1.

$z_1 = 3+2j \qquad z_2 = 2-5j$

a) $z_1^2 + jz_2 = 9 + 12j - 4 + 2j + 5 = 10 + 14j$

b) $Re(z_1 \cdot z_2) = Re(6 - 15j + 4j + 10) = Re(16 - 11j) = 16$

c) $|2\bar{z}_1 + Im(z_2)| = |6 - 4j - 5| = |1 - 4j| = \sqrt{1+4^2} = \sqrt{17}$

d) $\dfrac{\bar{z}_2 - j|z_1 + 2j|}{z_1} = \dfrac{2+5j - j|3+4j|}{3+2j} = \dfrac{2}{3+2j} \cdot \dfrac{3-2j}{3-2j} = \dfrac{6-4j}{13} = \dfrac{6}{13} - \dfrac{4}{13}j$

2. $z, z_1, z_2 \in \mathbb{C}$

a) $|z_1 + z_2| \stackrel{?}{=} |z_1| + |z_2|$

niech

$L = |1+2j + 3+4j| = |4+6j| = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$

$R = |1+2j| + |3+4j| = \sqrt{5} + 5$

wtedy

$L \neq R$

więc nie zachodzi dla dowolnych liczb

b) $Re(z_1 \cdot z_2) \stackrel{?}{=} Re(z_1) \cdot Re(z_2)$

niech

$L = Re((1+2j)\cdot(3+4j)) = Re(3 + 4j + 6j - 8) = Re(-5 + 10j) = -5$

$R = Re(1+2j) \cdot Re(3+4j) = 1 \cdot 3 = 3$

$L \neq R$ więc nie jest tożsamością

c) $z - \bar{z} \stackrel{?}{=} 2j\,Im(z)$

$L = (a+bj) - (a-bj) = a + bj - a + bj = 2bj = 2j\,Im(z) = R$

3.

a) $-e = e(\cos(\pi) + j\sin(\pi)) = e\,e^{\pi j}$

$\pi j = \pi(\cos(\frac{\pi}{2}) + j\sin(\frac{\pi}{2})) = \pi e^{\frac{\pi}{2}j}$

b) $3 - 3j = 3\sqrt{2}(\cos(-\frac{\pi}{4}) + j\sin(-\frac{\pi}{4})) = 3\sqrt{2}\,e^{-\frac{\pi}{4}j}$

$5\sqrt{2}j - 5\sqrt{2} = 10(\cos(\frac{3\pi}{4}) + j\sin(\frac{3\pi}{4})) = 10\,e^{\frac{3\pi}{4}j}$

c) $1 + j\sqrt{3} = 2(\cos(\frac{\pi}{3}) + j\sin(\frac{\pi}{3})) = 2e^{\frac{\pi}{3}j}$

$\sqrt{6} - j\sqrt{2} = 2\sqrt{2}(\frac{\sqrt{3}}{2} - \frac{1}{2}j) = 2\sqrt{2}(\cos(-\frac{\pi}{6}) + j\sin(-\frac{\pi}{6})) = 2\sqrt{2}\,e^{-\frac{\pi}{6}i}$

4.

a) $4(\cos(7\pi) + j\sin(7\pi)) = 4(-1 + 0j) = -4$

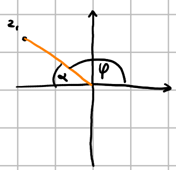
$\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + j\sin\left(\frac{7\pi}{4}\right)\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j\right) = 1 - j$

b) $2\left(\cos\left(-\frac{16\pi}{3}\right) + j\sin\left(-\frac{16\pi}{3}\right)\right) = 2e^{\frac{2\pi}{3}} = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}j\right) = -1 + \sqrt{3}j$

5.

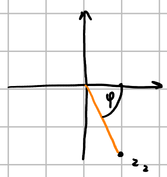
a) $z_1 = 3j - 4$ $\quad |z_1| = \sqrt{(-4)^2 + (3)^2} = 5$

$\alpha = \arctan\left(\frac{3}{4}\right)$ $\quad \varphi = \pi - \alpha = \pi - \arctan\left(\frac{3}{4}\right)$



b) $z_2 = 5 - 12j$ $\quad |z_2| = \sqrt{5^2 + (-12)^2} = 13$

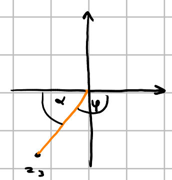
$\varphi = \arctan\left(-\frac{12}{5}\right)$



c) $z_3 = -6 - 8j$ $\quad |z_3| = \sqrt{(-6)^2 + (-8)^2} = 10$

$\alpha = \arctan\left(\frac{8}{6}\right) = \arctan\left(\frac{4}{3}\right)$

$\varphi = -\pi + \alpha = \arctan\left(\frac{4}{3}\right) - \pi$



d) $z_4 = \sqrt{2 - \sqrt{2}} + j\sqrt{2 + \sqrt{2}}$

$z_4^2 = \left(\sqrt{2-\sqrt{2}} + j\sqrt{2+\sqrt{2}}\right)^2 = (2 - \sqrt{2}) + 2j\sqrt{(2-\sqrt{2})(2+\sqrt{2})} - (2 + \sqrt{2})$

$z_4^2 = -2\sqrt{2} + 2\sqrt{2}j$

$\arg z_4^2 = \frac{3\pi}{4}$ $\quad \arg z_4^2 = 2\arg z_4 + 2k\pi$

$\frac{3\pi}{4} = 2\varphi + 2k\pi$ $\quad \frac{3\pi}{8} = \varphi + k\pi$

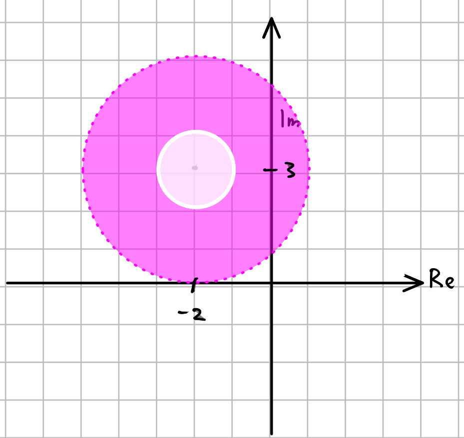
$\varphi = \frac{3\pi}{8} - k\pi \rightarrow \frac{3\pi}{8}$

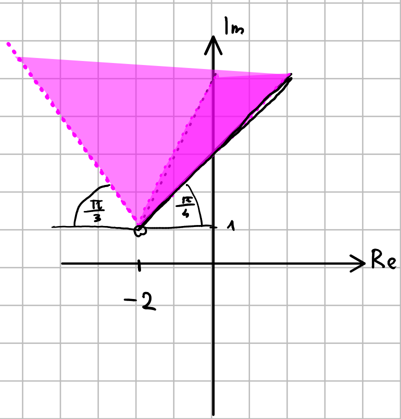
$|z_4^2| = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = 4$

$|z_4^2| = |z_4|^2$ $\quad |z_4| = 2$

6.

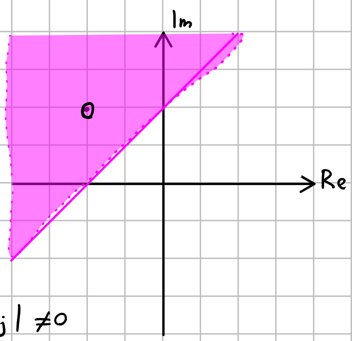
a) $1 \le |z + 2 - 3j| < 3$ $\qquad$ b) $\frac{\pi}{4} \le \arg(z + 2 - j) < \frac{2\pi}{3}$





argument nie jest zdefiniowany dla $z = 0$

c) $\left|\frac{z}{z+2-2j}\right| \geq 1$

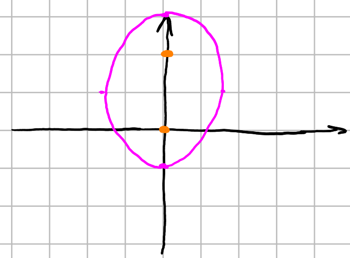


$|z| \geq |z+2-2j|$

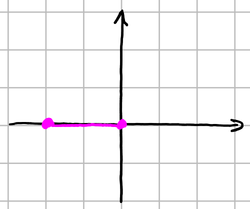
$|z| \geq |z-(-2+2j)|$

$|z+2-2j| \neq 0$

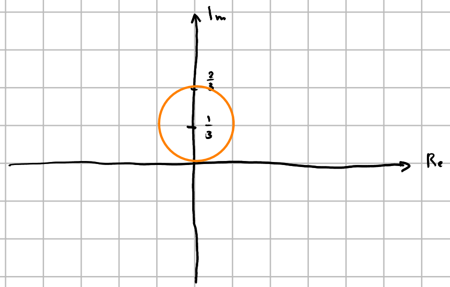
d) $|z-2j|+|z|=4$



e) $|z+2|+|z|=2$



f) $|z-j| = |2jz+1|$



$\sqrt{4x^2+4y^2-4y+1} = \sqrt{x^2+y^2-2y+1}$

$4x^2+4y^2-4y+1 = x^2+y^2-2y+1$

$3x^2+3y^2-2y=0$

$|2jz+1| = |2j(x+yj)+1| = |2xj-2y+1|$

$\sqrt{(1-2y)^2+(2x)^2} = \sqrt{4x^2+4y^2-4y+1}$

$|z-j| = |x+yj-j| = \sqrt{x^2+(y-1)^2} = \sqrt{x^2+y^2-2y+1}$

$x^2+y^2-\frac{2}{3}y=0$

$x^2+y^2-2\cdot y\cdot\frac{1}{3}+\frac{1}{9}=\frac{1}{9}$

$x^2+(y-\frac{1}{3})^2=(\frac{1}{3})^2$

g) $0 \leq \arg(jz^3) < \pi$

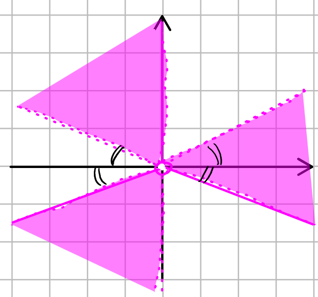
$\arg(jz^3) = \arg(j) + 3\arg(z) + 2k\pi = 3\varphi + \frac{\pi}{2} + 2k\pi$

$0 \leq 3\varphi+\frac{\pi}{2}+2k\pi<\pi$

$-\frac{\pi}{2} - 2k\pi \leq 3\varphi < \frac{\pi}{2} - 2k\pi$

$-\frac{\pi}{6} - \frac{2k\pi}{3} \leq \varphi < \frac{\pi}{6} - \frac{2k\pi}{3}$

$[-\frac{5\pi}{6}, -\frac{3\pi}{6}) \cup [-\frac{\pi}{6},\frac{\pi}{6}) \cup [\frac{3\pi}{6},\frac{5\pi}{6})$



h) $\mathrm{Im}(z^4) < 0$

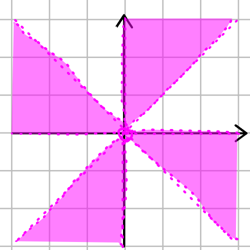
$\Leftrightarrow -\pi < \arg(z^4) < 0$

$\Leftrightarrow -\pi < 4\arg(z) + 2k\pi < 0, \quad k\in\mathbb{Z}$

$-\pi - 2k\pi < 4\arg(z) < -2k\pi$

$-\frac{\pi}{4} - \frac{k\pi}{2} < \arg(z) < -\frac{k\pi}{2}$

$\varphi \in (-\frac{3\pi}{4}, -\frac{\pi}{2}) \cup (-\frac{\pi}{4}, 0) \cup (\frac{\pi}{4},\frac{\pi}{2})\cup(\frac{3\pi}{4},\pi)$



i) $\operatorname{Re}\left(\frac{z+j}{z-j}\right) \leq 0$

$$\frac{z+j}{z-j} = \frac{x+j(y+1)}{x+j(y-1)} \cdot \frac{x-j(y-1)}{x-j(y-1)} = \frac{x^2 - jx(y-1) + jx(y+1) + (y-1)(y+1)}{x^2+(y-1)^2}$$

$$\frac{x^2+y^2-1 + jx(-y+1+y+1)}{x^2+y^2-1} = \frac{x^2+y^2-1 + 2jx}{x^2+y^2-1} = 1 + \frac{2x}{x^2+y^2-1}\, j$$

$\operatorname{Re}\left(1 + \frac{2x}{x^2+y^2-1}\, j\right) = 1 \qquad 1 \leq 0 \;\rightarrow\; \emptyset$