1.

$$a_n = \frac{2n-3}{5n+1} = \frac{5n+1-3n-4}{5n+1} = 1 + \frac{-3n-4}{5n+1} = 1 + \frac{1}{\frac{5n+1}{-3n-4}} \to -\frac{5}{3}$$

$$n = \frac{5n+1}{-3n-4} \cdot \frac{-3n-4}{5n+1} \cdot n = \overset{-\frac{5}{3}}{\frac{5n+1}{-3n-4}} \cdot \overset{-\infty}{\frac{-3n^2-4n}{5n+1}}$$

$$\lim_{n\to\infty} \Bigg(\underbrace{\frac{2n-3}{5n+1}}_{\to \frac{2}{5}}\Bigg)^{\overset{\infty}{n}} = 0$$

2.

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x-1}\right) = \lim_{x\to 0} \frac{e^x-1-x}{x(e^x-1)} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{e^x-1}{e^x-1+xe^x} \overset{\frac{0}{0}}{\underset{H}{=}} \lim_{x\to 0} \frac{e^x}{e^x+xe^x+e^x} = \lim_{x\to 0} \frac{1}{2+x} = \frac{1}{2}$$

3.

$$\int \frac{\operatorname{arccot}(e^x)}{e^{2x}}\,dx = \left| \begin{array}{l} t = e^x \\ dt = e^x\,dx \\ dx = \frac{dt}{t} \end{array} \right| = \int \frac{\operatorname{arccot}(t)}{t^3}\,dt = \left| \begin{array}{ll} f = \operatorname{arccot}(t) & g' = t^{-3} \\ f' = \frac{-1}{1+t^2} & g = -\frac{1}{2}t^{-2} \end{array} \right| = -\frac{\operatorname{arccot}(t)}{2t^2} - \frac{1}{2}\int \frac{dt}{t^2(1+t^2)}$$

$$\frac{1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} = \frac{At(t^2+1) + B(t^2+1) + Ct^3 + Dt^2}{t^2(t^2+1)}$$

$$1 = At^3 + At + Bt^2 + B + Ct^3 + Dt^2$$

$$1 = (A+C)t^3 + (B+D)t^2 + At + B$$

$$\begin{cases} 0 = A+C \\ 0 = B+D \\ 0 = A \\ 1 = B \end{cases} \qquad \begin{cases} A = 0 \\ B = 1 \\ C = 0 \\ D = -1 \end{cases}$$

$$\frac{1}{t^2(t^2+1)} = \frac{1}{t^2} - \frac{1}{t^2+1}$$

$$\int \frac{dt}{t^2(t^2+1)} = \int \frac{dt}{t^2} - \int \frac{dt}{t^2+1} = -t^{-1} - \arctan(t) + C$$

$$\int \frac{\operatorname{arccot}(e^x)}{e^{2x}}\,dx = -\frac{\operatorname{arccot}(t)}{2t^2} + \frac{1}{2t} + \frac{\arctan(t)}{2} + C = -\frac{\operatorname{arccot}(e^x)}{2e^{2x}} + \frac{1}{2e^x} + \frac{\arctan(e^x)}{2} + C$$

4.

$$\int \frac{4x}{x^2+3}\,dx = 2\int \frac{2x}{x^2+3}\,dx = \left| \begin{array}{l} t = x^2+3 \\ dt = 2x\,dx \end{array} \right| = 2\int \frac{dt}{t} = 2\ln|x^2+3|$$

$$\int_{-\infty}^{+\infty} \frac{4x}{x^2+3}\,dx = \int_{-\infty}^{0} \frac{4x}{x^2+3}\,dx + \int_{0}^{+\infty} \frac{4x}{x^2+3}\,dx = \lim_{S\to-\infty}\left[2\ln(3) - 2\ln(S^2+3)\right] + \lim_{T\to+\infty}\left[2\ln(T^2+3) - 2\ln(3)\right]$$

$$= \underbrace{2\lim_{T\to+\infty} \ln(T^2+3) - 2\lim_{S\to-\infty} \ln(S^2+3)}_{\infty - \infty}$$

obie rozbieżne więc $\int_{-\infty}^{\infty} f(x)$ rozbieżna

5.

$$y'' - 3y' + 2y = 3e^x$$

$$r^2 - 3r + 2 = r^2 - 2r - r + 2 = r(r-2) - 1(r-2) = (r-1)(r-2) = 0$$

$$r_1 = 1 \quad r_2 = 2$$

$$y = C_1 e^x + C_2 e^{2x}$$

$$\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 3e^x \end{bmatrix}$$

$$C_1' = \frac{-3e^x e^{2x}}{2e^{2x}e^x - e^x e^{2x}} = \frac{-3e^{3x}}{e^{3x}} = -3$$

$$C_1 = \int -3\,dx = -3x + C_1$$

$$C_2' = \frac{3e^x e^x}{e^{3x}} = 3e^{-x}$$

$$C_2 = 3\int e^{-x}\,dx = -3e^{-x} + C_2$$

$$y(x) = (-3x + C_1)e^x + (-3e^{-x} + C_2)e^{2x}$$

$$y(x) = C_1 e^x + C_2 e^{2x} - 3xe^x - 3e^x$$

$$y' = -3(xe^x + e^x) - 3e^x = -3xe^x - 6e^x$$

$$y'' = -3(xe^x + e^x) - 6e^x = -3xe^x - 9e^x$$

$$y'' - 3y' + 2y = -3xe^x - 9e^x + 9xe^x + 18e^x - 6xe^x - 6e^x$$

$$= 3e^x$$

6.

$$\sum_{n=2}^{\infty} \sin\left(\frac{\pi}{3\cdot 2^n}\right) \qquad \frac{2}{\pi}x \le \sin(x) \le x$$

$$\frac{2}{\pi}\cdot\frac{\pi}{3\cdot 2^n} \le \sin\left(\frac{\pi}{3\cdot 2^n}\right) \le \frac{\pi}{3\cdot 2^n}$$

$$\sum_{n=2}^{\infty} \frac{\pi}{3\cdot 2^n} = \frac{\pi}{3}\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^n$$ zbieżny (szereg geometryczny)

więc $\sum_{n=2}^{\infty} \sin\left(\frac{\pi}{3\cdot 2^n}\right)$ zbieżny z kryterium porównawczego