Bilinear Transformation

Let $H_c(s)$ denote the transfer function of a continuous-time filter. If we replace s by

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{— ①}$$

we obtain the transfer function of a discrete-time filter.

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \quad \text{— ②}$$

The transformation of equation ① is known as the bilinear transformation. If we begin the design procedure from a set of discrete-time specifications, the parameter T_d cancels in the procedure. We can therefore choose $T_d = 2$, and the bilinear transformation is

$$s = \left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{--- (3)}$$

and $H(z)$ is given by

$$H(z) = H_c\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{--- (4)}$$

Solving equation (3) for z gives

$$z = \frac{1+s}{1-s} \quad \text{--- (5)}$$

If we substitute $s = \sigma + j\omega$ into equation (5) we obtain

$$z = \frac{1+\sigma+j\omega}{1-\sigma-j\omega} \quad \text{--- (6)}$$

If $\sigma < 0$ then $|z| < 1$.

If $\sigma > 0$ then $|z| > 1$.

If $H_c(s)$ is the transfer function of a causal stable continuous-time filter, then all the poles of $H_c(s)$ lie in the left half of the s-plane. The image of a pole in the left half of the s-plane will be inside the unit circle in the z-plane. Therefore the discrete-time filter will be causal and stable.

If we substitute $s = j\omega$ into equation (5) we obtain

$$z = \frac{1 + j\omega}{1 - j\omega} \qquad — (7)$$

From equation (7) we see that $|z| = 1$ for all values of s on the $j\omega$ axis. The $j\omega$ axis in the s-plane maps onto the unit circle in the z-plane.

$$e^{j\Omega} = \frac{1 + j\omega}{1 - j\omega} \qquad — (8)$$

Substituting $z = e^{j\Omega}$ in equation (3) we obtain

$$s = \sigma + j\omega = \frac{1 - e^{-j\Omega}}{1 + e^{-j\Omega}}$$

$$= \frac{e^{-j\left(\frac{\Omega}{2}\right)}\left(e^{j\left(\frac{\Omega}{2}\right)} - e^{-j\left(\frac{\Omega}{2}\right)}\right)}{e^{-j\left(\frac{\Omega}{2}\right)}\left(e^{j\left(\frac{\Omega}{2}\right)} + e^{-j\left(\frac{\Omega}{2}\right)}\right)}$$

$$= \frac{2j \sin\left(\frac{\Omega}{2}\right)}{2 \cos\left(\frac{\Omega}{2}\right)}$$

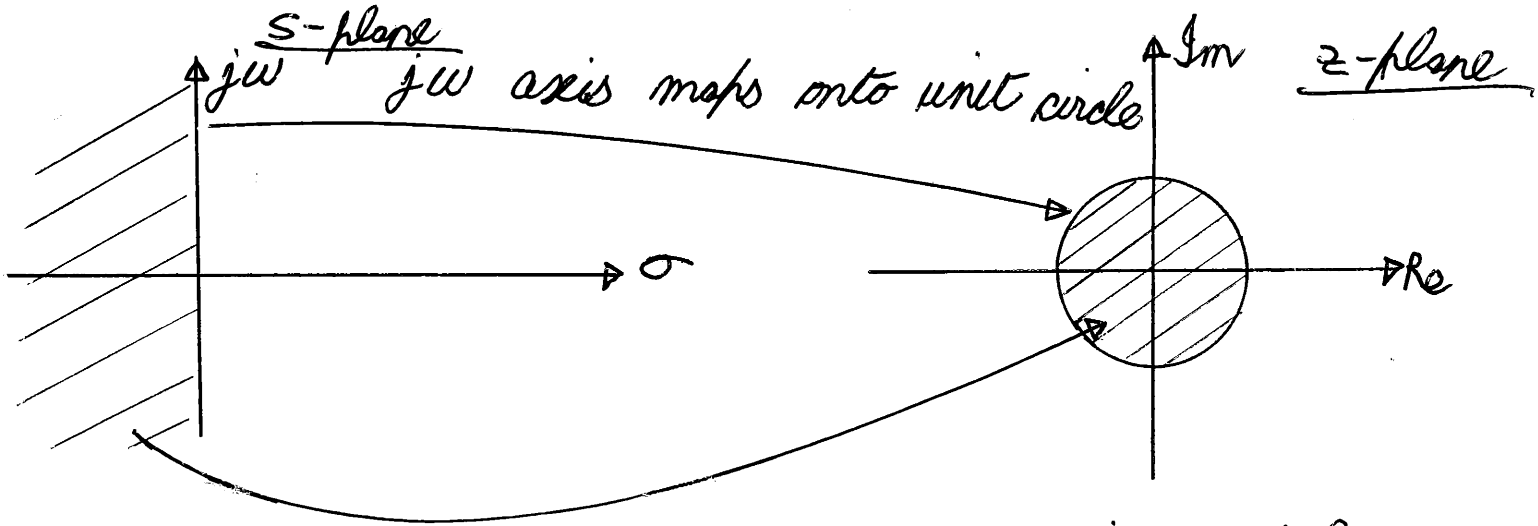
$$= j \tan\left(\frac{\Omega}{2}\right) \qquad \text{— (9)}$$

From equation (9) we have

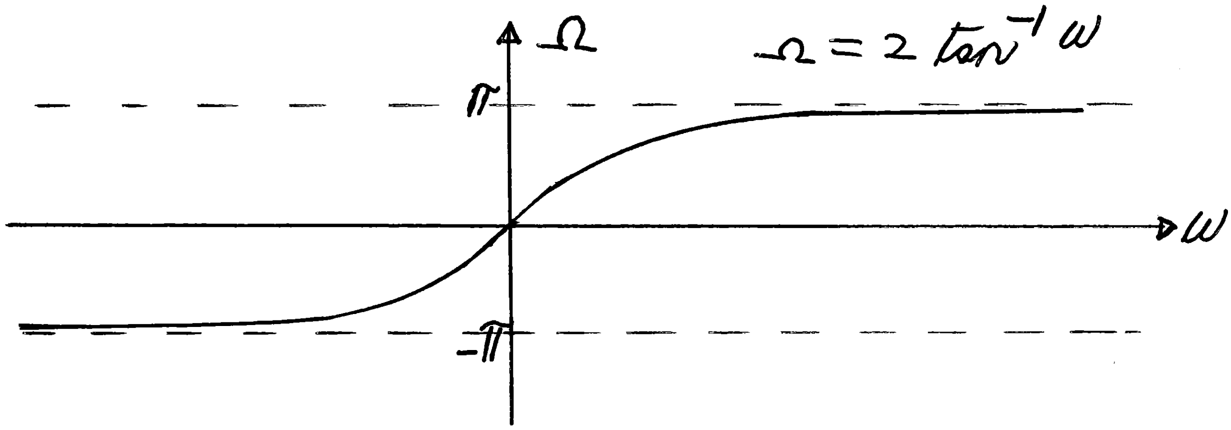
$$\omega = \tan\left(\frac{\Omega}{2}\right) \qquad \text{— (10)}$$

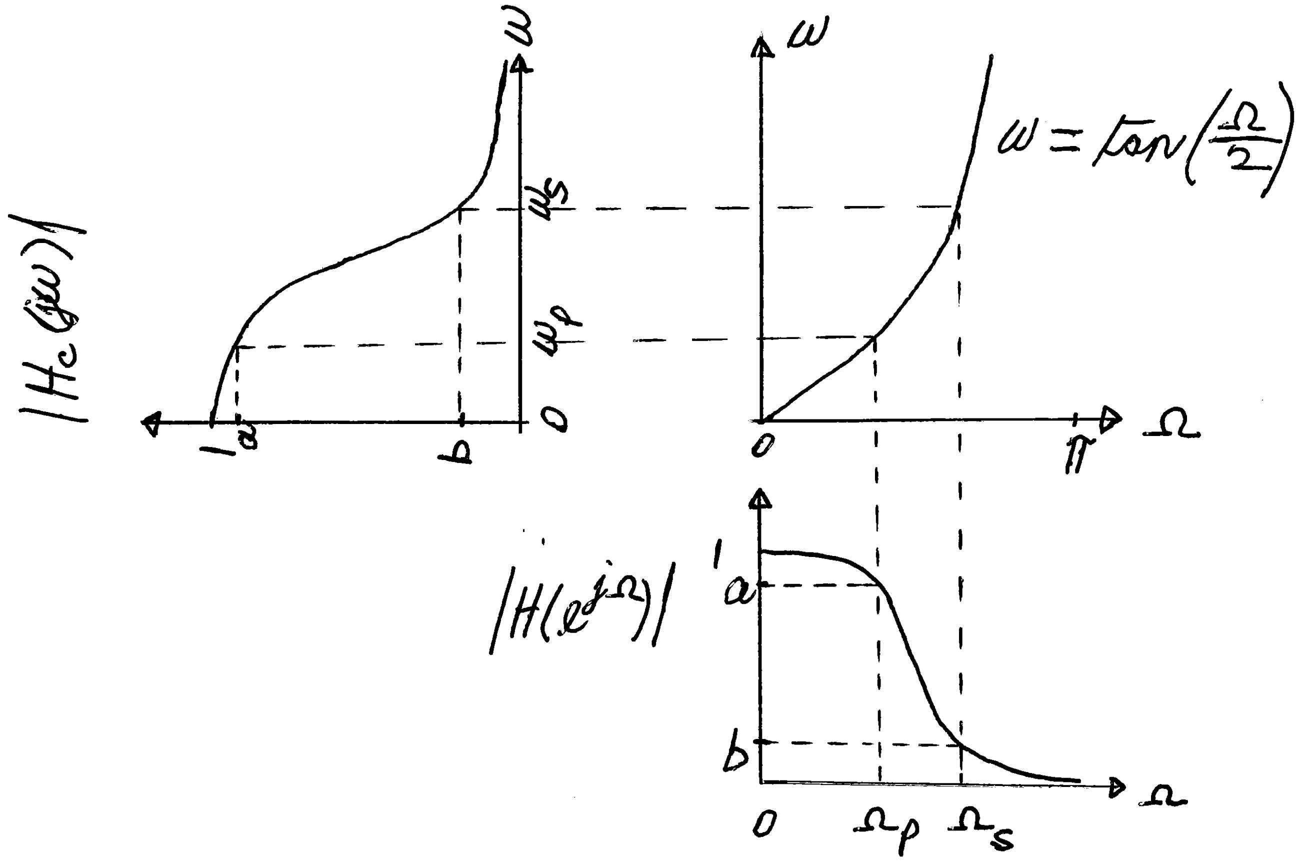
$$\text{or } \Omega = 2 \tan^{-1} \omega \qquad \text{— (11)}$$

The range of frequencies $0 \leq \omega \leq \infty$ maps to $0 \leq \Omega \leq \pi$, and the range of frequencies $-\infty \leq \omega \leq 0$ maps to $-\pi \leq \Omega \leq 0$.



left half plane maps inside unit circle.





Design of discrete-time filters using the bilinear transformation.

Example

Design a discrete-time low-pass filter by applying the bilinear transformation to the transfer function of an appropriate Butterworth continuous-time filter. The filter should meet the following specifications:

attenuation ≤ 1 dB for $|\Omega| \leq 0.3\pi$

and

attenuation ≥ 15 dB for $0.7\pi \leq |\Omega| \leq \pi$.

i.e.

$$0.89125 \leq |H(e^{j\Omega})| \leq 1, \quad |\Omega| \leq 0.3\pi$$

$$|H(e^{j\Omega})| \leq 0.17783, \quad 0.7\pi \leq |\Omega| \leq \pi$$

Step 1 Prewarp the critical discrete-time frequencies $\Omega_1 = 0.3\pi$, and $\Omega_2 = 0.7\pi$, to the corresponding analog frequencies.

$$w_1 = \tan\left(\frac{\Omega_1}{2}\right) = \tan(0.15\pi)$$

$$w_2 = \tan\left(\frac{\Omega_2}{2}\right) = \tan(0.35\pi)$$

Step 2 Determine the transfer function of the analog filter that meets the following specifications:

$$0.89125 \leq |H_c(jw)| \leq 1, \quad |w| \leq w_1$$

$$|H_c(jw)| \leq 0.17783, \quad w_2 \leq |w| \leq \infty$$

Butterworth low-pass filter response:

$$|H_c(jw)|^2 = \frac{1}{1 + \left(\frac{w}{w_0}\right)^{2N}}$$

$$1 + \left(\frac{\tan(0.35\pi)}{w_0}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

$$1 + \left(\frac{\tan(0.15\pi)}{w_0}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2$$

$$\left(\frac{\tan(0.35\pi)}{\tan(0.15\pi)}\right)^{2N} = 118.26$$

$$N = 1.7696$$

The order of the filter must be an integer. $\underline{\underline{N=2}}$

$$1 + \left(\frac{\tan(0.15\pi)}{W_o}\right)^4 = \left(\frac{1}{0.89125}\right)^2$$

$$W_o = 0.714$$

The 2nd. order Butterworth low-pass prototype filter transfer function is given by

$$H_p(S) = \frac{1}{S^2 + \sqrt{2}\,S + 1}$$

$$H_c(S) = H_p\left(\frac{S}{W_o}\right) = \frac{1}{\left(\frac{S}{0.714}\right)^2 + \sqrt{2}\left(\frac{S}{0.714}\right) + 1}$$

$$H_c(S) = \frac{0.510}{S^2 + 1.01\,S + 0.51}$$

<u>Step 3</u> Apply the bilinear transformation to $H_c(S)$.

$$H(z) = \frac{0.51}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.01\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.51}$$

$$H(z) = \frac{0.202 + 0.405 z^{-1} + 0.202 z^{-2}}{1 - 0.389 z^{-1} + 0.198 z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) - 0.389 z^{-1} Y(z) + 0.198 z^{-2} Y(z)$$
$$= 0.202 X(z) + 0.405 z^{-1} X(z) + 0.202 z^{-2} X(z)$$

$$y[n] = 0.389 y[n-1] - 0.198 y[n-2] + 0.202 x[n]$$
$$+ 0.405 x[n-1] + 0.202 x[n-2]$$

Selection of the Filter Type

1. An FIR filter can be designed to have a (generalized) linear phase response.

2. An FIR filter is always stable.

3. For most practical filter specifications the order of an FIR filter is considerably higher than the order of an equivalent IIR filter meeting the same magnitude specifications. The number of multiplications and memory required for the FIR filter is greater than for the IIR filter.