Computation of the Discrete Fourier Transform

The sequence $x[n]$ is zero outside the interval $0 \leq n \leq N-1$. The N-point DFT of $x[n]$ is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] \, W_N^{kn} \quad , \quad k = 0, 1, \cdots, N-1 \quad — (1)$$

where $W_N = e^{-j\left(\frac{2\pi}{N}\right)}$.

In eqn. (1) $x[n]$ may be complex.

For each value of k, the direct computation of $X[k]$ requires N complex multiplications and $N-1$ complex additions. The direct computation of the DFT for all N values of k requires N^2 complex multiplications and $N(N-1)$ complex additions.

Highly efficient algorithms for

the computation of the N-point DFT are known as fast Fourier transform (FFT) algorithms.

Decimation-in-Time FFT Algorithm

The DFT computation is decomposed into successively smaller DFT computations. The algorithm exploits the symmetry and periodicity of the complex exponential

$$W_N^{kn} = e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$W_N^{k(N-n)} = \left(W_N^{kn}\right)^* \quad -(2)$$

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n} \quad -(3)$$

We will consider the special case of N an integer power of 2, i.e. $N = 2^\nu$.

Since N is an even integer we

can compute $X[k]$ by separating $x[n]$ into two $\left(\frac{N}{2}\right)$-point sequences consisting of the even-numbered points and the odd-numbered points in $x[n]$.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}$$

$$0 \le k \le N-1. \qquad — (4)$$

Eqn. (4) may be written as

$$X[k] = \sum_{r=0}^{\left(\frac{N}{2}\right)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{\left(\frac{N}{2}\right)-1} x[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{n=0}^{\left(\frac{N}{2}\right)-1} x[2r] W_N^{2rk} + W_N^{k} \sum_{r=0}^{\left(\frac{N}{2}\right)-1} x[2r+1] W_N^{2rk} \qquad — (5)$$

note that $W_N^2 = e^{-2j\left(\frac{2\pi}{N}\right)} = e^{-j\left(\frac{2\pi}{N/2}\right)}$

$$= W_{\frac{N}{2}}$$

Eqn. (5) may be rewritten as

$$X[k] = \sum_{r=0}^{(\frac{N}{2})-1} x[2r] W_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{(\frac{N}{2})-1} x[2r+1] W_{\frac{N}{2}}^{rk}$$

$$= G[((k))_{\frac{N}{2}}] + W_N^k H[((k))_{\frac{N}{2}}], \quad 0 \leq k \leq N-1.$$

(6)

$G[k]$ is the $(\frac{N}{2})$-point DFT of the even-numbered points of the original sequence.

$H[k]$ is the $(\frac{N}{2})$-point DFT of the odd-numbered points of the original sequence.

The computation of the two $\frac{N}{2}$-point DFTs in eqn. (6) requires $2\left(\frac{N}{2}\right)^2$ complex multiplications if the direct method is used to compute the DFTs. To combine the two $(\frac{N}{2})$-point DFTs requires N complex multiplications.

The computation of eqn. ⑥ for all values of k requires $N + 2\left(\frac{N}{2}\right)^2$ complex multiplications.

$$N + \frac{N^2}{2} < N^2 \text{ if } N > 2.$$

Each of the $\frac{N}{2}$ - point DFT's can be decomposed into two $\frac{N}{4}$ -point DFT's, which are then combined to yield the $\frac{N}{2}$ - point DFTs.

$$g[n] = x[2n], \quad h[n] = x[2n+1], \quad 0 \le n \le \frac{N}{2} - 1.$$

$$G[k] = \sum_{r=0}^{\frac{N}{2}-1} g[r]\, W_{\frac{N}{2}}^{rk}$$

$$= \sum_{l=0}^{\frac{N}{4}-1} g[2l]\, W_{\frac{N}{4}}^{lk} + W_{\frac{N}{2}}^{k} \sum_{l=0}^{\frac{N}{4}-1} g[2l+1]\, W_{\frac{N}{4}}^{lk}$$

The $\frac{N}{2}$ -point DFT $G[k]$ can be obtained by combining the $\frac{N}{4}$ - point DFT's of the sequences $g[2l]$ and $g[2l+1]$

The process of breaking the DFT into

successively smaller DFTS is continued until left with only 2-point DFTS.

The process of decomposing the N-point DFT into smaller DFTS can be done at most $\nu = \log_2 N$ times.

Each stage requires N complex multiplications and N complex additions. There are $\log_2 N$ stages. Without further simplification the computation of the DFT requires $N \log_2 N$ complex multiplications and $N \log_2 N$ complex additions.

Computation of the N-point DFT using simplified butterfly computations requires $\left(\frac{N}{2}\right) \log_2 N$ complex multiplications and $N \log_2 N$ complex additions.

For example, if $N = 1024$, $N^2 = 1,048,576$ and $\left(\frac{N}{2}\right) \log_2 N = 5,120$.

Two Point DFT

The two point DFT of the sequence $r[n]$ is given by

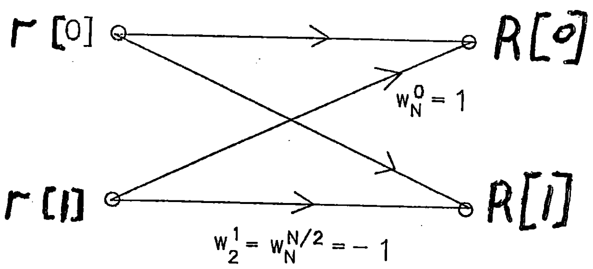
$$R[k] = \sum_{n=0}^{1} r[n]\, W_2^{kn}, \quad k = 0, 1.$$

$$R[0] = r[0]\, W_2^0 + r[1]\, W_2^0$$

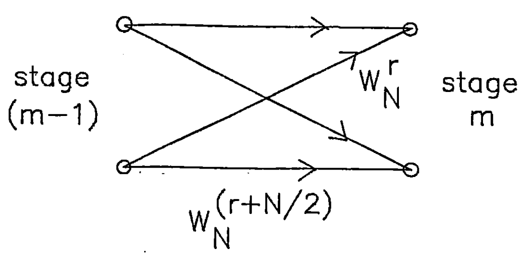
$$R[1] = r[0]\, W_2^0 + r[1]\, W_2$$

Note that $W_2^0 = 1$.

and $W_2^1 = W_N^{N/2} = -1$.



Butterfly Computation



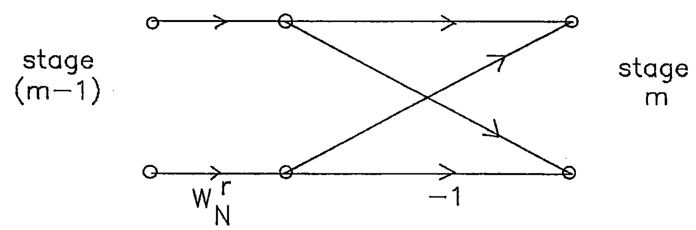
This butterfly computation requires two complex multiplications.

Note that $W_N^{(r+\frac{N}{2})} = W_N^r \cdot W_N^{N/2}$

$$W_N^{N/2} = e^{-j\left(\frac{2\pi}{N}\right)\left(\frac{N}{2}\right)} = e^{-j\pi} = -1.$$

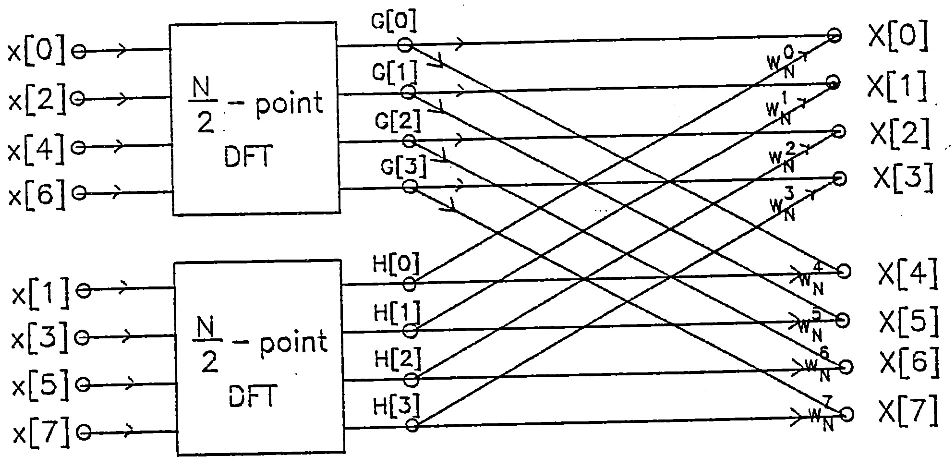
and $W_N^{(r+N/2)} = -1 \cdot W_N^r$

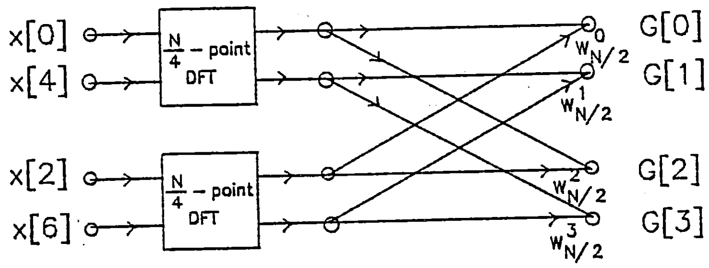
The butterfly computation may be simplified as shown below.

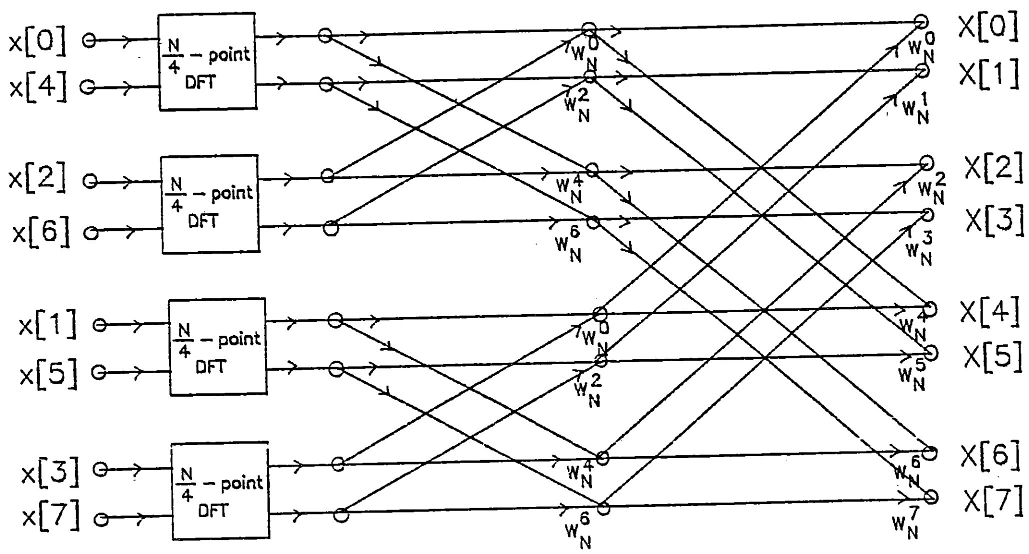


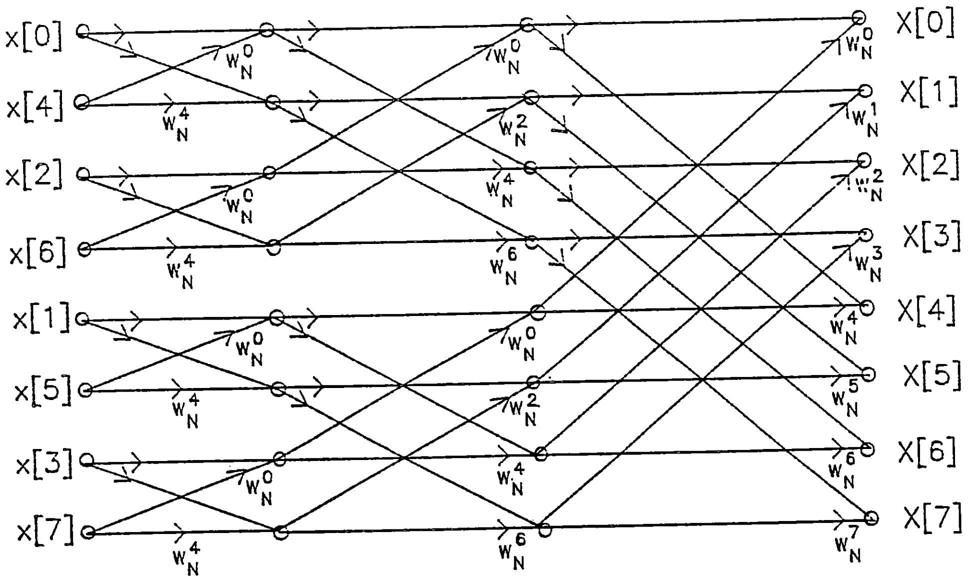
The simplified butterfly computation requires only one complex multiplication.

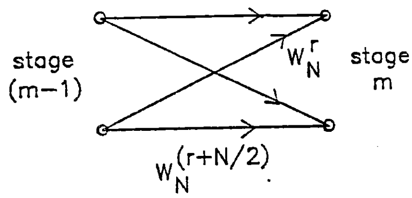
Decimation-in-Time FFT Algorithm

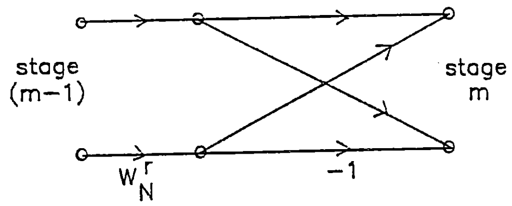


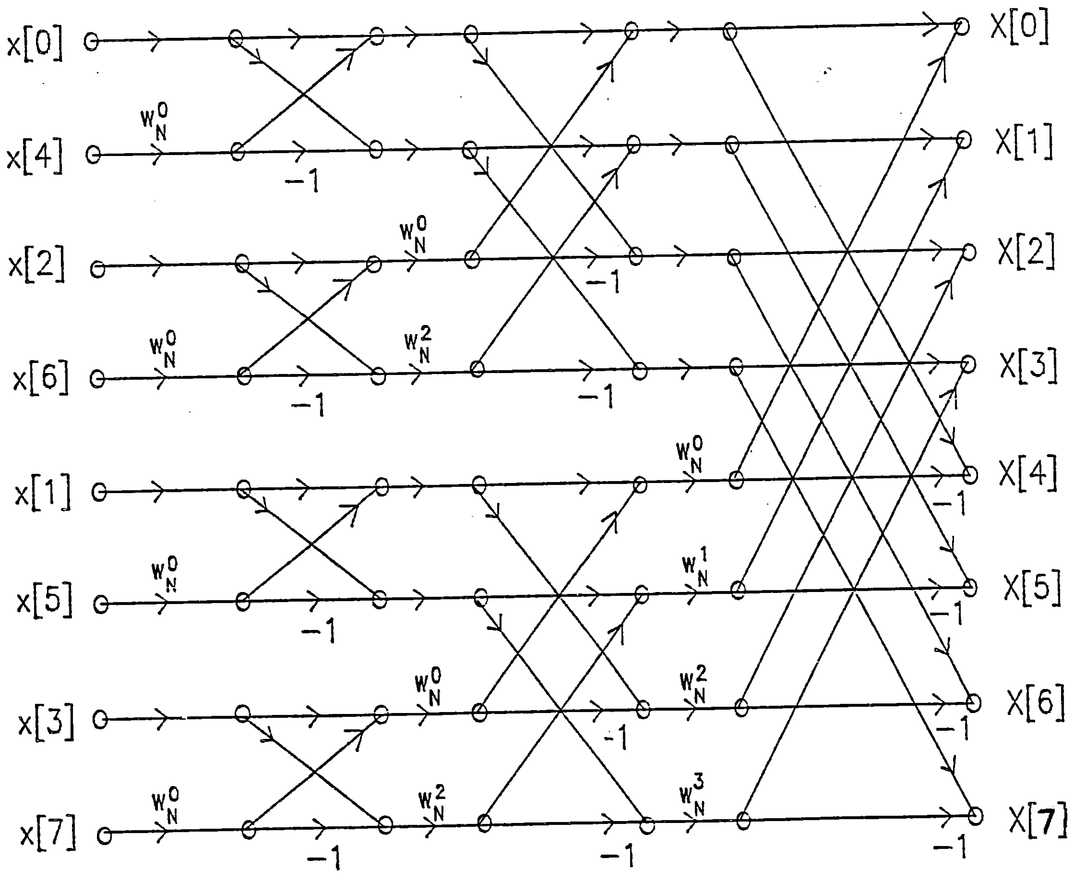





x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

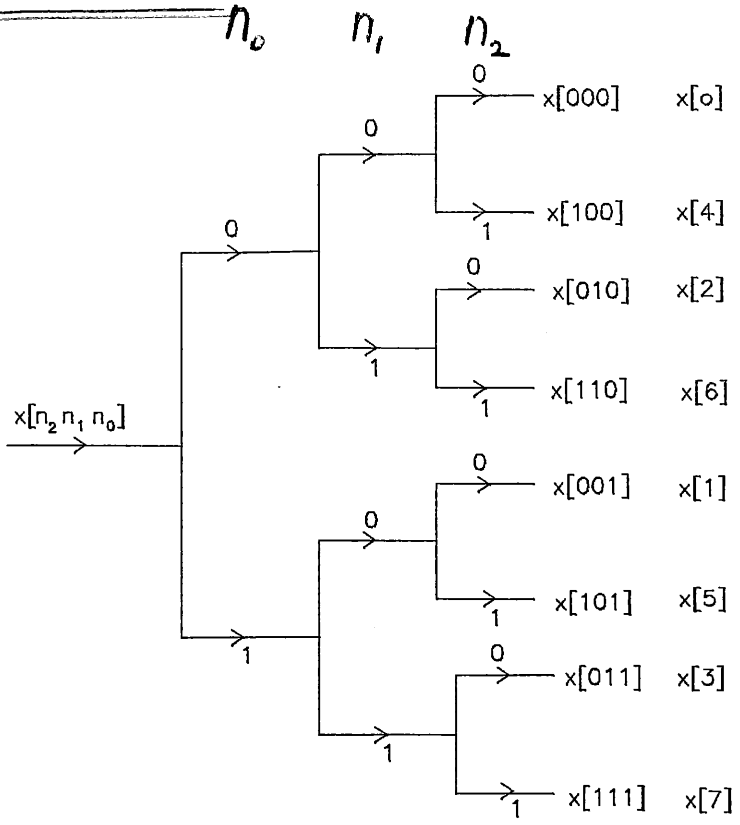

stage
(m−1)
stage
m
W_N^r
W_N^(r+N/2)


stage
(m−1)
stage
m
W_N^r
−1


x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

Bit Reversed Order

x[0]	x[000]
x[1]	x[001]
x[2]	x[010]
x[3]	x[011]
x[4]	x[100]
x[5]	x[101]
x[6]	x[110]
x[7]	x[111]



indices in binary form

In Place Computations

If the input is stored in bit-reversed order, the computation may be done in place — only one complex array of N storage registers is necessary to implement the complete computation.