The Response of LTI Systems to Complex Exponentials

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau
\end{aligned}$$

$$y(t) = e^{st} H(s)$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$.

$H(s)$ is referred to as the system function.

Complex exponentials are eigenfunctions of

LTI systems. $H(s)$ is a constant for a specific value of s. $H(s)$ is the eigenvalue associated with the eigenfunction e^{st}.

If the input to the LTI system is

$$x(t) = \sum_k a_k e^{s_k t}$$

then the output is

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

Fourier Series and LTI Systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

If $x(t) = e^{j\omega t}$ then $y(t) = e^{j\omega t} H(j\omega)$

where $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$

$H(j\omega)$ is called the frequency response of the system.

$$H(j\omega) = |H(j\omega)| e^{j\phi(\omega)}$$

$$y(t) = e^{j\omega t} |H(j\omega)| e^{j\phi(\omega)}$$

$$= |H(j\omega)| e^{j(\omega t + \phi(\omega))}$$

$H(j\omega)$ gives the change in magnitude and phase of the complex exponential $e^{j\omega t}$.

If the input $x(t)$ is a periodic signal with the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

the response $y(t)$ is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

$y(t)$ is also periodic with the same fundamental frequency as $x(t)$.

Example

$$x(t) = \cos w_0 t$$

$$h(t) = e^{-t} u(t)$$

Determine the output $y(t)$.

$$H(jw) = \int_0^\infty e^{-t} e^{-jwt} \, dt$$

$$= -\frac{1}{1+jw} e^{-t} e^{-jwt} \Big|_0^\infty$$

$$= \frac{1}{1+jw}$$

$$x(t) = \frac{1}{2}\left(e^{jw_0 t} + e^{-jw_0 t}\right)$$

$$y(t) = \frac{1}{2} H(jw_0) e^{jw_0 t} + \frac{1}{2} H(-jw_0) e^{-jw_0 t}$$

$$= \frac{1}{2}\left[\frac{1}{1+j\omega_0} e^{j\omega_0 t} + \frac{1}{1-j\omega_0} e^{-j\omega_0 t}\right]$$

$$y(t) = \frac{1}{2}\left[\frac{e^{j\theta}}{\sqrt{1+\omega_0^2}} e^{j\omega_0 t} + \frac{e^{-j\theta}}{\sqrt{1+(-\omega_0)^2}} e^{-j\omega_0 t}\right]$$

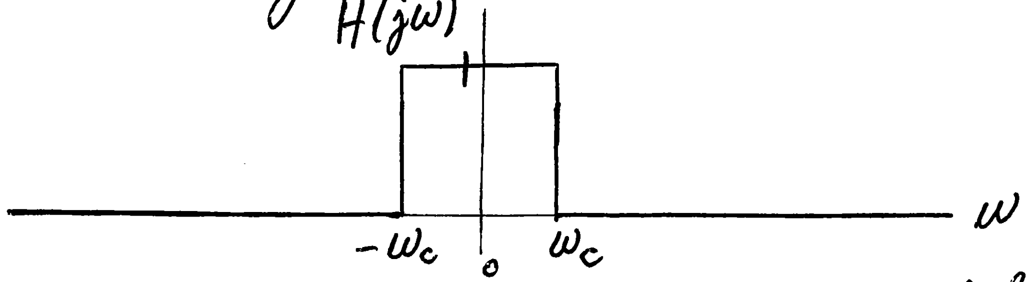
where $\theta = -\tan^{-1}(\omega_0)$

$$y(t) = \frac{1}{\sqrt{1+\omega_0^2}} \cdot \frac{1}{2}\left[e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}\right]$$

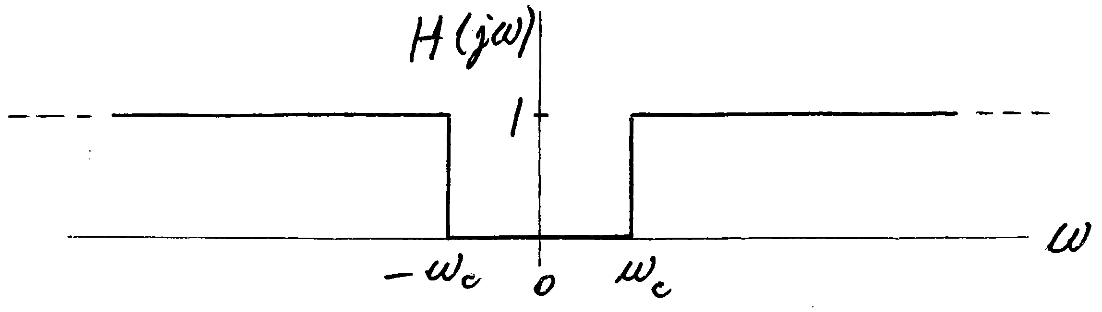
$$= \frac{1}{\sqrt{1+\omega_0^2}} \cos(\omega_0 t + \theta)$$

[Note that $\frac{1}{\sqrt{1+\omega_0^2}} = |H(j\omega_0)|$ and $\theta = \angle H(j\omega_0)$.]

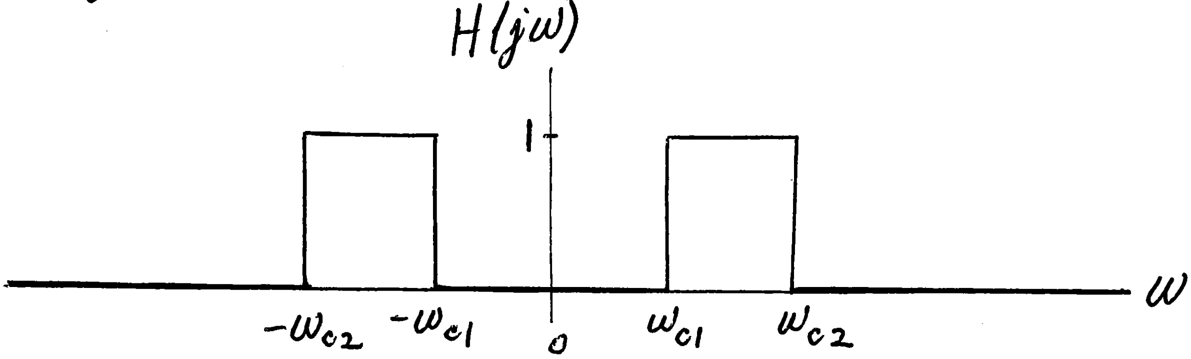
Ideal Frequency Selective Filters



Frequency response of an ideal lowpass filter.

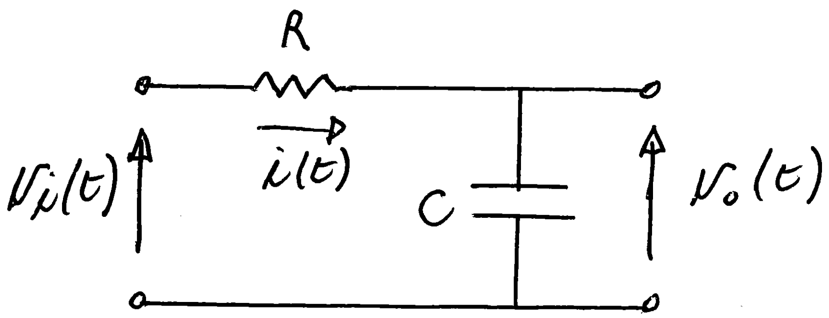


Frequency response of an ideal highpass filter.



Frequency response of an ideal bandpass filter.

First - order RC lowpass filter



$$i(t) = C \frac{d v_o(t)}{dt}$$

$$v_i(t) = R\, i(t) + v_o(t)$$

$$= RC \frac{d v_o(t)}{dt} + v_o(t) \qquad \text{— (1)}$$

Assuming initial rest the system described by eqn. ① is LTI. Let $H(j\omega)$ denote its frequency response. If the input voltage $v_i(t) = e^{j\omega t}$, then the output voltage.

$$v_o(t) = H(j\omega)\, e^{j\omega t}.$$

Substituting these expressions for $v_i(t)$ and $v_o(t)$ in eqn. ① we obtain

$$e^{j\omega t} = RC \frac{d}{dt}\left[H(j\omega)\, e^{j\omega t}\right] + H(j\omega)\, e^{j\omega t}$$

$$e^{j\omega t} = j\omega RC\, H(j\omega)\, e^{j\omega t} + H(j\omega)\, e^{j\omega t}$$

$$e^{j\omega t} = H(j\omega)\, e^{j\omega t}\left[1 + j\omega RC\right]$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$\underline{/H(j\omega)} = -\tan^{-1}(\omega RC)$$