1.

$$G(x) = \int_{-1}^{x} f(t)\,dt \qquad f(t) = \begin{cases} t\arctan\left(\frac{1}{t^2}\right) & \text{dla } t \neq 0 \\ 0 & \text{dla } t = 0 \end{cases}$$

$$\lim_{t\to 0} t\arctan\left(\frac{1}{t^2}\right) = \lim_{t\to 0} t \cdot \lim_{t\to 0} \arctan\left(\frac{1}{t^2}\right) = 0 \cdot \frac{\pi}{2} = 0 = f(0)$$

f jest ciągła w $\mathbb{R}$

$$\frac{d}{dx} G(x) = f(x) = \begin{cases} x\arctan\left(\frac{1}{x^2}\right) & \text{dla } x \neq 0 \\ 0 & \text{dla } x = 0 \end{cases}$$

2.

$$y = \ln(x) \qquad \ln(x) = \ln^2(x)$$
$$y = \ln^2(x) \qquad \ln^2(x) - \ln(x) = 0$$
$$\ln(x)[\ln(x) - 1] = 0$$
$$\ln(x) = 0 \ \lor \ \ln(x) = 1$$
$$x = 1 \ \lor \ x = e$$

$$\forall_{x\in[1,e]}\ \ln(x) \in [0,1] \implies \forall_{x\in[1,e]}\ \ln^2(x) < \ln(x)$$

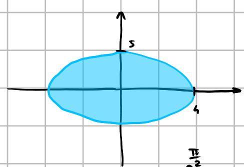
$$\int \ln^2(x)\,dx = \begin{vmatrix} f = \ln^2(x) & g' = 1 \\ f' = 2\ln(x)\cdot\frac{1}{x} & g = x \end{vmatrix} = x\ln^2(x) - 2\int \ln(x)\,dx = \begin{vmatrix} f = \ln(x) & g' = 1 \\ f' = \frac{1}{x} & g = x \end{vmatrix} = x\ln^2(x) - 2x\ln(x) + 2\int 1\,dx = x\ln^2(x) - 2x\ln(x) + 2x + C$$

$$\int [\ln(x) - \ln^2(x)]\,dx = x\ln(x) - x - x\ln^2(x) + 2x\ln(x) - 2x + C = x[3\ln(x) - \ln^2(x) - 3] + C$$

$$|D| = \int_1^e [\ln(x) - \ln^2(x)]\,dx = x[3\ln(x) - \ln^2(x) - 3]\Big|_1^e = e(3 - 1 - 3) - (0 - 0 - 3) = 3 - e$$

3.

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \qquad \begin{cases} x = 4\cos(\alpha) \\ y = 5\sin(\alpha) \\ \alpha \in [0, 2\pi] \end{cases}$$



$$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$$
$$\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$$

$$|S| = 4\int_0^4 y\,dx = \begin{vmatrix} y = 5\sin(\alpha) \\ dx = -4\sin(\alpha)\,d\alpha \\ x = 0 \to \alpha = \frac{\pi}{2} \\ x = 4 \to \alpha = 0 \end{vmatrix} = 4\int_{\frac{\pi}{2}}^{0} 5\sin(\alpha)\cdot(-4)\sin(\alpha)\,d\alpha = 80\int_0^{\frac{\pi}{2}} \sin^2(\alpha)\,d\alpha = 40\int_0^{\frac{\pi}{2}} [1 - \cos(2\alpha)]\,d\alpha$$

$$= 40 \cdot \alpha - \frac{1}{2}\sin(2\alpha)\Big|_0^{\frac{\pi}{2}} = 40\cdot\left[\frac{\pi}{2} - 0 - 0 + 0\right] = 20\pi = \pi \cdot 4 \cdot 5$$

4.

$$\int_1^{\infty} \frac{1}{\sqrt[3]{(9x-3)^2}}\,dx = \lim_{T\to+\infty} \int_1^T \frac{dx}{\sqrt[3]{(9x-3)^2}} = \lim_{T\to+\infty} \int_1^T (9x-3)^{-\frac{2}{3}}\,dx = \begin{vmatrix} t = 9x-3 \\ dt = 9dx \end{vmatrix} = \frac{1}{9}\lim_{T\to+\infty} \int_6^{9T-3} t^{-\frac{2}{3}}\,dt = \frac{1}{9}\lim_{T\to+\infty} \frac{1}{\frac{1}{3}} t^{\frac{1}{3}}\Big|_6^{9T-3} = \frac{1}{3}\lim_{T\to+\infty} \sqrt[3]{t}\Big|_6^{9T-3}$$

$$= \frac{1}{3}\lim_{T\to+\infty}\left[\sqrt[3]{9T-3} - \sqrt[3]{6}\right] = +\infty \quad \text{całka rozbieżna}$$

5.

$$\int_{\frac{1}{2}}^{\infty} \frac{\arctan(4x)}{x^3}\,dx = \lim_{T\to+\infty}\int_{\frac{1}{2}}^{T}\frac{\arctan(4x)}{x^3}\,dx$$

$$\int \frac{\arctan(4x)}{x^3}\,dx = \left| \begin{array}{ll} f=\arctan(4x) & g'=x^{-3} \\ f'=\frac{4}{1+16x^2} & g=-\frac{1}{2} x^{-2}\end{array}\right| = -\frac{\arctan(4x)}{2x^2} - \int \frac{4}{1+16x^2}\cdot\frac{-1}{2x^2}\,dx = -\frac{\arctan(4x)}{2x^2} + 2\int\frac{dx}{x^2(16x^2+1)} = -\frac{\arctan(4x)}{2x^2} + \frac{1}{8} \int \frac{dx}{x^2(x^2+\frac{1}{16})}$$

$$\frac{1}{x^2(x^2+\frac{1}{16})} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+\frac{1}{16}} = \frac{Ax(x^2+\frac{1}{16})+B(x^2+\frac{1}{16})+Cx^3+Dx^2}{x^2(x^2+\frac{1}{16})}$$

$$1 = Ax^3 + \frac{1}{16}Ax + Bx^2 + \frac{1}{16}B + Cx^3 + Dx^2$$
$$1 = (A+C)x^3 + (B+D)x^2 + \frac{1}{16}Ax + \frac{1}{16}B$$

$$\begin{cases} 0 = A+C \\ 0 = B+D \\ 0 = \frac{1}{16}A \\ 1 = \frac{1}{16}B \end{cases} \qquad \begin{array}{l} A=0 \\ B=16 \\ C=0 \\ D=-16 \end{array}$$

$$\frac{1}{x^2(x^2+\frac{1}{16})} = \frac{16}{x^2} - \frac{16}{x^2+\frac{1}{16}}$$

$$\int \frac{dx}{x^2(x^2+\frac{1}{16})} = 16\int\frac{dx}{x^2} - 16\int\frac{dx}{x^2+\frac{1}{16}} = \frac{-16}{x} - 64\arctan(4x)+C$$

$$\int\frac{dx}{x^2+\frac{1}{16}} = \left|\begin{array}{l} t=4x \\ dt=4dx \end{array}\right| = \frac{1}{4}\int\frac{dt}{\frac{1}{16}t^2+\frac{1}{16}} = 4\int\frac{dt}{t^2+1} = 4\arctan(4x)+C$$

$$\int\frac{\arctan(4x)}{x^3}\,dx = -\frac{\arctan(4x)}{2x^2} - \frac{2}{x} - 8\arctan(4x)+C = -\frac{\arctan(4x)+4x+16x^2\arctan(4x)}{2x^2}+C = -\frac{(16x^2+1)\arctan(4x)+4x}{2x^2}+C$$

$$\int_{\frac{1}{2}}^{\infty}\frac{\arctan(4x)}{x^3}\,dx = \lim_{T\to\infty} -\frac{(16T^2+1)\arctan(4T)+4T}{2T^2} + 10\arctan(2)+4 = -4\pi + 10\arctan(2)+4$$

$$\lim_{T\to+\infty} -\frac{16\overset{\frac{\pi}{2}}{\arctan(4T)}\,T^2+4T+\overset{\frac{\pi}{2}}{\arctan(4T)}}{2T^2} = \lim_{T\to+\infty} -8\arctan(4T) - \frac{2}{T} - \frac{\arctan(4T)}{2T^2} = -8\cdot\frac{\pi}{2} = -4\pi$$

6.

$$\int_{-2}^{-1}\frac{dx}{\sqrt{x(-2-x)}} = \lim_{a\to-2^+}\int_a^{-1}\frac{dx}{\sqrt{-x^2-2x}} = \lim_{a\to-2^+}\arcsin(x+1)\Big|_a^{-1} = \lim_{a\to-2^+}\arcsin(0)-\arcsin(a+1) = \arcsin(0)-\arcsin(-1) = 0-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}$$

$$x(-2-x) = -x(x+2)>0 \iff x\in(-2,0)$$

$$\int\frac{dx}{\sqrt{-x^2-2x}} = \int\frac{dx}{\sqrt{-x^2-2x-1+1}} = \int\frac{dx}{\sqrt{1-(x+1)^2}} = \arcsin(x+1)+C$$