3C1 Homework No. 3 — Solutions

1. $$Y(j\omega) = \mathcal{F}\{y(t)\}$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[e^{-t}\mu(t) - e^{-2t}\mu(t) \right] e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt - \int_{0}^{\infty} e^{-2t} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt - \int_{0}^{\infty} e^{-(2+j\omega)t}\, dt$$

$$= -\frac{e^{-t} e^{-j\omega t}}{1+j\omega} \Bigg|_{0}^{\infty} - \left[-\frac{e^{-2t} e^{-j\omega t}}{2+j\omega} \Bigg|_{0}^{\infty} \right]$$

$$= \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$$= \frac{1}{(1+j\omega)(2+j\omega)}$$

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

$$\frac{1}{(1+j\omega)(2+j\omega)} = X(j\omega)\left[\frac{1}{1+j\omega}\right]$$

$$X(j\omega) = \frac{1}{2+j\omega}$$

$$x(t) = \mathfrak{F}^{-1}\left\{\frac{1}{2+j\omega}\right\}$$

$$x(t) = e^{-2t}\, \mu(t).$$

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2. $$p(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

$$y(t) = x(t)\, p(t)$$
$$= x(t) \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

$X(j\omega) = \mathfrak{F}\{x(t)\}$, $\qquad Y(j\omega) = \mathfrak{F}\{y(t)\}$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left\{\sum_{k=-\infty}^{\infty} x(t)\, a_k\, e^{jk\omega_0 t}\right\} e^{-j\omega t}\, dt$$

Interchanging the order of integration and summation we get

$$Y(j\omega) = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - k\omega_0)t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k\, X\big(j(\omega - k\omega_0)\big)$$

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3. (i) See notes.

(ii) $$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\omega - \frac{2\pi k}{T}\right)\right)$$