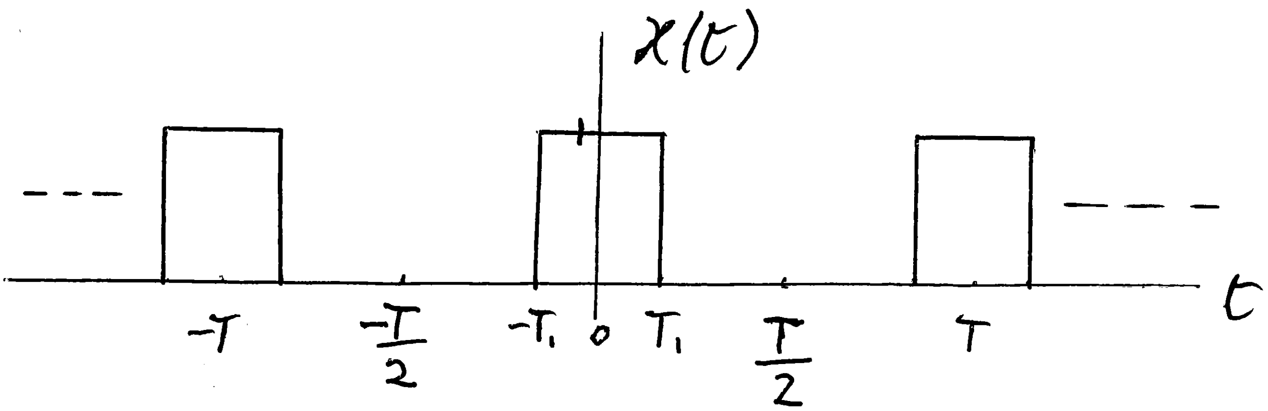
The Fourier Transform

Consider a periodic signal $x(t)$ which is defined over one period as follows:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$

where T is the fundamental period.



The Fourier series coefficients a_k for $x(t)$ are

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

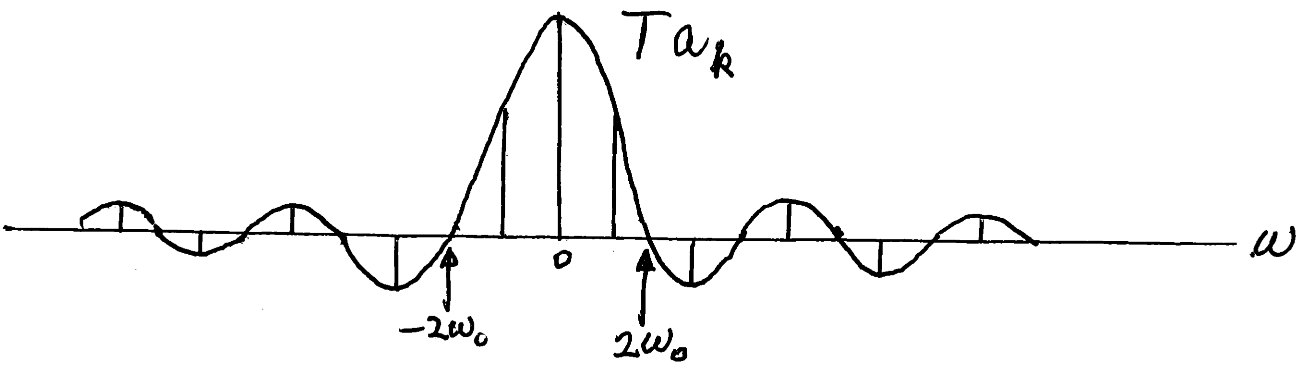
$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

where $\omega_0 = \frac{2\pi}{T}$.

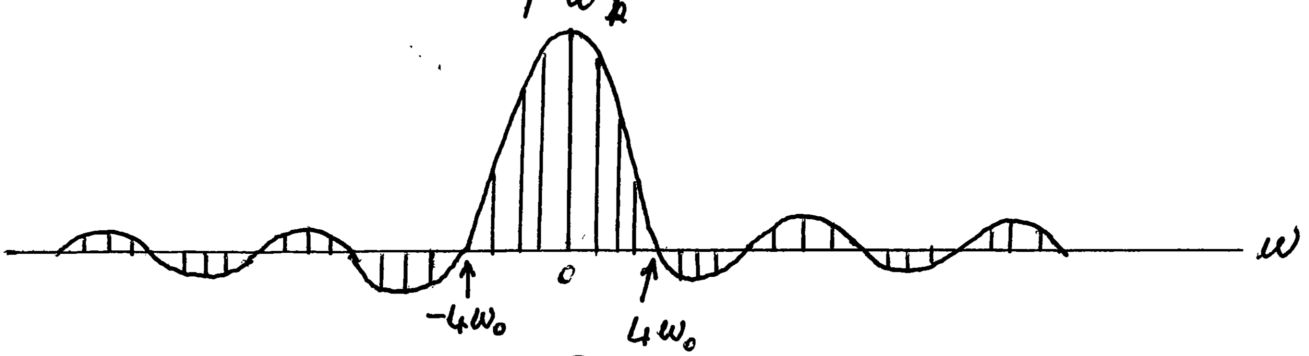
$$T a_k = \frac{2 \sin(k \omega_0 T_1)}{k \omega_0}$$

$$= \left. \frac{2 \sin(\omega T_1)}{\omega} \right|_{\omega = k\omega_0}$$

The function $2 \sin(\omega T_1)/\omega$ is the envelope of $T a_k$ and is independent of T.



$T = 4T_1$



$T = 8T_1$

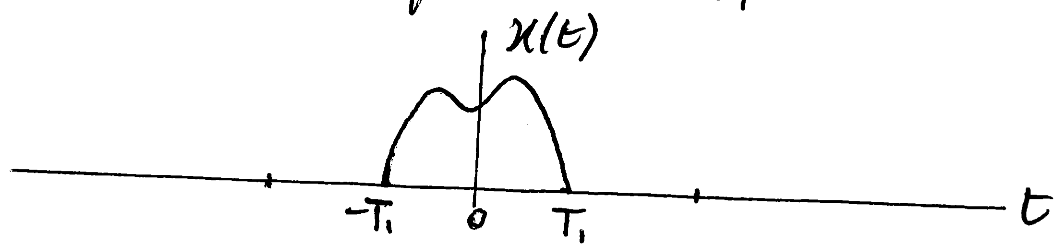
As T increases, the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases and the envelope is sampled with a closer spacing.

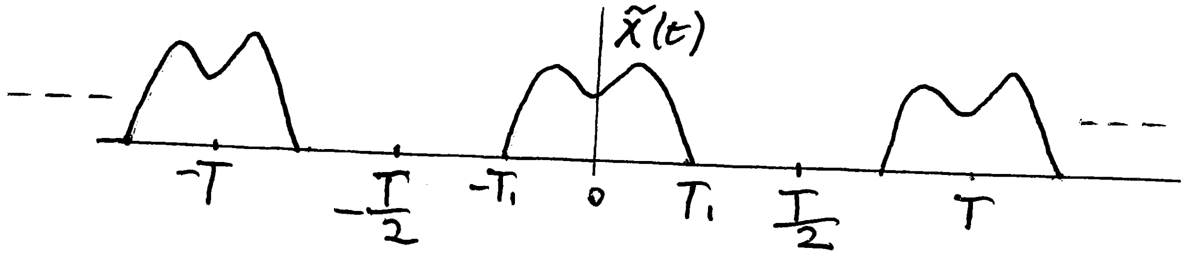
An aperiodic signal can be thought of as the limit of a periodic signal as $T \to \infty$.

We will examine the limiting behaviour of the Fourier series representation for this signal.

Consider a signal $x(t)$ that is of finite duration.

$$x(t) = 0 \text{ if } |t| > T_1.$$





$$\tilde{x}(t) = x(t) \text{ for } |t| < \frac{T}{2}.$$

The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{— (1)}$$

where

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t}\, dt$$

The envelope of $T a_k$ is

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

The coefficients a_k can be written as

$$a_k = \frac{1}{T} X(jk\omega_0) \quad \text{— (2')}$$

Combining eqns. (1') and (2') we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} \quad \text{— (3')}$$

Note that $\omega_0 = \frac{2\pi}{T}$ and $\frac{1}{T} = \frac{\omega_0}{2\pi}$.

Eqn. (3) can be rewritten as

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 \quad \text{— (4')}$$

as $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$ and eqn. (4') becomes a representation of $x(t)$.

Each term in the summation on the right hand side of eqn. (4') is the area of a rectangle of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0.

As $T \to \infty$, $\omega_0 \to 0$ and the right hand side of eqn. 4 becomes an integral.

<u>Fourier transform pair</u>

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \text{— } (5)$$

(synthesis eqn.)

and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{— } (6)$$

(analysis eqn.)

Convergence of Fourier transforms

We derived the Fourier transform pair for a signal $x(t)$ that is of finite duration. However eqns. (5) and (6) are also valid for many signals of infinite duration.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$

The error between $\hat{x}(t)$ and $x(t)$ is

$$e(t) = \hat{x}(t) - x(t).$$

When is $\hat{x}(t)$ a valid representation of $x(t)$?

If $x(t)$ is square integrable, so that

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$$

then $X(j\omega)$ is finite and $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$.

If $x(t)$ satisfies the <u>Dirichlet conditions</u> then $\hat{x}(t) = x(t)$ for any t except at a discontinuity where $\hat{x}(t)$ is equal to the average value on either side of the discontinuity.

1. $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$.
2. $x(t)$ has a finite number of maxima and minima in any finite interval.
3. $x(t)$ has a finite number of finite discontinuities within any finite interval.

<u>Example</u>

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Example

$$x(t) = e^{-at} u(t), \quad a > 0$$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} \, dt$$

$$= -\frac{1}{a+j\omega} e^{-at} e^{-j\omega t} \Big|_0^{\infty}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

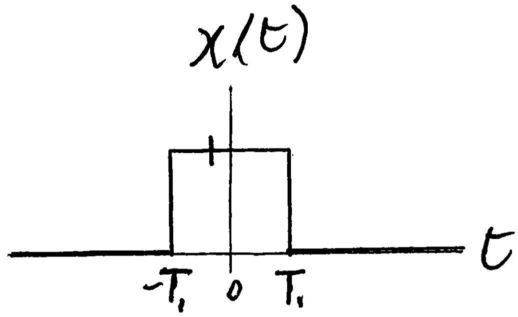
$$X(j\omega) = |X(j\omega)| e^{j\phi(\omega)}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\phi(\omega) = -\angle a+j\omega = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Example

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

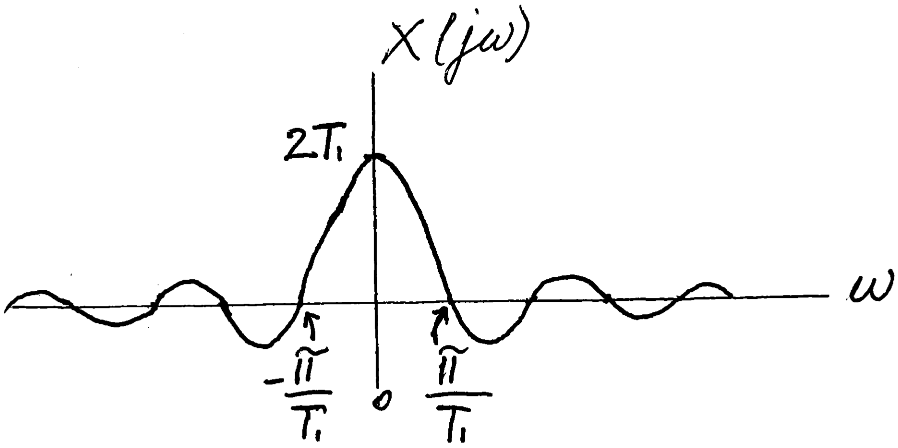


$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1}$$

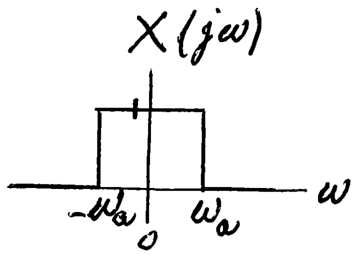
$$= -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1} \right]$$

$$= \frac{2 \sin(\omega T_1)}{\omega}$$



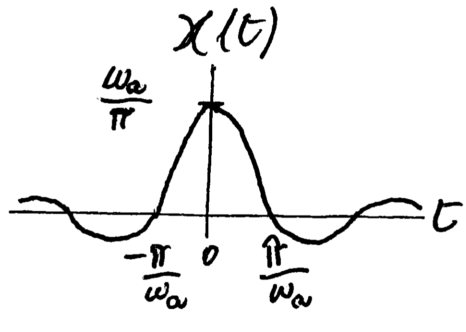
Example

$$X(j\omega) = \begin{cases} 1, & |\omega| < \omega_a \\ 0, & |\omega| > \omega_a \end{cases}$$



$$x(t) = \frac{1}{2\pi} \int_{-\omega_a}^{\omega_a} (1)\, e^{j\omega t}\, d\omega$$

$$= \frac{\sin(\omega_a t)}{\pi t}$$



Fourier transform of a periodic signal

Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ given by

$$X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$x(t) = \mathcal{F}^{-1}\left[X(j\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

$$= e^{j\omega_0 t}$$

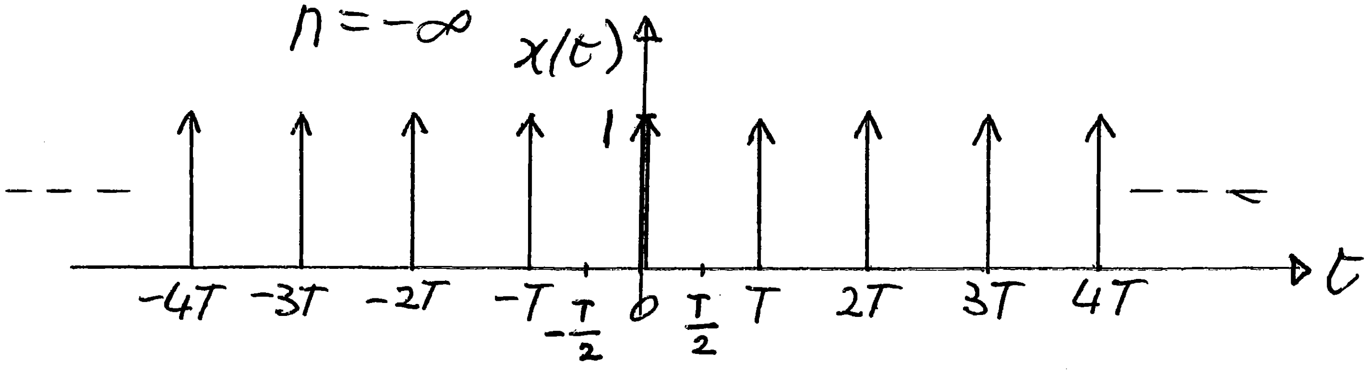
If $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0)$

then $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$

This corresponds exactly to the Fourier series representation of a periodic signal with fundamental period $T = \frac{2\pi}{\omega_0}$

Fourier transform of a periodic impulse train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



The Fourier series coefficients a_k are given by:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}$$

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$