

1.  $u = a + bj$   $z = c + dj$

a)  $|u+z|^2 = |u|^2 + |z|^2 + 2\operatorname{Re}(u\bar{z})$

$$\text{LHS} = |a + bj + c + dj|^2 = (\sqrt{(a+c)^2 + (b+d)^2})^2 = a^2 + b^2 + c^2 + d^2 + 2ac + 2bd$$

$$\begin{aligned} \text{RHS} &= |a + bj|^2 + |c + dj|^2 + 2\operatorname{Re}((a + bj)(c - dj)) \\ &= (\sqrt{a^2 + b^2})^2 + (\sqrt{c^2 + d^2})^2 + 2\operatorname{Re}(ac - adj + cbj + bd) \\ &= a^2 + b^2 + c^2 + d^2 + 2ac + 2bd = \text{LHS} \end{aligned}$$

Alternatywny sposób

$$|z|^2 = z \cdot \bar{z} \quad 2\operatorname{Re}(u) = u + \bar{u}$$

$$\begin{aligned} |u|^2 + |z|^2 + 2\operatorname{Re}(u\bar{z}) &= u \cdot \bar{u} + z \cdot \bar{z} + u\bar{z} + \bar{u}z \\ &= (z + u)(\bar{z} + \bar{u}) = (z + u)\overline{(z + u)} = |u + z|^2 \end{aligned}$$

b)  $|u - z|^2 = |u|^2 + |z|^2 - 2\operatorname{Re}(u\bar{z})$

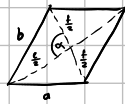
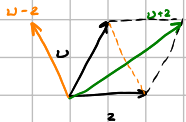
$$\text{LHS} = |a + bj - c - dj|^2 = (a - c)^2 + (b - d)^2 = a^2 + b^2 + c^2 + d^2 - 2ac - 2bd$$

$$\text{RHS} = a^2 + b^2 + c^2 + d^2 - 2ac - 2bd = \text{LHS}$$

$$\begin{aligned} |u + (-z)|^2 &= |u|^2 + |-z|^2 + 2\operatorname{Re}(u \cdot \overline{(-z)}) \\ &= |u|^2 + |z|^2 - 2\operatorname{Re}(u\bar{z}) \end{aligned}$$

c)  $|u+z|^2 + |u-z|^2 = |u|^2 + |z|^2 + 2\operatorname{Re}(u\bar{z}) + |u|^2 + |z|^2 - 2\operatorname{Re}(u\bar{z}) = 2|u|^2 + 2|z|^2$

tożsamość równoległoboku



$$b^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 + 2 \cdot \frac{1}{4}ef \cos(\alpha)$$

$$b^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 + \frac{1}{2}ef \cos(\alpha)$$

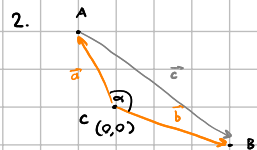
$$a^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 + \frac{1}{2}ef \cos(\pi - \alpha)$$

$$a^2 = \frac{1}{4}e^2 + \frac{1}{4}f^2 - \frac{1}{2}ef \cos(\alpha)$$

$$a^2 + b^2 = \frac{1}{2}e^2 + \frac{1}{2}f^2$$

$$e^2 + f^2 = 2a^2 + 2b^2$$

$$|u+z|^2 + |u-z|^2 = 2|u|^2 + 2|z|^2$$



$$c^2 = a^2 + b^2 - 2ab \cos(\alpha)$$

$$\begin{aligned} \cos(\alpha) &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a} - \vec{b}|^2}{2|\vec{a}| \cdot |\vec{b}|} = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a}|^2 - |\vec{b}|^2 + 2\operatorname{Re}(a\bar{b})}{2|\vec{a}| \cdot |\vec{b}|} \\ &= \frac{\operatorname{Re}((x_A + jy_A)(x_B - jy_B))}{|\vec{a}| \cdot |\vec{b}|} = \frac{x_A x_B + y_A y_B}{|\vec{a}| \cdot |\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \end{aligned}$$

$$\vec{c} = \vec{b} - \vec{a} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$$

$$|\vec{c}|^2 = |\vec{b} - \vec{a}|^2 = x_A^2 + x_B^2 + y_A^2 + y_B^2 - 2x_A x_B - 2y_A y_B$$

$$\cos(\alpha) = \frac{\operatorname{Re}(u\bar{z})}{|u||z|}$$

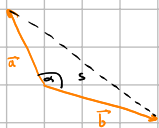
$$|u||z| = |u||z| = |u||z| = |u\bar{z}|$$

$$\sin(\alpha) = \sqrt{\frac{|u||z| - \operatorname{Re}(u\bar{z})}{|u||z|}} = \sqrt{\frac{|u||z| - \operatorname{Re}(u\bar{z})}{|u||z|}}$$

$$S = \frac{1}{2} ab \sin(\alpha)$$

$$\begin{aligned} \sin(\alpha) &= \sqrt{1 - \cos^2(\alpha)} = \sqrt{\frac{(x_A^2 + y_A^2)(x_B^2 + y_B^2) - (x_A x_B + y_A y_B)^2}{(x_A^2 + y_A^2)(x_B^2 + y_B^2)}} \\ \alpha &\in [0, \pi] \end{aligned}$$

$$= \sqrt{\frac{\operatorname{Re}(u\bar{z})^2 + |\operatorname{Im}(u\bar{z})|^2 - \operatorname{Re}(u\bar{z})^2}{|u||z|}} = \frac{|\operatorname{Im}(u\bar{z})|}{|u||z|}$$



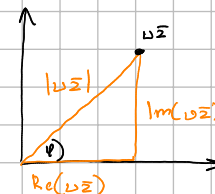
3.  $\sin(\alpha) = \frac{|x_A y_B - x_B y_A|}{\sqrt{x_A^2 + y_A^2} \cdot \sqrt{x_B^2 + y_B^2}}$

$$= \frac{|x_A y_B - x_B y_A|}{\sqrt{x_A^2 + y_A^2} \cdot \sqrt{x_B^2 + y_B^2}}$$

$$S = \frac{1}{2} ab \sin(\alpha) = \frac{1}{2} \sqrt{x_A^2 + y_A^2} \sqrt{x_B^2 + y_B^2} \cdot \frac{|x_A y_B - x_B y_A|}{\sqrt{x_A^2 + y_A^2} \cdot \sqrt{x_B^2 + y_B^2}} = \frac{1}{2} |x_A y_B - x_B y_A|$$

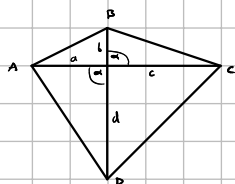
$$S = \frac{1}{2} ab \sin(\alpha) = \frac{1}{2} |u| \cdot |z| \cdot \frac{|\operatorname{Im}(u\bar{z})|}{|u||z|} = \frac{1}{2} |\operatorname{Im}((x_A + jy_A)(x_B - jy_B))| = \frac{1}{2} |-x_A y_B + x_B y_A| = \frac{1}{2} |x_A y_B - x_B y_A|$$

$$S = \frac{1}{2} \begin{vmatrix} x_A & y_A \\ x_B & y_B \end{vmatrix}$$



4.

$$|AB|^2 \cdot |CD|^2 = |AD|^2 \cdot |BC|^2 \iff AC \perp BD$$



$$\begin{aligned} |AB|^2 &= a^2 + b^2 + 2ab \cos(\alpha) & |AD|^2 &= a^2 + d^2 - 2ad \cos(\alpha) \\ |CD|^2 &= c^2 + d^2 + 2cd \cos(\alpha) & |BC|^2 &= b^2 + c^2 - 2bc \cos(\alpha) \\ a^2 + b^2 + c^2 + d^2 + 2 \cos(\alpha)(ab + cd) &= a^2 + b^2 + c^2 + d^2 - 2 \cos(\alpha)(ad + bc) \\ \cos(\alpha)(ab + cd) &= -\cos(\alpha)(ad + bc) \end{aligned}$$

$$1^\circ \cos(\alpha) = 0 :$$

$$0 = 0 \quad \checkmark$$

$$\cos(\alpha) = 0 \iff AC \perp BD$$

$$2^\circ \cos(\alpha) \neq 0 :$$

$$\underbrace{ab + cd}_{>0} = - \underbrace{(ad + bc)}_{>0}$$

sprzeczność

$$|a+d|^2 + |b+c|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2\operatorname{Re}(ad) + 2\operatorname{Re}(bc)$$

$$|a+b|^2 + |c+d|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2\operatorname{Re}(ab) + 2\operatorname{Re}(cd)$$

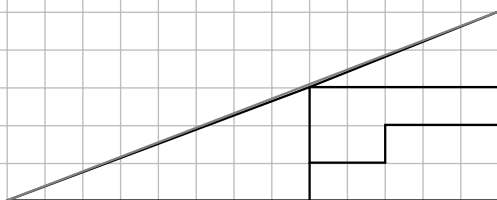
5.

$$(0,0) \quad (3,3) \quad (13,5)$$

$$S = \frac{1}{2} |x_1 y_2 - x_2 y_1|$$

$$S = \frac{1}{2} |3 \cdot 5 - 13 \cdot 3| = \frac{1}{2} \quad \text{-- różnica pola trójkąta i czworokąta wklęsłego na trójkąt}$$

stąd bierzemy stąd iluzję w zadaniu brakującego kwadratu



6.

$$D = W + a \quad \text{rotacja o } 90^\circ \circlearrowleft$$

$$T_1 = D + a \cdot (-j)$$

$$S = W + b \quad \text{rotacja o } 90^\circ \circlearrowright$$

$$T_2 = S + b \cdot j$$

$$X = \frac{T_1 + T_2}{2} = \frac{W + a - aj + W + b + bj}{2}$$

$$= \frac{W + (D - W) - (D - W)j + W + (S - W) \cdot (S - W)j}{2}$$

$$= \frac{D - Dj + Wj + S + Sj - Wj}{2} = \frac{D(1-j) + S(1+j)}{2}$$

