System Function

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

L.T.I. system

$$x(t) \longleftrightarrow X(s)$$

$$h(t) \longleftrightarrow H(s)$$

$$y(t) \longleftrightarrow Y(s)$$

$$y(t) = x(t) * h(t)$$

Taking the Laplace transform of both sides we have

$$Y(s) = X(s)\,H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$H(s)$, the Laplace transform of the impulse response $h(t)$, is called the system function or transfer function of the LTI system.

$$H(s) = \mathcal{L}\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-st} \, dt$$

Causal System

$$h(t) = 0 \quad \text{for } t < 0.$$

The impulse response is right-sided.
The ROC of $H(s)$ is of the form $\mathrm{Re}\{s\} > \sigma_{max}$, i.e. to the right of all the poles of $H(s)$.

If $H(s)$ is rational and its ROC is to the right of its rightmost pole then the impulse response of the system is causal.

Stable System

An LTI system is stable in the bounded-input bounded-output sense if and only if its impulse response is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty.$$

This is the Dirichlet condition for $h(t)$ to possess a convergent Fourier transform, $H(j\omega)$ (except for pathological cases).

$H(j\omega)$ is equal to the system function $H(s)$ evaluated along the $j\omega$-axis.

$$H(j\omega) = H(s)\Big|_{s=j\omega}$$

An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis.

Causal and Stable LTI system with rational system function $H(s)$.

Since the system is causal the ROC of $H(s)$ is to the right of the rightmost pole of $H(s)$.

Since the system is stable the ROC of $H(s)$ must include the $j\omega$-axis. The rightmost pole of $H(s)$

must therefore be to the left of the $j\omega$-axis.

A causal LTI system with rational system function $H(s)$ is stable if and only if all the poles of $H(s)$ lie in the left half of the s-plane.

The real part of each pole must be negative.

LTI systems characterised by linear constant-coefficient differential equations

Most LTI systems of practical interest can be described by finite-order linear differential equations with constant coefficients of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Taking the Laplace transform of both sides we have

$$\mathcal{L}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{L}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

Using the linearity property we obtain

$$\sum_{k=0}^{N} a_k \mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{L}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

Using the differentiation property we obtain

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

This equation may be rewritten as follows

$$Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{k=0}^{M} b_k s^k$$

The system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Example

The input $x(t)$ and output $y(t)$ of a causal LTI system satisfy the following differential equation

$$\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

(i) Determine the system function $H(s)$.

(ii) Plot the pole and zero of $H(s)$ in the s-plane and indicate the ROC of $H(s)$.

(iii) Plot the magnitude of the frequency response of the system.

(i)
Taking the Laplace transform of both sides of the differential equation we have

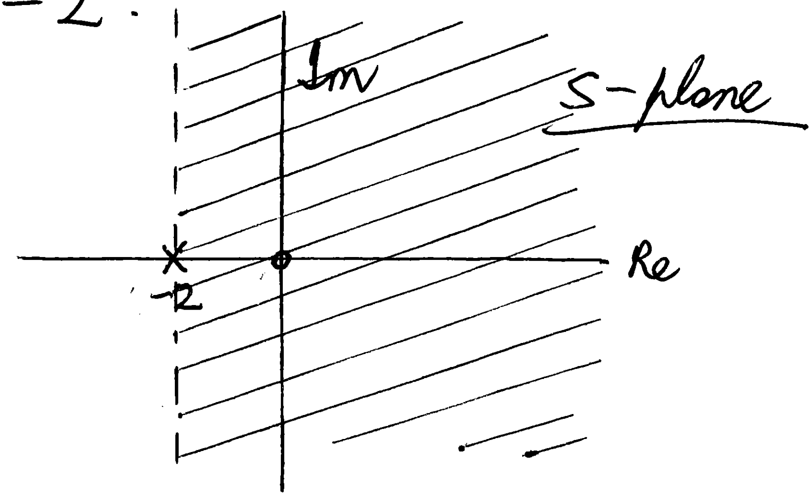
$$sY(s) + 2Y(s) = sX(s)$$

$$Y(s)[s+2] = sX(s)$$

The algebraic expression for $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s+2}$$

(ii) There is a pole at $s=-2$ and a zero at $s=0$. Since the system is causal the ROC of $H(s)$ is $Re\{s\} > -2$.



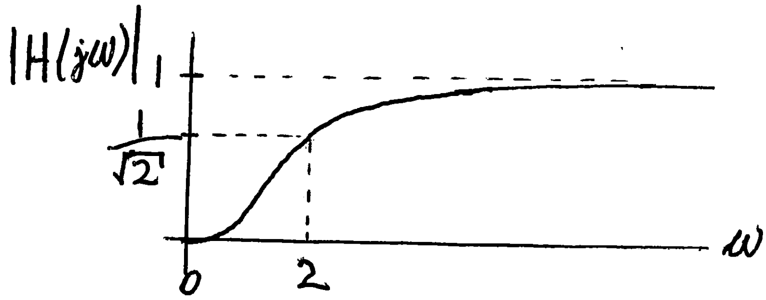
The ROC is the shaded area.

(iii) The frequency response $H(j\omega)$ is

$$H(j\omega) = \frac{j\omega}{j\omega+2}$$

$$|H(j\omega)| = \frac{|j\omega|}{|2+j\omega|} = \frac{\sqrt{\omega^2}}{\sqrt{4+\omega^2}} = \frac{1}{\sqrt{1+\frac{4}{\omega^2}}}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{2}{\omega}\right)^2}}$$



Geometric Evaluation of the Frequency Response

Consider the system function $H(s)$ given by

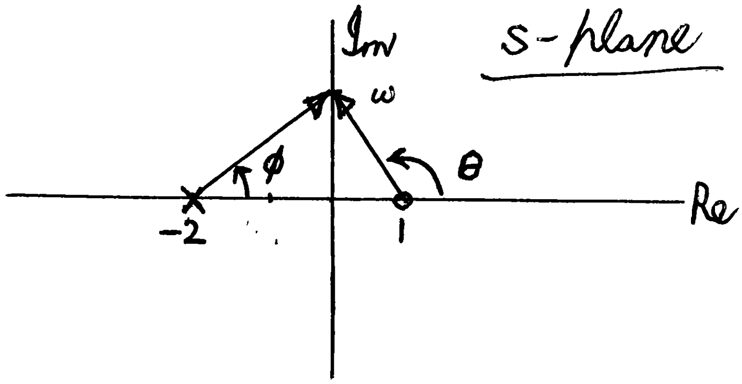
$$H(s) = \frac{s-1}{s+2}$$

There is a pole at $s=-2$ and a zero at $s=1$. The frequency response $H(j\omega)$ is

$$H(j\omega) = \frac{j\omega - 1}{j\omega + 2}$$

The magnitude of the frequency response is

$$|H(j\omega)| = \frac{\sqrt{1+\omega^2}}{\sqrt{2^2+\omega^2}}.$$



$\sqrt{1+\omega^2}$ = length of the vector from the zero at $z=1$ to the point $j\omega$.

$\sqrt{2^2+\omega^2}$ = length of the vector from the pole at $z=-2$ to the point $j\omega$.

The phase of the frequency response is

$$\angle H(j\omega) = \angle j\omega - 1 - \angle j\omega + 2$$

$$= \tan^{-1}(-\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$= \theta - \phi .$$

The system function $H(s)$ for a system described by a linear differential equation with constant coefficients is of the form

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

$H(s)$ can be factored into the form

$$H(s) = c \frac{\prod_{k=1}^{M} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)}$$

where $c = b_M / a_N$.

The frequency response $H(j\omega)$ is

$$H(j\omega) = c \frac{\prod_{k=1}^{M} (j\omega - z_k)}{\prod_{k=1}^{N} (j\omega - p_k)}$$

The magnitude of the frequency response $|H(j\omega)|$ is

$$|H(j\omega)| = |c| \frac{\prod_{k=1}^{M} |j\omega - z_k|}{\prod_{k=1}^{N} |j\omega - p_k|}$$

The magnitude response at the frequency ω equals the magnitude of the scale factor c times the product of the lengths of the zero vectors divided by the product of the lengths of the pole vectors.

The phase response $\angle H(j\omega)$ is

$$\angle H(j\omega) = \angle c + \sum_{k=1}^{M} \angle j\omega - z_k - \sum_{k=1}^{N} \angle j\omega - p_k$$

$\angle H(j\omega)$ is the angle of c plus the sum of the angles of the zero vectors minus the sum of the angles of the pole vectors.