1.

O – liczba orłów $\qquad O \sim B(n, p=\frac{1}{3})$

K – liczba oczek

k	1	2	3
$P(K=k)$	$1/3$	$1/3$	$1/3$

a) $P(O=2) = P(O=2 \mid K=1)P(K=1) + P(O=2 \mid K=2)P(K=2) + P(O=2 \mid K=3)P(K=3)$

$$= 0 \cdot \frac{1}{3} + \binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^0 \cdot \frac{1}{3} + \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 \cdot \frac{1}{3}$$

$$= \frac{1}{9} \cdot \frac{1}{3} + \cancel{3} \cdot \frac{1}{9} \cdot \frac{2}{3} \cdot \cancel{\frac{1}{3}} = \frac{1}{27} + \frac{2}{27} = \frac{3}{27} = \frac{1}{9}$$

b) $$P(K=3 \mid O=2) = \frac{P(O=2 \mid K=3)P(K=3)}{P(O=2)} = \frac{\frac{2}{27}}{\frac{3}{27}} = \frac{2}{3}$$

2.

$$f_{XY}(x,y) = \frac{1}{4\pi} \exp\left(-\frac{1}{4}\left[2(x-1)^2 + 2(x-1)(y+1) + (y+1)^2\right]\right)$$

$$\sqrt{\det C} = 2 \Rightarrow \det C = 4$$

$$f_{XY}(x,y) = \frac{1}{2\pi \cdot 2} \exp\left(-\frac{1}{2 \cdot 4}\left[4(x-1)^2 - 2 \cdot (-2)(x-1)(y+1) + 2(y+1)^2\right]\right)$$

$$(X,Y) \sim N\left(\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{m}, \underbrace{\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}}_{C}\right)$$

a) $E(X(X+2Y)) = E(X^2 + 2XY) = EX^2 + 2EXY = 3 + 2 \cdot (-3) = -3$

$VX = EX^2 - (EX)^2 \Longrightarrow EX^2 = VX + (EX)^2 = 2 + 1^2 = 3$

$cov(X,Y) = EXY - EX\,EY \Longrightarrow EXY = cov(X,Y) + EX\,EY = -2 + (1)(-1) = -3$

b) $$\begin{bmatrix} Z \\ T \end{bmatrix} = \begin{bmatrix} X-Y-1 \\ X+3Y+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}}_{A} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b}$$

$$m^* = Am + b = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$C^* = ACA^T = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -4 & 10 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -14 \\ -14 & 26 \end{bmatrix}$$

$$(Z,T) \sim N(m^*, C^*)$$

c) $$Y = \sum_{k=1}^{200} X_k \simeq N(200\,EX, 200\,VX) = N(200, 400)$$

$P(|Y-180| > 20) = P(Y-180 > 20 \vee Y-180 < -20)$

$= P(Y > 200 \vee Y < 160) = P(Y > 200) + P(Y < 160)$

$= 1 - F(200) + F(160) = 1 - \phi\left(\frac{200-200}{20}\right) + \phi\left(\frac{160-200}{20}\right)$

$= 1 - \phi(0) + \phi(-2) = 1 - \phi(0) + 1 - \phi(2)$

$\simeq 0.5228$

3.

$X \backslash Y$	-1	0	1	
-2	.3	.2	0	.5
0	0	.3	.2	.5
	.3	.5	.2	

$P(X^2+Y^2 \le 1) = 0 + 0.3 + 0.2 = 0.5$

$\rho(X,Y) = \dfrac{cov(X,Y)}{\sqrt{VXVY}} = \dfrac{0.5}{\sqrt{0.49}} = \dfrac{5}{7}$

$EX = -2 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = -1$

$EX^2 = 4 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 2$

$VX = 2 - (-1)^2 = 1$

$EY = -1 \cdot \frac{3}{10} + 0 \cdot \frac{5}{10} + 1 \cdot \frac{2}{10} = -\frac{1}{10}$

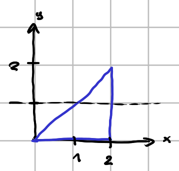
$EY^2 = 1 \cdot \frac{3}{10} + 0 \cdot \frac{5}{10} + 1 \cdot \frac{2}{10} = \frac{1}{2}$

$VY = \frac{1}{2} - (-\frac{1}{10})^2 = 0.49$

$EXY = 2 \cdot \frac{3}{10} = \frac{6}{10}$

$cov(X,Y) = \frac{6}{10} - (-1)(-\frac{1}{10}) = \frac{1}{2}$

4.



$(X,Y) \sim U(D) \qquad |D| = \frac{1}{2} \cdot 2 \cdot 2 = 2$

$f_{XY}(x,y) = \frac{1}{2} \mathbb{1}_D(x,y)$

$D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le x\}$

$\quad = \{(x,y) : 0 \le y \le 2,\ y \le x \le 2\}$

a) dla $1 \le t < 2$

$P(X \le t \mid Y < 1)$

$= \dfrac{P(X \le t,\ Y < 1)}{P(Y<1)} = \dfrac{\frac{1}{2}\left(\frac{1}{2} \cdot 1 \cdot 1 + 1 \cdot (t-1)\right)}{\frac{3}{4}} = \dfrac{\frac{1}{2}(\frac{1}{2} + t - 1)}{\frac{3}{4}}$

$= \frac{4}{3} \cdot \frac{1}{2}\left(t - \frac{1}{2}\right) = \frac{2}{3}\left(t - \frac{1}{2}\right)$

b) $f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} \int_y^2 \frac{1}{2}\,dx = \frac{1}{2}(2-y) & y \in [0,2] \\ 0 & \text{w p.p.} \end{cases}$

$f(x \mid y) = \begin{cases} \dfrac{\frac{1}{2}}{\frac{1}{2}(2-y)} = \dfrac{1}{2-y} & x \in [0,2] \text{ dla } y \in [0,2] \\ 0 & \text{w p.p.} \end{cases}$

$E(X \mid Y=1) = \int_1^2 x \cdot \frac{1}{2-1}\,dx = \frac{1}{2}x^2 \Big|_1^2 = \frac{3}{2}$

5.

$$f(x, a) = \begin{cases} (2a+1)\, x^{2a} & x \in [0,1] \\ 0 & \text{w p.p.} \end{cases}$$

$$L = \prod^{n} f(x_i, a)$$

$$\ln L = \ln\left(\prod^{n} f(x_i,a)\right) = \sum^{n} \ln(f(x_i,a))$$

$$= \sum^{n} \ln((2a+1)\, x_i^{2a})$$

$$= \sum^{n} \ln(2a+1) + 2a\ln(x_i)$$

$$= \sum^{n} \ln(2a+1) + \sum^{n} 2a \ln(x_i)$$

$$= n\ln(2a+1) + 2a\sum^{n}\ln(x_i)$$

$$= n\ln(2a+1) + 2a\ln\left(\prod^{n} x_i\right)$$

$$\frac{\partial \ln L}{\partial a} = \frac{2n}{2a+1} + 2\ln\left(\prod^{n} x_i\right) \qquad \frac{\partial^2 \ln L}{\partial a^2} = -\frac{4n}{(2a+1)^2} < 0$$

$$\frac{2n}{2a+1} + 2\ln\left(\prod^{n} x_i\right) = 0$$

$$\frac{n}{2a+1} = -\ln\left(\prod^{n} x_i\right)$$

$$n = -2a\ln\left(\prod^{n} x_i\right) - \ln\left(\prod^{n} x_i\right)$$

$$a = -\frac{n + \ln\left(\prod^{n} x_i\right)}{2\ln\left(\prod^{n} x_i\right)}$$

Dla podanej próby

$n = 10$

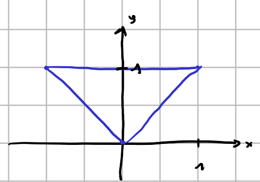
$\prod^{n} x_i \simeq 0.073$

$\ln\left(\prod^{n} x_i\right) \simeq -2.62$

$a \simeq 1.4$

6.

$$f(x,y) = \frac{15}{2} x^2 y \cdot \mathbb{1}_D(x,y)$$



$D = \{(x,y): \ -1 \le x \le 1, \ |x| \le y \le 1\}$

$\{(x,y): \ 0 \le y \le 1, \ -y \le x \le y\}$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \begin{cases} \int_{-y}^{y} \frac{15}{2} x^2 y\,dx = \frac{5}{2} y \cdot x^3\Big|_{-y}^{y} = \frac{5}{2} y(y^3 - (-y^3)) = 5y^4 & y \in [0,1] \\ 0 & \text{w p.p.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} \int_{|x|}^{1} \frac{15}{2} x^2 y\,dy = \frac{15}{4} x^2 \cdot y^2\Big|_{|x|}^{1} = \frac{15}{4} x^2(1-x^2) & x \in [-1,1] \\ 0 & \text{w p.p.} \end{cases}$$

$$F_Y(y) = \int f_Y(y)\,dy = \begin{cases} 0 & y < 0 \\ \int 5y^4 = y^5 & y \in [0,1] \\ 1 & y > 1 \end{cases}$$

$$EX = \int_{-1}^{1} \frac{15}{4} x^3 (1-x^2)\,dx = \frac{15}{4}\int_{-1}^{1} x^3 - x^5\,dx = 0$$

$$EY = \int_{0}^{1} 5y^5\,dy = \frac{5}{6} y^6\Big|_0^1 = \frac{5}{6}$$

$$EXY = \iint_{\mathbb{R}^2} xy\, f(x,y)\,dx\,dy = \iint_D \frac{15}{2} x^3 y^2\,dx\,dy = \frac{15}{2}\int_0^1 \int_{-y}^{y} x^3 y^2\,dx\,dy$$

$$= \frac{15}{2}\int_0^1 \frac{1}{4} y^2 x^4\Big|_{-y}^{y}\,dy = \frac{15}{8}\int_0^1 y^2\left(y^4 - (-y)^4\right)dy = \frac{15}{8}\int_0^1 0\,dy = 0$$

$$\operatorname{cov}(X,Y) = 0 - 0 \cdot \frac{5}{6} = 0$$

7.

$X \sim N(0,4)$
$Y \sim N(-1,9)$
niezależne

a) $P(X>0,\ |Y|<5) = P(X>0)P(|Y|<5)$

$= (1-P(X\leq 0))P(-5<Y<5)$

$= [1-\Phi(0)][\Phi(\frac{5+1}{3})-\Phi(\frac{-5+1}{3})]$

$= \frac{1}{2}(\Phi(2)-\Phi(-\frac{4}{3}))$

$= \frac{1}{2}(\Phi(2)-1+\Phi(\frac{4}{3}))$

$\simeq 0.4427$

b) $(X,Y) \sim N\left(\underbrace{\begin{bmatrix}0\\-1\end{bmatrix}}_{m}, \underbrace{\begin{bmatrix}4&0\\0&9\end{bmatrix}}_{C}\right)$

$$\begin{bmatrix}Z\\T\end{bmatrix} = \begin{bmatrix}X-Y+4\\2X+Y-5\end{bmatrix} = \underbrace{\begin{bmatrix}1&-1\\2&1\end{bmatrix}}_{A}\cdot\begin{bmatrix}X\\Y\end{bmatrix} + \underbrace{\begin{bmatrix}4\\-5\end{bmatrix}}_{m}$$

$m^* = Am+b = \begin{bmatrix}1&-1\\2&1\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}+\begin{bmatrix}4\\-5\end{bmatrix} = \begin{bmatrix}5\\-6\end{bmatrix}$

$C^* = ACA^T = \begin{bmatrix}1&-1\\2&1\end{bmatrix}\begin{bmatrix}4&0\\0&9\end{bmatrix}\begin{bmatrix}1&2\\-1&1\end{bmatrix} = \begin{bmatrix}4&-9\\8&9\end{bmatrix}\begin{bmatrix}1&2\\-1&1\end{bmatrix} = \begin{bmatrix}13&-1\\-1&25\end{bmatrix}$

$(Z,T) \sim N(m^*, C^*)$

$\det C^* = 324$

$$f_{ZT}(z,t) = \frac{1}{2\pi\sqrt{324}}\exp\left(-\frac{1}{2\cdot 324}\left[25(z-5)^2+2(z-5)(t+6)+13(t+6)^2\right]\right)$$

8.

$S_X = \{0,1,2\}$ $S_Y = \{-1,1\}$

$X \backslash Y$	-1	1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{8}$	0
	$\frac{1}{2}$	$\frac{1}{2}$

$P(Y=-1) = P(Y=1) = \frac{1}{2}$

$P(X=0, Y=-1) = P(X=0|Y=-1)P(Y=-1) = \frac{1}{4}\cdot\frac{1}{2}$

$= P(X=2, Y=-1) = P(X=1|Y=1)$

$S_{X|Y=1} = \{0,1\} \Rightarrow P(X=2, Y=1) = 0$

$P(X=1, Y=-1) = \frac{1}{2}-\frac{1}{8}-\frac{1}{8} = \frac{1}{4}$

$P(X=0, Y=1) = \frac{1}{2}-\frac{1}{8} = \frac{3}{8}$

$A = |X|+|Y| \geq 2$

$P(A) = \frac{1}{4}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$

$P(X=1|A) = \frac{\frac{1}{4}+\frac{1}{8}}{\frac{1}{2}} = \frac{3}{4}$

$P(X=2|A) = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

$F_{X|A}(x) = \begin{cases} 0 & x<1 \\ \frac{3}{4} & x\in[1,2) \\ 1 & x\geq 2\end{cases}$

$E(X|A) = 1\cdot\frac{3}{4}+2\cdot\frac{1}{4} = \frac{5}{4}$

$E(X^2|A) = 1\cdot\frac{3}{4}+4\cdot\frac{1}{4} = \frac{7}{4}$

$V(X|A) = \frac{7}{4}-\frac{25}{16} = \frac{3}{16}$

9.

B_1 – biała w I losowaniu
B_2 – biała w II losowaniu
X – liczba białych w dwu losowaniach
$S_X = \{0,1,2\}$

$P(B_2) = P(B_2|B_1)P(B_1)+P(B_2|B_1')P(B_1')$

$= \frac{5}{7}\cdot\frac{4}{6}+\frac{4}{7}\cdot\frac{2}{6} = \frac{14}{21}$

$P(B_1|B_2) = \frac{P(B_2|B_1)P(B_1)}{P(B_2)} = \frac{\frac{10}{21}}{\frac{14}{21}} = \frac{5}{7}$

$P(X=0) = P(X=0|B_1)P(B_1)+P(X=0|B_1')P(B_1')$

$= 0 + P(B_2'|B_1')P(B_1') = \frac{3}{7}\cdot\frac{2}{6} = \frac{1}{7}$

$P(X=1) = P(X=1|B_1)P(B_1)+P(X=1|B_1')P(B_1')$

$= P(B_2'|B_1)P(B_1) + P(B_2|B_1')P(B_1')$

$= \frac{2}{7}\cdot\frac{4}{6}+\frac{4}{7}\cdot\frac{2}{6} = \frac{4}{21}+\frac{4}{21} = \frac{8}{21}$

$P(X=2) = P(X=2|B_1)P(B_1)+P(X=2|B_1')P(B_1')$

$= P(B_2|B_1)P(B_1) + 0$

$= \frac{5}{7}\cdot\frac{4}{6} = \frac{10}{21}$

$E(X^2-X+1) = EX^2-EX+1 = \frac{48-28+21}{21} = \frac{41}{21}$

$EX = 0\cdot\frac{1}{7}+1\cdot\frac{8}{21}+2\cdot\frac{10}{21} = \frac{28}{21}$

$EX^2 = 0\cdot\frac{1}{7}+1\cdot\frac{8}{21}+4\cdot\frac{10}{21} = \frac{48}{21}$

10.

$X \backslash Y$	-1	0	3	
1	1/6	1/3	1/6	2/3
2	1/6	1/6	0	1/3
	1/3	1/2	1/6	

$Z = |X| - |Y|$

$X \backslash Y$	-1	0	3
1	0	1	-2
2	1	2	-1

z	-2	0	1	2
$P(Z=z)$	1/6	1/6	1/2	1/6

$$F_Z(z) = \begin{cases} 0 & z < -2 \\ \frac{1}{6} & z \in [-2,0) \\ \frac{2}{6} & z \in [0,1) \\ \frac{5}{6} & z \in [1,2) \\ 1 & z \geq 2 \end{cases}$$

a) $V(3X - 6Y - 10) = V(3X - 6Y) = V(3X) - 2\mathrm{cov}(3X, 6Y) + V(6Y)$

$= 9VX - 36\,\mathrm{cov}(X,Y) + 36VY = 9 \cdot \frac{2}{9} - 36 \cdot \frac{2}{9} + 36 \cdot \frac{65}{36}$

$= 2 - 8 + 65 = 59$

$EX = 1 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{4}{3}$

$EX^2 = 1 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = 2$

$VX = 2 - \frac{16}{9} = \frac{2}{9}$

$EY = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6} = \frac{1}{6}$

$EY^2 = 1 \cdot \frac{1}{3} + 9 \cdot \frac{1}{6} = \frac{11}{6}$

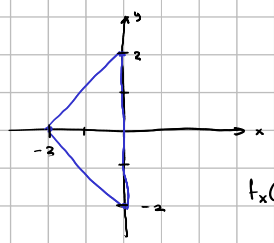
$VY = \frac{11}{6} - \frac{1}{36} = \frac{65}{36}$

$EXY = -1 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} - 2 \cdot \frac{1}{6} = 0$

$\mathrm{cov}(X,Y) = 0 - \frac{4}{3} \cdot \frac{1}{6} = \frac{2}{9}$

11.

$f(x,y) = \frac{3}{16}(x+2) \cdot \mathbb{1}_D(x,y)$



$D = \{(x,y) : -2 \leq x \leq 0,\ -x-2 \leq y \leq x+2\}$

$= \{(x,y) : -2 \leq y \leq 2,\ |y| - 2 \leq x \leq 0\}$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} \int_{-x-2}^{x+2} \frac{3}{16}(x+2)\,dy = \frac{3}{8}(x+2)^2 & x \in [-2,0] \\ 0 & \text{w pp} \end{cases}$$

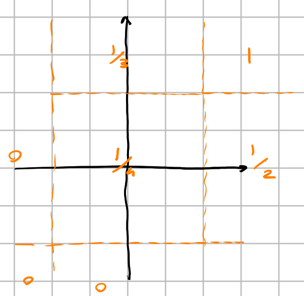
$$f(y|x) = \begin{cases} \dfrac{\frac{3}{16}(x+2)}{\frac{3}{8}(x+2)^2} = \dfrac{1}{2x+4} & y \in [-2,2] \text{ dla } x \in [-2,0] \\ 0 & \text{w pp} \end{cases}$$

$E(Y|X=-1) = \int_{-1}^{1} y f(y|-1)\,dy = \int_{-1}^{1} y \cdot \frac{1}{2}\,dy = \frac{1}{4} y^2 \Big|_{-1}^{1} = 0$

$E(Y^2|X=-1) = \int_{-1}^{1} y^2 f(y|-1)\,dy = \int_{-1}^{1} y^2 \cdot \frac{1}{2}\,dy = \frac{1}{6} y^3 \Big|_{-1}^{1} = \frac{1}{3}$

$V(Y|X=-1) = \frac{1}{3} - 0^2 = \frac{1}{3}$

12.



$P(X=-1, Y=-1) = \frac{1}{4} - 0 - 0 + 0 = \frac{1}{4}$

$P(X=1, Y=-1) = \frac{1}{2} - \frac{1}{4} - 0 + 0 = \frac{1}{4}$

$P(X=-1, Y=1) = \frac{1}{3} - 0 - \frac{1}{4} + 0 = \frac{1}{12}$

$P(X=1, Y=1) = 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{5}{12}$

$X \backslash Y$	-1	1	
-1	1/4	1/12	1/3
1	1/4	5/12	2/3
	1/2	1/2	

$X + Y = 0$

$EX = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$EX^2 = \frac{2}{3} + \frac{1}{3} = 1$

$VX = 1 - \frac{1}{9} = \frac{8}{9}$

$EY = \frac{1}{2} - \frac{1}{2} = 0$

$EY^2 = \frac{1}{2} + \frac{1}{2} = 1$

$VY = 1 - 0^2 = 1$

$EXY = \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{12} + \frac{5}{12} = \frac{1}{3}$

$\mathrm{cov}(X,Y) = \frac{1}{3} - \frac{1}{3} \cdot 0 = \frac{1}{3}$

$\rho(X,Y) = \dfrac{\frac{1}{3}}{\sqrt{\frac{8}{9} \cdot 1}} = \dfrac{1}{\sqrt{8}}$

$P(X+Y=0) = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$

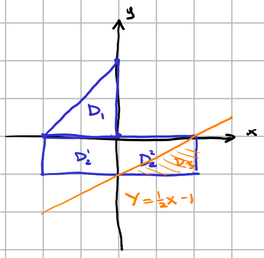
$P(Y=-1 \mid X+Y=0) = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$

$P(Y=1 \mid X+Y=0) = \dfrac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$

$E(Y \mid X+Y=0) = -1 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = -\frac{1}{2}$

13.

$$f(x,y) = \begin{cases} ax^2 & (x,y)\in D_1 \\ b & (x,y)\in D_2 \\ 0 & \text{u p.p.} \end{cases} \qquad P(Y<0) = \frac{1}{2}$$



$$D_1 = \{(x,y):\ -2 \le x \le 0,\ 0 \le y \le x+2\}$$
$$= \{(x,y):\ 0 \le y \le 2,\ y-2 \le x \le 0\}$$

$$P(Y<0) = \iint_{Y<0} f(x,y)\,dx\,dy = \iint_{D_2} b\,dx\,dy = |D_2|\cdot b = 4b$$

$$\frac{1}{2} = 4b \Rightarrow b = \frac{1}{8}$$

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1 \Longrightarrow \iint_{D_1} f(x,y)\,dx\,dy = \frac{1}{2}$$

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_{-2}^{0}\int_{0}^{x+2} ax^2\,dy\,dx = \int_{-2}^{0} ax^2(x+2)\,dx = a\int_{-2}^{0} x^3+2x^2\,dx$$

$$= a\left[\frac{1}{4}x^4 + \frac{2}{3}x^3\right]_{-2}^{0} = a\left(\frac{1}{4}\cdot 0 + \frac{2}{3}\cdot 0 - \frac{1}{4}(-2)^4 - \frac{2}{3}(-2)^3\right)$$

$$= a\left(-\frac{16}{4} + \frac{16}{3}\right) = a\,\frac{64-48}{12} = \frac{16}{12}a = \frac{4}{3}a$$

$$\frac{1}{2} = \frac{4}{3}a \Longrightarrow a = \frac{3}{8}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} \int_{0}^{x+2} \frac{3}{8}x^2\,dy + \int_{-1}^{0}\frac{1}{8}\,dy = \frac{3}{8}x^2(x+2) + \frac{1}{8} & x\in[-2,0] \\ \int_{-1}^{0}\frac{1}{8}\,dy = \frac{1}{8} & x\in(0,2] \\ 0 & \text{u p.p.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \begin{cases} \int_{y-2}^{0} \frac{3}{8}x^2\,dx = \frac{1}{8}x^3\Big|_{y-2}^{0} = -\frac{1}{8}(y-2)^3 & y\in[0,2] \\ \int_{-2}^{2}\frac{1}{8}\,dx = \frac{1}{2} & y\in[-1,0) \\ 0 & \text{u p.p.} \end{cases}$$

$$P\left(Y > \frac{1}{2}X - 1\right) = 1 - P\left(Y < \frac{1}{2}X - 1\right) = 1 - \frac{1}{8}|D_3| = 1 - \frac{1}{8}\,\frac{1}{2}\cdot 2\cdot 1 = \frac{7}{8}$$

14.

$S_{100} \sim B(n=100, p=0.1)$ dwumianowy

$S_{100} \simeq P(\lambda=10)$ $\lambda = np$

$S_{100} \approx N(10, 9)$

$$P(7 < S_{100} < 20) = P(8 \le S_{100} \le 19) = \Phi\left(\frac{19-10}{3}\right) - \Phi\left(\frac{8-10}{3}\right)$$
$$= \Phi(3) - \Phi\left(-\frac{2}{3}\right) = \Phi(3) + \Phi\left(\frac{2}{3}\right) - 1$$
$$\simeq 0.7441$$

$$P(S_{100} > M) = 1 - P(S_{100} \le M) = 1 - \Phi\left(\frac{M-10}{3}\right) = \Phi\left(\frac{10-M}{3}\right) = 0.9099$$
$$\frac{10-M}{3} = 1.34 \implies M = 5.98$$

15.

a) $X \sim B(n=800, p=0.4) \simeq N(320, 192)$
$$P(300 < X \le 400) = \Phi(5.77) - \Phi(-1.44) = \Phi(5.77) + \Phi(1.44) - 1$$
$$\simeq \Phi(1.44) \simeq 0.9251$$

b) $$P(X > N) = 1 - P(X \le N) = 1 - \Phi\left(\frac{N-320}{\sqrt{192}}\right) = 0.0228$$
$$\Phi\left(\frac{N-320}{\sqrt{192}}\right) = 0.9772 \simeq \Phi(2)$$
$$\frac{N-320}{\sqrt{192}} = 2 \implies N \simeq 347$$

16.

Z - złapane myszy

z	1	2	3
$P(Z=z)$	$1/6$	$1/2$	$1/3$

U - liczba myszy, które uciekły $U \sim B(z, \frac{1}{2})$

$$P(X=0) = P(U=1 \mid Z=1)P(Z=1) + P(U=2 \mid Z=2)P(Z=2) + P(U=3 \mid Z=3)P(Z=3)$$
$$= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{12} + \frac{1}{8} + \frac{1}{24} = \frac{6}{24}$$

$$P(X=1) = P(U=0 \mid Z=1)P(Z=1) + P(U=1 \mid Z=2)P(Z=2) + P(U=2 \mid Z=3)P(Z=3)$$
$$= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2}\binom{2}{1} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3}\binom{3}{2}\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{12} + \frac{1}{4} + \frac{1}{8} = \frac{11}{24}$$

$$P(X=2) = P(U=0 \mid Z=2)P(Z=2) + P(U=1 \mid Z=3)P(Z=3)$$
$$= \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{3}\binom{3}{1}\frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8} + \frac{1}{8} = \frac{6}{24}$$

$$P(X=3) = P(U=0 \mid Z=3)P(Z=3)$$
$$= \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{24}$$

$$P(Z=3 \mid X=1) = \frac{P(X=1 \mid Z=3)P(Z=3)}{P(X=1)} = \frac{P(U=2 \mid Z=3)P(Z=3)}{P(X=1)} = \frac{\frac{1}{8}}{\frac{11}{24}} = \frac{3}{11}$$

17.

$$P(X=x) = \begin{cases} 2p & x=1 \\ 1-5p & x=2 \\ 3p & x=3 \end{cases}$$

$$p \in (0, \tfrac{1}{5})$$

$$L = \prod^{n} P(X=x_i) = (2p)^{n_1} \cdot (1-5p)^{n_2} \cdot (3p)^{n_3}$$

$$\ln L = n_1 \ln(2p) + n_2 \ln(1-5p) + n_3 \ln(3p)$$

$$\frac{\partial \ln L}{\partial p} = \frac{n_1}{p} - \frac{5n_2}{1-5p} + \frac{n_3}{p}$$

$$\frac{\partial^2 \ln L}{\partial p^2} = -\frac{n_1}{p^2} - \frac{25n_2}{(1-5p)^2} - \frac{n_3}{p^2} < 0$$

$$\frac{n_1}{p} - \frac{5n_2}{1-5p} + \frac{n_3}{p} = 0 \implies \frac{n_1+n_3}{p} = \frac{5n_2}{1-5p}$$

$$n_1 - 5pn_1 + n_3 - 5pn_3 = 5n_2 p$$

$$n_1 + n_3 = 5pn_1 + 5pn_2 + 5pn_3$$

$$p = \frac{n_1 + n_3}{5(n_1+n_2+n_3)}$$

Dla podanej próby

$n_1 = 5 \quad n_2 = 3 \quad n_3 = 12$

$$p = \frac{5+12}{5 \cdot 20} = 0.17$$