Fourier Representation of Finite-Duration Sequences – The Discrete Fourier Transform

The discrete Fourier transform (DFT) is a frequency domain representation of <u>finite-duration sequences</u>. Consider a finite-duration sequence $x[n]$ that is zero outside $0 \le n \le N-1$. The corresponding periodic sequence $\tilde{x}[n]$, of period N, for which $x[n]$ is one period is given by

$$\tilde{x}[n] = x[((n))_N]$$

$$x[n] = \tilde{x}[n] R_N[n]$$

where
$$R_N[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Consider $\tilde{X}[k]$, the DFS coefficients of $\tilde{x}[n]$.

Form a finite-duration sequence $X[k]$, corresponding to one period of $\tilde{X}[k]$.

$$X[k] = \tilde{X}[k]\, R_N[k]$$

The operation is invertible since $\tilde{X}[k]$ can be obtained from $X[k]$ by

$$\tilde{X}[k] = X[((k))_N]$$

The sequence $x(n)$ is related to $X[k]$ by

$$x[n] \leftrightarrow \tilde{x}[n] \leftrightarrow \tilde{X}[k] \leftrightarrow X[k]$$

where "$\leftrightarrow$" denotes an invertible operation:

$\tilde{X}[k]$ and $\tilde{x}[n]$ are related by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

The DFT pair is given by

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\left(\frac{2\pi}{N}\right)kn}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

The sequence $X[k]$ is called the DFT of $x[n]$, and $x[n]$ is called the inverse DFT of $X[k]$.

For a finite-duration sequence $x[n]$ that is zero outside $0 \le n \le N-1$, the N-point DFT $X[k]$ is related to the discrete-time

Fourier Transform $X(e^{j\Omega})$

$$X[k] = X(e^{j\Omega})\Big|_{\Omega=\left(\frac{2\pi}{N}\right)k}, \quad 0 \le k \le N-1$$

Properties of the DFT

Linearity

Consider two finite-duration sequences, $x_1(n)$ that is zero outside the interval $0 \le n \le N_1-1$, and $x_2(n)$ that is zero outside $0 \le n \le N_2-1$.

$$x_3[n] = a\, x_1(n) + b\, x_2(n)$$

$x_3(n)$ is zero outside $0 \le n \le N_3-1$ where $N_3 = \max[N_1, N_2]$.

The DFT of $x_3[n]$ is

$$X_3[k] = a\, X_1[k] + b\, X_2[k]$$

where $$X_1[k] = \sum_{n=0}^{N-1} x_1[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

and $$X_2[k] = \sum_{n=0}^{N-1} x_2[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

The DFTs must be computed with $N \geq N_3$.

Circular Convolution

Consider two finite-duration sequences $x_1[n]$ and $x_2[n]$ that are zero outside the interval $0 \leq n \leq N-1$.

$$\tilde{x}_1[n] = x_1[((n))_N]$$

$$\tilde{x}_2[n] = x_2[((n))_N]$$

$x_1[n] \circledast x_2[n]$, the N-point circular convolution of $x_1[n]$ and $x_2[n]$, is defined by

$$x_1[n] \circledast x_2[n] = \left[\tilde{x}_1[n] \circledast \tilde{x}_2[n]\right] R_N[n]$$

$$x_1[n] \longleftrightarrow X_1[k]$$

$$x_2[n] \longleftrightarrow X_2[k]$$

$$\tilde{x}_1[n] \longleftrightarrow \tilde{X}_1[k]$$

$$\tilde{x}_2[n] \longleftrightarrow \tilde{X}_2[k]$$

$$\tilde{x}_1[n] \circledast \tilde{x}_2[n] \longleftrightarrow \tilde{X}_1[k]\, \tilde{X}_2[k]$$

$$\left[\tilde{x}_1[n] \circledast \tilde{x}_2[n]\right] R_N[n] \longleftrightarrow \tilde{X}_1[k]\, \tilde{X}_2[k]\, R_N[k]$$

Noting that $\tilde{X}_1[k]\, \tilde{X}_2[k]\, R_N[k] = \tilde{X}_1[k] R_N[k] \tilde{X}_2[k] R_N[k]$,

$X_1[k] = \tilde{X}_1[k] R_N[k]$ and $X_2[k] = \tilde{X}_2[k] R_N[k]$,

we have

$$\left[\tilde{x}_1[n] \circledast \tilde{x}_2[n]\right] R_N[n] \longleftrightarrow X_1[k]\, X_2[k]$$

$$x_1[n] \circledast x_2[n] \longleftrightarrow X_1[k]\, X_2[k]$$

Multiplication of Sequences

Consider two finite duration sequences $x_1[n]$ and $x_2[n]$ that are zero outside the interval $0 \leq n \leq N-1$.

$$\tilde{x}_1[n] = x_1[((n))_N]$$

$$\tilde{x}_2[n] = x_2[((n))_N]$$

$$x_1[n]\, x_2[n] = \tilde{x}_1[n] R_N[n]\, \tilde{x}_2[n] R_N[n]$$

$$= \tilde{x}_1(n)\, \tilde{x}_2(n)\, R_N[n]$$

$$\tilde{x}_1(n) \longleftrightarrow \tilde{X}_1[k]$$

$$\tilde{x}_2[n] \longleftrightarrow \tilde{X}_2[k]$$

$$x_1[n] \longleftrightarrow X_1[k]$$

$$x_2[n] \longleftrightarrow X_2[k]$$

$$\tilde{x}_1[n]\,\tilde{x}_2[n] \longleftrightarrow \frac{1}{N}\left[\tilde{X}_1[k] \circledast \tilde{X}_2[k]\right]$$

$$\tilde{x}_1[n]\,\tilde{x}_2[n]\,R_N[n] \longleftrightarrow \frac{1}{N}\left[\tilde{X}_1[k] \circledast \hat{X}_2[k]\right]R_N[k]$$

$$x_1[n]\,x_2[n] \longleftrightarrow \frac{1}{N}\left[X_1[k] \circledast X_2[k]\right]$$

Symmetry Property

The N-point DFT of a sequence $x[n]$ that is zero outside the interval $0 \leq n \leq N-1$, is given by

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

If $x(n)$ is real, then $X[k] = X^*[N-k]$, $1 \leq k \leq N-1$.

For $1 \leq k \leq N-1$, we have

$$X[N-k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)(N-k)n}$$

$$X^*[N-k] = \sum_{n=0}^{N-1} x[n]\, e^{j\left(\frac{2\pi}{N}\right)(N-k)n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$X^*[N-k] = X[k], \quad 1 \leq k \leq N-1.$$

Linear Convolution Using the DFT

Consider two finite-duration sequences $x_1[n]$ and $x_2[n]$. If $x_1[n]$ is zero outside the interval $0 \leq n \leq N_1-1$, and $x_2[n]$ is zero outside the interval $0 \leq n \leq N_2-1$, then $x_3[n] = x_1(n) * x_2[n]$

is zero outside the interval $0 \leq n \leq N_1 + N_2 - 2$

$x_3[n]$ is of duration $N_1 + N_2 - 1$ samples.

The N-point circular convolution $x_1[n] \circledast x_2[n]$ is identical to the linear convolution $x_1[n] * x_2[n]$, if $N \geq N_1 + N_2 - 1$.

An alternative way to perform the linear convolution of $x_1[n]$ and $x_2[n]$ is to determine the N-point DFTs $X_1[k]$ and $X_2[k]$, multiply $X_1[k]$ and $X_2[k]$, and then compute the inverse DFT of $X_1[k]\, X_2[k]$. The result is the linear convolution of $x_1[n]$ and $x_2[n]$, provided that $N \geq N_1 + N_2 - 1$.

The validity of this procedure can also be seen by noting that

$$x_3[n] = x_1[n] * x_2[n]$$

$$X_3(e^{j\Omega}) = X_1(e^{j\Omega})\, X_2(e^{j\Omega})$$

$x_3[n]$ is zero outside the interval $0 \leq n \leq N_1 + N_2 - 2$ and can therefore be represented by an N-point DFT where $N \geq N_1 + N_2 - 1$.

$$X_3[k] = \left[X_3(e^{j\Omega}) \Big|_{\Omega = \frac{2\pi k}{N}} \right] R_N[k]$$

$$= \left[X_1(e^{j\Omega}) \Big|_{\Omega = \frac{2\pi k}{N}} \right] R_N(k) \left[X_2(e^{j\Omega}) \Big|_{\Omega = \frac{2\pi k}{N}} \right] R_N(k)$$

$$X_3[k] = X_1[k]\, X_2[k]$$

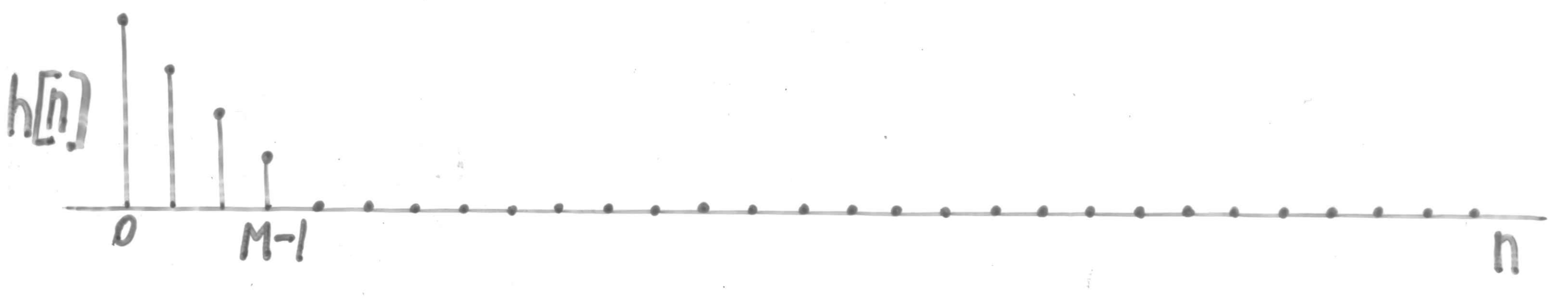
$$x_3[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X_1[k] X_2[k] e^{j\left(\frac{2\pi}{N}\right)kn}, & 0 \le n \le N-1. \\ 0, & \text{otherwise} \end{cases}$$

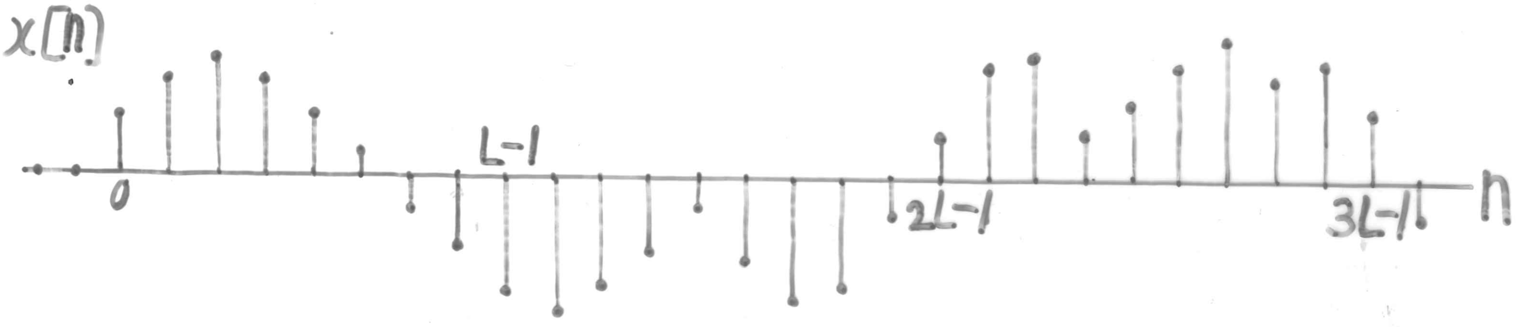
If fast Fourier transform (FFT) algorithms are used to compute the DFT and inverse DFT, this approach sometimes requires less computation than performing the linear convolution directly.

Sectioned Convolutions

If the DFT is to be used to convolve a finite-duration sequence with a sequence of indefinite duration, then the signal to be filtered must be segmented into sections of length L. Each section can then be convolved with the finite-duration unit-sample response and the filtered sections fitted together.

Consider the unit-sample response $h[n]$ of length M and the signal $x[n]$ shown below.





$x[n]$ can be segmented into sections having only L nonzero points, with the k^{th} section denoted by $x_k[n]$.

$$x_k[n] = \begin{cases} x[n], & kL \leq n \leq (k+1)L - 1 \\ 0, & \text{otherwise} \end{cases}$$

$x[n]$ can be written as

$$x(n) = \sum_{k=0}^{\infty} x_k[n]$$

Convolving $x(n)$ with $h(n)$ and using the distributive property of convolution, we obtain

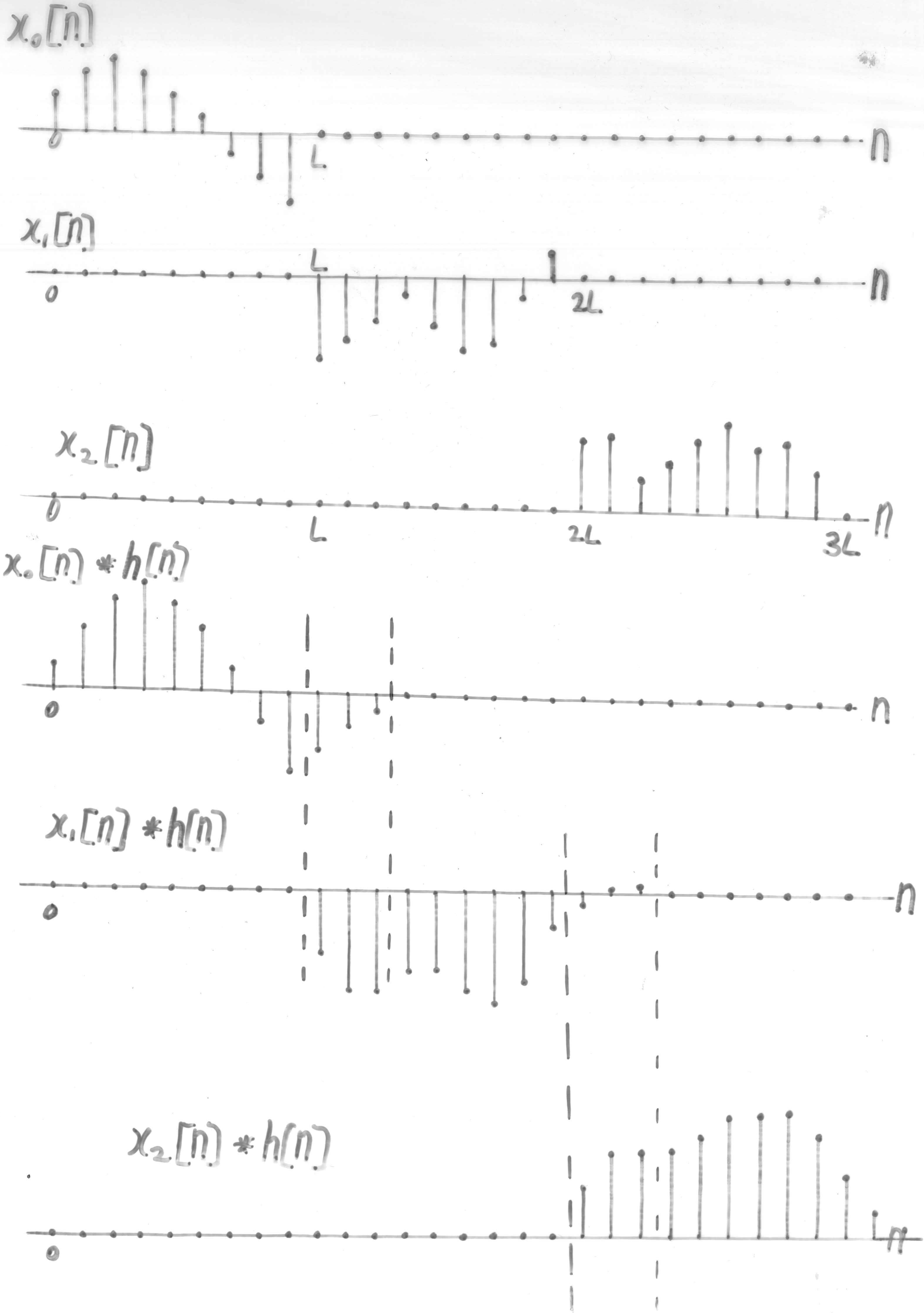
$$x[n] * h[n] = \left(\sum_{k=0}^{\infty} x_k[n] \right) * h[n]$$

$$= \sum_{k=0}^{\infty} \left(x_k[n] * h[n] \right)$$

$x[n]$ convolved with $h[n]$ is equal to the sum of the $x_k[n]$ convolved with $h[n]$. Since each of the terms $[x_k[n] * h[n]]$ in the sum is of length $L+M-1$, the linear convolution $x_k[n] * h[n]$ can be obtained using an N-point DFT, where $N \geq (L+M-1)$. Since each of the sections $x_k[n]$ has at most L nonzero points, the filtered sections

$x_k[n] * h[n]$ can have at most $(L+M-1)$ nonzero points. Therefore, the filtered sections will overlap by $(M-1)$ points in carrying out the sum $\sum_{k=0}^{\infty} x_k[n] * h[n]$.

Since the filtered sections are overlapped and added to construct the output, this procedure is referred to as the overlap - add method.



$x_0[n]$
0
L
n
$x_1[n]$
0
L
2L
n
$x_2[n]$
0
L
2L
3L
n
$x_0[n] * h(n)$
0
n
$x_1[n] * h(n)$
0
n
$x_2[n] * h(n)$
0
n

Parseval's relation for the DFT

If $x[n]$ is an N-point sequence and $X[k]$ is its N-point DFT, then

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Proof

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\left(\frac{2\pi}{N}\right)nk}$$

$$x^*[n] = \frac{1}{N} \sum_{r=0}^{N-1} X^*[r] e^{-j\left(\frac{2\pi}{N}\right)nr}$$

$$|x[n]|^2 = x[n]\, x^*[n]$$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} X[k] X^*[r] e^{j\left(\frac{2\pi}{N}\right)(k-r)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{r=0}^{N-1} X^*[r] \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)(k-r)n} \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] X^*[k] = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 .$$

Fourier Representation of Finite-Duration Sequences – The Discrete Fourier Transform

The discrete Fourier transform (DFT) is a frequency domain representation of finite-duration sequences. Consider a finite-duration sequence $x[n]$ that is zero outside $0 \leq n \leq N-1$. The corresponding periodic sequence $\tilde{x}[n]$, of period N, for which $x[n]$ is one period is given by

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where $$R_N[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$