1.

a) $w(z) = z^{17} + z^{12} - z^7 - z^2$

$z^{12}(z^5+1) - z^2(z^5+1)$

$(z^5+1)(z^{12}-z^2)$

$z^2(z^5+1)(z^{10}-1)$

$z^2(z^5+1)(z^5+1)(z^5-1)$

$z^2(z^5+1)^2(z^5-1)$

$z^5+1=0$

$z^5=-1$

$r^5 e^{5\varphi i} = e^{\pi i}$

$r^5=1 \quad r=1$

$5\varphi = \pi + 2k\pi$

$\varphi = \frac{\pi+2k\pi}{5}$

$z^5-1=0$

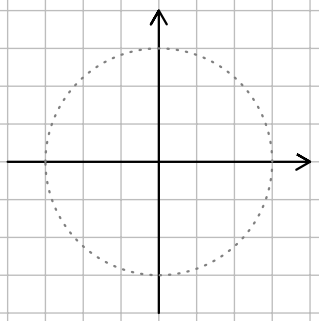
$z^5=1$

$r^5 e^{5\varphi i} = e^{0i}$

$r^5=1 \quad r=1$

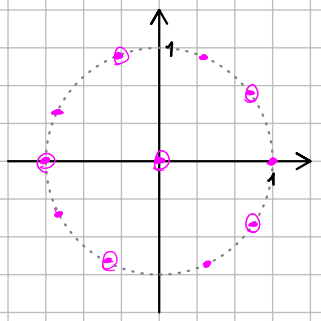
$5\varphi = 2k\pi$

$\varphi = \frac{2k\pi}{5}$



$\{ 0, e^{\frac{\pi}{5}i}, e^{\frac{3\pi}{5}i}, -1, e^{-\frac{\pi}{5}i}, e^{-\frac{3\pi}{5}i}, 1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{-\frac{2\pi}{5}i}, e^{-\frac{4\pi}{5}i} \}$

dwukrotne jednokrotne



3 różne p. rzeczywiste, 11 różnych p. zespolonych

⊙ pierwiastek 2-krotny

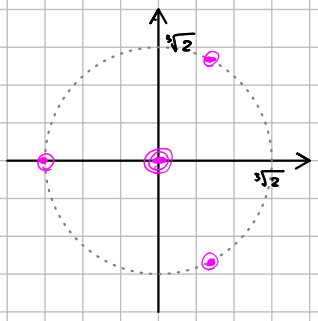
b) $w(z) = z^9 + 4z^6 + 4z^3 = z^3(z^6+4z^3+4) = z^3(z^3+2)^2$

$\sqrt[3]{-2} = \{ \sqrt[3]{2}\, e^{\frac{\pi+2k\pi}{3}i} : k=0,1,2 \}$

$\{ 0, \sqrt[3]{2}\, e^{\frac{\pi}{3}i}, -\sqrt[3]{2}, \sqrt[3]{2}\, e^{-\frac{\pi}{3}i} \}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$3 \quad 2 \quad 2 \quad 2$ krotność



2 różne rzeczywiste

4 różne zespolone

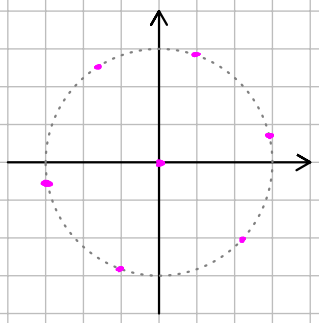
c) $w(z) = z^7 - iz = z(z^6 - i)$

$z^6 - i = 0 \rightarrow \sqrt[6]{i} = \sqrt[6]{e^{\frac{\pi}{2}i}} = \{ e^{\frac{\frac{\pi}{2}+2k\pi}{6}i} : k=0,1,2,3,4,5 \}$

$\{ 0, e^{\frac{\pi}{12}i}, e^{\frac{5\pi}{12}i}, e^{\frac{9\pi}{12}i}, e^{-\frac{3\pi}{12}i}, e^{-\frac{7\pi}{12}i}, e^{-\frac{11\pi}{12}i} \}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$ krotności



1 rzeczywisty, 7 różnych zespolonych

2.

$w(z) = z^4 + 2z^3 + 2z^2 + 10z + 25$

$z_0 = 1 + 2i$ $\quad w(z_0) = 0 \iff w(\overline{z_0}) = 0$ bo współczynniki są rzeczywiste

$z_1 = 1 - 2i$

$w(z) = (z - 1 - 2i)(z - 1 + 2i)(z^2 + bz + c)$

$= (z^2 - z + 2iz - z + 1 - 2i - 2iz + 2i + 4)(z^2 + bz + c)$

$(z^2 - 2z + 5)(z^2 + bz + c)$

$z^4 + bz^3 + cz^2 - 2z^3 - 2bz^2 - 2cz + 5z^2 + 5bz + 5c$

$z^4 + (b-2)z^3 + (c - 2b + 5)z^2 + (5b - 2c)z + 5c$

$2 = b - 2 \quad \rightarrow \quad b = 4$

$2 = c - 2b + 5$

$10 = 5b - 2c$

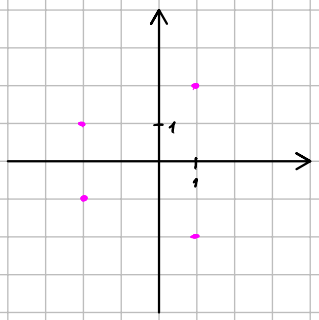
$25 = 5c \quad \rightarrow \quad c = 5$

$z^2 + 4z + 5 \qquad \Delta = 4^2 - 4 \cdot 5 = -4$

$\delta = 2i \in \sqrt{\Delta}$

$z_2 = \frac{-4 - 2i}{2} = -2 - i$

$z_3 = \frac{-4 + 2i}{2} = -2 + i$



$\{1 + 2i,\ 1 - 2i,\ -2 - i,\ -2 + i\}$

$$
\begin{array}{rl}
 & z^2 + 4z + 5 \\
\hline
z^4 + 2z^3 + 2z^2 + 10z + 25 & \big(z^2 - 2z + 5 \\
\ominus\ z^4 - 2z^3 + 5z^2 & \\
\hline
4z^3 - 3z^2 + 10z & \\
\ominus\ 4z^3 - 8z^2 + 20z & \\
\hline
5z^2 - 10z + 25 & \\
\ominus\ 5z^2 - 10z + 25 & \\
\hline
0 &
\end{array}
$$

3.

a) $v(z) = z^4 + 6z^2 + 25$

$= (z^2 - (-3-4i))(z^2 - (-3+4i))$

$\Delta = 36 - 100 = -64 \quad \delta = 8i \in \sqrt{\Delta}$

$p_1 = \frac{-6-8i}{2} = -3-4i$

$p_2 = -3+4i$

$z^2 = -3-4i \quad \vee \quad z^2 = -3+4i$

1) $z = x+yi \qquad x^2+y^2 = |z|^2 = |z^2| = 5$

$x^2 - y^2 = -3 \qquad 2x^2 = 2$

$2xyi = -4i \qquad x = 1 \vee x = -1$

$x^2 + y^2 = 5 \qquad y = -2 \quad y = 2$

2) $z = x+yi$

$x^2 - y^2 = -3 \qquad 2x^2 = 2$

$2xy = 4 \qquad x = 1 \quad \vee \quad x = -1$

$x^2 + y^2 = 5 \qquad y = 2 \qquad y = -2$

$z_0 = 1-2i \qquad z_1 = -1+2i \qquad z_2 = 1+2i \qquad z_3 = -1-2i$

$v(z) = (z-1+2i)(z+1-2i)(z-1-2i)(z+1+2i)$

b) $v(z) = z^4 - (3+2i)^4 = [z^2 - (3+2i)^2][z^2 + (3+2i)^2]$

$= (z-3-2i)(z+3+2i)(z-2+3i)(z+2-3i)$

$(a+bi)(a-bi) = a^2 - abi + abi - bi^2 = a^2 + b^2$

c) $v(z) = (1+\sqrt{3}i)z^4 + 8 = 2e^{\frac{\pi}{3}i} z^4 + 8$

$z^4 = \frac{-8}{2e^{\frac{\pi}{3}i}} = \frac{8e^{\pi i}}{2e^{\frac{\pi}{3}i}} = 4e^{\frac{2\pi}{3}i}$

$\sqrt[4]{4} = 2^{\frac{2}{4}} = \sqrt{2}$

$\sqrt[4]{4e^{\frac{2\pi}{3}i}} = \left\{ \sqrt[4]{4}\, e^{\frac{\frac{2\pi}{3} + 2k\pi}{4}i},\ k = 0,1,2,3 \right\}$

$\left\{ \sqrt{2}\, e^{\frac{\pi}{6}i}, \sqrt{2}\, e^{\frac{2\pi}{3}i}, \sqrt{2}e^{-\frac{\pi}{3}i}, \sqrt{2}\, e^{-\frac{5\pi}{6}i} \right\}$

$\left\{ \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right), \sqrt{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \sqrt{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), \sqrt{2}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) \right\}$

$\left\{ \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2} \right\}$

$v(z) = \left(z - \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}\right)\left(z + \frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}\right)\left(z - \frac{\sqrt{3}}{2} + i\frac{\sqrt{6}}{2}\right)\left(z + \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}\right)$

4.

a) $\frac{2x^2-7x}{(x-3)^3(x^2+4)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$

b) $\frac{3x^4-2}{x^4-2x^3+2x-1} = \frac{3x^4-2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} + E$

Rozkład jest tylko dla funkcji wymiernych właściwych

$x^4-1-2x^3+2x = (x^2+1)(x^2-1) - 2x(x^2-1) = (x^2-1)(x^2-2x+1) = (x+1)(x-1)(x-1)^2 = (x+1)(x-1)^3$

c) $\frac{5x^3}{(x^4+4)^2} = \frac{5x^3}{(x^2-2x+2)^2(x^2+2x+2)^2} = \frac{Ax+B}{x^2-2x+2} + \frac{Cx+D}{(x^2-2x+2)^2} + \frac{Ex+F}{x^2+2x+2} + \frac{Gx+H}{(x^2+2x+2)^2}$

$x^4+4 = x^4+4x^2+4-4x^2 = (x^2+2)^2-4x^2 = (x^2-2x+2)(x^2+2x+2)$

$\Delta = -4$ $\Delta = -4$

5.

a) $\frac{4x-10}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$4x-10 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$

dla $x=3$ $12-10 = C\cdot 2\cdot 1$ $C=1$

dla $x=2$ $8-10 = B\cdot 1\cdot(-1)$ $B=2$

dla $x=1$ $4-10 = A(-1)(-2)$ $A=-3$

$\frac{-3}{x-1} + \frac{2}{x-2} + \frac{1}{x-3}$

b) $\frac{8}{x^2-4x+2} = \frac{8}{(x-2+\sqrt{2})(x-2-\sqrt{2})} = \frac{A}{x-2+\sqrt{2}} + \frac{B}{x-2-\sqrt{2}} = \frac{A(x-2-\sqrt{2})+B(x-2+\sqrt{2})}{(x-2+\sqrt{2})(x-2-\sqrt{2})}$

$\Delta = 16-8 = 8$ $\sqrt{\Delta} = 2\sqrt{2}$ $8 = A(x-2-\sqrt{2})+B(x-2+\sqrt{2})$

$x_1 = \frac{4-2\sqrt{2}}{2} = 2-\sqrt{2}$

$x_2 = 2+\sqrt{2}$

dla $x = 2+\sqrt{2}$ $8 = B(2+\sqrt{2}-2+\sqrt{2})$ $B = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$

dla $x = 2-\sqrt{2}$ $8 = A(2-\sqrt{2}-2-\sqrt{2})$ $A = \frac{8}{-2\sqrt{2}} = -2\sqrt{2}$

$\frac{2\sqrt{2}}{x-2-\sqrt{2}} - \frac{2\sqrt{2}}{x-2+\sqrt{2}}$

c) $\frac{x^2-3x+6}{x^4-5x^2+4} = \frac{x^2-3x+6}{(x+1)(x-1)(x-2)(x+2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x+2}$

$x^4-5x^2+4 = x^4-4x^2-x^2+4 = x^2(x^2-4)-(x^2-4) = (x^2-1)(x^2-4) = (x+1)(x-1)(x-2)(x-2)$

$x^2-3x+6 = A(x-1)(x-2)(x+2)+B(x+1)(x-2)(x+2)+C(x+1)(x-1)(x+2)+D(x-1)(x+1)(x-2)$

dla $x=-1$ $1+3+6 = A(-2)(-3)(1)$ $A = \frac{10}{6} = \frac{5}{3}$

dla $x=1$ $1-3+6 = B(2)(-1)(3)$ $B = \frac{4}{-6} = -\frac{2}{3}$

dla $x=2$ $4-6+6 = C(3)(1)(4)$ $C = \frac{4}{12} = \frac{1}{3}$

dla $x=-2$ $4+6+6 = D(-3)(-1)(-4)$ $D = \frac{16}{-12} = -4/3$

$\frac{5/3}{x+1} - \frac{2/3}{x-1} + \frac{1/3}{x-2} - \frac{4/3}{x+2}$

d) $$\frac{2x^3+5x^2+3x+1}{x^2(x^2+1)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{(x^2+1)^2}$$

$$2x^3+5x^2+3x+1=Ax(x^2+1)^2+B(x^2+1)^2+(Cx+D)x^2(x^2+1)+(Ex+F)x^2$$

dla $x=0 \quad 1=B$ $\qquad (x^2+1)^2=x^4+2x^2+1$

$$Ax^5+2Ax^3+Ax+Bx^4+2Bx^2+B+Cx^5+Cx^3+Dx^4+Dx^2+Ex^3+Fx^2$$

$$(A+C)x^5+(B+D)x^4+(2A+C+E)x^3+(2B+D+F)x^2+Ax+B$$

$$\begin{cases} 0=A+C \\ 0=B+D \\ 2=2A+C+E \\ 5=2B+D+F \\ 3=A \\ 1=B \end{cases} \qquad \begin{matrix} A=3 \\ B=1 \\ C=-3 \\ D=-1 \\ E=-1 \\ F=4 \end{matrix}$$

$$\frac{3}{x}+\frac{1}{x^2}-\frac{3x+1}{x^2+1}-\frac{x-4}{(x^2+1)^2}$$

albo można podstawić $x=j$
(pierwiastki zespolone wielokrotne)

6.

a) $$\frac{4z+3j}{z^2-2jz+3}=\frac{4z+3j}{(z+j)(z-3j)}=\frac{A}{z+j}+\frac{B}{z-3j}$$

$$z^2-3jz+jz+3=z(z-3j)+j(z-3j)=(z+j)(z-3j)$$

b) $$\frac{z^4+3z-2}{z^2(z^2+9)(z+3j)^2}=\frac{z^4+3z-2}{z^2(z+3j)^3(z-3j)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z-3j}+\frac{D}{z+3j}+\frac{E}{(z+3j)^2}+\frac{F}{(z+3j)^3}$$

c) $$\frac{z^7+z^2}{(z^4+4)^2}=\frac{A}{z-1-j}+\frac{B}{(z-1-j)^2}+\frac{C}{z+1+j}+\frac{D}{(z+1+j)^2}+\frac{E}{z-1+j}+\frac{F}{(z-1+j)^2}+\frac{G}{z+1-j}+\frac{H}{(z+1-j)^2}$$

$$(z^4+4)^2=\left[(z^2-2j)(z^2+2j)\right]^2=(z-1-j)^2(z+1+j)^2(z-1+j)^2(z+1-j)^2$$

$$\sqrt{2j}=\sqrt{2e^{\frac{\pi}{2}j}}=\left\{\sqrt{2}e^{\frac{\pi}{4}j},\ \sqrt{2}e^{-\frac{3\pi}{4}j}\right\}=\{1+j,\ -1-j\}$$

$$\sqrt{-2j}=\sqrt{2e^{-\frac{\pi}{2}j}}=\left\{\sqrt{2}e^{-\frac{\pi}{4}j},\ \sqrt{2}e^{\frac{3\pi}{4}j}\right\}=\{1-j,\ -1+j\}$$

7.

a) $\frac{4z^2+1}{z^4-1}$ $f(z)=4z^2+1$ $g(z)=z^4-1$ $g'(z)=4z^3$ $z^4-1=(z^2-1)(z^2+1)=(z-1)(z+1)(z-i)(z+i)$

$a(z)=\frac{4z^2+1}{4z^3}$ $\frac{1}{i}\cdot\frac{i}{i}=\frac{i}{-1}=-i$

$a(1)=\frac{5}{4}$ $a(-1)=-\frac{5}{4}$ $a(i)=\frac{-3}{-4i}=-\frac{3}{4}i$ $a(-i)=\frac{-3}{-i\cdot(-1)-4}=-\frac{3}{4i}=\frac{3}{4}i$

$$\frac{4z^2+1}{z^4-1}=\frac{5/4}{z-1}-\frac{5/4}{z+1}-\frac{i\,3/4}{z-i}+\frac{i\,3/4}{z+i}$$

b) $\frac{z^{2020}}{z^{2021}+1}=\frac{f(z)}{g(z)}$ $g'(z)=2021\,z^{2020}$ $\frac{f(z)}{g'(z)}=\frac{z^{2020}}{2021\,z^{2020}}=\frac{1}{2021}$

$z^{2021}+1=0 \rightarrow \sqrt[2021]{-1}=\sqrt[2021]{e^{\pi i}}=\left\{ e^{\frac{\pi+2k\pi}{2021}i} : k\in[0,2020]\cap\mathbb{Z}\right\}$

$$\frac{z^{2020}}{z^{2021}+1}=\sum_{i=0}^{2020}\frac{1/2021}{z-e^{\frac{\pi+2k\pi}{2021}i}}$$