1

$$\lim_{n\to\infty} \sqrt[n]{4n - \cos(n)} = ?$$

$$\forall_{n\in\mathbb{N}} \quad \sqrt[n]{4n-1} \le a_n \le \sqrt[n]{4n+1}$$

$$\forall_{n\in\mathbb{N}} \quad \sqrt[n]{4n-1} \geq \sqrt[n]{4n-n} = \sqrt[n]{3n} \qquad \forall_{n\in\mathbb{N}} \quad \sqrt[n]{4n+1} \le \sqrt[n]{4n+n} = \sqrt[n]{5n}$$

$$\lim_{n\to\infty} \sqrt[n]{3n} = 1 \qquad \lim_{n\to\infty} \sqrt[n]{5n} = 1$$

$\lim_{n\to\infty} \sqrt[n]{4n-\cos(n)} = 1$ z twierdzenia o 3 ciągach

2.

$$2x \le \ln\left(\frac{1+x}{1-x}\right) \quad \forall x\in[0,1) \iff \forall x\in[0,1)\ f(x) \geq 0$$

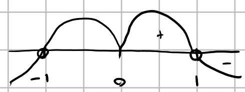
$$f(x) = \ln\left(\frac{1+x}{1-x}\right) - 2x$$

$$f'(x) = \frac{1}{\frac{1+x}{1-x}} \cdot \frac{1\cdot(1-x) - (1+x)\cdot(-1)}{(1-x)^2} - 2 = \frac{\cancel{1-x}}{1+x} \cdot \frac{1-x+1+x}{(1-x)^{\cancel{2}}} - 2$$

$$= \frac{2}{(1+x)(1-x)} - 2 = \frac{2}{1-x^2} - 2 = \frac{2-2+2x^2}{1-x^2} = \frac{2x^2}{1-x^2} = \frac{2x^2}{(1-x)(1+x)}$$

$$f'(x) > 0 \iff 2x^2(1-x)(1+x) > 0$$

$$\left.\begin{array}{l} f \nearrow \text{ w } [0,1) \\ f(0) = \ln(1) - 0 = 0 \end{array}\right\} \Rightarrow \forall x\in[0,1)\ f(x)\geq 0 \iff \forall x\in[0,1)\ \ln\left(\tfrac{1+x}{1-x}\right) \geq 2x$$



3.

$$\int \sin(\ln(x))\,dx = \left| \begin{array}{l} x = e^t \\ \ln(x) = t \\ dx = e^t\,dt \end{array} \right| = \int \sin(t)e^t\,dt = \left| \begin{array}{ll} f = \sin(t) & g' = e^t \\ f' = \cos(t) & g = e^t \end{array}\right| = \sin(t)e^t - \int \cos(t)e^t\,dt = \left| \begin{array}{ll} f = \cos(t) & g' = e^t \\ f' = -\sin(t) & g = e^t \end{array}\right| =$$

$$\int \sin(t)e^t dt = \sin(t)e^t - \cos(t)e^t - \int \sin(t)e^t\,dt$$

$$2\int \sin(t)e^t dt = e^t[\sin(t) - \cos(t)]$$

$$\int \sin(t)\,e^t dt = \frac{1}{2}e^t[\sin(t) - \cos(t)] + C$$

$$\int \sin(\ln(x))\,dx = \frac{1}{2}x\left[\sin(\ln(x)) - \cos(\ln(x))\right] + C$$

4.

$$f(x,y) = \begin{cases} \dfrac{2-x+y^3}{\sqrt{(x-2)^2+y^2}} & \text{dla } (x,y)\neq(2,0) \\ 0 & \text{dla } (x,y) = (2,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(2,0) = \lim_{\Delta x\to 0}\frac{f(2+\Delta x,0) - f(2,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{2-2-\Delta x+0}{\sqrt{(2+\Delta x-2)^2+0}} - 0}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{-\Delta x}{|\Delta x|}}{\Delta x} = \lim_{\Delta x\to 0}\frac{-1}{|\Delta x|}$$

nie istnieje bo granice jednostronne są różne

$$\lim_{\Delta x\to 0^+}\frac{-1}{|\Delta x|} = \lim_{\Delta x\to 0^+}\frac{-1}{\Delta x} = -\infty \neq \lim_{\Delta x\to 0^-}\frac{-1}{|\Delta x|} = \lim_{\Delta x\to 0^-}\frac{-1}{-\Delta x} = +\infty$$

$$\frac{\partial f}{\partial y}(2,0) = \lim_{\Delta y\to 0}\frac{f(2,\Delta y) - f(2,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{\frac{2-2+\Delta y^3}{\sqrt{(2-2)^2+\Delta y^2}} - 0}{\Delta y} = \lim_{\Delta y\to 0}\frac{\Delta y^2}{|\Delta y|} = 0$$

$$\lim_{\Delta y\to 0^+}\frac{\Delta y^2}{|\Delta y|} = \lim_{\Delta y\to 0^+}\frac{\Delta y^2}{\Delta y} = 0 = \lim_{\Delta y\to 0^-}\frac{\Delta y^2}{|\Delta y|} = \lim_{\Delta y\to 0^-}\frac{\Delta y^2}{-\Delta y} = 0$$

5.

$$\sum_{n=2}^{\infty} (-1)^n \cdot \frac{\ln^2(n)}{n}$$

$$\int \frac{\ln^2(x)}{x}\,dx = \left| \begin{matrix} t = \ln(x) \\ dt = \frac{1}{x}dx \end{matrix} \right| = \int t^2\,dt = \frac{1}{3}t^3 + C = \frac{1}{3}\ln^3(x) + C$$

$$\int_2^{\infty} \frac{\ln^2(x)}{x}\,dx = \lim_{T\to+\infty} \frac{1}{3}\ln^3(T) - \frac{1}{3}\ln^3(2) = +\infty$$

...

6.

$$y'' - 5y' + 6y = xe^{2x}$$

$$r^2 - 5r + 6 = r^2 - 2r - 3r + 6 = r(r-2) - 3(r-2) = (r-3)(r-2) = 0$$

$r_1 = 2 \quad r_2 = 3$

$\to \; y_0 = C_1 e^{2x} + C_2 e^{3x}$

$$\begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ xe^{2x} \end{bmatrix}$$

$$C_1' = \frac{-e^{3x} \cdot xe^{2x}}{3e^{3x} \cdot e^{2x} - e^{3x} \cdot 2e^{2x}} = \frac{-xe^{5x}}{3e^{5x} - 2e^{5x}} = \frac{-xe^{5x}}{e^{5x}} = -x$$

$$C_1(x) = \int -x\,dx = -\frac{1}{2}x^2 + C_1$$

$$C_2' = \frac{e^{2x} \cdot xe^{2x}}{e^{5x}} = \frac{xe^{4x}}{e^{5x}} = xe^{-x}$$

$$C_2(x) = \int xe^{-x}\,dx = \left| \begin{matrix} f = x & g' = e^{-x} \\ f' = 1 & g = -e^{-x} \end{matrix} \right| = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C_2 = -e^{-x}(x+1) + C_2$$

$$y(x) = \left(-\frac{1}{2}x^2 + C_1\right)e^{2x} + \left(-xe^{-x} - e^{-x} + C_2\right) \cdot e^{3x}$$

$$y(x) = C_1 e^{2x} + C_2 e^{3x} - \frac{1}{2}x^2 e^{2x} - xe^{2x} - e^{2x}$$

$$y(x) = C_1 e^{2x} + C_2 e^{3x} - e^{2x}\left(\frac{1}{2}x^2 + x + 1\right) \qquad C_1, C_2 \in \mathbb{R}$$