Properties of the Fourier transform

Linearity

$$x_1(t) \longleftrightarrow X_1(j\omega)$$

$$x_2(t) \longleftrightarrow X_2(j\omega)$$

$$a\,x_1(t) + b\,x_2(t) \longleftrightarrow a\,X_1(j\omega) + b\,X_2(j\omega)$$

for arbitrary constants a and b.

Time Shift

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega)$$

Proof

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

Let $\tau = t - t_0$.

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau+t_0)}\, d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= e^{-j\omega t_0} X(j\omega)$$

Time and Frequency Scaling

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

where a is a real constant.

Proof

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $\tau = at$. For $a > 0$ we have

$$\mathcal{F}\{x(at)\} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{j\omega}{a}\right).$$

For $a < 0$ we have

$$\mathcal{F}\{x(at)\} = \frac{1}{a}\int_{\infty}^{-\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= -\frac{1}{a} X\left(\frac{j\omega}{a}\right).$$

Combining the two results for $a > 0$ and $a < 0$ we can write

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Differentiation and Integration

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

Differentiating both sides of the equation we obtain

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t}\, d\omega$$

$$\frac{d\,x(t)}{dt} \longleftrightarrow j\omega X(j\omega)$$

The frequency-domain operation corresponding to time-domain integration is multiplication by $\frac{1}{j\omega}$. However, an additional term is needed to account for a possible dc component in the integrator output.

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega).$$

Convolution of Signals

$$h(t) * x(t) \longleftrightarrow H(j\omega)\, X(j\omega)$$

Proof

$$y(t) = h(t) * x(t)$$

$$= x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

Interchanging the order of integration, we have

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Note that the Fourier transform of $h(t-\tau)$ is $e^{-j\omega\tau} H(j\omega)$.

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} H(j\omega)\, d\tau$$

$$= H(j\omega) \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

Multiplication of Signals

$$x(t)\, y(t) \longleftrightarrow \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$$

Proof

$$w(t) = x(t)\, y(t)$$

$$W(j\omega) = \int_{-\infty}^{\infty} w(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t}\, dt$$

$x(t)$ may be written as $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, e^{j\theta t}\, d\theta$.

$$W(j\omega) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, e^{j\theta t}\, d\theta\right] y(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration we have

$$W(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\left[\int_{-\infty}^{\infty} y(t)\, e^{-j(\omega-\theta)t}\, dt\right] d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, Y(j(\omega-\theta))\, d\theta$$

$$W(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$$

Conjugation Property

$$x^*(t) \longleftrightarrow X^*(-j\omega)$$

Proof

$$\mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt$$

$$= \left[\int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt\right]^*$$

$$= X^*(-j\omega)$$

If $x(t)$ is a real-valued signal then $x^*(t) = x(t)$.
Hence $X(j\omega) = X^*(-j\omega)$. $X(j\omega)$ has conjugate symmetry.

$$X(-j\omega) = X^*(j\omega), \qquad x(t) \text{ real.}$$

In polar form $X(j\omega) = |X(j\omega)|\, e^{j\phi(\omega)}$.

$$X^*(j\omega) = |X(j\omega)|\, e^{-j\phi(\omega)}$$

$$X(-j\omega) = |X(-j\omega)|\, e^{j\phi(-\omega)}$$

If $x(t)$ is real-valued, then $X(-j\omega) = X^*(j\omega)$.

$$|X(-j\omega)|\, e^{j\phi(-\omega)} = |X(j\omega)|\, e^{-j\phi(\omega)}$$

$$|X(j\omega)| = |X(-j\omega)|$$

and

$$\phi(\omega) = -\phi(-\omega).$$

If $x(t)$ is real-valued, then $|X(j\omega)|$ is an even function of ω and $\phi(\omega)$ is an odd function of ω.

<u>Parseval's Relation</u>

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

Proof
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right]^* dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t} d\omega \right] dt$$

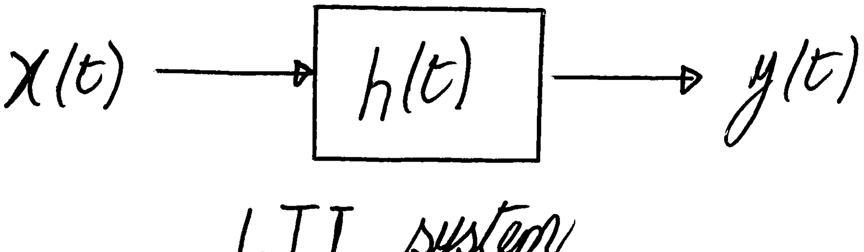
Interchanging the order of integration we have

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Frequency Response of an LTI System



LTI system

$$y(t) = x(t) * h(t)$$
$$= h(t) * x(t)$$

If $x(t) = e^{jwt}$ then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{jw(t-\tau)}\, d\tau$$

$$= e^{jwt} \int_{-\infty}^{\infty} h(\tau)\, e^{-jw\tau}\, d\tau$$

$$y(t) = e^{jwt} H(jw)$$

where $H(jw) = \int_{-\infty}^{\infty} h(\tau)\, e^{-jw\tau}\, d\tau.$

$H(jw)$ is called the frequency response of the system.

$H(jw)$ is the Fourier transform of the impulse response of the system, $h(t)$.