Random Variables

A real random variable X is completely characterised by its probability density function $p_X(x)$.

$$\Pr(x_1 < X \leq x_2) = \int_{x_1}^{x_2} p_X(x)\,dx.$$

$$\int_{-\infty}^{\infty} p_X(x)\,dx = 1$$

The expectation of a function of a random variable X, $E[f(X)]$, is defined by

$$E[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$$

The mathematical expectation is a linear operator.

$$E[f(X) + g(X)] = E[f(X)] + E[g(X)]$$

and $$E[c f(X)] = c E[f(X)]$$

where c is any scalar constant.

The mean of X, m_X, is defined by:

$$m_X = E[X] = \int_{-\infty}^{\infty} x \, p_X(x) \, dx$$

Mean-square value of X:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) \, dx$$

Variance of X:

$$\begin{aligned} Var[X] &= E[(X - m_X)^2] \\ &= \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x) \, dx \end{aligned}$$

Standard deviation of X:

$$\sigma_X = (Var[X])^{1/2}$$

$$\sigma_X^2 = E[(X-m_X)^2] = E[X^2 - 2m_X X + m_X^2]$$

$$= E[X^2] - 2m_X E[X] + m_X^2$$

$$= E[X^2] - m_X^2$$

Two random variables X and Y are completely characterised by their joint probability density function $p_{XY}(x,y)$.

The random variables are said to be statistically independent if

$$p_{XY}(x,y) = p_X(x)\, p_Y(y) \text{ for all } (x,y).$$

The correlation of two random variables X and Y is defined by

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x y\, p_{XY}(x,y)\, dx\, dy$$

The two random variables X and Y are uncorrelated if and only if $E[XY] = E[X]\,E[Y]$.

The covariance of two random variables X and Y is defined by

$$\mathrm{Cov}[XY] = E[(X-E[X])(Y-E[Y])].$$

Letting $m_X = E[X]$ and $m_Y = E[Y]$ we have

$$\mathrm{Cov}[XY] = E[XY] - m_X m_Y.$$

The correlation coefficient of X and Y, ρ_{XY}, is defined by

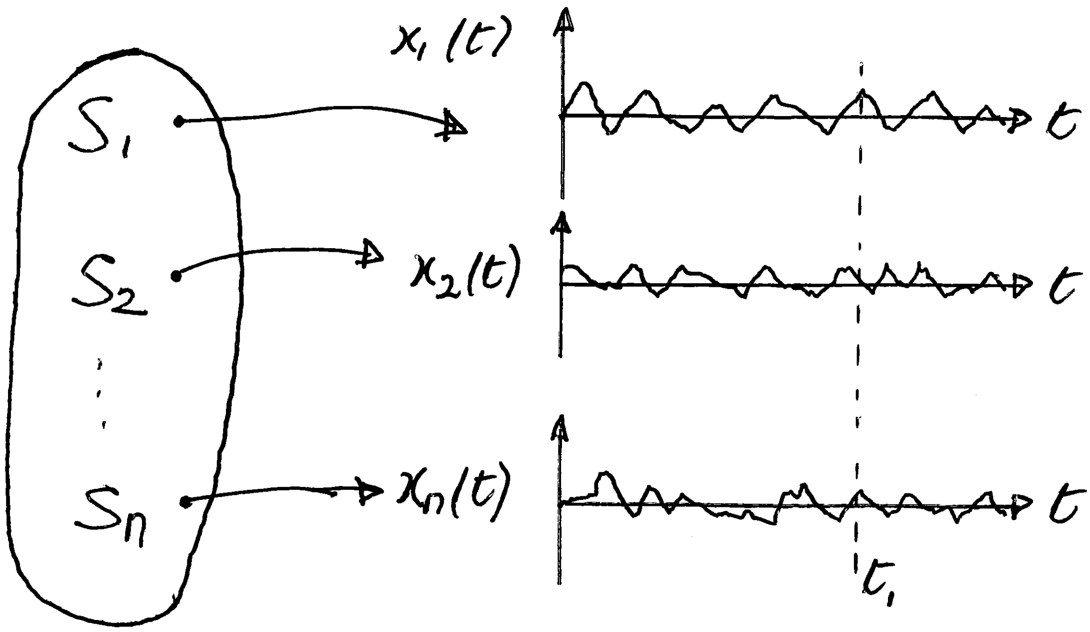
$$\rho_{XY} = \frac{\mathrm{Cov}[XY]}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y denote the standard deviation of X and Y respectively.

<u>Random Process</u>

A random process is a collection of time functions and an associated probability description.

Sample space S



Consider a random process $X(t)$ represented by the set of sample functions $\{x_j(t)\}$, $j = 1, 2, ---, n$. Each sample function $x_j(t)$ has a probability of occurrence, $P(s_j)$, $j = 1, 2, ---, n$. If we observe the set of waveforms $\{x_j(t)\}$, $j = 1, 2, —, n$, simultaneously at some time instant, $t = t_1$, then each sample point s_j of the sample space S has associated with it a number $x_j(t_1)$ and a probability $P(s_j)$. The resulting collection of numbers $\{x_j(t_1)\}$, $j = 1, 2, \ldots, n$, forms a random variable, $X(t_1)$. A different random variable $X(t_2)$ is obtained by observing the set of waveforms simultaneously a a second time instant t_2.

Random Variable → The outcome of an experiment is mapped into a number,

Random Process → The outcome of an experiment is mapped into a waveform that is a function of time.

Let $X(t_1), X(t_2), \ldots, X(t_k)$ denote the random variables obtained by observing the random process at time instants $t_1, t_2, \ldots, t_k$. The k random variables are completely characterised by their joint probability density function

$$p_{X(t_1), X(t_2), \ldots, X(t_k)}(x_1, x_2, \ldots, x_k)$$

<u>Stationarity</u>

The random process $X(t)$ is said to be <u>strictly stationary</u> if the joint probability density function is invariant under shifts of the time origin, that is, if the equality

$$p_{X(t_1), X(t_2), \ldots X(t_k)}(x_1, x_2, \ldots x_k) = p_{X(t_1+T), X(t_2+T), \ldots X(t_k+T)}(x_1, x_2, \ldots, x_k)$$

holds for every finite set of time instants $\{t_j\}$, $j = 1, 2, \cdots, k$, and for every time shift T.

A random process which is not stationary is called a <u>nonstationary</u> random process.

<u>Mean and Autocorrelation function</u>

The mean of a real-valued random process $X(t)$ is defined as

$$m_X(t_k) = E[X(t_k)]$$

where $X(t_k)$ is the random variable obtained by observing the random process at time t_k.

$$m_X(t_k) = \int_{-\infty}^{\infty} x \, p_{X(t_k)}(x) \, dx$$

The autocorrelation function of the random process $X(t)$ is a function of two time variables, t_k and t_i, and is defined as

$$R_X(t_k, t_i) = E[X(t_k)X(t_i)]$$

where $X(t_k)$ and $X(t_i)$ are random variables obtained by observing the random process $X(t)$ at times t_k and t_i respectively.

If the random process is strictly stationary then the mean of the random process is a constant.

$$m_X(t_k) = m_X \text{ , for all } t_k.$$

and the autocorrelation function.

$R_X(t_k, t_i)$, depends only on the time difference $t_i - t_k$.

The autocorrelation function of a stationary random process $X(t)$ may be written as

$$R_X(\tau) = E[X(t)X(t+\tau)].$$

<u>Wide-Sense Stationary random process</u>

A random process $X(t)$ is said to be wide-sense stationary if (i) the mean value of any random variable, $X(t_1)$, is independent of the choice of t_1 <u>and</u> (ii) the autocorrelation function, $R_X(t_1, t_2)$ depends only on the time difference $t_2 - t_1$.

Time Averages and Ergodic Processes

The mean of a stationary process is defined by

$$m_X = E[X(t)]$$

$$= \int_{-\infty}^{\infty} x \, p_{X(t)}(x) \, dx$$

and the autocorrelation function is defined by

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x y \, p_{X(t),X(t+\tau)}(x,y) \, dx \, dy$$

m_X and $R_X(\tau)$ are <u>ensemble averages</u> obtained by averaging over all the sample functions.

The time-averaged mean value of the sample function $x(t)$ of the random process $X(t)$ is defined as

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

The time-averaged autocorrelation function of the sample function $x(t)$ is defined as

$$\langle x(t)\, x(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) x(t+\tau)\, dt$$

In general, time averages and ensemble averages are not equal except for a class of random processes known as ergodic processes. A random process is <u>ergodic</u> if all time averages of sample functions equal the corresponding ensemble averages. It is necessary that a random process be stationary in the strict sense for it to be ergodic. Not all stationary processes are ergodic.

Wiener - Khinchin theorem

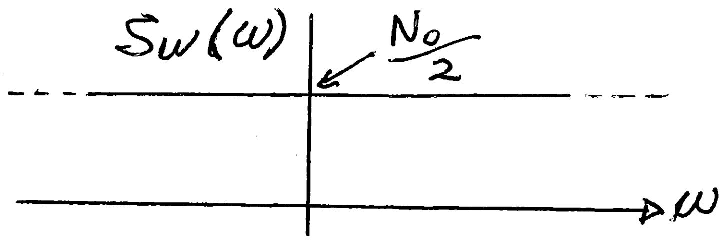
The power spectral density of a wide-sense stationary random process is given by the Fourier transform of the autocorrelation function.

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, e^{j\omega\tau}\, d\omega$$

White Noise

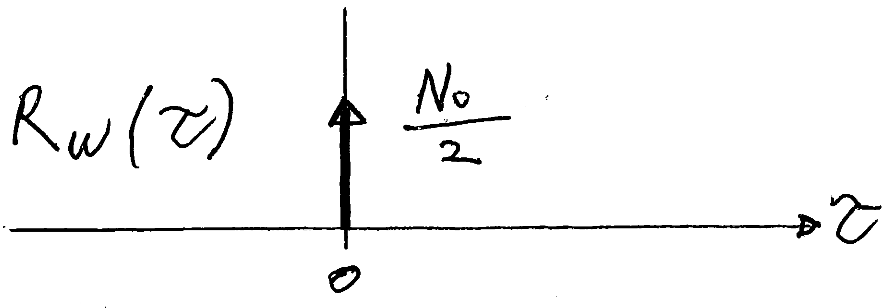
The power spectral density of a white-noise process, $W(t)$, is independent of frequency. $S_W(\omega) = \frac{N_0}{2}$



The autocorrelation function $R_W(\tau)$ is the inverse Fourier transform of the power spectral density.

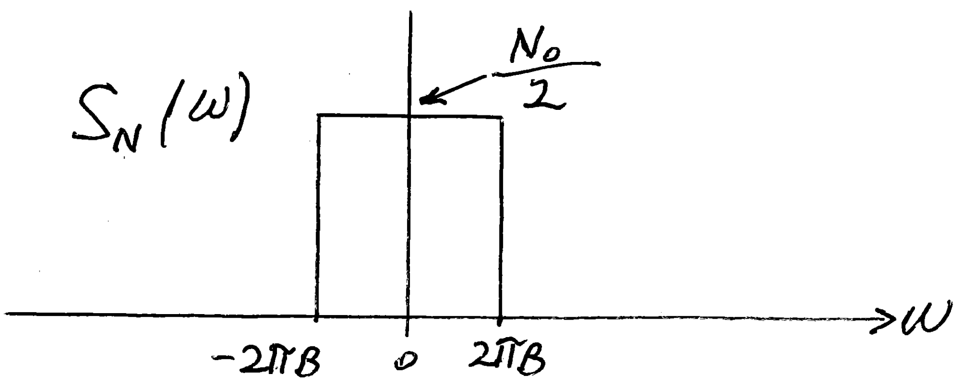
$$R_W(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} e^{j\omega\tau}\, d\omega$$

$$= \frac{N_0}{2}\,\delta(\tau)$$



<u>Band - limited white noise</u>

$$S_N(\omega) = \begin{cases} \frac{N_0}{2}, & -2\pi B < \omega < 2\pi B \\ 0, & |\omega| > 2\pi B \end{cases}$$



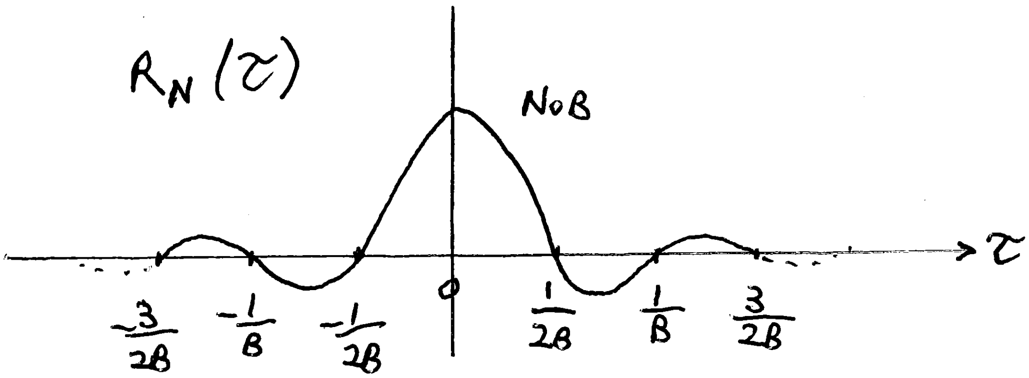
The average noise power, P_N, is

$$P_N = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_N(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \frac{N_0}{2}\, d\omega$$

$$= N_0 B$$

The autocorrelation function, $R_N(\tau)$, is shown below:



<u>Mean Square Value from Spectral Density</u>

Let $X(t)$ denote a wide sense stationary random process.

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}\,d\omega$$

$$E[X^2(t)] = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\,d\omega$$

Properties of the autocorrelation function of a stationary random process.

1. $$R_X(0) = E[X^2(t)]$$

2. $$R_X(\tau) = R_X(-\tau)$$

3. $$|R_X(\tau)| \leq R_X(0)$$

Properties of the Power Spectral Density of a WSS random process $X(t)$.

1. $$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

2. $$E[X^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)\, d\omega$$

3. $S_X(\omega) \geq 0$ for all ω.

4. If $X(t)$ is a real-valued random process.
$$S_X(-\omega) = S_X(\omega)$$