3C1 Homework No. 2 Solutions

1. $$h(t) = 2e^{-2t} u(t)$$

(i) $$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} 2e^{-2t} e^{-j\omega t} dt$$

$$= 2\int_{0}^{\infty} e^{-(2+j\omega)t} dt$$

$$= -\frac{2}{2+j\omega} e^{-2t} e^{-j\omega t}\Big|_0^{\infty}$$

$$= \frac{2}{2+j\omega}$$

(ii) $$x(t) = 1 + \cos(\omega_0 t) - 2\cos(2\omega_0 t)$$

$$= 1 + \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) - \left(e^{j2\omega_0 t} + e^{-j2\omega_0 t}\right)$$

$$y(t) = 1.\, H(j0) + \frac{1}{2} e^{j\omega_0 t} H(j\omega_0) + \frac{1}{2} e^{-j\omega_0 t} H(-j\omega_0)$$
$$- e^{j2\omega_0 t} H(j2\omega_0) - e^{-2j\omega_0 t} H(-j2\omega_0)$$

$$y(t) = 1 + e^{j\omega_0 t}\left[\frac{1}{2+j\omega_0}\right] + e^{-j\omega_0 t}\left[\frac{1}{2-j\omega_0}\right]$$

$$- e^{j2\omega_0 t}\left[\frac{2}{2+j2\omega_0}\right] - e^{-j2\omega_0 t}\left[\frac{2}{2-j2\omega_0}\right]$$

---

2. $$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

(i) Let $x(t) = e^{j\omega t}$

Then $y(t) = e^{j\omega t} H(j\omega)$.

$$\frac{d}{dt}\left[e^{j\omega t} H(j\omega)\right] + 3e^{j\omega t} H(j\omega) = e^{j\omega t}$$

$$j\omega e^{j\omega t} H(j\omega) + 3e^{j\omega t} H(j\omega) = e^{j\omega t}$$

$$H(j\omega)\left[j\omega + 3\right] = 1$$

$$H(j\omega) = \frac{1}{3+j\omega}$$

(ii)

$$x(t) = \cos(\omega_0 t)$$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$y(t) = \frac{1}{2} e^{j\omega_0 t} H(j\omega_0) + \frac{1}{2} e^{-j\omega_0 t} H(-j\omega_0)$$

$$= \frac{1}{2} e^{j\omega_0 t}\left[\frac{1}{3+j\omega_0}\right] + \frac{1}{2} e^{-j\omega_0 t}\left[\frac{1}{3-j\omega_0}\right]$$

$$= \left(\frac{1}{6+j2\omega_0}\right) e^{j\omega_0 t} + \left(\frac{1}{6-j2\omega_0}\right) e^{-j\omega_0 t}$$

3. Fundamental frequency $\omega_0 = \frac{2\pi}{T} = 8\pi$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk8\pi t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk8\pi) e^{jk8\pi t}$$

$H(jk8\pi) = 0$ if $|k8\pi| > 50\pi$.

i.e. $H(jk8\pi) = 0$ if $|k| > 6$.

$y(t) = x(t)$

$\Rightarrow a_k = 0$ for $|k| > 6$.

---

4. (i) $x_1(t) = e^{-at}\mu(t), \quad a>0.$

$$X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= -\frac{1}{a+j\omega} e^{-at} e^{-j\omega t}\Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}$$

(ii) $x_2(t) = e^{at}\mu(-t), \quad a>0.$

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a-j\omega} e^{at} e^{-j\omega t}\Big|_{-\infty}^{0}$$

$$= \frac{1}{a-j\omega}$$

(iii) $x_3(t) = e^{-a|t|}, \quad a > 0.$

$$x_3(t) = \begin{cases} e^{-at} u(t), & t > 0 \\ e^{at} u(-t), & t < 0 \end{cases}$$

$$X_3(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{a+j\omega + a-j\omega}{(a-j\omega)(a+j\omega)}$$

$$X_3(j\omega) = \frac{2a}{a^2+\omega^2}$$

---

5. (i) $X(j\omega) = 2\pi\delta(\omega) + \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{2\pi\delta(\omega) + \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)\} e^{j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega + \frac{1}{2}\int_{-\infty}^{\infty} \delta(\omega+\omega_0) e^{j\omega t} d\omega + \frac{1}{2}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega$$

$$= e^{0} + \frac{1}{2} e^{-j\omega_0 t} + \frac{1}{2} e^{j\omega_0 t}$$

$$x(t) = 1 + \cos(\omega_0 t)$$

(ii) $$X(j\omega) = \begin{cases} 1, & |\omega| < \omega_a \\ 0, & |\omega| > \omega_a \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_a}^{\omega_a} 1 .\ e^{j\omega t}\, d\omega$$

$$= \left. \frac{e^{j\omega t}}{2\pi j t} \right|_{-\omega_a}^{\omega_a}$$

$$= \frac{e^{j\omega_a t} - e^{-j\omega_a t}}{2\pi j t}$$

$$= \frac{2j \sin(\omega_a t)}{2\pi j t} = \frac{\sin(\omega_a t)}{\pi t}$$