

1.

$$a) \frac{(-2 + 2\sqrt{3}j)^{21}}{(2 - 2j)^{24}} = \frac{(4e^{\frac{2\pi}{3}j})^{21}}{(2\sqrt{2}e^{-\frac{\pi}{4}j})^{24}} = \frac{2^{42}e^{14\pi j}}{2^{36}e^{-\frac{13\pi}{2}j}} = 2^3e^{\frac{41}{2}\pi j} = 8e^{\frac{\pi}{2}j} = 8j$$

$$b) \left(\frac{2j-4}{j+3}\right)^{101} = \left(\frac{-4+2j}{3+j} \cdot \frac{3-j}{3-j}\right)^{101} = \left(\frac{-12+4j+6j+2}{9+1}\right)^{101} = \left(\frac{-10+10j}{10}\right)^{101} = (-1+j)^{101}$$

$$= (\sqrt{2}e^{\frac{3\pi}{4}j})^{101} = 2^{50}\sqrt{2}e^{\frac{303\pi}{4}j} = 2^{50}\sqrt{2}e^{-\frac{\pi}{4}j} = 2^{50}\sqrt{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) = 2^{50} - j2^{50}$$

$$c) \sqrt{5-12j} \quad |5-12j|=13$$

$$\begin{aligned} t &= a+bj \\ t^2 &= a^2-b^2+2abj \\ |t|^2 &= |5-12j| \end{aligned} \quad \begin{cases} a^2-b^2=5 \\ 2ab=-12 \\ a^2+b^2=13 \end{cases} \quad \begin{aligned} 2a^2 &= 18 \\ a^2 &= 9 \\ 1) \ a &= 3 \\ 6b &= -12 \\ b &= -2 \end{aligned} \quad \begin{aligned} 2) \ a &= -3 \\ -6b &= -12 \\ b &= 2 \end{aligned}$$

$$\sqrt{5-12j} = \{3-2j, -3+2j\}$$

$$d) \sqrt[4]{-\sqrt{3}+3j}$$

$$z = -\sqrt{3}+3j \quad |z| = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$$

$$\arg z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$t_n \rightarrow \sqrt[4]{12} e^{\frac{2\pi + 2kn}{4}j}$$

$$|t_n| = \sqrt[4]{|z|} = \sqrt[4]{12}$$



$$\alpha = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$\left\{ \sqrt[4]{12} e^{\frac{\pi}{6}j}, \sqrt[4]{12} e^{\frac{5\pi}{6}j}, \sqrt[4]{12} e^{\frac{5\pi}{2}j}, \sqrt[4]{12} e^{\frac{7\pi}{6}j} \right\}$$

$$\left\{ \sqrt[4]{12} \cdot \frac{\sqrt{3}}{2} + j\sqrt[4]{12} \cdot \frac{1}{2}, -\frac{1}{2}\sqrt[4]{12} + j\frac{\sqrt{3}}{2}\sqrt[4]{12}, \frac{1}{2}\sqrt[4]{12} - j\frac{\sqrt{3}}{2}\sqrt[4]{12}, -\frac{\sqrt{3}}{2}\sqrt[4]{12} - j\frac{1}{2}\sqrt[4]{12} \right\}$$

$$e) \sqrt[4]{8j}$$

$$z = 8j = 8e^{\frac{\pi}{2}j}$$

$$t_n \rightarrow \sqrt[4]{8} e^{\frac{\frac{\pi}{2} + 2kn}{4}j} = \sqrt[4]{8} e^{\left(\frac{\pi}{8} + \frac{kn}{2}\right)j}$$

$$\left\{ \sqrt[4]{8} e^{\frac{\pi}{8}j}, \sqrt[4]{8} e^{\frac{5\pi}{8}j}, \sqrt[4]{8} e^{\frac{9\pi}{8}j}, \sqrt[4]{8} e^{\frac{13\pi}{8}j} \right\}$$

$$f) \sqrt[3]{16+16j}$$

$$z = 16+16j = 16\sqrt{2}e^{\frac{\pi}{4}j} = 2^{\frac{9}{2}}e^{\frac{\pi}{4}j}$$

$$t_n \rightarrow 2^{\frac{3}{2}}e^{\frac{\frac{\pi}{4} + 2kn}{3}j} = 2\sqrt{2}e^{\frac{\pi + 8kn}{12}j}$$

$$\left\{ 2\sqrt{2}e^{\frac{\pi}{12}j}, 2\sqrt{2}e^{\frac{5\pi}{12}j}, 2\sqrt{2}e^{\frac{9\pi}{12}j} \right\}$$

2.

$$z = \cos(x) + j \sin(x) \quad z = a + bj$$

$$z^5 = \cos(5x) + j \sin(5x)$$

$$(a + bj)^5 = (a^3 + 3a^2bj + 3ab^2j^2 + b^3j^3)(a^2 + 2abj + b^2j^2)$$

$$(a^3 + 3a^2bj - 3ab^2 - b^3j)(a^2 - b^2 + 2abj)$$

$$a^5 - a^3b^2 + 2a^4bj + 3a^4bj - 3a^2b^3j + 6a^2b^2j^2 - 3a^3b^2 + 3ab^4 - 6a^2b^3j - a^2b^3j + b^5j - 2ab^4j^2$$

$$a^5 - a^3b^2 - 6a^3b^2 - 3a^3b^2 + 3ab^4 + 2ab^4 + 2a^4bj + 3a^4bj - 3a^2b^3j - 6a^2b^3j - a^2b^3j + b^5j$$

$$(a^5 - 10a^3b^2 + 5ab^4) + j(b^5 - 10a^2b^3 + 5a^4b)$$

$$\cos(5x) = \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x)$$

$$\sin(5x) = \sin^5(x) - 10\sin^3(x)\cos^2(x) + 5\sin(x)\cos^4(x)$$

3.

$$a) \quad z^4 \cdot |z| = -8(\bar{z})^2$$

$$\frac{d}{d\alpha} z \neq 0$$

$$r^4 e^{4j\varphi} \cdot r = 8 e^{j\pi} \cdot r^2 e^{-2j\varphi}$$

$$r^5 e^{4j\varphi} = 8r^2 e^{(j\pi - 2j\varphi)}$$

$$r^5 = 8r^2$$

$$4\varphi = \pi - 2\varphi + 2k\pi$$

$$r^3 = 8$$

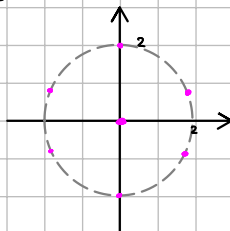
$$6\varphi = \pi + 2k\pi$$

$$r = 2$$

$$\varphi = \frac{\pi + 2k\pi}{6}$$

$$\frac{d}{d\alpha} z = 0$$

$$0 = 0 \quad \checkmark$$



$$b) \quad z^2 - z = \frac{2j - 14}{j + 3}$$

$$\frac{-14 + 2j}{3 + j} \cdot \frac{3 - j}{3 - j} = \frac{-42 + 14j + 6j + 2}{9 + 1} = \frac{-40 + 20j}{10} = -4 + 2j$$

$$z^2 - z + (4 - 2j) = 0$$

$$\Delta = (-1)^2 - 4(1)(4 - 2j) = 1 - 4(4 - 2j) = 1 - 16 + 8j = 8j - 15$$

$$t = a + bj$$

$$a^2 - b^2 = -15$$

$$t^2 = a^2 - b^2 + 2abj$$

$$2ab = 8$$

$$|t|^2 = a^2 + b^2$$

$$a^2 + b^2 = 17$$

$$|\Delta| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$$

$$2a^2 = 2$$

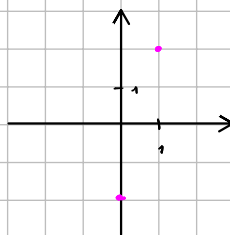
$$1) \quad a = 1$$

$$b = 4$$

$$t_1 = 1 + 4j$$

$$z_1 = \frac{1 - 1 - 4j}{2} = -2j$$

$$z_2 = \frac{1 + 1 + 4j}{2} = 1 + 2j$$

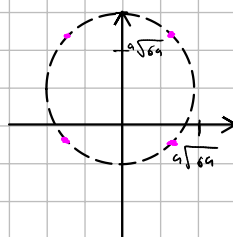


$$c) \quad (2 - j)^4 = (\sqrt{3} + j)^6$$

$$(\sqrt{3} + j)^6 = (2 e^{j\pi/6})^6 = 2^6 e^{j\pi} = -64$$

$$(2 - j)^4 = -64 \quad \sqrt[4]{64 e^{j\pi}} \rightarrow \sqrt[4]{64} e^{j\frac{\pi + 2k\pi}{4}}$$

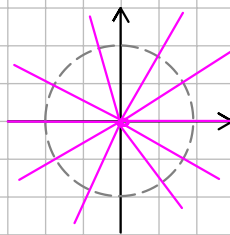
$$z - j \in \left\{ \sqrt[4]{64} e^{j\frac{\pi}{4}}, \sqrt[4]{64} e^{j\frac{3\pi}{4}}, \sqrt[4]{64} e^{j\frac{5\pi}{4}}, \sqrt[4]{64} e^{j\frac{7\pi}{4}} \right\}$$



d)  $z^5 = \bar{z}^5$   
 dla  $z \neq 0$   
 $r^5 e^{5\varphi j} = r^5 e^{-5\varphi j}$   
 $\forall r \in \mathbb{R}_+, r^5 = r^5$

dla 0  
 $0^5 = \bar{0}^5 \quad \checkmark$

$5\varphi = -5\varphi + 2k\pi$   
 $10\varphi = 2k\pi$   
 $\varphi = \frac{2k\pi}{10} = \frac{\pi}{5}k$



e)  $\operatorname{Re}(z^4(3-j\sqrt{3})) = 0$

$\iff$

$\arg(z^4(3-j\sqrt{3})) \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$

$\arg(z^4(3-j\sqrt{3})) = 4\arg(z) + \arg(3-j\sqrt{3}) + 2k\pi$

$\arg(3-j\sqrt{3}) = \arctan(-\frac{\sqrt{3}}{3}) = -\frac{\pi}{6}$

$4\varphi - \frac{\pi}{6} + 2k\pi = -\frac{\pi}{2} \quad \vee \quad 4\varphi - \frac{\pi}{6} + 2k\pi = \frac{\pi}{2}$

$4\varphi = -\frac{\pi}{3} - 2k\pi$

$\varphi = -\frac{\pi}{12} - \frac{k\pi}{2}$

$4\varphi = \frac{2\pi}{3} - 2k\pi$

$\varphi = \frac{\pi}{6} - \frac{k\pi}{2}$

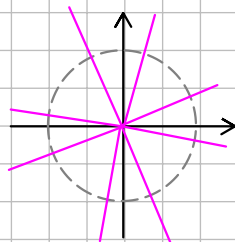
$\varphi \in \{\frac{2\pi}{3}, \frac{\pi}{6}, -\frac{\pi}{3}, -\frac{5\pi}{6}, \frac{11\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{12}, -\frac{7\pi}{12}\}$

$\operatorname{Re}(r^4 e^{4\varphi j} \cdot 2\sqrt{3} e^{-\frac{\pi}{6}j}) = 0$

$\operatorname{Re}(2\sqrt{3} r^4 e^{(4\varphi - \frac{\pi}{6})j}) = 0$

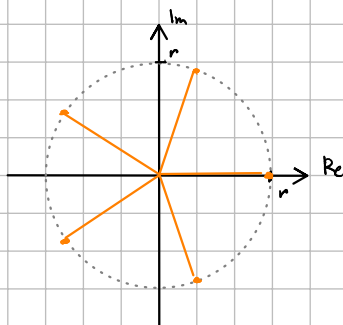
$2\sqrt{3} r^4 \cos(4\varphi - \frac{\pi}{6}) = 0$

$\cos(4\varphi - \frac{\pi}{6}) = 0$



f)  $z^4 = (j2+1)^4 \quad z^4 - (j2+1)^4 = 0$   
 $(z^2 - (j2+1)^2)(z^2 + (j2+1)^2) = 0$

4.



Wielokąt można przesunąć, tak że środek drugiego znajduje się w  $z = 0 + 0j$

Wielokąt można obrócić tak, że 1 z wierzchołków to  $z = r + 0j$

Pozostałe wierzchołki mają postać  $rw_k$

Dwa sąsiadujące wierzchołki:

$re^{j\varphi}$  i  $re^{j(\varphi + \frac{2\pi}{n})}$

Obrót  $\rightarrow n |re^{j(\varphi + \frac{2\pi}{n})} - re^{j\varphi}| = n |re^{j\varphi}(e^{j\frac{2\pi}{n}} - 1)| = nr |e^{j\frac{2\pi}{n}} - 1|$

$nr |\cos(\frac{2\pi}{n}) - 1 + j\sin(\frac{2\pi}{n})|$

$nr \sqrt{(\cos(\frac{2\pi}{n}) - 1)^2 + \sin^2(\frac{2\pi}{n})}$

$nr \sqrt{\cos^2(\frac{2\pi}{n}) - 2\cos(\frac{2\pi}{n}) + 1 + \sin^2(\frac{2\pi}{n})}$

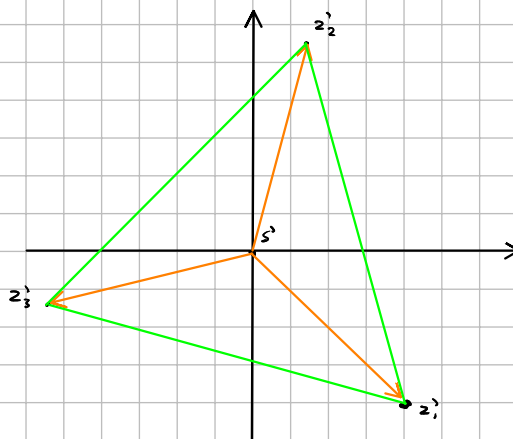
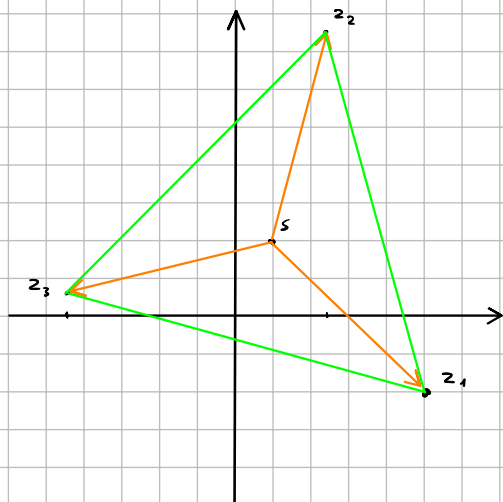
$nr \sqrt{2 - 2\cos(\frac{2\pi}{n})} = nr \sqrt{4\sin^2(\frac{\pi}{n})} = 2nr \sin(\frac{\pi}{n})$

$2 - 2\cos(\frac{2\pi}{n}) = 2 - 2(2\cos^2(\frac{\pi}{n}) - 1)$

$2 - 4\cos^2(\frac{\pi}{n}) + 2 = 4(1 - \cos^2(\frac{\pi}{n})) = 4\sin^2(\frac{\pi}{n})$

Pole  $\rightarrow n \cdot \frac{1}{2} \cdot |re^{j\varphi}| \cdot |re^{j(\varphi + \frac{2\pi}{n})}| \cdot \sin(\frac{2\pi}{n}) = \frac{1}{2} nr^2 \sin(\frac{2\pi}{n})$

5.



$$z_n' = z_n - s \quad z_1' = 5 - 2j - 1 - 2j = 4 - 4j = 4\sqrt{2} e^{-\frac{\pi}{4}j}$$

$$z_2' = z_1' \cdot e^{\frac{2\pi}{3}j} = 4\sqrt{2} e^{\frac{5\pi}{12}j} = 4\sqrt{2} \left( \frac{\sqrt{3}-1}{2\sqrt{2}} + j \frac{1+\sqrt{3}}{2\sqrt{2}} \right) = 2\sqrt{3} - 2 + j(2+2\sqrt{3})$$

$$z_3' = z_1' \cdot e^{-\frac{2\pi}{3}j} = 4\sqrt{2} e^{-\frac{11\pi}{12}j} = 4\sqrt{2} \left( -\frac{1+\sqrt{3}}{2\sqrt{2}} - j \frac{\sqrt{3}-1}{2\sqrt{2}} \right) = -2 - 2\sqrt{3} - j(2\sqrt{3}-2)$$

$$z_2 = z_2' + s = 2\sqrt{3} - 2 + 2j + 2\sqrt{3}j + 1 + 2j = (2\sqrt{3} - 1) + j(4 + 2\sqrt{3})$$

$$z_3 = z_3' + s = -2 - 2\sqrt{3} - 2\sqrt{3}j + 2j + 1 + 2j = (-1 - 2\sqrt{3}) + j(4 - 2\sqrt{3})$$

6.

$$\sqrt[8]{7-3j}$$

$$z = 7-3j = r e^{j\varphi} \quad \text{mit } t_0 = z$$

$$t_k = \sqrt[8]{r} e^{\frac{\varphi+2k\pi}{8}j}$$

$$\prod_{k=0}^7 t_k = \sqrt[8]{r} e^{\frac{\varphi}{8}j} \cdot \sqrt[8]{r} e^{\frac{\varphi+2\pi}{8}j} \cdot \dots \cdot \sqrt[8]{r} e^{\frac{\varphi+14\pi}{8}j}$$

$$= (\sqrt[8]{r})^8 \cdot e^{\frac{\varphi+\varphi+2\pi+\varphi+4\pi+\dots+\varphi+14\pi}{8}j}$$

$$= r e^{\frac{8\varphi + \sum_{k=1}^7 2\pi k}{8}j} = r e^{\frac{8\varphi + 56\pi}{8}j} = r e^{(\varphi+7\pi)j} = r e^{(\varphi+\pi)j}$$

$$= -z = -7+3j$$



$$\sum_{k=0}^7 t_k = \sqrt[8]{r} e^{\frac{\varphi}{8}j} + \sqrt[8]{r} e^{\frac{\varphi+2\pi}{8}j} + \dots + \sqrt[8]{r} e^{\frac{\varphi+14\pi}{8}j}$$

$$= \sqrt[8]{r} e^{\frac{\varphi}{8}j} (e^{0j} + e^{\frac{2\pi}{8}j} + e^{\frac{4\pi}{8}j} + \dots + e^{\frac{14\pi}{8}j})$$

$$= \sqrt[8]{r} e^{\frac{\varphi}{8}j} \cdot 0 = 0$$

