Representation of Periodic Sequences – The Discrete Time Fourier Series

Consider a sequence $\tilde{x}[n]$ that is periodic with period N.

$$\tilde{x}[n] = \tilde{x}[n+N]$$

$\tilde{x}[n]$ can be represented by a Fourier Series, that is, by a sum of complex exponential sequences with frequencies that are integer multiples of the fundamental frequency $\frac{2\pi}{N}$.

Discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical.

$$e^{j\left(\frac{2\pi}{N}\right)kn} = e^{j\left(\frac{2\pi}{N}\right)(k+mN)n}, \quad m = \pm 1, \pm 2, \ldots$$

There are only N distinct discrete-time complex exponentials that are periodic of period N samples. The Fourier series representation of $\tilde{x}[n]$ need only contain N of these complex exponentials and has the form

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \text{— (1)}$$

where we have used the notation $k = \langle N \rangle$ to indicate that k varies over a range of N successive integers. To obtain a relation for a_k in terms of $\tilde{x}[n]$ we use the fact that

$$\sum_{n=0}^{N-1} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise} \end{cases} \qquad \text{— (2)}$$

The left hand side of this equation is the sum of a finite number of terms in a geometric series. It is of the form

$$\sum_{n=0}^{N-1} \alpha^n \quad \text{with } \alpha = e^{jk\left(\frac{2\pi}{N}\right)}$$

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \end{cases}$$

$e^{jk\left(\frac{2\pi}{N}\right)} = 1$ only when $k = 0, \pm N, \pm 2N, \cdots$

$$\sum_{n=0}^{N-1} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N & k = 0, \pm N, \pm 2N, \cdots \\ \dfrac{1 - e^{jk\left(\frac{2\pi}{N}\right)N}}{1 - e^{jk\left(\frac{2\pi}{N}\right)}}, & \text{otherwise} \end{cases}$$

which reduces to equation ② since

$$e^{jk\left(\frac{2\pi}{N}\right)N} = e^{jk2\pi} = 1.$$

To obtain a relation for a_k in terms of $\tilde{x}[n]$ we multiply both sides of equation ① by $e^{-jr\left(\frac{2\pi}{N}\right)n}$ and sum over N terms, giving

$$\sum_{n=\langle N\rangle} \tilde{x}[n]\, e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

Interchanging the order of summation on

the right-hand side of the equation,

$$\sum_{n=\langle N\rangle} \tilde{x}[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n} \quad \text{— (3)}$$

From the identity in equation (2) we have,

$$\sum_{n=\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & (k-r) = 0, \pm N, \pm 2N \cdots \\ 0, & \text{otherwise} \end{cases}$$

If we choose the values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side of equation (3) equals N if $k=r$ and 0 if $k \neq r$. The right-hand side of equation (3) reduces to $N a_r$, and we have

$$a_r = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n]\, e^{-jr\left(\frac{2\pi}{N}\right)n}$$

Discrete-time Fourier series pair:

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

The discrete-time Fourier series pair can also be written as:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Properties of the discrete-time Fourier Series (DFS)

Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both of period N samples.

$$\tilde{x}_3[n] = a\,\tilde{x}_1[n] + b\,\tilde{x}_2[n]$$

The coefficients in the DFS representation of $\tilde{x}_3[n]$ are given by

$$\tilde{X}_3[k] = a\,\tilde{X}_1[k] + b\,\tilde{X}_2[k]$$

where $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$ are the coefficients in the DFS representation of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ respectively.

Periodic Convolution

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences of period N samples.

$$\tilde{X}_1[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\, e^{-j\left(\frac{2\pi}{N}\right)mk}$$

$$\tilde{X}_2[k] = \sum_{r=0}^{N-1} \tilde{x}_2[r]\, e^{-j\left(\frac{2\pi}{N}\right)rk}$$

let $\tilde{X}_3[k] = \tilde{X}_1[k]\,\tilde{X}_2[k]$

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \sum_{r=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[r]\, e^{-j\left(\frac{2\pi}{N}\right)k(m+r)}$$

$$\tilde{x}_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_3[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{r=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[r]\, e^{j\left(\frac{2\pi}{N}\right)k(n-m-r)}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \sum_{r=0}^{N-1} \tilde{x}_2[r] \left[\frac{1}{N} \sum_{k=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)k(n-m-r)} \right]$$

Note that

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)k(n-m-r)} = \begin{cases} 1, \text{ for } r=(n-m)+lN \\ 0. \text{ otherwise} \end{cases}$$

where l is any integer.

Therefore, we have

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[n-m]$$

This type of convolution is known as a <u>periodic convolution.</u>

$$\tilde{x}_3[n] = \tilde{x}_1[n] \circledast \tilde{x}_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\,\tilde{x}_2[n-m]$$

$$\tilde{x}_3[n] = \tilde{x}_2[n] \circledast \tilde{x}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\,\tilde{x}_1[n-m]$$

Multiplication of Sequences

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both of period N samples, with the discrete-time Fourier series coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively.

$$\tilde{x}_1[n] = \frac{1}{N} \sum_{m=0}^{N-1} \tilde{X}_1[m]\, e^{j\left(\frac{2\pi}{N}\right)nm}$$

$$\tilde{x}_2[n] = \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}_2[r]\, e^{j\left(\frac{2\pi}{N}\right)nr}$$

$$\tilde{x}_1[n]\, \tilde{x}_2[n] = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{r=0}^{N-1} \tilde{X}_1[m]\, \tilde{X}_2[r]\, e^{j\left(\frac{2\pi}{N}\right)n(m+r)}$$

$$\tilde{x}_3[n] = \tilde{x}_1[n]\, \tilde{x}_2[n]$$

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}_3[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{r=0}^{N-1} \tilde{X}_1[m]\, \tilde{X}_2[r]\, e^{-j\left(\frac{2\pi}{N}\right)n(k-m-r)}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \tilde{X}_1[m] \sum_{r=0}^{N-1} \tilde{X}_2[r] \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)n(k-m-r)} \right]$$

Note that $\frac{1}{N} \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)n(k-m-r)}$

$$= \begin{cases} 1, & \text{for } r = (k-m) + lN \\ 0, & \text{otherwise} \end{cases}$$

where l is any integer.

This results in

$$\tilde{X}_3[k] = \frac{1}{N} \sum_{m=0}^{N-1} \tilde{X}_1[m]\, \tilde{X}_2[k-m]$$

$$= \frac{1}{N}\left[\tilde{X}_1[k] \circledast \tilde{X}_2[k] \right]$$

<u>Sampling the Fourier Transform</u>

Consider an aperiodic sequence $x[n]$ with Fourier Transform $X(e^{j\Omega})$.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \qquad \text{— ①}$$

Let $$\tilde{X}[k] = X(e^{j\Omega})\Big|_{\Omega=\left(\frac{2\pi}{N}\right)k}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \qquad \text{— ②}$$

$X(e^{j\Omega})$ is periodic in Ω with period 2π.

$\tilde{X}[k]$ is periodic in k with period N.

Consider the periodic sequence $\tilde{x}[n]$ with the DFS coefficients $\tilde{X}[k]$:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn} \quad — (3)$$

Substitute the values of $\tilde{X}[k]$ from eqn. (2) into eqn. (3):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]\, e^{-j\left(\frac{2\pi}{N}\right)km}\right] e^{j\left(\frac{2\pi}{N}\right)kn} \quad — (4)$$

Interchanging the order of summation yields

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)k(n-m)}\right]$$

Note that

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)k(n-m)} = \sum_{r=-\infty}^{\infty} \delta[n-m+rN]$$

Therefore, we have

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\sum_{r=-\infty}^{\infty} \delta[n-m+rN]\right]$$

$$= \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m+rN]$$

$$= \sum_{r=-\infty}^{\infty} x[n+rN] \qquad \text{—— (5)}$$

If $x[n]$ is of duration less than N, it can be recovered exactly from $\tilde{x}[n]$ by extracting one period of $\tilde{x}[n]$.

If $x[n]$ is of duration greater than N, "aliasing" occurs and $x[n]$ cannot be recovered from $\tilde{x}[n]$.