1.

rysunek 1 $\rightarrow$ $r = 0.38$
rysunek 2 $\rightarrow$ $r = -0.05$
rysunek 3 $\rightarrow$ $r = 0$
rysunek 4 $\rightarrow$ $r = -1$
rysunek 5 $\rightarrow$ $r = -0.8$

2.

$$\sum_{k=1}^{n}(x_k-\bar{x})(y_k-\bar{y}) \stackrel{?}{=} \sum_{k=1}^{n}(x_k-\bar{x})\,y_k \stackrel{?}{=} \sum_{k=1}^{n}(y_k-\bar{y})\,x_k$$

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n}x_k \qquad \bar{y} = \frac{1}{n}\sum_{k=1}^{n}y_k \qquad n\bar{x} = \sum x_k$$

$$\sum_{k=1}^{n}(x_k-\bar{x})(y_k-\bar{y}) = \sum_{k=1}^{n} x_k y_k - x_k\bar{y} - y_k\bar{x} + \bar{x}\bar{y}$$

$$\sum_{k=1}^{n} x_k(y_k-\bar{y}) + \bar{x}(\bar{y}-y_k)$$

$$\sum_{k=1}^{n}\bar{x}(y_k-\bar{y}) = \bar{x}\cdot\sum_{k=1}^{n}(y_k-\bar{y}) = \bar{x}\cdot\left[\sum_{k=1}^{n}y_k - \sum_{k=1}^{n}\bar{y}\right] = \bar{x}\,(n\bar{y}-n\bar{y}) = 0$$

3.

$S_x = \{-2, -1, 0, 1, 2\}$

$\forall_{k \in S_x} \; P(X=k) = \frac{1}{5}$

$Y = X^2$

$\mathrm{cov}(X, Y) = E(X \cdot Y) - EX \cdot EY = E(X^3) - EX \cdot E(X^2)$

$EX = \frac{1}{5}(-2-1+0+1+2) = 0$

$EX^2 = \frac{1}{5}(4+1+0+1+4) = 2$

$EX^3 = \frac{1}{5}(-8-1+0+1+8) = 0$

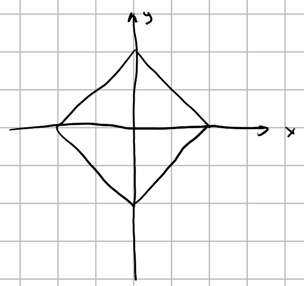
$\mathrm{cov}(X, Y) = 0 - 0 \cdot 2 = 0 \Rightarrow X$ i Y są nieskorelowane

$\left.\begin{array}{l} P(X=0, Y=4) = 0 \\ P(X=0)\,P(Y=4) = \frac{1}{5} \cdot \frac{2}{5} \end{array}\right\}$ nie są niezależne

4.

$D = \{(x,y): |x|+|y| \le 1\}$ $(X,Y) \sim U(D)$

$E(2X+3Y)$ $V(X+Y)$ $|D| = 2$



$f_{XY}(x,y) = \frac{1}{2} \mathbb{1}_D(x,y)$

(EX, EY) – środek ciężkości

$(EX, EY) = (0,0)$

$E(2X+3Y) = 2EX + 3EY = 0$

$V(X+Y) = VX + VY + 2\operatorname{cov}(X,Y)$

$$EX^2 = \iint_{\mathbb{R}^2} x^2 f_{XY}(x,y)\,dxdy = \iint_D \frac{1}{2}x^2\,dxdy = \int_{-1}^{1}\int_{|x|-1}^{1-|x|} \frac{1}{2}x^2\,dy\,dx = \int_{-1}^{1} \frac{1}{2}x^2(1-|x|-|x|+1)\,dx$$

$$= \int_{-1}^{1} x^2(1-|x|)\,dx = \int_{-1}^{1} x^2 - x^2|x|\,dx$$

$$= 2\int_0^1 x^2 - x^3\,dx = 2\left(\left(\frac{1}{3}x^3\right)\Big|_0^1 - \left(\frac{1}{4}x^4\right)\Big|_0^1\right)$$

$$= 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

$$E(XY) = \iint_{\mathbb{R}^2} xy f_{XY}(x,y)\,dxdy = \int_{-1}^{1}\int_{|x|-1}^{1-|x|} \frac{1}{2}xy\,dydx = 0$$

nieparzysta

$VX = \frac{1}{6} - 0^2 = \frac{1}{6}$

$\operatorname{cov}(X,Y) = EXY - EXEY = 0$

$V(X+Y) = \frac{1}{6} + \frac{1}{6} + 0 = \frac{1}{3}$

5.

$(X,Y) \sim N\left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}\right)$

$EX = 5 \quad VX = 3$

$EY = 1 \quad VY = 4$

$\operatorname{cov}(X,Y) = -1$

$E(X(X+3Y)) = E(X^2 + 3XY) = EX^2 + 3EXY = 28 + 3 \cdot 4 = 40$

$\operatorname{cov}(X,Y) = EXY - EX \cdot EY$

$EXY = \operatorname{cov}(X,Y) + EX\,EY = -1 + 5 \cdot 1 = 4$

$VX = EX^2 - (EX)^2$

$EX^2 = VX + (EX)^2 = 3 + 25 = 28$

$V(6X + Y - 3) = V(6X) + 2\operatorname{cov}(6X, Y-3) + V(Y-3)$

$= 36VX + 2 \cdot 6 \operatorname{cov}(X,Y) + VY = 36 \cdot 3 + 12 \cdot (-1) + 4 = 100$

$\operatorname{cov}(X + 4Y, 2X - 4Y) =$

$= \operatorname{cov}(X, 2X) + \operatorname{cov}(X, -4Y) + \operatorname{cov}(4Y, 2X) + \operatorname{cov}(4Y, -4Y)$

$= 2\operatorname{cov}(X,X) - 4\operatorname{cov}(X,Y) + 8\operatorname{cov}(Y,X) - 16\operatorname{cov}(Y,Y)$

$= 2VX + 4\operatorname{cov}(X,Y) - 16VY$

$= 2 \cdot 3 + 4 \cdot (-1) - 16 \cdot 4 = -62$

Albo używając wzoru na przekształcenie liniowe

$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} X + 4Y \\ 2X - 4Y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & -4 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$

$C_{U,V} = A\,C_{X,Y}\,A^T$

$C_{U,V} = \begin{bmatrix} 1 & 4 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 4 & -4 \end{bmatrix} = \ldots$

6.

$X \backslash Y$	0	1	2	
1	.1	.1	.2	.4
2	.1	.2	.3	.6
	.2	.3	.5	

$cov(X,Y) = EXY - EX\,EY = 2.1 - 1.6 \cdot 1.3 = 0.02 > 0$ skorelowane dodatnio

$EXY = .1 + .4 + .4 + 1.2 = 2.1$

$EX = .4 + 1.2 = 1.6$ $\quad EX^2 = .4 + 2.4 = 2.8$

$EY = .3 + 1 = 1.3$ $\quad EY^2 = .3 + 2 = 2.3$

$VX = 2.8 - (1.6)^2 = 0.24$

$VY = 2.3 - (1.3)^2 = 0.61$

$\rho(X,Y) = \dfrac{cov(X,Y)}{\sqrt{VX\ VY}} \approx 0.05$ słaba korelacja

$V(X+Y) = VX + VY + 2cov(X,Y) = 0.89$

$E(X+Y) = EX + EY = 2.9$

7.

$\{0, 1, 2, 3\}$

a) losujemy ze zwracaniem $\Rightarrow$ X i Y niezależne $\Rightarrow$ $\rho_{X,Y} = 0$

b) losujemy bez zwracania $P(X=k, Y=l) = \begin{cases} 0 & k = l \\ \frac{1}{4}\cdot\frac{1}{3} & k \neq l \end{cases}$, $k, l \in \{0, 1, 2, 3\}$

$x \backslash y$	0	1	2	3	
0	0	1/12	1/12	1/12	1/4
1	1/12	0	1/12	1/12	1/4
2	1/12	1/12	0	1/12	1/4
3	1/12	1/12	1/12	0	1/4
	1/4	1/4	1/4	1/4	

$EX = EY = \frac{1}{4}(0+1+2+3) = \frac{3}{2}$

$EX^2 = \frac{1}{4}(0+1+4+9) = \frac{7}{2}$

$VX = VY = \frac{14}{4} - \frac{9}{4} = \frac{5}{4}$

$EXY = 2 \cdot \frac{1}{12}(2+3+6) = \frac{11}{6}$

$cov(X,Y) = EXY - EXEY = \frac{11}{6} - \frac{9}{4} = -\frac{5}{12}$

$\rho = \frac{-\frac{5}{12}}{\sqrt{\frac{5}{4}\cdot\frac{5}{4}}} = -\frac{1}{3}$

8.

X - przychód
$EX = 30\,000$
$\sqrt{VX} = 3000$

Y - koszt
$EY = 20\,000$
$\sqrt{VY} = 4000$

$Z = X - Y$

$\rho_{XY} = 0.75$

$EZ = EX - EY = 10\,000$

$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{VX \cdot VY}} \Rightarrow cov(X,Y) = 0.75 \cdot 3000 \cdot 4000 = 9000\,000$$

$VZ = VX - 2\,cov(X,Y) + VY = 9e+6 - 18e+6 + 16e+6 = 7e+6$

$\sigma_Z = \sqrt{7} \cdot 1000$