty}^{0} f_X(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} a\,dx = 0 + 1\cdot a \Rightarrow a = \frac{1}{6}$$

$$1 = \int_{-\infty}^{+\infty} f_X(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} \frac{1}{6}\,dx + \int_{0}^{1} b(x^2+x)\,dx + \int_{1}^{+\infty} 0\,dx = 0 + \frac{1}{6} + b\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 + 0$$

$$\frac{5}{6} = \frac{5}{6}b \Rightarrow b = 1$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx = \begin{cases} \int_{-\infty}^{t} 0\,dx = 0 & t<-1 \\ 0 + \int_{-1}^{t} \frac{1}{6}\,dx = \frac{1}{6}(t+1) & -1 \le t < 0 \\ 0 + \int_{-1}^{0} \frac{1}{6}\,dx + \int_{0}^{t} (x^2+x)\,dx = \frac{1}{6} + \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^t = \frac{1}{6} + \frac{1}{3}t^3 + \frac{1}{2}t^2 & 0 \le t \le 1 \\ 0 + \frac{1}{6} + \int_{0}^{1} (x^2+x)\,dx + \int_{1}^{t} 0\,dx = \frac{1}{6} + \frac{5}{6} + 0 = 1 & t > 1 \end{cases}$$

b)

$$P(|X| > \tfrac{1}{2}) = P(X > \tfrac{1}{2} \vee X < -\tfrac{1}{2}) = P(X > \tfrac{1}{2}) + P(X < -\tfrac{1}{2})$$

$$= \lim_{x\to+\infty} F_X(x) - F_X(\tfrac{1}{2}) + F_X(-\tfrac{1}{2}) - \lim_{x\to-\infty} F_X(x)$$

$$= 1 - \left(\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{4}\right) + \frac{1}{6}\left(-\frac{1}{2}+1\right) - 0 = \frac{3}{4}$$

4.

$P(O) = p \in (0, 1)$

Rzucamy monetą do momentu wyrzucenia orła lub dwóch reszek z rzędu

X – liczba rzutów

$\Omega = \{ O, RO, RR \}$

$EX = 1 \cdot P(O) + 2 \cdot P(RO) + 2 \cdot P(RR) = p + 2(1-p)p + 2(1-p)^2 = 2 - p$

5.

$4 \times B$ losujemy jedną, odkładamy z powrotem 2 takie same
$2 \times C$

X – liczba białych kul w 2 losowaniach

$S_X = \{0, 1, 2\}$

$P(X=0) = P(C_1) \cdot P(C_2 | C_1) = \frac{2}{6} \cdot \frac{3}{7} = \frac{3}{21}$

$P(X=1) = P(C_1) \cdot P(B_2 | C_1) + P(B_1) \cdot P(C_2 | B_1) = \frac{2}{6} \cdot \frac{4}{7} + \frac{4}{6} \cdot \frac{2}{7} = \frac{8}{21}$

$P(X=2) = P(B_1) \cdot P(B_2 | B_1) = \frac{4}{6} \cdot \frac{5}{7} = \frac{10}{21}$

x_i	0	1	2
$p_X(x_i)$	$\frac{3}{21}$	$\frac{8}{21}$	$\frac{10}{21}$

$\sum_{x_i \in S_X} p_X(x_i) = 1$

$E(X^2 - X + 1) = (0-0+1)\frac{3}{21} + (1-1+1)\frac{8}{21} + (4-2+1)\frac{10}{21} = \frac{3}{21} + \frac{8}{21} + \frac{30}{21} = \frac{41}{21}$

6.

$\Omega = \{(x,y): x, y \in [0,1]\}$

P – prawdopodobieństwo geometryczne

$T(x,y) = |x-y| \in [0,1]$

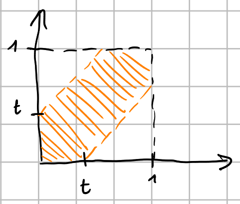
$$F_T(t) = \begin{cases} 0 & t < 0 \\ 1-(1-t)^2 & t \in [0,1] \\ 1 & t > 1 \end{cases}$$

$F_T(t) = P(T \le t) = P(\{(x,y) \in \Omega : |x-y| < t\})$

$|y-x| \le t$

$-t \le y-x \le t$

$x-t \le y \le x+t$



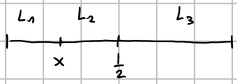
$F_T(t) = 1 - 2 \cdot \frac{1}{2}(1-t)^2$

$f_T(t) = \frac{d}{dt} F_T(t) = 2(1-t) \cdot \mathbb{1}_{[0,1]}(t)$

$ET = \int_{-\infty}^{+\infty} t\, f_T(t)\,dt = \int_0^1 2t - 2t^2\, dt = \frac{1}{3}$

7.

$$f(x) = \mathbb{1}_{[0,1]}(x)$$



$$L_1 = \begin{cases} x & x \leq \frac{1}{2} \\ \frac{1}{2} & x > \frac{1}{2} \end{cases}$$

$$L_2 = \left|x - \frac{1}{2}\right|$$

$$E\, g(x) = \int_{-\infty}^{+\infty} g(x) f(x)\, dx \qquad L_3 = 1 - L_1 - L_2$$

$$EL_1 = \int_0^{\frac{1}{2}} x\, dx + \int_{\frac{1}{2}}^{1} \frac{1}{2}\, dx = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

$$EL_2 = \int_{-\infty}^{+\infty} \left|x - \frac{1}{2}\right| f(x) dx = \int_0^{\frac{1}{2}} \left(\frac{1}{2} - x\right) dx + \int_{\frac{1}{2}}^{1} \left(x - \frac{1}{2}\right) dx = \frac{1}{4} - \frac{1}{8} + \frac{3}{8} - \frac{1}{4} = \frac{2}{8}$$

$$EL_3 = 1 - EL_1 - EL_2 = \frac{3}{8}$$