1.

a) $\dfrac{(-2+2\sqrt{3}j)^{21}}{(2-2j)^{26}} = \dfrac{\left(4e^{\frac{2\pi}{3}j}\right)^{21}}{\left(2\sqrt{2}e^{-\frac{\pi}{4}j}\right)^{26}} = \dfrac{2^{42}\, e^{14\pi j}}{2^{39}\, e^{-\frac{13\pi}{2}j}} = 2^3 e^{\frac{41}{2}\pi j} = 8e^{\frac{\pi}{2}j} = 8j$

b) $\left(\dfrac{2j-4}{j+3}\right)^{101} = \left(\dfrac{-4+2j}{3+j}\cdot\dfrac{3-j}{3-j}\right)^{101} = \left(\dfrac{-12+4j+6j+2}{9+1}\right)^{101} = \left(\dfrac{-10+10j}{10}\right)^{101} = (-1+j)^{101}$

$= \left(\sqrt{2}\,e^{\frac{3\pi}{4}j}\right)^{101} = 2^{50}\sqrt{2}\,e^{\frac{303\pi}{4}j} = 2^{50}\sqrt{2}\,e^{-\frac{\pi}{4}j} = 2^{50}\sqrt{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) = 2^{50} - j\,2^{50}$

c) $\sqrt{5-12j}$ $\qquad |5-12j| = 13$

$t = a+bj$
$t^2 = a^2 - b^2 + 2abj$
$|t|^2 = |5-12j|$

$\begin{cases} a^2 - b^2 = 5 \\ 2ab = -12 \\ a^2 + b^2 = 13 \end{cases}$

$2a^2 = 18$
$a^2 = 9$

1) $a = 3$
$6b = -12$
$b = -2$

2) $a = -3$
$-6b = -12$
$b = 2$

$\sqrt{5-12j} = \{3-2j,\ -3+2j\}$

d) $\sqrt[4]{-\sqrt{3}+3j}$

$z = -\sqrt{3}+3j \qquad |z| = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

$\arg z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$|t_n| = \sqrt[4]{|z|} = \sqrt[8]{12}$ $\qquad \alpha = \arctan(\sqrt{3}) = \frac{\pi}{3}$

$t_n \to \sqrt[8]{12}\, e^{\frac{\frac{2\pi}{3}+2k\pi}{4}j}$

$\left\{ \sqrt[8]{12}\,e^{\frac{\pi}{6}j},\ \sqrt[8]{12}\,e^{\frac{2\pi}{3}j},\ \sqrt[8]{12}\,e^{-\frac{\pi}{3}j},\ \sqrt[8]{12}\,e^{-\frac{5\pi}{6}j} \right\}$

$\left\{ \sqrt[8]{12}\cdot\frac{\sqrt{3}}{2} + \sqrt[8]{12}\cdot\frac{1}{2}j,\ -\frac{1}{2}\sqrt[8]{12} + j\frac{\sqrt{3}}{2}\sqrt[8]{12},\ \frac{1}{2}\sqrt[8]{12} - j\frac{\sqrt{3}}{2}\sqrt[8]{12},\ -\frac{\sqrt{3}}{2}\sqrt[8]{12} - j\frac{1}{2}\sqrt[8]{12} \right\}$

e) $\sqrt[4]{8j}$

$z = 8j = 8e^{\frac{\pi}{2}j}$

$t_n \to \sqrt[4]{8}\, e^{\frac{\frac{\pi}{2}+2k\pi}{4}j} = \sqrt[4]{8}\, e^{\left(\frac{\pi}{8}+\frac{k\pi}{2}\right)j}$

$\left\{ \sqrt[4]{8}\,e^{\frac{\pi}{8}j},\ \sqrt[4]{8}\,e^{\frac{5\pi}{8}},\ \sqrt[4]{8}\,e^{\frac{-3\pi}{8}},\ \sqrt[4]{8}\,e^{\frac{-7\pi}{8}} \right\}$

f) $\sqrt[3]{16+16j}$

$z = 16+16j = 16\sqrt{2}\,e^{\frac{\pi}{4}j} = 2^{\frac{9}{2}}\,e^{\frac{\pi}{4}j}$

$t_n \to 2^{\frac{3}{2}}\, e^{\frac{\frac{\pi}{4}+2k\pi}{3}j} = 2\sqrt{2}\, e^{\frac{\pi+8k\pi}{12}j}$

$\left\{ 2\sqrt{2}\,e^{\frac{\pi}{12}},\ 2\sqrt{2}\,e^{\frac{3\pi}{4}},\ 2\sqrt{2}\,e^{-\frac{7\pi}{12}} \right\}$

2.

$z = \cos(x) + j\sin(x) \qquad z = a + bj$

$z^5 = \cos(5x) + j\sin(5x)$

$(a+bj)^5 = (a^3 + 3a^2bj + 3ab^2j^2 + b^3j^3)(a^2 + 2abj + b^2j^2)$

$(a^3 + 3a^2bj - 3ab^2 - b^3j)(a^2 - b^2 + 2abj)$

$a^5 - a^3b^2 + 2a^4bj + 3a^4bj - 3a^2b^3j + 6a^3b^2j^2 - 3a^3b^2 + 3ab^4 - 6a^2b^3j - a^2b^3j + b^5j - 2ab^4j^2$

$a^5 - a^3b^2 - 6a^3b^2 - 3a^3b^2 + 3ab^4 + 2ab^4 + 2a^4bj + 3a^4bj - 3a^2b^3j - 6a^2b^3j - a^2b^3j + b^5j$

$(a^5 - 10a^3b^2 + 5ab^4) + j(b^5 - 10a^2b^3 + 5a^4b)$

$\cos(5x) = \cos^5(x) - 10\cos^3(x)\sin^2(x) + 5\cos(x)\sin^4(x)$

$\sin(5x) = \sin^5(x) - 10\sin^3(x)\cos(x) + 5\sin(x)\cos^4(x)$

3.

a) $z^4 \cdot |z| = -8(\bar{z})^2$

dla $z \neq 0$

$r^4 e^{4\varphi j} \cdot r = 8e^{\pi j} \cdot r^2 e^{-2\varphi j}$

$r^5 e^{4\varphi j} = 8r^2 e^{(\pi - 2\varphi)j}$

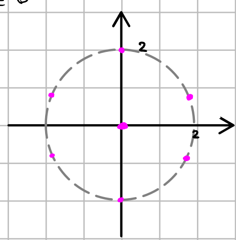
$r^5 = 8r^2 \qquad 4\varphi = \pi - 2\varphi + 2k\pi$

$r^3 = 8 \qquad 6\varphi = \pi + 2k\pi$

$r = 2 \qquad \varphi = \frac{\pi + 2k\pi}{6}$

dla $z = 0$

$0 = 0$ ✓



b) $z^2 - z = \frac{2j - 14}{j + 3}$

$\frac{-14 + 2j}{3 + j} \cdot \frac{3 - j}{3 - j} = \frac{-42 + 14j + 6j + 2}{9 + 1} = \frac{-40 + 20j}{10} = -4 + 2j$

$z^2 - z + (4 - 2j) = 0$

$\Delta = (-1)^2 - 4(1)(4 - 2j) = 1 - 4(4 - 2j) = 1 - 16 + 8j = 8j - 15$

$t = a + bj \qquad a^2 - b^2 = -15$

$t^2 = a^2 - b^2 + 2abj \qquad 2ab = 8$

$|t|^2 = a^2 + b^2 \qquad a^2 + b^2 = 17$

$|\Delta| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$

$2a^2 = 2$

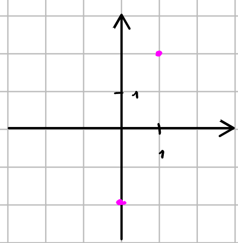
1) $a = 1$

$b = 4$

$t_0 = 1 + 4j$

$z_1 = \frac{1 - 1 - 4j}{2} = -2j$

$z_2 = \frac{1 + 1 + 4j}{2} = 1 + 2j$

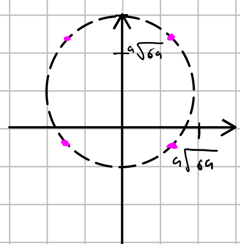


c) $(z - j)^4 = (\sqrt{3} + j)^6$

$(\sqrt{3} + j)^6 = (2e^{\frac{\pi}{6}j})^6 = 2^6 e^{\pi j} = -64$

$(z - j)^4 = -64 \qquad \sqrt[4]{64e^{\pi j}} \longrightarrow \sqrt[4]{64}\, e^{\frac{\pi + 2k\pi}{4}j}$

$z - j \in \{ \sqrt[4]{64}\, e^{\frac{\pi}{4}j}, \sqrt[4]{64}\, e^{\frac{3\pi}{4}j}, \sqrt[4]{64}\, e^{-\frac{3\pi}{4}j}, \sqrt[4]{64}\, e^{-\frac{\pi}{4}j} \}$



d) $z^5 = \bar{z}^5$

dla 0

$0^5 = \bar{0}^5$ ✓

dla $z \neq 0$

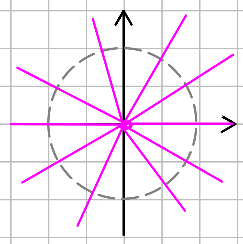
$r^5 e^{5\varphi j} = r^5 e^{-5\varphi j}$

$\forall r \in \mathbb{R}_+ \; r^5 = r^5$

$5\varphi = -5\varphi + 2k\pi$

$10\varphi = 2k\pi$

$\varphi = \frac{2k\pi}{10} = \frac{\pi}{5}k$



e) $Re(z^4(3 - j\sqrt{3})) = 0$

$\iff$

$\arg(z^4(3 - j\sqrt{3})) \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$

$Re(r^4 e^{4\varphi j} \cdot 2\sqrt{3} e^{-\frac{\pi}{6}j}) = 0$

$Re(2\sqrt{3} r^4 e^{(4\varphi - \frac{\pi}{6})j}) = 0$

$2\sqrt{3} r^4 \cos(4\varphi - \frac{\pi}{6}) = 0$

$\cos(4\varphi - \frac{\pi}{6}) = 0$

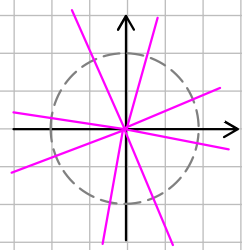
$\arg(z^4(3 - j\sqrt{3})) = 4\arg(z) + \arg(3 - j\sqrt{3}) + 2k\pi$

$\arg(3 - j\sqrt{3}) = \arctan(-\frac{\sqrt{3}}{3}) = -\frac{\pi}{6}$

$4\varphi - \frac{\pi}{6} + 2k\pi = -\frac{\pi}{2} \quad \vee \quad 4\varphi - \frac{\pi}{6} + 2k\pi = \frac{\pi}{2}$

$4\varphi = -\frac{\pi}{3} - 2k\pi \qquad 4\varphi = \frac{2\pi}{3} - 2k\pi$

$\varphi = -\frac{\pi}{12} - \frac{k\pi}{2} \qquad \varphi = \frac{\pi}{6} - \frac{k\pi}{2}$

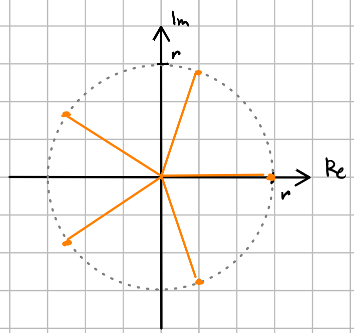


$\varphi \in \{\frac{2\pi}{3}, \frac{\pi}{6}, -\frac{\pi}{3}, -\frac{5\pi}{6}, \frac{11\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{12}, -\frac{7\pi}{12}\}$

f) $z^4 = (jz + 1)^4 \qquad z^4 - (jz+1)^4 = 0$

$(z^2 - (jz+1)^2)(z^2 + (jz+1)^2) = 0$

4.



Wielokąt można przesunąć, tak że środek okręgu znajduje się w $z = 0 + 0j$

Wielokąt można obrócić tak, że 1 z wierzchołków to $z = r + 0j$

Pozostałe wierzchołki mają postać $r\omega_k$

Dwa sąsiednie wierzchołki:

$re^{\varphi j}$ i $re^{(\varphi + \frac{2\pi}{n})j}$

Obwód $\rightarrow \quad n|re^{(\varphi + \frac{2\pi}{n})j} - re^{\varphi j}| = n|re^{\varphi j}(e^{\frac{2\pi}{n}j} - 1)| = nr|e^{\frac{2\pi}{n}j} - 1|$

$nr|\cos(\frac{2\pi}{n}) - 1 + j\sin(\frac{2\pi}{n})|$

$nr\sqrt{(\cos(\frac{2\pi}{n}) - 1)^2 + \sin^2(\frac{2\pi}{n})}$

$nr\sqrt{\cos^2(\frac{2\pi}{n}) - 2\cos(\frac{2\pi}{n}) + 1 + \sin^2(\frac{2\pi}{n})}$

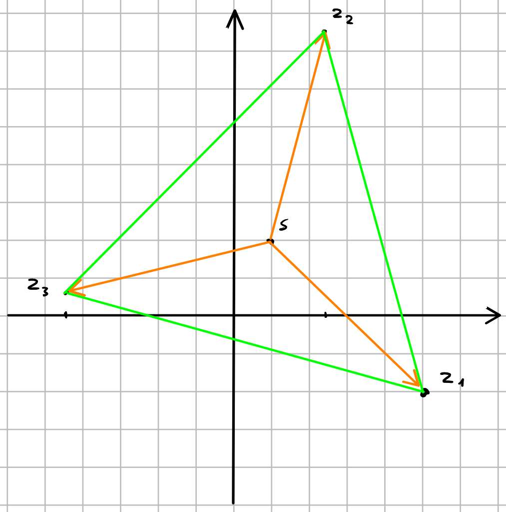
$nr\sqrt{2 - 2\cos(\frac{2\pi}{n})} = nr\sqrt{4\sin^2(\frac{\pi}{n})} = 2nr\sin(\frac{\pi}{n})$

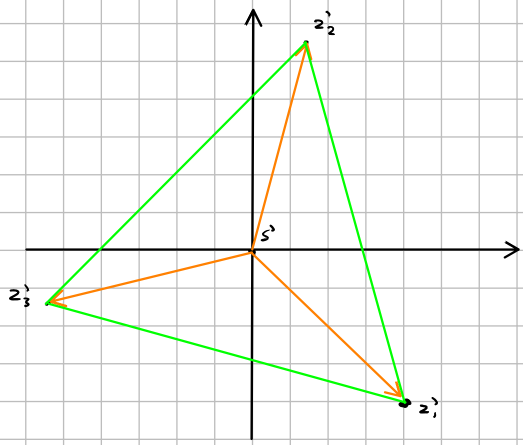
$2 - 2\cos(\frac{2\pi}{n}) = 2 - 2(2\cos^2(\frac{\pi}{n}) - 1)$

$2 - 4\cos^2(\frac{\pi}{n}) + 2 = 4(1 - \cos^2(\frac{\pi}{n})) = 4\sin^2(\frac{\pi}{n})$

Pole $\rightarrow \quad n \cdot \frac{1}{2} \cdot |re^{\varphi j}| \cdot |re^{(\varphi + \frac{2\pi}{n})j}| \cdot \sin(\frac{2\pi}{n}) = \frac{1}{2} nr^2 \sin(\frac{2\pi}{n})$

5.





$$z_n' = z_n - s \qquad z_1' = 5 - 2j - 1 - 2j = 4 - 4j = 4\sqrt{2}\, e^{-\frac{\pi}{4} j}$$

$$z_2' = z_1' \cdot e^{\frac{2\pi}{3} j} = 4\sqrt{2}\, e^{\frac{5\pi}{12} j} = 4\sqrt{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}} + j\,\frac{1+\sqrt{3}}{2\sqrt{2}}\right) = 2\sqrt{3} - 2 + j(2 + 2\sqrt{3})$$

$$z_3' = z_1' \cdot e^{-\frac{2\pi}{3} j} = 4\sqrt{2}\, e^{-\frac{11\pi}{12} j} = 4\sqrt{2}\left(-\frac{1+\sqrt{3}}{2\sqrt{2}} - j\,\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = -2 - 2\sqrt{3} - j(2\sqrt{3} - 2)$$

$$z_2 = z_2' + s = 2\sqrt{3} - 2 + 2j + 2\sqrt{3}j + 1 + 2j = (2\sqrt{3} - 1) + j(4 + 2\sqrt{3})$$

$$z_3 = z_3' + s = -2 - 2\sqrt{3} - 2\sqrt{3}j + 2j + 1 + 2j = (-1 - 2\sqrt{3}) + j(4 - 2\sqrt{3})$$

6.

$$\sqrt[8]{7 - 3j}$$

$z = 7 - 3j = re^{\varphi j}$ niech $t_0^8 = z$ g

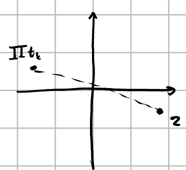
$$t_k = \sqrt[8]{r}\, e^{\frac{\varphi + 2k\pi}{8} j}$$

$$\prod_{k=0}^{7} t_k = \sqrt[8]{r}\, e^{\frac{\varphi}{8} j} \cdot \sqrt[8]{r}\, e^{\frac{\varphi + 2\pi}{8} j} \cdot \ldots \cdot \sqrt[8]{r}\, e^{\frac{\varphi + 14\pi}{8} j}$$

$$= (\sqrt[8]{r})^8 \cdot e^{\frac{\varphi + \varphi + 2\pi + \varphi + 4\pi + \ldots + \varphi + 14\pi}{8} j}$$

$$= r\, e^{\frac{8\varphi + \sum_{i=0}^{7} 2\pi i}{8} j} = r\, e^{\frac{8\varphi + 56\pi}{8} j} = r\, e^{(\varphi + 7\pi) j} = r\, e^{(\varphi + \pi) j}$$

$$= -z = -7 + 3j$$



$$\sum_{k=0}^{7} t_k = \sqrt[8]{r}\, e^{\frac{\varphi}{8} j} + \sqrt[8]{r}\, e^{\frac{\varphi + 2\pi}{8} j} + \ldots + \sqrt[8]{r}\, e^{\frac{\varphi + 14\pi}{8} j}$$

$$= \sqrt[8]{r}\, e^{\frac{\varphi}{8} j}\left(e^{0j} + e^{\frac{2\pi}{8} j} + e^{\frac{4\pi}{8} j} + \ldots + e^{\frac{14\pi}{8} j}\right)$$

$$= \sqrt[8]{r}\, e^{\frac{\varphi}{8} j} \cdot 0 = 0$$