1.

a) $\lim\limits_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{e^x\left(1-\frac{1}{e^{2x}}\right)}{e^x\left(1+\frac{1}{e^{2x}}\right)} = \lim\limits_{x\to\infty} \frac{1-\frac{1}{e^{2x}}}{1+e^{2x}} = 1$

b) $\lim\limits_{x\to0} \frac{\cos(x)-1+\frac12 x^2}{x^4} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{\frac{d}{dx}\left[\cos(x)-1+\frac12 x^2\right]}{\frac{d}{dx}x^4} = \lim\limits_{x\to0} \frac{-\sin(x)-0+x}{4x^3} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{-\cos(x)+1}{12x^2} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{\sin(x)}{24x} = \frac{1}{24}$

c) $\lim\limits_{x\to\frac{\pi}{2}^-} \frac{\tan(x)}{\ln(\frac{\pi}{2}-x)} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim\limits_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\cos^2(x)}}{\frac{1}{\frac{\pi}{2}-x}\cdot(-1)} = \lim\limits_{x\to\frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cos^2(x)} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to\frac{\pi}{2}^-} \frac{1}{2\cos(x)\cdot(-\sin(x))} = \lim\limits_{x\to\frac{\pi}{2}^-} \frac{1}{-\sin(2x)} = -\infty$

d) $\lim\limits_{x\to-\infty} x^2 e^{2x} = \lim\limits_{x\to-\infty} \frac{x^2}{e^{-2x}} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim\limits_{x\to-\infty} \frac{2x}{-2e^{-2x}} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim\limits_{x\to-\infty} \frac{2}{4e^{-2x}} = 0$

e) $\lim\limits_{x\to0^+} (\arcsin(x))^{\tan(x)} = \lim\limits_{x\to0^+} e^{\tan(x)\ln(\arcsin(x))} = e^0 = 1$

$\lim\limits_{x\to0^+} \tan(x)\cdot\ln(\arcsin(x)) = \lim\limits_{x\to0^+} \frac{\ln(\arcsin(x))}{\cot(x)} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim\limits_{x\to0^+} \frac{\frac{1}{\arcsin(x)}\cdot\frac{1}{\sqrt{1-x^2}}}{\frac{-1}{\sin^2(x)}} = \lim\limits_{x\to0^+} -\frac{1}{\sqrt{1-x^2}}\cdot\frac{\sin^2(x)}{\arcsin(x)} = -1\cdot0 = 0$

$\lim\limits_{x\to0^+} \frac{\sin^2(x)}{\arcsin(x)} \overset{\frac00}{=} \lim\limits_{x\to0^+} \frac{2\sin(x)\cos(x)}{\frac{1}{\sqrt{1-x^2}}} = 0$

f) $\lim\limits_{x\to\infty} (\ln(2x))^{\log_x e} = \lim\limits_{x\to\infty} e^{\ln\left[\ln(2x)^{\frac{1}{\ln(x)}}\right]} = \lim\limits_{x\to\infty} e^{\frac{\ln(\ln(2x))}{\ln(x)}} = e^0 = 1$

$\log_x(e) = \frac{\ln(e)}{\ln(x)} = \frac{1}{\ln(x)}$

$\lim\limits_{x\to\infty} \frac{\ln(\ln(2x))}{\ln(x)} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim\limits_{x\to\infty} \frac{\frac{1}{\ln(2x)}\cdot\frac{1}{2x}\cdot2}{\frac1x} = \lim\limits_{x\to\infty} \frac{2x}{2x\ln(x)} = \lim\limits_{x\to\infty} \frac{1}{\ln(x)} = 0$

g) $\lim\limits_{x\to0} \left(\frac{\sin(x)}{x}\right)^{\frac{1}{x^2}} = \lim\limits_{x\to0} e^{\frac{1}{x^2}\ln\left(\frac{\sin(x)}{x}\right)} = e^{-\frac16} = \frac{1}{\sqrt[6]{e}}$

$\lim\limits_{x\to0} \frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2} \overset{\frac00}{=} \lim\limits_{x\to0} \frac{\frac{x}{\sin(x)}\cdot\frac{\cos(x)x-\sin(x)}{x^2}}{2x} = \lim\limits_{x\to0} \frac{x\cos(x)-\sin(x)}{2x^2\sin(x)} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{-x\sin(x)+\cos(x)-\cos(x)}{2x^2\cos(x)+4x\sin(x)}$

$\lim\limits_{x\to0} \frac{-x\sin(x)}{2x^2\cos(x)+4x\sin(x)} = \lim\limits_{x\to0} \frac{-\sin(x)}{2x\cos(x)+4\sin(x)} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{-\cos(x)}{-2x\sin(x)+2\cos(x)+4\cos(x)} = \frac{-1}{0+2+4} = -\frac16$

h) $\lim\limits_{x\to0} \left(\frac1x - \frac{1}{\sin(x)}\right) = \lim\limits_{x\to0} \frac{\sin(x)-x}{x\sin(x)} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{\cos(x)-1}{x\cos(x)+\sin(x)} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to0} \frac{-\sin(x)}{\cos(x)-x\sin(x)+\cos(x)} = \frac{0}{1-0+1} = \frac02 = 0$

i) $\lim\limits_{x\to\infty} (x-\ln(x)) = \lim\limits_{x\to\infty} (\ln(e^x)-\ln(x)) = \lim\limits_{x\to\infty} \ln\left(\frac{e^x}{x}\right)$

j) $\lim\limits_{x\to-\infty} x\cdot(\pi+2\arctan(x)) = \lim\limits_{x\to-\infty} \frac{\pi+2\arctan(x)}{1/x} \overset{\frac00}{\underset{H}{=}} \lim\limits_{x\to-\infty} \frac{\frac{2}{1+x^2}}{-\frac{1}{x^2}} = \lim\limits_{x\to-\infty} \frac{-2x^2}{1+x^2} = \lim\limits_{x\to-\infty} \frac{-2}{(1+\frac{1}{x^2})} = -2$

k) $\lim\limits_{x\to4^+} \left(\frac{2x+1}{3x-1}\right)^{\frac{4}{x-4}} = 0$

$\lim\limits_{x\to4^+} \frac{4}{x-4} = +\infty \qquad \lim\limits_{x\to4^+} \frac{2x+1}{3x-1} = \frac{9}{11} \qquad \left|\frac{9}{11}\right| < 1$

l) $\lim\limits_{x\to0^+} \sqrt[x]{3x+1} = \lim\limits_{x\to0^+} (3x+1)^{\frac1x} = \lim\limits_{x\to0^+} (1+3x)^{\frac{1}{3x}\cdot3} = e^3$

m) $\lim_{x\to 0^+} (\tan(x))^{x^2-x} = \lim_{x\to 0^+} e^{(x^2-x)\ln(\tan(x))}$

$\lim_{x\to 0^+} (x^2-x)\ln(\tan(x)) = \lim_{x\to 0^+} \frac{\ln(\tan(x))}{\frac{1}{x^2-x}} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0^+} \frac{\frac{1}{\tan(x)} \cdot \frac{1}{\cos^2(x)}}{\frac{-2x+1}{(x^2-x)^2}} = \lim_{x\to 0^+} \frac{(x^2-x)^2}{(-2x+1)\tan(x)\cos^2(x)}$

$\lim_{x\to 0^+} \frac{x^2(x-1)^2}{(-2x+1)\frac{\sin(x)}{\cos(x)}\cos^2(x)} = \lim_{x\to 0^+} \frac{x}{\sin(x)} \cdot \frac{(x-1)^2}{(-2x+1)\cos(x)} \cdot x = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$1 \quad 1 \quad 0$

n) $\lim_{x\to 0} \frac{2e^x - 2e^{-x} - 4x}{\sin(x) - x} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{2e^x + 2e^{-x} - 4}{\cos(x) - 1} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{2e^x - 2e^{-x}}{-\sin(x)} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{2e^x + 2e^{-x}}{-\cos(x)} = \frac{2+2}{-1} = -4$

o) $\lim_{x\to\infty} \left(\frac{\ln(x)}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to\infty} e^{\frac{1}{x^2}\ln\left(\frac{\ln(x)}{x}\right)}$

$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \frac{\frac{1}{x}}{1} = 0$

$\lim_{x\to\infty} \frac{1}{x^2} \ln\left(\frac{\ln(x)}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{\ln(x)}{x}\right)}{x^2} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{\frac{1}{\frac{\ln(x)}{x}} \cdot \frac{\frac{1}{x}\cdot x - \ln(x)}{x^2}}{2x} = \lim_{x\to\infty} \frac{1-\ln(x)}{2x^2\ln(x)} \overset{\frac{\infty}{\infty}}{\underset{H}{=}}$

$\lim_{x\to\infty} \frac{-\frac{1}{x}}{4x\ln(x) + 2x} = \lim_{x\to\infty} \frac{-1}{4x^2\ln(x) + 2x} = 0$

p) $\lim_{x\to\infty} \left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right)^x = \lim_{x\to\infty} e^{x\ln\left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right)}$

$\lim_{x\to\infty} x\ln\left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{2^{\frac{1}{x}} + 3^{\frac{1}{x}}}{2}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to\infty} \frac{\frac{2}{2^{\frac{1}{x}} + 3^{\frac{1}{x}}} \cdot \frac{-\frac{1}{x^2}\ln(2)2^{\frac{1}{x}} - \frac{1}{x^2}\ln(3)3^{\frac{1}{x}}}{2}}{\frac{-1}{x^2}}$

$= \lim_{x\to\infty} \frac{\cancel{\left(\frac{-1}{x^2}\right)}\left[\ln(2)2^{\frac{1}{x}} + \ln(3)3^{\frac{1}{x}}\right]}{\cancel{\left(\frac{-1}{x^2}\right)}\left(2^{\frac{1}{x}} + 3^{\frac{1}{x}}\right)} = \frac{\ln(2) + \ln(3)}{2} = \frac{1}{2}\ln(6) = \ln(\sqrt{6})$

q) $\lim_{x\to\pi^+} (1 + 2\sin(x))^{\frac{1}{\pi - x}} = \lim_{x\to\pi^+} \left[(1 + 2\sin(x))^{\frac{1}{2\sin(x)}}\right]^{\frac{2\sin(x)}{\pi - x}} = \lim_{x\to\pi^+} \left\{(1+2\sin(x))^{\frac{1}{2\sin(x)}}\right\}^{\frac{\sin(\pi - x)}{\pi - x}\cdot 2} = e^{1\cdot 2} = e^2$

r) $\lim_{x\to 1^+} \left(\frac{x+1}{2x}\right)^{\frac{3}{x-1}} = \lim_{x\to 1^+} \left(\frac{2x + 1 - x}{2x}\right)^{\frac{3}{x-1}} = \lim_{x\to 1^+} \left[\left(1 + \frac{1-x}{2x}\right)^{\frac{2x}{x-1}}\right]^{\frac{3}{2x}} = e^{-\frac{3}{2}}$

2.

$f(x) = \frac{x^2+1}{x-1} \qquad \frac{df}{dx} = \frac{2x(x-1) - (x^2+1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}$

$x + y = 5 \rightarrow y = -x + 5 \qquad D = \mathbb{R} \setminus \{1\}$

$f'(x) = -1 \Leftrightarrow \frac{x^2 - 2x - 1}{(x-1)^2} = -1$

$x^2 - 2x - \cancel{1} = -x^2 + 2x - \cancel{1}$

$2x^2 - 4x = 0$

$x(x-2) = 0 \qquad x = 0 \quad \vee \quad x = 2$

1) $y = f'(0)(x - 0) + f(0) = -x - 1$

2) $y = f'(2)(x - 2) + f(2) = -(x - 2) + 5 = -x + 7$

3.

a) $f(x) = \arcsin\left(\frac{1+x}{1-x}\right)$

$D: \frac{1+x}{1-x} \geq -1 \quad \wedge \quad \frac{1+x}{1-x} \leq 1$

$\frac{1+\cancel{x}+1-\cancel{x}}{1-x} \geq 0 \quad \wedge \quad \frac{\cancel{1}+x-\cancel{1}+x}{1-x} \leq 0$

$2x(1-x) \leq 0$

$2(1-x) \geq 0$ $\quad$ $x \in (-\infty, 0] \cup [1, +\infty)$

$x \leq 0$

$D_f = (-\infty, 0]$

$\lim\limits_{x\to 0^-} \arcsin\left(\frac{1+x}{1-x}\right) = \arcsin(1) = \frac{\pi}{2}$

brak asymptot pionowych

$\lim\limits_{x\to -\infty} \frac{\arcsin\left(\frac{1+x}{1-x}\right)}{x} = 0$

$\lim\limits_{x\to -\infty} \frac{1+x}{1-x} = \lim\limits_{x\to -\infty} \frac{x\left(1+\frac{1}{x}\right)}{x\left(-1+\frac{1}{x}\right)} = -1$

$\lim\limits_{x\to -\infty} \arcsin\left(\frac{1+x}{1-x}\right) = \lim\limits_{t\to -1^+} \arcsin(t) = -\frac{\pi}{2}$

$y = -\frac{\pi}{2}$ asymptota pozioma lewostronna

b) $f(x) = e^{\frac{1}{x^2-4}} = e^{\frac{1}{(x-2)(x+2)}}$

$D_f = \mathbb{R} - \{-2, 2\}$

$\lim\limits_{x\to -2^-} f(x) = \left\{ e^{\frac{1}{-4 \cdot 0^-}} = e^{\infty} \right\} = +\infty$

$\lim\limits_{x\to -2^+} f(x) = \left\{ e^{\frac{1}{-4 \cdot 0^+}} = e^{-\infty} \right\} = 0$

$x = -2$ asymptota pionowa lewostronna

$\lim\limits_{x\to 2^-} e^{\frac{1}{(x-2)(x+2)}} = \left\{ e^{\frac{1}{0^- \cdot 4}} = e^{-\infty} \right\} = 0$

$\lim\limits_{x\to 2^+} f(x) = \left\{ e^{\frac{1}{0^+ \cdot 4}} = e^{\infty} \right\} = +\infty$

$x = 2$ asymptota pionowa prawostronna

$\lim\limits_{x\to +\infty} \frac{e^{\frac{1}{x^2-4}}}{x} = \left\{ \frac{e^0}{\infty} = \frac{1}{\infty} \right\} = 0$

$\lim\limits_{x\to +\infty} e^{\frac{1}{x^2-4}} = e^0 = 1$

$\lim\limits_{x\to -\infty} \frac{e^{\frac{1}{x^2-4}}}{x} = 0$

$\lim\limits_{x\to -\infty} e^{\frac{1}{x^2-4}} = e^0 = 1$

$y = 1$ asymptota pozioma obustronna

c) $f(x) = \sqrt{x^2+2x} + x = \sqrt{x(x+2)} + x = |x|\sqrt{1+\frac{2}{x}} + x$

$\lim\limits_{x\to +\infty} \frac{f(x)}{x} = \lim\limits_{x\to +\infty} \frac{x\sqrt{1+\frac{2}{x}} + x}{x} = \lim\limits_{x\to +\infty} \sqrt{1+\frac{2}{x}} + 1 = 2$

$\lim\limits_{x\to +\infty} \left[f(x) - 2x\right] = \lim\limits_{x\to +\infty} \left[\sqrt{x^2+2x} - x\right] = \lim\limits_{x\to +\infty} \frac{\left(\sqrt{x^2+2x} - x\right)\left(\sqrt{x^2+2x} + x\right)}{\sqrt{x^2+2x} + x} = \lim\limits_{x\to +\infty} \frac{\cancel{x^2} + 2x - \cancel{x^2}}{x\left(\sqrt{1+\frac{2}{x}} + 1\right)} = \frac{2}{2} = 1$

$\lim\limits_{x\to -\infty} \frac{f(x)}{x} = \lim\limits_{x\to -\infty} \frac{-x\sqrt{1+\frac{2}{x}} + x}{x} = \lim\limits_{x\to -\infty} \left[-\sqrt{1+\frac{2}{x}} + 1\right] = 0$

$\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} \frac{\left(\sqrt{x^2+2x} + x\right)\left(\sqrt{x^2+2x} - x\right)}{\sqrt{x^2+2x} - x} = \lim\limits_{x\to -\infty} \frac{\cancel{x^2} + 2x - \cancel{x^2}}{-x\left(\sqrt{1+\frac{2}{x}} + 1\right)} = \frac{2}{-2} = -1$

$x(x+2) \geq 0$

$D = (-\infty, -2] \cup [0, +\infty)$

$\lim\limits_{x\to -2^-} f(x) = f(-2) = -2$

$\lim\limits_{x\to 0^+} f(x) = f(0) = 0$

brak asymptot pionowych

$y = 2x + 1$ asymptota ukośna prawostronna

$y = -1$ asymptota pozioma lewostronna

$f(x) = x + \sin(x) \qquad D = \mathbb{R}$

$\lim\limits_{x\to +\infty} \frac{f(x)}{x} = \lim\limits_{x\to +\infty} \frac{x + \sin(x)}{x} = \lim\limits_{x\to +\infty} \frac{\sin(x)}{x} + 1 = 0 + 1 = 1$

$\lim\limits_{x\to +\infty} x + \sin(x) - x = \lim\limits_{x\to +\infty} \sin(x)$ nie istnieje