$c = 7 \quad d = 5$

1.

a) $m = 2\cdot 5 + 3 = 13$

$$\left| (5+5j)e^{\frac{\pi}{7}j} \right| \cdot z^6 = j^{13} \cdot |z^7| \qquad z = 0 \ \checkmark$$

dla $z \neq 0 \quad r > 0$ $\qquad j^{13} = j^{12}\cdot j = 1 \cdot j = j$

$$\left| 5\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j\right)\cdot e^{\frac{\pi}{7}j} \right| \cdot r^6 e^{6\varphi j} = j\,|r^7 e^{7\varphi j}|$$

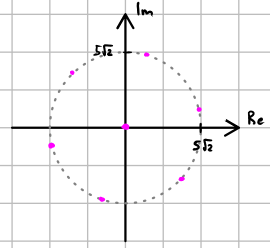
$$\left| 5\sqrt{2}\cdot e^{(\frac{\pi}{7}+\frac{\pi}{4})j} \right| \cdot r^6 e^{6\varphi j} = r^7 j$$

$$5\sqrt{2}\cdot r^6 e^{6\varphi j} = r^7 e^{\frac{\pi}{2}j}$$

$$5\sqrt{2}\, e^{6\varphi j} = r e^{\frac{\pi}{2}j}$$

$$5\sqrt{2} = r \ \wedge \ 6\varphi = \frac{\pi}{2} + 2k\pi$$

$$\varphi = \frac{\pi}{12} + \frac{k\pi}{3} = \frac{\pi + 4k\pi}{12} \qquad k \in \mathbb{Z}$$



$$5\sqrt{2}\, e^{\frac{9\pi}{12}j} = 5\sqrt{2}\, e^{\frac{3\pi}{4}j} = 5\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = -5+5j$$

$$5\sqrt{2}\, e^{-\frac{3\pi}{12}j} = 5\sqrt{2}\, e^{-\frac{\pi}{4}j} = 5\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) = 5-5j$$

b) $k = 7+5+1 = 13$

$$B = \left\{ z \in \mathbb{C} : \ \mathrm{Re}\left(2e^{\frac{7\pi}{3}j}\right) \leq |z + 2j^{13}| \leq \left| \frac{50j}{(1+2j)^4} \right| \ \vee \ \arg(3-3j) < \arg(z + 2j^{13} - 2) < \arg(2+2j) \right\}$$

$$\mathrm{Re}\left(2e^{\frac{7\pi}{3}j}\right) = \mathrm{Re}\left(2e^{\frac{\pi}{3}j}\right) = \mathrm{Re}\left(2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)j\right) = \mathrm{Re}(1+\sqrt{3}j) = 1$$

$$\left| \frac{50j}{(1+2j)^4} \right| = \frac{|50j|}{|1+2j|^4} = \frac{50}{(\sqrt{5})^4} = \frac{50}{25} = 2$$

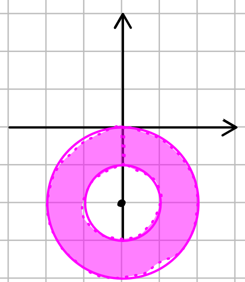
$$2j^{13} = 2j^{12}j = 2(j^4)^3 j = 2j$$

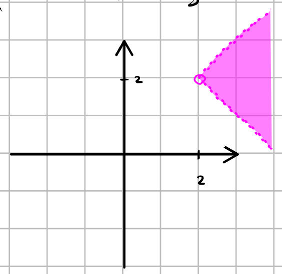
$$\arg(3-3j) = \arg\left(3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right)\right) = \arg\left(3\sqrt{2}\, e^{-\frac{\pi}{4}j}\right) = -\frac{\pi}{4}$$

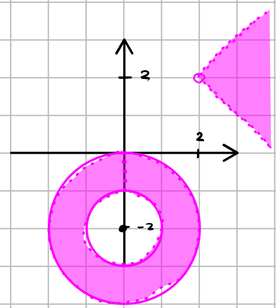
$$\arg(2+2j) = \arg\left(2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)\right) = \arg\left(2\sqrt{2}\, e^{\frac{\pi}{4}j}\right) = \frac{\pi}{4}$$

$$2j^{13} - 2 = -2 + 2j$$

$B_1 = \{ z \in \mathbb{C} : \ 1 \leq |z - (-2j)| \leq 2 \}$ $\qquad B_2 = \{ z \in \mathbb{C} : \ -\frac{\pi}{4} < \arg(z - (2-2j)) < \frac{\pi}{4} \}$ $\qquad B = B_1 \cup B_2$







2. $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 10x + 15y - 5z \\ 5y \\ 10x + 30y - 5z \end{bmatrix} = \begin{bmatrix} 10 & 15 & -5 \\ 0 & 5 & 0 \\ 10 & 30 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad M_{E_3}^{E_3}(\varphi) = \begin{bmatrix} 10 & 15 & -5 \\ 0 & 5 & 0 \\ 10 & 30 & -5 \end{bmatrix}$$

$$M_{E_3}^{A}(\varphi) = M_{E_3}^{E_3}(\varphi)\cdot M_{E_3}^{A}(id) = \begin{bmatrix} 10 & 15 & -5 \\ 0 & 5 & 0 \\ 10 & 30 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & -5 \\ 0 & 5 & -15 \end{bmatrix}$$

$$\begin{array}{cc} M_{E_3}(A) & I_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_3 - 2w_1]{} \left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow[-1\cdot w_2]{w_1 + w_3} \left[\begin{array}{cccccc} 1 & 0 & -3 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow[w_3 + 3w_2]{w_1 + 3w_2} \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{\cdot(-1)} \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$M_A(E_3)$

$$M_{A}^{A}(\varphi) = M_{A}^{E_3}(id)\cdot M_{E_3}^{E_3}(\varphi)\cdot M_{E_3}^{A}(id) = (M_{E_3}(A))^{-1}\cdot M_{E_3}^{A}(\varphi) = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & -5 \\ 0 & 5 & -15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 10 & 15 & -5 & 0 \\ 0 & 5 & 0 & 0 \\ 10 & 30 & -5 & 0 \end{array}\right] \xrightarrow[\substack{\frac{1}{5}w_2 \\ w_3 - w_1}]{} \left[\begin{array}{cccc} 10 & 15 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{array}\right] \xrightarrow{w_1 - 15w_2} \left[\begin{array}{cccc} 10 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccc} x & y & \uparrow \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \qquad t \in \mathbb{R}$$

$\dim \operatorname{Ker}\varphi = 1$

$\dim \operatorname{Im}\varphi = \operatorname{rank}(\varphi) = 2$

$\dim \mathbb{R}^3 = 3 = 2 + 1$

$$M_{E_3}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & -3 \end{bmatrix} \qquad \operatorname{Ker}\varphi = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in \operatorname{Ker}\varphi$$

reszta nie daje się wyrazić jako kombinacja $(\frac{1}{2}, 0, 1)$

$$\left[\begin{array}{ccc|c|c|c} 10 & 15 & -5 & 1 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 & -1 \\ 10 & 30 & -5 & 2 & 1 & -3 \end{array}\right] \xrightarrow[\;v_3 - v_1]{3 \cdot v_2} \left[\begin{array}{ccc|c|c|c} 10 & 15 & -5 & 1 & 1 & 0 \\ 0 & 15 & 0 & 0 & 0 & -3 \\ 0 & 15 & 0 & 1 & 0 & -3 \end{array}\right] \xrightarrow[\frac{1}{15} v_2]{\frac{1}{10} v_1} \left[\begin{array}{ccc|c|c|c} 1 & \frac{3}{2} & -\frac{1}{2} & \times & \frac{1}{10} & 0 \\ 0 & 1 & 0 & \times & 0 & -\frac{1}{5} \end{array}\right] \quad \left[\begin{array}{ccc|c|c|c} 1 & 0 & -0.5 & \times & 0.1 & 0.3 \\ 0 & 1 & 0 & \times & 0 & -0.2 \end{array}\right]$$

$M_{E_3}^{E_3}(\varphi)$ $\quad v_1 \quad v_2 \quad v_3$ $\qquad \cancel{v_1}$

$$v_2' = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix} \qquad v_3' = \begin{bmatrix} 0.3 \\ -0.2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix}$$

$\operatorname{rank}(A|v_2) = \operatorname{rank}(A|v_3) = \operatorname{rank}(A) = 2$

$v_1 \notin \operatorname{Im}\varphi$

$v_2, v_3 \in \operatorname{Im}\varphi$

3.

$$\left[\begin{array}{ccc|c} 3 & a^2 & 4-a^2 & a \\ 1 & 2 & a & a^2+1 \\ 2 & 1 & 2a & 2a^2-1 \end{array}\right] \qquad \begin{vmatrix} 3 & a^2 & 4-a^2 \\ 1 & 2 & a \\ 2 & 1 & 2a \end{vmatrix} \underset{v_3 - 2v_2}{=} \begin{vmatrix} 3 & a^2 & 4-a^2 \\ 1 & 2 & a \\ 0 & -3 & 0 \end{vmatrix} = - \begin{vmatrix} 3 & 4-a^2 & a^2 \\ 1 & a & 2 \\ 0 & 0 & -3 \end{vmatrix} = - \begin{vmatrix} 3 & 4-a^2 \\ 1 & a \end{vmatrix} \cdot (-3) = (3a - 4 + a^2) \cdot 3 = 3(a^2 + 4a - a - 4) = 3(a+4)(a-1)$$

1° $a \in \mathbb{R} \setminus \{-4, 1\}$

$\operatorname{Rank}(A|B) = \operatorname{Rank}(A) = 3$

dokładnie 1 rozwiązanie

2° $a = -4$

$$\left[\begin{array}{ccc|c} 3 & 16 & -12 & -4 \\ 1 & 2 & -4 & 17 \\ 2 & 1 & -8 & 31 \end{array}\right] \xrightarrow[v_2 - 2v_1]{v_1 - 3v_2} \left[\begin{array}{ccc|c} 0 & 10 & 0 & -55 \\ 1 & 2 & -4 & 17 \\ 0 & -3 & 0 & -3 \end{array}\right]$$

układ sprzeczny $\quad \operatorname{rank}(A) = 2 \quad \operatorname{rank}(A|B) = 3$

3° $a = 1$

$$\left[\begin{array}{ccc|c} 3 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{array}\right] \xrightarrow[v_2 - 2v_1]{v_1 - 3v_2} \left[\begin{array}{ccc|c} 0 & -5 & 0 & -5 \\ 1 & 2 & 1 & 2 \\ \cancel{0} & \cancel{-3} & \cancel{0} & \cancel{-3} \end{array}\right] \xrightarrow{-\frac{1}{5} v_1} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$x \quad y \quad t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad t \in \mathbb{R}$$

$\operatorname{rank}(A|B) = \operatorname{rank}(A) = 2$

nieskończenie wiele rozwiązań, 1 parametr

4.

$$A = \left\{ x \in \mathbb{R} : \exists n \in \mathbb{N} \;\; 2 + \frac{3}{n} \le x < 5 + \frac{8}{\sqrt{n}} \right\} = \bigcup_{i=1}^{\infty} P_i = (2, 13)$$

$$B = \left\{ x \in \mathbb{R} : \forall n \in \mathbb{N} \;\; 2 + \frac{3}{n} \le x < 5 + \frac{8}{\sqrt{n}} \right\} = \bigcap_{i=1}^{\infty} P_i = \{5\}$$

$$C = \{ x \in \mathbb{R} : x < 8 \Rightarrow x \in B \} = \{ x \in \mathbb{R} : x \ge 8 \vee x \in B \} = \{5\} \cup [8, +\infty)$$

$P_i = \{ x \in \mathbb{R} : 2 + \frac{3}{n} \le x < 5 + \frac{8}{\sqrt{n}} \}$

$P_1 = [5, 13)$

$P_4 = [2.75, 9)$

$P_n \to (2, 5]$

5.

$$W = \{ (x, y, z, t, v) : |x - 2z + 3t| + |3z - 2t - v| = 0 \}$$

$$|x - 2z + 3t| + |3z - 2t - v| = 0 \iff x - 2z + 3t = 0 \wedge 3z - 2t - v = 0 \iff \begin{array}{l} x = 2z - 3t \\ v = 3z - 2t \end{array}$$

$$\begin{bmatrix} x \\ y \\ z \\ t \\ v \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$v = (x, y, z, t, v) \qquad \alpha v = (\alpha x, \alpha y, \alpha z, \alpha t, \alpha v)$

$$|\alpha x - 2\alpha z + 3\alpha t| + |3\alpha z - 2\alpha t - \alpha v| = |\alpha| \cdot |x - 2z + 3t| + |\alpha| \cdot |3z - 2t - v|$$

$$= |\alpha| \cdot \left[|x - 2z + 3t| + |3z - 2t - v| \right] = |\alpha| \cdot 0 = 0 \;\rightarrow\; \alpha v \in W$$

$$v_1 + v_2 \;\rightarrow\; |x_1 + x_2 - 2z_1 - 2z_2 + 3t_1 + 3t_2| + |3z_1 + 3z_2 - 2t_1 - 2t_2 - v_1 - v_2|$$

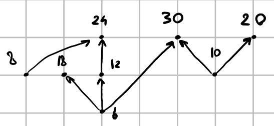
$$= |(x_1 - 2z_1 + 3t_1) + (x_2 - 2z_2 + 3t_2)| + |(3z_1 - 2t_1 - v_1) + (3z_2 - 2t_2 - v_2)|$$

$$= |0 + 0| + |0 + 0| = 0$$

$$\dim W = \operatorname{rank} \begin{bmatrix} 0 & 2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix} = 3$$

6.

$(\{6, 8, 10, 12, 18, 20, 24, 30\}, |)$



elementy maksymalne $18, 20, 24, 30$

elementy minimalne $6, 8, 10$

$\sup(A) = 120 = \operatorname{lcm}(20, 24, 30)$

$\inf(A) = 2 = \gcd(6, 8, 10)$

antyłańcuch maksymalnej długości

$8, 12, 18, 20, 30$

7.

$A \in M_{3\times 3}(\mathbb{C}) \quad \det A = 3 + \cos(\frac{5\pi}{2}) - 3 + \cos(\frac{\pi}{2}) = 3$

$\det(A + A) = \det(2A) = \det(2 \cdot I_3 \cdot A) = \det(2I_3)\det A = 2^3 \cdot 3 = 24$

$\det(A^3) = (\det A)^3 = 27$

$\det(-jA) = \det(-jI_3) \cdot \det A = (-j)^3 \cdot 3 = -1 \cdot j^2 \cdot j \cdot 3 = 3j$

$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{3}$

$\operatorname{rank}(4 \cdot A^6) = 3$ bo $\det(4A^6) = 4^3(\det A)^6 = 6^6 \neq 0$