Zestaw 1.

1.

$A = (x_A, y_A) \qquad B = (x_B, y_B)$

$$\begin{vmatrix} 1 & x & y \\ 1 & x_A & y_A \\ 1 & x_B & y_B \end{vmatrix} \begin{aligned} &= x_A y_B + x y_A + y x_B - y x_A - x_B y_A - x y_B \\ &= (y_A - y_B)x + (x_B - x_A)y + (x_A y_B - x_B y_A) \\ &= C_1 x + C_2 y + C_3 \end{aligned}$$

dla $x_B \neq x_A \qquad y = \frac{y_A - y_B}{x_B - x_A} x + x_A y_B - x_B y_A$

to jest równanie prostej

podstawiając A

$$\begin{vmatrix} 1 & x_A & y_A \\ 1 & x_A & y_A \\ 1 & x_B & y_B \end{vmatrix}$$ liniowo zależne $= 0$

podstawiając B

$$\begin{vmatrix} 1 & x_B & y_B \\ 1 & x_A & y_A \\ 1 & x_B & y_B \end{vmatrix}$$ liniowo zależne $= 0$

prosta przechodzi przez A i B

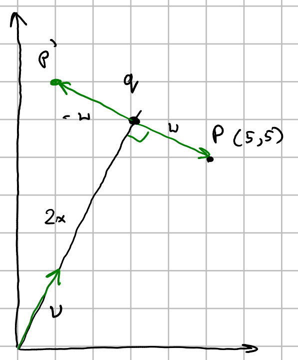
można to uogólnić do równania płaszczyzny w $\mathbb{R}^3$ przechodzącej przez 3 punkty A, B, C

$$\begin{vmatrix} 1 & x & y & z \\ 1 & a_x & a_y & a_z \\ 1 & b_x & b_y & b_z \\ 1 & c_x & c_y & c_z \end{vmatrix} = 0$$

rozwinięcie Laplace'a względem 1. kolumny

$\alpha x + \beta y + \gamma z + \delta = 0$

2.



$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad q = \alpha v = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} \qquad w = p - q$

$q \perp w$

$$\begin{aligned} q \cdot w &= 0 \\ \alpha v \cdot (p - q) &= 0 \\ \alpha v \cdot (p - \alpha v) &= 0 \\ v \cdot p + v \cdot (-\alpha v) &= 0 \\ v \cdot p &= \alpha v \cdot v \\ v \cdot p &= \alpha |v|^2 \end{aligned}$$

$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad p = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \alpha = \frac{15}{5} = 3 \quad \alpha = \frac{v \cdot p}{|v|^2}$

$q = 3v = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad p'' = q - w = q - p + q = 2q - p = \begin{bmatrix} 6 \\ 12 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

3. $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 18 + 0 + 3 - 0 - (-2) - (-3) = 26$

$a = [2, 1, 3]$
$b = [-1, 3, 1]$
$c = [0, -1, 3]$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = i - 3j + 6k + k - 2j - 9i = \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 5 + 21 = 26$$

$$b \times c = \begin{vmatrix} i & j & k \\ -1 & 3 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 9i + k + 3j + i = \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \qquad a \cdot (b \times c) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} = 26$$

$$c \times a = \begin{vmatrix} i & j & k \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{vmatrix} = -3i + 6j + 2k - 3i = \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix} \qquad b \cdot (c \times a) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix} = 26$$

rozwinięcie Laplace'a

$$(u \times v) \cdot w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \left(\hat{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$(u \times v) \cdot w = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$ zwinięcie Laplace'a

permutacja parzysta na wierszach nie zmienia znaku wyznacznika

4. $$a \times (b \times c) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 10 & 3 & 1 \end{vmatrix} = \hat{i} + 30\hat{j} + 6\hat{k} - 10\hat{k} - 9\hat{i} - 2\hat{j} = \begin{bmatrix} -8 \\ 28 \\ -4 \end{bmatrix}$$

$$(a \cdot c)b - (a \cdot b)c = \left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 8 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 28 \\ -4 \end{bmatrix}$$

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

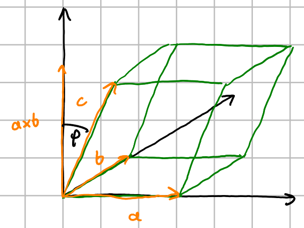
$$a \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a \times (b_2c_3\hat{i} + b_3c_1\hat{j} + b_1c_2\hat{k} - b_2c_1\hat{k} - b_3c_2\hat{i} - b_1c_3\hat{j})$$

$$= a \times \begin{bmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= (a_2b_1c_2 - a_2b_2c_1)\hat{i} + (a_3b_2c_3 - a_3b_3c_2)\hat{j} + (a_1b_3c_1 - a_1b_1c_3)\hat{k}$$
$$- (a_3b_3c_1 - a_3b_1c_3)\hat{i} - (a_1b_1c_2 - a_1b_2c_1)\hat{j} - (a_2b_2c_3 - a_2b_3c_2)\hat{k}$$

$$= \begin{bmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{bmatrix}$$

$$(a \cdot c)b - (a \cdot b)c = (a_1c_1 + a_2c_2 + a_3c_3)b - (a_1b_1 + a_2b_2 + a_3b_3)c = \begin{bmatrix} \cancel{a_1b_1c_1} + a_2b_1c_2 + a_3b_1c_3 \\ a_1b_2c_1 + \cancel{a_2b_2c_2} + a_3b_2c_3 \\ a_1b_3c_1 + a_2b_3c_2 + \cancel{a_3b_3c_3} \end{bmatrix} - \begin{bmatrix} \cancel{a_1b_1c_1} + a_1b_1c_2 + a_1b_1c_3 \\ a_1b_1c_2 + \cancel{a_2b_2c_2} + a_3b_3c_2 \\ a_1b_1c_3 + a_2b_2c_3 + \cancel{a_3b_3c_3} \end{bmatrix} = a \times (b \times c)$$



orientacja bazy

znak wyznacznika wektorów bazy

$S = |a \times b|$

$V = |a \times b| \cdot |c| \cos(\varphi)$

5.

$O = (0,0,0)$

$A = (2,1,3)$

$B = (-1,3,1)$

$C = (0,-1,3)$

$$n = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & -2 \\ -2 & -2 & 0 \end{vmatrix} = \hat{i}\cdot(-4) - \hat{j}(-4) + \hat{k}\cdot 10 = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}$$

normalny do płaszczyzny ABC

$$(X - A)\cdot n = 0 \qquad \begin{bmatrix} x-2 \\ y-1 \\ z-3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix} = 0 \qquad -4(x-2) + 4(y-1) + 10(z-3) = 0$$

$$-4x + 4y + 10z - 26 = 0$$

$$n \cdot X = 26$$

$$P_{ABC} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}|n| = \frac{1}{2}\sqrt{(-4)^2 + 4^2 + 10^2} = \frac{1}{2}\sqrt{132} = \sqrt{33}$$

$$d(O, ABC) = \left|\frac{n \cdot 26}{|n|^2}\right| = \frac{26}{|n|} = \frac{26}{2\sqrt{33}} = \frac{13}{\sqrt{33}} \qquad d(O, ABC) = \frac{|(P_0 - P)\cdot n|}{|n|}$$

$$V_{ABCO} = \frac{1}{3} P_{ABC} \cdot d(O, ABC) = \frac{1}{3} \cdot \frac{1}{2}|n| \cdot \frac{26}{|n|} = \frac{13}{3}$$

$$V_{ABCO} = \frac{1}{6}\left|(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AO}\right| = \frac{1}{6}\begin{vmatrix} -3 & 2 & -2 \\ -2 & -2 & 0 \\ -2 & -1 & -3 \end{vmatrix} = \frac{13}{3}$$