Discrete - time Random Process

A discrete - time random process consists of a collection of random variables $X[n]$ for all integer values of n. The random variables $X[n_1], X[n_2], ---, X[n_k]$ are completely characterised by their joint probability density function

$$p_{X[n_1], X[n_2], ---, X[n_k]}(x_1, x_2, ---, x_k)$$

for all k and all integers $n_1, n_2, ---, n_k$.

Mean of the process

$$m_{X[n]} = E\left[X[n]\right] = \int_{-\infty}^{\infty} x \, p_{X[n]}(x) \, dx$$

Mean - Square Value

$$E\left[X[n]^2\right] = \int_{-\infty}^{\infty} x^2 \, p_{X[n]}(x)\, dx$$

Variance

$$\sigma_{X[n]}^2 = E\left[\left(X[n] - m_{X[n]}\right)^2\right]$$

$$= E\left[X[n]^2\right] - \left(m_{X[n]}\right)^2$$

Autocorrelation Sequence

The autocorrelation sequence of a real-valued random process is defined as

$$\phi_{XX}[n+m, n] = E\left[X[n+m]\, X[n]\right]$$

If the random process is wide-sense stationary then

$$m_X = E[X[n]]$$

$$\sigma_X^2 = E[(X[n] - m_x)^2]$$

and

$$\phi_{XX}[n+m, n] = \phi_{XX}[m]$$

The mean of a wide-sense stationary random process is independent of time, n, and the autocorrelation sequence is a function of the time difference m.

The mean-square value of a wide-sense stationary random process is given by

$$E[X[n]^2] = \phi_{XX}[0]. \qquad -(1)$$

<u>Ergodic Process</u>

a random process for which time averages equal ensemble averages is called an <u>ergodic process</u>. For any single sample sequence, $x[n]$, of an ergodic process, $X(t)$, we have

$$\langle x[n] \rangle = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n] = m_x$$

and

$$\langle x[n+m]\, x[n] \rangle = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n+m]\, x[n] = \phi_{xx}[m]$$

<u>Power Spectral Density</u>

The discrete-time Fourier transform of the autocorrelation sequence $\phi_{xx}[m]$ is given by

$$\Phi_{xx}(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]\, e^{-j\Omega m}$$

The inverse Fourier transform of $\Phi_{XX}(e^{j\Omega})$

is
$$\phi_{XX}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{j\Omega}) e^{j\Omega m} \, d\Omega$$
— (2)

From equations (1) and (2) we have

$$E[X[n]^2] = \phi_{XX}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{XX}(e^{j\Omega}) \, d\Omega \qquad \text{—(3)}$$

If we define
$$S_{XX}(\Omega) = \Phi(e^{j\Omega})$$

equation (3) can be written as
$$E[X[n]^2] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega) \, d\Omega \qquad \text{—(4)}$$

$S_{XX}(\Omega)$ is referred to as the power spectral density of the wide sense stationary random process $X(t)$. $S_{XX}(\Omega)$ is the D.T.F.T. of $\phi_{XX}[m]$.

The area under $S_{XX}(\Omega)$ for $-\pi \leq \Omega \leq \pi$ is proportional to the average power in the signal. The integral of $S_{XX}(\Omega)$ over a band of frequencies is proportional to the power of the signal in that band.

White Noise

The power spectral density, $S_{XX}(\Omega)$, of a white noise process is a constant.

LTI Filtering of WSS Process

Consider a linear time-invariant filter with a real-valued unit sample response $h[n]$. The input to the filter is a real-valued wide-sense stationary random process, $X[n]$.

We wish to determine the statistical properties of the output random process, $Y[n]$

$$X[n] \longrightarrow \underbrace{\boxed{h[n]}}_{\text{L.T.I.}} \longrightarrow Y[n]$$

$$Y[n] = \sum_{k=-\infty}^{\infty} h[k]\, X[n-k]$$

The mean of the output random process is

$$m_Y = E\left[Y[n]\right] = E\left[\sum_{k=-\infty}^{\infty} h[k]\, X[n-k]\right]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, E\left[X[n-k]\right]$$

$$= m_X \sum_{k=-\infty}^{\infty} h[k]$$

$$= m_X\, H(e^{j0}) \qquad -(5)$$

Since the input random process is WSS the mean of the output random process is a constant independent of the time index n.

The autocorrelation sequence of the output random process is

$$\phi_{YY}[n+m, n] = E\left[Y[n+m]\,Y[n]\right]$$

$$= E\left[\sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} h[k]\,h[r]\,X[n+m-k]\,X[n-r]\right]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\, E\left[X[n+m-k]\,X[n-r]\right]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\, \phi_{XX}[m+r-k] \qquad -(6)$$

$$= \phi_{YY}[m]$$

The output autocorrelation sequence depends on the difference, m, of the time indices n+m and n. The output random process is also WSS.

Substituting $l = k - r$ in Eq. (6) we obtain

$$\phi_{YY}[m] = \sum_{k=-\infty}^{\infty} h[k] \sum_{l=-\infty}^{\infty} h[k-l]\, \phi_{XX}[m-l]$$

$$= \sum_{l=-\infty}^{\infty} \phi_{XX}[m-l] \sum_{k=-\infty}^{\infty} h[k]\, h[k-l]$$

$$= \sum_{l=-\infty}^{\infty} \phi_{XX}[m-l]\, v[l]$$

where $v[l] = h[l] * h[-l]$ —(7)

$$\phi_{YY}[m] = \phi_{XX}[m] * v[m] \qquad \text{—(8)}$$

Let $\Phi_{YY}(e^{j\Omega})$, $V(e^{j\Omega})$, and $H(e^{j\Omega})$ denote the DTFT of the sequences $\phi_{YY}[m]$, $v[m]$ and $h[m]$

respectively. Since $h[n]$ is a real-valued sequence the DTFT of $h[-n]$ is $H^*(e^{j\Omega})$.

Taking the DTFT of both sides of Eq.(7) we have

$$V(e^{j\Omega}) = H(e^{j\Omega})H^*(e^{j\Omega})$$

$$= |H(e^{j\Omega})|^2 \qquad -(9)$$

Taking the DTFT of both sides of Eq.(8) we obtain

$$\Phi_{YY}(e^{j\Omega}) = \Phi_{XX}(e^{j\Omega})V(e^{j\Omega}) \qquad -(10)$$

Substituting Eq.(9) in Eq.(10) we have

$$\Phi_{YY}(e^{j\Omega}) = \Phi_{XX}(e^{j\Omega})|H(e^{j\Omega})|^2 \qquad -(11)$$

Using the notations $S_{XX}(\Omega)$ and $S_{YY}(\Omega)$ to denote the input and output power spectral densities, $\Phi_{XX}(e^{j\Omega})$ and $\Phi_{YY}(e^{j\Omega})$, respectively, we can rewrite Eq. (11) as

$$S_{YY}(\Omega) = \left|H(e^{j\Omega})\right|^2 S_{XX}(\Omega)$$