1.

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2} = \frac{(x+y)(x-y)}{x^2+y^2}$$

1° $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$

$$f(x_n, y_n) = \frac{\left(\frac{1}{n}+\frac{1}{n}\right)\left(\frac{1}{n}-\frac{1}{n}\right)}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{\frac{2}{n}\cdot 0}{\frac{2}{n^2}} = 0$$

2° $(\tilde{x}_n, \tilde{y}_n) = \left(\frac{2}{n}, \frac{1}{n}\right) \to (0,0)$

$$f(\tilde{x}_n, \tilde{y}_n) = \frac{\left(\frac{2}{n}+\frac{1}{n}\right)\left(\frac{2}{n}-\frac{1}{n}\right)}{\frac{4}{n^2}+\frac{1}{n^2}} = \frac{\frac{3}{n^2}}{\frac{5}{n^2}} = \frac{3}{5}$$

Granica nie istnieje

3.

$$f(x,y) = \begin{cases} \frac{1}{(1-x)(1-y)} & \text{dla } x \neq 1 \wedge y \neq 1 \\ 0 & \text{dla } x = 1 \vee y = 1 \end{cases}$$

$$\frac{\partial f}{\partial x}(1,1) = \lim_{\Delta x \to 0} \frac{f(1+\Delta x, 1) - f(1,1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$\frac{\partial f}{\partial y}(1,1) = \lim_{\Delta y \to 0} \frac{f(1, 1+\Delta y) - f(1,1)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

4.

$$f(x,y) = \begin{cases} \frac{\sqrt{x^6+6y^4}}{x^2+y^2} & \text{dla } (x,y) \neq (0,0) \\ 0 & \text{dla } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\sqrt{\Delta x^6 + 0}}{\Delta x^2 + 0} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x^3|}{\Delta x^3} = \lim_{\Delta x \to 0} \operatorname{sgn}(\Delta x^3) \quad \text{nie istnieje}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{\frac{\sqrt{6\Delta y^4}}{\Delta y^2}}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{6}\,|\Delta y^2|}{\Delta y^3} = \lim_{\Delta y \to 0} \frac{\sqrt{6}}{\Delta y} \quad \text{nie istnieje}$$

5.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{\frac{\sqrt{2\Delta x^6}}{\Delta x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{2}\,|\Delta x^3|}{\Delta x^3} \quad \text{nie istnieje}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{\frac{\sqrt{5\Delta y^4}}{\Delta y^2}}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{5}}{\Delta y} \quad \text{nie istnieje}$$

6. $f(x,y,z) = 4x\cos(3y) + 2\ln(2xyz) + \frac{4x}{z}$

$$\frac{\partial f}{\partial x} = 4\cos(3y) + 2\cdot\frac{1}{2xyz}\cdot 2yz + \frac{4}{z} = 4\cos(3y) + \frac{2}{x} + \frac{4}{z}$$

$$\frac{\partial f}{\partial y} = -12x\sin(3y) + 2\,\frac{1}{2xyz}\cdot 2xz = -12x\sin(3y) + \frac{2}{y}$$

$$\frac{\partial f}{\partial z} = 2\,\frac{1}{2xyz}\cdot 2xy + 4x\cdot(-z^{-2}) = \frac{2}{z} - \frac{4x}{z^2}$$

$$\nabla f(x,y,z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

7. Ciągłość i istnienie pochodnych cząstkowych są od siebie niezależne