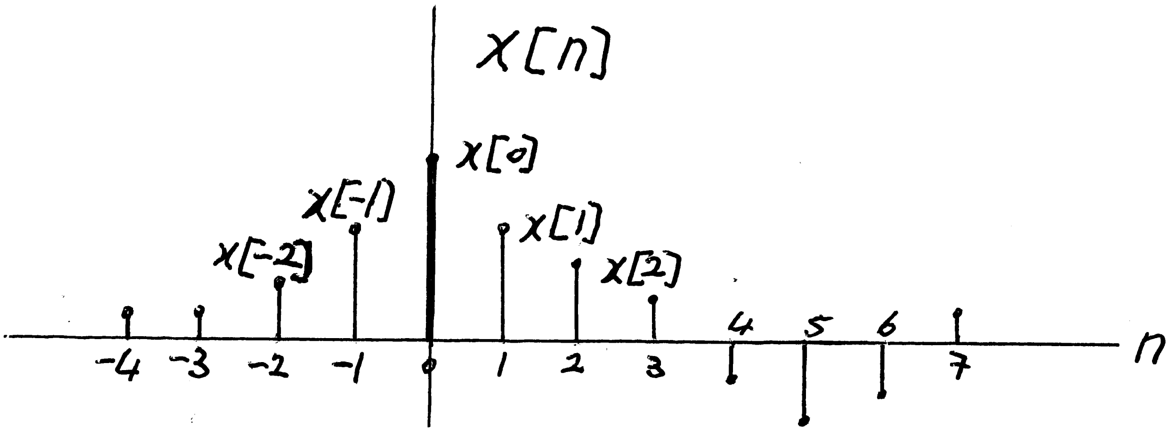
Discrete - Time Signals and Systems

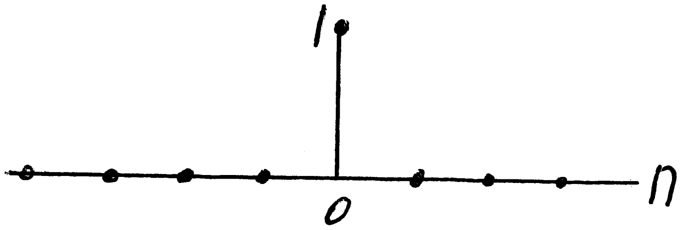
Discrete - time signals

Discrete - time signals are represented as sequences of numbers. The discrete - time signal $x[n]$ is defined only for integer values of n.



Unit - sample sequence, $\delta[n]$

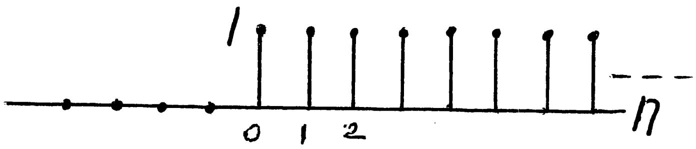
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



$\delta[n]$ is also referred to as the "unit impulse".

Unit-step sequence, $u[n]$

$$u[n] = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$$



Real exponential sequence, a^n

$$x[n] = a^n$$

where a is a real number.

A sequence $x[n]$ is defined to be <u>periodic</u> with <u>period N</u> if $x[n] = x[n+N]$ for all n.

An arbitrary sequence $x[n]$ can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Linear Time-Invariant Systems

Let $y_1[n]$ be the response to $x_1[n]$ and

$$x[n] \longrightarrow \boxed{\text{Discrete - Time System}} \longrightarrow y[n]$$

$y_2[n]$ be the response to $x_2[n]$. The discrete - time system is linear if the response to $a\,x_1[n] + b\,x_2[n]$ is $a\,y_1[n] + b\,y_2[n]$ for arbitrary constants a and b.

Let $h_k[n]$ be the response of the system to $\delta[n-k]$.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

The response of the system to an arbitrary input $x[n]$ can be expressed as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

A *time-invariant system* has the property that if $h[n]$ is the response to $\delta[n]$, then $h[n-k]$ is the response to $\delta[n-k]$

If the system is both linear and time-invariant the output $y[n]$ is given by:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

A linear time-invariant system is completely characterized by its *unit-sample response* $h[n]$.

$y[n]$ is the *convolution* of $x[n]$ with $h[n]$.

$$y[n] = x[n] * h[n]$$

Convolution is a *commutative* operation, that is,

$$x[n] * h[n] = h[n] * x[n]$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

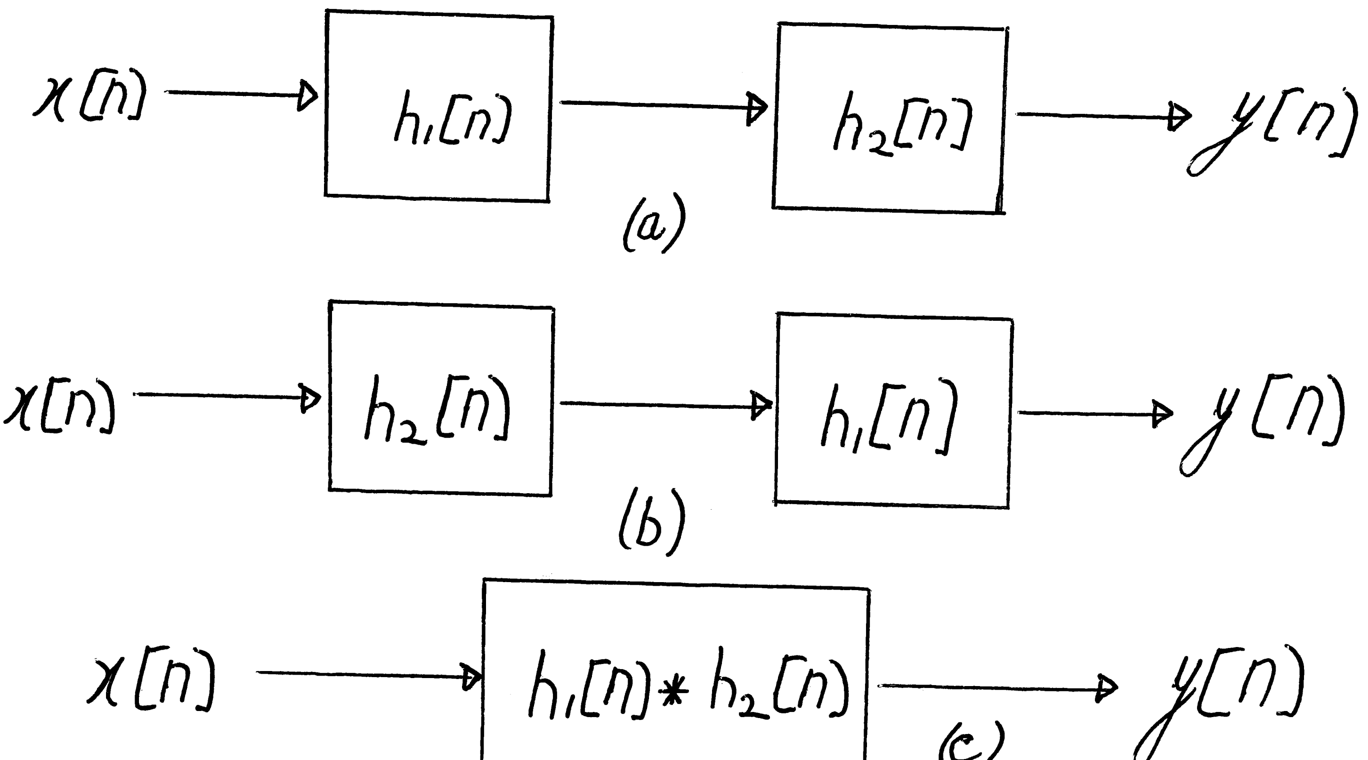
Letting $r = n-k$, or equivalently $k = n-r$,

we have, $$x[n] * h[n] = \sum_{r=-\infty}^{\infty} x[n-r]h[r]$$

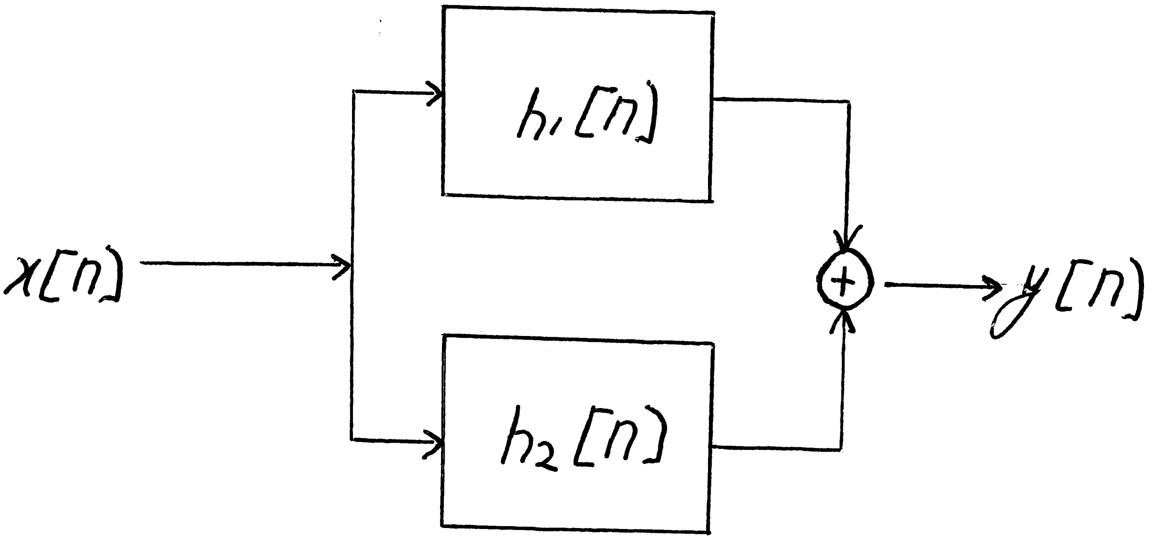
$$= h[n] * x[n].$$

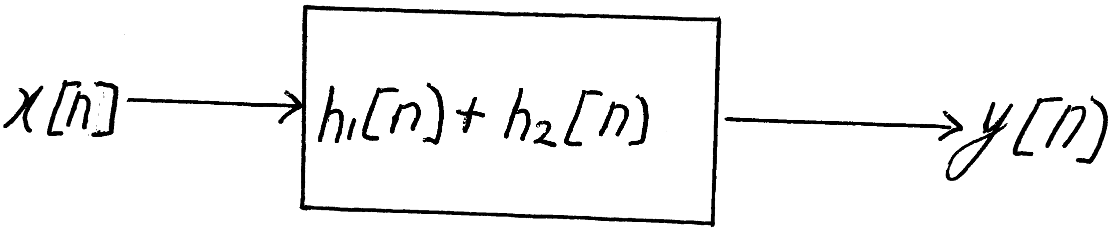
The convolution operation is also associative, that is,

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$



Series combination of the LTI systems in (a) or (b) and the equivalent system (c).





Parallel combination of LTI systems and the equivalent system.

Stability and Causality

A system is stable if every bounded input produces a bounded output.

LTI systems are stable if and only if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

a discrete-time LTI system is causal if $h[n] = 0$ for $n < 0$.

Linear Constant-Coefficient Difference Equations

An important class of LTI discrete-time systems is that for which the input $x[n]$ and output $y[n]$ are related through a linear constant-coefficient difference equation of the form:

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This equation can be rearranged in the form:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

If the system is causal we can specify initial rest conditions, so that if $x[n] = 0$, $n < n_0$, then $y[n] = 0$, $n < n_0$.

Example

Consider the recursion formula:

$$y[n] = a\,y[n-1] + x[n]$$

To obtain the unit-sample response, let $x[n] = \delta[n]$, and assume that $y[n] = 0$ for $n < 0$.

$$h[n] = 0, \qquad n < 0$$

$$h[0] = a\,h[-1] + 1 = 1$$

$$h[1] = a\,h[0] = a$$

$$h[2] = a\,h[1] = a^2$$

$$\vdots$$

$$h[n] = a\,h[n-1] = a^n$$

$$h[n] = a^n u[n]$$

The unit-sample response is of infinite duration and the system is referred to as an <u>infinite impulse response (IIR) system</u>

Consider the non-recursive equation:

$$y[n] = \sum_{k=0}^{M} b_k \, x[n-k]$$

The impulse response of this system is

$$h[n] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

The impulse response for this system has finite duration. The system is referred to as a <u>finite impulse response (FIR)</u> system

Frequency-Domain Representation of Discrete-Time Signals and Systems

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

If $x[n] = e^{j\Omega n}, \quad -\infty < n < \infty,$

then $$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega k}$$

If we define $$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega k}$$

we can write

$$y[n] = H(e^{j\Omega})\, e^{j\Omega n}$$

$H(e^{j\Omega})$ is called the <u>frequency response</u> of the system. $H(e^{j\Omega})$ describes the change in complex amplitude of a complex exponential as a function of the frequency Ω.

$H(e^{j\Omega})$ is a periodic function of Ω with period 2π, and can be represented as a Fourier series:

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n}$$

The Fourier coefficients correspond to the unit-sample response $h[n]$.

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

Discrete-Time Fourier transform

The Fourier transform of a sequence $x[n]$ is defined as: $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$

This equation will converge either if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, or if $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$.

The <u>inverse discrete-time Fourier transform</u> is defined as:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega.$$

Properties of the discrete-time Fourier transform

<u>Sequence</u>	<u>Fourier transform</u>
$a x_1[n] + b x_2[n]$	$a X_1(e^{j\Omega}) + b X_2(e^{j\Omega})$
$x[n-n_0]$	$e^{-j\Omega n_0} X(e^{j\Omega})$
$x_1[n] * x_2[n]$	$X_1(e^{j\Omega}) X_2(e^{j\Omega})$
$x_1[n]\, x_2[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta}) X_2(e^{j(\Omega-\theta)})\, d\theta$
$x_2[n] = e^{j\alpha n} x_1[n]$	$X_2(e^{j\Omega}) = X_1(e^{j(\Omega-\alpha)})$

<u>Hermitian symmetry</u>

The DTFT of a <u>real</u> sequence $x[n]$

is conjugate - symmetric i.e. $X(e^{j\Omega}) = X^*(e^{-j\Omega})$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$X^*(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\Omega n}$$

$$X^*(e^{-j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$= X(e^{j\Omega})$$

$X(e^{j\Omega})$ can be expressed in terms of its real and imaginary parts as:

$$X(e^{j\Omega}) = X_R(e^{j\Omega}) + j\, X_I(e^{j\Omega})$$

$$X(e^{-j\Omega}) = X_R(e^{-j\Omega}) + j\, X_I(e^{-j\Omega})$$

$$X^*(e^{-j\Omega}) = X_R(e^{-j\Omega}) - j\, X_I(e^{-j\Omega})$$

$$X(e^{j\Omega}) = X^*(e^{-j\Omega})$$

$$\Rightarrow X_R(e^{j\Omega}) = X_R(e^{-j\Omega})$$

and $$X_I(e^{j\Omega}) = -X_I(e^{-j\Omega})$$

If $x[n]$ is a real sequence, the real part of its DTFT is an even function of Ω, and the imaginary part of the DTFT is an odd function of Ω. This condition is known as Hermitian symmetry.

Also note that if $x[n]$ is real,

then $$|X(e^{j\Omega})| = |X(e^{-j\Omega})|$$

and $$\underline{/X(e^{j\Omega})} = -\underline{/X(e^{-j\Omega})}$$