Stability for LTI Systems

Theorem

A linear time-invariant system is stable if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty .$$

Proof

If $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ and x is bounded, i.e. $|x[n]| < M$ for all n, for some $M \geq 0$, then

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]|\, |x[n-k]|$$

$$|y[n]| \leq M \sum_{k=-\infty}^{\infty} |h[k]| < \infty \text{ for all } n.$$

Thus y is bounded and the system is <u>stable</u>.

Conversely, if $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, a bounded input can be found that will cause an unbounded output.

Consider the input sequence $x[n]$ with values:

$$x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|}, & h[-n] \neq 0 \\ 0, & h[-n] = 0 \end{cases}$$

$|x[n]| \leq 1$ for all n.

The value of the output at $n=0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} x[-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} = \infty.$$

The bounded input sequence causes an unbounded output sequence. The system is unstable.