The Laplace Transform

The Laplace transform of the signal $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad — ①$$

$X(s)$ is a function of the complex variable $s = \sigma + j\omega$. The range of values of s for which the integral in eqn. ① converges is referred to as the <u>region of convergence</u> (ROC) of the Laplace transform.

When $s = j\omega$ eqn. ① becomes

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

This is the Fourier transform of $x(t)$.

$$X(s)\Big|_{s=j\omega} = \mathcal{F}\{x(t)\} \qquad — ②$$

With $s = \sigma + j\omega$ we may write eqn. ① as follows

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t} \right] e^{-j\omega t}\, dt \quad \text{—③}$$

The r.h.s. of eqn. ③ is the Fourier transform of $x(t)\, e^{-\sigma t}$.

<u>Example 1</u> $\quad x(t) = e^{-at} u(t)$

The Fourier transform of $x(t)$ converges for $a > 0$ and is given by

$$X(j\omega) = \frac{1}{a + j\omega}, \quad a > 0.$$

The Laplace transform of $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt$$

$$X(s) = \int_0^\infty e^{-(s+a)t}\, dt$$

With $s = \sigma + j\omega$

$$X(\sigma + j\omega) = \int_0^\infty e^{-(\sigma + a)t}\, e^{-j\omega t}\, dt$$

which corresponds to the Fourier transform of $e^{-(\sigma+a)t}\, u(t)$.

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0$$

Since $s = \sigma + j\omega$ and $\sigma = Re\{s\}$ we have

$$X(s) = \frac{1}{s+a}, \quad Re\{s\} > -a.$$

If $a > 0$ then $X(s)$ converges for $\sigma = 0$.

$$X(0 + j\omega) = \frac{1}{a + j\omega}$$

With $\sigma = 0$, the Laplace transform is equal to the Fourier transform. If $a \leq 0$, the Laplace transform exists but the Fourier transform does not.

<u>Example 2</u>

$$x(t) = -e^{-at}\,u(-t)$$

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}\,u(-t)\,e^{-st}\,dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}\,dt$$

$$X(s) = \frac{1}{s+a}, \quad Re\{s\} < -a.$$

Note that the algebraic expression for the Laplace transform of $e^{-at}u(t)$ is also $\frac{1}{s+a}$. However the ROC for the Laplace transform of $e^{-at}u(t)$ is different. To specify the Laplace transform of a signal both the algebraic expression and the ROC are required.

Example 3

$$x(t) = e^{-t} u(t) + e^{-3t} u(t)$$

$$X(s) = \int_{-\infty}^{\infty} \left[e^{-t} u(t) + e^{-3t} u(t) \right] e^{-st} \, dt$$

$$= \int_{0}^{\infty} e^{-(s+1)t} \, dt + \int_{0}^{\infty} e^{-(s+3)t} \, dt$$

$$= \frac{1}{s+1} + \frac{1}{s+3}, \quad Re\{s\} > -1.$$

$$= \frac{s+3+s+1}{(s+1)(s+3)}, \quad Re\{s\} > -1.$$

$$X(s) = \frac{2(s+2)}{(s+1)(s+3)}, \quad Re\{s\} > -1.$$

Pole / zero Diagram

In examples 1, 2, and 3 the Laplace transform is a ratio of polynomials in the complex variable s. that is

$$X(s) = \frac{N(s)}{D(s)}, \text{ for } s \text{ in ROC.}$$

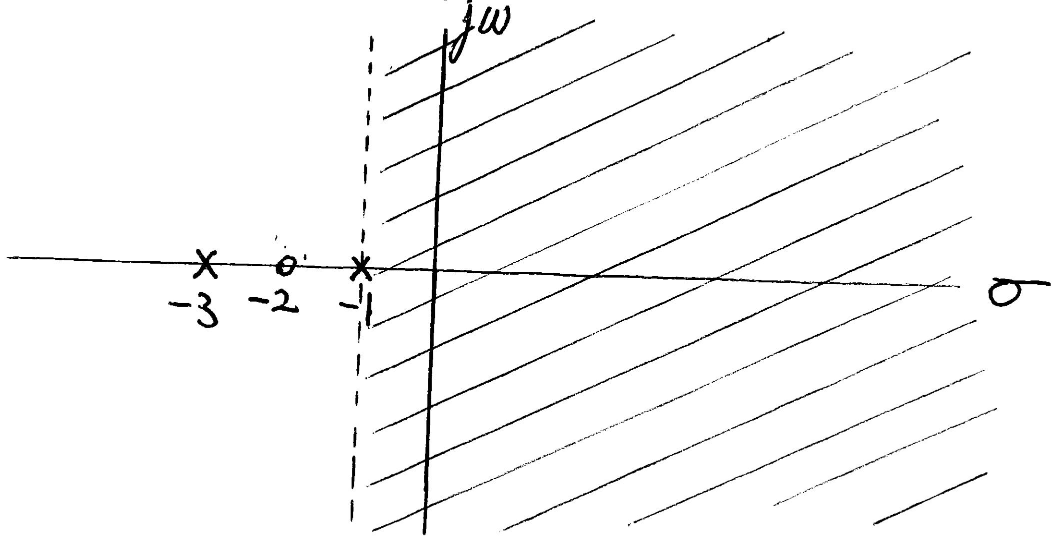
The roots of the numerator polynomial $N(s)$ are called the <u>zeros</u> of the Laplace transform because $X(s) = 0$ for those values of s.

The roots of the denominator polynomial $D(s)$ are called the <u>poles</u> of $X(s)$ and for those values of s, $X(s)$ is infinite.

From example 3 we have

$$X(s) = \frac{2(s+2)}{(s+1)(s+3)}, \quad Re\{s\} > -1.$$

The pole / zero plot of $X(s)$ is shown below.



The shaded region is the ROC.

ROC for rational Laplace transforms

Right - Sided Signals

A right - sided signal satisfies the condition

$$x(t) = 0, \quad t < t_1 \quad \text{for some finite value of } t_1.$$

If $X(s)$ exists the ROC is of the form

$$Re\{s\} > \sigma_{max}$$

where σ_{max} is the maximum real part of the poles of $X(s)$. $X(s)$ converges to the right of the vertical line $Re\{s\} = \sigma_{max}$ in the s-plane. The ROC is the region in the s-plane to the right of the rightmost pole.

Left - Sided Signals

A left - sided signal satisfies the condition

$$x(t) = 0, \quad t > t_2 \quad \text{for some finite value } t_2.$$

If $X(s)$ exists the ROC is of the form

$$Re\{s\} < \sigma_{min}$$

where σ_{min} is the minimum real part of the poles of $X(s)$. $X(s)$ converges to the left of the vertical line $Re\{s\} = \sigma_{min}$ in the s-plane. The ROC is the region in the s-plane to the left of the leftmost pole.

Two-Sided Signals

A signal that is neither right-sided nor left-sided is called two-sided. If $X(s)$ exists the ROC is of the form

$$\sigma_1 < Re\{s\} < \sigma_2$$

where σ_1 and σ_2 are the real parts of two of the poles. The ROC is a vertical strip in the s-plane between the vertical lines $Re\{s\} = \sigma_1$ and $Re\{s\} = \sigma_2$.

Let $x_1(t) = x(t)\,u(t-t_0)$ and $x_2(t) = x(t)\,u(-t+t_0)$ for any value of t_0 where an impulse does not occur. $x_1(t)$ is a right-sided signal and $X_1(s)$ converges for $Re\{s\} > \sigma_1$.

$x_2(t)$ is a left-sided signal and $X_2(s)$ converges for $Re\{s\} < \sigma_2$.

$X(s) = X_1(s) + X_2(s)$ converges if $\sigma_2 > \sigma_1$.
The ROC is $\sigma_1 < Re\{s\} < \sigma_2$.

<u>The Inverse Laplace Transform</u>

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\omega t} dt$$

$$= \mathcal{F}\left\{ x(t) e^{-\sigma t} \right\}$$

$$x(t) e^{-\sigma t} = \mathcal{F}^{-1}\left\{ X(\sigma + j\omega) \right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega$$

Multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega .$$

Make the substitution $s = \sigma + j\omega$ and $ds = j\,d\omega$.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

This is a contour integral in the s-plane. The contour of integration is a vertical line from $\sigma - j\infty$ to $\sigma + j\infty$. This vertical line must be in the ROC of $X(s)$.
For rational Laplace transforms we do not need to evaluate this integral. Instead we will obtain a partial fraction for $X(s)$ and determine the inverse Laplace transform of each of the lower order terms.

Example 4

Invert each of the following Laplace transforms:

(a) $$X_1(s) = \frac{1}{s^2+3s+2}, \quad Re\{s\} > -1$$

(b) $$X_2(s) = \frac{1}{s^2+3s+2}, \quad -2 < Re\{s\} < -1$$

(c) $$X_3(s) = \frac{1}{s^2+3s+2}, \quad Re\{s\} < -2.$$

(a) $$X_1(s) = \frac{1}{(s+1)(s+2)}, \quad Re\{s\} > -1$$

$$X_1(s) = \frac{A}{s+1} + \frac{B}{s+2}, \quad Re\{s\} > -1$$

where $$A = \left[(s+1)X_1(s)\right]\Big|_{s=-1} = 1$$

and $$B = \left[(s+2)X_1(s)\right]\Big|_{s=-2} = -1$$

$$X_1(s) = \frac{1}{s+1} - \frac{1}{s+2}, \quad Re\{s\} > -1$$

$$x_1(t) \longleftrightarrow X_1(s)$$

$x_1(t)$ is a right-sided signal.

From Example 1 we have

$$e^{-t} u(t) \longleftrightarrow \frac{1}{s+1}, \quad Re\{s\} > -1$$

$$e^{-2t} u(t) \longleftrightarrow \frac{1}{s+2}, \quad Re\{s\} > -2$$

$$x_1(t) = \left[e^{-t} - e^{-2t}\right] u(t)$$

(b) $$X_2(s) = \frac{1}{s+1} - \frac{1}{s+2}, \quad -2 < Re\{s\} < -1$$

$$x_2(t) \longleftrightarrow X_2(s)$$

$x_2(t)$ is a two-sided signal.

$$-e^{-2t} u(t) \longleftrightarrow -\frac{1}{s+2}, \quad Re\{s\} > -2$$

From Example 2 we have

$$-e^{-t}u(-t) \longleftrightarrow \frac{1}{s+1}, \quad Re\{s\} < -1$$

$$x_2(t) = -e^{-2t}u(t) - e^{-t}u(-t)$$

(c) $$X_3(s) = \frac{1}{s+1} - \frac{1}{s+2}, \qquad Re\{s\} < -2$$

$$x_3(t) \longleftrightarrow X_3(s)$$

$x_3(t)$ is a left-sided signal.

$$-e^{-t}u(-t) \longleftrightarrow \frac{1}{s+1}, \quad Re\{s\} < -1$$

$$e^{-2t}u(-t) \longleftrightarrow -\frac{1}{s+2}, \quad Re\{s\} < -2$$

$$x_3(t) = \left[e^{-2t} - e^{-t}\right]u(-t)$$

Properties of the Laplace Transform

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R$$

$$x_1(t) \longleftrightarrow X_1(s) \text{ with ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \text{ with ROC} = R_2$$

Linearity

$$a\,x_1(t) + b\,x_2(t) \longleftrightarrow a X_1(s) + b X_2(s)$$

with ROC containing $R_1 \cap R_2$.

Time Shift

$$x(t - t_0) \longleftrightarrow e^{-st_0} X(s) \text{ with ROC} = R$$

Time Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \text{ with ROC} = aR$$

Convolution of Signals

$$x_1(t) * x_2(t) \longleftrightarrow X_1(s) X_2(s) \text{ with ROC}$$

containing $R_1 \cap R_2$

Modulation

$$e^{s_0 t} x(t) \longleftrightarrow X(s-s_0) \text{ with ROC} = R + Re\{s_0\}$$

Differentiation

$$\frac{d\, x(t)}{dt} \longleftrightarrow s X(s) \text{ with ROC containing } R.$$

Integration

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{s} X(s) \text{ with ROC}$$

$$\text{containing } R \cap \{Re\{s\} > 0\}$$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} s X(s)$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)$$