1.

$\Omega = \{0, \dots 9\}$ $\quad \forall_{k \in \Omega} \; P(\{k\}) = 0.1$

$X(k) = \cos(k\pi)$ $\qquad S_x = \{1, -1\}$

$Y(k) = \sin(\frac{k\pi}{2})$ $\qquad S_X = \{0, 1, -1\}$

x \ y	-1	0	1
-1	.2	0	.3
1	0	.5	0

k	0	1	2	3	4	5	6	7	8	9
X	1	-1	1	-1	1	-1	1	-1	1	-1
Y	0	1	0	-1	0	1	0	-1	0	1

$P(X = Y) = 0.2 + 0 = 0.2$

2.

X \ Y	-1	0	1	
-1	$a-\frac{1}{16}$	$\frac{1}{4}-a$	0	$3/16$
0	$\frac{1}{8}$	$3/16$	$1/8$	$7/16$
1	$a+1/16$	$1/16$	$\frac{1}{4}-a$	$6/16$
	$2a+\frac{1}{8}$	$\frac{1}{2}-a$	$\frac{3}{8}-a$	

$$p(x,y) := P(X=x,\ Y=y)$$

a) $\begin{cases} a-\frac{1}{16} \geq 0 \\ a+\frac{1}{16} \geq 0 \\ \frac{1}{4}-a \geq 0 \end{cases} \implies \begin{cases} a \geq \frac{1}{16} \\ a \geq -\frac{1}{16} \\ a \leq \frac{1}{4} \end{cases} \implies a \in \left[\frac{1}{16}, \frac{4}{16}\right]$

$\sum_{x,y} p(x,y) = 1$ (nie zależy od a)

b) $P(X>2Y) = \frac{7}{16}$

$= p(-1,-1) + p(0,-1) + p(1,-1) + p(1,0)$

$= a-\frac{1}{16} + \frac{1}{8} + a + \frac{1}{16} + \frac{1}{16}$

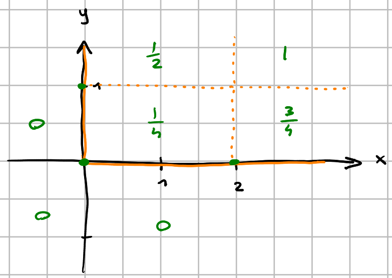
$= 2a + \frac{3}{16}$

$2a + \frac{3}{16} = \frac{7}{16} \implies a = \frac{1}{8} \in \left[\frac{1}{16}, \frac{4}{16}\right]$

c) $F_{XY}(0,1) = P(X \leq 0,\ Y \leq 1) = \frac{3}{16} + \frac{7}{16} = \frac{10}{16} = \frac{5}{8}$

$F_{XY}\left(-\frac{1}{2}, \frac{1}{2}\right) = P\left(X \leq -\frac{1}{2},\ Y \leq \frac{1}{2}\right) = p(-1,-1) + p(-1,0) = a - \frac{1}{16} + \frac{1}{4} - a = \frac{3}{16}$

3.



$S_{xy} = \{(0,0),\ (0,1),\ (2,0)\}$

$x \backslash y$	0	1	
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
2	$\frac{1}{2}$	0	$\frac{1}{2}$
	$\frac{3}{4}$	$\frac{1}{4}$	

$P(X=a, Y=b) = F(a,b) - F(a^-,b) - F(a,b^-) + F(a^-,b^-)$

$P(X=0, Y=0) = \frac{1}{4} - 0 - 0 + 0 = \frac{1}{4}$

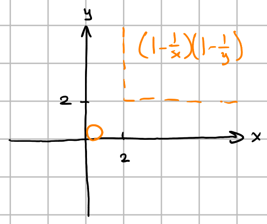
$P(X=0, Y=1) = \frac{1}{2} - 0 - \frac{1}{4} + 0 = \frac{1}{4}$

$P(X=2, Y=0) = \frac{3}{4} - \frac{1}{4} - 0 + 0 = \frac{1}{2}$

$P(X=2, Y=1) = 1 - \frac{1}{2} - \frac{3}{4} + \frac{1}{4} = 0$

4.

$$F(x,y) = \begin{cases} \left(1-\frac{1}{x}\right)\cdot\left(1-\frac{1}{y}\right) & x \geq 2 \ \wedge \ y \geq 2 \\ 0 & x < 2 \ \vee \ y < 2 \end{cases}$$



$$F_X(x) = \lim_{y \to +\infty} F(x,y) = \left(1-\frac{1}{x}\right)\cdot \mathbb{1}_{[2,+\infty)}(x)$$

$$F_Y(y) = \lim_{x \to +\infty} F(x,y) = \left(1-\frac{1}{y}\right)\cdot \mathbb{1}_{[2,+\infty)}(y)$$

$$P(X>2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - \left(1-\frac{1}{2}\right) = \frac{1}{2}$$

$$P(1<X\leq 3,\ 1<Y\leq 4) = F(3,4) - F(1,4) - F(3,1) + F(1,1) = \left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) - 0 - 0 + 0 = \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}$$

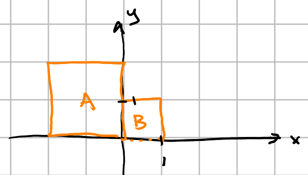
$$P(X=2,\ Y=2) = F(2,2) - F(2^-,2) - F(2,2^-) + F(2^-,2^-) = \frac{1}{2}\cdot\frac{1}{2} - 0 - 0 + 0 = \frac{1}{4}$$

5.

$$f_{xy}(x,y) = \begin{cases} a & -2 \le x \le 0 \ \wedge \ 0 \le y \le 2 \\ \frac{1}{2} & 0 < x \le 1 \ \wedge \ 0 < y \le 1 \\ 0 & \text{w p.p.} \end{cases}$$

$A = [-2,0] \times [0,2]$

$B = (0,1] \times (0,1]$



$a \in \mathbb{R}$

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{xx}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2} f_{xx}(x,y)\,dx\,dy = \iint_A f_{xx}(x,y)\,dx\,dy + \iint_B f_{xy}(x,y)\,dx\,dy + 0$$

$$\iint_A f_{xx}(x,y)\,dx\,dy = \int_{-2}^{0}\left[\int_0^2 a\,dy\right]dx = \int_{-2}^{0} 2a\,dx = 2\cdot 2a = 4a$$

$$\iint_B f_{xy}(x,y)\,dx\,dy = \int_0^1\left[\int_0^1 \tfrac{1}{2}\,dy\right]dx = \int_0^1 \tfrac{1}{2}\,dx = 1\cdot\tfrac{1}{2} = \tfrac{1}{2}$$

objętości prostopadłościanów

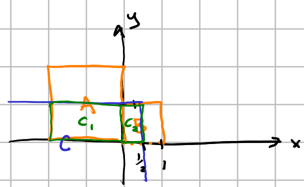
$$1 = \tfrac{1}{2} + 4a \Rightarrow a = \tfrac{1}{8}$$

$$F_{xy}\left(\tfrac{1}{2},1\right) = \int_{-\infty}^{\frac{1}{2}}\int_{-\infty}^{1} f_{xy}(x,y)\,dy\,dx = \iint_C f_{xy}(x,y)\,dx\,dy$$

$C = \left(-\infty, \tfrac{1}{2}\right] \times (-\infty, 1]$

$$= \iint_{C_1} f_{xy}(x,y)\,dx\,dy + \iint_{C_2} f_{xy}(x,y)\,dx\,dy$$

$$= \tfrac{1}{8}\cdot(2\cdot 1) + \tfrac{1}{2}\cdot\left(\tfrac{1}{2}\cdot 1\right) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$



6.

$$f_{xy}(x,y) = \begin{cases} e^{-y} & x \geq 0 \wedge y \geq x \\ 0 & \text{w p.p.} \end{cases}$$

$$D = \{ (x,y) : 0 \leq x \ , \ x \leq y \}$$

$$= \{ (x,y) : 0 \leq y \ , \ 0 \leq x \leq y \}$$

$S_X = [0, +\infty) \qquad S_Y = [0, +\infty)$

dla $x \in S_X$: $\quad f_X(x) = \int_{-\infty}^{+\infty} f_{xy}(x,y)\,dy = \int_x^{+\infty} e^{-y}\,dy = -e^{-y}\Big|_x^{\infty} = e^{-x}$

dla $y \in S_Y$: $\quad f_Y(y) = \int_{-\infty}^{+\infty} f_{xy}(x,y)\,dx = \int_0^y e^{-y}\,dx = e^{-y} \int_0^y 1\,dx = y e^{-y}$

$$P(X+Y \leq 2) = \iint_{x+y \leq 2} f_{xy}(x,y)\,dx\,dy = \int_0^1 \left[\int_x^{2-x} e^{-y}\,dy \right] dx = \int_0^1 (-e^{-y})\Big|_x^{2-x} dx = \int_0^1 -e^{x-2} + e^{-x}\,dx$$

$$= -e^{x-2}\Big|_0^1 - e^{-x}\Big|_0^1 = -e^{-1} + e^{-2} + -e^{-1} + e^{0} = (1-e^{-1})^2 > 0$$