1.

a)

$$\begin{bmatrix} -1 & 3j & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3j \\ 4 \end{bmatrix} = \begin{bmatrix} (-1)(-1) + (3j)(3j) + (4)(4) \end{bmatrix} = \begin{bmatrix} 1 - 9 + 16 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

b)

$$\begin{bmatrix} -1 \\ 3j \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3j & 4 \end{bmatrix} = \begin{bmatrix} (-1)(-1) & (-1)(3j) & (-1)(4) \\ (3j)(-1) & (3j)(3j) & (3j)(4) \\ (4)(-1) & (4)(3j) & (4)(4) \end{bmatrix} = \begin{bmatrix} 1 & -3j & -4 \\ -3j & -9 & 12j \\ -4 & 12j & 16 \end{bmatrix}$$

c)

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B^2 = \begin{bmatrix} (1)(1)+(1)(0) & (1)(1)+(1)(1) \\ (0)(1)+(1)(0) & (0)(1)+(1)(1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$a_1 = 1 \quad b_1 = 1 \quad c_1 = 0 \quad d_1 = 1$$

$$B^3 = B \cdot B^2 = \begin{bmatrix} (1)(1)+(1)(0) & (1)(2)+(1)(1) \\ (0)(1)+(1)(0) & (0)(2)+(1)(1) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$a_n = a_{n-1} + c_{n-1} = a_{n-1} = a_1 = 1$$
$$b_n = b_{n-1} + d_{n-1} = b_{n-1} + 1 \implies b_n = n$$
$$c_n = c_{n-1} = c_1 = 0$$
$$d_n = d_{n-1} = d_1 = 1$$

$$B^n = B \cdot B^{n-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

d)

$$C \cdot D = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\cos(\alpha + b) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

e)

$$C^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}$$

dowód

$$\left(C^{n-1} = \begin{bmatrix} \cos((n-1)\alpha) & -\sin((n-1)\alpha) \\ \sin((n-1)\alpha) & \cos((n-1)\alpha) \end{bmatrix} \implies C^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix} \right) \wedge C^2 = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix}$$

$$C^n = C \cdot C^{n-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \cos((n-1)\alpha) & -\sin((n-1)\alpha) \\ \sin((n-1)\alpha) & \cos((n-1)\alpha) \end{bmatrix} = \begin{bmatrix} \cos(\alpha + (n-1)\alpha) & -\sin(\alpha + (n-1)\alpha) \\ \sin(\alpha + (n-1)\alpha) & \cos(\alpha + (n-1)\alpha) \end{bmatrix} = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}$$

$$C^2 = C \cdot C = \begin{bmatrix} \cos(\alpha+\alpha) & -\sin(\alpha+\alpha) \\ \sin(\alpha+\alpha) & \cos(\alpha+\alpha) \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix}$$

więc $\forall_{n \geq 2} \; C^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}$

2.

a) $A=\begin{bmatrix} j & 3 \\ 1+j & 4-3j \end{bmatrix}$ $\quad \det A = (j)(4-3j) - (3)(1+j) = 3+4j-3-3j = j \neq 0$

$\frac{1}{\det A} = \frac{1}{j} = \frac{-j}{j\cdot(-j)} = \frac{-j}{(-1)(j^2)} = -j$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -j\begin{bmatrix} 4-3j & -3 \\ -1-j & j \end{bmatrix} = \begin{bmatrix} -3-4j & 3j \\ -1+j & 1 \end{bmatrix}$$

b) $B=\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

$\det B = (1)(-2)(1) + (2)(0)(2) + (-1)(3)(-3)$
$\quad - (-1)(-2)(2) - (-3)(0)(1) - (1)(3)(2)$
$\quad = -2+9-4-6 = -3$

$$B^D = \begin{bmatrix} \begin{vmatrix} -2 & 0 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -2 & -3 & -5 \\ 1 & 3 & 7 \\ -2 & -3 & -8 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 & -2 \\ -3 & 3 & -3 \\ -5 & 7 & -8 \end{bmatrix}$$

$$B^{-1} = -\frac{1}{3}B^D = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 1 & -1 & 1 \\ 5/3 & -7/3 & 8/3 \end{bmatrix}$$

c) $C=\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 0 \\ 4 & 0 & -1 \end{bmatrix}$ $\quad \det C = 2-8+6=0$

nie istnieje macierz odwrotna

d) $D=\begin{bmatrix} 0 & 0 & 71 & 70 \\ 0 & 0 & 70 & 69 \\ 69 & 70 & 0 & 0 \\ 70 & 71 & 0 & 0 \end{bmatrix}$

$$\det D = (-1)^2 \begin{vmatrix} 71 & 70 & 0 & 0 \\ 70 & 69 & 0 & 0 \\ 0 & 0 & 69 & 70 \\ 0 & 0 & 70 & 71 \end{vmatrix} = \begin{vmatrix} 71 & 70 \\ 70 & 69 \end{vmatrix} \cdot \begin{vmatrix} 69 & 70 \\ 70 & 71 \end{vmatrix} = \left[(70+1)(70-1) - 70^2\right]\cdot\left[(70-1)(70+1) - 70^2\right] = (-1)\cdot(-1) = 1$$

$k_1 \leftrightarrow k_3 \quad k_2 \leftrightarrow k_4$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 71 & 70 & 1 & 0 & 0 & 0 \\ 0 & 0 & 70 & 69 & 0 & 1 & 0 & 0 \\ 69 & 70 & 0 & 0 & 0 & 0 & 1 & 0 \\ 70 & 71 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \underset{w_4 - w_3}{\left[\begin{array}{cccc|cccc} 0 & 0 & 71 & 70 & 1 & 0 & 0 & 0 \\ 0 & 0 & 70 & 69 & 0 & 1 & 0 & 0 \\ 69 & 70 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]} \longrightarrow \underset{w_3 - 69\, w_4}{\left[\begin{array}{cccc|cccc} 0 & 0 & 71 & 70 & 1 & 0 & 0 & 0 \\ 0 & 0 & 70 & 69 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 70 & -69 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]} \longrightarrow \underset{w_1 - w_2}{\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 70 & 69 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 70 & -69 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]}$$

$$\underset{w_2 - 69\, w_1}{\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -69 & 70 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 70 & -69 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]} \longrightarrow \underset{w_4 - w_3 \qquad w_1 - w_2}{\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 70 & -71 & 0 & 0 \\ 0 & 0 & 1 & 0 & -69 & 70 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 70 & -69 \\ 1 & 0 & 0 & 0 & 0 & 0 & -71 & 70 \end{array}\right]} \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -71 & 70 \\ 0 & 1 & 0 & 0 & 0 & 0 & 70 & -69 \\ 0 & 0 & 1 & 0 & -69 & 70 & 0 & 0 \\ 0 & 0 & 0 & 1 & 70 & -71 & 0 & 0 \end{array}\right]$$

$$D^{-1} = \begin{bmatrix} 0 & 0 & -71 & 70 \\ 0 & 0 & 70 & -69 \\ -69 & 70 & 0 & 0 \\ 70 & -71 & 0 & 0 \end{bmatrix}$$

3.

$A \in \mathbb{M}_{3\times 3}(\mathbb{C}) \quad B \in \mathbb{M}_{4\times 4}(\mathbb{C})$

$\det A = 2j \quad \det B = -3$

- $\det(3jA) = \det(3jI_3)\cdot\det A = (3j)^3\cdot 2j = -27j\cdot 2j = 54$

- $\det(jA^2) = \det(jI_3)\cdot(\det A)^2 = j^3\cdot(2j)^2 = 4j$

- $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{2j} = \frac{j}{-2}$

- $\det((-B)^T) = \det(-B) = \det(-1\cdot I_4)\cdot\det B = (-1)^4\cdot(-3) = -3$

- $\det(\pi B^3) = \det(\pi I_4)\cdot(\det B)^3 = \pi^4\cdot(-3)^3 = -27\pi^4$

- nieprawda że $\det(A\cdot B) = \det A\cdot\det B$, $A\cdot B$ nie istnieje bo macierze mają niezgodne wymiary

4.

a)

$$\begin{vmatrix} 54 & 55 & 56 & 57 \\ 55 & 56 & 57 & 58 \\ 56 & 57 & 58 & 59 \\ 57 & 58 & 59 & 60 \end{vmatrix} = \begin{vmatrix} 54 & 55 & 56 & 57 \\ 55 & 56 & 57 & 58 \\ 56 & 57 & 58 & 59 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 54 & 55 & 56 & 57 \\ 55 & 56 & 57 & 58 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$w_4 - w_3$

b)

$$\begin{vmatrix} 1 & 1 & 1 & x+1 \\ 2 & x+2 & 2 & 2 \\ 3 & 3 & x+3 & 3 \\ x+4 & 4 & 4 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -x & x+1 \\ -x & x+2 & 0 & 2 \\ 0 & 3 & x & 3 \\ x & 4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -x & -x & x+1 \\ -x & x & 0 & 2 \\ 0 & 0 & x & 3 \\ x & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -x & -x & x+1 \\ 0 & x & 0 & 6 \\ 0 & 0 & x & 3 \\ x & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & x+10 \\ 0 & x & 0 & 6 \\ 0 & 0 & x & 3 \\ x & 0 & 0 & 4 \end{vmatrix} = -\begin{vmatrix} x+10 & 0 & 0 & 0 \\ 6 & x & 0 & 0 \\ 3 & 0 & x & 0 \\ 4 & 0 & 0 & x \end{vmatrix} = -x^3(x+10)$$

$k_1 -= k_2 \quad k_3 -= k_4 \qquad k_2 -= k_4 \qquad w_2 += w_4 \qquad w_1 += w_2 + w_3 \qquad k_1 \longleftrightarrow k_4$

c)

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -5 & -8 & -9 \\ 0 & -4 & -8 & -10 & -14 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -5 & -8 & -9 \\ 0 & 0 & -4/3 & 2/3 & -2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -5 & -8 & -9 \\ 0 & 0 & -4/3 & 2/3 & -2 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 3/2 \end{vmatrix} = (1)(\cancel{-3})\left(-\frac{4}{\cancel{3}}\right)\overset{2}{\cancel{(4)}}\left(\frac{3}{\cancel{2}}\right) = 24$$

$w_2 -= 2w_1 \quad w_3 -= 3w_1 \qquad w_3 += -\frac{4}{3}w_2 \qquad w_5 -= \frac{1}{2}w_4$

5.

a)
$$A=\begin{bmatrix}2&3&4&5&6&7\\7&6&5&4&3&2\\12&13&14&15&16&17\\17&16&15&14&13&12\end{bmatrix}$$

$$\operatorname{rank}(A)=\operatorname{rank}\left(\begin{bmatrix}2&3&4&5&6&7\\7&6&5&4&3&2\\19&19&19&19&19&19\\\cancel{19}&\cancel{19}&\cancel{19}&\cancel{19}&\cancel{19}&\cancel{19}\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}2&3&4&5&6&7\\9&9&9&9&9&9\\\cancel{19}&\cancel{19}&\cancel{19}&\cancel{19}&\cancel{19}&\cancel{19}\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}2&3&4&5&6&7\\9&9&9&9&9&9\end{bmatrix}\right)$$

$v_3 += v_2 \quad v_4 += v_1 \qquad\qquad v_2 += v_1$

$\operatorname{rank} A \le 2 \qquad \begin{vmatrix}2&3\\9&9\end{vmatrix}=18-27=-9\neq 0 \qquad \operatorname{rank} A = 2$

b)
$$B=\begin{bmatrix}1&2&3&1&5\\0&4&7&1&2\\1&2&3&4&6\\-1&-2&-3&5&-3\end{bmatrix}$$

$\operatorname{rank} B \le 4$

$$\operatorname{rank} B=\operatorname{rank}\left(\begin{bmatrix}1&2&3&1&5\\0&4&7&1&2\\0&0&0&3&1\\\cancel{0}&\cancel{0}&\cancel{0}&\cancel{6}&\cancel{2}\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}1&2&3&1&5\\0&4&7&1&2\\0&0&0&3&1\end{bmatrix}\right)$$

$v_3 -= v_1 \quad v_4 += v_1$

$\operatorname{rank} B \le 3$

ze schodków wychodzi minor maksymalny $\begin{vmatrix}1&2&1\\0&4&1\\0&0&3\end{vmatrix}$

$\begin{vmatrix}1&2&3\\0&4&7\\0&0&0\end{vmatrix}=0 \qquad \begin{vmatrix}2&3&1\\4&7&1\\0&0&3\end{vmatrix}=42-36=6\neq 0 \qquad \operatorname{rank} B = 3$

c)
$$C=\begin{bmatrix}3&1&6&2&1\\2&1&4&2&2\\3&1&3&1&3\\2&1&2&1&4\end{bmatrix}$$

$\operatorname{rank} C \le 4$

$$\operatorname{rank} C=\operatorname{rank}\left(\begin{bmatrix}1&1&4&1&0\\0&1&2&1&1\\1&1&1&0&2\\0&1&0&0&3\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}1&0&3&1&0\\0&1&2&1&1\\1&0&0&0&2\\0&1&0&0&3\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}1&0&1&1&0\\0&1&0&1&0\\1&0&0&0&2\\0&1&0&0&2\end{bmatrix}\right)$$

$k_1 -= k_2 \quad k_3 -= k_2 \quad k_4 -= k_2 \qquad k_2 -= k_1 \quad k_3 -= k_1 \qquad k_3 -= 2k_4 \quad k_5 -= k_2$

$$\operatorname{rank}\left(\begin{bmatrix}1&0&1&1&0\\0&1&0&1&0\\0&0&-1&-1&2\\0&0&0&-1&2\end{bmatrix}\right)=4 \quad \text{(macierz schodkowa)}$$

$v_3 -= v_1 \quad v_4 -= v_2$

d)
$$D=\begin{bmatrix}1&1&1&p\\1&1&p&p\\1&p&p&p\end{bmatrix}$$

$\forall_{p\in\mathbb{R}} \operatorname{rank} D \le 3$

$$\operatorname{rank} D=\operatorname{rank}\left(\begin{bmatrix}1&1&1&p-1\\1&1&p&0\\1&p&p&0\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}1&1&0&p-1\\1&1&p-1&0\\1&p&0&0\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}1&0&0&p-1\\1&0&p-1&0\\1&p-1&0&0\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}p-1&0&0&1\\0&p-1&0&1\\0&0&p-1&1\end{bmatrix}\right)$$

$k_4 -= k_3 \qquad k_3 -= k_2 \qquad k_2 -= k_1 \qquad k_1 \leftrightarrow k_4 \quad k_2 \leftrightarrow k_3$

$\begin{vmatrix}p-1&0&0\\0&p-1&0\\0&0&p-1\end{vmatrix}=(p-1)^3 \qquad (p-1)^3\neq 0 \iff p\neq 1$

dla $p=1$ $\quad \operatorname{rank} D=\operatorname{rank}\left(\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\right)=\operatorname{rank}([1])=1$

$$\operatorname{rank} D=\begin{cases}3 & p\neq 1\\1 & p=1\end{cases}$$

e)
$$E=\begin{bmatrix}1&1&p\\3&p&3\\2p&2&2\end{bmatrix}$$

$\operatorname{rank} E \le 3$

$\det E = 2p+6p+6p-2p^3-6-6=-2p^3+14p-12$

$\det E = 0 \iff p^3-7p+6=0$

$\iff (p-1)(p-2)(p+3)=0$

$\iff p\in\{1,2,-3\} \implies \operatorname{rank} E\neq 3$

$$\operatorname{rank} E=\begin{cases}3 & p\in\{-3,1,2\}\\2 & p\notin\{-3,1,2\}\end{cases}$$

dla $p=-3$ $\quad \begin{vmatrix}1&-3\\-3&3\end{vmatrix}=3-9=-6\neq 0 \qquad \operatorname{rank} E=2$

dla $p=1$ $\quad \begin{vmatrix}1&1\\3&1\end{vmatrix}=1-3=-2\neq 0 \qquad \operatorname{rank} E=2$

dla $p=2$ $\quad \begin{vmatrix}1&1\\3&2\end{vmatrix}=2-3=-1\neq 0 \qquad \operatorname{rank} E=2$