FREQUENCY DOMAIN FILTERS

ICT4201: DIP

FREQUENCY DOMAIN ANALYSIS

A signal is analyzed with respect to time, but in frequency domain we analyze signal with respect to frequency.

Difference between spatial domain and frequency domain

- In spatial domain, we deal with images as it is. The value of the pixels of the image change with respect to scene. Whereas in frequency domain, we deal with the rate at which the pixel values are changing in spatial domain.
- In simple spatial domain, we directly deal with the image matrix. Whereas in frequency domain, we deal an image like this.

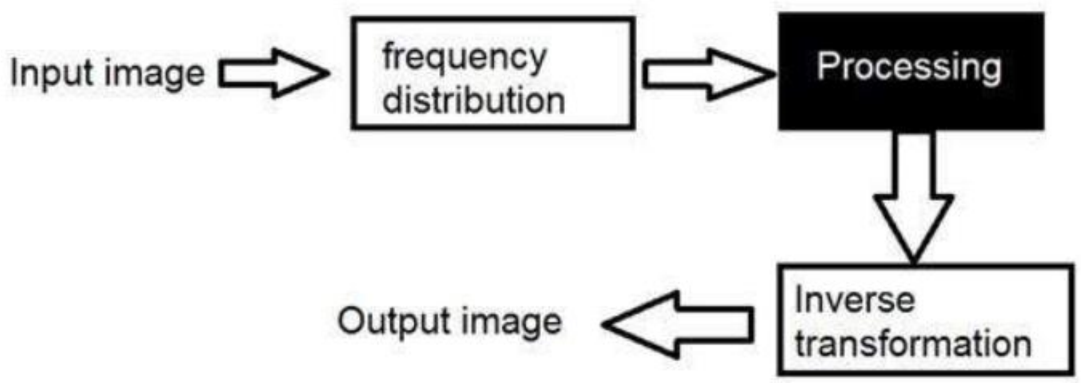
FREQUENCY DOMAIN

We first transform the image to its frequency distribution.

Then our black box system perform what ever processing it has to performed, and the output of the black box in this case is not an image, but a transformation.

After performing inverse transformation, it is converted into an image which is then viewed in spatial domain.

What is a transformation here?



TRANSFORMATION OF SIGNAL/IMAGE

A signal can be converted from time domain into frequency domain using mathematical operators called transforms. There are many kind of transformation that does this. Some of them are given below.

1. Fourier Series
2. Fourier transformation
3. Laplace transform
4. Z transform

We can consider image as signal here.

For DIP, we will discuss about Fourier series and Fourier transformation in details.

FREQUENCY COMPONENTS

Any image in spatial domain can be represented in a frequency domain. But what do this frequencies actually mean.

We will divide frequency components into two major components.

- High frequency components
  - High frequency components correspond to edges in an image.
- Low frequency components
  - Low frequency components in an image correspond to smooth regions.

FOURIER SERIES

Fourier series simply states that, periodic signals can be represented into sum of sines and cosines when multiplied with a certain weight. It further states that periodic signals can be broken down into further signals with the following properties.

- The signals are sines and cosines
- The signals are harmonics of each other

How it is calculated?

- Since as we have seen in the frequency domain, that in order to process an image in frequency domain, we need to first convert it using into frequency domain and we have to take inverse of the output to convert it back into spatial domain. That’s why both Fourier series and Fourier transform has two formulas. One for conversion and one converting it back to the spatial domain.

Fourier Series

Continuous signals are classified as periodic and non-periodic. A periodic signal of period T can be expressed as summation of an average/dc value, a sinusoidal wave of period T called fundamental component and infinite number sinusodial waves of period of $T/2$, $T/3$, $T/4$,… … … called harmonies.

Such presentation of periodic wave is done by Fourier series. For any periodic signal $x(t)$ of period T, its Fourier series expansion is given as,

$$x(t) = \frac{a_0}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{2\pi n}{T}t\right)$$

$$x(t) = \frac{\tau}{T} + \frac{2\tau}{T}\sum_{n=1}^{\infty}\left(\sin c\left(n\frac{\tau}{T}\right)\cos\frac{2\pi n}{T}t\right)$$

$$= d + 2d\sum_{n=1}^{\infty}\left(\sin c(nd)\cos\frac{2\pi n}{T}t\right)$$

Here dc component is d, the peak of fundamental component is $2d\text{sinc}(d)$, the peak of 2^{nd} harmonic is $2d\text{sinc}(2d)$, that of 3^{rd} harmonic is $2d\text{sinc}(3d)$ and so on…

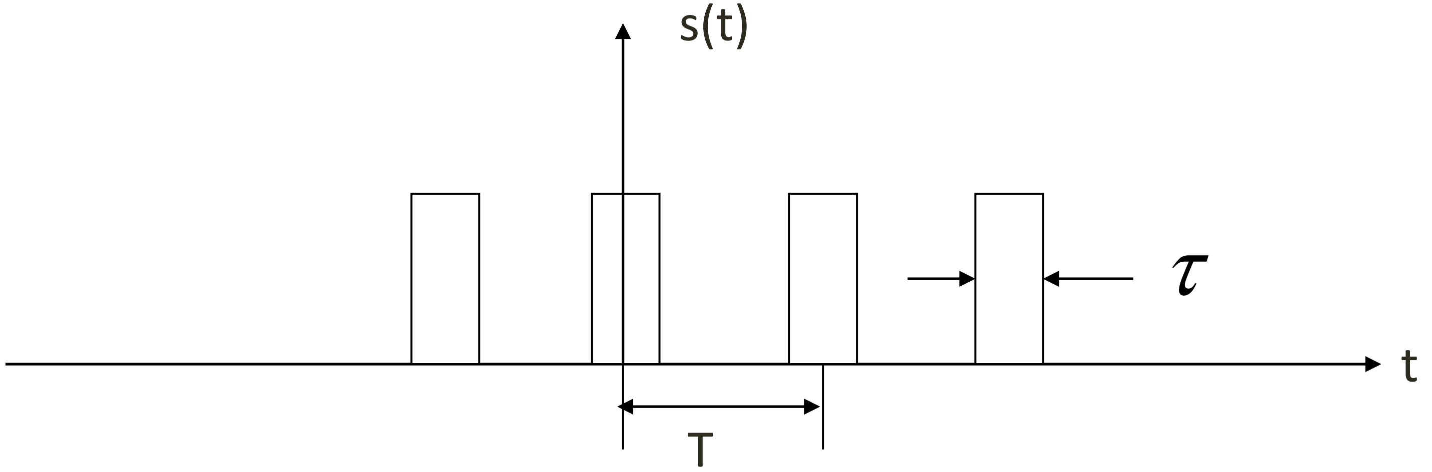
;where the constants a_n and b_n give the peak value of frequency components of n/T Hz, is given as,

$$a_n = \int_{-T/2}^{T/2} x(t)\cos\frac{2\pi nt}{T}\,dt$$

$$b_n = \int_{-T/2}^{T/2} x(t)\sin\frac{2\pi nt}{T}\,dt$$

$$a_0 = Lt_{n\to 0}\, a_n$$

$$x(t) = \frac{a_0}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi n}{T}t + b_n\sin\frac{2\pi n}{T}t\right)$$



$$a_n = \int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$= \int_{-\tau/2}^{\tau/2} 1.\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$= \left[\frac{T}{2\pi n}\sin\left(\frac{2\pi nt}{T}\right)\right]_{-\tau/2}^{\tau/2}$$

$$= \tau\sin\left(\frac{\pi n\tau}{T}\right) / \frac{\pi n\tau}{T}$$

$$= \tau \sin c\left(\frac{n\tau}{T}\right) = \tau\sin(nd)$$

$$a_0 = Lt_{n\to 0} a_n = Lt_{n\to 0}\tau \sin c\left(\frac{n\tau}{T}\right) = \tau.1 = \tau$$

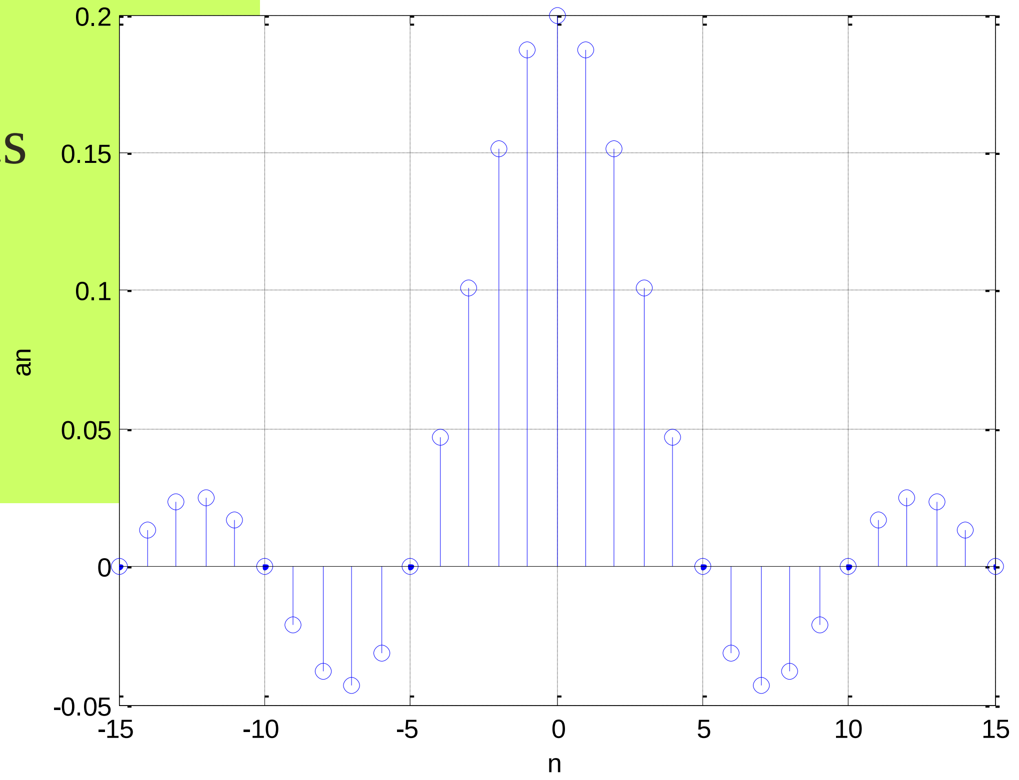
τ

$$b_n = \int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$= \int_{-\tau/2}^{\tau/2} 1.\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$= 0$$

```
n=-15:1:15;
A=1;
T=1;
tau =0.2;
d =tau/T; %dc components
an=A*tau*sinc(n*d);
%fundamental or harmonics
stem(n, an)
grid on
 xlabel('n')
ylabel('an')
```



FOURIER TRANSFORM

The Fourier transform simply states that that the non periodic signals whose area under the curve is finite can also be represented into integrals of the sines and cosines after being multiplied by a certain weight.

The Fourier transform has many wide applications that include, image compression (e.g JPEG compression), filtering and image analysis.

Difference between Fourier series and transform

- Although both Fourier series and Fourier transform are given by Fourier , but the difference between them is Fourier series is applied on periodic signals and Fourier transform is applied for non periodic signals

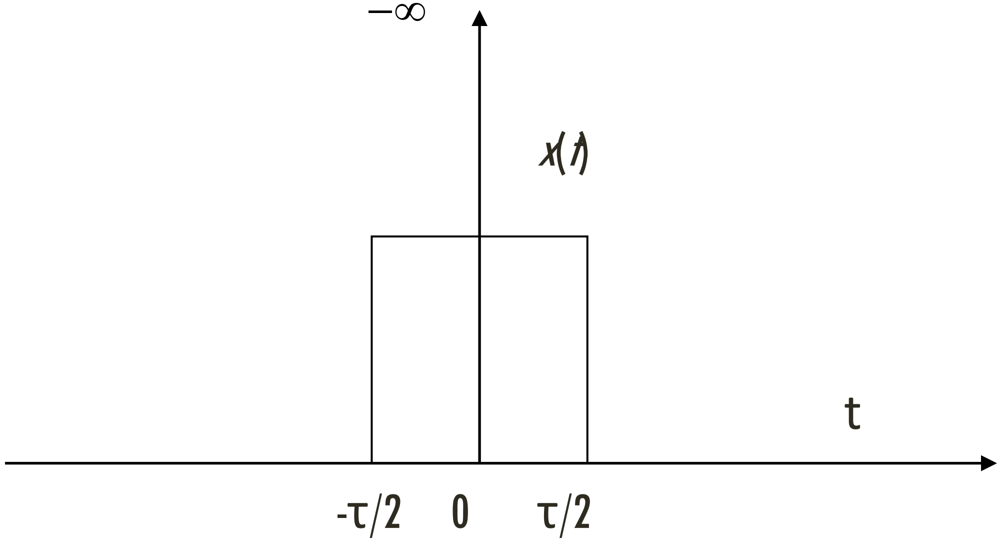
Which one is applied on images

- Now the question is that which one is applied on the images , the Fourier series or the Fourier transform. Well, the answer to this question lies in the fact that what images are. Images are non – periodic. And since the images are non periodic, so Fourier transform is used to convert them into frequency domain.

Fourier Transform

Spectrum of any non-periodic signal is analyzed by Fourier transform/integral instead of Fourier series. Let us now consider a non-periodic signal *x*(*t*), its Fourier integral/transform is expressed as,

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

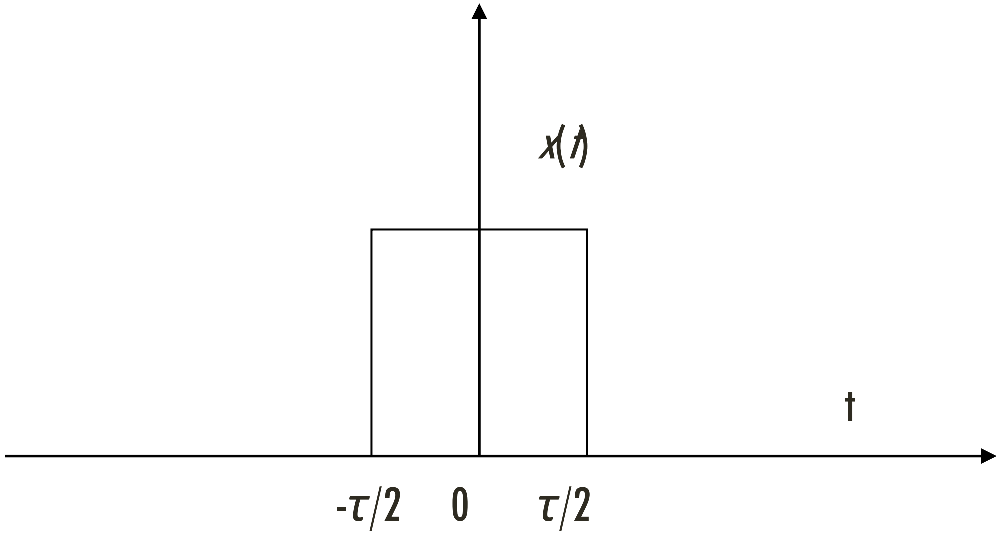


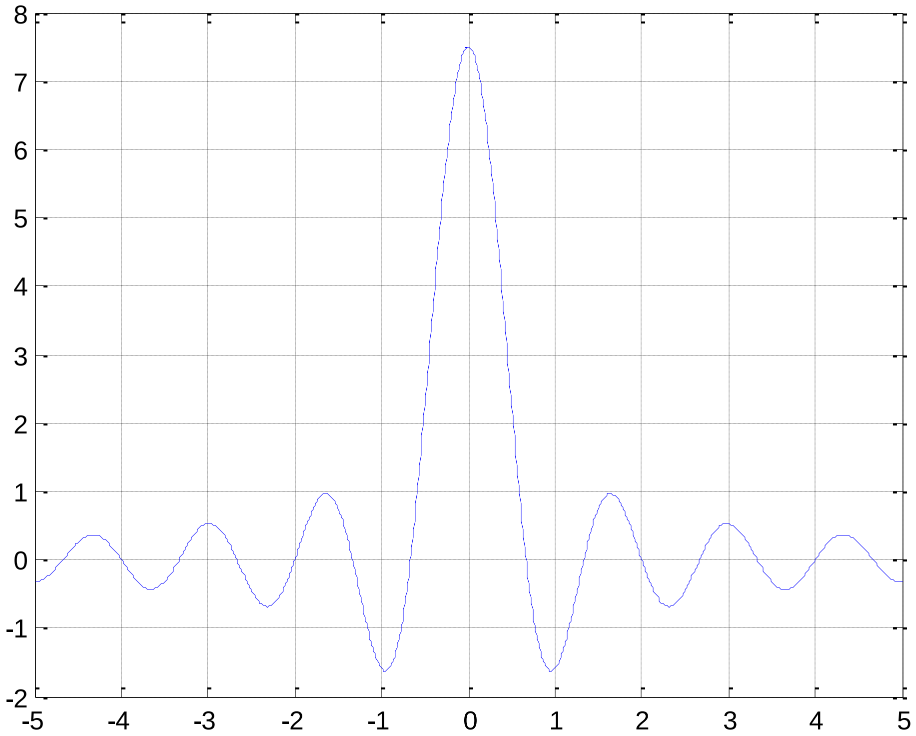
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$= \int_{-\tau/2}^{\tau/2} A e^{-j2\pi ft} dt$$

$$= A\tau \sin c(f\tau)$$

```
f=-5:0.01:5;
A=5;
tau=1.5;
F=A*tau*sinc(f*tau);
plot(f, F)
grid on
```





DISCRETE FOURIER TRANSFORM

Since we are dealing with images, and in fact digital images, so for digital images we will be working on discrete Fourier transform,

Consider the above Fourier term of a sinusoid. It include three things.

- Spatial Frequency
- Magnitude
- Phase

The spatial frequency directly relates with the brightness of the image. The magnitude of the sinusoid directly relates with the contrast. Contrast is the difference between maximum and minimum pixel intensity. Phase contains the color information.

2D DFT

The formula for 2 dimensional discrete Fourier transform is given below.

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

The discrete Fourier transform is actually the sampled Fourier transform, so it contains some samples that denotes an image. In the above formula f(x,y) denotes the image, and F(u,v) denotes the discrete Fourier transform. The formula for 2 dimensional inverse discrete Fourier transform is given below.

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

The inverse discrete Fourier transform converts the Fourier transform back to the image

ALGORITHM FOR FREQUENCY DOMAIN FILTERING

For larger mask (more than 9×9) frequency domain would be consider for image enhancement.

The process of filtering in the frequency domain can be summarized as follows:

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding sizes P and Q using Eqs. (4-102) and (4-103); that is, $P = 2M$ and $Q = 2N$.
2. Form a padded image $f_p(x, y)$ of size $P \times Q$ using zero-, mirror-, or replicate padding (see Fig. 3.39 for a comparison of padding methods).
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center the Fourier transform on the $P \times Q$ frequency rectangle.
4. Compute the DFT, $F(u,v)$, of the image from Step 3.
5. Construct a real, symmetric filter transfer function, $H(u,v)$, of size $P \times Q$ with center at $(P/2, Q/2)$.
6. Form the product $G(u,v) = H(u,v)F(u,v)$ using elementwise multiplication; that is, $G(i,k) = H(i,k)F(i,k)$ for $i = 0, 1, 2, \ldots, M - 1$ and $k = 0, 1, 2, \ldots, N - 1$.
7. Obtain the filtered image (of size $P \times Q$) by computing the IDFT of $G(u,v)$:
$$g_p(x, y) = (real[\text{IDFT}\{G(u, v)\}])(-1)^{x+y}$$
8. Obtain the final filtered result, $g(x, y)$, of the same size as the input image, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

IDEAL LOWPASS FILTERS

A 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius from the origin, and "cuts off" all frequencies outside this, circle is called an *ideal lowpass filter* (ILPF);

it is specified by the transfer function $H(u,v) = \begin{cases} 1 & if\ D(u,v) \leq D_0 \\ 0 & if\ D(u,v) > D_0 \end{cases}$

- where D_0 is a positive constant which is called "cut off frequency", and $D(u,v)$ is the distance between a point (u,v) in the frequency domain and the center of the $P \times Q$ frequency rectangle; so $D(u,v)$ can be expressed as
- $D(u,v) = \sqrt{\left(u - \frac{P}{2}\right)^2 + \left(v - \frac{Q}{2}\right)^2}$; as image size is $M \times N$, in frequency domain, $(u,v) \equiv (P/2, Q/2)$
- It's a smoothing operation.

- Here the effective approach is to use a single filter and apply it radially on all the frequencies of the image.
- Mask can be circular, rectangular or any other customize shape.
- Radial distance $D(u,v)$ need to be calculated.

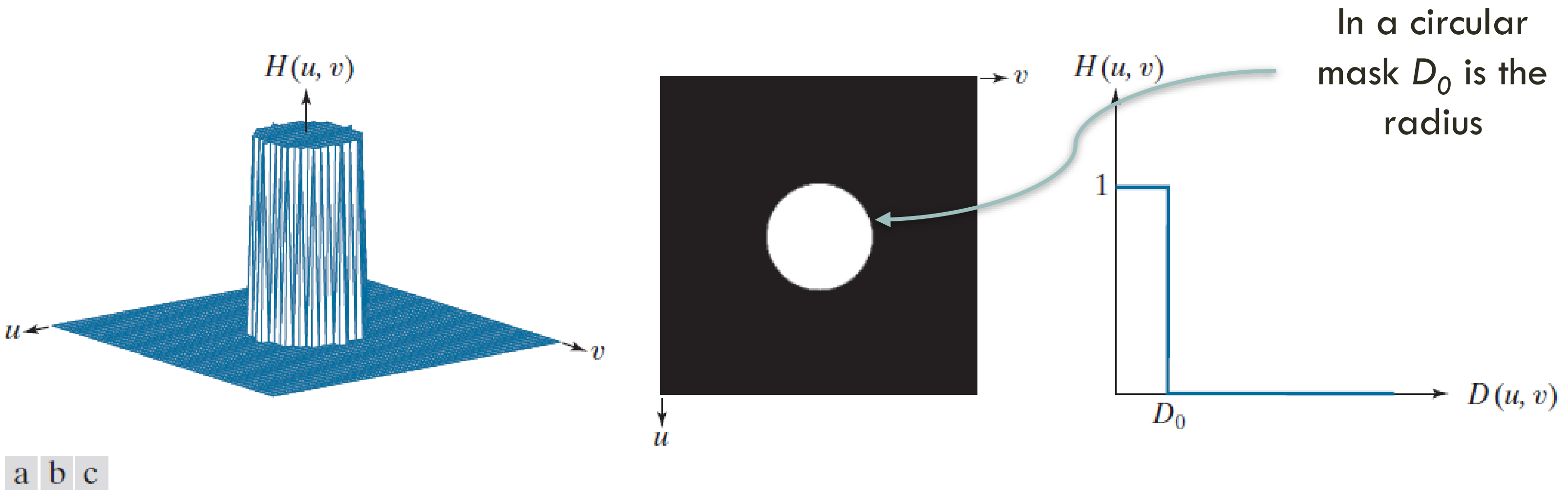


FIGURE 4.39 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross section.

GAUSSIAN LOWPASS FILTERS

GLPF can minimize the ringing effect or Gibb's phenomenon.

These filters do not have any steep profile like ideal filters.

Gaussian lowpass filter (GLPF) transfer functions have the form

$$H(u,v) = e^{-D_2(u,v)/2\sigma^2} \equiv e^{-D_2(u,v)/2D_0^2};$$

- D(u,v) is the distance from the center of the P ×Q frequency rectangle to any point, (u,v), contained by the rectangle.
- σ is the a measure of spread about the center.
- We put $\sigma = D_0 = D(u,v)$, the cut off frequency.
- GLPF transfer function is down to 0.607 of its maximum value of 1.0.

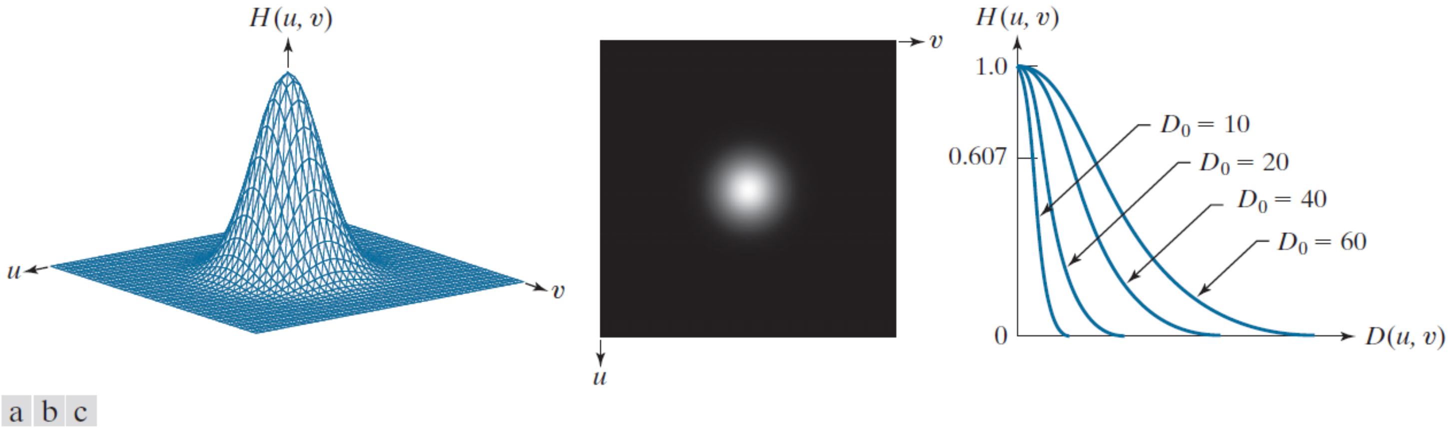


FIGURE 4.43 (a) Perspective plot of a GLPF transfer function. (b) Function displayed as an image. (c) Radial cross sections for various values of D_0.

BUTTERWORTH LOWPASS FILTERS

BLPF shows the lowest or no ringing effect for order 1.

The transfer function of a Butterworth lowpass filter (BLPF) of order *n*, with cutoff frequency at a distance D_0 from the center of the frequency rectangle is defined as:

$$H(u,v) = \frac{1}{1 + \left[D(u,v)/D_0\right]^{2n}}$$

BLPF function can be controlled to approach the characteristics of the ILPF using higher values of *n*, and the GLPF for lower values of *n*, while providing a smooth transition in from low to high frequencies.

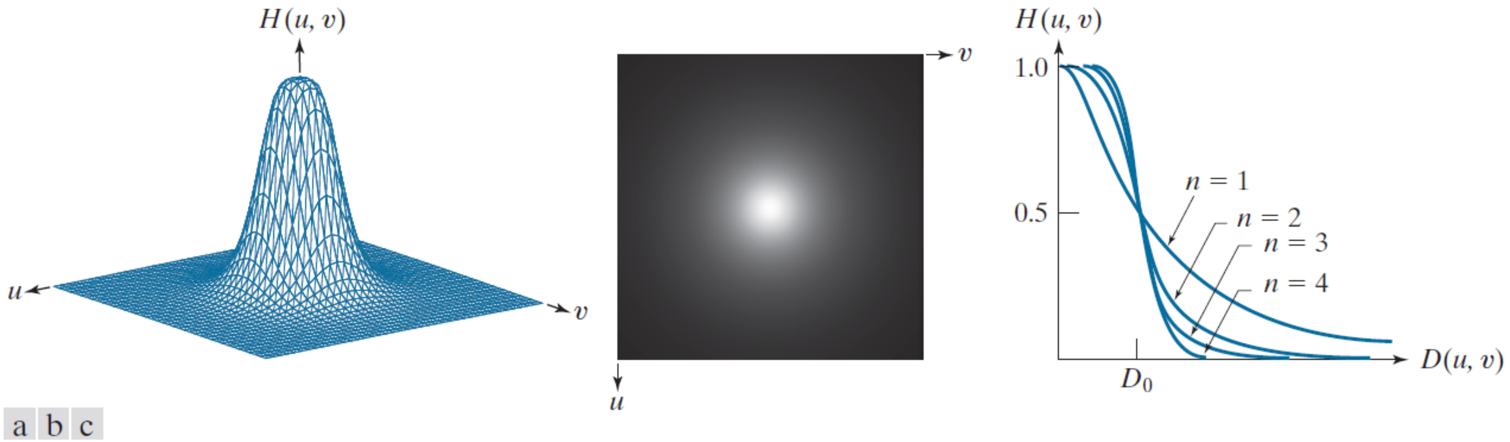


FIGURE 4.45 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross sections of BLPFs of orders 1 through 4.

- H is the mask magnitude ranged from 0 to 1
- Higher the n, filter becomes more sharper, n=0, no ringing effect, i.e. works as ideal filter.

HIGHPASS FILTERS FOR IMAGE SHARPENING

We can design high pass filter from low pass filters by subtracting a lowpass filter transfer function from 1 yields the corresponding highpass filter transfer function in the frequency domain:

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

- Where $H_{lp}(u,v)$ is the transfer function of a lowpass filter, IHPF transfer function can given as:

$$H(u,v) = \begin{cases} 1 \ \ for\ D(u,v) \le D_0 \\ 0 \ \ for D(u,v) > D_0 \end{cases}$$ and $D(u,v)$ is the distance from the center of the $P \times Q$ frequency rectangle.

In this way we can design Ideal high pass filter, Butterworth and Gaussian high pass filters.

These filters attenuates the lower frequency components and allow high frequency components like edges, boundaries and other abrupt changes in any image.

So GHPF transfer function can be given as: $H(u,v) = 1 - e^{-D_2(u,v)/2\sigma^2}$

BHPF transfer function can be given as: $H(u,v) = 1 - \frac{1}{1+\left[{D(u,v)}/{D_0}\right]^{2n}}$

- The order *n* determines the sharpness of the cut-off value and the amount of ringing.
- The higher the *n* value, smother the filter.

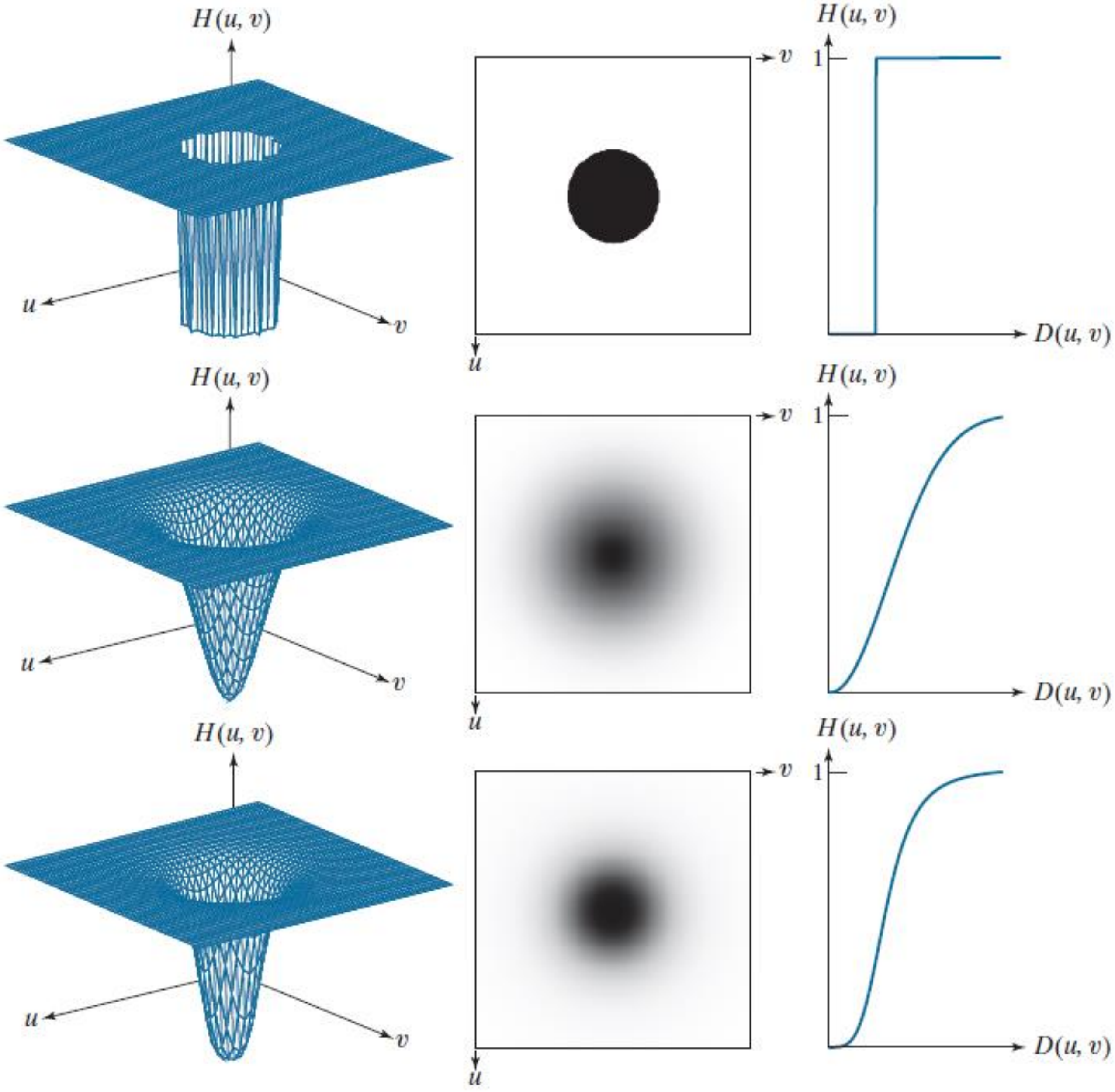


FIGURE 4.51 Top row: Perspective plot, image, and, radial cross section of an IHPF transfer function. Middle and bottom rows: The same sequence for GHPF and BHPF transfer functions. (The thin image borders were added for clarity. They are not part of the data.)

THE LAPLACIAN IN THE FREQUENCY DOMAIN

Laplacian can also be done in frequency domain.

Transfer function can be expressed as: $H(u,v) = -4\pi^2(u^2 + v^2)$

or, with respect to the center of the frequency rectangle, using the transfer function

$$\begin{aligned} H(u,v) &= -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right] \\ &= -4\pi^2 D^2(u,v) \end{aligned} \tag{4-124}$$

where $D(u,v)$ is the distance function defined in Eq. (4-112). Using this transfer function, the Laplacian of an image, $f(x,y)$, is obtained in the familiar manner:

$$\nabla^2 f(x,y) = \mathfrak{I}^{-1}\left[H(u,v)F(u,v)\right] \tag{4-125}$$

where $F(u,v)$ is the DFT of $f(x,y)$. As in Eq. (3-54), enhancement is implemented using the equation

$$g(x,y) = f(x,y) + c\nabla^2 f(x,y) \tag{4-126}$$

UNSHARP MASKING, HIGH-BOOST FILTERING, AND HIGH FREQUENCY-EMPHASIS FILTERING

Using frequency domain methods, the mask defined in Eq. (3-55) is given by

$$g_{\text{mask}}(x,y) = f(x,y) - f_{\text{LP}}(x,y) \tag{4-128}$$

with

$$f_{\text{LP}}(x,y) = \mathfrak{I}^{-1}\left[H_{\text{LP}}(u,v)F(u,v)\right] \tag{4-129}$$

where $H_{\text{LP}}(u,v)$ is a lowpass filter transfer function, and $F(u,v)$ is the DFT of $f(x,y)$. Here, $f_{\text{LP}}(x,y)$ is a smoothed image analogous to $\bar{f}(x,y)$ in Eq. (3-55). Then, as in Eq. (3-56),

$$g(x,y) = f(x,y) + kg_{\text{mask}}(x,y) \tag{4-130}$$

This expression defines *unsharp masking* when $k = 1$ and high-boost filtering when $k > 1$. Using the preceding results, we can express Eq. (4-130) entirely in terms of frequency domain computations involving a lowpass filter:

$$g(x,y) = \mathfrak{I}^{-1}\left\{\left(1 + k\left[1 - H_{\text{LP}}(u,v)\right]\right)F(u,v)\right\} \tag{4-131}$$

CONTINUOUS ……

We can express this result in terms of a highpass filter using Eq. (4-118):

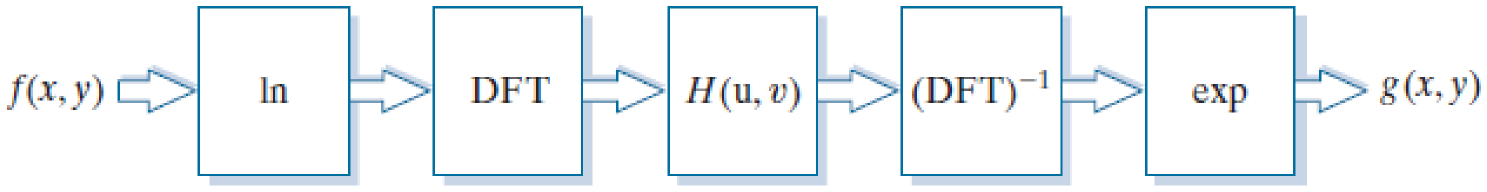
$$g(x,y) = \Im^{-1}\left\{\left[1 + kH_{\mathrm{HP}}(u,v)\right]F(u,v)\right\} \tag{4-132}$$

The expression contained within the square brackets is called a *high-frequency-emphasis filter transfer function*. As noted earlier, highpass filters set the dc term to zero, thus reducing the average intensity in the filtered image to 0. The high-frequency-emphasis filter does not have this problem because of the 1 that is added to the highpass filter transfer function. Constant k gives control over the proportion of high frequencies that influences the final result. A slightly more general formulation of high-frequency-emphasis filtering is the expression

$$g(x,y) = \Im^{-1}\left\{\left[k_1 + k_2H_{\mathrm{HP}}(u,v)\right]F(u,v)\right\} \tag{4-133}$$

HOMOMORPHIC FILTERING

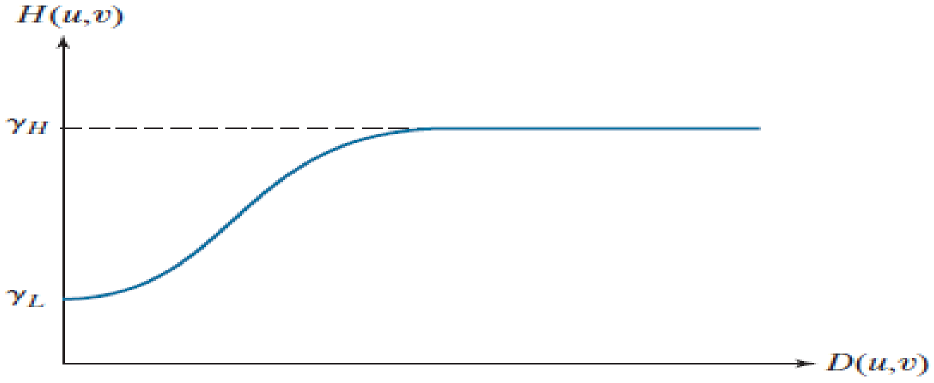
FIGURE 4.58 Summary of steps in homomorphic filtering.



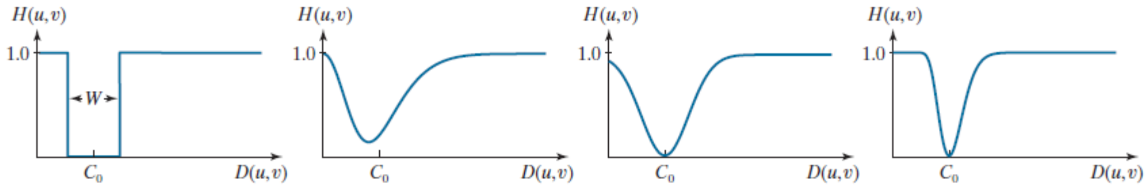
The shape of the function in Fig. 4.59 can be approximated using a highpass filter transfer function. For example, using a slightly modified form of the GHPF function yields the homomorphic function

$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-cD^2(u,v)/D_0^2}\right] + \gamma_L \tag{4-147}$$

where $D(u,v)$ is defined in Eq. (4-112) and constant c controls the sharpness of the slope of the function as it transitions between γ_L and γ_H. This filter transfer function is similar to the high-frequency-emphasis function discussed in the previous section.



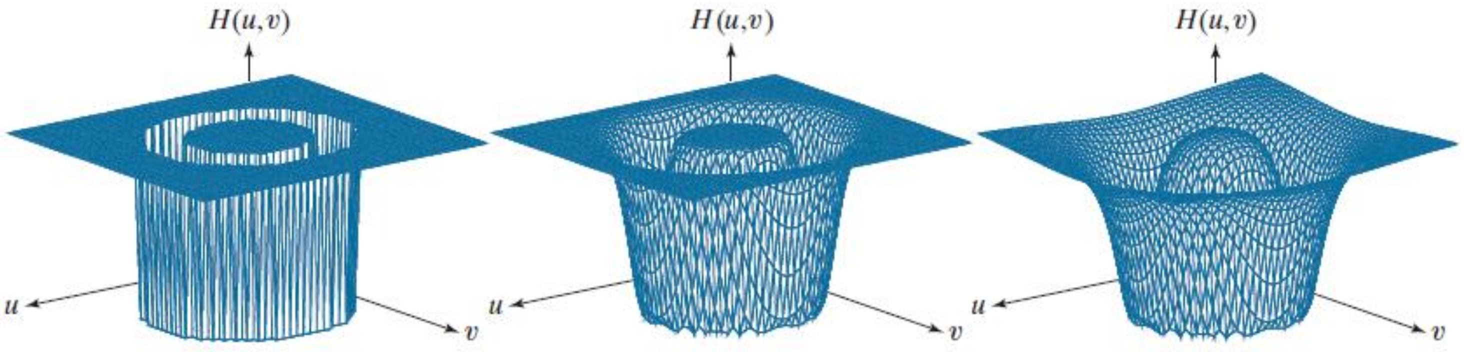
BANDREJECT AND BANDPASS FILTERS



a b c d

FIGURE 4.61 Radial cross sections. (a) Ideal bandreject filter transfer function. (b) Bandreject transfer function formed by the sum of Gaussian lowpass and highpass filter functions. (The minimum is not 0 and does not align with C_0.) (c) Radial plot of Eq. (4-149). (The minimum is 0 and is properly aligned with C_0, but the value at the origin is not 1.) (d) Radial plot of Eq. (4-150); this Gaussian-shape plot meets all the requirements of a bandreject filter transfer function.

Ideal (IBRF)	Gaussian (GBRF)	Butterworth (BBRF)
$H(u,v) = \begin{cases} 0 & \text{if } C_0 - \frac{W}{2} \le D(u,v) \le C_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = 1 - e^{-\left[\frac{D^2(u,v) - C_0^2}{D(u,v)W}\right]^2}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{D(u,v)W}{D^2(u,v) - C_0^2}\right]^{2n}}$



a b c

FIGURE 4.62 Perspective plots of (a) ideal, (b) modified Gaussian, and (c) modified Butterworth (of order 1) bandreject filter transfer functions from Table 4.7. All transfer functions are of size 512×512 elements, with $C_0 = 128$ and $W = 60$.