Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

FACULTY OF ENGINEERING, MATHEMATICS & SCIENCE

SCHOOL OF ENGINEERING

Electronic & Electrical Engineering

Engineering **Hilary Term, 2017**
Senior Sophister
Annual Examinations

Digital Signal Processing (4C5)

6th January 2017 **Venue: Goldsmith Hall** **Time: 09.30 – 11.30**

Dr. W. Dowling

Instructions to Candidates:

Answer THREE questions. All questions carry equal marks.

Materials permitted for this examination:

Mathematical Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.

Please indicate the make and model of your calculator on each answer book used.

Q.1 (a) A continuous-time signal $x_a(t)$ has the Fourier transform $X_a(j\omega)$.

The discrete-time signal $x[n]$ is derived from $x_a(t)$ by periodic sampling:

$$x[n] = x_a(nT), \text{where } T \text{ is a constant.}$$

Let $X(e^{j\Omega})$ denote the discrete-time Fourier transform of $x[n]$. Show that

$$X(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\frac{\Omega}{T} - \frac{2\pi k}{T}\right)\right).$$

[11 marks]

(b) The continuous-time signal $y(t)$ is given by

$$y(t) = \sum_{n=-\infty}^{\infty} x[n]h(t-nT),$$

where

$$h(t) = \begin{cases} 1, & 0 < t < T, \\ 0.5, & t = 0, T, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y(j\omega)$ and $H(j\omega)$ denote the Fourier transforms of $y(t)$ and $h(t)$ respectively.

(i) Obtain an expression for $H(j\omega)$ and sketch $|H(j\omega)|$ for $|\omega| \leq \dfrac{2\pi}{T}$. **[5 marks]**

(ii) Show that $Y(j\omega) = X(e^{j\Omega})|_{\Omega=\omega T}\, H(j\omega)$ **[4 marks]**

Q.2 (a) Show that the bilinear transformation, $s = (1 - z^{-1})/(1 + z^{-1})$, has the following properties:

(i) The imaginary axis in the s-plane maps to the unit circle in the z-plane.

[4 marks]

(ii) The left half of the s-plane maps to the inside of the unit circle in the z-plane.

[4 marks]

(b) A discrete-time bandpass filter with frequency response, $H(e^{j\Omega})$, is to be designed to meet the following specifications:

$$\frac{1}{\sqrt{2}} \leq |H(e^{j\Omega})| \leq 1, \qquad 0.4\pi \leq |\Omega| \leq 0.6\pi,$$

$$|H(e^{j\Omega})| \leq 0.2, \quad 0.9\pi \leq |\Omega| \leq \pi,$$

$$\text{and } |H(e^{j\Omega})| \leq 0.2, \quad |\Omega| \leq 0.1\pi.$$

The filter is to be designed by applying the bilinear transformation to the transfer function of an appropriate Butterworth continuous-time filter. Verify that a second order filter is sufficient to meet the specifications. Determine the transfer function, $H(z)$, of the discrete-time filter.

Note that the transfer function of a first order Butterworth lowpass prototype filter is

$$H(s) = \frac{1}{s+1}$$

and the lowpass to bandpass transformation for a continuous-time filter is

$$s \rightarrow \frac{s^2 + \omega_1\omega_2}{s(\omega_2 - \omega_1)}$$

where ω_1 and ω_2 are the lower and upper cut-off frequencies respectively.

[12 marks]

Q.3 (a) A discrete-time filter has a unit sample response, $h[n]$, that is zero for $n < 0$ and for $n > N-1$. If $h[n] = h[N-1-n]$ and N is odd, show that the filter has a frequency response with generalized linear phase. **[8 marks]**

(b) An ideal discrete-time highpass filter has a frequency response, $H_{id}(e^{j\Omega})$, given by

$$H_{id}\left(e^{j\Omega}\right) = \begin{cases} 0, & |\Omega| < \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} < |\Omega| < \pi. \end{cases}$$

Obtain an expression for the unit-sample response of this filter. **[7 marks]**

(c) An 11-point Hamming window, $w_H[n]$, is given by

$$w_H[n] = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{\pi n}{5}\right), & -5 \le n \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Using the Hamming window, design an 11-point finite impulse response filter that approximates the magnitude response of the ideal highpass filter in part (b). **[5 marks]**

Q.4 (a) Let $x[n]$ denote an infinite length sequence with discrete-time Fourier transform $X(e^{j\Omega})$ and let $X_1[k]$ denote the N-point discrete Fourier transform (DFT) of the N-point sequence $x_1[n]$. Determine the relation between $x_1[n]$ and $x[n]$ if $X_1[k]$ and $X(e^{j\Omega})$ are related by

$$X_1[k] = X(e^{j2\pi k/N}), \qquad k = 0, 1, \ldots, N-1.$$

[8 marks]

(b) The finite length sequences, $x[n]$ and $y[n]$, are zero for $n < 0$ and for $n > N-1$. Let $X[k]$ and $Y[k]$ denote the N-point DFT of $x[n]$ and $y[n]$ respectively. The sequence $g[n]$ is the N-point circular convolution of $x[n]$ and $y[n]$. Let $G[k]$ denote the N-point DFT of $g[n]$. Show that $G[k] = X[k]Y[k]$.

[7 marks]

(c) A 12,500 point sequence is to be linearly convolved with a sequence that is 100 points long. The convolution is to be implemented using DFTs and inverse DFTs of length 1024. If the overlap-add method is used, what is the minimum number of 1024-point DFTs and the minimum number of 1024-point inverse DFTs needed to implement the convolution for the entire 12,500 point sequence? **[5 marks]**

Q.5 (a) Consider a stable, linear, shift-invariant system with unit-sample response $h[n]$. Let $x[n]$ be a real input sequence that is a sample sequence of a wide-sense stationary discrete-time random process. Let $y[n]$ denote the output sequence. Show that the input and output autocorrelation sequences, $\phi_{XX}[m]$ and $\phi_{YY}[m]$, respectively, are related by

$$\phi_{YY}[m] = \sum_{l=-\infty}^{\infty} v[l]\phi_{XX}[m-l],$$

where

$$v[l] = \sum_{k=-\infty}^{\infty} h[k]h[l+k].$$

[8 marks]

(b) Let $x[n]$ be a real white-noise sequence with zero mean and autocorrelation sequence $\phi_{XX}[m] = \sigma_X^2\,\delta[m]$, where $\delta[m]$ is the unit-sample sequence. The sequence $x[n]$ is the input to a linear shift-invariant system with unit-sample response $h[n] = a^n u[n]$, where $|a| < 1$ and $u[n]$ is the unit-step sequence.

(i) Find an expression for the output autocorrelation sequence, $\phi_{YY}[m]$.

[4 marks]

(ii) Express the power spectral density, $S_{YY}(\Omega)$, of the output process in terms of the magnitude of the frequency response of the system. **[6 marks]**

(iii) Determine the mean, m_Y, and the variance, σ_Y^2, of the output process.

[2 marks]