

1.

$$a_n = \frac{2n-3}{5n+1} = \frac{5n+1-3n-4}{5n+1} = 1 + \frac{-3n-4}{5n+1} = 1 + \frac{1}{\frac{5n+1}{-3n-4}} \rightarrow -\frac{5}{3}$$

$$n = \frac{5n+1}{-3n-4} \cdot \frac{-3n-4}{5n+1} \cdot n = \frac{5n+1}{-3n-4} \cdot \frac{-3n^2-4n}{5n+1}$$

$$\lim_{n \rightarrow \infty} \left(\frac{2n-3}{5n+1} \right)^n = 0$$

$$2. \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{e^x-1} \right) = \lim_{x \rightarrow 0} \frac{e^x-1-x}{x(e^x-1)} \stackrel{0}{=} \lim_{x \rightarrow 0} \frac{e^x-1}{e^x-1+xe^x} \stackrel{0}{=} \lim_{x \rightarrow 0} \frac{e^x}{e^x+xe^x+e^x} = \lim_{x \rightarrow 0} \frac{1}{2+x} = \frac{1}{2}$$

$$3. \int \frac{\operatorname{arccot}(e^x)}{e^{2x}} dx = \left| \begin{array}{l} t = e^x \\ dt = e^x dx \\ dx = \frac{dt}{t} \end{array} \right| = \int \frac{\operatorname{arccot}(t)}{t^3} dt = \left| \begin{array}{l} t = \operatorname{arccot}(t) \\ t' = \frac{-1}{1+t^2} \\ g = -\frac{1}{2}t^{-2} \end{array} \right| = -\frac{\operatorname{arccot}(t)}{2t^2} - \frac{1}{2} \int \frac{dt}{t^3(1+t^2)}$$

$$\frac{1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} = \frac{At(t^2+1) + B(t^2+1) + Ct^3 + Dt^2}{t^2(t^2+1)}$$

$$\begin{aligned} 1 &= At^3 + At + Bt^2 + B + Ct^3 + Dt^2 \\ 1 &= (A+C)t^3 + (B+D)t^2 + At + B \end{aligned} \quad \begin{cases} 0 = A+C \\ 0 = B+D \\ 0 = A \\ 1 = B \end{cases} \quad \begin{cases} A = 0 \\ B = 1 \\ C = 0 \\ D = -1 \end{cases}$$

$$\frac{1}{t^2(t^2+1)} = \frac{1}{t^2} - \frac{1}{t^2+1}$$

$$\int \frac{dt}{t^2(t^2+1)} = \int \frac{dt}{t^2} - \int \frac{dt}{t^2+1} = -t^{-1} - \arctan(t) + C$$

$$\int \frac{\operatorname{arccot}(e^x)}{e^{2x}} dx = -\frac{\operatorname{arccot}(t)}{2t^2} + \frac{1}{2t} + \frac{\arctan(t)}{2} + C = -\frac{\operatorname{arccot}(e^x)}{2e^{2x}} + \frac{1}{2e^x} + \frac{\arctan(e^x)}{2} + C$$

$$4. \int \frac{4x}{x^2+3} dx = 2 \int \frac{2x}{x^2+3} dx = \left| \begin{array}{l} t = x^2+3 \\ dt = 2x dx \end{array} \right| = 2 \int \frac{dt}{t} = 2 \ln|x^2+3|$$

$$\int_{-\infty}^{+\infty} \frac{4x}{x^2+3} dx = \int_{-\infty}^0 \frac{4x}{x^2+3} dx + \int_0^{+\infty} \frac{4x}{x^2+3} dx = \lim_{s \rightarrow -\infty} [2 \ln(3) - 2 \ln(s^2+3)] + \lim_{T \rightarrow +\infty} [2 \ln(T^2+3) - 2 \ln(3)]$$

$$= 2 \lim_{T \rightarrow +\infty} \ln(T^2+3) - 2 \lim_{s \rightarrow -\infty} \ln(s^2+3) \quad \text{obě rozložíme užší} \int_{-\infty}^{+\infty} f(x) \text{ rozložena}$$

5.

$$y'' - 3y' + 2y = 3e^x$$

$$r^2 - 3r + 2 = r^2 - 2r - r + 2 = r(r-2) - 1(r-2) = (r-1)(r-2) = 0$$

$$r_1 = 1 \quad r_2 = 2$$

$$y = C_1 e^x + C_2 e^{2x}$$

$$\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3e^x \end{bmatrix}$$

$$C_1' = \frac{-3e^x e^{2x}}{2e^{2x}e^x - e^x e^{2x}} = \frac{-3e^{3x}}{e^{3x}} = -3$$

$$C_1 = \int -3 dx = -3x + C_1$$

$$C_2' = \frac{3e^x e^x}{e^{3x}} = 3e^{-x}$$

$$C_2 = 3 \int e^{-x} dx = -3e^{-x} + C_2$$

$$y(x) = (-3x + C_1)e^x + (-3e^{-x} + C_2)e^{2x}$$

$$y(x) = C_1 e^x + C_2 e^{2x} - 3xe^x - 3e^{-x}$$

$$y' = -3(xe^x + e^x) - 3e^{-x} = -3xe^x - 6e^{-x}$$

$$y'' = -3(xe^x + e^x) - 6e^{-x} = -3xe^x - 9e^{-x}$$

$$y'' - 3y' + 2y = -3xe^x - 9e^{-x} + 3xe^x + 18e^{-x} - 6xe^x - 6e^{-x} = 3e^x$$

6.

$$\sum_{n=2}^{\infty} \sin\left(\frac{\pi}{3 \cdot 2^n}\right)$$

$$\frac{2}{\pi} x \leq \sin(x) \leq x$$

$$\frac{2}{\pi} \cdot \frac{\pi}{3 \cdot 2^n} \leq \sin\left(\frac{\pi}{3 \cdot 2^n}\right) \leq \frac{\pi}{3 \cdot 2^n}$$

$$\sum_{n=2}^{\infty} \frac{\pi}{3 \cdot 2^n} = \frac{\pi}{3} \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \text{ zbieżny (szereg geometryczny)}$$

$$\text{Wtgc } \sum_{n=2}^{\infty} \sin\left(\frac{\pi}{3 \cdot 2^n}\right) \text{ zbieżny z kryterium porównawczego}$$