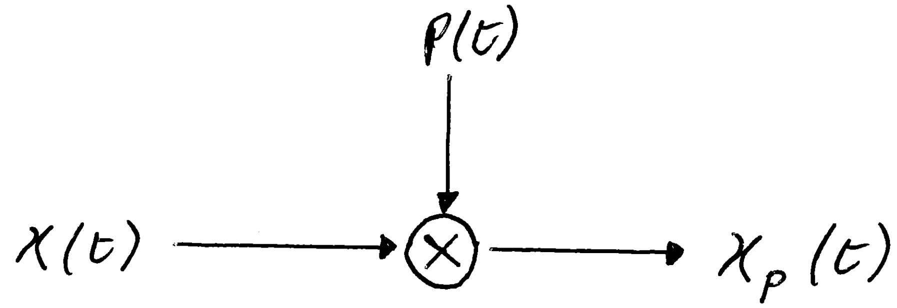
# Sampling of Analogue Signals

## Impulse - Train Sampling




$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

where $\omega_s = \frac{2\pi}{T}$

$$x_p(t) = x(t)\,p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} x(t)\, e^{jk\omega_s t}$$

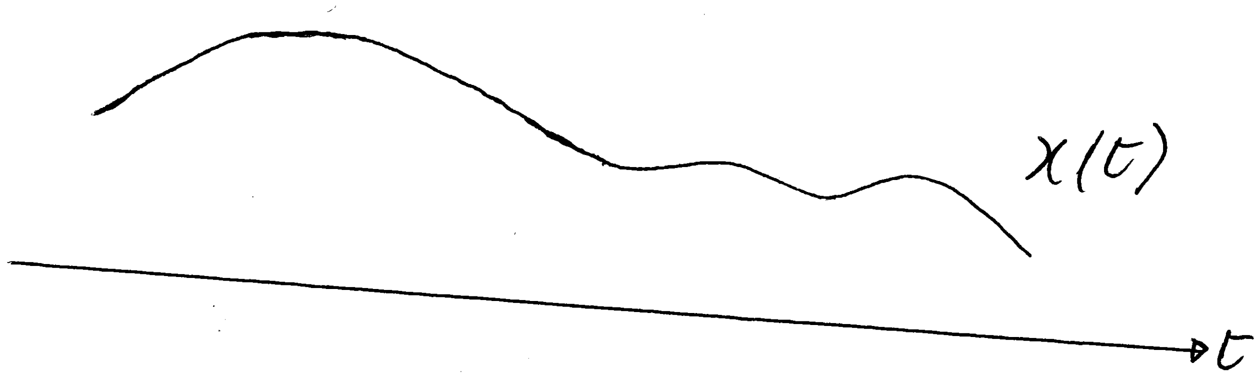
$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

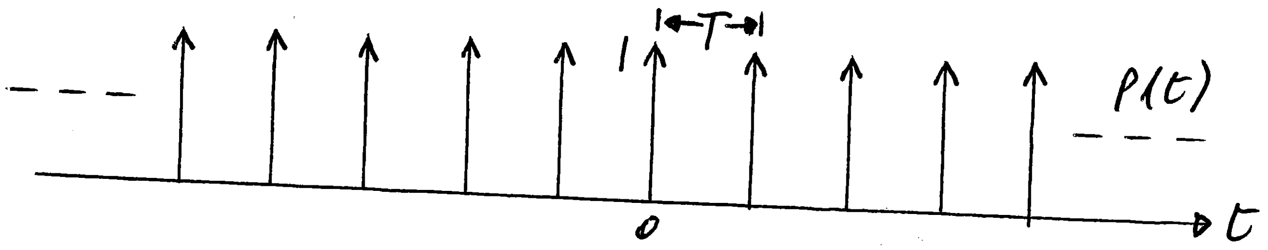
$$X_p(j\omega) = \mathcal{F}\{x_p(t)\} = \int_{-\infty}^{\infty} x(t)\, p(t)\, e^{-j\omega t}\, dt$$

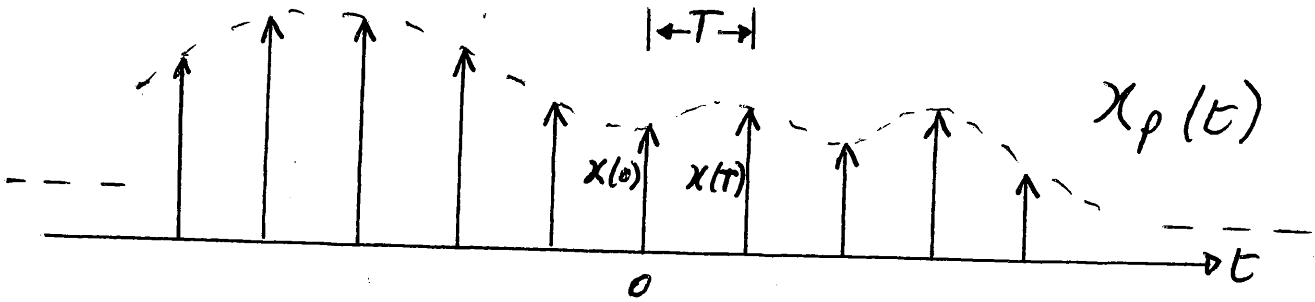
$$X_p(j\omega) = \int_{-\infty}^{\infty} \frac{1}{T} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t} e^{-j\omega t} dt$$

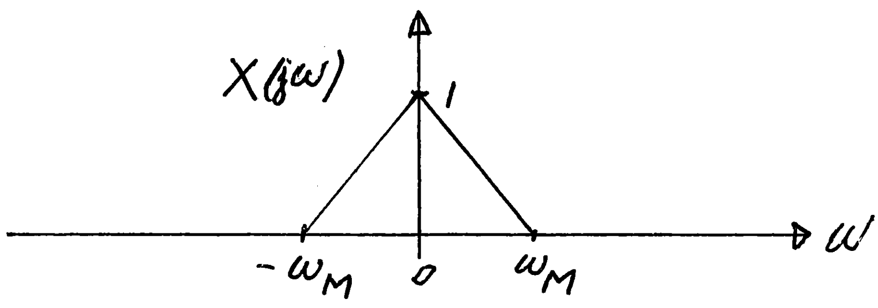
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(t) e^{-j(\omega - k\omega_s)t} dt \right\}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

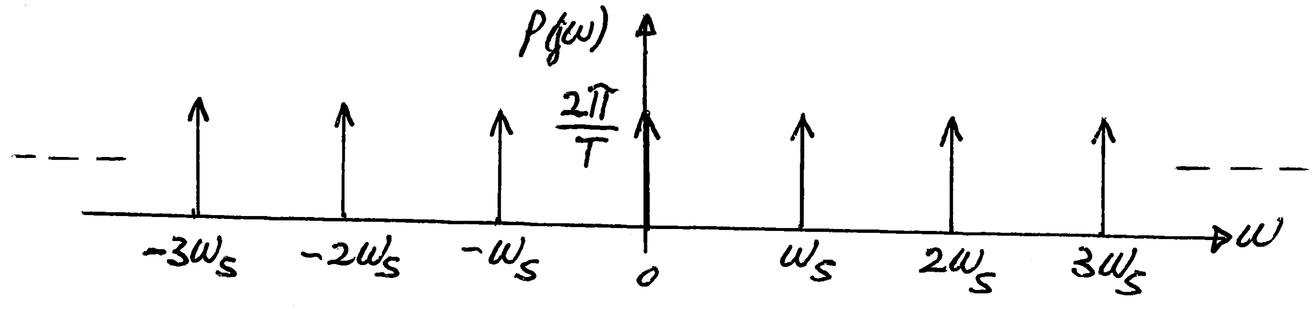


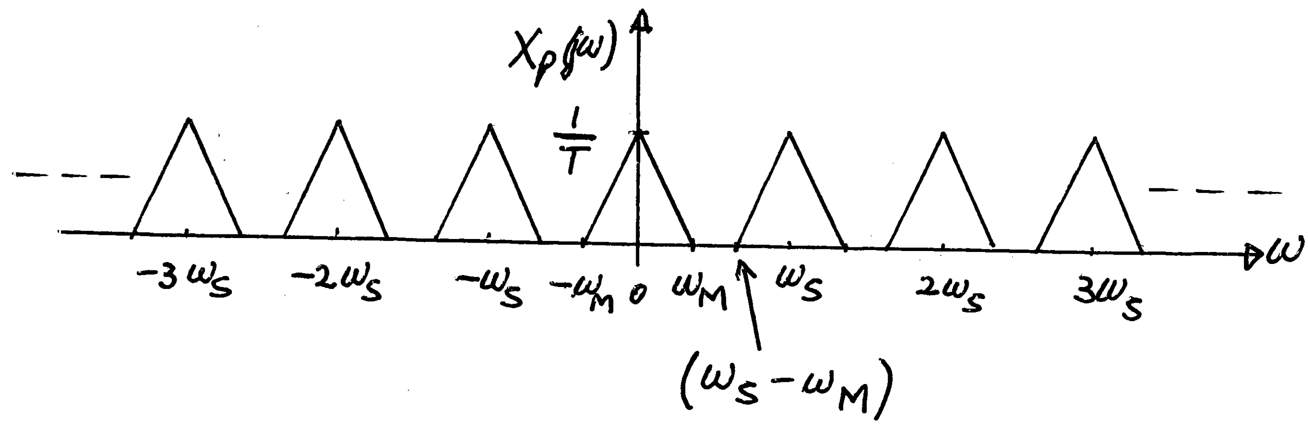





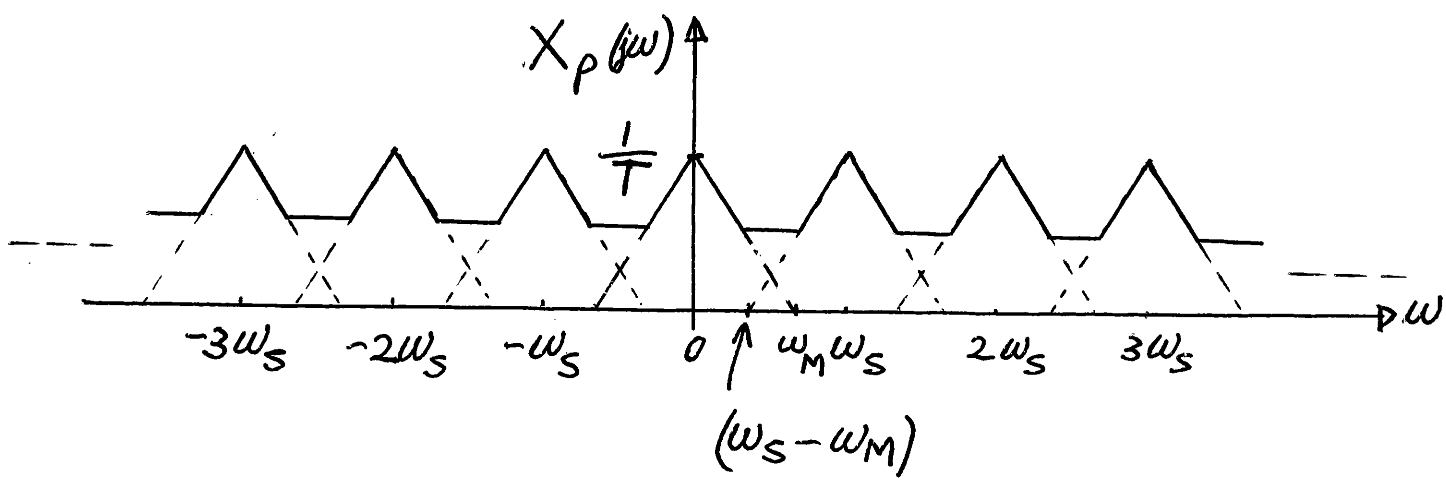

Spectrum of original signal $x(t)$




Spectrum of sampling function $p(t)$



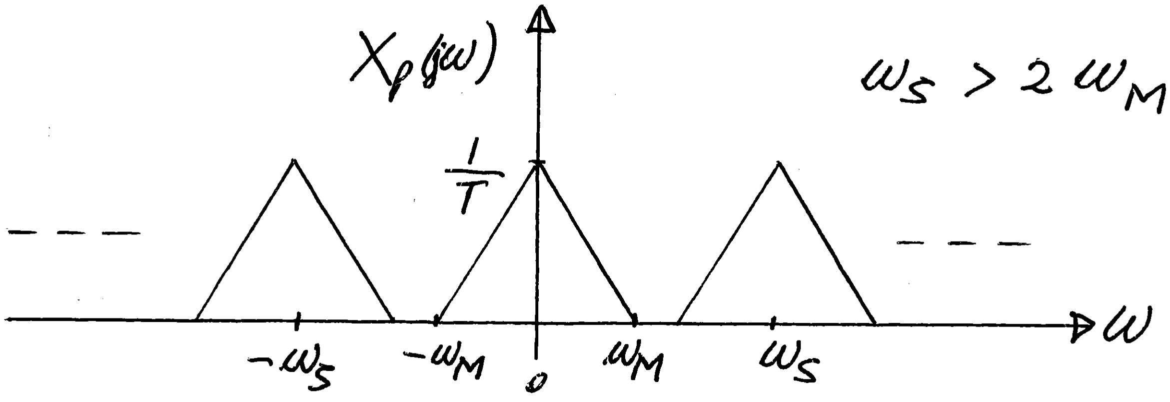

Spectrum of sampled signal with $\omega_s > 2\omega_M$

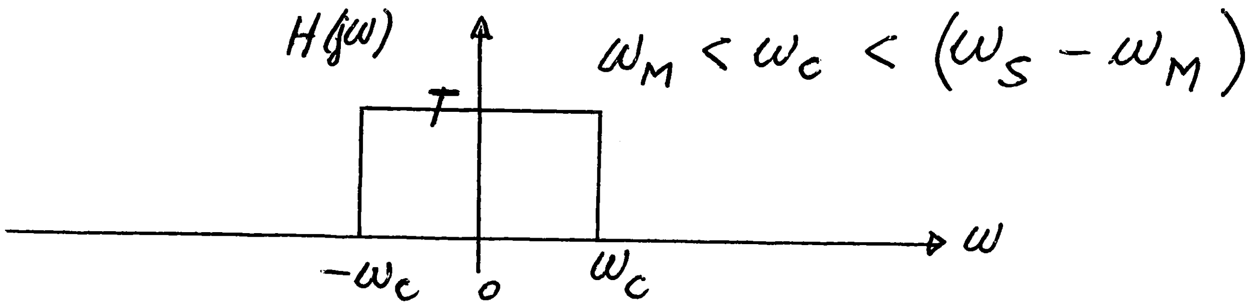


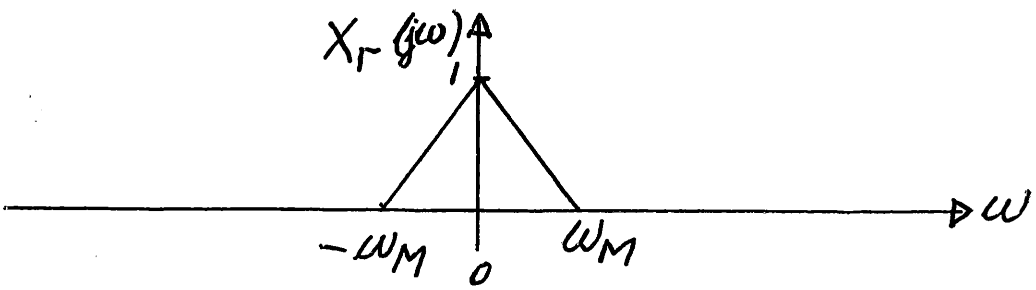

Spectrum of sampled signal with $\omega_s < 2\omega_M$.

$X_p(j\omega)$ is a periodic function of frequency consisting of a sum of shifted replicas of $X(j\omega)$, scaled by $\frac{1}{T}$. If $\omega_M < (\omega_S - \omega_M)$ or equivalently $\omega_S > 2\omega_M$ there is no overlap between the shifted replicas of $X(j\omega)$ and $x(t)$ can be recovered exactly from $x_p(t)$ by means of an ideal low-pass filter with gain $T$ and a cut-off frequency greater than $\omega_M$ and less than $\omega_S - \omega_M$.

If $\omega_S < 2\omega_M$ the shifted replicas of $X(j\omega)$ overlap and it is therefore not possible to recover the original signal. This overlapping of spectra is referred to as <u>aliasing</u>.




Exact recovery of a continuous-time signal from the sampled signal $x_p(t)$ using an ideal low-pass filter.

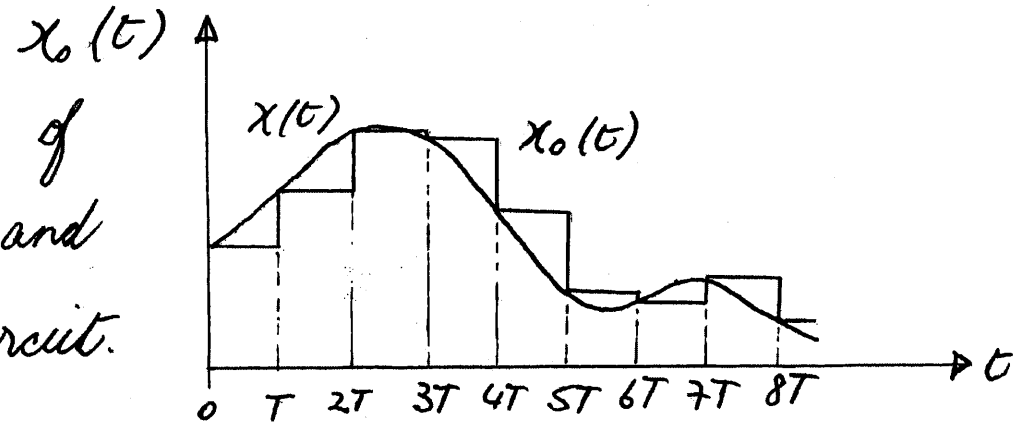
## Sampling Theorem

If a signal $x(t)$ has a bandlimited Fourier transform $X(j\omega)$, that is $X(j\omega) = 0$ for $|\omega| > 2\pi f_M$, then $x(t)$ can be uniquely reconstructed without error from equally spaced samples $x(nT)$, $-\infty < n < \infty$, if $f_s > 2f_M$ where $f_s = \frac{1}{T}$ is the sampling frequency.

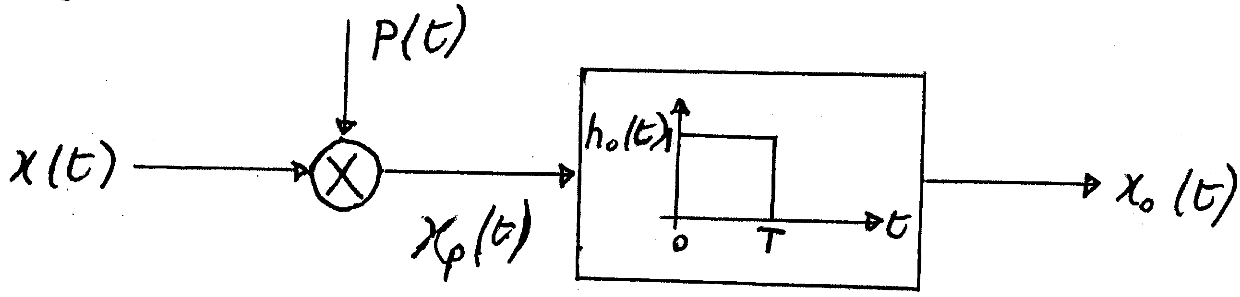
## Signal Reconstruction

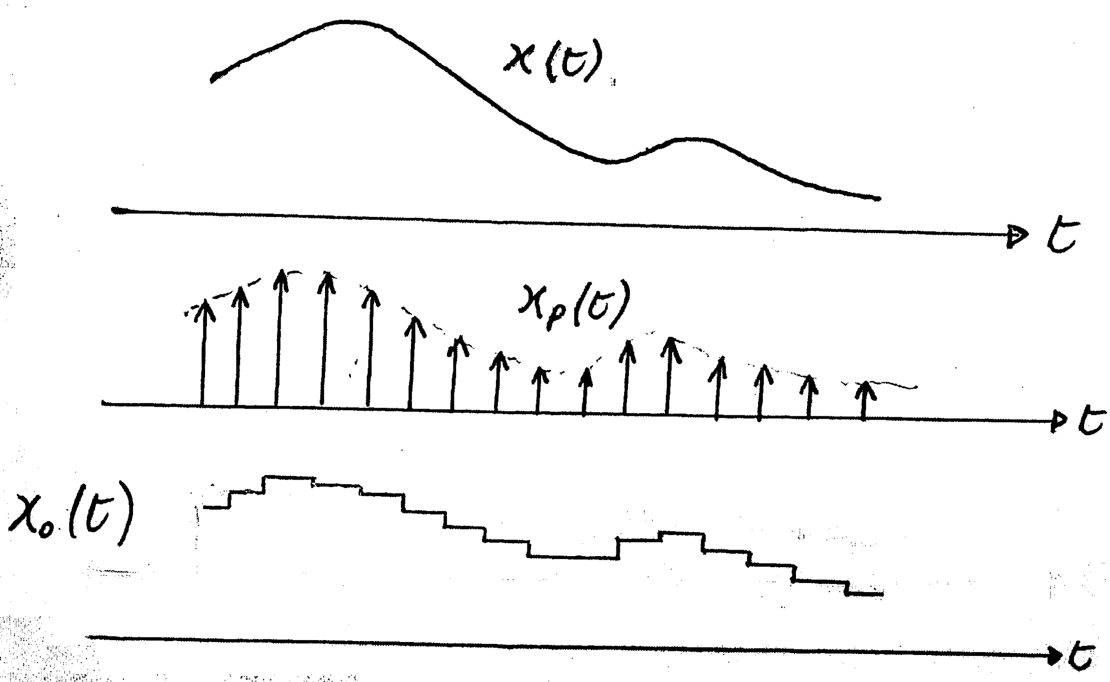
In practice a "sample and hold" circuit followed by a low-pass filter is used to reconstruct $x(t)$ from the samples $x(nT)$. The output $x_0(t)$ of the sample and zero-order hold circuit can be generated by

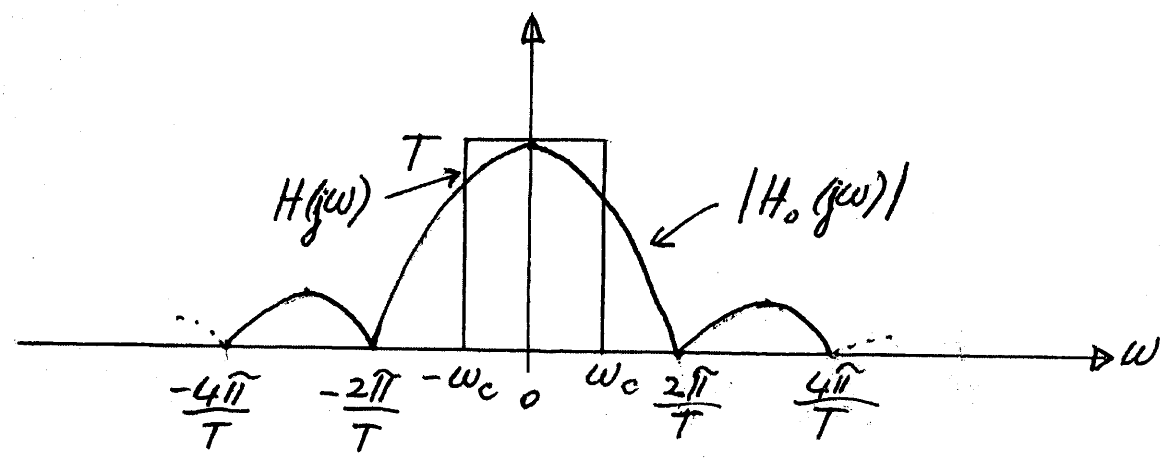
Output of sample and hold circuit.




impulse - train sampling followed by a linear time - invariant system with a rectangular impulse response as shown below.

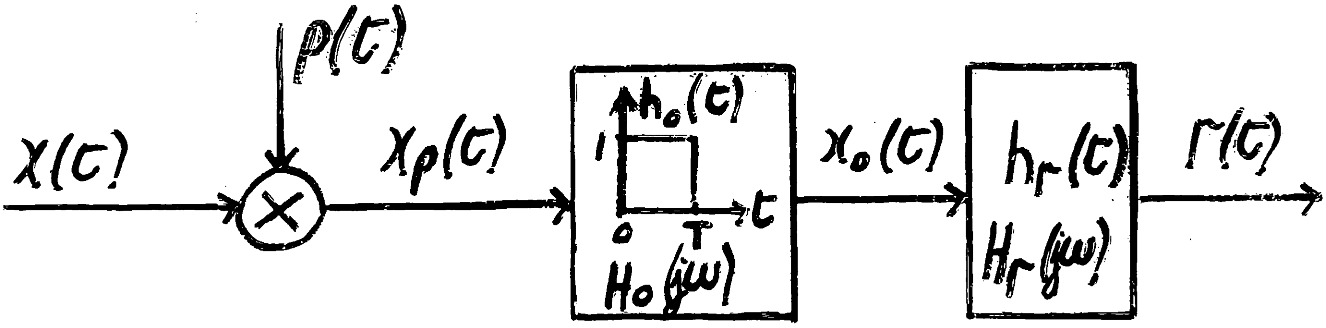





Gain characteristics of the transfer functions for the zero order hold ($H_0(j\omega)$) and for the ideal low-pass interpolating filter ($H(j\omega)$).

$$H_0(j\omega) = T e^{-j\left(\frac{\omega T}{2}\right)} \left[ \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \right]$$




The representation of the sample and zero-order hold operation followed by a low-pass reconstruction filter.

$$r(t) = x(t) \text{ if } \omega_s > 2\omega_M \text{ and}$$

$$H_0(j\omega)\, H_r(j\omega) = T, \quad |\omega| \leq \omega_M$$

$$= 0, \quad |\omega| \geq \omega_s - \omega_M$$

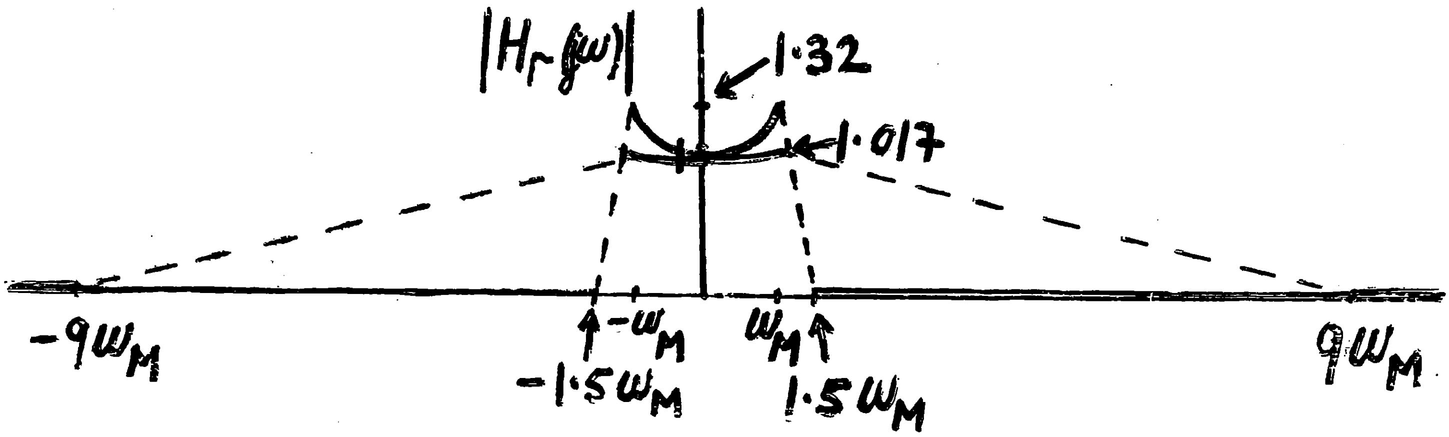
$$H_r(j\omega) = \frac{T}{H_0(j\omega)} = e^{j\left(\frac{\omega T}{2}\right)}\left[\frac{\left(\frac{\omega T}{2}\right)}{\sin\left(\frac{\omega T}{2}\right)}\right], \quad |\omega| \leq \omega_M$$

$$= 0, \quad |\omega| \geq (\omega_s - \omega_M)$$

$$|H_r(j\omega)| = \frac{\left(\frac{\omega T}{2}\right)}{\sin\left(\frac{\omega T}{2}\right)}, \quad |\omega| \le \omega_M$$

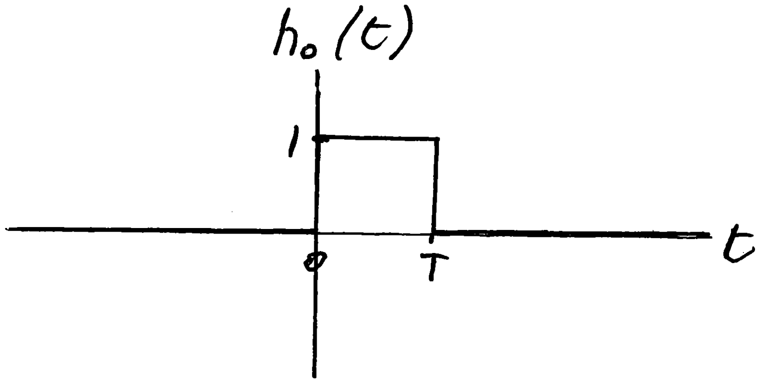
$$= 0, \quad |\omega| \geq (\omega_S - \omega_M)$$

$|H_r(j\omega)|$ for (i) $\omega_S = 2.5\,\omega_M$ and (ii) $\omega_S = 10\,\omega_M$




The reconstruction filter should have a <u>linear phase response</u> for $|\omega| \le \omega_M$.

Consider a linear time-invariant system with impulse response $h_0(t)$ shown below:




Let $H_0(j\omega)$ denote the frequency response of the system.

$$H_0(j\omega) = \mathcal{F}\{h_0(t)\} = \int_{-\infty}^{\infty} h_0(t) e^{-j\omega t} dt$$

$$= \int_0^T e^{-j\omega t} dt$$

$$= -e^{-j\omega t}/j\omega \Big|_0^T$$

$$= \frac{1 - e^{-j\omega T}}{j\omega}$$

$$= \frac{e^{-j(\omega T/2)}\left[e^{j(\omega T/2)} - e^{-j(\omega T/2)}\right]}{j\omega}$$

$$= \frac{e^{-j\left(\frac{\omega T}{2}\right)}\left[2j\sin\left(\frac{\omega T}{2}\right)\right]}{j\omega}$$

$$H_0(j\omega) = T\, e^{-j\left(\frac{\omega T}{2}\right)}\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right]$$

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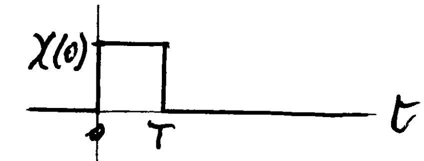
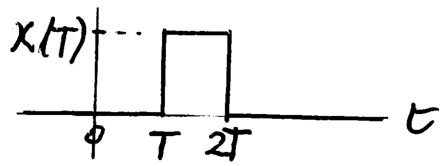
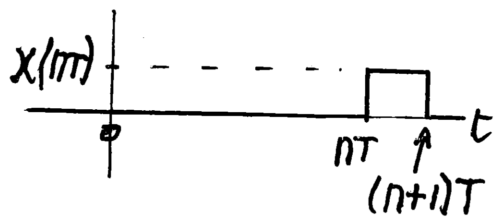
$$x(0)\,\delta(t) \longrightarrow \boxed{h_0(t)} \longrightarrow x(0)\,h_0(t)$$




$$x(T)\,\delta(t-T) \longrightarrow \boxed{h_0(t)} \longrightarrow x(T)\,h_0(t-T)$$




$$x(nT)\,\delta(t-nT) \longrightarrow \boxed{h_0(t)} \longrightarrow x(nT)\,h_0(t-nT)$$




$$\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT) \longrightarrow \boxed{h_0(t)} \longrightarrow \sum_{n=-\infty}^{\infty} x(nT)\,h_0(t-nT)$$

$$\sum_{n=-\infty}^{\infty} x(nT)\,h_0(t-nT)$$