The z - Transform

z - Transform

The z-transform $X(z)$ of a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable. For any given sequence the set of values of z for which the z-transform converges is called the <u>region of convergence</u>. In general, this region is of the form $R_1 < |z| < R_2$. A sufficient condition for convergence is

$$\sum_{n=-\infty}^{\infty} | x(n)\, z^{-n} | < \infty$$

<u>Example</u> Let $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = 1$$

The region of convergence is the entire z-plane

Example Consider the sequence $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

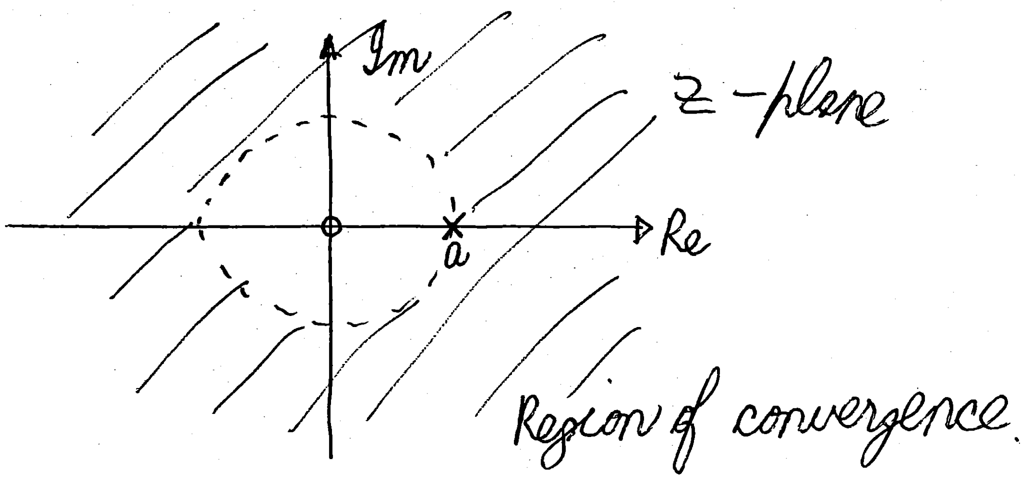
$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

which converges to $\dfrac{1}{1-az^{-1}}$, for $|z| > |a|$.

Values of z for which $X(z)$ is infinite are referred to as the poles of $X(z)$. Values of z for which $X(z) = 0$ are referred to as the zeros of $X(z)$.

$$X(z) = \frac{z}{z-a}$$

$X(z)$ has a zero at $z=0$ and a pole at $z=a$.



<u>z-transform of a causal sequence</u>

$$x(n) = 0 \text{ for } n<0$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \qquad \text{— } (1)$$

The ROC is of the form $|z| > R_1$.

To see that this is true, suppose that the series is absolutely convergent for $z = z_1$, so that

$$\sum_{n=0}^{\infty} |x(n)\, z_1^{-n}| < \infty \qquad — ②$$

Consider the series $\sum_{n=0}^{\infty} |x(n)\, z^{-n}|$ — ③

Note that if $|z| > |z_1|$, then each term is smaller than in the series of equation ②, and thus

$$\sum_{n=0}^{\infty} |x(n)\, z^{-n}| < \infty \text{ for } |z| > |z_1|$$

If R_1 is the smallest value of $|z|$ for which the series of eqn. ① converges, then the series converges for $|z| > R_1$. The ROC of the z-transform of a causal sequence is the exterior of a circle.

Relation between the z-transform and the Fourier transform of a sequence.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}}$$

Inverse z-transform

The Cauchy integral theorem states that

$$\frac{1}{2\pi j} \oint_C z^{k-1}\, dz = \begin{cases} 1, & k=0 \\ 0, & k \neq 0. \end{cases} \quad \text{— (1)}$$

where C is a counter-clockwise contour that encircles the origin.

The z-transform of the sequence $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad — ②$$

If we multiply both sides of eqn. ② by z^{k-1} and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely in the region of convergence of $X(z)$, we obtain

$$\frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz = \frac{1}{2\pi j} \oint_C \sum_{n=-\infty}^{\infty} x[n] z^{-n+k-1} dz \quad — ③$$

Interchanging the order of integration and summation on the right-hand side of eqn. ③ we obtain

$$\frac{1}{2\pi j}\oint_C X(z) z^{k-1} dz = \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j}\oint_C z^{-n+k-1} dz \quad — (4)$$

which from equation (1) becomes

$$\frac{1}{2\pi j}\oint_C X(z) z^{k-1} dz = x[k] \quad — (5)$$

The inverse z-transform relation is given by the contour integral

$$x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz \quad — (6)$$

where C is a counter-clockwise closed contour in the region of convergence of X(z) and encircling the origin in the z-plane.

Contour integrals of the form of eqn. (6) can be evaluated using the residue theorem.

Partial Fraction Expansion

Consider the z-transform

$$X(z) = \frac{3 - z^{-1}}{(1-0.25z^{-1})(1-0.5z^{-1})}, \; |z| > 0.5$$

$x[n]$ is a causal sequence since $X(z)$ converges at $z = \infty$.

$X(z)$ can be expanded in a partial fraction expansion.

$$X(z) = \frac{1}{1-0.25z^{-1}} + \frac{2}{1-0.5z^{-1}}$$

$$x[n] = (0.25)^n u[n] + 2(0.5)^n u[n]$$

Properties of the z-transform

Linearity

Consider two sequences $x[n]$ and

$y[n]$ with z-transforms $X(z)$ and $Y(z)$ respectively.

$$x[n] \longleftrightarrow X(z), \quad ROC: R_x$$

$$y[n] \longleftrightarrow Y(z), \quad ROC: R_y$$

$$a\,x[n] + b\,y[n] \longleftrightarrow a\,X(z) + b\,Y(z)$$

ROC: at least $R_x \cap R_y$

<u>Shift of a sequence</u>

$$x[n] \longleftrightarrow X(z), \quad ROC: R_x$$

$$x[n-n_0] \longleftrightarrow z^{-n_0}\,X(z)$$

ROC: R_x with the possible exception of $z=0$ or $z=\infty$.

Differentiation of $X(z)$

$$x[n] \longleftrightarrow X(z), \quad ROC: R_x$$

$$n\,x[n] \longleftrightarrow -z\frac{dX(z)}{dz}, \quad ROC: R_x$$

Initial Value Theorem

If $x[n]$ is zero for $n<0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Convolution of Sequences

If $x[n] \longleftrightarrow X(z)$, $ROC: R_x$

$y[n] \longleftrightarrow Y(z)$, $ROC: R_y$

$$w[n] = \sum_{k=-\infty}^{\infty} x[k]\,y[n-k]$$

then $W(z) = X(z)Y(z)$, ROC: at least $R_x \cap R_y$

The convolution property can be derived as follows:

$$W(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]\, y[n-k] \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]\, z^{-n}$$

Let $m = n-k$,

$$W(z) = \sum_{k=-\infty}^{\infty} x[k] \left[\sum_{m=-\infty}^{\infty} y[m]\, z^{-m} \right] z^{-k}$$

For values of z inside the regions of convergence of both $X(z)$ and $Y(z)$ we can write

$$W(z) = X(z)\, Y(z)$$

Multiplication of Sequences

Let $w[n] = x[n]y[n]$

so that

$$W(z) = \sum_{n=-\infty}^{\infty} x[n]\,y[n]\,z^{-n}$$

But $x[n] = \dfrac{1}{2\pi j}\oint_{C_1} X(v)\,v^{n-1}\,dv$

where C_1 is a counter-clockwise contour within the ROC of $X(v)$.

Then $$W(z) = \frac{1}{2\pi j}\sum_{n=-\infty}^{\infty} y[n]\oint_{C_1} X(v)\left(\frac{z}{v}\right)^{-n} v^{-1}\,dv$$

$$= \frac{1}{2\pi j}\oint_{C_1}\left[\sum_{n=-\infty}^{\infty} y[n]\left(\frac{z}{v}\right)^{-n}\right] v^{-1}X(v)\,dv$$

or $$W(z) = \frac{1}{2\pi j}\oint_{C_1} X(v)\,Y\!\left(\frac{z}{v}\right) v^{-1}\,dv \qquad \text{— (1)}$$

where C_1 is a closed contour in the overlap of the regions of convergence of $X(v)$ and $Y\left(\frac{z}{v}\right)$.

Equation ① is called the <u>complex convolution theorem</u>.

<u>Note</u> If $X(z)$ and $Y(z)$ converge on the unit circle, we can chose $z = e^{j\Omega}$ and $v = e^{j\theta}$. Equation ① then becomes

$$W(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y\left(e^{j(\Omega-\theta)}\right) d\theta$$

<u>System Function</u>

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$, the z-transform of the unit-sample response, is referred to as the system function or transfer function of the system.

For z evaluated on the unit circle ($z = e^{j\Omega}$), $H(z)$ reduces to the frequency response of the system provided that the unit circle is in the ROC for $H(z)$.

If the system is <u>stable</u>, the impulse response is absolutely summable and the Fourier transform of $h[n]$ converges.

Since $H(e^{j\Omega}) = H(z)\Big|_{z=e^{j\Omega}}$, this implies that for a stable system, the ROC of $H(z)$ must include the unit circle.

For a stable and causal system, the ROC of $H(z)$ includes the unit circle and the entire z-plane outside the unit circle, including $z=\infty$.

For a system that is both causal and stable, all the poles of $H(z)$ must be inside the unit circle.

Example A digital filter has an impulse response $h[n] = (0.25)^n u[n]$. The input

$$x[n] = (0.5)^n u[n]$$

Determine the response $y[n]$.

$$X(z) = \frac{1}{1-0.5z^{-1}} \quad |z| > 0.5$$

$$H(z) = \frac{1}{1-0.25z^{-1}} \quad |z| > 0.25$$

$$Y(z) = X(z)H(z) = \frac{1}{(1-0.5z^{-1})(1-0.25z^{-1})}, \; |z| > 0.5$$

$$Y(z) = \frac{2}{1-0.5z^{-1}} - \frac{1}{1-0.25z^{-1}}$$

$$y[n] = 2(0.5)^n u[n] - (0.25)^n u[n]$$

Systems Characterised by Linear Constant-Coefficient Difference Equations

Consider an LTI system for which the

input and output satisfy a difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Applying the z-transform to each side of this equation we have

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\text{or } Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

An additional constraint such as causality or stability of the system is required to specify the region of convergence of $H(z)$

<u>Example</u> The input and output of a stable and causal LTI system satisfy the linear constant coefficient difference equation

$$y[n] = x[n] + 0.75\, y[n-1]$$

Determine $H(z)$ and sketch the frequency response of the system.

$$Y(z) = X(z) + 0.75 z^{-1} Y(z)$$

$$Y(z)\left[1 - 0.75 z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.75 z^{-1}}, \quad |z| > 0.75$$

$$H(e^{j\Omega}) = \frac{1}{1-0.75e^{-j\Omega}}$$

$$|H(e^{j\Omega})| = \frac{1}{|1-0.75e^{-j\Omega}|}$$

$$= \frac{1}{|(1-0.75\cos\Omega)+j0.75\sin\Omega|}$$

$$\angle H(e^{j\Omega}) = -\angle 1-0.75e^{-j\Omega}$$

$$= -\angle(1-0.75\cos\Omega)+j0.75\sin\Omega$$

$$\angle H(e^{j\Omega}) = -\tan^{-1}\left[\frac{0.75\sin\Omega}{1-0.75\cos\Omega}\right]$$