1.

a) $\int (x^2+1)\cos(x)\, dx =$

$f = x^2+1 \quad g = \sin(x)$
$f' = 2x \quad g' = \cos(x)$

$= (x^2+1)\sin(x) - \int 2x\sin(x)\, dx$

$f = 2x \quad g = -\cos(x)$
$f' = 2 \quad g' = \sin(x)$

$= (x^2+1)\sin(x) - \left[-2x\cos(x) - \int -2\cos(x)\, dx\right]$

$= (x^2+1)\sin(x) + 2x\cos(x) + 2\int \cos(x) dx$

$= (x^2+1)\sin(x) + 2x\cos(x) + 2\sin(x) = (x^2+3)\sin(x) + 2x\cos(x) + C$

b) $\int \arcsin(x) dx$

$f = \arcsin(x) \quad g = x$
$f' = \frac{1}{\sqrt{1-x^2}} \quad g' = 1$

$= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\, dx$

$u = -x^2+1$
$du = -2x\, dx$
$-\frac{1}{2} du = x dx$

$= x\arcsin(x) + \frac{1}{2}\int \frac{dt}{\sqrt{u}}$

$= x\arcsin(x) + \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = x\arcsin(x) + \frac{1}{2}\cdot\frac{\sqrt{u}}{\frac{1}{2}} = x\arcsin(x) + \sqrt{1-x^2} + C$

c) $\int x\ln^2(x) dx$

$f = \ln^2(x) \quad g = \frac{1}{2}x^2$
$f' = 2\ln(x)\frac{1}{x} \quad g' = x$

$= \frac{1}{2}x^2\ln^2(x) - \int \cancel{2}\ln(x)\cancel{\tfrac{1}{x}}\cdot\frac{1}{\cancel{2}}x^{\cancel{2}}\, dx = \frac{1}{2}x^2\ln^2(x) - \int x\ln(x)\, dx$

$f = \ln(x) \quad g = \frac{1}{2}x^2$
$f' = \frac{1}{x} \quad g' = x$

$= \frac{1}{2}x^2\ln^2(x) - \frac{1}{2}x^2\ln(x) + \int \frac{1}{x}\cdot\frac{1}{2}x^2\, dx$
$= \frac{1}{2}x^2\ln^2(x) - \frac{1}{2}x^2\ln(x) + \frac{1}{2}\int x\, dx$
$= \frac{1}{2}x^2\ln^2(x) - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C$
$= \frac{1}{4}x^2(2\ln^2(x) - 2\ln(x) + 1) + C$

d) $\int \sqrt{x}\arctan(\sqrt{x})\, dx$

$f = \arctan(\sqrt{x}) \quad g = \frac{2}{3}x^{\frac{3}{2}}$
$f' = \frac{1}{2\sqrt{x}(1+x)} \quad g' = \sqrt{x}$

$= \frac{2}{3}x^{\frac{3}{2}}\arctan(\sqrt{x}) - \int \frac{1}{\cancel{2}\cancel{\sqrt{x}}(1+x)}\cdot\frac{\cancel{2}}{3}x\cancel{\sqrt{x}}\, dx$

$\frac{d}{dx}\arctan(\sqrt{x}) = \frac{1}{1+\sqrt{x}^2}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$

$= \frac{2}{3}x\sqrt{x}\arctan(\sqrt{x}) - \frac{1}{3}\int\frac{x}{1+x}\, dx$

$f = x \quad g = \ln(x+1)$
$f' = 1 \quad g' = \frac{1}{1+x}$

$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$

$\int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}$

$= \frac{2}{3}x\sqrt{x}\arctan(\sqrt{x}) - \frac{1}{3}x\ln(x+1) + \frac{1}{3}\int\ln(x+1)dx$

$t = x+1 \quad dt = dx$

$= \frac{2}{3}x\sqrt{x}\arctan(\sqrt{x}) - \frac{1}{3}x\ln(x+1) + \frac{1}{3}[(x+1)\ln(x+1) - x -] + C$

$= \frac{1}{3}\left[2x\sqrt{x}\arctan(\sqrt{x}) + \ln(x+1) - x\right] + C$

e) $\int \ln(1+\frac{2}{x})\, dx$

$f = \ln(1+\frac{2}{x}) \quad g = x$
$f' = -2\frac{1}{x(x+2)} \quad g' = 1$

$= x\ln(1+\frac{2}{x}) - \int -2\frac{x}{x(x+2)}\, dx$

$\frac{d}{dx}\ln(1+\frac{2}{x}) = \frac{d}{dx}\ln(\frac{x+2}{x}) = \frac{x}{x+2}\cdot\frac{\cancel{x}-\cancel{x}-2}{x^2} = \frac{-2}{x(x+2)}$

$= x\ln(1+\frac{2}{x}) + 2\int\frac{dx}{x+2}$

$t = x+2 \quad dt = dx$

$= x\ln(1+\frac{2}{x}) + 2\int\frac{dt}{t} = x\ln(1+\frac{2}{x}) + 2\ln|x+2| + C$

2.

a) $\int \frac{x^3}{\sqrt{(1-x^2)^3}}\,dx = \left| \begin{array}{l} t = 1-x^2 \\ dt = -2x\,dx \\ dx = \frac{1}{-2x}\,dt \end{array} \right| = \int \frac{x^3}{\sqrt{t^3}} \cdot \frac{dt}{-2x} = -\frac{1}{2} \int \frac{x^2}{\sqrt{t^3}}\,dt = -\frac{1}{2} \int \frac{1-t}{\sqrt{t^3}}\,dt = -\frac{1}{2}\left[\int t^{-\frac{3}{2}}\,dt - \int t^{-\frac{1}{2}}\,dt \right]$

$= -\frac{1}{2} \cdot \frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{t} + \frac{1}{\sqrt{t}} + C = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} + C$

b) $\int e^{\sqrt{x}}\,dx = \int 2t e^t\,dt = 2t e^t - \int 2 e^t\,dt = 2t e^t - 2e^t + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$

$t = \sqrt{x}$ $\quad dt = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2t\,dt$

c) $\int \frac{\cos(\ln(x))}{x}\,dx = \int \cos(t)\,dt = \sin(t) + C = \sin(\ln(x)) + C$

$t = \ln(x)$ $\quad dt = \frac{1}{x}\,dx$

d) $\int \arcsin(x)\,dx = \int t\cos(t)\,dt = t\sin(t) + \int -\sin(t)\,dt = t\sin(t) + \cos(t) + C$

$x = \sin(t)$ $\quad f = t$ $\quad g = \sin(t)$

$t = \arcsin(x)$ $\quad f' = 1$ $\quad g' = \cos(t)$

$dx = \cos(t)\,dt$

$= x\arcsin(x) + \sqrt{1-\sin^2(t)} + C = x\arcsin(x) + \sqrt{1-x^2} + C$

e) $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx = \int \frac{6t^5}{t^3 + t^2}\,dt = 6\int \frac{t^5}{t^3+t^2}\,dt = 6\int \frac{t^3}{t+1}\,dt$

$t = x^{\frac{1}{6}}$ $\quad dt = \frac{1}{6} x^{-\frac{5}{6}}\,dx$

$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{3}{6}} + x^{\frac{2}{6}}}$ $\quad dt = \frac{1}{6} t^{-5}\,dx$

$6\,dt\,t^5 = dx$

$t^3 : (t+1) = t^2 - t + 1$

$\ominus\ t^3 + t^2$

$-t^2$

$\ominus\ -t^2 - t$

t

$\ominus\ t + 1$

-1

$6\int t^2 - t + 1 - \frac{1}{t+1}\,dx = 6\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 + t - \ln|t+1|\right] + C$

$(t^2 - t + 1)(t+1) - 1 = t^3 \cancel{-t^2} \cancel{+t} \cancel{+t^2} \cancel{-t} \cancel{+1} \cancel{-1} = t^3$

$2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x} + 1| + C$

f) $\int x^3 e^{x^2}\,dx = \frac{1}{4}\int e^{\sqrt{t}}\,dt = \frac{1}{4}\int 2u e^u\,du = \frac{1}{2}\int u e^u\,du = \frac{1}{2}\left[u e^u - \int e^u\,du\right]$

$t = x^4$ $\quad dt = 4x^3\,dx$ $\quad u = \sqrt{t}$ $\quad du = \frac{1}{2\sqrt{t}}\,dt$ $\quad f = u$ $\quad g = e^u$

$\sqrt{t} = x^2$ $\quad x^3\,dx = \frac{1}{4}\,dt$ $\quad 2u\,du = dt$ $\quad f' = 1$ $\quad g' = e^u$

$= \frac{1}{2}\left[u e^u - e^u\right] + C = \frac{1}{2}e^u(u-1) + C = \frac{1}{2}e^{x^2}(x^2 - 1) + C$

3.

a) $\int \sin^2(x)\,dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx = \frac{1}{2}x - \frac{1}{2}\int \cos(2x)\,dx + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

$\cos(2x) = 2\cos^2(x) - 1 = 2(1-\sin^2(x)) - 1 = 2 - 1 - 2\sin^2(x) = 1 - 2\sin^2(x)$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\frac{d}{dx}\sin(2x) = 2\cos(2x)$

b) $\int \sin(4x)\cos(6x)\,dx = \int \frac{1}{2}[\sin(-2x) + \sin(10x)]dx = \frac{1}{2}\int \sin(-2x)dx + \frac{1}{2}\int \sin(10x)\,dx$

$= -\frac{1}{2}\int \sin(2x)dx + \frac{1}{2}\int \sin(10x)\,dx = \frac{1}{2}\int -\sin(2x)\,dx + \frac{1}{2}\int \sin(10x)\,dx$

$= \frac{1}{4}\cos(2x) - \frac{1}{20}\cos(10x) + C$ $\qquad \frac{d}{dx}\frac{1}{10}\cos(10x) = \frac{1}{10}\cdot(-\sin(10x))\cdot 10 = -\sin(10x)$

c) $\int \sin^4(x)\cos^3(x)\,dx = \int \sin^4(x)(1-\sin^2(x))\cos(x)\,dx = \left|\begin{matrix} t = \sin(x) \\ dt = \cos(x)dx \end{matrix}\right| = \int t^4(1-t^2)\,dt = \int -t^6 + t^4\,dt$

$= -\int t^6\,dt + \int t^4\,dt = -\frac{1}{7}t^7 + \frac{1}{5}t^5 + C = -\frac{1}{7}\sin^7(x) + \frac{1}{5}\sin^5(x) + C$

d) $\int \frac{1}{\sin(2x)}\,dx = \int \frac{1}{2\sin(x)\cos(x)}\,dx = \int \frac{\sin^2(x) + \cos^2(x)}{2\sin(x)\cos(x)}\,dx = \int \frac{\sin^2(x)}{2\sin(x)\cos(x)}\,dx + \int \frac{\cos^2(x)}{2\sin(x)\cos(x)}\,dx = \frac{1}{2}\int \tan(x)\,dx + \frac{1}{2}\int \cot(x)\,dx$

$= -\frac{1}{2}\ln|\cos(x)| + \frac{1}{2}\ln|\sin(x)| + C$

4.

a) $\int \frac{1}{x^2-6x+13}\,dx = \int \frac{dx}{(x-3)^2+4} = \left|\begin{matrix} t = \frac{1}{2}(x-3) \\ dt = \frac{1}{2}dx \\ dx = 2dt \end{matrix}\right| = \int \frac{2\,dt}{(2t)^2+4} = \int \frac{2\,dt}{4t^2+4} = \frac{1}{2}\int \frac{dt}{1+t^2} =$

$\Delta = 36 - 52 < 0$

$\frac{1}{2}\arctan(t) + C = \frac{1}{2}\arctan\left(\frac{1}{2}x - \frac{3}{2}\right) + C$

b) $\int \frac{x+1}{x^2+8x+25}\,dx = \frac{1}{2}\int \frac{2x+8}{x^2+8x+25}\,dx - 3\int \frac{1}{x^2+8x+25}\,dx = \frac{1}{2}\ln|x^2+8x+25| - \arctan\left(\frac{x+4}{3}\right) + C$

$\Delta < 0$

$\int \frac{2x+8}{x^2+8x+25}\,dx = \left|\begin{matrix} t = x^2+8x+25 \\ dt = (2x+8)\,dx \end{matrix}\right| = \int \frac{dt}{t} = \ln|t| + C = \ln|x^2+8x+25| + C$

$\int \frac{1}{x^2+8x+25}\,dx = \int \frac{1}{(x+4)^2+9}\,dx = \left|\begin{matrix} t = \frac{1}{3}(x+4) \\ dt = \frac{1}{3}dx \\ dx = 3\,dt \end{matrix}\right| = \int \frac{3\,dt}{(3t)^2+9} = \frac{3}{9}\int \frac{dt}{t^2+1} = \frac{1}{3}\arctan(t) + C = \frac{1}{3}\arctan\left(\frac{1}{3}x + \frac{4}{3}\right) + C$

c) $\int \frac{1}{x(x+1)^2}\,dx = ?$

$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$

$x = 0 \to 1 = A$

$x = -1 \to 1 = -C \quad C = -1$

$1 = x^2 + 2x + 1 + Bx^2 + Bx - x$

$0 = (B+1)x^2 + (B+1)x$

$B + 1 = 0 \quad B = -1$

$$\int \frac{1}{x(x+1)^2}\,dx = \int\left[\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$$

$$\int \frac{1}{(x+1)^2}\,dx = \left|\begin{array}{l} t = x+1 \\ dt = dx \end{array}\right| = \int \frac{1}{t^2}\,dt = -\frac{1}{t} = \frac{-1}{x+1}$$

$$\int \frac{1}{x(x+1)^2}\,dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$$

d) $\displaystyle\int \frac{1}{x^4+1}\,dx$ $\qquad x^4+1=0,\quad x^4=-1$ $\qquad \sqrt[4]{-1} = \left\{ e^{\frac{\pi+2k\pi}{4}i},\ k=0,1,2,3 \right\}$

$$\left\{ e^{\frac{\pi}{4}i},\ e^{\frac{3\pi}{4}i},\ e^{-\frac{3\pi}{4}i},\ e^{-\frac{\pi}{4}i} \right\}$$

$$x^4+1 = (x - e^{\frac{\pi}{4}i})(x - e^{-\frac{\pi}{4}i})(x - e^{\frac{3\pi}{4}i})(x - e^{-\frac{3\pi}{4}i})$$

$$= \left[x^2 - x\left(\tfrac{\sqrt2}{2} - \tfrac{\sqrt2}{2}i\right) - x\left(\tfrac{\sqrt2}{2} + \tfrac{\sqrt2}{2}i\right) + 1\right]\left[x^2 - x\left(-\tfrac{\sqrt2}{2} - \tfrac{\sqrt2}{2}i\right) - x\left(-\tfrac{\sqrt2}{2} + \tfrac{\sqrt2}{2}i\right) + 1\right]$$

$$= \left(x^2 - \tfrac{\sqrt2}{2}x + \cancel{\tfrac{\sqrt2}{2}ix} - \tfrac{\sqrt2}{2}x - \cancel{\tfrac{\sqrt2}{2}ix} + 1\right)\left(x^2 + \tfrac{\sqrt2}{2}x + \cancel{\tfrac{\sqrt2}{2}ix} + \tfrac{\sqrt2}{2}x - \cancel{\tfrac{\sqrt2}{2}ix} + 1\right)$$

$$= (x^2 - \sqrt2x + 1)(x^2 + \sqrt2x + 1)$$

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt2x+1} + \frac{Cx+D}{x^2+\sqrt2x+1} = \frac{Ax(x^2+\sqrt2x+1) + B(x^2+\sqrt2x+1) + Cx(x^2-\sqrt2x+1) + D(x^2-\sqrt2x+1)}{x^4+1}$$

$$1 = Ax^3 + \sqrt2Ax^2 + Ax + Bx^2 + \sqrt2Bx + B + Cx^3 - \sqrt2Cx^2 + Cx + Dx^2 - \sqrt2Dx + D$$

$$1 = (A+C)x^3 + (\sqrt2A + B - \sqrt2C + D)x^2 + (A + \sqrt2B + C - \sqrt2D)x + (B+D)$$

$$\begin{cases} A+C=0 \\ \sqrt2A+B-\sqrt2C+D=0 \\ A+C+\sqrt2B-\sqrt2D=0 \\ B+D=1 \end{cases} \qquad \begin{cases} C=-A \\ D=1-B \\ 2\sqrt2A+1=0 \\ \sqrt2(2B-1)=0 \end{cases} \qquad \begin{array}{l} A=-\frac{1}{2\sqrt2} \\ B=\frac12 \\ C=\frac{1}{2\sqrt2} \\ D=\frac12 \end{array}$$

$$\int \frac{1}{x^4+1}\,dx = \int\left[\frac{-\frac{1}{2\sqrt2}x+\frac12}{x^2-\sqrt2x+1} + \frac{\frac{1}{2\sqrt2}x+\frac12}{x^2+\sqrt2x+1}\right]dx$$

$$\int \frac{-\frac{1}{2\sqrt2}x+\frac12}{x^2-\sqrt2x+1}\,dx = \frac{-1}{4\sqrt2}\int\frac{2x-2\sqrt2}{x^2-\sqrt2x+1}\,dx = \frac{-1}{4\sqrt2}\left[\int\frac{2x-\sqrt2}{x^2-\sqrt2x+1}\,dx - \sqrt2\int\frac{1}{x^2-\sqrt2x+1}\,dx\right]$$

$$\int\frac{2x-\sqrt2}{x^2-\sqrt2x+1}\,dx = \left|\begin{array}{l} t = x^2-\sqrt2x+1 \\ dt = (2x-\sqrt2)\,dx \end{array}\right| = \int\frac{dt}{t} = \ln|x^2-\sqrt2x+1| + C$$

$$\int\frac{1}{x^2-\sqrt2x+1}\,dx = \int\frac{1}{(x-\frac{\sqrt2}{2})^2+\frac12}\,dx = \left|\begin{array}{l} t = \sqrt2(x-\frac{\sqrt2}{2}) \\ dt = \sqrt2\,dx \end{array}\right| = \frac{1}{\sqrt2}\int\frac{1}{(\frac{1}{\sqrt2}t)^2+\frac12}\,dt = \frac{1}{\sqrt2}\int\frac{1}{\frac12(t^2+1)}\,dt = \sqrt2\arctan(\sqrt2x-1)+C$$

$$\int \frac{\frac{1}{2\sqrt2}x+\frac12}{x^2+\sqrt2x+1}\,dx = \frac{1}{4\sqrt2}\int\frac{2x+2\sqrt2}{x^2+\sqrt2x+1}\,dx = \frac{1}{4\sqrt2}\left[\int\frac{2x+\sqrt2}{x^2+\sqrt2x+1}\,dx + \sqrt2\int\frac{1}{x^2+\sqrt2x+1}\,dx\right]$$

$$\int\frac{2x+\sqrt2}{x^2+\sqrt2x+1}\,dx = \left|\begin{array}{l} t = x^2+\sqrt2x+1 \\ dt = (2x+\sqrt2)\,dx \end{array}\right| = \int\frac{dt}{t} = \ln|x^2+\sqrt2x+1| + C$$

$$\int\frac{1}{x^2+\sqrt2x+1}\,dx = \int\frac{1}{(x+\frac{\sqrt2}{2})^2+\frac12}\,dx = \left|\begin{array}{l} t = \sqrt2(x+\frac{\sqrt2}{2}) \\ dt = \sqrt2\,dx \end{array}\right| = \frac{1}{\sqrt2}\int\frac{1}{(\frac{1}{\sqrt2}t)^2+\frac12}\,dt = \frac{1}{\sqrt2}\int\frac{1}{\frac12(t^2+1)}\,dt = \sqrt2\arctan(\sqrt2x+1)+C$$

$$\int\frac{1}{x^4+1}\,dx = -\frac{1}{4\sqrt2}\left[\ln|x^2-\sqrt2x+1| - 2\arctan(\sqrt2x-1)\right] + \frac{1}{4\sqrt2}\left[\ln|x^2+\sqrt2x+1| + 2\arctan(\sqrt2x+1)\right] + C$$

$$= \frac{1}{4\sqrt2}\left[\ln|x^2+\sqrt2x+1| - \ln|x^2-\sqrt2x+1|\right] + \frac{\sqrt2}{4}\arctan(\sqrt2x+1) - \frac{\sqrt2}{4}\arctan(\sqrt2x-1) + C$$

$$= \frac{1}{4\sqrt2}\ln\left|\frac{x^2+\sqrt2x+1}{x^2-\sqrt2x+1}\right| + \frac{\sqrt2}{4}\arctan(\sqrt2x+1) - \frac{\sqrt2}{4}\arctan(\sqrt2x-1) + C$$

5.

a) $\int \frac{1}{\sqrt{4x+x^2}}dx = \int \frac{1}{\sqrt{(x+2)^2-4}}dx = \left| \begin{matrix} t = \frac{1}{2}(x+2) \\ dt = \frac{1}{2}dx \end{matrix} \right| = 2\int \frac{1}{\sqrt{4t^2-4}}dt = \int \frac{1}{\sqrt{t^2-1}}dt =$

$\ln|t+\sqrt{t^2-1}|+C = \ln\left|\frac{x+2}{2}+\sqrt{\frac{x^2+4x}{4}}\right|+C = \ln|x+2+\sqrt{x^2+4x}|+C$

b) $\int \sqrt{x^2-2x+3}\,dx = \int \frac{x^2-2x+3}{\sqrt{x^2-2x+3}} = (Ax+B)\sqrt{x^2-2x+3} + \lambda \int \frac{dx}{\sqrt{x^2-2x+3}} \quad / \frac{d}{dx}$

$\frac{x^2-2x+3}{\sqrt{x^2-2x+3}} = A\sqrt{x^2-2x+3} + \frac{(Ax+B)(2x-2)}{2\sqrt{x^2-2x+3}} + \frac{\lambda}{\sqrt{x^2-2x+3}} \quad / \cdot \sqrt{x^2-2x+3}$

$x^2-2x+3 = A(x^2-2x+3) + (Ax+B)(x-1) + \lambda$

$x^2-2x+3 = Ax^2 - 2Ax + 3A + Ax^2 - Ax + Bx - B + \lambda$

$x^2-2x+3 = (2A)x^2 + (B-3A)x + (3A+\lambda-B)$

$\begin{cases} 1 = 2A \\ -2 = B-3A \\ 3 = \lambda - B + 3A \end{cases} \longrightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \\ \lambda = 1 \end{cases}$

$\int \frac{dx}{\sqrt{x^2-2x+3}} = \int \frac{dx}{\sqrt{(x-1)^2+2}} = \left| \begin{matrix} t = x-1 \\ dt = dx \end{matrix} \right| = \int \frac{dt}{\sqrt{t^2+2}} = \ln|t+\sqrt{t^2+2}|+C = \ln|x-1+\sqrt{x^2-2x+3}|+C$

$\int \sqrt{x^2-2x+3} = \frac{1}{2}(x-1)\sqrt{x^2-2x+3} + \ln|x-1+\sqrt{x^2-2x+3}|+C$

? c) $\int \frac{1}{\sqrt{4x-x^2}}dx = \int \frac{1}{\sqrt{-(x-2)^2+4}}dx = \left| \begin{matrix} t = \frac{1}{2}(x-2) \\ dt = \frac{1}{2}dx \end{matrix} \right| = 2\int \frac{dt}{\sqrt{4-4t^2}} = \int \frac{dt}{\sqrt{1-t^2}}$

$= \arcsin(t)+C = \arcsin\left(\frac{x-2}{2}\right)+C$

d) $\int \frac{x^2}{\sqrt{2x+x^2}} = (Ax+B)\sqrt{x^2+2x} + \lambda \int \frac{dx}{\sqrt{x^2+2x}}$

$\frac{x^2}{\sqrt{x^2+2x}} = \frac{(Ax+B)(2x+2)}{2\sqrt{x^2+2x}} + A\sqrt{x^2+2x} + \frac{\lambda}{\sqrt{x^2+2x}}$

$x^2 = (Ax+B)(x+1) + A(x^2+2x) + \lambda$

$x^2 = Ax^2 + Ax + Bx + B + Ax^2 + 2Ax + \lambda$

$x^2 = (2A)x^2 + (3A+B)x + (B+\lambda)$

$\begin{cases} 1 = 2A \\ 0 = 3A+B \\ 0 = B+\lambda \end{cases} \quad \begin{cases} A = \frac{1}{2} \\ B = -\frac{3}{2} \\ \lambda = \frac{3}{2} \end{cases}$

$\int \frac{dx}{\sqrt{x^2+2x}} = \int \frac{dx}{\sqrt{(x+1)^2-1}} = \left| \begin{matrix} t = x+1 \\ dt = dx \end{matrix} \right| = \int \frac{dt}{\sqrt{t^2-1}} =$

$\ln|t+\sqrt{t^2-1}|+C = \ln|x+1+\sqrt{x^2+2x}|+C$

$\int \frac{x^2}{\sqrt{2x+x^2}} = \frac{1}{2}(x-3)\sqrt{x^2+2x} + \frac{3}{2}\ln|x+1\sqrt{x^2+2x}|+C$

? e) $\int x\sqrt{6x-x^2}\,dx = \int \frac{x(6x-x^2)}{\sqrt{6x-x^2}} = \int \frac{-x^3+6x^2}{\sqrt{6x-x^2}} = (Ax^2+Bx+C)\sqrt{6x-x^2} + \lambda \int \frac{dx}{\sqrt{6x-x^2}}$

$$\frac{-x^3+6x^2}{\sqrt{6x-x^2}} = \frac{(Ax^2+Bx+C)(6-2x)}{2\sqrt{6x-x^2}} + (2Ax+B)\sqrt{6x-x^2} + \frac{\lambda}{\sqrt{6x-x^2}}$$

$-x^3+6x^2 = (Ax^2+Bx+C)(-x+3) + (2Ax+B)(-x^2+6x) + \lambda$

$-x^3+6x^2 = -Ax^3+3Ax^2-Bx^2+3Bx-Cx+3C-2Ax^3+12Ax^2-Bx^2+6Bx+\lambda$

$-x^3+6x^2 = (-3A)x^3+(15A-2B)x^2+(9B-C)x+3C+\lambda$

$$\begin{cases} -1 = -3A \\ 6 = 15A-2B \\ 0 = 9B-C \\ 0 = 3C-\lambda \end{cases} \qquad \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{2} \\ C = -\frac{9}{2} \\ \lambda = -\frac{27}{2} \end{cases}$$

$$\int \frac{dx}{\sqrt{6x-x^2}} = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \left| \begin{matrix} t = \frac{1}{3}(x-3) \\ dt = \frac{1}{3}dx \end{matrix} \right| = 3\int \frac{dt}{\sqrt{9-9t^2}} = \int \frac{dt}{\sqrt{1-t^2}}$$

$= \arcsin(t) + C = \arcsin\left(\frac{x-3}{3}\right) + C$

$$\int x\sqrt{6x-x^2}\,dx = \left(\frac{1}{3}x^2-\frac{1}{2}x-\frac{9}{2}\right)\sqrt{6x-x^2} - \frac{27}{2}\arcsin\left(\frac{x-3}{3}\right) + C$$

6

a) $\int \frac{\arctan(x)}{(x+1)^2}dx = \left| \begin{matrix} f = \arctan(x) & g' = \frac{1}{(x+1)^2} \\ f' = \frac{1}{1+x^2} & g = -\frac{1}{1+x} \end{matrix} \right| = -\frac{\arctan(x)}{1+x} + \int \frac{1}{(1+x^2)(1+x)}dx$

$$\frac{1}{(1+x^2)(1+x)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx(x+1)+C(x+1)}{(x^2+1)(x+1)}$$

$1 = Ax^2+A+Bx^2+Bx+Cx+C$

$1 = (A+B)x^2+(B+C)x+(A+C)$

$$\begin{cases} 0 = A+B \\ 0 = B+C \\ 1 = A+C \end{cases} \qquad \begin{cases} B = -A \\ C = A \\ 2A = 1 \end{cases} \qquad \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \\ C = \frac{1}{2} \end{cases}$$

$$\int \frac{1}{(x^2+1)(x+1)}dx = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{2x-2}{x^2+1}dx = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\arctan(x) + C$

$$\int \frac{\arctan(x)}{(x+1)^2}dx = -\frac{\arctan(x)}{1+x} + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\arctan(x) + C$$

? b) $\int \frac{4}{x(x+2\sqrt{x}+4)}dx = \left| \begin{matrix} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}dx \\ dx = 2t\,dt \end{matrix} \right| = \int \frac{8t\,dt}{t^2(t^2+2t+4)} = 8\int \frac{dt}{t(t^2+2t+4)} =$

$$\frac{1}{t(t^2+2t+4)} = \frac{A}{t} + \frac{Bt+C}{t^2+2t+4} = \frac{A(t^2+2t+4)+Bt^2+Ct}{t(t^2+2t+4)}$$

$1 = At^2+2At+4A+Bt^2+Ct$

$1 = (A+B)t^2+(2A+C)t+(4A)$

$$\begin{cases} 0 = A+B \\ 0 = 2A+C \\ 1 = 4A \end{cases} \qquad \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = -\frac{1}{2} \end{cases}$$

$$\int \frac{dt}{t(t^2+2t+4)} = \frac{1}{4}\int \frac{dt}{t} - \frac{1}{8}\int \frac{2t+4}{t^2+2t}dt$$

$$\int \frac{2t+4}{t^2+2t}\,dt = \int \frac{2t+2}{t^2+2t}\,dt + 2\int \frac{dt}{t^2+2t}$$

$$\int \frac{2t+2}{t^2+2t}\,dt = \left| \begin{array}{l} u = t^2+2t \\ du = (2t+2)\,dt \end{array} \right| = \int \frac{du}{u} = \ln|t^2+2t| + C = \ln|x+2\sqrt{x}| + C$$

$$\frac{1}{t^2+2t} = \frac{1}{t(t+2)} = \frac{A}{t} + \frac{B}{t+2} = \frac{A(t+2)+Bt}{t(t+2)}$$

$t=-2 \rightarrow 1 = -2B \quad B = -\frac{1}{2}$

$t=0 \rightarrow 1 = 2A \quad A = \frac{1}{2}$

$$\int \frac{dt}{t^2+2t} = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2} = \frac{1}{2}\ln|\sqrt{x}| - \frac{1}{2}\ln|\sqrt{x}+2| + C$$

$$\int \frac{4}{x(x+2\sqrt{x}+4)}\,dx = 8\left[\frac{1}{4}\ln|\sqrt{x}| - \frac{1}{8}\left(\ln|x+2\sqrt{x}| + 2\left(\frac{1}{2}\ln|\sqrt{x}| - \frac{1}{2}\ln|\sqrt{x}+2|\right)\right)\right] + C$$

$$= 2\ln|\sqrt{x}| - \ln|x+2\sqrt{x}| + \ln|\sqrt{x}| - \ln|\sqrt{x}+2| + C$$

$$= 3\ln|\sqrt{x}| - \ln|x+2\sqrt{x}| - \ln|\sqrt{x}+2| + C$$

? c) $$\int \frac{1}{2\cos(x)+\sin(2x)}\,dx = \left| \begin{array}{l} t = \tan(\frac{x}{2}) \\ dx = \frac{2dt}{t^2+1} \end{array} \right| = \int \frac{\frac{2}{t^2+1}}{2\frac{1-t^2}{1+t^2} + 2\frac{1-t^2}{1+t^2}\cdot\frac{2t}{1+t^2}}\,dt = \int \frac{1}{1-t^2+\frac{2t(1-t^2)}{1+t^2}}\,dt$$

$\sin(x) = \frac{2t}{1+t^2} \qquad \cos(x) = \frac{1-t^2}{1+t^2}$

$$\int \frac{1}{\frac{(1-t^2)(1+t^2)+2t(1-t^2)}{1+t^2}}\,dt = \int \frac{1+t^2}{(1-t^2)(t+1)^2}\,dt = \int \frac{dt}{1-t^2} = -\int \frac{dt}{(t-1)(t+1)}$$

$$\frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1)+B(t-1)}{(t-1)(t+1)}$$

$t=-1 \rightarrow 1 = -2B \quad B = -\frac{1}{2}$

$t=1 \rightarrow 1 = 2A \quad A = \frac{1}{2}$

$$\int \frac{1}{2\cos(x)+\sin(2x)}\,dx = -\left[\frac{1}{2}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1}\right] = \frac{1}{2}\left[\ln|t+1| - \ln|t-1|\right] + C = \frac{1}{2}\ln\left|\frac{\tan(\frac{x}{2})+1}{\tan(\frac{x}{2})-1}\right| + C$$

d) $$\int \frac{1}{\sin(x)+2\cos(x)+3}\,dx = \left| \begin{array}{l} t = \tan(\frac{x}{2}) \\ dx = \frac{2\,dt}{1+t^2} \\ \sin(x) = \frac{2t}{1+t^2} \\ \cos(x) = \frac{1-t^2}{1+t^2} \end{array} \right| = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2} + 2\frac{1-t^2}{1+t^2} + 3}\,dt = \int \frac{\frac{2}{1+t^2}}{\frac{2t+2-2t^2+3+3t^2}{1+t^2}}\,dt = \int \frac{2}{t^2+2t+5}\,dt$$

$$= 2\int \frac{dt}{(t+1)^2+4} = \left| \begin{array}{l} u = \frac{1}{2}(t+1) \\ du = \frac{1}{2}dt \end{array} \right| = 4\int \frac{du}{4u^2+4} = \int \frac{du}{1+u^2} = \arctan(u) + C = \arctan\left(\frac{\tan(\frac{x}{2})+1}{2}\right) + C$$

e) $$\int \frac{e^{2x}+e^x}{\sqrt{4-e^{2x}}}\,dx = \int \frac{(e^x+1)e^x\,dx}{\sqrt{4-e^{2x}}} = \left| \begin{array}{l} t = e^x \\ dt = e^x dx \end{array} \right| = \int \frac{t+1}{\sqrt{4-t^2}}\,dt = \int \frac{t}{\sqrt{4-t^2}}\,dt + \int \frac{dt}{\sqrt{4-t^2}}$$

$$\int \frac{t}{\sqrt{4-t^2}}\,dt = \left| \begin{array}{l} u = 4-t^2 \\ du = -2t\,dt \end{array} \right| = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot\frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C = -\sqrt{u} + C = -\sqrt{4-t^2} + C$$

$$\int \frac{dt}{\sqrt{4-t^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{1-\frac{t^2}{4}}} = \left| \begin{array}{l} u = \frac{t}{2} \\ du = \frac{1}{2}dt \end{array} \right| = \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C = \arcsin\left(\frac{t}{2}\right) + C$$

$$\int \frac{e^{2x}+e^x}{\sqrt{4-e^{2x}}}\,dx = -\sqrt{4-e^{2x}} + \arcsin\left(\frac{1}{2}e^x\right) + C$$

7.

$$\int \sin^n(x)\,dx \stackrel{?}{=} -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \qquad \Big/ \frac{d}{dx}$$

$$\Longleftrightarrow$$

$$\sin^n(x) = -\frac{1}{n}\frac{d}{dx}\left[\sin^{n-1}(x)\cos(x)\right] + \frac{n-1}{n}\cdot\sin^{n-2}(x)$$

$$\sin^n(x) = -\frac{1}{n}\left[(n-1)\sin^{n-2}(x)\cos^2(x) - \sin(x)\sin^{n-1}(x)\right] + \frac{n-1}{n}\sin^{n-2}(x)$$

$$\sin^n(x) = -\frac{n-1}{n}\sin^{n-2}(x)\cos^2(x) + \frac{1}{n}\sin^n(x) + \frac{n-1}{n}\sin^{n-2}(x)$$

$$\sin^n(x) = \frac{n-1}{n}\sin^{n-2}(x)\left[1-\cos^2(x)\right] + \frac{1}{n}\sin^n(x)$$

$$\sin^n(x) = \frac{n-1}{n}\sin^{n-2}(x)\sin^2(x) + \frac{1}{n}\sin^n(x) = \frac{n-1+1}{n}\sin^n(x) = \sin^n(x)$$

LHS = RHS