Continuous - Time Systems

$$x(t) \xrightarrow{\text{Excitation}} \boxed{\text{System}} \xrightarrow{\text{Response}} y(t)$$

(Input Signal) (Output Signal)

System Block Diagram

Causal System

A system is causal if its output $y(t)$ at any time $t = t_0$ depends only on the input $x(t)$ for $t \leq t_0$.

Linear System

<u>Input</u>		<u>Output</u>
$x_1(t)$	$\longrightarrow$	$y_1(t)$
$x_2(t)$	$\longrightarrow$	$y_2(t)$

The system is linear if the response to

$a\, x_1(t) + b\, x_2(t)$ is $a\, y_1(t) + b\, y_2(t)$ for all $x_1(t)$ and $x_2(t)$, and for any complex constants a and b.

Time - Invariant System

A system is time - invariant if a time - shift in the input signal results in an identical time - shift in the output signal.

Input		Output
$x(t)$	$\longrightarrow$	$y(t)$
$x(t-t_0)$	$\longrightarrow$	$y(t-t_0)$

for any $x(t)$ and any t_0.

Stable System

a system is stable in the bounded-input, bounded-output (BIBO) sense if any bounded input $x(t)$ satisfying

$$|x(t)| \leq B_1$$

for some finite constant B_1, produces a bounded output $y(t)$ satisfying

$$|y(t)| \leq B_2$$

for some finite constant B_2.

Invertible System

a system is invertible if distinct inputs lead to distinct outputs.

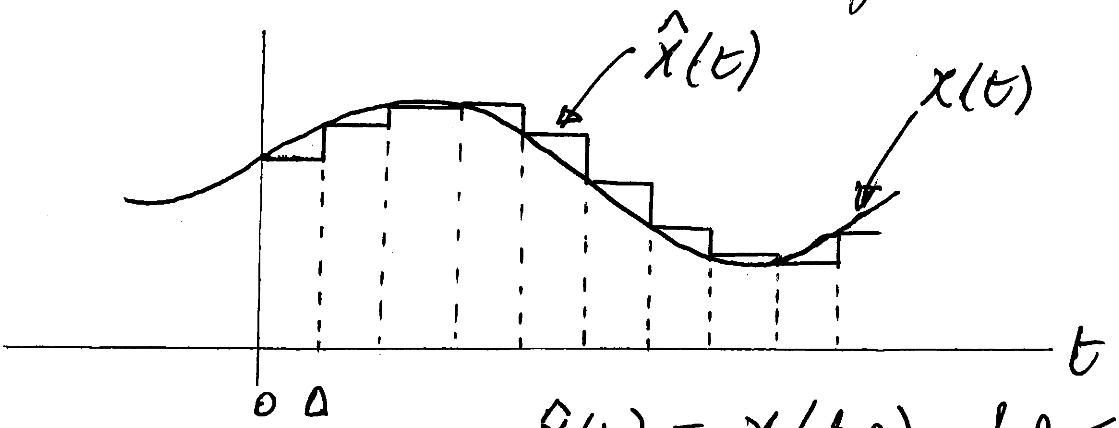
If a system is invertible then an inverse system exists that has the output $x(t)$ when the input is $y(t)$.

$$x(t) \longrightarrow \boxed{\text{Invertible System}} \longrightarrow y(t) \longrightarrow \boxed{\text{Inverse System}} \longrightarrow x(t)$$

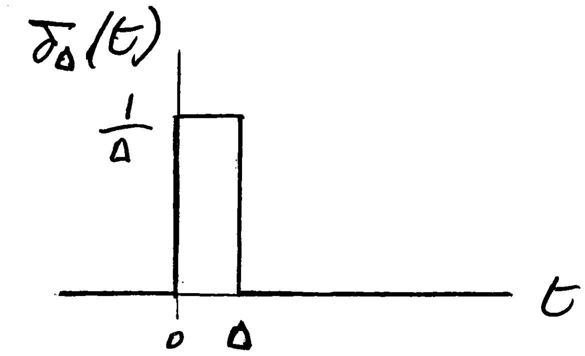
Invertible System and its Inverse System

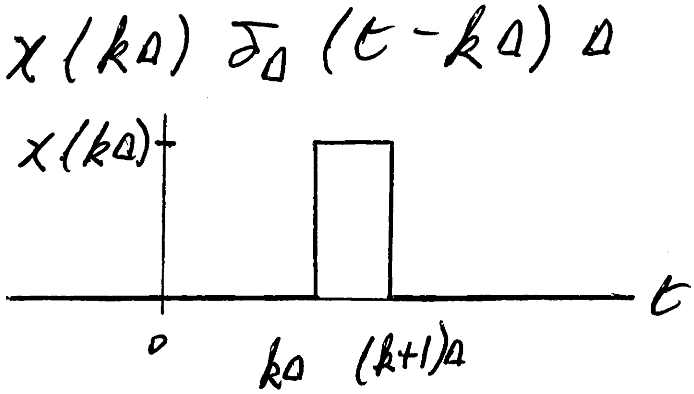
Representation of Signals in terms of Impulses

Let $\hat{x}(t)$ denote the "staircase" approximation to the continuous-time signal $x(t)$.



$$\hat{x}(t) = x(k\Delta), \quad k\Delta < t < (k+1)\Delta$$





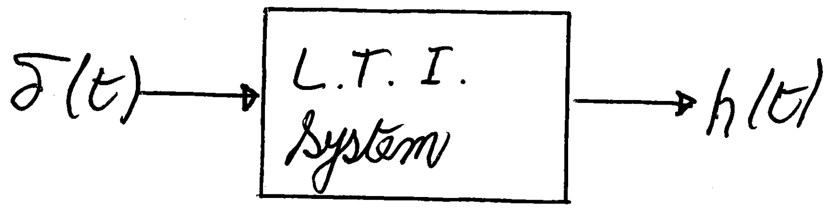
$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

This relationship is called the sifting property of the unit impulse function.

Impulse Response, $h(t)$



Unit Impulse Response, $h(t)$

Response of a Continuous-Time L.T.I. System

Input		Output
$\delta(t)$	$\longrightarrow$	$h(t)$
$x(t)$	$\longrightarrow$	$y(t)$
$\delta_\Delta(t)$	$\longrightarrow$	$h_\Delta(t)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \rightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$$

Letting $\Delta \rightarrow 0$, $h_\Delta(t)$ approaches the impulse response $h(t)$ since $\delta_\Delta(t)$ approaches $\delta(t)$ and $\hat{y}(t) \rightarrow y(t)$ since $\hat{x}(t) \rightarrow x(t)$.

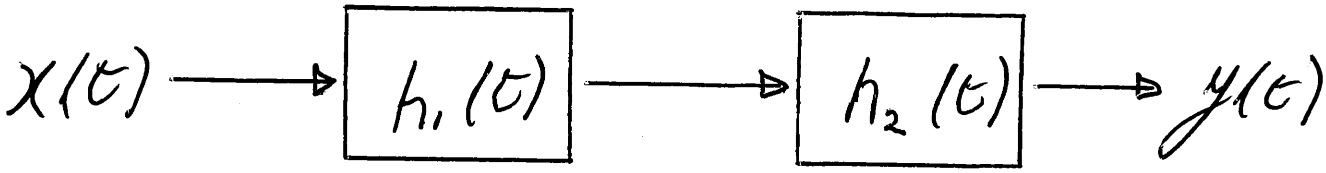
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

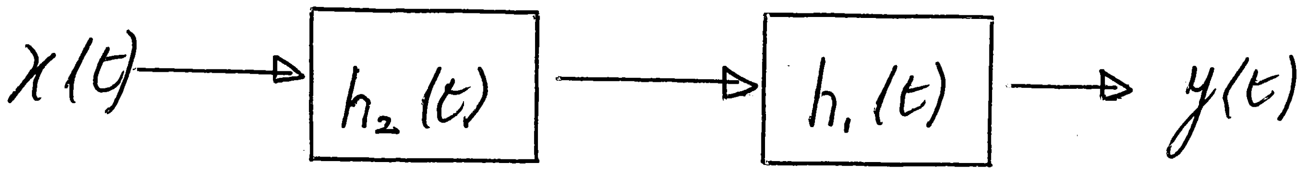
$$y(t) = x(t) * h(t)$$

The output of any L.T.I. system is the convolution of the input $x(t)$ and the unit impulse response of the system $h(t)$.

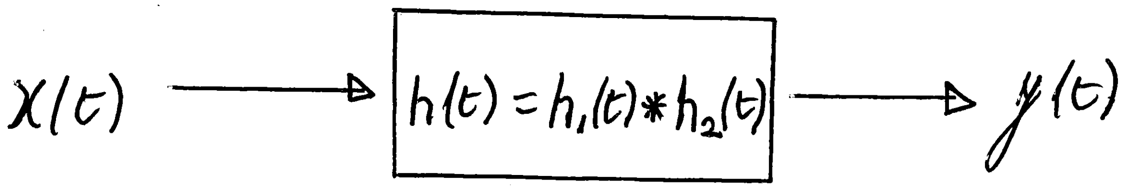
Series interconnection of LTI systems



(a)



(b)



(c)

Series combination of the LTI systems in (a) or (b) and the equivalent system (c).

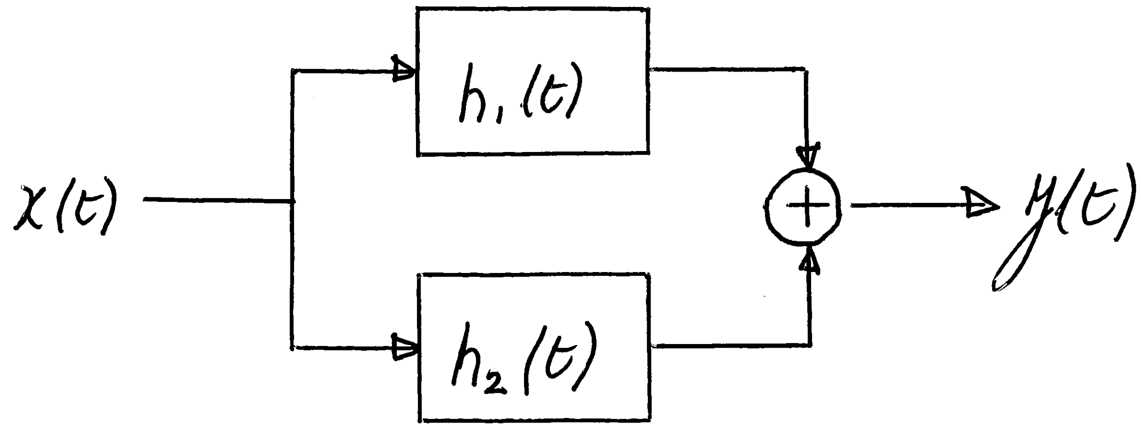
$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

$$= x(t) * [h_1(t) * h_2(t)] \quad \text{(Associative Property)}$$

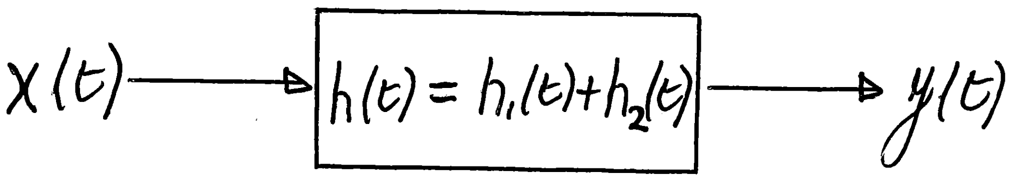
$$= x(t) * [h_2(t) * h_1(t)] \quad \text{(Commutative Property)}$$

$$= [x(t) * h_2(t)] * h_1(t) \quad \text{(Associative Property)}$$

Parallel interconnection of LTI systems



(a)



(b)

Parallel combination of LTI systems (a) and the equivalent system (b).

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) * [h_1(t) + h_2(t)] \quad \text{(Distributive Property)}$$

Unit - Step Response

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow s(t)$$

L.T.I. system

Let s(t) denote the unit - step response of the system.

$$
\begin{aligned}
s(t) &= u(t) * h(t) \\
&= h(t) * u(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau \\
&= \int_{-\infty}^{t} h(\tau)\, d\tau
\end{aligned}
$$

Note that

$$h(t) = \frac{d\, s(t)}{dt}$$

Causal System

The system is causal if $h(t) = 0, \quad t < 0.$

Stability for LTI systems

Theorem

A linear time-invariant system is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty .$$

Proof If $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$ and x is bounded, i.e. $|x(t)| \leq M$ for all t, then

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\, d\tau$$

$$|y(t)| \leq M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \text{ for all } t.$$

Thus y is bounded and the system is stable.

Conversely, if $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$, a bounded input can be found that will cause an unbounded output.

Consider the input signal $x(t)$ given by

$$x(t) = \begin{cases} \dfrac{h^*(-t)}{|h(-t)|}, & h(-t) \neq 0 \\ 0, & h(-t) = 0 \end{cases}$$

x is bounded since $|x(t)| \leq 1$ for all t.

The value of the output at $t = 0$ is

$$y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{|h(\tau)|^2}{|h(\tau)|}\, d\tau = \infty.$$

The bounded input signal causes an unbounded output signal. The system is <u>unstable</u>.