Even & Odd Signals
Periodic & Non Periodic Signals

IT3105: Signals and Systems

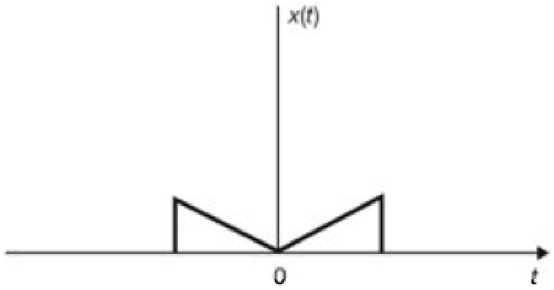
Even & Odd signals

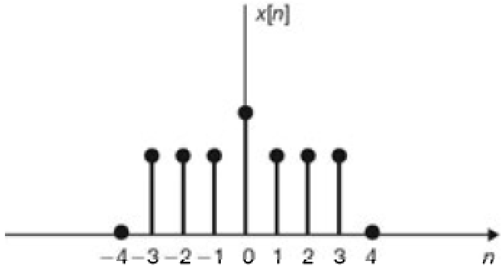
- Even signals remains identical under folding/time reversal/refection operation.

$$x(t) \xrightarrow{T.R} x(-t) = x(t)$$

Example: Prove $x(t) = cos\omega t$ is even.
$x(-t) = \cos(-\omega t) = cos\omega t = x(t)$

we know, $\cos(-\theta) = cos\theta$, so we can say given signal is even as $x(-t) = x(t)$



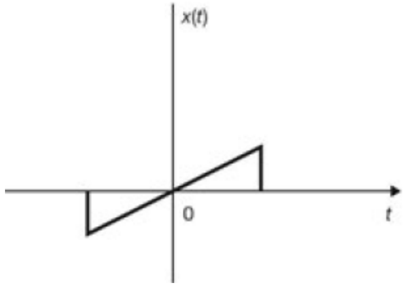


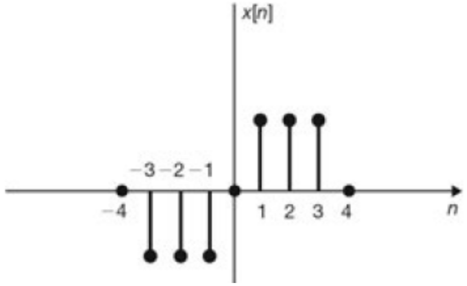
- Odd signals don't remains identical under folding/time reversal/refection operation.

$$x(-t) \neq x(t); x(-t) = -x(t)$$

Example: Prove $x(t) = sin\omega t$ is odd.
$x(-t) = \sin(-\omega t) = -\ sin\omega t$

we know, $\sin(-\theta) = -sin\theta$, so we can say given signal is odd as $x(-t) = -x(t)$





Even and Odd Component of a Signal

- Any continuous time signal can be represented as the sum of even and odd components.
- If $x(t)$ is a CTS with even component, $x_e(t)$ and odd component, $x_o(t)$ then we can express as,

  $x(t) = x_e(t) + x_o(t)$ ------(1)
- Now putting $t = -t$, we get $x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$----(2)
- (1)+(2) we get, $x(t) + x(-t) = 2x_e(t)$, even component $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$
- (1) – (2) we get, $x(t) - x(-t) = 2x_o(t)$, odd component $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$
- Example Problem: Find the even and odd components of the signal $x(t) = e^{-2t}cost$.
- Solution: Replacing t with –t into $x(t)$, we get $x(-t) = e^{-2(-t)} \cos(-t) = e^{2t} \cos t$
- Even component,

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[e^{-2t}cost + e^{2t} \cos t] = \frac{1}{2}[e^{-2t} + e^{2t}]cost = \cosh 2t \cos t$$

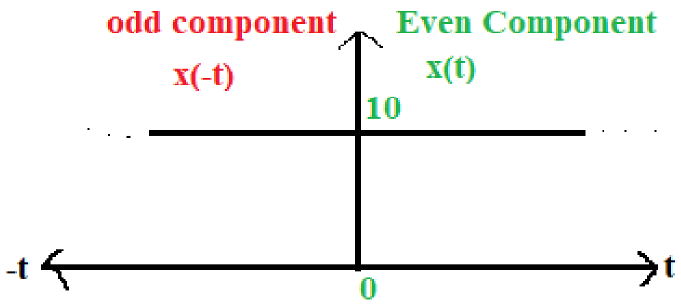
- Odd Component,

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^{-2t}cost - e^{2t} \cos t] = \frac{1}{2}[e^{-2t} - e^{2t}] \cos t = -\sinh(2t) \cos t$$

According to Hyperbolic definitions, $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$

Properties of Even and Odd Signals

- Let us consider any signal $x(t)$=10; where 10 is the dc value.
  - dc value is the magnitude of the signal at t=0;
  - If we apply time reversal on $x(t)$ as $x(t) \xrightarrow{T.R.} x(-t)$
  - For the example signal shown here dc value =10



- **Property 1:** $x(t)$= $x(-t)$ =10, i.e. any CTS signal represented by dc value is even.
- **Property 2:** Odd component of any signal is zero and even component is the dc value, i.e. $x_e(t)$ = dc value and $x_o(t) = 0$.
  - Proof:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[10 + 10] = 10$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[10 - 10] = 0$$

Properties of Even and Odd Signals

- **Property 3:** Summation of dc value and even signal or summation of two even signals are even.
  - Proof by example:
    - Let $x(t) = 10 + t^2$; After T.R. $x(-t) = 10 + (-t^2) = 10 + t^2 = x(t)$
    - t^2 is an even signal, so we can say dc value + even = even/ even +even= even
- **Property 4:** Summation of dc value and odd signal or summation of two odd signals are odd.
  - Proof by example:
    - Let $x(t) = 10 + t^3$; After T.R. $x(-t) = 10 + (-t^3) = 10 - t^3 \neq -x(t) \neq x(t)$; where t^3 is an odd signal. Any signal is odd if $x(-t) = -x(t)$.
    - Sum of dc value and odd signal is neither even nor odd. According to definition any general signal is neither even nor odd. In this way we can proof any general signal can be represented as the sum of even and odd components.

Properties of Even and Odd Signals

- **Property 4:** Multiplication of two even signals are even; E×E=E.
  - Proof by example:
    - Let $x(t) = t^2 \times t^4 = t^6, x(-t) = (-t^6) = t^6 = x(t)$; where t^2 and t^4 are two even signals and after multiplying them we get another even signal t^6
- **Property 5:** Multiplication of two odd signals are even. O×O=E.
  - Proof by example:
    - Let $x(t) = t^3 \times t^5 = t^8, x(-t) = (-t^8) = t^8 = x(t)$; where t^3 and t^5 are two odd signals and after multiplying them we get an even signal t^8
- **Property 6:** Multiplication of an odd and an even signals are odd. O×E=O.
  - Proof by example:
    - Let $x(t) = t^3 \times t^6 = t^9, x(-t) = (-t^9) = -t^9 = -x(t)$; where t^3 is an odd signal and t^6 is an even signal and after multiplying them we get an odd signal $-t^9$.

Properties of Even and Odd Signals

- **Property 7:** Differentiation of an even signal is odd; $\frac{d}{dt}(E) = O$; which is not valid for dc value as, $\frac{d(10)}{dt} = 0$. Zero is neither even nor odd.
- **Property 8:** Differentiation of an odd signal is even; $\frac{d}{dt}(O) = E$.
- **Property 9:** Integration of an even signal is zero; $\int E dt = 0$
- **Property 10:** Integration of an odd signal is even; $\int O dt = E$
- **Property 11:** Inverse of an odd signal is odd; $\frac{1}{O} = O$
- **Property 12:** Inverse of an even signal is even; $\frac{1}{E} = E$

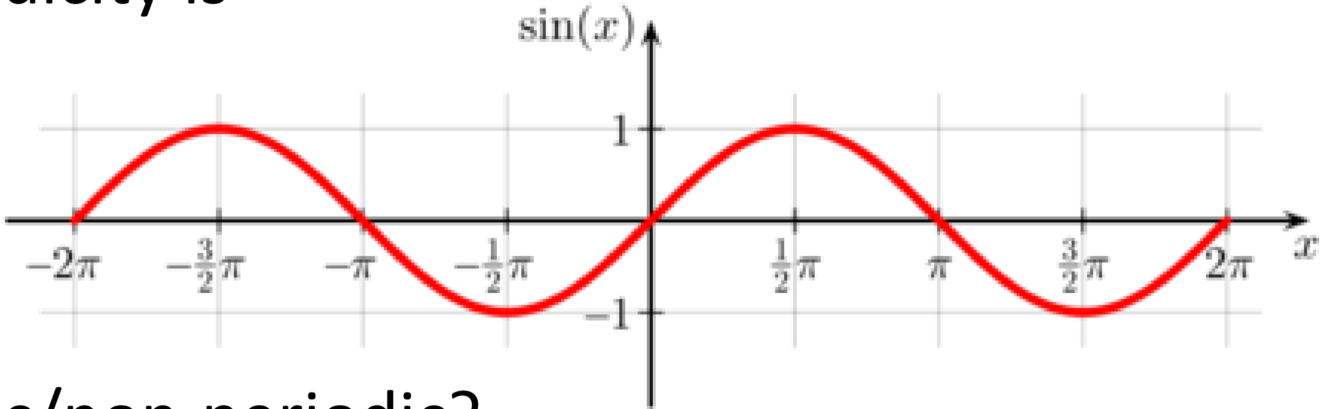
Periodic & Non-periodic/Aperiodic Signals

- A is periodic if it repeats itself after a regular interval of time, i.e. a signal will have the same value after that particular time. Otherwise the signal is non-periodic or aperiodic.
  - For example, let time interval of any signal is Δt=5 sec. According the definition we can write:

    $x(t)=x(t+5)=x(t+10)=\ldots\ldots\ldots\ldots x[t+(n-1)5]=x(t+n5)=x(t+nT_0)$, where T_0 is the fundamental period.
- Finally, for any continuous time signal the condition of periodicity can be expressed as: $x(t) = x(t \pm nT_0)$; where *n* is an integer and T_0 is the fundamental time period (FTP).
  - Definition of FTP: T_0 is the smallest positive value of time for which signal is periodic and it is fixed for that signal.
  - Fundamental period repeats the same value for both increment and decrement of time, so we have put $\pm$ here.

Periodic & Non-periodic/Aperiodic Signals

- Let us consider the basic sine signal as shown in fig.
  - If x=0 then sin(x)=0, if x=π/2, sin(x)=1,
  - If x= π then sin(x)=0 if x=3π/2, sin(x)=-1
  - So the fundamental period is 2π, not π for basic sine wave.
    - $\sin x = \sin(x \pm n2\pi)$ or $T_0 = 2\pi$
  - Fundamental frequency, $f_0 = 1/T_0$ Hz.
  - Fundamental angular frequency $\omega_0 = 2\pi f_0 = 2\pi / T_0$ rad/sec.
- For DTS, the condition of Periodicity is
  - $x[n]=x[n \pm mN]$
  - Where m is an integer
  - N is FTP and also an integer.



- Question: Is dc value is periodic/non-periodic?

Calculation of FTP

- We know the condition of periodicity as $x(t) = x(t \pm nT_0)$;
- Let *n*=1 and T_0 is adding to *t*, then we can write $x(t) = x(t + T_0)$
- Assume, $x(t) = Ae^{j\omega_0 t}$, then we can write $x(t + T_0) = Ae^{j\omega_0(t+T_0)}$

Now $x(t) = x(t + T_0)$

$\Rightarrow Ae^{j\omega_0 t} = Ae^{j\omega_0(t+T_0)}$

$\Rightarrow \cancel{Ae^{j\omega_0 t}} = \cancel{A(e^{j\omega_0 t}}.e^{j\omega_0 T_0})$

$\Rightarrow e^{j\omega_0 T_0} = 1$------(1)

We know by the Euler's equation:

$e^{jx} = \cos x + j\sin x$; putting $x = 2\pi k$ we get, $e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k$; where $\cos 2\pi k$=1 and $\sin 2\pi k = 0$ we get $e^{j\,2\pi k} = 1$ -----(2)

By comparing equation (1) and (2), we get $\omega_0 T_0 = 2\pi k$; $\therefore T_0 = \frac{2\pi}{\omega_0}$; for *k=1*

This the fundamental period of non- composite signals, i.e. signals having single function of *t*.

Properties of FTP

1. FTP is independent of time shifting.

Proof: Let $x_1(t) = sin2\pi t \equiv \sin\omega_0 t,\ \omega_0 = 2\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$ sec.

$$x_2(t) = x_1(t+2) = \sin[2\pi(t+2)] = \sin(2\pi t + 4\pi) \equiv \sin(\omega_0 t + \theta)$$

$\therefore\ \omega_0 = 2\pi$; which is same as $x_1(t)$, i.e. $T_0 = 1$ sec

2. FTP is not independent of time scaling.

Proof: Let $x_3(t) = x_1(2t) = sin4\pi t$, we can write $\omega_0 = 4\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$ sec

So we can say if scaling factor is α, then, for any signal $x(\alpha t)$ the $T_0 = \frac{1}{|\alpha|}$; where $\alpha \neq 0$

3. FTP is independent of time reversal.

Proof: Let $x_4(t) = x_1(-t) = -sin2\pi t;$ where $\omega_0 = 2\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$ sec.

Properties of FTP

4. FTP is independent of phase shifting.

Proof: Let $x_4(t) = x_1(t + 45^0) = \sin(2\pi t + 45^0)$, $\omega_0 = 2\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$ sec

5. FTP is independent of amplitude shifting.

Proof: $x_5(t) = 1 + x_1(t) = 1 + sin2\pi t;\ \omega_0 = 2\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$ sec

6. FTP is independent of amplitude scaling.

Proof: $x_6(t) = 2x_1(t) = 2sin2\pi t;\ \omega_0 = 2\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$ sec

7. FTP is independent of amplitude reversal.

Proof: $x_7(t) = -x_1(t) = -sin2\pi t;\ \omega_0 = 2\pi;\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = 1$ sec

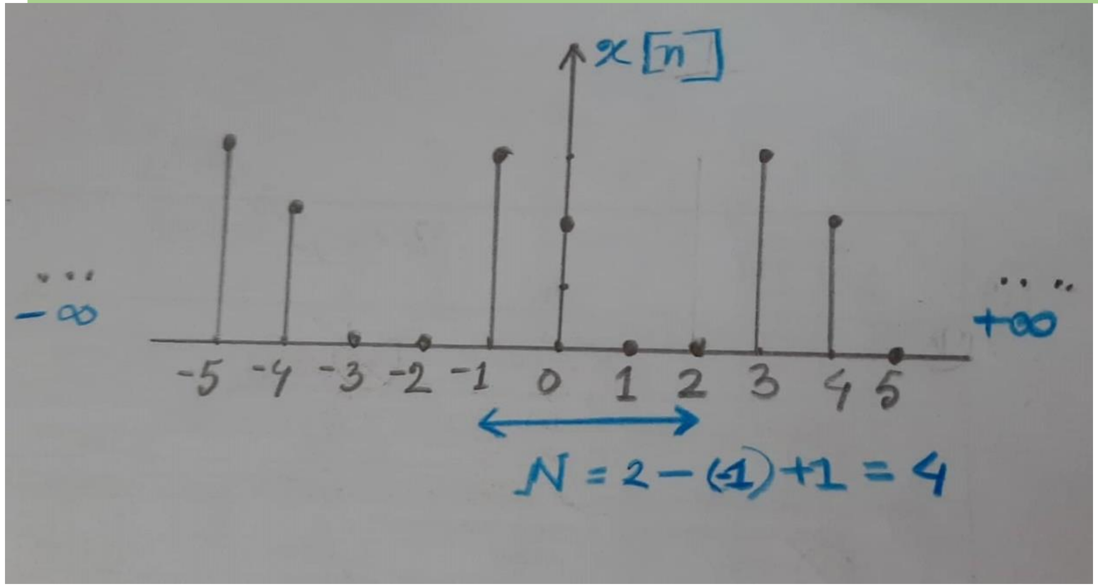
Review: Priority Order

- We do multiple transformation based on time axis. So the priority of operations on time axis is follows:
  1. Time Reversal
  2. Time Shifting
  3. Time Scaling
  4. Amplitude scaling (if present)

Periodic Discrete Signals

- According the definition any discrete time signals is said to be periodic if the signal remains same after performing left or right shifting of that signal using the same parameters such as ‘m’ times; m is an integer.
- If the fundamental time period of a DTS, $x[n]$, is N then the condition for periodicity would be $x[n] = x[n \pm mN]$.
- For DTS we may have multiple FTP, so we denote the minimum value of FTP as N_0
  - For any non-composite signal, if $\frac{2\pi}{\omega_0}$ is rational then we can say the given signal is periodic, otherwise it would be aperiodic.
  - For any composite signal, we have to find separate FTP for every part of the signal then the final FTP would be the LCM of the intermediate FTPs, such as if $x[n] = x_1[n] + x_2[n]$ and N_1 and N_2 are the respective FTPs, then x[n] is periodic if and only if N_1/N_2 is rational, otherwise aperiodic.
  - As both N_1 and N_2 are integers, so N_1/N_2 must be rational and $N_0 = LCM(N_1, N_2)$

Example



- The given signal is periodic but to get the FTP we have to take minimum 4 samples, So the minimum FTP N_0 = 4, i.e. m=4, shifting to left of right should be by 4.

Example

- Find N for $x[n] = e^{j2n}$.
  - For given signal, $\omega_0 = 2$ so $\frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$; which is not rational, so x[n] is no periodic.
- $x[n] = \sin\left[\frac{3\pi}{4}n\right] + \cos\left[\frac{5\pi}{7}n\right]$, N=?
  - Let $x_1[n] = \sin\left[\frac{3\pi}{4}n\right]$ and $x_2[\mathrm{n}] = \cos\left[\frac{5\pi}{7}n\right]$ and respective FTPs are N_1 and N_2 and $\omega_1 = \frac{3\pi}{4}$ and $\omega_2 = \frac{5\pi}{7}$
  - For $x_1[n]$, $\frac{2\pi}{\omega_1} = \frac{2\pi\times4}{3\pi} = \frac{8}{3}$ and $N_1 = \frac{2\pi m}{\omega_1} = \frac{8}{3}m$; minimum value of m=3, and N_1 = 8
  - For $x_2[\mathrm{n}]$, $\frac{2\pi}{\omega_2} = \frac{2\pi\times7}{5\pi} = \frac{14}{5}$ and $N_1 = \frac{2\pi m}{\omega_2} = \frac{14}{5}m$; minimum value of m=5, and N_1 = 14
  - Finally N = LCM (N_1 , N_2)= 56

Example from Haykin

Example 1.6 A pair of sinusoidal signals with a common angular frequency is defined by

$$x_1[n] = \sin[5\pi n]$$

and

$$x_2[n] = \sqrt{3}\cos[5\pi n]$$

(a) Specify the condition which the period N of both $x_1[n]$ and $x_2[n]$ must satisfy for them to be periodic.

(b) Evaluate the amplitude and phase angle of the composite sinusoidal signal

$$y[n] = x_1[n] + x_2[n]$$

In the form $y[n] = A\cos(\omega_0 n + \varphi)$, and find magnitude *A* and phase φ

Problem 1.16 Determine the FTP of the sinusoidal signal $x[n] = 10\cos\left[\frac{4\pi}{31}n + \frac{\pi}{5}\right]$