Computer Graphics

2D Transformations

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Lecture Outlines

- Transformation
- Types of 2D Transformation
  - ✓Geometric
  - ✓Coordinate
  - ✓Composite
  - ✓Instance
- Matrix Revisit
  - ✓Use of Matrix in 2D Transformation

What is Transformation?

- The geometrical changes of an object from a current state to modified state is referred to as Transformation. It allows us to change the -
  - ✓ Position;
  - ✓ Size;
  - ✓ Orientation of the objects.
- Why it is needed?
  - ✓ To manipulate the initially created object;
  - ✓ To display the modified object without having to redraw it.

Two Dimensional Transformation

- There are two complementary points of view for describing object movement -
  - ✓ The first is that the object itself is moved relative to a stationary coordinate system or background **[Geometric Transformations].**
  - ✓ The second point of view holds that the object is held stationary while the coordinate system is moved relative to the object **[Coordinate Transformations].**

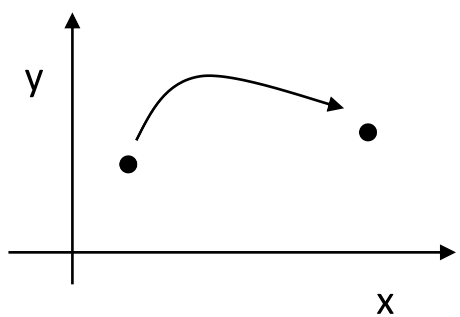
Two Dimensional Transformation

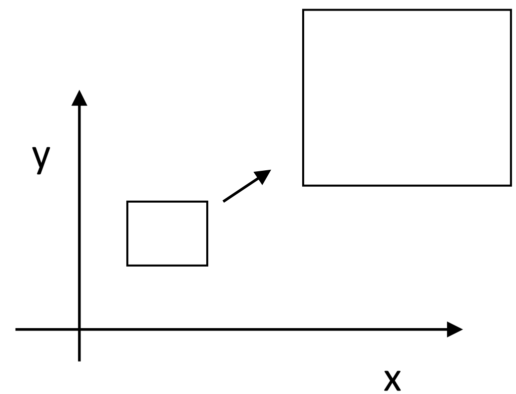
- An example involves the motion of an automobile against a scenic background.
  - ✓ We can simulate this by moving the automobile while keeping the background fixed **[Geometric Transformations].**
  - ✓ We can also keep the automobile fixed while moving the background scenery **[Coordinate Transformations].**

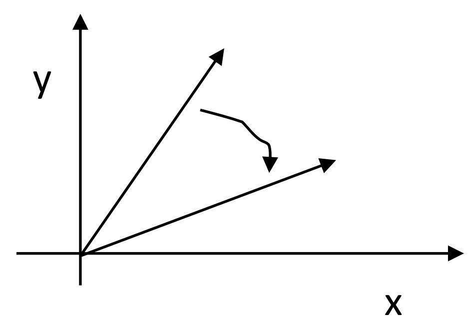
2D Transformation

- Two ways -
  - ❑ Object Transformation -
    - ✓Alter the coordinate of an object;
    - ✓Translation, rotation, scaling etc.
    - ✓Coordinate system unchanged.
  - ❑ Coordinate Transformation -
    - ✓Produce a different coordinate system.

Examples of 2D Transformations







Geometric Transformations

- Let us impose a coordinate system on a plane.
- An object *Obj* in the plane can be considered as a set of points.
- Every object point P has coordinates (x, y), and so the object is the sum total of all its coordinate points.
- If the object is moved to a new position, it can be regarded as a new object Obj’, all of whose coordinate point P’ can be obtained from the original points P by the application of a geometric transformation.

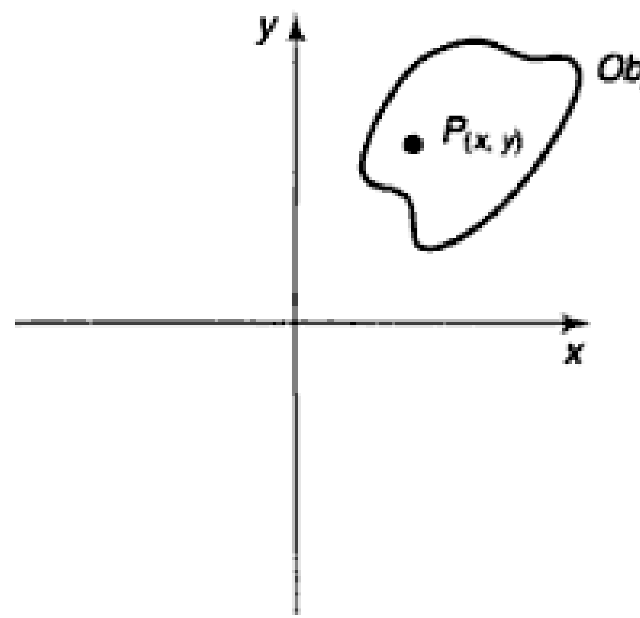


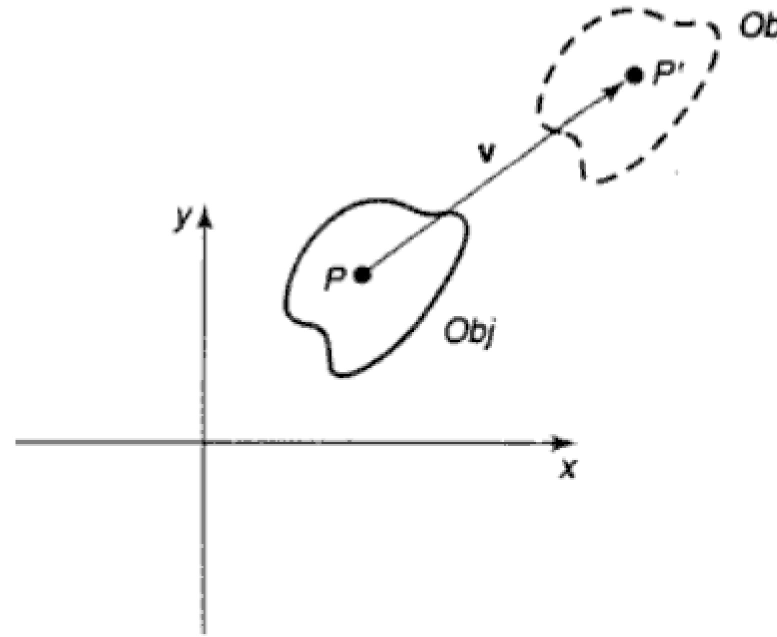
Fig. 4.1

Geometric Transformations

- Translation;
- Rotation about the Origin;
- Scaling with Respect to the Origin;
- Mirror Reflection about an Axis.

Translation

- In translation, an object is displaced a given distance and direction from its original position.
- If the displacement is given by the vector $v = t_x\mathbf{I} + t_y\mathbf{j}$ the new object point $P'(x', y')$ can be found by applying the transformation T_v to $P(x, y)$



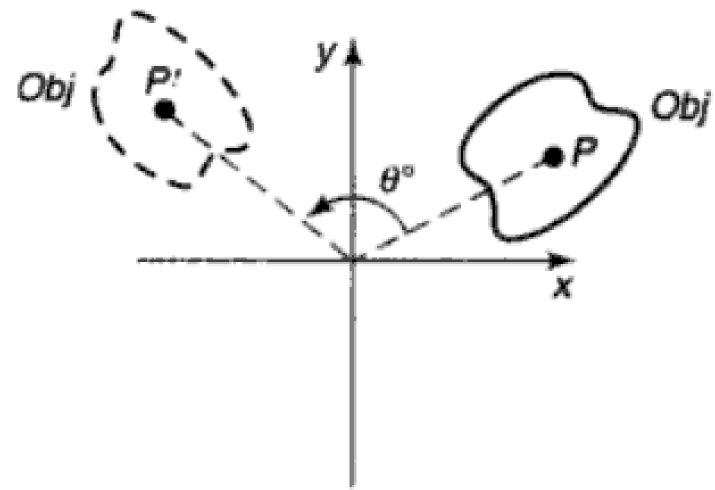
Now,

$$P' = T_v(P)$$

Where, $x' = x + t_x$ and $y' = y + t_y$

Rotation about the Origin

- In rotation, the object is rotated $\theta°$ about the origin.
- The convention is that the direction of rotation is counterclockwise if θ is a positive angle and clockwise if θ is a negative angle.
- The transformation of rotation R_θ is -



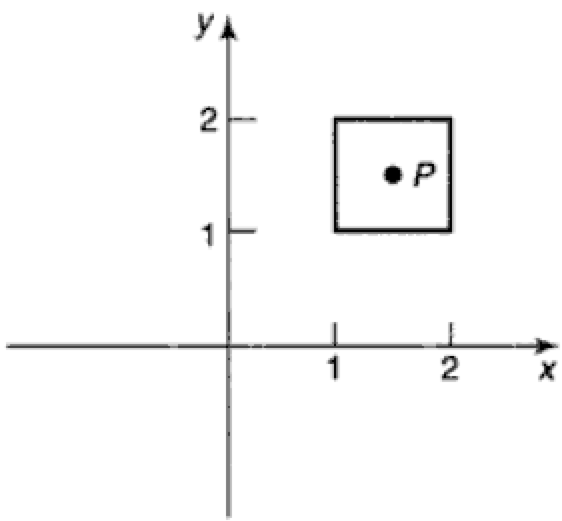
$$P' = R_\theta(P)$$

where $x' = x\cos(\theta) - y\sin(\theta)$

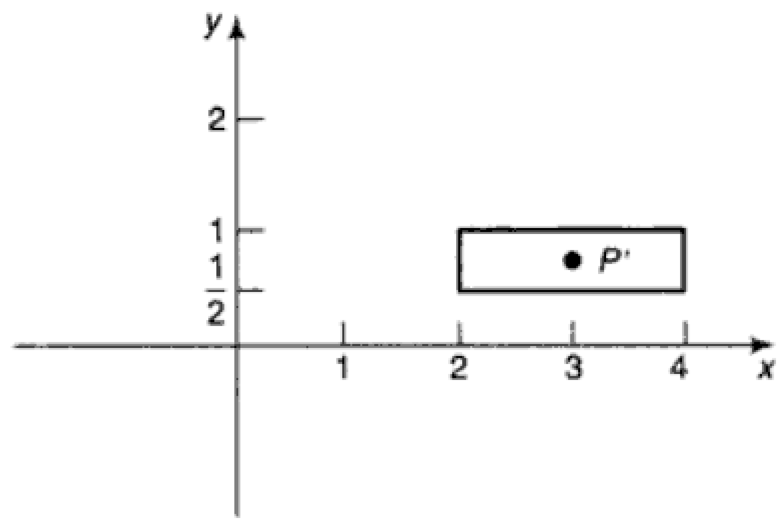
and $y' = x\sin(\theta) + y\cos(\theta)$

Scaling with Respect to the origin

- Scaling is the process of expanding or compressing the dimension of an object.
- Positive scaling constants s_x and s_y are used to describe changes in length with respect to the x direction and y direction, respectively.
- A scaling constant greater than one indicates an expansion of length, and less than one, compression of length.



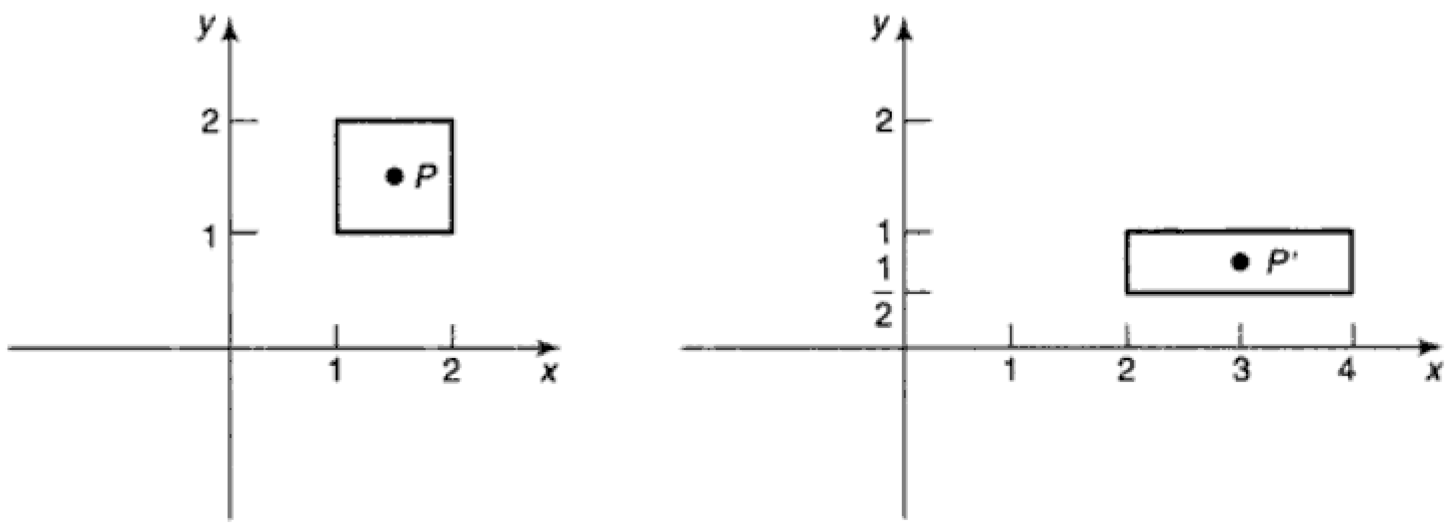
(a)



(b)

Scaling with Respect to the origin

- The scaling transformation $S_{Sx,Sy}$ is given by $P' = S_{Sx,Sy}(P)$ where, $x' = s_x x$ and $y' = s_y y$.
- After a scaling transformation is performed, the new object is located at a different position relative to the origin.
- In fact, in a scaling transformation, the only point that remains fixed is the origin.



(a) Original Object

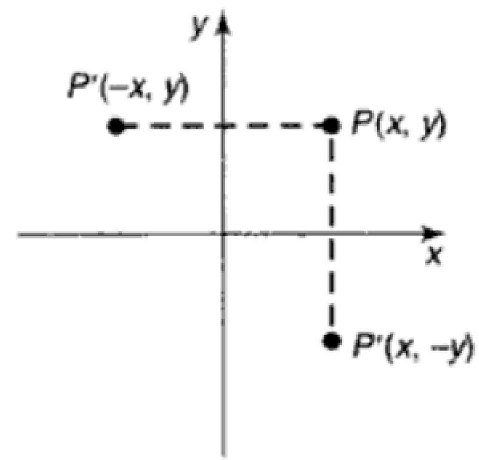
(b) Scaling factors s_x =2
Scaling factors s_y =1/2

Mirror Reflection about an Axis

- If either the x and y axis is treated as a mirror, the object has a mirror image or reflection.
- Since the reflection P' of an object point P is located the same distance from the mirror as P, the mirror reflection transformation M_x about the x-axis is given by $P' = M_x(P)$

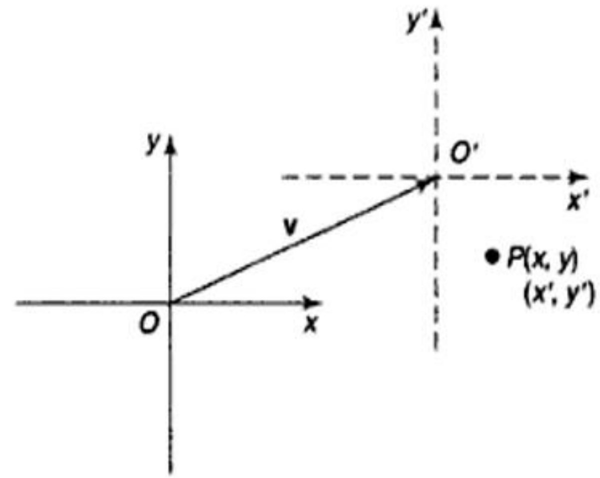
  where $x' = x$ and $y' = -y$.
- Similarly, the mirror reflection about the y-axis is $P' = M_y(P)\,y$

  where, $x' = -x$ and $y' = y$.



Coordinate Transformations

- Suppose that we have two coordinate systems in the plane. The first system is located at origin O and has coordinates axes xy.
- The second coordinate system is located at origin O' and has coordinate axes x’y’.
- Now each point in the plane has two coordinate descriptions: (x,y) or (x',/), depending on
- which coordinate system is used. If we think of the second system x'y' as arising from a transformation applied to the first system xy, we say that a coordinate transformation has been applied. We can describe this transformation by determining how the (x’,y’) coordinates of a point P are related to the (x,y) coordinates of the same point.



Coordinate Transformations

- Translation;
- Rotation about the Origin;
- Scaling with Respect to the Origin;
- Mirror Reflection about an Axis.

Translation

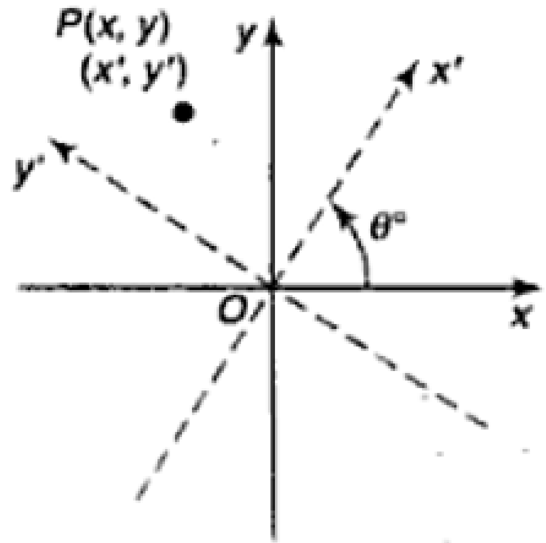
- If the xy coordinate system is displaced to a new position, where the direction and distance of the displacement is given by the vector $v = t_x I + t_y J$, the coordinates of a point in both systems are related by the translation transformation T_v:

$$(x', y') = T_v\,(x, y)$$

where $x' = x - t_x$ and $y' = y - t_y$

Rotation about the Origin

- The xy system is rotated $\theta°$ about the origin.
- Then the coordinates of a point in both systems are related by the rotation transformation R_θ:



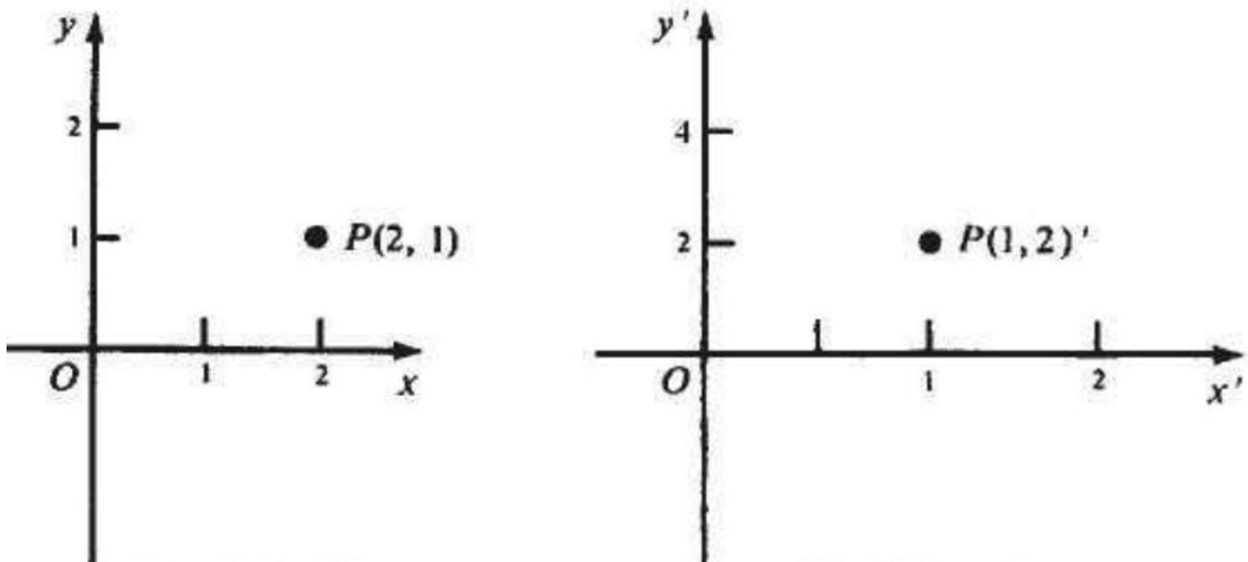
- $(x', y') = R_\theta\,(x, y)$
- $x' = x\cos(\theta) + y\sin(\theta)$
- $y' = — x\sin(\theta) + y\cos(\theta)$.

Scaling with Respect to the Origin

- Suppose that a new coordinate system is formed by leaving the origin and coordinate axes unchanged, but introducing different units of measurement along the x and y axes.
- If the new units are obtained from the old units by a scaling of s_x along the x axis and s_y along the y axis, the coordinates in the new system are related to coordinates in the old system through the scaling transformation Ss_xs_x :
- where x' = $\{1/s_x)x$ and y' = $\{1/s_y)y$.

Continue…

- Figure shows coordinate scaling transformation using scaling factors $s_x = 2$ and $s_y = ½$.



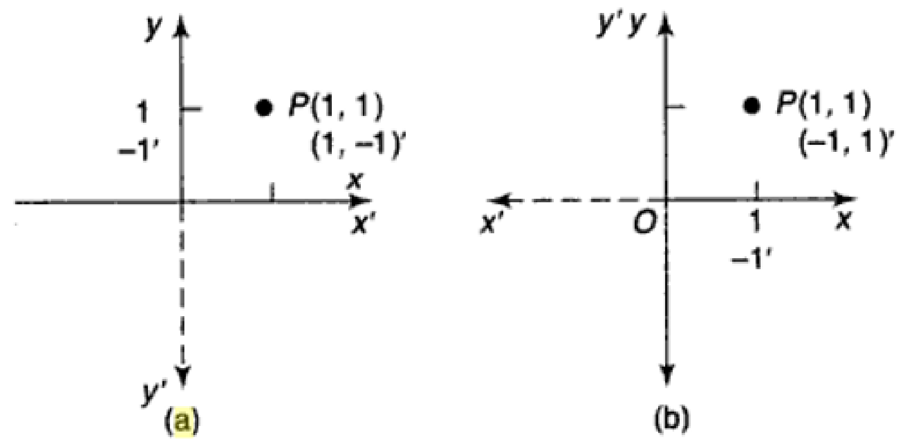
(a) Old units. (b) New units.

Mirror Reflection about an Axis

- If the new coordinate system is obtained by reflecting the old system about either x or y axis, the relationship between coordinates is given by the coordinate transformations M_x and M_y.
- Reflection about the x axis [Fig. (a)]:

$$(x', y') = M_x(x, y);$$

  - where $x' = x$ and $y' = -y$.



- Reflection about the y axis [Fig. (b)]:

$$(x',y') = M_y(x,y);$$

  - where $x' = -x$ and $y' = y$.

Composite Transformation

- More complex geometric and coordinate transformations can be built from the basic transformations described above by using the process of composition of functions.
- For example, such operations as rotation about a point other than the origin or reflection about lines other than the axes can be constructed from the basic transformations.

Continue…

- If we want to apply a series of transformation T_1, T_2, T_3 to a set of points, we can do it like below-
  - Calculate, $\mathbf{T} = T_1 \times T_2 \times T_3$

    then $\mathbf{P'} = \mathbf{T} \times \mathbf{P}$
- This method saves large number of adds and multiplications.

Example - 01

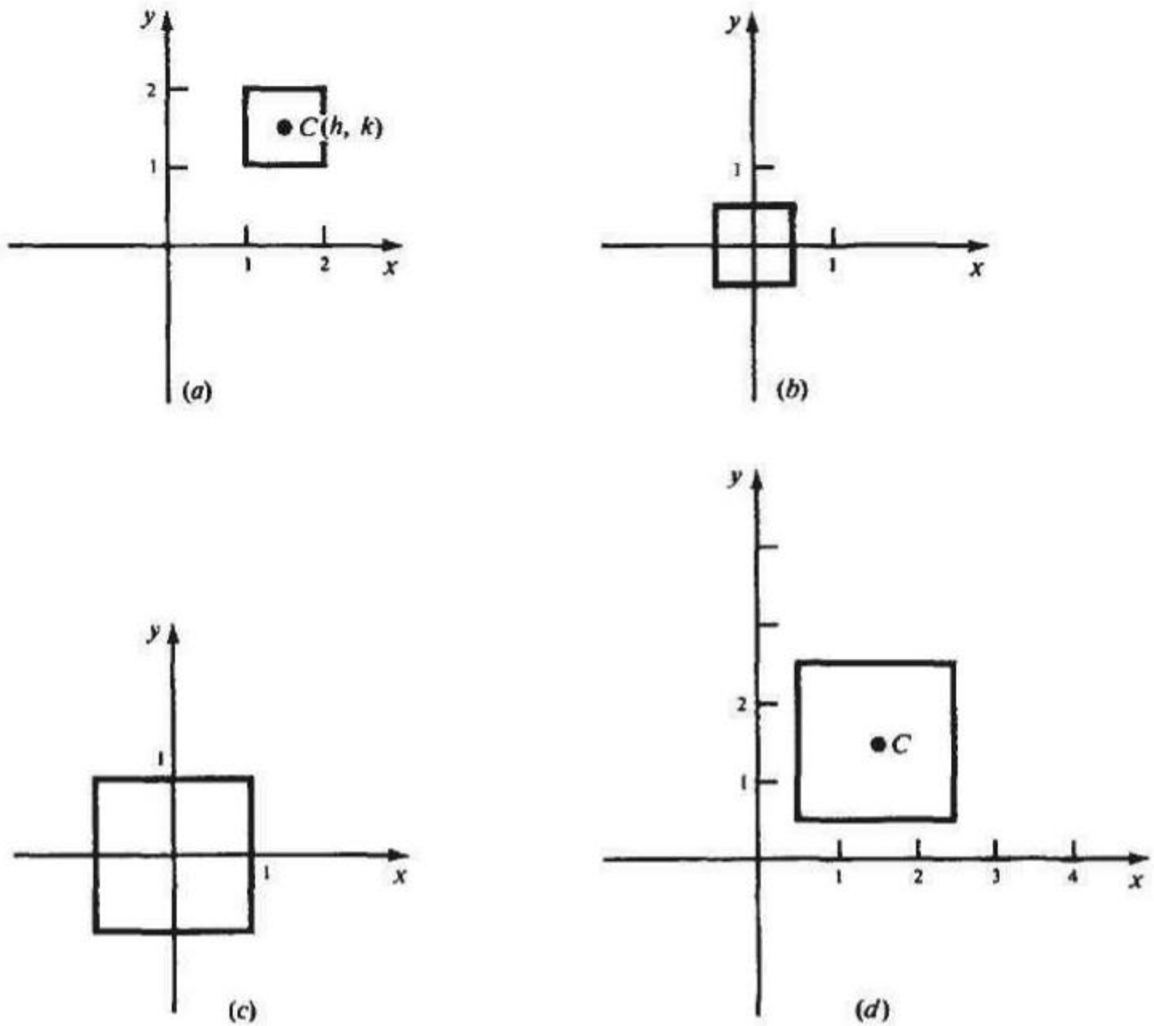
- Magnification of an object while keeping its center fixed:
- Let the geometric center be located at $C(h, k)$. Choosing a magnification factor $s > 1$, we construct the transformation by performing the following sequence of basic transformations:

  (1) Translate the object so that its center coincides with the origin;

  (2) Scale the object with respect to the origin;

  (3) Translate the scaled object back to the original position.

Continue…



Continue…

- The required transformation $S_{S,C}$ can be formed by compositions:

$$S_{S,C} = T_v . S_{S,S} . T_v^{-1} \qquad \text{- where } v = hI + kJ.$$

- By using composition, we can build more general scaling, rotation, and reflection transformations.

- For these transformations, we shall use the following notations:

(1) $S_{Sx,\, Sy,\, P}$ —scaling with respect to a fixed point P;

(2) $R_{\theta,\, p}$ —rotation about a point P;

(3) M_L —reflection about a line L.

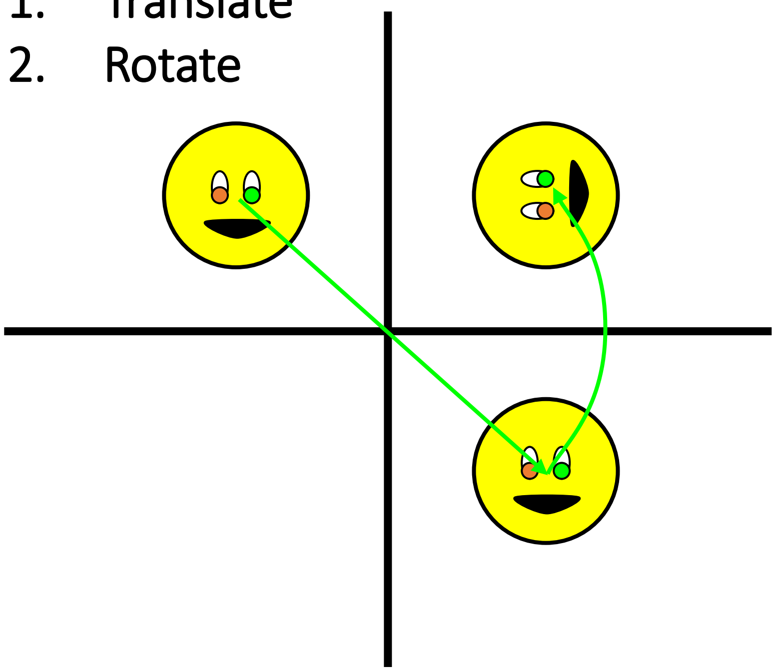
Transformations are NOT Commutative

- If we scale /rotate and then translate is that equivalent to translate first and then scale/rotate?
- No, because in general case result of matrix multiplication depends on the order.
- So, the order of transformation has to be maintained .

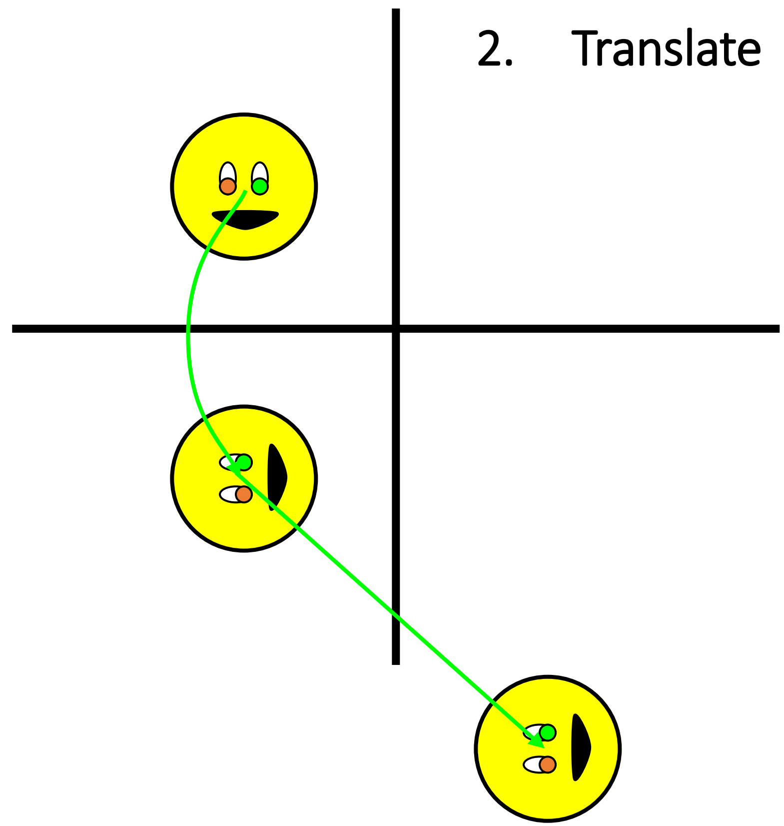
Order of operations

It does matter. Let's look at an example:

1. Translate
2. Rotate



1. Rotate
2. Translate

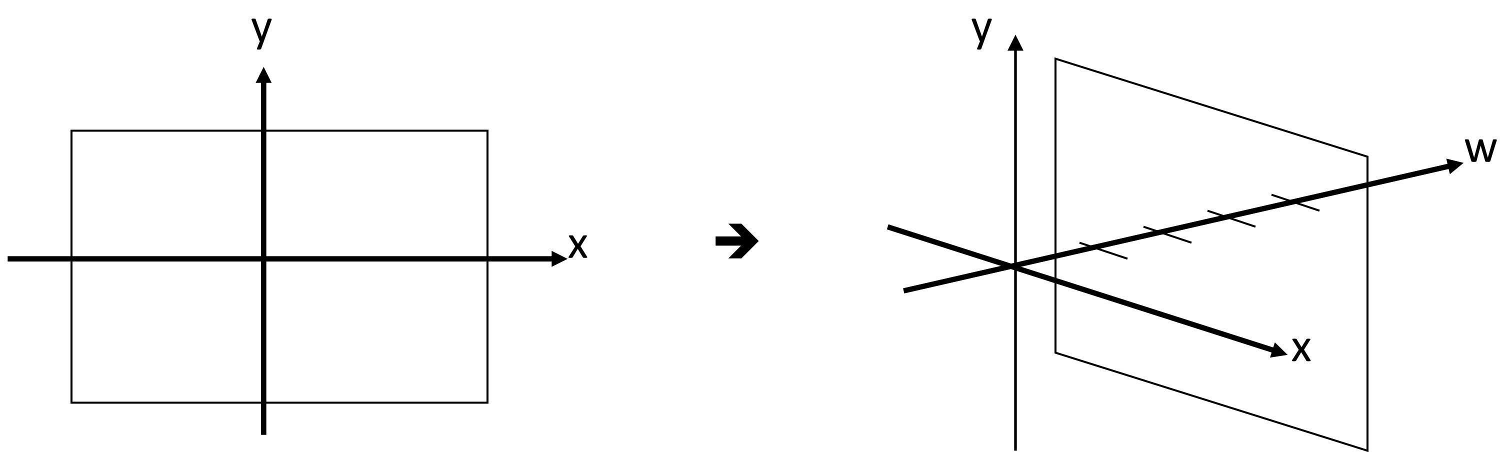


Matrix Description of the Basic Transformations

- The transformations of rotation, scaling, and reflection can be represented as matrix functions:

Geometric transformations	Coordinate transformations
$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$	$\bar{R}_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$
$S_{s_x, s_y} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$	$\bar{S}_{s_x, s_y} = \begin{pmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{s_y} \end{pmatrix}$
$M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\bar{M}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\bar{M}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Homogenous Coordinates



- Let’s move our problem into 3D.
- Let point (x, y) in 2D be represented by point (x, y, 1) in the new space.
- Scaling our new point by any value a puts us somewhere along a particular line: (ax, ay, a).
- A point in 2D can be represented in many ways in the new space.
- (2, 4) ---------→ (8, 16, 4) or (6, 12, 3) or (2, 4, 1) or etc.

Continue…

- We can always map back to the original 2D point by dividing by the last coordinate
- (15, 6, 3) ---→ (5, 2).
- (60, 40, 10) -→ ?.

- Why do we use 1 for the last coordinate?

- The fact that all the points along each line can be mapped back to the same point in 2D gives this coordinate system its name – homogeneous coordinates.

Matrix Representation

- Point (x, y) in column matrix:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Our point now has three coordinates. So our matrix is needs to be 3x3.
- **Translation:**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Continue…

- Rotation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Scaling:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation about an Arbitrary Point P

- To rotate an object about a point $P(x, y)$ we need to follow the following steps:
  - Step 1: Translate by $(-x,-y)$
  - Step 2: Rotate
  - Step 3: Translate by (x,y)

Continue…

- From Step 1 we get-

$$T_3(-x,-y) = \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}$$

- From Step 2 we get-

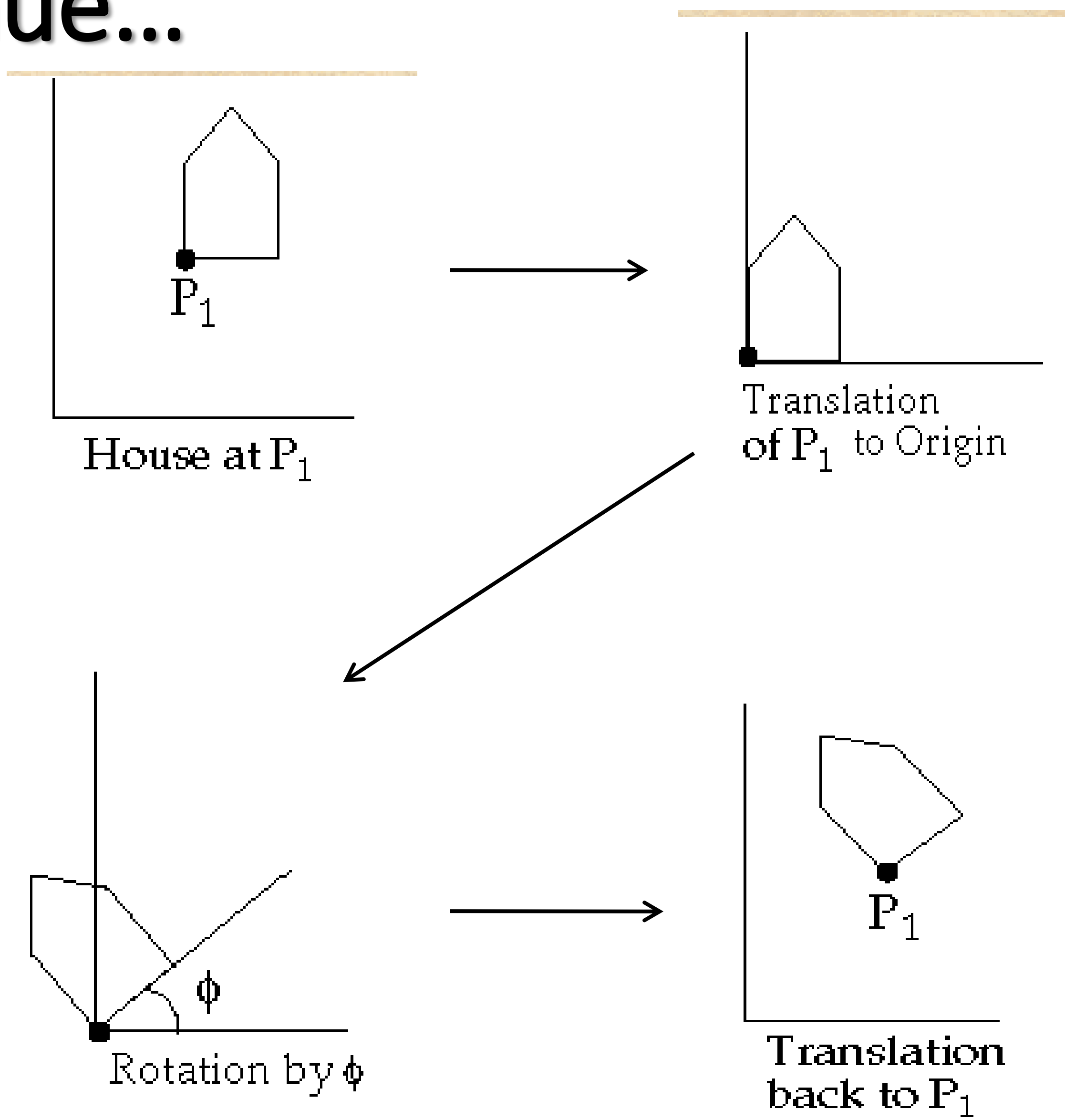
$$R(\Theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- From Step 3 we get-

$$T_1(x,y) = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

- So, $T = T_1(x,y) * R(\Theta) * T_3(-x,-y)$

Continue…



P_1
House at P_1
Translation
of P_1 to Origin
ϕ
Rotation by ϕ
P_1
Translation
back to P_1

Example - 02

- Perform a 45° rotation of triangle A (0, 0), B (1, 1), C (5, 2)
  - (a) about the origin, and (b) about P(-l, -l).

SOLUTION

We represent the triangle by a matrix formed from the homogeneous coordinates of the vertices:

$$\begin{array}{ccc} A & B & C \end{array}$$
$$\begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

(*a*) The matrix of rotation is

$$R_{45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the coordinates $A'B'C'$ of the rotated triangle ABC can be found as

$$[A'B'C'] = R_{45^\circ} \cdot [ABC] = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{7\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Thus $A' = (0, 0)$, $B' = (0, \sqrt{2})$, and $C' = (\frac{3}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$.

Continue…

(b) about P(-l, -l).

$$R_{45^\circ,P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{pmatrix}$$

Now

$$[A'B'C'] = R_{45^\circ,P} \cdot [ABC] = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & (\frac{3}{2}\sqrt{2}-1) \\ (\sqrt{2}-1) & (2\sqrt{2}-1) & (\frac{9}{2}\sqrt{2}-1) \\ 1 & 1 & 1 \end{pmatrix}$$

So $A' = (-1, \sqrt{2}-1)$, $B' = (-1, 2\sqrt{2}-1)$, and $C' = (\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1)$.

Example - 03

- Example
  - Perform 60° rotation of a point P(2, 5) about a pivot point (1,2). Find P'?

$$P' = (-1, 4)$$

Composite Transformation Matrix

General Fixed-Point Scaling

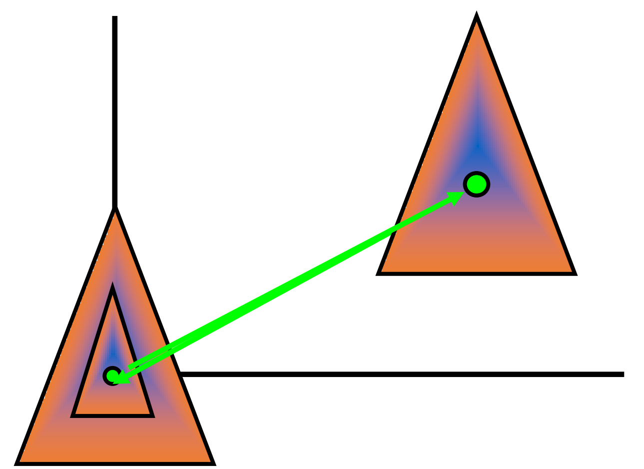
Operation :-

1. Translate (fixed point is moved to origin)
2. Scale with respect to origin
3. Translate (fixed point is returned to original position)

T(fixed) • S(scale) • T(–fixed)

Find the matrix that represents scaling of an object with respect to any fixed point?

Given P(6, 8) , Sx = 2, Sy = 3 and fixed point (2, 2). Use that matrix to find P'?



Answer

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} Sx & 0 & -t_x Sx \\ 0 & Sy & -t_y Sy \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} Sx & 0 & -t_x Sx + t_x \\ 0 & Sy & -t_y Sy + t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$x = 6$, $y = 8$, $Sx = 2$, $Sy = 3$, $t_x = 2$, $t_y = 2$

$$\begin{pmatrix} 2 & 0 & -2(2) + 2 \\ 0 & 3 & -2(3) + 2 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 6 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \\ 1 \end{pmatrix}$$

Practice Problem

Solved problems from Chapter-4:

- 4.2, 4.4 to 4.9.

Thank You!