3.5

$$f(x) = \arctan\left(\frac{2+x}{2-x}\right) - \arctan\left(\frac{x}{2}\right)$$

$$D_f = \mathbb{R}\setminus\{2\}$$

$$\frac{df}{dx} = \frac{1}{1+\left(\frac{2+x}{2-x}\right)^2}\cdot\frac{(1)(2-x)-(-1)(2+x)}{(2-x)^2} - \frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$

$$= \frac{1}{\frac{4-4x+x^2+4+4x+x^2}{(2-x)^2}}\cdot\frac{2-x+2+x}{(2-x)^2} - \frac{1}{\frac{4+x^2}{4}}\cdot\frac{1}{2}$$

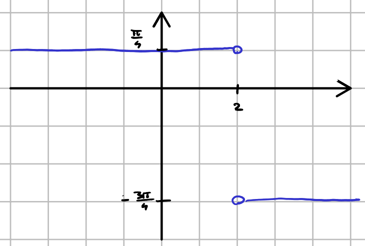
$$= \frac{(2-x)^2}{2x^2+8}\cdot\frac{4}{(2-x)^2} - \frac{2}{x^2+4} = \frac{2}{x^2+4} - \frac{2}{x^2+4} = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \underbrace{\arctan\left(\frac{2+x}{2-x}\right)}_{\downarrow\, -\infty} - \arctan\left(\frac{x}{2}\right) = -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \underbrace{\arctan\left(\frac{2+x}{2-x}\right)}_{\downarrow\, +\infty} - \arctan\left(\frac{x}{2}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \arctan\left(\frac{x+2}{-x+2}\right) - \arctan\left(\frac{x}{2}\right) = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \arctan\left(\frac{x+2}{-x+2}\right) - \arctan\left(\frac{x}{2}\right) = -\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$$



brak asymptot pionowych

asymptoty poziome $y = -\frac{3\pi}{4}$ $y = \frac{\pi}{4}$

3.4a

$$f(x) = \sqrt{1+x^2} \qquad f(x) \simeq f(0) + \frac{\frac{df}{dx}(0)}{1!}x + \frac{\frac{d^2f}{dx^2}(0)}{2!}x^2 + \frac{\frac{d^3f}{dx^3}(0)}{3!}x^3 + \frac{\frac{d^4f}{dx^4}(0)}{4!}x^4$$

$$\frac{df}{dx} = \frac{1}{2\sqrt{1+x^2}}\cdot 2x = \frac{x}{\sqrt{1+x^2}}$$

$$\frac{d^2f}{dx^2} = \frac{(1)\sqrt{1+x^2} - x\frac{x}{\sqrt{1+x^2}}}{1+x^2} = \frac{\frac{1+x^2}{\sqrt{1+x^2}} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)\sqrt{1+x^2}} = (1+x^2)^{-\frac{3}{2}}$$

$$\frac{d^3f}{dx^3} = -\frac{3}{2}(1+x^2)^{-\frac{5}{2}}\cdot 2x = -3x(1+x^2)^{-\frac{5}{2}}$$

$$\frac{d^4f}{dx^4} = -3(1+x^2)^{-\frac{5}{2}} + (-3x)\cdot\left(-\frac{5}{2}\right)(1+x^2)^{-\frac{7}{2}}\cdot 2x = -3(1+x^2)^{-\frac{5}{2}} + 15x^2(1+x^2)^{-\frac{7}{2}}$$

$$f(0) = \sqrt{1} = 1$$
$$\frac{df}{dx}(0) = \frac{0}{1} = 0$$
$$\frac{d^2f}{dx^2}(0) = 1^{-\frac{3}{2}} = 1$$
$$\frac{d^3f}{dx^3}(0) = -3\cdot 0\cdot 1^{-\frac{5}{2}} = 0$$
$$\frac{d^4f}{dx^4}(0) = -3(1)^{-\frac{5}{2}} + 0 = -3$$

$$f(x) \simeq 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4$$

6-7.1c

$$\int \sqrt{x}\arctan(\sqrt{x})\,dx = \left| \begin{array}{ll} f = \arctan(\sqrt{x}) & g' = \sqrt{x} \\ f' = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}} & g = \frac{2}{3}x\sqrt{x} \end{array} \right|$$

$$= \frac{2}{3}x\sqrt{x}\arctan(\sqrt{x}) - \int \frac{2}{3}\,\frac{x\sqrt{x}}{2\sqrt{x}(x+1)}\,dx = \frac{2}{3}x\sqrt{x}\arctan(\sqrt{x}) - \frac{1}{3}\int\frac{x}{x+1}\,dx$$

$$\int\frac{x}{x+1}\,dx = \int\frac{x+1-1}{x+1}\,dx = \int 1\,dx - \int\frac{1}{x+1}\,dx = x - \ln|x+1| + C$$

$$= \frac{2}{3}x\sqrt{x}\arctan(\sqrt{x}) - \frac{1}{3}x + \frac{1}{3}\ln|x+1| + C$$

6-7.2

b) $\int e^{\sqrt{x}}\,dx = \left| \begin{array}{l} t=\sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}dx \\ dx = 2t\,dt \end{array} \right| = 2\int t e^t\,dt = \left| \begin{array}{ll} f=t & g'=e^t \\ f'=1 & g=e^t \end{array} \right| = 2te^t - 2\int e^t dt = 2te^t - 2e^t + c = 2e^{\sqrt{x}}(\sqrt{x}-1)+C$

c) $\int \arcsin(x)\,dx = \left| \begin{array}{l} x=\sin(t) \\ dx=\cos(t)dt \end{array} \right| = \int \arcsin(\sin(t))\cos(t)dt = \int t\cos(t)\,dt = \left| \begin{array}{ll} f=t & g'=\cos(t) \\ f'=1 & g=\sin(t) \end{array} \right| = t\sin(t) - \int \sin(t)\,dt = t\sin(t)+\cos(t)+C$

$t=\arcsin(x)$

$= x\arcsin(x) + \cos(\arcsin(x)) + C = x\arcsin(x) + \sqrt{1-x^2}+C$

$\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} = \sqrt{1-x^2}$

f) $\int x^3 e^{x^2}\,dx = \left| \begin{array}{l} t=x^2 \\ dt=2x\,dx \end{array} \right| = \frac{1}{2}\int te^t\,dt = \left| \begin{array}{ll} t & e^t \\ 1 & e^t \end{array} \right| = \frac{1}{2}te^t - e^t + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}+C = \frac{1}{2}e^{x^2}(x^2-1)+C$

6-7.3

b) $\int \sin(4x)\cos(6x)\,dx = \int \frac{1}{2}\left[\sin((4-6)x) + \sin((4+6)x)\right]dx$

$= \frac{1}{2}\int \sin(-2x)\,dx + \frac{1}{2}\int \sin(10x)\,dx$

$= -\frac{1}{2}\int \sin(2x)\,dx + \frac{1}{2}\int \sin(10x)\,dx$

$= -\frac{1}{2}\left[-\frac{1}{2}\cos(2x) - \left(-\frac{1}{10}\cos(10x)\right)\right] + C$

$= \frac{1}{4}\cos(2x) - \frac{1}{20}\cos(10x) + C$

c) $\int \sin^4(x)\cos^3(x)\,dx = \int \sin^4(x)\cdot(1-\sin^2(x))\cdot\cos(x)\,dx = \left| \begin{array}{l} t=\sin(x) \\ dt=\cos(x)dx \end{array} \right| = \int t^4(1-t^2)\,dt = \int[-t^6+t^4]\,dt = -\frac{1}{7}\sin^7(x) + \frac{1}{5}\sin^5(x)+C$

6-7.4

b) $\int \frac{x+1}{x^2+8x+25}dx = \frac{1}{2}\int \frac{2x+8-6}{x^2+8x+25}dx = \frac{1}{2}\int\frac{2x+8}{x^2+8x+25}dx - 3\int\frac{dx}{x^2+8x+25}$

$\int \frac{2x+8}{x^2+8x+25}dx = \left| \begin{array}{l} t=x^2+8x+25 \\ dt=(2x+8)dx \end{array} \right| = \int \frac{dt}{t} = \ln|x^2+8x+25|+C$

$\int \frac{dx}{x^2+8x+25} = \int \frac{dx}{(x+4)^2+9} = \left| \begin{array}{l} t=\frac{1}{3}(x+4) \\ dt=\frac{1}{3}dx \end{array} \right| = 3\int\frac{dt}{9t^2+9} = \frac{1}{3}\int\frac{dt}{t^2+1} = \frac{1}{3}\arctan\left(\frac{x+4}{3}\right)+C$

$\int \frac{x+1}{x^2+8x+25}dx = \frac{1}{2}\ln|x^2+8x+25| - \arctan\left(\frac{x+4}{3}\right)+C$

6-7.5 e

$\int x\sqrt{6x-x^2}\,dx = \int \frac{x(6x-x^2)}{\sqrt{-x^2+6x}}dx = \int\frac{-x^3+6x^2}{\sqrt{-x^2+6x}} = \left[Ax^2+Bx+C\right]\sqrt{-x^2+6x} + \lambda\int\frac{dx}{\sqrt{-x^2+6x}}$

$\frac{-x^3+6x^2}{\sqrt{-x^2+6x}} = \left[2Ax+B\right]\sqrt{-x^2+6x} + \frac{\left[Ax^2+Bx+C\right]}{2\sqrt{-x^2+6x}}\cdot(-2x+6) + \frac{\lambda}{\sqrt{-x^2+6x}}$

$-x^3+6x^2 = (2Ax+B)(-x^2+6x) + (Ax^2+Bx+C)(-x+3) + \lambda$

$-x^3+6x^2 = -2Ax^3 + 12Ax^2 - Bx^2 + 6Bx - Ax^3 + 3Ax^2 - Bx^2 + 3Bx - Cx + 3C + \lambda$

$-x^3+6x^2 = (-3A)x^3 + (15A-2B)x^2 + (9B-C)x + (3C+\lambda)$

$\begin{cases} -1 = -3A \\ 6 = 15A-2B \\ 0 = 9B-C \\ 0 = 3C+\lambda \end{cases}$ $\quad A=\frac{1}{3},\ B=-\frac{1}{2},\ C=-\frac{9}{2},\ \lambda=\frac{27}{2}$

$\int\frac{dx}{\sqrt{-x^2+6x}} = \int\frac{dx}{\sqrt{-x^2+6x-9+9}} = \int\frac{dx}{\sqrt{9-(x-3)^2}} = \left| \begin{array}{l} t=\frac{1}{3}(x-3) \\ dt=\frac{1}{3}dx \\ dx=3dt \end{array} \right| = \int\frac{3\,dt}{\sqrt{9-9t^2}} = \int\frac{dt}{\sqrt{1-t^2}}$

$= \arcsin\left(\frac{x-3}{3}\right)+C$

$\int x\sqrt{6x-x^2}\,dx = \left(\frac{1}{3}x^2 - \frac{1}{2}x - \frac{9}{2}\right)\sqrt{6x-x^2} + \frac{27}{2}\arcsin\left(\frac{x-3}{3}\right)+C$

6-7.6

d) $$\int \frac{1}{\sin(x)+2\cos(x)+3}\,dx = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}+2\,\frac{1-t^2}{1+t^2}+3\,\frac{1+t^2}{1+t^2}}\,dt = \int \frac{\frac{2}{1+t^2}}{\frac{2t+2(1-t^2)+3(1+t^2)}{1+t^2}}\,dt = 2\int\frac{dt}{2t+2-2t^2+3+3t^2} = 2\int \frac{dt}{t^2+2t+5}$$

$$\left|\begin{array}{l} t=\tan(\frac{x}{2}) \\ dx=\frac{2}{1+t^2}\,dt \\ \sin(x)=\frac{2t}{1+t^2} \\ \cos(x)=\frac{1-t^2}{1+t^2}\end{array}\right|$$

$$=2\int\frac{dt}{(t+1)^2+4} = \left|\begin{array}{l} u=\frac12(t+1) \\ du=\frac12\,dt\end{array}\right| = 4\int\frac{du}{4u^2+4} = \int\frac{du}{u^2+1} = \arctan(u)+c = \arctan\left(\frac{t+1}{2}\right)+c = \arctan\left(\frac{\tan(\frac{x}{2})+1}{2}\right)+c$$

e) $$\int\frac{e^{2x}+e^x}{\sqrt{4-e^{2x}}}\,dx = \left|\begin{array}{l} t=e^x \\ dt=e^x\,dx\end{array}\right| = \int\frac{t+1}{\sqrt{4-t^2}}\,dt = \int\frac{t}{\sqrt{4-t^2}}\,dt + \int\frac{dt}{\sqrt{4-t^2}}$$

$$\int\frac{t}{\sqrt{4-t^2}}\,dt = \left|\begin{array}{l} u=4-t^2 \\ du=-2t\,dt\end{array}\right| = -\frac12\int\frac{du}{\sqrt{u}} = -\frac12\cdot\frac{1}{\frac12}\cdot u^{\frac12}+C = -\sqrt{u}+C = -\sqrt{4-t^2}+C$$

$$\int\frac{dt}{\sqrt{4-t^2}} = \left|\begin{array}{l} u=\frac12 t \\ du=\frac12\,dt\end{array}\right| = 2\int\frac{du}{\sqrt{4-4u^2}} = \int\frac{du}{\sqrt{1-u^2}} = \arcsin(u)+c = \arcsin\left(\frac{t}{2}\right)+c$$

$$\int\frac{e^{2x}+e^x}{\sqrt{4-e^{2x}}}\,dx = \arcsin\left(\tfrac12 e^x\right) - \sqrt{4-e^{2x}} + C$$

8-9.1c

$$-f(x)=f(-x)$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

$$\int_{-a}^{0} f(x)\,dx = \left|\begin{array}{l} t=-x \\ dt=-dx\end{array}\right| = \int_{a}^{0} -f(-t)\,dt = \int_{a}^{0} f(t)\,dt = -\int_{0}^{a} f(t)\,dt$$

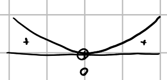
8-9.2b

$$F(x) = \int_{x}^{2x}\frac{e^t}{t}\,dt = G(2x)-G(x)$$

$$G(t) = \int\frac{e^t}{t}\,dt$$

$$\frac{dF}{dx} = G'(2x)\cdot 2 - G'(x) = \frac{e^{2x}}{2x}\cdot 2 - \frac{e^x}{x} = \frac{e^{2x}-e^x}{x} = e^x\,\frac{e^x-1}{x}$$

$$\frac{dF}{dx}>0 \iff \underset{x=0}{x}\,\underset{>0}{e^x}\,\underset{x=0}{(e^x-1)}>0$$



F ↗ u $(-\infty,0)$, $(0,+\infty)$

8-9.3c

$$\lim_{x\to+\infty}\frac{\int_{e^x}^{e^{3x}}\frac{dt}{\ln^3(t)}}{e^{3x}} \overset{?\ \frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to+\infty}\frac{\frac{e^{3x}-9e^x}{9x^3}}{3e^{3x}} = \lim_{x\to+\infty}\frac{1-9e^{-2x}}{27x^3} = 0$$

$$G(t) = \int\frac{dt}{\ln^3(t)}$$

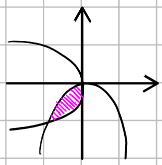
$$\frac{d}{dx}\int_{e^x}^{e^{3x}}\frac{dt}{\ln^3(t)} = \frac{d}{dx}G(e^{3x}) - \frac{d}{dx}G(e^x) = G'(e^{3x})\cdot e^{3x}\cdot 3 - G'(e^x)\,e^x$$

$$= \frac{3e^{3x}}{\ln^3(e^{3x})} - \frac{e^x}{\ln^3(e^x)} = \frac{3e^{3x}}{(3x)^3} - \frac{e^x}{x^3} = \frac{e^{3x}}{9x^3} - \frac{9e^x}{9x^3} = \frac{e^{3x}-9e^x}{9x^3}$$

8-9.4 a

$2y = -x^2 \to y = -\frac{1}{2}x^2$

$2x = -y^2 \to y = -\sqrt{-2x}$



$-\sqrt{-2x} = -\frac{1}{2}x^2$

$-2x = \frac{1}{4}x^4$

$\frac{1}{4}x^4 + 2x = 0$

$\frac{1}{4}x(x^3+8) = 0 \qquad \frac{1}{4}x(x+2)(x^2-2x+4) = 0$

$x = 0 \qquad x = -2$

$y = 0 \qquad y = -2$

$$|S| = \int_{-2}^{0} \left[-\frac{1}{2}x^2 - (-\sqrt{-2x})\right] dx$$

$$= \int_{-2}^{0} \left[-\frac{1}{2}x^2 + \sqrt{-2x}\right] dx = -\frac{1}{2}\int_{-2}^{0} x^2 dx + \int_{-2}^{0} \sqrt{-2x}\, dx$$

$$\int x^2 dx = \frac{1}{3}x^3 + C \qquad \int \sqrt{-2x}\, dx = \left| \begin{matrix} t = -2x \\ dt = -2dx \end{matrix} \right| = -\frac{1}{2}\int \sqrt{t}\, dt = -\frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}} = -\frac{1}{3}(-2x)\sqrt{-2x} = \frac{2}{3}x\sqrt{-2x}$$

$$= -\frac{1}{2}\cdot\frac{1}{3}\, x^3\Big|_{-2}^{0} + \frac{2}{3}\, x\sqrt{-2x}\,\Big|_{-2}^{0}$$

$$= -\frac{1}{6}\cdot(0-(-8)) + \frac{2}{3}(0-(-2\sqrt{4})) = -\frac{8}{6} + \frac{2}{3}\cdot 4 = \frac{16}{6} - \frac{8}{6} = \frac{8}{6} = \frac{4}{3}$$

$$\int \frac{dx}{\sqrt{x^2+k}} = \ln\left|x + \sqrt{x^2+k}\right| + C$$

$$\int \frac{W_n(x)}{\sqrt{ax^2+bx+c}}\, dx = W_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int R(\sin(x), \cos(x))\, dx = \left| \begin{matrix} t = \tan(\frac{x}{2}) \\ dx = \frac{2}{1+t^2}dt \\ \sin(x) = \frac{2t}{1+t^2} \\ \cos(x) = \frac{1-t^2}{1+t^2} \end{matrix} \right|$$

$$\int f(x) g'(x)\, dx = f(x)g(x) - \int f'(x) g(x)\, dx$$

$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin((a-b)x) + \sin((a+b)x)\right]$

$\sin(ax)\sin(ax) = \frac{1}{2}\left[\cos((a-b)x) - \cos((a+b)x)\right]$

$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos((a-b)x) + \cos((a+b)x)\right]$

$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\cos^2(x) = \frac{1+\cos(2x)}{2}$

$\int \ln(x)\, dx = x\ln(x) - x$

$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)|$

$$\int_{h(x)}^{g(x)} f(t)\, dt = F(g(x)) - F(h(x))$$

$$F(t) = \int f(t)\, dt$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\, dt = F'(g(x))g'(x) - F'(h(x))h'(x) = f(g(x))g'(x) - f(h(x))h'(x)$$

$$|V| = \pi \int_a^b f^2(x)\, dx$$

$$|L| = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$|S| = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx$$

Zestaw 2018-19 A

1. $$\int \frac{3x^2+3x+1}{\sqrt{x^2+4x+8}}dx = (Ax+B)\sqrt{x^2+4x+8} + \lambda \int \frac{dx}{\sqrt{x^2+4x+8}}$$

$$\frac{3x^2+3x+1}{\sqrt{x^2+4x+8}} = \frac{(Ax+B)(2x+4)}{2\sqrt{x^2+4x+8}} + A\sqrt{x^2+4x+8} + \frac{\lambda}{\sqrt{x^2+4x+8}}$$

$$3x^2+3x+1 = (Ax+B)(x+2) + A(x^2+4x+8) + \lambda$$
$$3x^2+3x+1 = Ax^2+2Ax+Bx+2B + Ax^2+4Ax+8A+\lambda$$
$$3x^2+3x+1 = 2Ax^2 + (6A+B)x + (8A+2B+\lambda)$$

$$\begin{cases} 3 = 2A \\ 3 = 6A+B \\ 1 = 8A+2B+\lambda \end{cases} \quad \begin{cases} A = \frac{3}{2} \\ B = -6 \\ \lambda = 1-12+12 = 1 \end{cases}$$

$$\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{dx}{\sqrt{x^2+4x+4+4}} = \int \frac{dx}{\sqrt{(x+2)^2+4}} = \left| \begin{matrix} t = \frac{1}{2}(x+2) \\ dt = \frac{1}{2}dx \end{matrix} \right| = \int \frac{2dt}{\sqrt{4t^2+4}} = \int \frac{dt}{\sqrt{t^2+1}} =$$

$$\ln|t+\sqrt{t^2+1}| + C = \ln\left|\frac{x+2}{2} + \sqrt{\left(\frac{x+2}{2}\right)^2+1}\right| + C = \ln|x+2+\sqrt{x^2+4x+8}| + C$$

$$\int \frac{3x^2+3x+1}{\sqrt{x^2+4x+8}}dx = \left(\frac{3}{2}x-6\right)\sqrt{x^2+4x+8} + \ln|x+2+\sqrt{x^2+4x+8}| + C$$

2? $$\int_0^{+\infty} \frac{e^x}{1+e^{2x}}dx = \lim_{T\to+\infty} \int_0^T \frac{e^x}{1+e^{2x}}dx \left| \begin{matrix} t = e^x \\ dt = e^x dx \end{matrix} \right| = \lim_{T\to+\infty} \int_1^{e^T} \frac{dt}{t^2+1} = \lim_{T\to+\infty} \arctan(t)\Big|_1^{e^T} = \lim_{T\to+\infty} \arctan(e^T) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$