1.

a) $2\arctan(x) + \arcsin\left(\frac{2x}{x^2+1}\right) = \pi$ dla $x \geq 1$

$\frac{2x}{x^2+1} \geq -1$

$\frac{2x+x^2+1}{x^2+1} \geq 0$

$(x+1)^2(x^2+1) \geq 0$

$x \neq -1$

$\frac{2x}{x^2+1} \leq 1$

$\frac{2x - x^2 - 1}{x^2+1} \leq 0$

$\frac{x^2-2x+1}{x^2+1} \geq 0$

$(x-1)^2(x^2+1) \geq 0$

$x \neq 1$

$D = \mathbb{R} \setminus \{-1, 1\}$

dla $x \geq 1$

$$\frac{df}{dx} = 2 \cdot \frac{1}{1+x^2} + \frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$$

$$\frac{df}{dx} = \frac{2}{1+x^2} + \frac{1}{\sqrt{\frac{x^4+2x^2+1-4x^2}{(x^2+1)^2}}} \cdot \frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$\frac{df}{dx} = \frac{2}{1+x^2} + \frac{1}{\sqrt{\frac{(x^2-1)^2}{(x^2+1)^2}}} \cdot \frac{-2x^2+2}{(x^2+1)^2}$$

$$\frac{df}{dx} = \frac{2}{x^2+1} + \frac{\cancel{x^2+1}}{x^2-1} \cdot \frac{-2x^2+2}{(x^2+1)^{\cancel{2}}}$$

$$\frac{df}{dx} = \frac{2x^2-2 \quad -2x^2+2}{(x^2+1)(x^2-1)} = 0$$

$f(1) = 2\arctan(1) + \arcsin(1) = 2 \cdot \frac{\pi}{4} + \frac{\pi}{2} = \pi$

$f(1) = \pi \wedge \forall_{x \geq 1}\ f'(x) = 0 \implies \forall_{x \geq 1}\ f(x) = \pi$

funkcja jest stała na $[1, +\infty)$

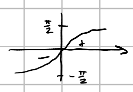
b) $2x\arctan(x) \geq \ln(x^2+1)$ dla $x \in \mathbb{R}$

$f(x) = 2x\arctan(x) - \ln(x^2+1) \geq 0$ dla $x \in \mathbb{R}$

$$\frac{df}{dx} = 2x \cdot \frac{1}{1+x^2} + 2\arctan(x) - \frac{1}{x^2+1} \cdot 2x$$

$$\frac{df}{dx} = \frac{2x}{x^2+1} - \frac{2x}{x^2+1} + 2\arctan(x) = 2\arctan(x)$$

$2\arctan(x) = 0 \iff x = 0$



minimum lokalne

$f(0) = 0 - \ln(1) = 0$

więc $\forall_{x \in \mathbb{R}}\ f(x) \geq 0 \iff \forall_{x \in \mathbb{R}}\ 2x\arctan(x) \geq \ln(x^2+1)$

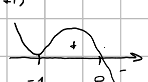
c) $\frac{x}{x+1} \leq \ln(x+1) \leq x$ dla $x \geq 0$

$f(x) = \frac{x}{x+1} - \ln(x+1) \leq 0$

$g(x) = \ln(x+1) - x \leq 0$

$$\frac{df}{dx} = \frac{x+1-x}{(x+1)^2} - \frac{1}{x+1} \cdot 1 = \frac{1}{(x+1)^2} - \frac{x+1}{(x+1)^2} = \frac{-x}{(x+1)^2}$$

$\frac{-x}{(x+1)^2} = 0 \iff x = 0$



maksimum lokalne i globalne

$f(0) = 0 - 0 = 0$

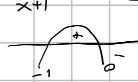
$f \searrow$ w $[0, +\infty)$

$\forall_{x \geq 0}\ f(x) \leq 0 \iff \forall_{x \geq 0}\ \frac{x}{x+1} \leq \ln(x+1)$

$$\frac{dg}{dx} = \frac{1}{x+1} - 1 = \frac{\cancel{1} - x \cancel{-1}}{x+1} = \frac{-x}{x+1}$$

$\frac{-x}{x+1} = 0 \iff x = 0$

$x = 0$ $\quad f \searrow$ w $[0, +\infty)$



maksimum globalne

$g(0) = 1 - 1 = 0$

$\forall_{x \geq 0}\ g(x) \leq 0 \iff \forall_{x \geq 0}\ \ln(x+1) \leq x$

d) $\ln(x) < 2\sqrt{x} \quad \forall_{x>0}$

$$\frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$f(x) = \ln(x) - 2\sqrt{x}$

$$\frac{df}{dx} = \frac{1}{x} - \frac{1}{\sqrt{x}} = \frac{1}{x} - \frac{\sqrt{x}}{x} = \frac{1-\sqrt{x}}{x}$$

$$f'(x) = 0 \iff \frac{1-\sqrt{x}}{x} = 0 \iff x = 1$$

$$f''(x) = \frac{-\frac{1}{2\sqrt{x}}\, x - (1-\sqrt{x})}{x^2} = \frac{-\frac{1}{2}\sqrt{x} - 1 + \sqrt{x}}{x^2} = \frac{\frac{1}{2}\sqrt{x} - 1}{x^2}$$

$f''(1) = \frac{\frac{1}{2}-1}{1} = -\frac{1}{2} < 0$ maksimum lokalne w $x=1$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln(x) - 2\sqrt{x} = \{-\infty - \infty\} = -\infty$$

dla $x>0$ $\quad f'(x) < 0 \iff \frac{1-\sqrt{x}}{x} < 0 \iff x > 1$

$f \searrow$ w $[1, +\infty) \implies$ maksimum globalne w $x = 1$

$f(1) = \ln(1) - 2 = -2 < 0$

$\forall_{x>0}\ f(x) < 0 \iff \forall_{x>0}\ \ln(x) < 2\sqrt{x}$

e) $2x < \ln\left(\frac{1+x}{1-x}\right) \quad \forall_{x\in(0,1)}$

$f(x) = \ln\left(\frac{1+x}{1-x}\right) - 2x > 0 \quad \forall_{x\in(0,1)}$

$$\lim_{x\to 0^+} f(x) = \ln(1) = 0 \qquad \lim_{x\to 1^-} f(x) = \{+\infty - 2\} = +\infty$$

$$\frac{df}{dx} = \frac{1}{\frac{1+x}{1-x}} \cdot \frac{1(1-x) - (1+x)(-1)}{(1-x)^2} - 2 = \frac{\cancel{1-x}}{1+x} \cdot \frac{1 \cancel{-x} + 1 \cancel{+x}}{(1-x)^{\cancel{2}}} - 2$$

$$= \frac{2}{(1+x)(1-x)} - 2 = \frac{2}{1-x^2} - \frac{2(1-x^2)}{1-x^2} = \frac{2-2+2x^2}{1-x^2} = \frac{2x^2}{1-x^2}$$

$\forall_{x\in(0,1)}\ f'(x) > 0$

f jest rosnąca i ograniczona z dołu przez 0 w $(0,1)$

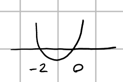
$\forall_{x\in(0,1)}\ f(x) > 0 \iff \forall_{x\in(0,1)}\ 2x < \ln\left(\frac{1+x}{1-x}\right)$

2.

a) $f(x) = x e^{-\frac{2}{x}} \qquad D = \mathbb{R} \setminus \{0\}$

$$\frac{df}{dx} = x \cdot \frac{2}{x^2} e^{-\frac{2}{x}} + e^{-\frac{2}{x}} = e^{-\frac{2}{x}}\left(1 + \frac{2}{x}\right)$$

$$\frac{df}{dx} > 0 \iff 1 + \frac{2}{x} > 0 \iff x(x+2) > 0$$



$x \in (-\infty, -2) \cup (0, +\infty)$

$f \nearrow (-\infty, -2],\ (0, +\infty)$

$f \searrow [-2, 0)$

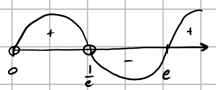
maksimum lokalne w $x = -2 \qquad f(-2) = -2e$

b) $f(x) = \dfrac{x}{(1+\ln(x))^2}$ $\qquad D = (0, +\infty) \setminus \{\frac{1}{e}\}$ $\qquad \ln(x) = -1 \iff x = \frac{1}{e}$

$$\frac{df}{dx} = \frac{(1+\ln x)^2 - x \frac{d}{dx}(1+\ln x)^2}{(1+\ln(x))} = \frac{(1+\ln(x))^2 - x\left(2(1+\ln(x)) \cdot \frac{1}{x}\right)}{(1+\ln(x))^4}$$

$$\frac{df}{dx} = \frac{[\ln(x)]^2 + 2\ln(x) + 1 - 2 - 2\ln(x)}{(1+\ln(x))^4} = \frac{[\ln(x)-1]\cancel{[\ln(x)+1]}}{[\ln(x)+1]^{\cancel{4}3}}$$

$f'(x) > 0 \iff [\ln(x)-1][\ln(x)+1]^3 > 0$



$f \nearrow$ w $(0, \frac{1}{e})$, $[e, +\infty)$

$f \searrow$ w $(\frac{1}{e}, e]$

minimum lokalne w e

$$f(e) = \frac{e}{(1+\ln(e))^2} = \frac{e}{4}$$

c) $f(x) = \arccos\left(\dfrac{1-x^2}{1+x^2}\right)$

$\dfrac{1-x^2}{1+x^2} \geqslant -1 \qquad \dfrac{1-\cancel{x^2}+1+\cancel{x^2}}{1+x^2} \geqslant 0 \qquad \dfrac{2}{1+x^2} \geqslant 0 \quad \forall_{x \in \mathbb{R}}$

$\dfrac{1-x^2}{1+x^2} \leqslant 1 \qquad \dfrac{\cancel{1}-x^2\ \cancel{-1}-x^2}{1+x^2} \leqslant 0 \qquad -2x^2(1+x^2) \leqslant 0 \quad \forall_{x \in \mathbb{R}}$

$D = \mathbb{R}$

$$\frac{df}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{-2x(1+x^2) - (1-x^2)\cdot 2x}{(1+x^2)^2} = \frac{-1}{\sqrt{\frac{\cancel{1}+2x^2+\cancel{x^4}\ \cancel{-1}+2x^2\ \cancel{-x^4}}{(1+x^2)^2}}} \cdot \frac{-2x\ \cancel{-2x^3}\ -2x\ \cancel{+2x^3}}{(1+x^2)^2}$$

$$\frac{df}{dx} = -\sqrt{\frac{(1+x^2)^2}{4x^2}} \cdot \frac{-4x}{(1+x^2)^2} = \frac{\cancel{1+x^2}}{2|x|} \cdot \frac{4x}{(1+x^2)^{\cancel{2}}} = \frac{2}{1+x^2}\operatorname{sgn}(x)$$

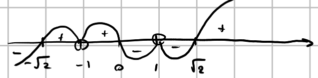
$f \nearrow$ w $[0, +\infty)$ $\qquad f \searrow$ w $(-\infty, 0]$ $\qquad f(0) = \arccos(1) = 0$

minimum lokalne w $x = 0$

d) $f(x) = \dfrac{e^{x^2+1}}{x^2-1} = \dfrac{e^{x^2+1}}{(x-1)(x+1)}$ $\qquad D = \mathbb{R} \setminus \{-1, 1\}$

$$\frac{df}{dx} = \frac{2x\,e^{x^2+1}(x^2-1) - e^{x^2+1} \cdot 2x}{(x^2-1)^2} = \frac{2x\,e^{x^2+1}(x^2-2)}{(x-1)^2(x+1)^2}$$

$f'(x) > 0 \iff 2x(x-\sqrt{2})(x+\sqrt{2})(x-1)^2(x+1)^2 \underbrace{e^{x^2+1}}_{>0} > 0$



$f \nearrow$ w $[-\sqrt{2}, -1)$, $(-1, 0]$, $[\sqrt{2}, +\infty)$

$f \searrow$ w $(-\infty, -\sqrt{2}]$, $[0, 1)$, $(1, \sqrt{2}]$

minima lokalne w $x = -\sqrt{2}$, $x = \sqrt{2}$

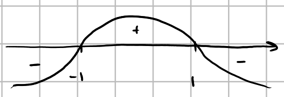
maksimum lokalne w $x = 0$

$f(-\sqrt{2}) = e^3 \qquad f(\sqrt{2}) = e^3 \qquad f(0) = -e$

e) $f(x) = 2\arctan(x) - x$ $\qquad D = \mathbb{R}$

$$\frac{df}{dx} = \frac{2}{1+x^2} - 1 = \frac{2-1-x^2}{1+x^2} = \frac{1-x^2}{1+x^2} = \frac{(1-x)(1+x)}{1+x^2}$$

$f'(x) > 0 \iff (1-x)(1+x)(1+x^2) > 0$



$f \searrow$ w $(-\infty, -1]$, $[1, +\infty)$

$f \nearrow$ w $[-1, 1]$

minimum lokalne w $x = -1$ $\quad f(-1) = 2\arctan(-1) + 1 = 1 - \frac{\pi}{2}$

maximum lokalne w $x = 1$ $\quad f(1) = 2\arctan(1) - 1 = \frac{\pi}{2} - 1$

3.

$f(x) = \arctan\left(\frac{2+x}{2-x}\right) - \arctan\left(\frac{x}{2}\right)$ $\qquad \arctan(-x) = -\arctan(x)$

$D = \mathbb{R} \setminus \{2\}$

$f(-x) = \arctan\left(\frac{2-x}{2+x}\right) - \arctan\left(\frac{-x}{2}\right) = \arctan\left(\frac{2-x}{2+x}\right) + \arctan\left(\frac{x}{2}\right)$

$f(-x) \neq f(x) \qquad f(-x) \neq -f(x)$

$\lim\limits_{x \to 2^+} f(x) = -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$ $\qquad \lim\limits_{t \to -\infty} \arctan(t) = -\frac{\pi}{2}$

$\lim\limits_{x \to 2^-} f(x) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ $\qquad \lim\limits_{x \to 2^+} \frac{2-x}{2+x} = \left\{\frac{4}{0^-}\right\} = -\infty$

brak asymptot pionowych

$\lim\limits_{x \to +\infty} \frac{f(x)}{x} = \left\{ \frac{-\frac{\pi}{4} - \frac{\pi}{2}}{+\infty} \right\} = 0$ $\qquad \lim\limits_{x \to +\infty} f(x) = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$

$\lim\limits_{x \to -\infty} \frac{f(x)}{x} = \left\{ \frac{-\frac{\pi}{4} + \frac{\pi}{2}}{-\infty} \right\} = 0$ $\qquad \lim\limits_{x \to -\infty} f(x) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$

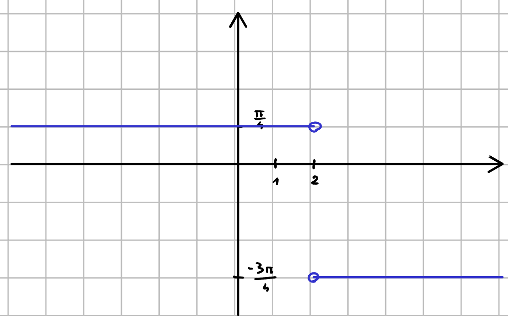
$y = -\frac{3\pi}{4}$ asymptota pozioma prawostronna

$y = \frac{\pi}{4}$ asymptota pozioma lewostronna

$$\frac{df}{dx} = \frac{1}{1+\left(\frac{2+x}{2-x}\right)^2} \cdot \frac{(2-x) - (2+x)(-1)}{(2-x)^2} - \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = \frac{1}{\frac{4 - \cancel{4x} + x^2 + 4 + \cancel{4x} + x^2}{(2-x)^2}} \cdot \frac{2 - \cancel{x} + 2 + \cancel{x}}{(2-x)^2} - \frac{1}{\frac{4+x^2}{4}} \cdot \frac{1}{2}$$

$$\frac{df}{dx} = \frac{\cancel{(2-x)^2}}{2x^2+8} \cdot \frac{4}{\cancel{(2-x)^2}} - \frac{2}{x^2+4} = \frac{2}{x^2+4} - \frac{2}{x^2+4} = 0$$

funkcja jest stała w $(-\infty, 2)$, $(2, +\infty)$



4.

a) $f(x) = \sqrt{1+x^2}$ $\qquad f(0) = \sqrt{1} = 1$

$$\frac{df}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}} \qquad \frac{df}{dx}(0) = 0$$

$$\frac{d^2f}{dx^2} = \frac{1 \cdot \sqrt{1+x^2} - x \cdot \frac{x}{\sqrt{1+x^2}}}{1+x^2} = \frac{\frac{1+x^2}{\sqrt{1+x^2}} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{\frac{1}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{\frac{3}{2}}} \qquad \frac{d^2f}{dx^2}(0) = \frac{1}{1^{\frac{3}{2}}} = 1$$

$$\frac{d^3f}{dx^3} = \frac{-\frac{3}{2}(1+x^2)^{\frac{1}{2}} \cdot 2x}{(1+x^2)^3} = -3x(1+x^2)^{-\frac{5}{2}} \qquad \frac{d^3f}{dx^3}(0) = 0$$

$$\frac{d^4f}{dx^4} = -3x \cdot \left(-\frac{5}{2}\right)(1+x^2)^{-\frac{7}{2}} + (-3)(1+x^2)^{-\frac{5}{2}} = \frac{15}{2}x(1+x^2)^{-\frac{7}{2}} - 3(1+x^2)^{-\frac{5}{2}} \qquad \frac{d^4f}{dx^4}(0) = 0 - 3 = -3$$

$$f(x) \approx 1 + \frac{x^2}{2} - \frac{3x^4}{4!} = 1 + \frac{x^2}{2} - \frac{x^4}{8}$$

b) $f(x) = \cos(x)$ $\qquad f(0) = 1$

$$\frac{df}{dx}(0) = -\sin(0) = 0 \qquad \frac{d^2f}{dx^2}(0) = -\cos(0) = -1$$

$$\frac{d^3f}{dx^3}(0) = \sin(0) = 0 \qquad \frac{d^4f}{dx^4}(0) = \cos(0) = 1$$

$$\frac{d^5f}{dx^5}(0) = -\sin(0) = 0 \qquad \frac{d^6f}{dx^6}(0) = -\cos(0) = -1$$

$$f(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

5.

$$\forall_{x \geq 0} \quad e^x \geq 1 + x + \frac{x^2}{2} \quad ?$$

$$f(x) = e^x \qquad \frac{d^nf}{dx^n} = e^x$$

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{e^{\theta x}}{3!}x^3, \qquad \theta \in (0,1)$$

$$r(x) = \frac{1}{6}e^{\theta x}x^3$$

$$\forall_{\theta \in (0,1)} \; r(0) = 0 \qquad \forall_{x > 0 \,\wedge\, \theta \in (0,1)} \; r(x) > 0$$

$$\Longrightarrow \quad 1 + x + \frac{x^2}{2} + r(x) \geq 1 + x + \frac{x^2}{2}$$

$$\Longleftrightarrow \quad e^x \geq 1 + x + \frac{x^2}{2} \quad \text{dla } x \geq 0$$