

1.

$$\lim_{n \rightarrow \infty} \left(1 + 3n^2 \sin\left(\frac{1}{n^3}\right) \right)^n$$

$$3n^2 \sin\left(\frac{1}{n^3}\right) = \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} \cdot \frac{1}{n^3} \cdot 3n^2 = \frac{3}{n} \cdot \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} = \frac{1}{\frac{n}{3} \cdot \frac{1}{\sin\left(\frac{1}{n^3}\right)}}$$

$$n = \frac{n}{3} \cdot \frac{1}{\frac{1}{n^3}} \cdot 3 \cdot \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}}$$

$$\lim_{n \rightarrow \infty} \left(1 + 3n^2 \sin\left(\frac{1}{n^3}\right) \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n}{3} \cdot \frac{1}{\sin\left(\frac{1}{n^3}\right)}} \right)^{\frac{n}{3} \cdot \frac{1}{\sin\left(\frac{1}{n^3}\right)}} = e^3$$

$$\lim_{n \rightarrow \infty} \frac{n}{3} \cdot \frac{1}{\sin\left(\frac{1}{n^3}\right)} = +\infty \quad \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} = 1 \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$$

2.

$$\lim_{x \rightarrow \frac{\pi}{2}^-} (\cot(x))^{\cot(x)} = \lim_{x \rightarrow \frac{\pi}{2}^-} e^{\ln(\cot(x) \cdot \cot(x))} = \lim_{x \rightarrow \frac{\pi}{2}^-} e^{\cot(x) \cdot \ln(\cot(x))} = e^0 = 1$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \cot(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cos(x)}{\sin(x)} = \left| \frac{0}{1} \right| = 0$$

$$\frac{d}{dx} \cot(x) = \frac{d}{dx} \frac{\cos(x)}{\sin(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$$

$$\frac{d}{dx} x^{-1} = -x^{-2}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \cot(x) \ln(\cot(x)) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\ln(\cot(x))}{\frac{1}{\cot(x)}} \stackrel{0}{=} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{1}{\cot(x)} \cdot \frac{-1}{\cot^2(x)}}{-\frac{1}{\cot^2(x)} \cdot \frac{-1}{\sin^2(x)}} = \lim_{x \rightarrow \frac{\pi}{2}^-} -\cot(x) = 0$$

3.

$$\int e^{\cos^2(x)} \sin(x) \cos(x) dx = \left| \begin{array}{l} t = \cos(x) \\ dt = -\sin(x) dx \end{array} \right| = - \int t e^{t^2} dt = \left| \begin{array}{l} u = t^2 \\ du = 2t dt \end{array} \right| = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{t^2} + C = -\frac{1}{2} e^{\cos^2(x)} + C$$

$$\int_0^{\infty} e^{\cos^2(x)} \sin(x) \cos(x) dx = \lim_{T \rightarrow \infty} \left[-\frac{1}{2} e^{\cos^2(T)} + \frac{1}{2} e^{\cos^2(0)} \right] = \frac{e}{2} - \frac{1}{2} \lim_{T \rightarrow \infty} e^{\cos^2(T)}$$

$\lim_{T \rightarrow \infty} \cos(T)$ nie istnieje więc całka jest nieskończona

$$T_n = 2\pi n \rightarrow \lim_{n \rightarrow \infty} \cos(T_n) = 1$$

$$T_n = 2\pi n + \frac{\pi}{2} \rightarrow \lim_{n \rightarrow \infty} \cos(T_n) = 0$$

?

5. $x y' = y \ln\left(\frac{y^2}{x^2}\right)$

$$y' = \frac{y}{x} \ln\left(\frac{y^2}{x^2}\right)$$

$$v = \frac{y}{x} \quad y' = \frac{d}{dx} vx = v'x + v$$

$$v'x + v = v \ln(v^2) = 2v \ln(v)$$

$$\frac{dv}{dx} = \frac{2v \ln(v) - v}{x} \quad \int \frac{dv}{2v \ln(v) - v} = \int \frac{dx}{x}$$

$$\int \frac{dv}{2v \ln(v) - v} = \int \frac{1}{2 \ln(v) - 1} \cdot \frac{dv}{v} = \left| \begin{array}{l} t = 2 \ln(v) - 1 \\ dt = 2 \frac{1}{v} dv \end{array} \right| = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |2 \ln(v) - 1| + C$$

$$\frac{1}{2} \ln |2 \ln(v) - 1| = \ln |x| + C$$

$$\ln |2 \ln(v) - 1| = 2 \ln |x| + C$$

$$\ln |2 \ln(v) - 1| - \ln |x^2| = C$$

$$\ln \left| \frac{2 \ln(v) - 1}{x^2} \right| = C$$

$$\frac{2 \ln(v) - 1}{x^2} = C$$

$$2 \ln(v) - 1 = C x^2$$

$$\ln(v) = \frac{C x^2 + 1}{2} = C x^2 + \frac{1}{2}$$

$$v = e^{C x^2 + \frac{1}{2}} \quad y = x e^{C x^2 + \frac{1}{2}}$$

sprawdzamy $C=1$

$$y = x e^{x^2 + \frac{1}{2}}$$

$$y' = x \cdot 2x e^{x^2 + \frac{1}{2}} = 2x^2 e^{x^2 + \frac{1}{2}}$$

$$y'x = 2x^3 e^{x^2 + \frac{1}{2}}$$

$$y \ln\left(\frac{y^2}{x^2}\right) = x e^{x^2 + \frac{1}{2}} \cdot \ln\left(\frac{x^2 e^{2x^2 + 1}}{x^2}\right) = x e^{x^2 + \frac{1}{2}} \ln(e^{2x^2 + 1})$$

$$= e^{x^2 + \frac{1}{2}} \cdot x (2x^2 + 1) = 2x^3 e^{x^2 + \frac{1}{2}} + x e^{x^2 + \frac{1}{2}}$$

$$6. \quad y'' - 4y' + 4y = \frac{e^{2x}}{x^2} + 2$$

$$r^2 - 4r + 4 = (r-2)^2 = 0 \quad r_0 = 2$$

$$y(x) = C_1 e^{2x} + C_2 x e^{2x} \quad \frac{d}{dx} x e^{2x} = x \cdot 2e^{2x} + e^{2x} = e^{2x}(2x+1)$$

$$\begin{bmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{2x}}{x^2} + 2 \end{bmatrix}$$

$$C_1' = \frac{-x e^{2x} \cdot \left(\frac{e^{2x}}{x^2} + 2\right)}{(2x+1)e^{2x} e^{2x} - x e^{2x} \cdot 2e^{2x}} = \frac{-\frac{e^{4x}}{x} - 2x e^{2x}}{2x e^{4x} + e^{4x} - 2x e^{4x}} = \frac{e^{4x} \left(-\frac{1}{x} - 2x e^{-2x}\right)}{e^{4x}} = -\frac{1}{x} - 2x e^{-2x}$$

$$C_2' = \frac{e^{2x} \left(\frac{e^{2x}}{x^2} + 2\right)}{e^{4x}} = \frac{1}{x^2} + 2e^{-2x}$$

$$C_1 = \int -\frac{1}{x} - 2x e^{-2x} dx = -\int \frac{dx}{x} - 2 \int x e^{-2x} dx$$

$$\int x e^{-2x} dx = \left| \begin{array}{l} t=x \\ t'=1 \end{array} \right. \begin{array}{l} y=e^{-2x} \\ y'=-\frac{1}{2}e^{-2x} \end{array} = -\frac{1}{2}x e^{-2x} + \frac{1}{2} \int e^{-2x} = -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}$$

$$C_1 = -\ln|x| + x e^{-2x} + \frac{1}{4} e^{-2x} + C_1$$

$$C_2 = \int \frac{1}{x^2} dx + 2 \int e^{-2x} dx = -\frac{1}{x} - e^{-2x} + C_2$$

$$\begin{aligned} y(x) &= \left(-\ln|x| + x e^{-2x} + \frac{1}{4} e^{-2x} + C_1\right) e^{2x} + \left(-\frac{1}{x} - e^{-2x} + C_2\right) x e^{2x} \\ &= -e^{2x} \ln|x| + x + \frac{1}{4} - e^{2x} - x + C_1 e^{2x} + C_2 x e^{2x} \\ &= C_1 e^{2x} + C_2 x e^{2x} - e^{2x} \ln|x| - e^{2x} + \frac{1}{4} \quad C_1, C_2 \in \mathbb{R} \end{aligned}$$