Periodic Signals and the Fourier Series

Let $x(t)$ denote a periodic signal with fundamental period T.

$$x(t) = x(t+T) \text{ for all } t.$$

$\omega_0 = \frac{2\pi}{T}$ is called the fundamental frequency.

The complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

is periodic with fundamental frequency ω_0 and fundamental period $T = \frac{2\pi}{\omega_0}$.

Let $\phi_k(t) = e^{jk\omega_0 t}$ where k is an integer.

Note that
$$\phi_k(t+T) = e^{jk\omega_0(t+T)}$$
$$= e^{jk\omega_0 t} \cdot e^{jk\omega_0 T}$$

$$= e^{jk\omega_0 t} \cdot e^{jk\left(\frac{2\pi}{T}\right)T}$$

$$= e^{jk\omega_0 t}$$

since $e^{jk2\pi} = 1$ for any integer k.

Consider a periodic signal $x(t)$ which can be expressed as a linear combination of harmonically related complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{— } ①$$

where $\omega_0 = \frac{2\pi}{T}$.

$x(t)$ is periodic with period $T = \frac{2\pi}{\omega_0}$.

The term for $k = 0$ is a constant.

The terms of the series for $k = \pm 1$ are

called the fundamental components.
The terms for $k = \pm 2$ are referred to as the second harmonic components.
The terms for $k = \pm N$ are called the $N^{th.}$ harmonic components.

The representation of the periodic signal in eqn. ① is called the Fourier Series.

Determining the Fourier Series Coefficients

If we multiply both sides of eqn. ① by $e^{-jn\omega_0 t}$, we obtain

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \quad \text{— ②}$$

Integrating both sides of eqn. ② over one period of $x(t)$ we have

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \int_T \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}\, dt \quad \text{— (3)}$$

Interchanging the order of integration and summation we obtain

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-n)\omega_0 t}\, dt \quad \text{— (4)}$$

Note that

$$\int_T e^{j(k-n)\omega_0 t}\, dt = \begin{cases} T, & k=n \\ 0, & k\neq n \end{cases}$$

Eqn. (4) may be written as follows

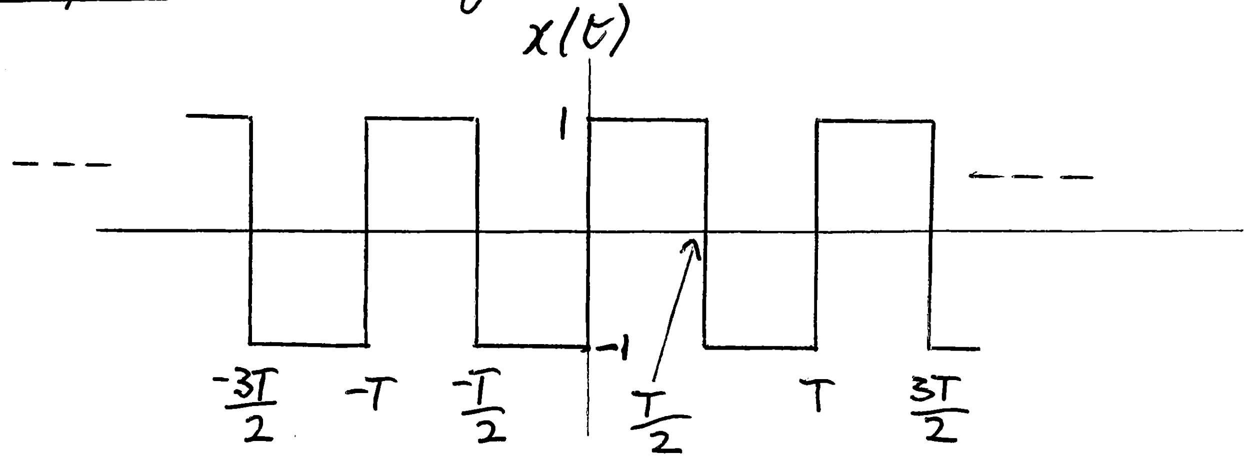
$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T \quad \text{— (5)}$$

Fourier Series Pair

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{(Synthesis eqn.)} \quad -(6)$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \quad \text{(Analysis eqn.)} \quad -(7)$$

Example **Square Wave $x(t)$**



$$x(t) = \begin{cases} 1, & 0 < t < \frac{T}{2} \\ -1, & \frac{T}{2} < t < T \end{cases}$$

The Fourier Series coefficients are given by

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} (1)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_{\frac{T}{2}}^{T} (-1)\, e^{-jk\omega_0 t}\, dt$$

$$= -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_0^{\frac{T}{2}} + \frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{\frac{T}{2}}^{T}$$

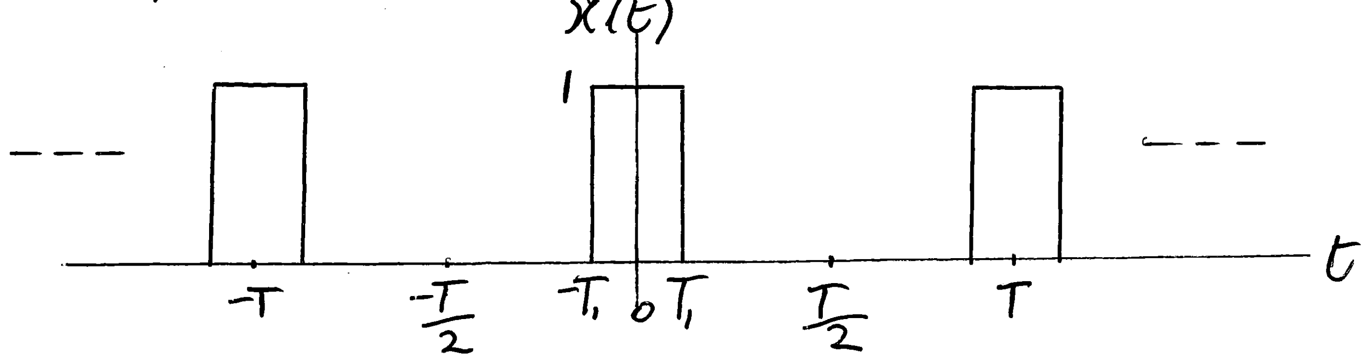
$$= -\frac{1}{jk2\pi}\left[\left(e^{-jk\pi} - e^{-j0}\right) - \left(e^{-jk2\pi} - e^{-jk\pi}\right)\right]$$

$$= \frac{1}{jk2\pi}\left[1 - \left(e^{-j\pi}\right)^k + 1 - \left(e^{-j\pi}\right)^k\right]$$

$$= \frac{2}{jk2\pi}\left[1 - (-1)^k\right]$$

$$a_k = \begin{cases} \frac{2}{jk\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

<u>Example</u> <u>Rectangular pulse train $x(t)$</u>



$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} (1)\, e^{-jk\omega_0 t}\, dt$$

$$= -\frac{1}{jk\omega_0 T}\, e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{jk\omega_0 T}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$= \frac{1}{jk2\pi}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right)$$

$$= \frac{\sin[k\omega_0 T_1]}{k\pi}$$

Convergence of the Fourier Series

Let $x_N(t)$ be the finite series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \quad \text{— (8)}$$

The approximation error $e_N(t)$ is

$$e_N(t) = x(t) - x_N(t)$$

$$= x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \quad \text{— (9)}$$

The mean-square-error (MSE) is

$$E_N = \frac{1}{T}\int_T |e_N(t)|^2 \, dt \quad \text{— (10)}$$

The coefficients a_k that minimize E_N are given by

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} \, dt \quad \text{— (11)}$$

The best approximation $x_N(t)$ is a truncated Fourier Series.

If $E_N \to 0$ as $N \to \infty$ the Fourier series is said to converge to $x(t)$.

The Fourier series converges if $x(t)$ is continuous or if $x(t)$ is square-integrable over a period T, i.e. if

$$\int_T |x(t)|^2 \, dt < \infty.$$

Note that MSE convergence of the Fourier series of $x(t)$ does <u>not</u> imply that $x(t)$ and $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ are equal at every value of t.

If a signal satisfies the Dirichlet conditions then $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ except at values of t for which $x(t)$ is discontinuous.

At each discontinuity the Fourier series converges to the average of the values on either side of the discontinuity.

Dirichlet Conditions

1. $\int_T |x(t)|\, dt < \infty$

2. There are no more than a finite number of maxima and minima in any finite interval of time.

3. There are no more than a finite number of finite discontinuities in any finite interval of time.

Gibbs phenomenon

The partial sum $\sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ exhibits an overshoot of 9% of the height of the discontinuity no matter how large N becomes.

Properties of Continuous-Time Fourier Series

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

Linearity

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

Let $z(t) = A\,x(t) + B\,y(t)$

$$z(t) \xleftrightarrow{FS} c_k$$

$$c_k = A\,a_k + B\,b_k$$

Time Shifting

Let $y(t) = x(t - t_0)$

$$y(t) \xleftrightarrow{FS} b_k$$

$$b_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

let $\tau = t - t_0$.

$$b_k = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau$$

$$= e^{-jk\omega_0 t_0}\, \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

$$= e^{-jk\omega_0 t_0}\, a_k$$

where a_k is the k th. Fourier series coefficient of $x(t)$.

$$x(t) \xleftrightarrow{FS} a_k$$

$$x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0}\, a_k$$

<u>Multiplication</u>

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

$$x(t)\, y(t) \xleftrightarrow{FS} \sum_{m=-\infty}^{\infty} a_m b_{k-m}$$