Filter Design by Impulse Invariance

Let $h_c(t)$ denote the impulse-response of the continuous-time filter.

The unit sample response $h[n]$ of the discrete-time filter is given by

$$h[n] = T_d \, h_c(nT_d)$$

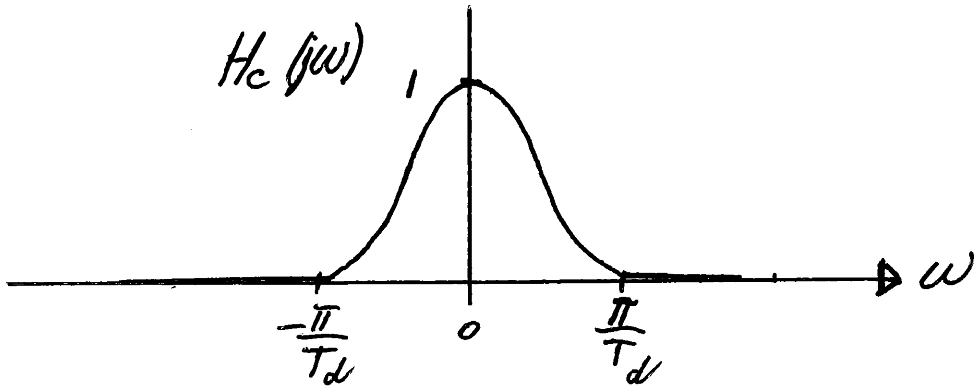
where T_d is the sampling period.

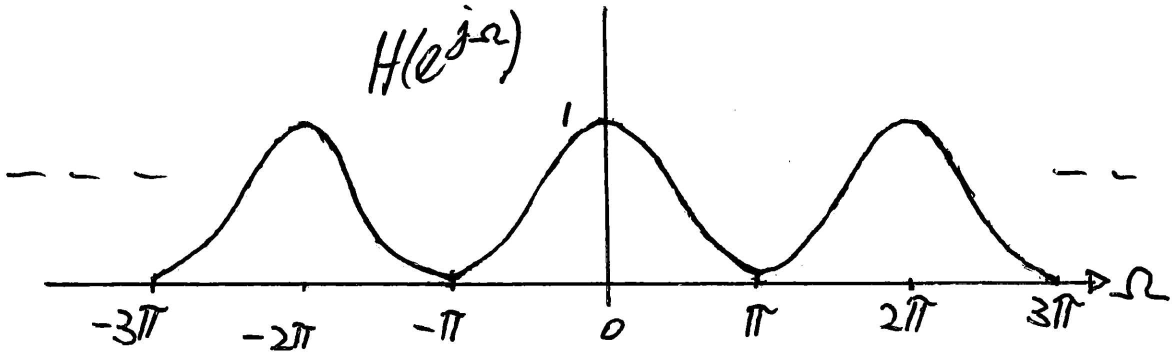
$$h_c(t) \longleftrightarrow H_c(j\omega)$$
$$h[n] \longleftrightarrow H(e^{j\Omega})$$

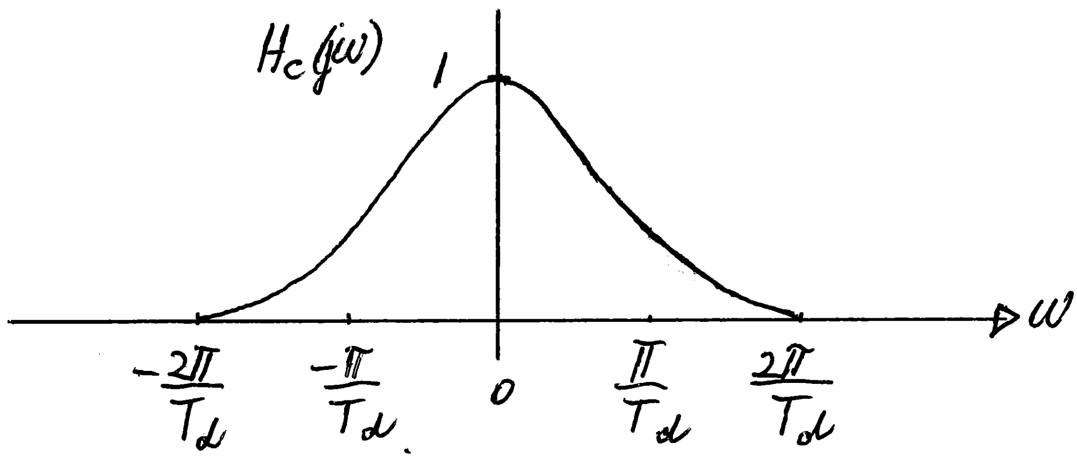
$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\Omega}{T_d} - \frac{2\pi k}{T_d}\right)\right)$$

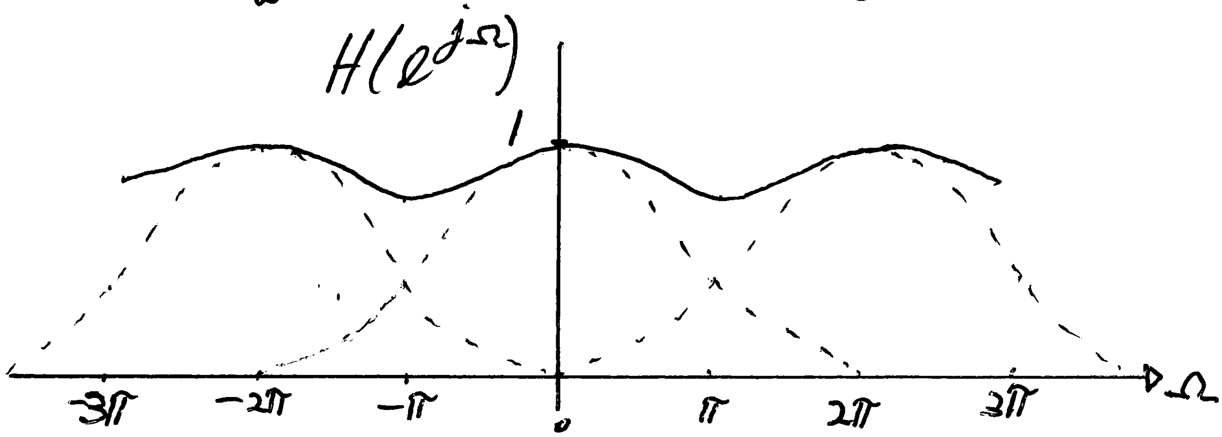
If $H_c(j\omega) = 0$ for $|\omega| \geq \frac{\pi}{T_d}$, then

$$H(e^{j\Omega}) = H_c\left(j\left(\frac{\Omega}{T_d}\right)\right), \quad |\Omega| \leq \pi$$


$H_c(j\omega)$
1
ω
$-\frac{\pi}{T_d}$
0
$\frac{\pi}{T_d}$


$H(e^{j\Omega})$
1
Ω
-3π
-2π
$-\pi$
0
π
2π
3π


$H_c(j\omega)$
1
ω
$-\frac{2\pi}{T_d}$
$-\frac{\pi}{T_d}$
0
$\frac{\pi}{T_d}$
$\frac{2\pi}{T_d}$


$H(e^{j\Omega})$
1
Ω
-3π
-2π
$-\pi$
0
π
2π
3π

Aliasing

If all the poles of $H_c(s)$ are single order, then $H_c(s)$ can be expressed in terms of a partial fraction expansion as follows:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

The corresponding impulse response is

$$h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

$$h[n] = T_d h_c(nT_d)$$

$$= \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n]$$

$$= \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$= \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

A pole at $s = s_k$ in the s-plane transforms to a pole at $e^{s_k T_d}$ in the z-plane.

$$s_k = \sigma_k + j\omega_k$$

$$e^{s_k T_d} = e^{\sigma_k T_d} e^{j\omega_k T_d}$$

$$|e^{s_k T_d}| = |e^{\sigma_k T_d}|$$

If the continuous-time filter is stable, σ_k is less than zero and $|e^{s_k T_d}| < 1$. The corresponding pole $z = e^{s_k T_d}$ is inside the unit circle. The causal discrete-time filter is also stable.

Since the design procedure for the discrete-time filter begins from a set of discrete-time specifications, the parameter T_d cancels in the procedure. We can therefore choose $T_d = 1$ so that $\Omega = \omega$.

<u>Example</u> Design a discrete-time filter to meet the following specifications:

$$0.89125 \leq |H(e^{j\Omega})| \leq 1, \quad 0 \leq |\Omega| \leq 0.1\pi$$

$$|H(e^{j\Omega})| \leq 0.17783, \quad 0.4\pi \leq |\Omega| \leq \pi$$

The filter is to be designed by applying impulse invariance to an appropriate Butterworth continuous-time filter.

Choose $T_d = 1$, so that $\Omega = \omega$.

We want to design a cts.-time Butterworth filter that meets the following specifications:

$$0.89125 \leq |H_c(j\omega)| \leq 1, \qquad 0 \leq |\omega| \leq 0.1\pi$$

$$|H_c(j\omega)| \leq 0.17783, \qquad 0.4\pi \leq |\omega| \leq \pi$$

Butterworth response

$$|H_c(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_0}\right)^{2N}}$$

where ω_0 is the 3-dB frequency.

The passband and stopband specifications of the cts.-time filter will be satisfied if

$$|H_c(j(0.1\pi))| \geq 0.89125$$

and

$$|H_c(j(0.4\pi))| \leq 0.17783$$

Using these equations with equality leads to the eqns.

$$1+\left(\frac{0.1\pi}{\omega_0}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2$$

and

$$1+\left(\frac{0.4\pi}{\omega_0}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

Solving these two eqns. we get $N=1.72$

The order of the filter must be an integer i.e. $N=2$.

With $N=2$, $\omega_0 = 0.44404$.

The transfer function of the 2nd. order Butterworth low-pass prototype filter is given by

$$H_p(s) = \frac{1}{s^2+\sqrt{2}\,s+1}$$

$$H_c(s) = H_p\left(\frac{s}{\omega_0}\right) = \frac{1}{\left(\frac{s}{0.4404}\right)^2 + \sqrt{2}\left(\frac{s}{0.4404}\right) + 1}$$

$$= \frac{0.1940}{s^2 + 0.6228\, s + 0.1940}$$

$$= \frac{0.1940}{(s-s_1)(s-s_2)}$$

$$= \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2}$$

<u>Ex.</u> Determine s_1, s_2, A_1 and A_2

$$H(z) = \frac{A_1}{1-e^{s_1} z^{-1}} + \frac{A_2}{1-e^{s_2} z^{-1}}$$

$$= \frac{0.14\, z^{-1}}{1 - 1.394\, z^{-1} + 0.536\, z^{-2}}$$

<u>note</u> If the discrete-time filter fails to meet the specifications due to aliasing, try again with a higher order filter.