

$$\begin{aligned}
 1. \quad \lim_{n \rightarrow \infty} \left[1 + \sin\left(\frac{2n+1}{n^2}\right) \right]^{n+1} &= \lim_{n \rightarrow \infty} \left[1 + \frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}} \cdot \frac{2n+1}{n^2} \right]^{n+1} = \lim_{n \rightarrow \infty} \left[1 + \frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}} \cdot \frac{1}{\frac{n^2}{2n+1}} \right]^{\frac{n^2}{2n+1} \cdot \frac{2n+1}{n^2} \cdot (n+1)} \\
 &= \lim_{n \rightarrow \infty} \left(\left[1 + \frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}} \cdot \frac{1}{\frac{n^2}{2n+1}} \right]^{\frac{n^2}{2n+1}} \right)^{\frac{2n^2+3n+1}{n^2}} = e^2 \\
 \left(1 + \frac{1}{n}\right)^n &\rightarrow e \quad \lim_{n \rightarrow \infty} \frac{2n+1}{n^2} = 0 \quad \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}} = 1 \\
 \lim_{n \rightarrow \infty} \frac{2n^2+3n+1}{n^2} &= 2 \quad \lim_{n \rightarrow \infty} \left[1 + \frac{\sin\left(\frac{2n+1}{n^2}\right)}{\frac{2n+1}{n^2}} \cdot \frac{1}{\frac{n^2}{2n+1}} \right]^{\frac{n^2}{2n+1}} = e
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \forall x > 3 \quad \sqrt[4]{x} - \sqrt[4]{3} < \sqrt[4]{x-3} \\
 \Leftrightarrow \forall x > 3 \quad \sqrt[4]{x} - \sqrt[4]{3} - \sqrt[4]{x-3} < 0 \\
 \Leftrightarrow \forall x > 3 \quad f(x) < 0 \\
 f(x) &= \sqrt[4]{x} - \sqrt[4]{3} - \sqrt[4]{x-3} \\
 f'(x) &= \frac{1}{4} x^{-\frac{3}{4}} - \frac{1}{4} (x-3)^{-\frac{3}{4}} \cdot 1 = 0 \\
 &= \frac{1}{4} \left[x^{-\frac{3}{4}} - (x-3)^{-\frac{3}{4}} \right] \\
 f'(x) \geq 0 &\Leftrightarrow x^{-\frac{3}{4}} - (x-3)^{-\frac{3}{4}} \geq 0 \\
 \frac{1}{x^{\frac{3}{4}}} - \frac{1}{(x-3)^{\frac{3}{4}}} &\geq 0 \quad \frac{(x-3)^{\frac{3}{4}} - x^{\frac{3}{4}}}{(x(x-3))^{\frac{3}{4}}} \geq 0
 \end{aligned}$$

$$\begin{aligned}
 x > 3 &\Rightarrow x(x-3) > 0 \quad (\sqrt[4]{x-3} - \sqrt[4]{x})((x-3)^{\frac{3}{4}} + ((x-3)x)^{\frac{1}{4}} + x^{\frac{3}{4}}) \geq 0 \\
 &\Leftrightarrow \sqrt[4]{x-3} - \sqrt[4]{x} \geq 0 \\
 &\quad \sqrt[4]{x-3} \geq \sqrt[4]{x} \\
 &\text{sprawdzamy dla } x \in (3, +\infty) \text{ bo } g(t) = \sqrt[4]{t} \text{ jest rosn\u0105c\u0105 w } \mathbb{R}, \\
 &\text{z\u0142\u0105c } f \text{ jest malej\u0105c\u0105 w } (3, +\infty)
 \end{aligned}$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \left[\sqrt[4]{x} - \sqrt[4]{3} - \sqrt[4]{x-3} \right] = 0 \quad \Rightarrow \quad \forall x > 3 \quad f(x) < 0$$

$$\begin{aligned}
 3. \quad \int \frac{e^{2x}}{1+e^x} dx &= \left| \frac{t=e^x}{dt=e^x dx} \right| = \int \frac{t}{1+t} dt = \int \left[1 - \frac{1}{1+t} \right] dt = \int 1 dt - \int \frac{1}{1+t} dt = t - \ln|1+t| + C \\
 &= e^x - \ln|1+e^x| + C
 \end{aligned}$$

$$4. \int \frac{\ln(x^4 + x^2)}{x^2} dx = \left| \begin{array}{l} f = \ln(x^4 + x^2) \quad g' = \frac{1}{x^2} \\ f' = \frac{4x^3 + 2x}{x^4 + x^2} \quad g = -\frac{1}{x} \end{array} \right| = -\frac{\ln(x^4 + x^2)}{x} + \int \frac{4x^2 + 2}{x^4 + x^2} dx$$

$$\frac{d}{dx} \ln(x^4 + x^2) = \frac{1}{x^4 + x^2} \cdot (4x^3 + 2x) = \frac{4x^3 + 2x}{x^4 + x^2} = \frac{4x^2 + 2}{x^3 + x}$$

$$\int x^{-2} dx = \frac{1}{-2+1} x^{-2+1} = -x^{-1} = -\frac{1}{x}$$

$$\frac{4x^2 + 2}{x^4 + x^2} = \frac{4x^2 + 2}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1} = \frac{Ax(x^2 + 1) + B(x^2 + 1) + Cx^3 + Dx^2}{x^4 + x^2}$$

$$4x^2 + 2 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$4x^2 + 2 = (A + C)x^3 + (B + D)x^2 + Ax + B$$

$$\begin{cases} 0 = A + C & A = 0 \\ 4 = B + D & B = 2 \\ 0 = A & C = 0 \\ 2 = B & D = 2 \end{cases}$$

$$\frac{4x^2 + 2}{x^4 + x^2} = \frac{2}{x^2} + \frac{2}{x^2 + 1}$$

$$\int \frac{\ln(x^4 + x^2)}{x^2} dx = -\frac{\ln(x^4 + x^2)}{x} + 2 \int \frac{dx}{x^2} + 2 \int \frac{dx}{x^2 + 1} = -\frac{\ln(x^4 + x^2)}{x} - \frac{2}{x} + 2 \arctan(x) + C$$

$$\int_1^{\infty} \frac{\ln(x^4 + x^2)}{x^2} dx = \lim_{T \rightarrow \infty} -\frac{\ln(T^4 + T^2)}{T} - \frac{2}{T} + 2 \arctan(T) + \ln(2) - 2 - 2 \arctan(1)$$

$$\lim_{T \rightarrow \infty} \frac{\ln(T^4 + T^2)}{T} \stackrel{H}{=} \lim_{T \rightarrow \infty} \frac{\frac{4x^2 + 2}{x^3 + x}}{1} = 0$$

$$\int_1^{\infty} \frac{\ln(x^4 + x^2)}{x^2} dx = 0 - 0 + 2 \cdot \frac{\pi}{2} + \ln(2) - 2 - 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} + \ln(2) - 2$$

$$5. f(x, y) = \begin{cases} \frac{x^2 y^3}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x^2 \cdot 0}{\Delta x^2 + 0} - 0}{\Delta x} = 0$$

$$\frac{\partial}{\partial x} \frac{x^2 y^3}{(x^2 + y^2)^2} = \frac{\partial}{\partial x} \frac{x^2 y^3}{x^4 + 2y^2 x^2 + y^4} = \frac{2y^3 x (x^2 + y^2)^2 - x^2 y^3 (4x^2 + 4y^2)}{(x^2 + y^2)^4} = \frac{2y^3 x^5 + 4y^5 x^3 + 2y^7 x - 4x^5 y^3 - 4x^3 y^5}{(x^2 + y^2)^4} = \frac{2y^7 x - 2y^3 x^5}{(x^2 + y^2)^4}$$

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{2y^7 x - 2y^3 x^5}{(x^2 + y^2)^4} & \text{dla } (x, y) \neq (0, 0) \\ 0 & \text{dla } (x, y) = (0, 0) \end{cases} \quad 2xy \frac{y^6 - y^2 x^4}{(x^2 + y^2)^4}$$

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \rightarrow (0, 0)$$

$$\frac{\partial f}{\partial x}(x_n, y_n) = \frac{2 \cdot \frac{1}{n^7} \cdot \frac{1}{n} - 2 \cdot \frac{1}{n^5} \cdot \frac{1}{n^5}}{\left(\frac{1}{n^2} + \frac{1}{n^2}\right)^4} = \frac{\frac{2}{n^8} - \frac{2}{n^8}}{\frac{16}{n^8}} = 0$$

$$(\tilde{x}_n, \tilde{y}_n) = \left(\frac{1}{n}, \frac{2}{n}\right)$$

$$\frac{\partial f}{\partial x}(\tilde{x}_n, \tilde{y}_n) = \frac{2 \cdot \frac{16}{n^7} \cdot \frac{1}{n} - 2 \cdot \frac{8}{n^5} \cdot \frac{1}{n^5}}{\left(\frac{1}{n^2} + \frac{4}{n^2}\right)^4} = \frac{\frac{256 - 16}{n^8}}{\frac{5^4}{n^8}} = \frac{240}{625} \neq 0$$

po pochodna w (0, 0) istnieje i jest nieciągła

6.

$$y'' - 4y' + 3y = 6e^{3x} + 9x$$

$$r^2 - 4r + 3 = r^2 - 3r - r + 3 = (r-1)(r-3)$$

$$r_1 = 1 \quad r_2 = 3$$

$$\rightarrow y_0(x) = C_1 e^x + C_2 e^{3x}$$

$$\begin{bmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6e^{3x} + 9x \end{bmatrix} \quad C_1 = \frac{\begin{vmatrix} 0 & e^{3x} \\ 6e^{3x} + 9x & 3e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix}} = \frac{-e^{3x}(6e^{3x} + 9x)}{3e^{4x} - e^{4x}} = \frac{-6e^{6x} - 9xe^{3x}}{2e^{4x}} = -3e^{2x} - \frac{9}{2}xe^{-x}$$

$$C_1(x) = \int -3e^{2x} dx - \frac{9}{2}xe^{-x} dx = -3 \int e^{2x} dx - \frac{9}{2} \int xe^{-x} dx = -3 \cdot \frac{1}{2}e^{2x} - \frac{9}{2} \cdot (-e^{-x})(x+1) = -\frac{3}{2}e^{2x} + \frac{9}{2}e^{-x}(x+1) + C_1$$

$$C_2 = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 6e^{3x} + 9x \end{vmatrix}}{2e^{4x}} = \frac{6e^{4x} + 9xe^x}{2e^{4x}} = 3 + \frac{9}{2}xe^{-3x}$$

$$\int xe^{-3x} dx = \left| \begin{matrix} t=x & g=e^{-3x} \\ t'=1 & g'=-\frac{1}{3}e^{-3x} \end{matrix} \right| = -\frac{1}{3}xe^{-3x} + \frac{1}{3} \int e^{-3x} dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}$$

$$C_2(x) = 3 \int 1 dx + \frac{9}{2} \int xe^{-3x} dx = 3x - \frac{3}{2}xe^{-3x} - \frac{1}{2}e^{-3x} + C_2$$

$$y(x) = \left[-\frac{3}{2}e^{2x} + \frac{9}{2}e^{-x}(x+1) + C_1 \right] e^x + \left[3x - \frac{3}{2}xe^{-3x} - \frac{1}{2}e^{-3x} + C_2 \right] e^{3x}$$

$$= C_1 e^x + C_2 e^{3x} - \frac{3}{2}e^{3x} + \frac{9}{2}x + \frac{9}{2} + 3xe^{3x} - \frac{3}{2}x - \frac{1}{2}$$

$$= C_1 e^x + C_2 e^{3x} + 3xe^{3x} - \frac{3}{2}e^{3x} + 3x + 4$$