Eigenvalues and Eigenvectors

Consider a square matrix n × n. If X is the non-trivial column vector solution of the matrix equation AX = λX, where λ is a scalar, then X is the eigenvector of matrix A and the corresponding value of λ is the eigenvalue of matrix A.

Suppose the matrix equation is written as A X – λ X = 0. Let I be the n × n identity matrix.

If I X is substituted by X in the equation above, we obtain A X – λ I X = 0.

The equation is rewritten as (A – λ I) X = 0.

The equation above consists of non-trivial solutions, if and only if, the determinant value of the matrix is 0.

The characteristic equation of A is Det (A – λ I) = 0.

'A' being an n × n matrix, if (A – λ I) is expanded, (A – λ I) will be the characteristic polynomial of A because it's degree is n.

Example 2: Find all eigenvalues and corresponding eigenvectors for the matrix A if

$$\begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution:

$$\det\left(\begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$\Downarrow$$

$$= \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & -3 & 0 \\ 2 & -5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$= \det \begin{pmatrix} 2-\lambda & -3 & 0 \\ 2 & -5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix} -5-\lambda & 0 \\ 0 & 3-\lambda \end{pmatrix} - (-3)\det\begin{pmatrix} 2 & 0 \\ 0 & 3-\lambda \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 2 & -5-\lambda \\ 0 & 0 \end{pmatrix}$$

$$= (2-\lambda)\left(\lambda^2+2\lambda-15\right)-(-3)\cdot 2(-\lambda+3)+0\cdot 0$$
$$= -\lambda^3+13\lambda-12$$
$$-\lambda^3+13\lambda-12=0$$
$$-(\lambda-1)(\lambda-3)(\lambda+4)=0$$

So, The eigenvalues are :

$\lambda = 1,\ \lambda = 3,\ \lambda = -4$

Eigenvectors for $\lambda = 1$

$$\begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix} - 1\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$(A-1I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{Bmatrix} x-3y=0 \\ z=0 \end{Bmatrix} = \begin{Bmatrix} z=0 \\ x=3y \end{Bmatrix}$$

Since, $y \neq 0$, Let y = 1

Then,

Eigenvectors for $\lambda = 1$ = $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Similarly

Eigenvectors for $\lambda = 3$: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Eigenvectors for $\lambda = -4$: $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

The eigenvectors for $\begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Example 3: Consider the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -5 & 3 & a \\ 4 & -2 & -1 \end{bmatrix}$$

for some variable 'a'. Find all values of 'a' which will prove that A has eigenvalues 0, 3, and −3.

Solution:

Let p (t) be the characteristic polynomial of A, i.e. let p (t) = det (A − tI) = 0. By expanding along the second column of A − tI, we can obtain the equation

$$p(t) = \det\left(\begin{bmatrix} -2 & 0 & 1 \\ -5 & 3 & a \\ 4 & -2 & -1 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}\right)$$

$$= \begin{vmatrix} -2-t & 0 & 1 \\ -5 & 3-t & a \\ 4 & -2 & -1-t \end{vmatrix}$$

$$= (3-t)\begin{vmatrix} -2-t & 1 \\ 4 & -1-t \end{vmatrix} + 2\begin{vmatrix} -2-t & 1 \\ -5 & a \end{vmatrix}$$

$= (3 - t) [(-2 -t) (-1 - t) - 4] + 2[(-2 - t) a + 5]$

$= (3 - t) (2 + t + 2t + t^2 -4) + 2 (-2a - ta + 5)$

$= (3 - t) (t^2 + 3t - 2) + (-4a -2ta + 10)$

$= 3t^2 + 9t - 6 - t^3 - 3t^2 + 2t - 4a - 2ta + 10$

$= -t^3 + 11t - 2ta + 4 - 4a$

$= -t^3 + (11 - 2a) t + 4 - 4a$

For the eigenvalues of A to be 0, 3 and −3, the characteristic polynomial p (t) must have roots at t = 0, 3, −3. This implies $p (t) = -t (t - 3) (t + 3) = -t(t^2 - 9) = -t^3 + 9t$

Therefore, $-t^3 + (11 - 2a) t + 4 - 4a = -t^3 + 9t$.

For this equation to hold, the constant terms on the left and right-hand sides of the above equation must be equal. This means that 4 − 4a = 0, which implies a = 1.

Hence, A has eigenvalues 0, 3, −3 precisely when a = 1.

Example 4: Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix}$$

Solution:

$$\det\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \quad = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{pmatrix}$$

$$= \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det\begin{pmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{pmatrix} - 0\cdot\det\begin{pmatrix} 0 & 4 \\ 0 & 9-\lambda \end{pmatrix} + 0\cdot\det\begin{pmatrix} 0 & 3-\lambda \\ 0 & 4 \end{pmatrix}$$

$$= (2-\lambda)\left(\lambda^2-12\lambda+11\right) - 0\cdot 0 + 0\cdot 0$$
$$= -\lambda^3+14\lambda^2-35\lambda+22$$
$$-\lambda^3+14\lambda^2-35\lambda+22=0$$
$$-(\lambda-1)(\lambda-2)(\lambda-11)=0$$

The eigenvalues are :
$\lambda=1,\ \lambda=2,\ \lambda=11$

Eigenvectors for $\lambda=1$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix} - 1\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix}$$

$$(A-1I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{Bmatrix} x=0 \\ y+2z=0 \end{Bmatrix}$$

Since, $z \neq 0$

Let $z = 1$

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$Similarly$

Eigenvectors for $\lambda = 2$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvectors for $\lambda = 11$: $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

The eigenvectors for $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Exercise

Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0 & 1 \\ -5 & 3 & a \\ 4 & -2 & -1 \end{bmatrix}$$

See and practice:

https://kanchiuniv.ac.in/coursematerials/Eigenvalues_and_Eigenvectors.pdf

https://www.varsitytutors.com/linear_algebra-help/eigenvalues-and-eigenvectors?page=5

Properties of Eigenvalues

Let A be a matrix with eigenvalues

$\lambda_1, \ldots, \lambda_n$

The following are the properties of eigenvalues.

1] The trace of A, defined as the sum of its diagonal elements, is also the sum of all eigenvalues,

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

2] The determinant of A is the product of all its eigenvalues,

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n.$$

3] The eigenvalues of the

k^{th}
power of A; that is the eigenvalues of
A^k
, for any positive integer k, are
$\lambda_1^k, \ldots, \lambda_n^k.$

4] The matrix A is invertible if and only if every eigenvalue is nonzero.

5] If A is invertible, then the eigenvalues of

A^{-1}
are
$\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$
and each eigenvalue's geometric multiplicity coincides. The characteristic polynomial of the inverse is the reciprocal polynomial of the original, the eigenvalues share the same algebraic multiplicity.

6] If A is equal to its conjugate transpose, or equivalently if A is Hermitian, then every eigenvalue is real. The same is true of any symmetric real matrix.

7] If A is not only Hermitian but also positive-definite, positive-semidefinite, negative-definite, or negative-semidefinite, then every eigenvalue is positive, non-negative, negative, or non-positive, respectively.

8] If A is unitary, every eigenvalue has absolute value

$|\lambda_i| = 1$

.

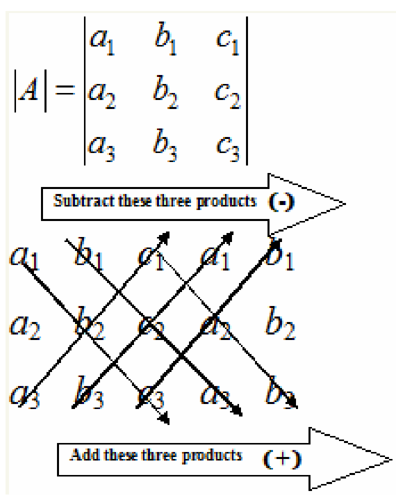
9] If A is a

$n \times n$
matrix and
$\{\lambda_1, \ldots, \lambda_k\}$
are its eigenvalues, then the eigenvalues of matrix I + A (where I is the identity matrix) are
$\{\lambda_1 + 1, \ldots, \lambda_k + 1\}$

The Sarrus Rule

This method only works for 3×3 matrices. Given a matrix A of order 3×3. To apply Sarrus rule, copy the first and second column of A to form fourth and fifth columns. The determinant of A is then obtained by adding the products of the three “DOWNWARD DIAGONALS” and subtracting the products of the three “UPWARD DIAGONALS” as shown

Thus, the determinant of 3 × 3 matrix A is given by the following
$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2\, a_1 - c_3\, a_2\, b_1$



Example:

$$\begin{vmatrix} 0 & 3 & 2 \\ 1 & 7 & 8 \\ 0 & 5 & 2 \end{vmatrix} = 0(14-40) - 3(2-0) + 2(5-0)$$

$$= 0 - 6 + 10 = 4$$

Method 2, is the Sarrus rule

$$\begin{array}{ccccc} \mathbf{0} & \mathbf{3} & \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{7} & \mathbf{8} & \mathbf{1} & \mathbf{7} \\ \mathbf{0} & \mathbf{5} & \mathbf{2} & \mathbf{0} & \mathbf{5} \end{array}$$

Multiply and add the elements of the corresponding arrows that go upwards:

$$(0 \times 7 \times 2) + (5 \times 8 \times 0) + (2 \times 1 \times 3) = 6$$

Multiply and add the elements of the corresponding arrows that go downwards:

$$(0 \times 7 \times 2) + (3 \times 8 \times 0) + (2 \times 1 \times 5) = 10$$

Determinant: Sum of lower arrows – sum of upper arrows = 10 – 6 = 4