1.

b) f jest parzysta

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx = 2\int_0^a f(x)\,dx$$

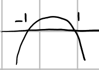
2.

a) $F(x) = \int_x^{2x} \frac{dt}{t^2+2t+2}$ $\qquad D_F = \mathbb{R}$

$F(x) = G(2x) - G(x)$ $\qquad G(x) = \int \frac{dt}{t^2+2t+2}$

$$F'(x) = 2\,\frac{1}{4x^2+4x+2} - \frac{1}{x^2+2x+2} = \frac{1}{2x^2+2x+1} - \frac{1}{x^2+2x+2}$$

$$= \frac{x^2+2x+2-2x^2-2x-1}{(2x^2+2x+1)(x^2+2x+2)} = \frac{(1-x)(1+x)}{(2x^2+2x+1)(x^2+2x+2)}$$



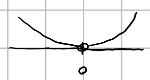
$F \searrow$ w $(-\infty,-1]$, $[1,+\infty)$
$F \nearrow$ w $[-1,1]$

b) $F(x) = \int_x^{2x} \frac{e^t}{t}\,dt$ $\qquad G(x) = \int \frac{e^x}{x}\,dx$ $\qquad x \neq 0$

$F(x) = G(2x) - G(x)$

$$F'(x) = \frac{d}{dx}G(2x) - \frac{d}{dx}G(x) = \frac{e^{2x}}{2x}\cdot 2 - \frac{e^x}{x} = \frac{e^{2x}-e^x}{x} = e^x\,\frac{e^x-1}{x}$$

$F'(x) > 0 \iff e^x(e^x-1)x > 0$



$F \nearrow$ w $(-\infty,0)$, $(0,+\infty)$

3

a)

$$\lim_{x\to0^+} \frac{\int_0^{\sin(x)} \sqrt{\tan(t)}\,dt}{\int_0^{\tan(x)} \sqrt{\sin(t)}\,dt} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to0^+} \frac{\sqrt{\tan(\sin(x))}\cos(x)}{\sqrt{\sin(\tan(x))}\cdot\frac{1}{\cos^2(x)}} = \lim_{x\to0^+} \sqrt{\frac{\tan(\sin(x))}{\sin(\tan(x))}}\cos^3(x) = 1$$

$x \to 0^+$

$$\frac{\tan(\sin(x))}{\sin(\tan(x))} = \frac{\frac{\tan(\sin(x))}{\sin(x)}}{\frac{\sin(\tan(x))}{\sin(x)}} = \frac{\frac{\tan(\sin(x))}{\sin(x)} \to 1}{\underbrace{\frac{\sin(\tan(x))}{\tan(x)}}_{\to 1}\cdot\frac{1}{\cos(x)}} \to 1$$

$G(x) = \int \sqrt{\tan(x)}\,dx$
$F(x) = G(\sin(x)) - G(0)$
$\frac{d}{dx}F(x) = G'(\sin(x))\cos(x) = \sqrt{\tan(\sin(x))}\cos(x)$

b)

$$\lim_{x\to\infty} \frac{\int_{x^2}^{x^2+1} \ln(t+1)\,dt}{x^2} \overset{\frac{?}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{\ln(x^2+2)\cdot 2x - \ln(x^2+1)\cdot 2x}{2x} = \lim_{x\to\infty} \ln\left(\frac{x^2+2}{x^2+1}\right) = \ln(1) = 0$$

c)

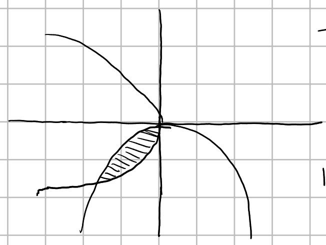
$$\lim_{x\to\infty} \frac{\int_{e^x}^{e^{3x}} \frac{dt}{\ln^3(t)}}{e^{3x}} \overset{\frac{?}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{\frac{1}{\ln^3(e^{3x})}\cdot e^{3x}\cdot 3 - \frac{1}{\ln^3(e^x)}\cdot e^x}{3e^{3x}} = \lim_{x\to\infty} \frac{\frac{3e^{3x}}{(3x)^3} - \frac{e^x}{x^3}}{3e^{3x}} = \lim_{x\to\infty} \frac{3e^{3x} - 27e^x}{81e^{3x}x^3}$$

$$= \lim_{x\to\infty} \frac{e^{2x}-9}{27x^3e^{3x}} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{2e^{2x}}{27\cdot3x^2e^{3x} + 27\cdot3e^{3x}x^3} = \frac{2}{81}\lim_{x\to\infty} \frac{e^{2x}}{e^{3x}(x^3+x^2)} = \frac{2}{81}\cdot\lim_{x\to\infty} \frac{1}{e^x(x^3+x^2)} = 0$$

4.

a) $2y = -x^2 \rightarrow y = -\frac{1}{2}x^2$

$2x = -y^2 \rightarrow y = \pm\sqrt{-2x} \qquad x = -\frac{1}{2}y^2$



$-\sqrt{-2x} = -\frac{1}{2}x^2$

$-2x = \frac{1}{4}x^4$

$-8 = x^3 \qquad x = -2$

$$|D| = \int_{-2}^{0}\left(-\frac{1}{2}x^2 - (-\sqrt{-2x})\right)dx$$

$$= \int_{-2}^{0}\left(-\frac{1}{2}x^2 + \sqrt{-2x}\right)dx = -\frac{1}{2}\int_{-2}^{0}x^2dx + \int_{-2}^{0}\sqrt{-2x}\,dx$$

$$\int_{-2}^{0}x^2dx = \frac{1}{3}x^3\Big|_{-2}^{0} = +\frac{8}{3}$$

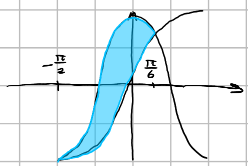
$$\int_{-2}^{0}\sqrt{-2x}\,dx = \left|\begin{matrix} t = -2x \\ dt = -2dx \end{matrix}\right| = -\frac{1}{2}\int_{4}^{0}\sqrt{t}\,dt = \frac{1}{2}\int_{0}^{4}\sqrt{t}\,dt = \frac{1}{2}\cdot\left(\frac{2}{3}t\sqrt{t}\right)\Big|_{0}^{4} = \frac{8}{3}$$

$$|D| = \left(-\frac{1}{2}\right)\left(\frac{8}{3}\right) + \frac{8}{3} = \frac{4}{3}$$

b) $y = \sin(x)$

$y = \cos(2x)$

$(0, \frac{1}{2})$



$\sin(x) = \cos(2x)$

$\sin(x) = 2\cos^2(x)+1 = 1-2\sin^2(x)$

$2\sin^2(x) + \sin(x) - 1 = 0$

$2\sin^2(x) + 2\sin(x) - \sin(x) - 1 = 0$

$2\sin(x)(\sin(x)+1) - (\sin(x)+1) = 0$

$(2\sin(x)-1)(\sin(x)+1) = 0$

$\sin(x) = \frac{1}{2} \lor \sin(x) = -1$

$x = \frac{\pi}{6} \lor x = -\frac{\pi}{2}$

$$|D| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}}[\cos(2x) - \sin(x)]\,dx = \frac{1}{2}\sin(2x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{6}} + \cos(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0\right) + \frac{\sqrt{3}}{2} - 0 = \frac{3\sqrt{3}}{4}$$

5.

a) $0 \le x \le \frac{\pi}{2}$

$0 \le y \le \sin(x) + \cos(x)$

$$|V| = \pi\int_{0}^{\frac{\pi}{2}}[\sin(x)+\cos(x)]^2dx = \pi\int_{0}^{\frac{\pi}{2}}[1+\sin(2x)]dx = \pi\left[x\Big|_{0}^{\frac{\pi}{2}} - \frac{1}{2}\cos(2x)\Big|_{0}^{\frac{\pi}{2}}\right] = \pi\left(\frac{\pi}{2} - \frac{1}{2}(-1-1)\right) = \pi\left(\frac{\pi}{2}+1\right)$$

b) $0 \le x \le \pi$

$\sin(x) \le y \le 2\sin(x)$

$$|V| = |V_2| - |V_1| = \pi\int_{0}^{\pi}[2\sin(x)]^2dx - \pi\int_{0}^{\pi}[\sin(x)]^2dx = 4\pi\int_{0}^{\pi}\sin^2(x)\,dx - \pi\int_{0}^{\pi}\sin^2(x)\,dx$$

$$= 3\pi\int_{0}^{\pi}\frac{1-\cos(2x)}{2}dx = \frac{3\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_{0}^{\pi} = \frac{3\pi^2}{2}$$

Kolos → zadania 3/25 → 4/28-9

6.

a) $y = x\sqrt{x} \qquad x \in [0,1]$

$$|L| = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$\frac{d}{dx} x\sqrt{x} = \frac{d}{dx} x^{\frac{3}{2}} = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$|L| = \int_0^1 \sqrt{1+(\tfrac{3}{2}\sqrt{x})^2}\,dx = \int_0^1 \sqrt{1+\tfrac{9}{4}x}\,dx = \left| \begin{matrix} t = \frac{9}{4}x+1 \\ dt = \frac{9}{4}dx \end{matrix} \right| = \frac{4}{9}\int_1^{\frac{13}{4}} \sqrt{t}\,dt = \frac{4}{9} \cdot \frac{2}{3} t\sqrt{t}\,\Big|_1^{\frac{13}{4}}$$

$$|L| = \frac{4}{9}\cdot\frac{2}{3}\left(\frac{13}{4}\cdot\frac{\sqrt{13}}{2} - 1\right) = \frac{8}{27}\cdot\frac{13\sqrt{13}-8}{8} = \frac{13\sqrt{13}-8}{27}$$

b) $y = \cosh(x) = \dfrac{e^x + e^{-x}}{2} \qquad \sinh = \dfrac{e^x - e^{-x}}{2} \qquad \cosh^2(x) - \sinh^2(x) = 1$

$x \in [0,1]$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$|L| = \int_0^1 \sqrt{1+\sinh^2(x)}\,dx = \int_0^1 \cosh(x)\,dx = \sinh(x)\,\Big|_0^1 = \frac{e - e^{-1}}{2}$$

7.

a) $$\int_0^{+\infty} x e^{-x^2}\,dx = \lim_{T\to+\infty} \int_0^T x e^{-x^2} dx \left| \begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix} \right| = \lim_{T\to\infty} \int_0^{T^2} e^{-t}\cdot\tfrac{1}{2}dt = \frac{1}{2}\lim_{T\to\infty} (-e^{-t})\Big|_0^{T^2}$$

$$= \frac{1}{2}\lim_{T\to\infty}(-e^{-T^2}+1) = \frac{1}{2}$$

b) $$\int e^x\cos(x)\,dx = \left| \begin{matrix} f = \cos(x) & g' = e^x \\ f' = -\sin(x) & g = e^x \end{matrix} \right| = e^x\cos(x) + \int e^x \sin(x)\,dx = \left| \begin{matrix} f = \sin(x) & g = e^x \\ f' = \cos(x) & g' e^x \end{matrix} \right| = e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)\,dx$$

$$2\int e^x\cos(x)\,dx = e^x\cos(x) + e^x\sin(x)$$

$$\int e^x\cos(x)\,dx = \frac{e^x}{2}(\cos(x)+\sin(x))$$

$$\int_{-\infty}^0 e^x\cos(x)\,dx = \lim_{s\to-\infty}\int_s^0 e^x\cos(x)\,dx = \frac{1}{2}\lim_{s\to-\infty} e^x[\cos(x)+\sin(x)]\Big|_s^0 = \frac{1}{2}\lim_{s\to-\infty}\Big[1 - \underbrace{e^s}_{\to 0}\underbrace{(\cos(s)+\sin(s))}_{\text{ograniczona}}\Big] = \frac{1}{2}$$

c) $$\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5} = \int_{-\infty}^0 \frac{dx}{x^2+4x+5} + \int_0^{+\infty}\frac{dx}{x^2+4x+5} = \lim_{s\to-\infty}\int_s^0 \frac{dx}{x^2+4x+5} + \lim_{T\to\infty}\int_0^T \frac{dx}{x^2+4x+5}$$

$$= \lim_{s\to-\infty} \arctan(x+2)\Big|_s^0 + \lim_{T\to\infty}\arctan(x+2)\Big|_0^T \qquad \int\frac{dx}{x^2+4x+5} = \int\frac{dx}{(x+2)^2+1} = \arctan(x+2)+C$$

$$= \lim_{s\to-\infty}\left[\arctan(2) - \arctan(s+2)\right] + \lim_{T\to\infty}\left[\arctan(T) - \arctan(2)\right]$$

$$= \arctan(2) - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \arctan(2) = \pi$$

8.

a) $$\int_1^e \frac{dx}{x\sqrt{\ln(x)}} = \lim_{a\to 1^+} \int_a^e \frac{dx}{x\sqrt{\ln(x)}} = \lim_{a\to 1^+} \left[2\sqrt{\ln(e)} - 2\sqrt{\ln(a)}\right] = 2$$

$$\int \frac{dx}{x\sqrt{\ln(x)}} = \left| \begin{matrix} t = \ln(x) \\ dt = \frac{1}{x}dx \end{matrix} \right| = \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{\ln(x)}$$

b) $$\int_1^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{a\to 1^+} \int_a^2 \frac{x}{\sqrt{x-1}}\,dx = \dots$$

$$\int \frac{x}{\sqrt{x-1}}\,dx = \left| \begin{matrix} t = x-1 \\ dt = dx \end{matrix} \right| = \int \frac{t+1}{\sqrt{t}}\,dt = \int\left[\sqrt{t} + \frac{1}{\sqrt{t}}\right]dt = \int\sqrt{t}\,dt + \int\frac{1}{\sqrt{t}}\,dt = \frac{3}{2}t\sqrt{t} + 2\sqrt{t} + C = \frac{3}{2}(x+1)\sqrt{x+1} + 2\sqrt{x+1} + C$$

c) $$f(x) = \frac{1}{x^2-4x+3} = \frac{1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{A(x-3)+B(x-1)}{(x-1)(x-3)} = \frac{-\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x-3}$$

$1 = -2A \quad A = -\frac{1}{2}$

$1 = 2B \quad B = \frac{1}{2}$

$$\int \frac{dx}{x^2-4x+3} = -\frac{1}{2}\int\frac{1}{x-1}\,dx + \frac{1}{2}\int\frac{1}{x-3}\,dx = \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + c$$

$$\int_0^2 \frac{dx}{x^2-4x+3} = \int_0^1 \frac{dx}{x^2-4x+3} + \int_1^2 \frac{dx}{x^2-4x+3} = \lim_{a\to 1^-}\int_0^a \frac{dx}{x^2-4x+3} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{x^2-4x+3}$$

$$= \lim_{a\to 1^-}\left[\frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1|\right]\Big|_0^a + \lim_{b\to 1^+}\frac{1}{2}\left[\ln|x-3| - \ln|x-1|\right]\Big|_b^2$$

$$= \frac{1}{2}\lim_{a\to 1^-}\left[\ln|a-3| - \underset{\downarrow \atop +\infty}{\ln|a-1|} - \ln|-3| + \ln|-1|\right] + \frac{1}{2}\lim_{b\to 1^+}\left[\ln|-1| - \ln|1| - \ln|b-3| + \underset{\downarrow \atop -\infty}{\ln|b-1|}\right]$$

całka nie istnieje