

1.

a) $W = \{(x, y) \in \mathbb{R}^2 : xy \leq 0\} \quad \mathbb{R}^2 \text{ nad } \mathbb{R}$

$$(-1, 2) + (2, -1) = (1, 1)$$

$$-2 \leq 0 \quad -2 \leq 0 \quad \underline{1 \leq 0}$$

nie jest podprzestrzenią

b) $W = \{(x, y, z) \in \mathbb{R}^3 : (x+y)^2 = z^2\} \quad \mathbb{R}^3 \text{ nad } \mathbb{R}$

$$v_1 = (2, 2, 4) \quad v_2 = (-2, -2, 4)$$

$v_1 - v_2$ jest zerem

$$v_1 + v_2 = (0, 0, 8) \notin W$$

W nie jest podprzestrzenią \mathbb{R}^3

c) $W = \{(x, y, z) \in \mathbb{R}^3 : (x+y)^2 + z^2 = 0\} \quad \mathbb{R}^3 \text{ nad } \mathbb{R}$

$$(x+y)^2 + z^2 = 0 \iff (x+y)^2 = 0 \wedge z^2 = 0$$

$$\iff x+y=0 \wedge z=0$$

$$\iff y=-x \wedge z=0$$

$$W = \{(x, -x, 0) : x \in \mathbb{R}\}$$

$$v_1 + v_2 = (x_1 + x_2, -x_1 - x_2, 0) = (x_1 + x_2, -(x_1 + x_2), 0) \in W$$

$$\alpha v = (\alpha x, -\alpha x, 0) \in W$$

jest podprzestrzenią

d) $W = \{z \in \mathbb{C} : |z| = 3\} \quad \mathbb{C} \text{ nad } \mathbb{R}$

$$3j + 3j = 6j \notin W$$

nie jest przestrzenią liniową

e) $W = \{u \in \mathbb{R}[x] : u'(1) = 0\} \quad \mathbb{R}[x] \text{ nad } \mathbb{R}$

$$(u_1 + u_2)'(1) = u_1'(1) + u_2'(1) = 0 \implies u_1 + u_2 \in W$$

$$(\alpha u)'(1) = \alpha \cdot u'(1) = \alpha \cdot 0 = 0 \implies \alpha u \in W$$

jest podprzestrzenią

$$W = \{u \in \mathbb{R}[x] : a_0 + a_2(x-1)^2 + a_3(x-1)^3 + \dots\}$$

$$\frac{du}{dx} = 2a_2(x-1) + 3a_3(x-1)^2 + \dots$$

dla $x=1$ to stę zeruje

2.

a) $A = ((2, 1, 1), (3, 0, 1), (0, 3, 1)) \quad \mathbb{R}^3 \text{ nad } \mathbb{R}$

1 sposób $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 2 - 6 - 3 = 0$ A nie jest bazą \mathbb{R}^3
i A jest liniowo zależny

2 sposób

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

$$\begin{cases} 2\alpha + 3\beta = 0 \\ \alpha + 3\gamma = 0 \\ \alpha + \beta + \gamma = 0 \end{cases} \Rightarrow \begin{cases} \alpha = -3\gamma \\ \beta = \frac{-2\alpha}{3} = \frac{6\gamma}{3} = 2\gamma \\ -3\gamma + 2\gamma + \gamma = 0 \end{cases} \Rightarrow \begin{cases} \alpha = -3\gamma \\ \beta = 2\gamma \\ \gamma \in \mathbb{R} \end{cases}$$

$$-3v_1 + 2v_2 + v_3 = 0 \quad v_3 = 3v_1 - 2v_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

A jest liniowo zależny

b) $A = (1-2j, 3+j) \quad \mathbb{C} \text{ nad } \mathbb{C}$

$$\frac{1-2j}{3+j} = \frac{(1-2j)(3-j)}{(3+j)(3-j)} = \frac{3-j-6j-2}{9-j^2} = \frac{1-7j}{10} = \frac{1}{10} - \frac{7}{10}j$$

$$v_1 = \left(\frac{1}{10} - \frac{7}{10}j\right)v_2 \quad \text{wektor jest liniowo zależny}$$

nie jest bazą \mathbb{C}

c) $A = (1-2j, 3+j) \quad \mathbb{C} \text{ nad } \mathbb{R}$

$$\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7 \neq 0 \quad \text{wektor jest bazą } \mathbb{C}$$

i jest liniowo niezależny

$(1, j)$ jest bazą \mathbb{C} nad \mathbb{R}

\hookrightarrow wektor 2 wektorów

\hookrightarrow wymiar = 2

\hookrightarrow wektor 2 niezależnych liniowo

wektorów jest bazą \mathbb{C} nad \mathbb{R}

$$\alpha(1-2j) + \beta(3+j) = 0$$

$$\begin{cases} \alpha + 3\beta = 0 \\ -2\alpha + \beta = 0 \end{cases} \rightarrow \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$\alpha v_1 + \beta v_2 = 0$$

$$\begin{cases} \alpha + 3\beta = 0 \\ -2\alpha + \beta = 0 \end{cases} \Rightarrow \begin{cases} \beta = -\frac{1}{3}\alpha \\ \beta = 2\alpha \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

d) $A = ((x+2)^2, x+2) \quad \mathbb{R}_2[x] \text{ nad } \mathbb{R}$

$$v_1 = x^2 + 4x + 4 \quad v_2 = x + 2$$

$$w = \alpha v_1 + \beta v_2$$

$$\alpha v_1 + \beta v_2 = 0$$

$$\alpha x^2 + 4\alpha x + 4\alpha + \beta x + 2\beta = 0$$

$$\alpha x^2 + (4\alpha + \beta)x + (4\alpha + 2\beta) = 0$$

$$\begin{cases} \alpha = 0 \\ 4\alpha + \beta = 0 \\ 4\alpha + 2\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

Wektor jest liniowo niezależny

A nie jest bazą $\mathbb{R}_2[x]$

$$w = x^2 = \alpha v_1 + \beta v_2$$

$$x^2 = \alpha x^2 + (4\alpha + \beta)x + (4\alpha + 2\beta)$$

$$\begin{cases} \alpha = 1 \\ 4\alpha + \beta = 0 \\ 4\alpha + 2\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = -4 \\ \beta = -2 \end{cases}$$

nie da się wyrazić jako kombinacja liniowa v_1 i v_2

$$\text{rank} \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 4 & 2 \end{bmatrix} = 2 = \text{liczba zmiennych}$$

czyli wektor jest liniowo niezależny

0 parametrów

Liczba wektorów \hookrightarrow wektorem $\rightarrow 2$

Liczba wymiarów $\mathbb{R}_2[x] \text{ nad } \mathbb{R} \rightarrow 3$

3. $v = \begin{bmatrix} -7 \\ \pi \\ \pi \end{bmatrix} \quad B = ((7, 8, 0), (0, 1, 8), (1, 0, \pi))$

1 способ

$$\begin{bmatrix} -7 \\ \pi \\ \pi \end{bmatrix} = a \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ \pi \end{bmatrix}$$

$$\begin{cases} -7 = 7a + c \\ \pi = 8a + b \\ \pi = 8b + \pi c \end{cases} \Rightarrow \begin{cases} c = -7a - 7 \\ \pi = 8a + b \\ \pi = 8b - 7\pi a - 7\pi \end{cases}$$

$$\begin{aligned} 8a + b &= 8b - 7\pi a - 7\pi \\ (8 + 7\pi)a &= 7b - 7\pi \\ a &= \frac{7b - 7\pi}{8 + 7\pi} \end{aligned}$$

$$\pi = 56 \frac{b - \pi}{8 + 7\pi} + b$$

$$\begin{aligned} 8\pi + 7\pi^2 &= 56b - 56\pi + 8b + 7\pi b \\ (64 + 7\pi)b &= 64\pi + 7\pi^2 \end{aligned}$$

$$b = \pi$$

$$a = \frac{7\pi - 7\pi}{8 + 7\pi} = 0$$

$$c = -7$$

$$\begin{cases} a = 0 \\ b = \pi \\ c = -7 \end{cases} \quad 0 \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} - 7 \begin{bmatrix} 1 \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} -7 \\ \pi \\ \pi \end{bmatrix}$$

2 вариант $((0, 1, 8), (1, 0, \pi), (7, 8, 0))$
на основе $(\pi, -7, 0)$

2 способ

$$\begin{bmatrix} -7 \\ \pi \\ \pi \end{bmatrix} = x \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} 7 & 0 & 1 \\ 8 & 1 & 0 \\ 0 & 8 & \pi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & 0 & 1 \\ 8 & 1 & 0 \\ 0 & 8 & \pi \end{bmatrix}^{-1} \begin{bmatrix} -7 \\ \pi \\ \pi \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 1 & -7 \\ 8 & 1 & 0 & \pi \\ 0 & 8 & \pi & \pi \end{bmatrix} \xrightarrow{u_2 - u_1} \begin{bmatrix} 7 & 0 & 1 & -7 \\ 1 & 1 & -1 & 7 + \pi \\ 0 & 8 & \pi & \pi \end{bmatrix} \xrightarrow{u_1 - 7u_2} \begin{bmatrix} 0 & -7 & 8 & -54 - 7\pi \\ 1 & 1 & -1 & 7 + \pi \\ 0 & 8 & \pi & \pi \end{bmatrix} \xrightarrow{-\frac{1}{7}u_1} \begin{bmatrix} 0 & 1 & -\frac{8}{7} & 8 + \pi \\ 1 & 1 & -1 & 7 + \pi \\ 0 & 8 & \pi & \pi \end{bmatrix} \xrightarrow{u_2 - u_1} \begin{bmatrix} 0 & 1 & -\frac{8}{7} & 8 + \pi \\ 1 & 0 & \frac{1}{7} & -1 \\ 0 & 0 & \frac{64}{7} + \pi & -64 - 7\pi \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\frac{8}{7} & 8 + \pi \\ 1 & 0 & \frac{1}{7} & -1 \\ 0 & 0 & \frac{64}{7} + \pi & -64 - 7\pi \end{bmatrix} \xrightarrow{u_1 - \frac{1}{7}u_2} \begin{bmatrix} 0 & 1 & -\frac{8}{7} & 8 + \pi \\ 1 & 0 & \frac{1}{7} & -1 \\ 0 & 0 & 1 & -7 \end{bmatrix} \xrightarrow{u_1 + \frac{8}{7}u_3} \begin{bmatrix} 0 & 1 & 0 & \pi \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pi \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

$$\frac{64 + 7\pi}{7} \cdot \frac{7}{64 + 7\pi} \cdot (-64 - 7\pi) = -7$$

$$(-7, \pi, \pi) = (0, \pi, -7)_B$$

$$4. \quad B = (x+1, 3x+4) \quad \mathbb{R}_2[x] \text{ nad } \mathbb{R}$$

$$A = (x+1, 3x+4, x^2, x^3) = (v_1, v_2, v_3, v_4)$$

$$\alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4 = 0$$

$$\delta x^3 + \gamma x^2 + (\alpha + 3\beta)x + (\alpha + 4\beta) = 0$$

$$\begin{cases} \delta = 0 \\ \gamma = 0 \\ \alpha + 3\beta = 0 \\ \alpha + 4\beta = 0 \end{cases} \Rightarrow \begin{cases} \delta = 0 \\ \gamma = 0 \\ \beta = 0 \\ \alpha = 0 \end{cases}$$

uklad je linearno nezávislý

$$u = \alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4$$

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = \delta x^3 + \gamma x^2 + (\alpha + 3\beta)x + (\alpha + 4\beta)$$

$$\begin{cases} \alpha = \delta \\ \beta = \gamma \\ \gamma = \alpha + 3\beta \\ \delta = \alpha + 4\beta \end{cases} \Rightarrow \begin{cases} \alpha = c - 3(d - c) = 4c - 3d \\ \beta = d - c \\ \gamma = b \\ \delta = a \end{cases}$$

každý vektor můžeme vyjádřit jako kombinaci vektorů ukřadu

A je báze

$$2x+2 = (2, 0, 0, 0)_A$$

$$x^3 + x^2 + x = (4, -3, 0, 0)_A = (4, -1, 1, 1)_A = 4x+4 - 3x - 4 + x^2 + x^3 = x^3 + x^2 + x$$

$$(2, 0, 1, -1)_A = 2x+2 + x^2 - x^3 = -x^3 + x^2 + 2x + 2$$

2 způsob

$$x+1 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad 3x+4 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\det B \neq 0 \rightarrow B$ je báze $\mathbb{R}_2[x]$ nad \mathbb{R}

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{v_4 - v_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array} \right] \xrightarrow{v_3 - 3v_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array} \right]$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$2x+2 = ax^3 + bx^2 + c(x+1) + d(3x+4)$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = B^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \rightarrow 2 \cdot (x+1)$$

$$x^3 + x^2 + x = B^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} = x^3 + x^2 + 4x + 4 - 3x - 4$$

$$B \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -3 \end{bmatrix} = 2 \cdot x^3 + 0 \cdot x^2 + 1(x+1) - 1(3x+4) = 2x^3 - 2x - 3$$

5.

$$a) V = \text{span} \{ (1, 0, 3, 1), (0, 1, 2, -1), (1, -1, 1, 2) \}$$

$$\dim V = \text{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ \text{---} & \text{---} & \text{---} \end{bmatrix} = 2$$

double 2 = 3 wektorów są bazą V $((1, 0, 3, 1), (0, 1, 2, -1))$

$$b) V = \text{span} \{ (1, 0, 2), (2, 1, 1), (3, 2, 0), (-1, 3, -11), (2, -1, 7) \}$$

$$\dim V = \text{rank} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 2 & 1 & 0 & -11 & 7 \end{bmatrix} \xrightarrow{v_3 - v_1} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -9 & 3 \end{bmatrix} \xrightarrow{v_3 + 3v_2} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 2$$

bazą V są na przykład $((1, 0, 2), (2, 1, 1))$

$$c) V = \text{span} \{ v_1 - v_2 + v_3 - v_4, v_1 + v_2 - v_3 + v_4, 2v_1 + v_2 - v_3 + v_4 \} \subseteq W$$

(v_1, v_2, v_3, v_4) jest bazą W

$$\dim V = \text{rank} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ \text{---} & \text{---} & \text{---} & \text{---} \\ -1 & 1 & -1 & 1 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ \text{---} & \text{---} & \text{---} \end{bmatrix} = 2$$

$(v_1 - v_2 + v_3 - v_4, v_1 + v_2 - v_3 + v_4)$ jest bazą V

$$6. \dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

$$U+W = \text{span} \{ (3, -1, 0, 0), (0, 0, 2, 1), (3, 0, -2, 0), (0, 1, 0, 1) \}$$

$$\dim(U+W) = \text{rank} \begin{bmatrix} 3 & 0 & 3 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}v_1, \frac{1}{2}v_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{v_2 + v_1, v_4 - v_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 3$$

$$\dim U = \text{rank} \begin{bmatrix} 3 & 0 \\ -1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = 2 \quad \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2 \neq 0 \quad \dim W = \text{rank} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ -2 & 0 \\ 0 & 1 \end{bmatrix} = 2 \quad \begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = -2 \neq 0$$

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W) = 1$$