

## Zestaw 1.

1.

$$A = (x_A, y_A) \quad B = (x_B, y_B)$$

$$\begin{cases} 1 & x & y & = & x_A y_B + x y_A + y x_B - y x_A - x_B y_A - x y_B \\ 1 & x_A & y_A & = & (y_A - y_B)x + (x_B - x_A)y + (x_A y_B - x_B y_A) \\ 1 & x_B & y_B & = & C_1 x + C_2 y + C_3 \end{cases} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{to jest równanie} \\ \text{prostej} \end{array}$$

$$\text{dla } x_B \neq x_A \quad y = \frac{y_A - y_B}{x_B - x_A} x + \frac{x_A y_B - x_B y_A}{x_B - x_A}$$

podstawiając A

$$\begin{bmatrix} 1 & x_A & y_A \\ 1 & x_A & y_A \\ 1 & x_B & y_B \end{bmatrix} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{liniowo zależne} \\ = 0 \end{array}$$

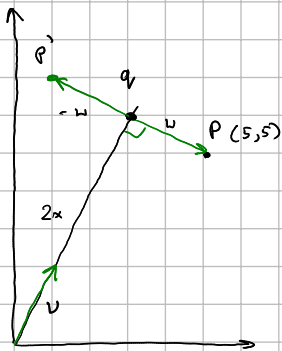
podstawiając B

$$\begin{bmatrix} 1 & x_B & y_B \\ 1 & x_A & y_A \\ 1 & x_B & y_B \end{bmatrix} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{liniowo zależne} \\ = 0 \end{array} \quad \text{prosta przechodzi przez A i B}$$

można to uogólnić do równania płaszczyzny w  $\mathbb{R}^3$   
przechodzącej przez 3 punkty A, B, C

$$\begin{bmatrix} 1 & x & y & z \\ 1 & a_x & a_y & a_z \\ 1 & b_x & b_y & b_z \\ 1 & c_x & c_y & c_z \end{bmatrix} = 0 \quad \begin{array}{l} \text{rozwinąć Laplace'a względem 1. kolumny} \\ x + by + cz + d = 0 \end{array}$$

2.



$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad q = \alpha v = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} \quad w = p - q$$

$$q \perp w$$

$$\begin{aligned} q \cdot w &= 0 \\ \alpha v \cdot (p - q) &= 0 \\ \alpha v \cdot (p - \alpha v) &= 0 \\ v \cdot p + v \cdot (-\alpha v) &= 0 \\ v \cdot p &= \alpha v \cdot v \\ v \cdot p &= \alpha |v|^2 \end{aligned}$$

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad p = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \alpha = \frac{15}{5} = 3 \quad \alpha = \frac{v \cdot p}{|v|^2}$$

$$q = 3v = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad p' = q - w = q - p + q = 2q - p = \begin{bmatrix} 6 \\ 12 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$3. \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 18 + 0 + 3 - 0 - (-2) - (-3) = 26$$

$$a = [2, 1, 3]$$

$$b = [-1, 3, 1]$$

$$c = [0, -1, 3]$$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = i - 3j + 6k + k - 2j - 9i = \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} \quad \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 5 + 21 = 26$$

$$b \times c = \begin{vmatrix} i & j & k \\ -1 & 3 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 9i + k + 3j + i = \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \quad a \cdot (b \times c) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} = 26$$

$$c \times a = \begin{vmatrix} i & j & k \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{vmatrix} = -3i + 6j + 2k - 3i = \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix} \quad b \cdot (c \times a) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 6 \\ 2 \end{bmatrix} = 26$$

$$(u \times v) \cdot w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \overset{\text{rozwiniecie Laplace'a}}{\left( \hat{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}} = w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \quad \text{zwiniecie Laplace'a} \quad \text{permutacja parzyste na wierszach nie zmienia znaku wyznacznika}$$

$$4. a \times (b \times c) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 10 & 3 & 1 \end{vmatrix} = \hat{i} + 30\hat{j} + 6\hat{k} - 10\hat{i} - 9\hat{j} - 2\hat{j} = \begin{bmatrix} -8 \\ 28 \\ -4 \end{bmatrix}$$

$$(a \cdot c)b - (a \cdot b)c = \left( \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right) \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - \left( \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 8 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 28 \\ -4 \end{bmatrix}$$

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

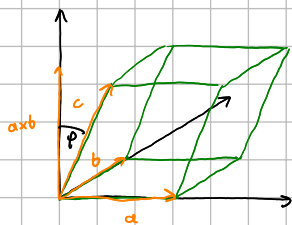
$$a \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a \times (b_2c_3\hat{i} + b_3c_1\hat{j} + b_1c_2\hat{k} - b_2c_1\hat{j} - b_3c_2\hat{i} - b_1c_3\hat{j})$$

$$= a \times \begin{bmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= (a_2b_1c_2 - a_3b_2c_1)\hat{i} + (a_3b_2c_3 - a_1b_3c_2)\hat{j} + (a_1b_3c_1 - a_2b_1c_3)\hat{k} - (a_3b_3c_1 - a_1b_1c_3)\hat{i} - (a_1b_1c_2 - a_2b_2c_1)\hat{j} - (a_2b_2c_3 - a_3b_3c_2)\hat{k}$$

$$= \begin{bmatrix} a_2b_1c_2 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_3 \\ a_3b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_1 \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \end{bmatrix}$$

$$(a \cdot c)b - (a \cdot b)c = (a_1c_1 + a_2c_2 + a_3c_3)b - (a_1b_1 + a_2b_2 + a_3b_3)c = \begin{bmatrix} a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 \\ a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 \\ a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 \end{bmatrix} - \begin{bmatrix} a_1b_1c_1 + a_1b_1c_2 + a_1b_1c_3 \\ a_1b_1c_2 + a_2b_1c_2 + a_3b_1c_2 \\ a_1b_1c_3 + a_2b_1c_3 + a_3b_1c_3 \end{bmatrix} = a \times (b \times c)$$



orientacja bazy  
znak wyznacznika wektorów bazy

$$S = |a \times b|$$

$$V = |a \times b| \cdot |c| \cos(\varphi)$$

5.

$$O = (0, 0, 0)$$

$$A = (2, 1, 3)$$

$$B = (-1, 3, 1)$$

$$C = (0, -1, 3)$$

normalny do  
płaszczyzny ABC

$$(X - A) \cdot n = 0$$

$$n = \vec{AB} \times \vec{AC} = \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & -2 \\ -2 & -2 & 0 \end{vmatrix} = \hat{i} \cdot (-4) - \hat{j} \cdot (-4) + \hat{k} \cdot 10 = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x - 2 \\ y - 1 \\ z - 3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix} = 0$$

$$-4(x - 2) + 4(y - 1) + 10(z - 3) = 0$$

$$-4x + 4y + 10z - 26 = 0$$

$$n \cdot X = 26$$

$$P_{ABC} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |n| = \frac{1}{2} \sqrt{(-4)^2 + 4^2 + 10^2} = \frac{1}{2} \sqrt{132} = \sqrt{33}$$

$$d(O, ABC) = \left| \frac{n \cdot 26}{|n|^2} \right| = \frac{26}{|n|} = \frac{26}{2\sqrt{33}} = \frac{13}{\sqrt{33}}$$

$$d(O, ABC) = \frac{|(P_0 - P) \cdot n|}{|n|}$$

$$V_{ABCO} = \frac{1}{3} P_{ABC} \cdot d(O, ABC) = \frac{1}{3} \cdot \frac{1}{2} |n| \cdot \frac{26}{|n|} = \frac{13}{3}$$

$$V_{ABCO} = \frac{1}{6} |(\vec{AB} \times \vec{AC}) \cdot \vec{AO}| = \frac{1}{6} \begin{vmatrix} -3 & 2 & -2 \\ -2 & -2 & 0 \\ -2 & -1 & -3 \end{vmatrix} = \frac{13}{3}$$