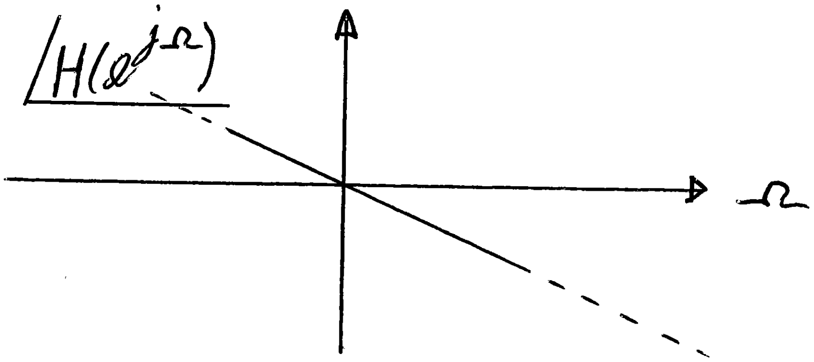
Linear Phase Systems

In general a linear phase discrete-time system has frequency response

$$H(e^{j\Omega}) = |H(e^{j\Omega})| \, e^{-j\Omega\alpha}$$

where α is a real constant.

$$\angle H(e^{j\Omega}) = -\Omega\alpha$$



Generalized Linear Phase

A system is said to be a generalized linear phase system if its frequency response can be expressed in the form

$$H(e^{j\Omega}) = A(\Omega) \, e^{-j(\alpha\Omega + \beta)}$$

where α and β are constants and $A(\Omega)$ is a real function of Ω. The phase of $H(e^{j\Omega})$ has the form

$$\underline{/H(e^{j\Omega})} = -(\alpha\Omega + \beta) \text{ when } A(\Omega) > 0$$

and

$$\underline{/H(e^{j\Omega})} = -(\alpha\Omega + \beta) - \pi \text{ when } A(\Omega) < 0$$

Generalized Linear Phase FIR systems.

Let $h[n]$ denote the unit-sample response of a causal, N-point, FIR discrete-time filter, whose unit-sample response begins at zero and ends at $N-1$. The FIR filter will have generalised linear phase if the unit-sample response $h[n]$, is symmetric about its midpoint.

$$h[n] = h[N-1-n] \quad \text{for } n = 0, 1, \cdots, N-1.$$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{N-1} h[n] e^{-j\Omega n}$$

N even

$$H(e^{j\Omega}) = \sum_{n=0}^{\left(\frac{N}{2}\right)-1} h[n] e^{-j\Omega n} + \sum_{n=\left(\frac{N}{2}\right)}^{N-1} h[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{\left(\frac{N}{2}\right)-1} h[n] \left\{ e^{-j\Omega n} + e^{-j\Omega(N-1-n)} \right\}$$

$$= e^{-j\Omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\left(\frac{N}{2}\right)-1} h[n] \left\{ e^{j\Omega\left(\frac{N-1}{2}-n\right)} + e^{-j\Omega\left(\frac{N-1}{2}-n\right)} \right\}$$

$$= e^{-j\Omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{\left(\frac{N}{2}\right)-1} 2h[n] \cos\left[\Omega\left(\frac{N-1}{2}-n\right)\right] \right\}$$

The sum in brackets is real.

<u>N odd</u> $$H(e^{j\Omega}) = \sum_{n=0}^{\left(\frac{N-1}{2}\right)-1} h[n] \left[e^{-j\Omega n} + e^{-j\Omega(N-1-n)} \right]$$

$$+ h\left(\frac{N-1}{2}\right) e^{-j\Omega\left(\frac{N-1}{2}\right)}$$

$$= e^{-j\Omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{\left(\frac{N-1}{2}\right)-1} 2h[n] \cos\left[\Omega\left(\frac{N-1}{2}-n\right)\right] + h\left(\frac{N-1}{2}\right) \right\}$$

The sum in brackets is real.

Design of FIR filters using windows

Let $H_d(e^{j\Omega})$ denote the ideal desired frequency response.

$$H_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h_d[n]\, e^{-j\Omega n}$$

where $h_d[n]$ is the unit sample response of the ideal filter. By inverse Fourier transforming $H_d(e^{j\Omega})$ we can determine the unit sample response $h_d[n]$.

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

In general, $h_d[n]$ has infinite duration. An FIR filter is obtained by multiplying $h_d[n]$ with a finite duration "window" $w[n]$

$$h[n] = h_d[n]\, w[n]$$

The frequency response of the FIR filter, $H(e^{j\Omega})$, is given by

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} h_d[n]\, w[n]\, e^{-j\Omega n}$$

Using the complex convolution theorem we have

$$H(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\Omega-\theta)})\, d\theta$$

where $W(e^{j\Omega})$ is the discrete-time Fourier transform of the window $w[n]$.

<u>Rectangular Window</u>

The N-point rectangular window

is given by

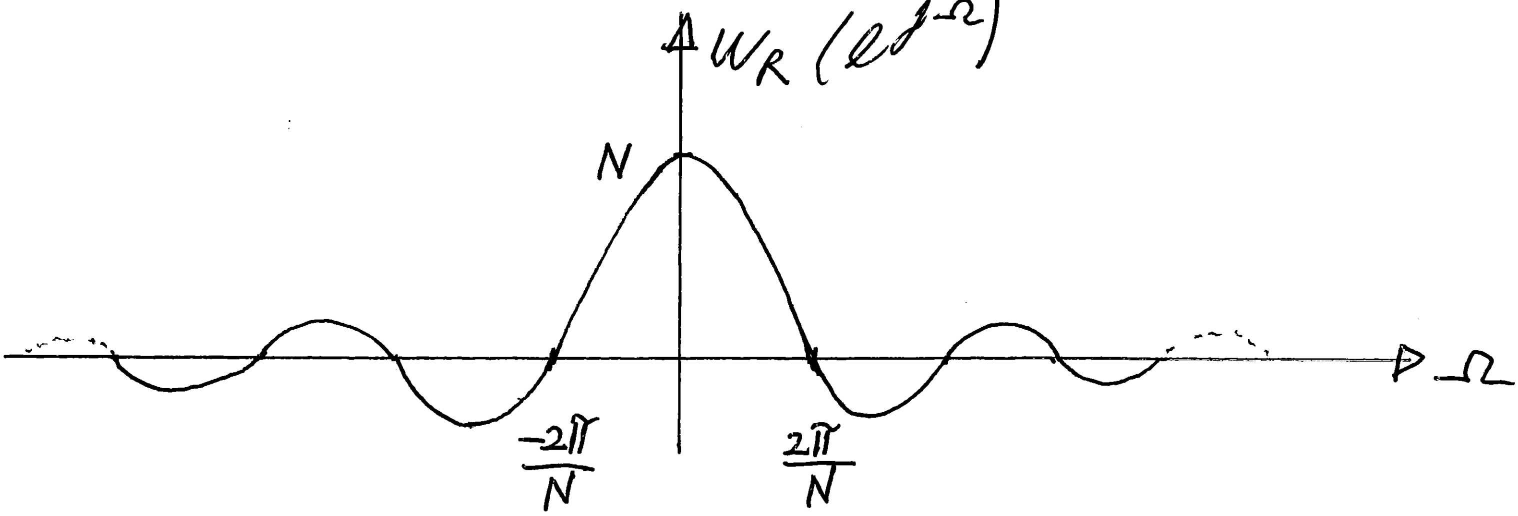
$$w_R[n] = \begin{cases} 1, & -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

$$W_R(e^{j\Omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\Omega n}$$

$$= \frac{e^{j\Omega\left(\frac{N-1}{2}\right)}\left(1-e^{-j\Omega N}\right)}{1-e^{-j\Omega}}$$

$$= \frac{e^{j\Omega\left(\frac{N}{2}\right)} - e^{-j\Omega\left(\frac{N}{2}\right)}}{e^{j\left(\frac{\Omega}{2}\right)} - e^{-j\left(\frac{\Omega}{2}\right)}}$$

$$W_R(e^{j\Omega}) = \frac{\sin(\Omega N/2)}{\sin(\Omega/2)}$$



Example

Design an FIR filter that approximates an ideal low-pass filter with a cut-off frequency $\Omega_c = \frac{\pi}{2}$. The length of the unit-sample response $N = 31$.

$$H_d(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \Omega_c < |\Omega| \leq \pi \end{cases}$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})e^{j\Omega n}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\,d\Omega$$

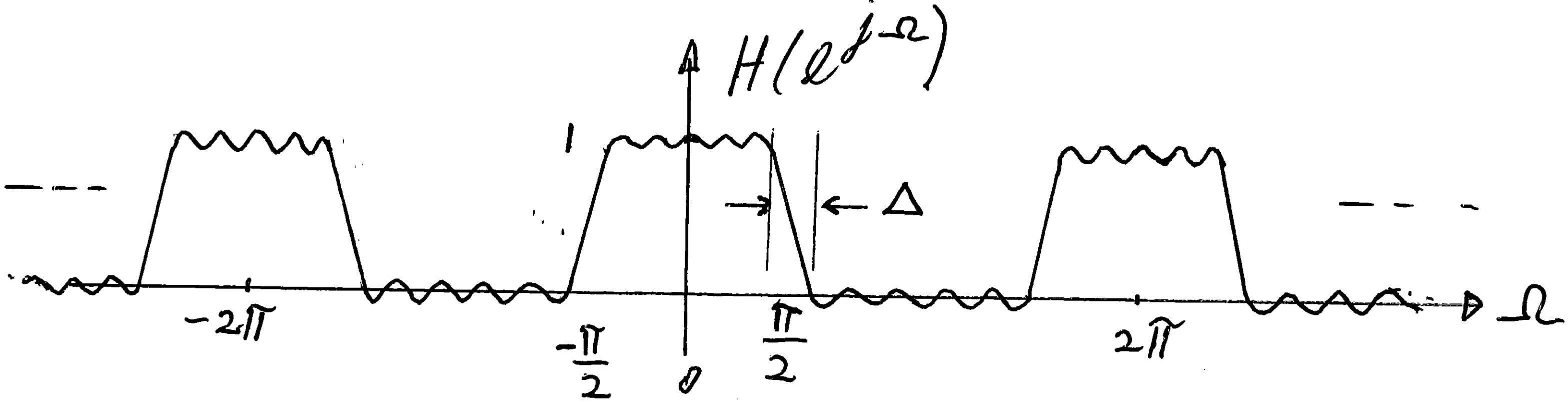
$$h_d[n] = \frac{\sin(\Omega_c n)}{\pi n}$$

Rectangular Window

$$w_R[n] = \begin{cases} 1, & -15 \leq n \leq 15 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = h_d[n]\, w_R[n]$$

$$h[n] = \begin{cases} \dfrac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}, & -15 \leq n \leq 15 \\ 0, & \text{otherwise} \end{cases}$$



Effects of windowing $h_d[n]$:

1) Discontinuities in $H_d(e^{j\Omega})$ become

transition bands between values on either side of the discontinuity. The width (Δ) of the transition bands depends on the width of the main lobe of $W_R(e^{j\Omega})$.

2) The ripples in the passband and stopband are caused by the side lobes of $W_R(e^{j\Omega})$.

In general, the desirable characteristics of a window $w[n]$ with DTFT, $W(e^{j\Omega})$, are:

1) Small width of the main lobe of $W(e^{j\Omega})$.

2) Sidelobes of $W(e^{j\Omega})$ that decrease in energy rapidly as Ω tends to π.

A number of windows with more favourable lobe characteristics than those of the rectangular window have been designed.

Window	Width of main lobe	Peak sidelobe amplitude (dB) relative to main lobe.	Minimum stopband attn. (dB) (lowpass filter)
Rectangular	$\frac{4\pi}{N}$	-13	21
Bartlett	$\frac{8\pi}{N}$	-25	25
Hamming	$\frac{8\pi}{N}$	-41	53

$$h[n] = \begin{cases} \dfrac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}, & -15 \le n \le 15 \\ 0, & \text{otherwise} \end{cases}$$

Causal filter

$$h_1[n] = h[n-15]$$

$$h_1[n] = \begin{cases} \dfrac{\sin\left[\frac{\pi}{2}(n-15)\right]}{\pi(n-15)}, & 0 \le n \le 30 \\ 0, & \text{otherwise} \end{cases}$$

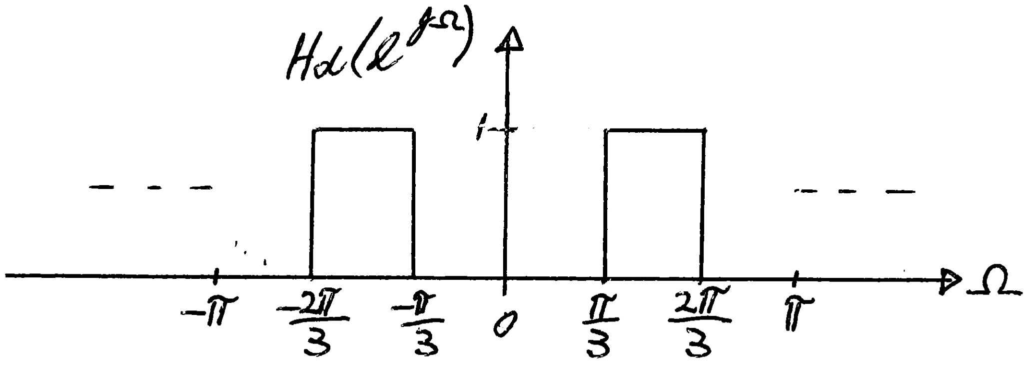
$$H_1(e^{j\Omega}) = e^{-j15\Omega} H(e^{j\Omega})$$

Example

An ideal discrete-time band-pass filter has a frequency response $H_d(e^{j\Omega})$ given by

$$H_d(e^{j\Omega}) = \begin{cases} 0, & |\Omega| < \frac{\pi}{3} \\ 1, & \frac{\pi}{3} < |\Omega| < \frac{2\pi}{3} \\ 0, & \frac{2\pi}{3} < |\Omega| \leq \pi \end{cases}$$

Using a rectangular window sequence, design a causal, 9 point FIR filter which approximates the magnitude response of the ideal band-pass filter.



$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})\, e^{j\Omega n}\, d\Omega$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} e^{j\Omega n}\, d\Omega + \frac{1}{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\Omega n}}{jn}\Bigg|_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} + \frac{e^{j\Omega n}}{jn}\Bigg|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \right)$$

$$= \frac{1}{2\pi}\left(\frac{e^{-j\frac{\pi}{3}n} - e^{-j\frac{2\pi}{3}n} + e^{j\frac{2\pi}{3}n} - e^{j\frac{\pi}{3}n}}{jn} \right)$$

$$h_d[n] = \frac{\sin\left(\frac{2\pi}{3}n\right)}{\pi n} - \frac{\sin\left(\frac{\pi}{3}n\right)}{\pi n}, \quad -\infty < n < \infty$$

$$w_R[n] = \begin{cases} 1, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = h_d[n]\, w_R[n] = \begin{cases} \frac{\sin\left(\frac{2\pi}{3}n\right) - \sin\left(\frac{\pi}{3}n\right)}{\pi n}, & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Let $h_1[n]$ denote the unit sample response of the causal 9 point FIR filter.

$$h_1[n] = h[n-4] = \begin{cases} \frac{\sin\left[\frac{2\pi}{3}(n-4)\right] - \sin\left[\frac{\pi}{3}(n-4)\right]}{\pi(n-4)}, & 0 \le n \le 8 \\ 0, & \text{otherwise} \end{cases}$$