

1.

$$d) \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim_{x \rightarrow \infty} \frac{e^x (1 - \frac{1}{e^{2x}})}{e^x (1 + \frac{1}{e^{2x}})} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = 1$$

$$b) \lim_{x \rightarrow 0} \frac{\cos(x) - 1 + \frac{1}{2}x^2}{x^4} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{\frac{d}{dx} [\cos(x) - 1 + \frac{1}{2}x^2]}{\frac{d}{dx} x^4} = \lim_{x \rightarrow 0} \frac{-\sin(x) - 0 + x}{4x^3} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{-\cos(x) + 1}{12x^2} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{\sin(x)}{24x} = \frac{1}{24}$$

$$c) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan(x)}{\ln(\frac{\pi}{2} - x)} \stackrel{\frac{\infty}{0}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\frac{\pi}{2} - x} \cdot (-1)} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos^2(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{2\cos(x) \cdot (-\sin(x))} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{-\sin(2x)} = -\infty$$

$$d) \lim_{x \rightarrow -\infty} x^2 e^{2x} = \lim_{x \rightarrow -\infty} \frac{x^2}{e^{-2x}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow -\infty} \frac{2x}{-2e^{-2x}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow -\infty} \frac{2}{4e^{-2x}} = 0$$

$$e) \lim_{x \rightarrow 0^+} (\arcsin(x))^{\tan(x)} = \lim_{x \rightarrow 0^+} e^{\tan(x) \ln(\arcsin(x))} = e^0 = 1$$

$$\lim_{x \rightarrow 0^+} \tan(x) \cdot \ln(\arcsin(x)) = \lim_{x \rightarrow 0^+} \frac{\ln(\arcsin(x))}{\cot(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{\arcsin(x)} \cdot \frac{1}{\sqrt{1-x^2}}}{\frac{-1}{\sin^2(x)}} = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{1-x^2}} \cdot \frac{\sin^2(x)}{\arcsin(x)} = -1 \cdot 0 = 0$$

$$\lim_{x \rightarrow 0^+} \frac{\sin^2(x)}{\arcsin(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0^+} \frac{2\sin(x)\cos(x)}{\frac{1}{\sqrt{1-x^2}}} = 0$$

$$f) \lim_{x \rightarrow \infty} (\ln(2x))^{\log_x e} = \lim_{x \rightarrow \infty} e^{\ln[(\ln(2x))^{\frac{1}{\ln(x)}}]} = \lim_{x \rightarrow \infty} e^{\frac{\ln(\ln(2x))}{\ln(x)}} = e^0 = 1$$

$$\log_x(e) = \frac{\ln(e)}{\ln(x)} = \frac{1}{\ln(x)}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(\ln(2x))}{\ln(x)} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln(2x)} \cdot \frac{1}{2x} \cdot 2}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{2x}{2x \ln(x)} = \lim_{x \rightarrow \infty} \frac{1}{\ln(x)} = 0$$

$$g) \lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} e^{\frac{1}{x^2} \ln\left(\frac{\sin(x)}{x}\right)} = e^{-\frac{1}{6}} = \frac{1}{6\sqrt{e}}$$

$$\lim_{x \rightarrow 0} \frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{x \cdot \frac{\cos(x)x - \sin(x)}{x^2}}{2x} = \lim_{x \rightarrow 0} \frac{x \cos(x) - \sin(x)}{2x^2 \sin(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{-x \sin(x) + \cos(x) - \cos(x)}{2x^2 \cos(x) + 4x \sin(x)}$$

$$\lim_{x \rightarrow 0} \frac{-x \sin(x)}{2x^2 \cos(x) + 4x \sin(x)} = \lim_{x \rightarrow 0} \frac{-\sin(x)}{2x \cos(x) + 4 \sin(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{-\cos(x)}{-2x \sin(x) + 2 \cos(x) + 4 \cos(x)} = \frac{-1}{0 + 2 + 4} = -\frac{1}{6}$$

$$h) \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) = \lim_{x \rightarrow 0} \frac{\sin(x) - x}{x \sin(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x \cos(x) + \sin(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{-\sin(x)}{\cos(x) - x \sin(x) + \cos(x)} = \frac{0}{1 - 0 + 1} = \frac{0}{2} = 0$$

$$i) \lim_{x \rightarrow \infty} (x - \ln(x)) = \lim_{x \rightarrow \infty} (\ln(e^x) - \ln(x)) = \lim_{x \rightarrow \infty} \ln\left(\frac{e^x}{x}\right)$$

$$j) \lim_{x \rightarrow -\infty} x - (\pi + 2 \arctan(x)) = \lim_{x \rightarrow -\infty} \frac{\pi + 2 \arctan(x)}{\frac{1}{x}} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow -\infty} \frac{\frac{2}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{-2x^2}{1+x^2} = \lim_{x \rightarrow -\infty} \frac{-2}{(1+\frac{1}{x^2})} = -2$$

$$k) \lim_{x \rightarrow 4^+} \left(\frac{2x+1}{3x-1} \right)^{\frac{1}{x-4}} = 0$$

$$\lim_{x \rightarrow 4^+} \frac{4}{x-4} = +\infty \quad \lim_{x \rightarrow 4^+} \frac{2x+1}{3x-1} = \frac{9}{11} \quad \left| \frac{9}{11} \right| < 1$$

$$l) \lim_{x \rightarrow 0^+} x \sqrt[3]{3x+1} = \lim_{x \rightarrow 0^+} (3x+1)^{\frac{1}{3}} = \lim_{x \rightarrow 0^+} \left(1 + 3x \right)^{\frac{1}{3x} \cdot 3} = e^3$$

$$m) \lim_{x \rightarrow 0^+} (\tan(x))^{x^2-x} = \lim_{x \rightarrow 0^+} e^{(x^2-x) \ln(\tan(x))}$$

$$\lim_{x \rightarrow 0^+} (x^2-x) \ln(\tan(x)) = \lim_{x \rightarrow 0^+} \frac{\ln(\tan(x))}{\frac{1}{x^2-x}} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{\tan(x)} \cdot \frac{1}{\cos^2(x)}}{\frac{-2x+1}{(x^2-x)^2}} = \lim_{x \rightarrow 0^+} \frac{(x^2-x)^2}{(-2x+1) \tan(x) \cos^3(x)}$$

$$\lim_{x \rightarrow 0^+} \frac{x^2(x-1)^2}{(-2x+1) \frac{\sin(x)}{\cos(x)} \cos^3(x)} = \lim_{x \rightarrow 0^+} \frac{x}{\sin(x)} \cdot \frac{(x-1)^2}{(-2x+1) \cos(x)} \cdot x = 0$$

$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$
 $1 \quad \quad \quad 1 \quad \quad \quad 0$

$$n) \lim_{x \rightarrow 0} \frac{2e^x - 2e^{-x} - 4x}{\sin(x) - x} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{2e^x + 2e^{-x} - 4}{\cos(x) - 1} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{2e^x - 2e^{-x}}{-\sin(x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{2e^x + 2e^{-x}}{-\cos(x)} = \frac{2+2}{-1} = -4$$

$$o) \lim_{x \rightarrow \infty} \left(\frac{\ln(x)}{x} \right)^{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} e^{\frac{1}{x^2} \ln\left(\frac{\ln(x)}{x}\right)}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} \stackrel{\frac{\infty}{\infty}}{=} \frac{1}{1} = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} \ln\left(\frac{\ln(x)}{x}\right) = \lim_{x \rightarrow \infty} \frac{\ln\left(\frac{\ln(x)}{x}\right)}{x^2} \stackrel{\frac{0}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\frac{\ln(x)}{x}} \cdot \frac{\frac{1}{x} \cdot x - \ln(x)}{x^2}}{2x} = \lim_{x \rightarrow \infty} \frac{1 - \ln(x)}{2x^2 \ln(x)} \stackrel{\frac{\infty}{\infty}}{=} \frac{0}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{-\frac{1}{x}}{4x \ln(x) + 2x} = \lim_{x \rightarrow \infty} \frac{-1}{4x^2 \ln(x) + 2x} = 0$$

$$p) \lim_{x \rightarrow \infty} \left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2} \right)^x = \lim_{x \rightarrow \infty} e^{x \ln\left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right)}$$

$$\lim_{x \rightarrow \infty} x \ln\left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right) = \lim_{x \rightarrow \infty} \frac{\ln\left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right)}{\frac{1}{x}} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow \infty} \frac{\frac{2}{2^{\frac{1}{x}+3^{\frac{1}{x}}}} \cdot \frac{-\frac{1}{x^2} \ln(2) 2^{\frac{1}{x}} - \frac{1}{x^2} \ln(3) 3^{\frac{1}{x}}}{2}}{\frac{-1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(\frac{-1}{x^2}\right) [\ln(2) 2^{\frac{1}{x}} + \ln(3) 3^{\frac{1}{x}}]}{\left(\frac{-1}{x^2}\right) (2^{\frac{1}{x}} + 3^{\frac{1}{x}})} = \frac{\ln(2) + \ln(3)}{2} = \frac{1}{2} (\ln(4) + \ln(9)) = \ln(\sqrt{6})$$

$$q) \lim_{x \rightarrow \pi^+} (1 + 2 \sin(x))^{\frac{1}{\pi-x}} = \lim_{x \rightarrow \pi^+} \left[(1 + 2 \sin(x))^{\frac{1}{2 \sin(x)}} \right]^{\frac{2 \sin(x)}{\pi-x}} = \lim_{x \rightarrow \pi^+} \left[(1 + 2 \sin(x))^{\frac{1}{2 \sin(x)}} \right]^{\frac{\sin(\pi-x) \cdot 2}{\pi-x}} = e^{1 \cdot 2} = e^2$$

$$r) \lim_{x \rightarrow 1^+} \left(\frac{x+1}{2x} \right)^{\frac{3}{x-1}} = \lim_{x \rightarrow 1^+} \left(\frac{2x+1-x}{2x} \right)^{\frac{3}{x-1}} = \lim_{x \rightarrow 1^+} \left[\left(1 + \frac{1-x}{2x} \right)^{\frac{2x}{x-1}} \right]^{\frac{3}{2x}} = e^{-\frac{3}{2}}$$

2.

$$f(x) = \frac{x^2+1}{x-1} \quad \frac{df}{dx} = \frac{2x(x-1) - (x^2+1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}$$

$$x+y=5 \rightarrow y = -x+5$$

$$D = \mathbb{R} \setminus \{1\}$$

$$f'(x) = -1 \Leftrightarrow \frac{x^2 - 2x - 1}{(x-1)^2} = -1$$

$$x^2 - 2x - 1 = -x^2 + 2x - 1$$

$$2x^2 - 4x = 0$$

$$x(x-2) = 0 \quad x = 0 \quad \vee \quad x = 2$$

$$1) \quad y = f'(0)(x-0) + f(0) = -x - 1$$

$$2) \quad y = f'(2)(x-2) + f(2) = -(x-2) + 5 = -x + 7$$

3.

a) $f(x) = \arcsin\left(\frac{1+x}{1-x}\right)$

$$D: \frac{1+x}{1-x} \geq -1 \quad \wedge \quad \frac{1+x}{1-x} \leq 1$$

$$\frac{1+x+1-x}{1-x} \geq 0 \quad \wedge \quad \frac{1+x-1-x}{1-x} \leq 0$$

$$\frac{2}{1-x} \geq 0 \quad 2x(1-x) \leq 0$$

$$2(1-x) \geq 0 \quad x \in (-\infty, 0] \cup [1, \infty)$$

$$x \leq 0$$

$$D_f = (-\infty, 0]$$

$$\lim_{x \rightarrow -\infty} \frac{\arcsin\left(\frac{1+x}{1-x}\right)}{x} = 0$$

$$\lim_{x \rightarrow 0} \arcsin\left(\frac{1+x}{1-x}\right) = \arcsin(1) = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \frac{1+x}{1-x} = \lim_{x \rightarrow -\infty} \frac{x(1+\frac{1}{x})}{x(-1+\frac{1}{x})} = -1$$

brak asymptot pionowych

$$\lim_{x \rightarrow -\infty} \arcsin\left(\frac{1+x}{1-x}\right) = \lim_{t \rightarrow -1^+} \arcsin(t) = -\frac{\pi}{2}$$

$$y = -\frac{\pi}{2} \text{ asymptota pozioma lewostronna}$$

b) $f(x) = \frac{1}{e^{x^2-4}} = \frac{1}{e^{(x-2)(x+2)}}$

$$D_f = \mathbb{R} \setminus \{-2, 2\}$$

$$\lim_{x \rightarrow -2^-} f(x) = \left\{ e^{\frac{1}{-4 \cdot 0^-}} = e^{-\infty} \right\} = +\infty$$

$$\lim_{x \rightarrow 2^-} e^{\frac{1}{(x-2)(x+2)}} = \left\{ e^{\frac{1}{0^- \cdot 4}} = e^{-\infty} \right\} = 0$$

$$\lim_{x \rightarrow -2^+} f(x) = \left\{ e^{\frac{1}{-4 \cdot 0^+}} = e^{-\infty} \right\} = 0$$

$$\lim_{x \rightarrow 2^+} f(x) = \left\{ e^{\frac{1}{0^+ \cdot 4}} = e^{+\infty} \right\} = +\infty$$

$x = -2$ asymptota pionowa lewostronna

$x = 2$ asymptota pionowa prawostronna

$$\lim_{x \rightarrow +\infty} \frac{e^{\frac{1}{x^2-4}}}{x} = \left\{ \frac{e^0}{\infty} \rightarrow \frac{1}{\infty} \right\} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{e^{\frac{1}{x^2-4}}}{x} = 0$$

$$\lim_{x \rightarrow +\infty} e^{\frac{1}{x^2-4}} = e^0 = 1$$

$$\lim_{x \rightarrow -\infty} e^{\frac{1}{x^2-4}} = e^0 = 1$$

$y = 1$ asymptota pozioma obustronna

c) $f(x) = \sqrt{x^2+2x} + x = \sqrt{x(x+2)} + x = |x|\sqrt{1+\frac{2}{x}} + x$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{x\sqrt{1+\frac{2}{x}} + x}{x} = \lim_{x \rightarrow +\infty} \sqrt{1+\frac{2}{x}} + 1 = 2$$

$$\lim_{x \rightarrow +\infty} [f(x) - 2x] = \lim_{x \rightarrow +\infty} [\sqrt{x^2+2x} - x] = \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2+2x} - x)(\sqrt{x^2+2x} + x)}{\sqrt{x^2+2x} + x} = \lim_{x \rightarrow +\infty} \frac{x^2+2x - x^2}{x(\sqrt{1+\frac{2}{x}} + 1)} = \frac{2}{2} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{1+\frac{2}{x}} + x}{x} = \lim_{x \rightarrow -\infty} [-\sqrt{1+\frac{2}{x}} + 1] = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2+2x} + x)(\sqrt{x^2+2x} - x)}{\sqrt{x^2+2x} - x} = \lim_{x \rightarrow -\infty} \frac{x^2+2x - x^2}{-x(\sqrt{1+\frac{2}{x}} + 1)} = \frac{2}{-2} = -1$$

$$x(x+2) \geq 0 \quad \begin{matrix} \diagup \\ -2 \end{matrix} \quad \begin{matrix} \diagdown \\ 0 \end{matrix}$$

$$D = (-\infty, -2] \cup [0, +\infty)$$

brak asymptot pionowych

$y = 2x + 1$ asymptota ukośna prawostronna

$y = -1$ asymptota pozioma lewostronna

$$\lim_{x \rightarrow -2^-} f(x) = f(-2) = -2$$

$$\lim_{x \rightarrow 0^+} f(x) = f(0) = 0$$

$$f(x) = x + \sin(x) \quad D = \mathbb{R}$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{x + \sin(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\sin(x)}{x} + 1 = 0 + 1 = 1$$

$$\lim_{x \rightarrow +\infty} x + \sin(x) - x = \lim_{x \rightarrow +\infty} \sin(x) \text{ nie istnieje}$$