Zestaw 0

1. $w = a + bj \quad z = c + dj$

a) $|w+z|^2 \stackrel{?}{=} |w|^2 + |z|^2 + 2\mathrm{Re}(w\bar{z})$

$LHS = |a+bj+c+dj|^2 = \left(\sqrt{(a+c)^2+(b+d)^2}\right)^2 = a^2+b^2+c^2+d^2+2ac+2bd$

$RHS = |a+bj|^2 + |c+dj|^2 + 2\mathrm{Re}\left((a+bj)(c-dj)\right)$

$= \left(\sqrt{a^2+b^2}\right)^2 + \left(\sqrt{c^2+d^2}\right)^2 + 2\mathrm{Re}\left(ac - adj + cbj + bd\right)$

$= a^2+b^2+c^2+d^2+2ac+2bd = LHS$

Alternatywny sposób

$|z|^2 = z\cdot\bar{z} \qquad 2\mathrm{Re}(w) = w + \bar{w}$

$|w|^2 + |z|^2 + 2\mathrm{Re}(w\bar{z}) = w\cdot\bar{w} + z\cdot\bar{z} + w\bar{z} + z\bar{w}$

$= (z+w)(\bar{z}+\bar{w}) = (z+w)\overline{(z+w)} = |w+z|^2$

b) $|w-z|^2 = |w|^2 + |z|^2 - 2\mathrm{Re}(w\bar{z})$

$LHS = |a+bj-c-dj|^2 = (a-c)^2 + (b-d)^2 = a^2+b^2+c^2+d^2-2ac-2bd$

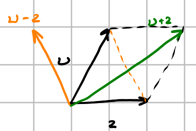
$RHS = a^2+b^2+c^2+d^2-2ac-2bd = LHS$

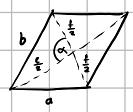
$|w+(-z)|^2 = |w|^2 + |-z|^2 + 2\mathrm{Re}(w\cdot\overline{(-z)})$

$= |w|^2 + |z|^2 - 2\mathrm{Re}(w\bar{z})$

c) $|w+z|^2 + |w-z|^2 = |w|^2+|z|^2+2\mathrm{Re}(w\bar{z}) + |w|^2+|z|^2 - 2\mathrm{Re}(w\bar{z}) = 2|w|^2 + 2|z|^2$

tożsamość równoległoboku




$b^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 + 2\cdot\frac{1}{4}ef\cos(\alpha)$

$b^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 + \frac{1}{2}ef\cos(\alpha)$

$a^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 + \frac{1}{2}ef\cos(\pi-\alpha)$

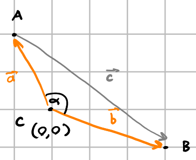
$a^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 - \frac{1}{2}ef\cos(\alpha)$

$a^2 + b^2 = \frac{1}{2}e^2 + \frac{1}{2}f^2$

$e^2 + f^2 = 2a^2 + 2b^2$

$|w+z|^2 + |w-z|^2 = 2|w|^2 + 2|z|^2$

2.




$c^2 = a^2 + b^2 - 2ab\cos(\alpha)$

$\cos(\alpha) = \frac{a^2+b^2-c^2}{2ab} = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a}-\vec{b}|^2}{2|\vec{a}|\cdot|\vec{b}|} = \frac{|\vec{a}|^2+|\vec{b}|^2-|\vec{a}|^2-|\vec{b}|^2+2\mathrm{Re}(a\bar{b})}{2|\vec{a}||\vec{b}|}$

$= \frac{\mathrm{Re}\left((x_A+y_Aj)(x_B-y_Bj)\right)}{|\vec{a}||\vec{b}|} = \frac{x_Ax_B + y_Ay_B}{|\vec{a}|\cdot|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

$\vec{c} = \vec{b} - \vec{a} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$

$|\vec{c}|^2 = |\vec{b}-\vec{a}|^2 = x_A^2 + x_B^2 + y_A^2 + y_B^2 - 2x_Ax_B - 2y_Ay_B$

$|wz| = |w||z| = |w||\bar{z}| = |w\bar{z}|$

$\cos(\alpha) = \frac{\mathrm{Re}(w\bar{z})}{|wz|}$

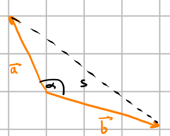
$\sin(\alpha) = \sqrt{\frac{|wz|^2 - \mathrm{Re}(w\bar{z})^2}{|wz|^2}} = \sqrt{\frac{|w\bar{z}|^2 - \mathrm{Re}(w\bar{z})^2}{|wz|^2}}$

$= \sqrt{\frac{\mathrm{Re}(w\bar{z})^2 + \mathrm{Im}(w\bar{z})^2 - \mathrm{Re}(w\bar{z})^2}{|wz|^2}} = \frac{|\mathrm{Im}(w\bar{z})|}{|wz|}$

$S = \frac{1}{2}ab\sin(\alpha)$

$\sin(\alpha) = \sqrt{1-\cos^2(\alpha)} = \sqrt{\frac{(x_A^2+y_A^2)(x_B^2+y_B^2) - (x_Ax_B+y_Ay_B)^2}{(x_A^2+y_A^2)(x_B^2+y_B^2)}}$

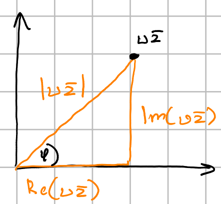
$\alpha \in [0, \pi]$




$= \frac{\sqrt{\cancel{x_A^2x_B^2} + x_A^2y_B^2 + x_B^2y_A^2 + \cancel{y_A^2y_B^2} - \cancel{x_A^2x_B^2} - \cancel{y_A^2y_B^2} - 2x_Ax_By_Ay_B}}{\sqrt{x_A^2+y_A^2}\cdot\sqrt{x_B^2+y_B^2}}$

$= \frac{\sqrt{(x_Ay_B - x_By_A)^2}}{\sqrt{x_A^2+y_A^2}\cdot\sqrt{x_B^2+y_B^2}} = \frac{|x_Ay_B - x_By_A|}{\sqrt{x_A^2+y_A^2}\cdot\sqrt{x_B^2+y_B^2}}$

3. $\sin(\alpha) = \frac{|x_Ay_B - x_By_A|}{\sqrt{x_A^2+y_A^2}\cdot\sqrt{x_B^2+y_B^2}}$



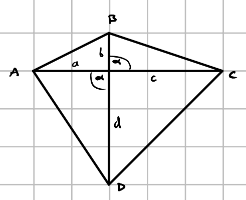

$S = \frac{1}{2}ab\sin(\alpha) = \frac{1}{2}\sqrt{x_A^2+y_A^2}\sqrt{x_B^2+y_B^2}\cdot\frac{|x_Ay_B - x_By_A|}{\sqrt{x_A^2+y_A^2}\cdot\sqrt{x_B^2+y_B^2}} = \frac{1}{2}|x_Ay_B - x_By_A|$

$S = \frac{1}{2}ab\sin(\alpha) = \frac{1}{2}|w|\cdot|z|\cdot\frac{\mathrm{Im}(w\bar{z})}{|wz|} = \frac{1}{2}\left|\mathrm{Im}\left((x_A+y_Aj)(x_B-y_Bj)\right)\right| = \frac{1}{2}|-x_Ay_B + x_By_A| = \frac{1}{2}|x_Ay_B - x_By_A|$

$S = \frac{1}{2}\left|\begin{vmatrix} x_A & x_B \\ y_A & y_B \end{vmatrix}\right|$

4.

$|AB|^2 + |CD|^2 = |AD|^2 + |BC|^2 \iff AC \perp BD$




$|AB|^2 = a^2 + b^2 + 2ab\cos(\alpha)$ $\quad |AD|^2 = a^2 + d^2 - 2ad\cos(\alpha)$

$|CD|^2 = c^2 + d^2 + 2cd\cos(\alpha)$ $\quad |BC|^2 = b^2 + c^2 - 2bc\cos(\alpha)$

$a^2 + b^2 + c^2 + d^2 + 2\cos(\alpha)(ab+cd) = a^2 + b^2 + c^2 + d^2 - 2\cos(\alpha)(ad+bc)$

$\cos(\alpha)(ab+cd) = -\cos(\alpha)(ad+bc)$

$1^\circ$ $\cos(\alpha) = 0$ :

$0 = 0$ ✓

$\cos(\alpha) = 0 \iff AC \perp BD$

$2^\circ$ $\cos(\alpha) \neq 0$ :

$\underbrace{ab+cd}_{>0} = -\underbrace{(ad+bc)}_{>0}$

sprzeczność

$$|a+d|^2 + |b+c|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2\mathrm{Re}(a\bar{d}) + 2\mathrm{Re}(b\bar{c})$$
$$|a+b|^2 + |c+d|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2\mathrm{Re}(a\bar{b}) + 2\mathrm{Re}(c\bar{d})$$

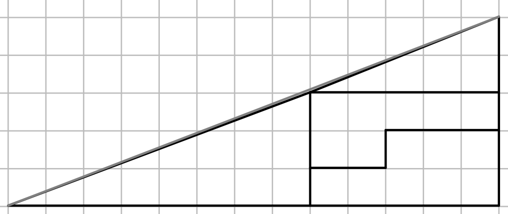
5.

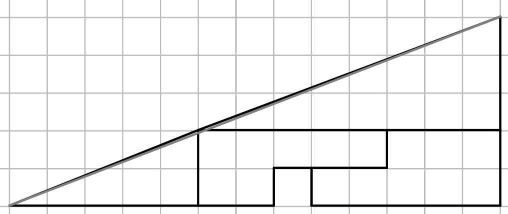
$(0,0)$ $(8,3)$ $(13,5)$

$S = \frac{1}{2}|x_A y_B - x_B y_A|$

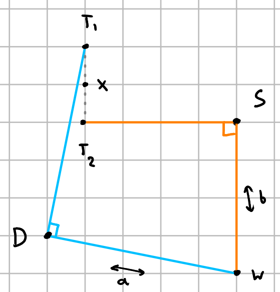
$S = \frac{1}{2}|8\cdot 5 - 13\cdot 3| = \frac{1}{2}$ – różnica pola trójkąta i czworokąta wyglądającego na trójkąt

stąd bierze się iluzja w zagadce brakującego kwadratu





6.




$D = W + a$ ↷ obrót o 90° ↻

$T_1 = D + a\cdot(-j)$

$S = W + b$ ↷ obrót o 90° ↺

$T_2 = S + b\cdot j$

$$X = \frac{T_1 + T_2}{2} = \frac{W + a - aj + W + b + bj}{2}$$
$$= \frac{W + (D-W) - (D-W)j + W + (S-W) + (S-W)j}{2}$$
$$= \frac{D - Dj + Wj + S + Sj - Wj}{2} = \frac{D(1-j) + S(1+j)}{2}$$