Basic Operations on Signals

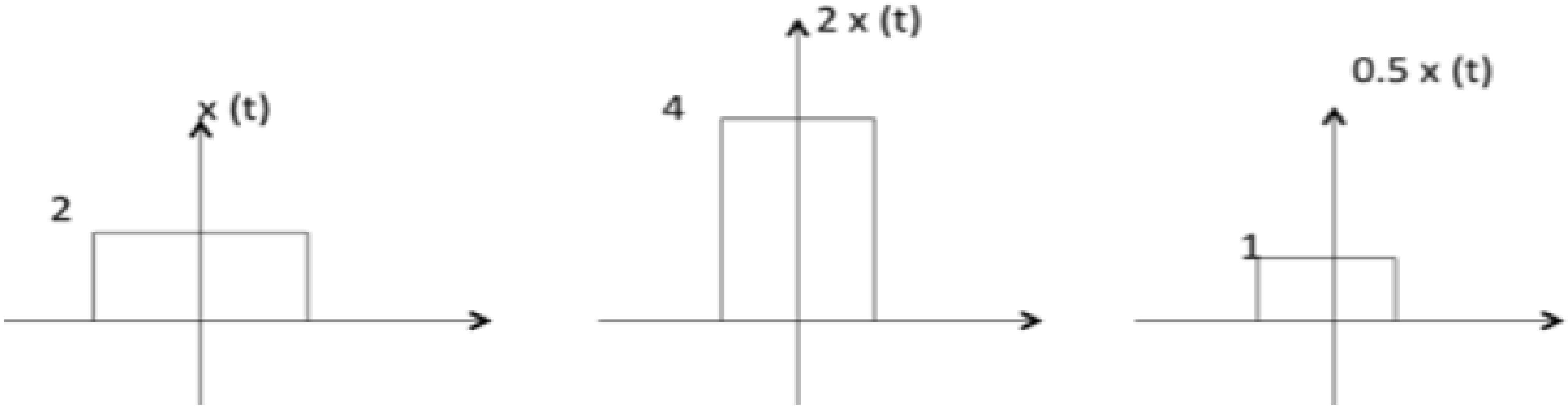
IT 3105 Signals and Systems

Basic Operations on Signals

- There are two basic variable parameters of signals as follows
    1. Amplitude
    2. Time
- Following operations can be performed on Amplitude such as
    - Amplitude scaling
    - Amplitude shifting
    - Addition
    - Subtraction
    - Multiplication
    - Amplitude Reversal
- Following operations can be performed with time such as:
    - Time Shifting
    - Time scaling
    - Time Reversal

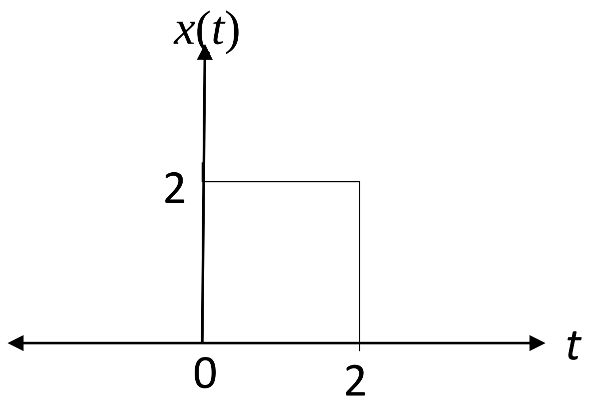
Amplitude Scaling

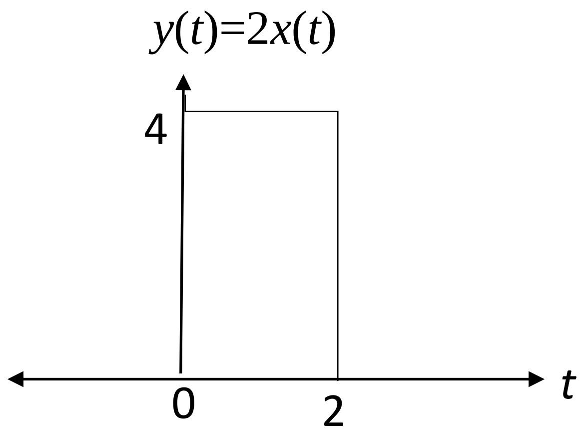
- $Cx(t)$ is a amplitude scaled version of $x(t)$ whose amplitude is scaled by a factor C; $x(t) \xrightarrow{Amp.Scaling} y(t) = Cx(t)$

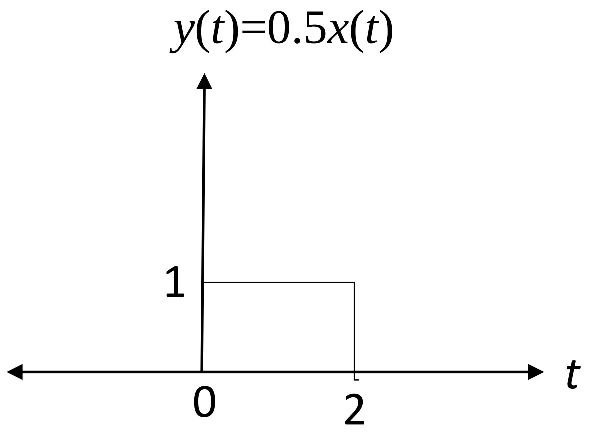


Amplitude Scaling

- **Case I**: Where $|C| > 1$, then the scaling is called amplification.
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \leq t \leq 2 \\ 0; & t > 2 \end{cases}$; For C=2, $y(t) = 2x(t) = \begin{cases} 0; & t < 0 \\ 4; & 0 \leq t \leq 2 \\ 0; & t > 2 \end{cases}$
- **Case II:** Where $|C| < 1$, then the scaling is called reduction
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \leq t \leq 2 \\ 0; & t > 2 \end{cases}$; For C=0.5, $y(t) = 2x(t) = \begin{cases} 0; & t < 0 \\ 1; & 0 \leq t \leq 2 \\ 0; & t > 2 \end{cases}$

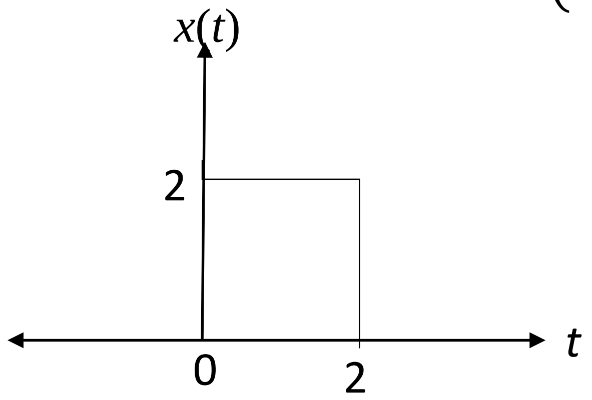


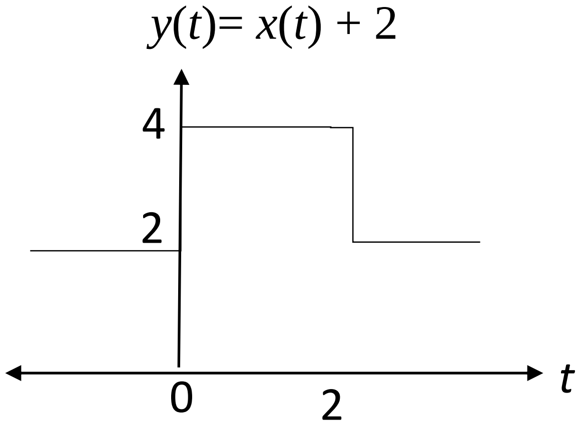


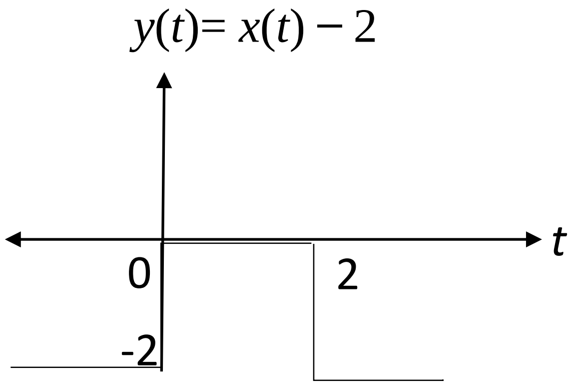


Amplitude Shifting

- ***Case I:*** When k >1; the amplitude of a signal is shifted upward.
  - $x(t) \xrightarrow{Amp.Shifting} y(t) = x(t) + k$
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases}$; For k=2, $y(t) = x(t) + 2 = \begin{cases} 2; & t < 0 \\ 4; & 0 \le t \le 2 \\ 2; & t > 2 \end{cases}$
- ***Case II:*** When k < 1; the amplitude of a signal is shifted downward.
  - $x(t) \xrightarrow{Amp.Shifting} y(t) = x(t) - k$
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases}$; For k=2, $y(t) = x(t) - 2 = \begin{cases} -2; & t < 0 \\ 0; & 0 \le t \le 2 \\ -2; & t > 2 \end{cases}$

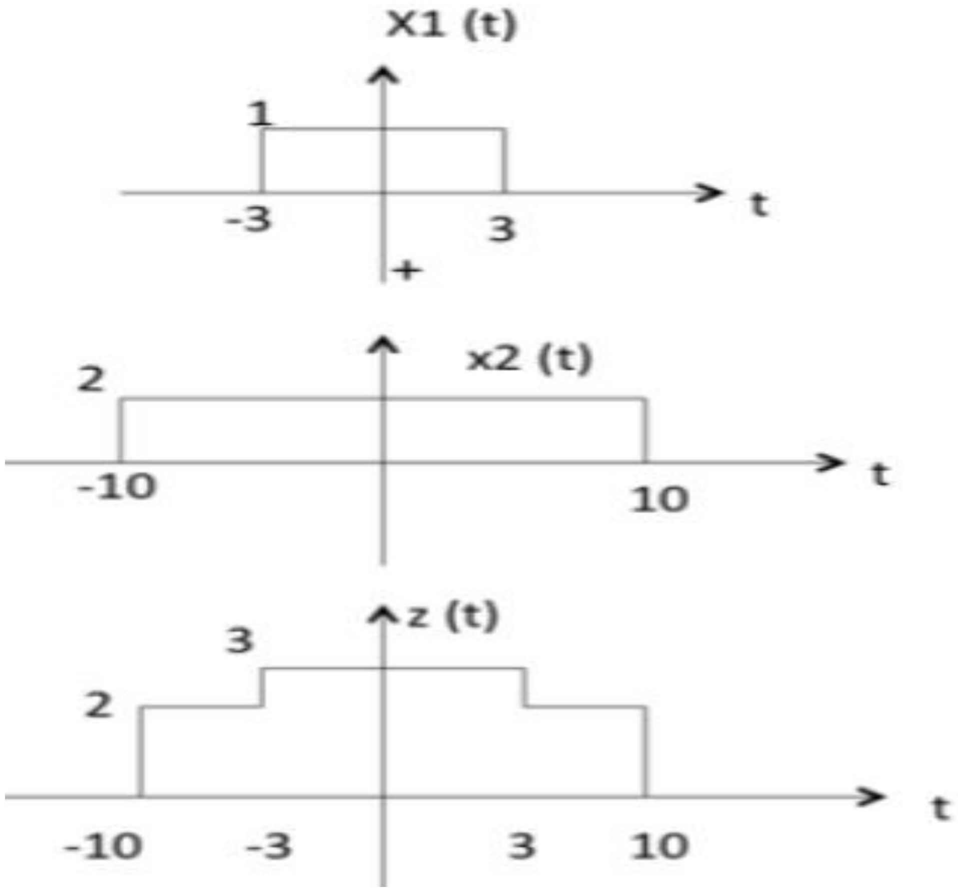






Addition

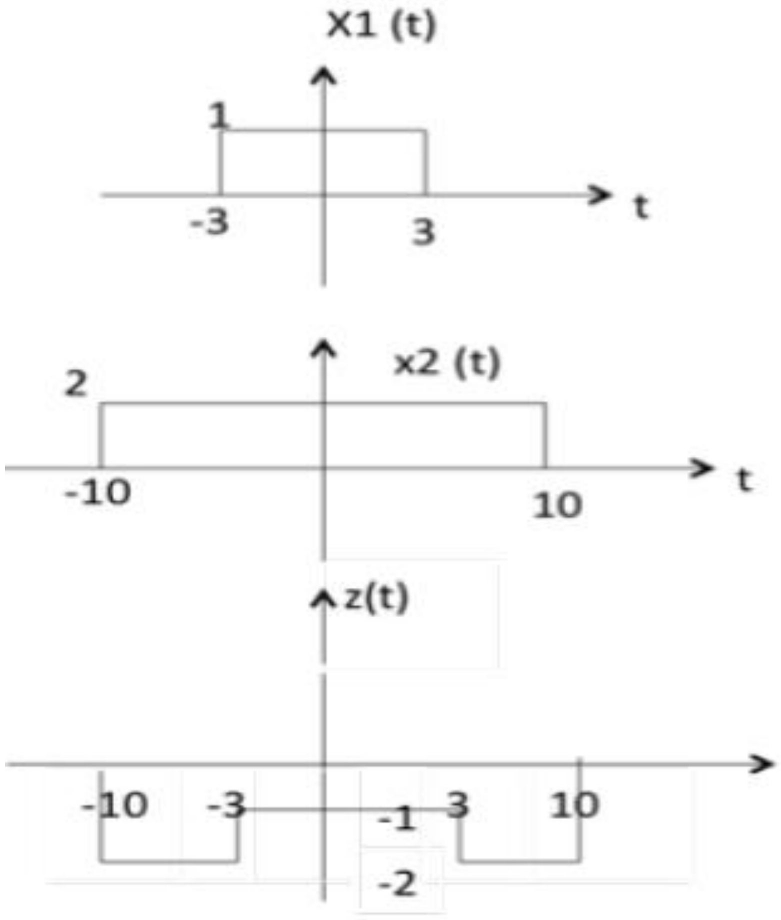
- Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:



- As seen from the diagram above,
  - -10 < t < -3 amplitude of *z* = *x*1*(t)* + *x*2*(t)* = 0 + 2 = 2
  - -3 < t < 3 amplitude of *z* = *x*1*(t)* + *x*2*(t)* = 1 + 2 = 3
  - 3 < t < 10 amplitude of *z* = *x*1*(t)* + *x*2*(t)* = 0 + 2 = 2]

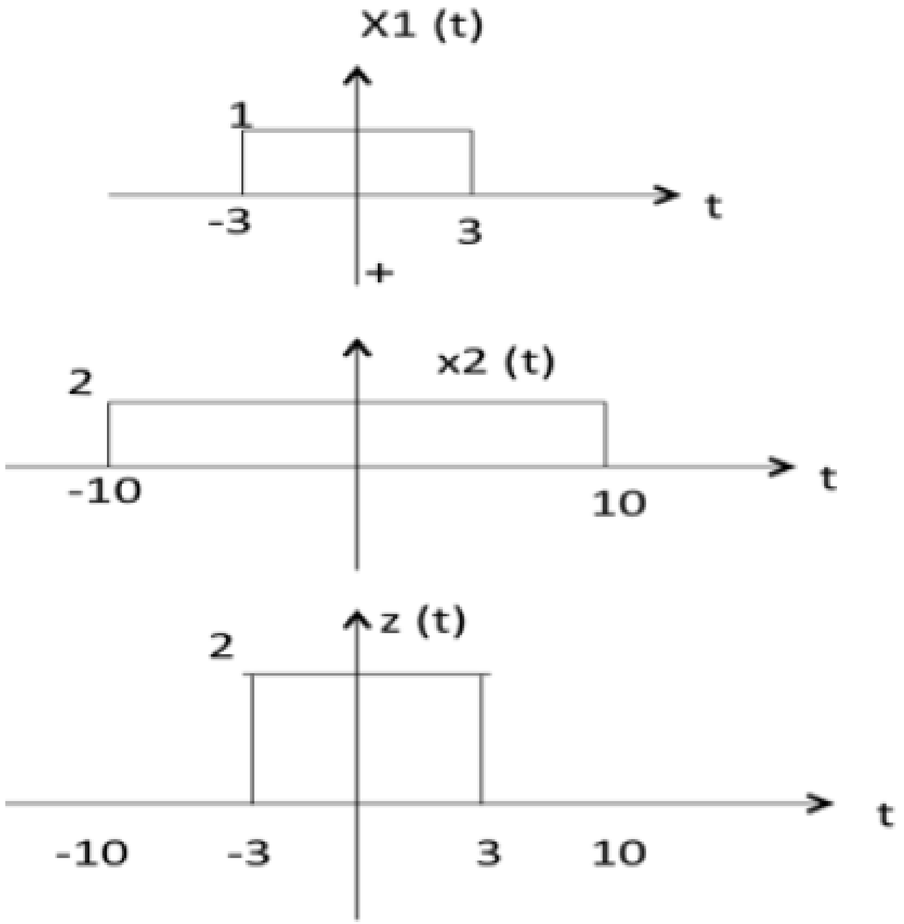
Subtraction

- Subtraction of two signals is nothing but subtraction of their corresponding amplitudes. This can be best explained by the following example:
- As seen from the diagram above,
  - -10 < t < -3 amplitude of $z = x1(t) - x2(t) = 0 - 2 = -2$
  - -3 < t < 3 amplitude of $z = x1(t) - x2(t) = 1 - 2 = -1$
  - 3 < t < 10 amplitude of $z = x1(t) - x2(t) = 0 - 2 = -2$



Multiplication

- Multiplication of two signals is nothing but multiplication of their corresponding amplitudes. This can be best explained by the following example:
- As seen from the diagram above,
  - -10 < t < -3 amplitude of $z = x1(t) \times x2(t) = 0 \times 2 = 0$
  - -3 < t < 3 amplitude of $z = x1(t) \times x2(t) = 1 \times 2 = 2$
  - 3 < t < 10 amplitude of $z = x1(t) \times x2(t) = 0 \times 2 = 0$



Example Problem

- Using the discrete-time signals $x_1[n]$ and $x_2[n]$ shown in Fig. (i), represent each of the following signals by a graph and by a sequence of numbers.

$$\text{(a) } y_1[n] = x_1[n] + x_2[n]; \quad \text{(b) } y_2[n] = 2x_1[n]; \text{ (c) } y_3[n] = x_1[n] * x_2[n]$$

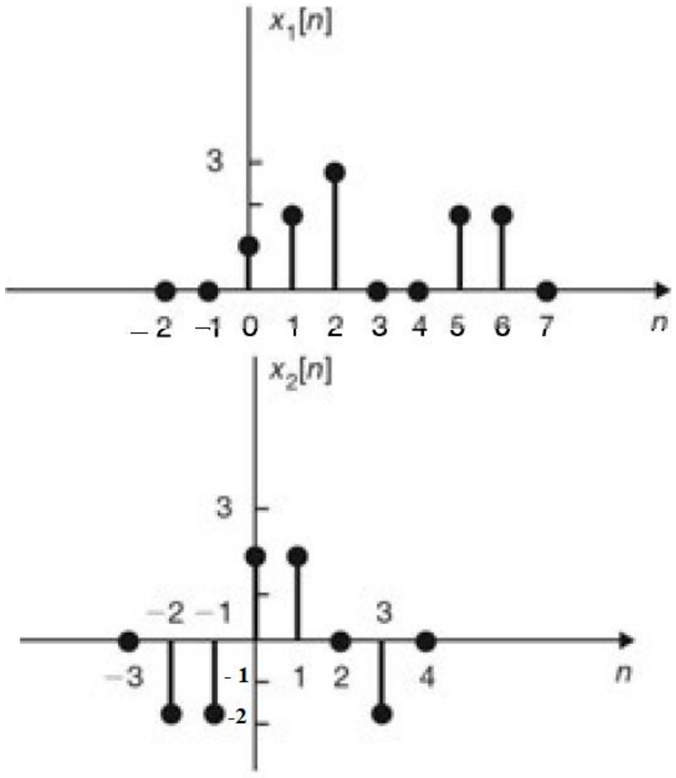


Fig. (i)

(a) $y_1[n] = x_1[n] + x_2[n]$

$y_1[-3] = 0+0=0$

$y_1[-2] = 0+(-2)= -2$

$y_1[-1] = 0+(-2)= -2$

$y_1[0] = 1+2= 3$

$y_1[1] = 2+2= 4$

$y_1[2] = 3+0= 3$

$y_1[3] = 0+(-2)= -2$

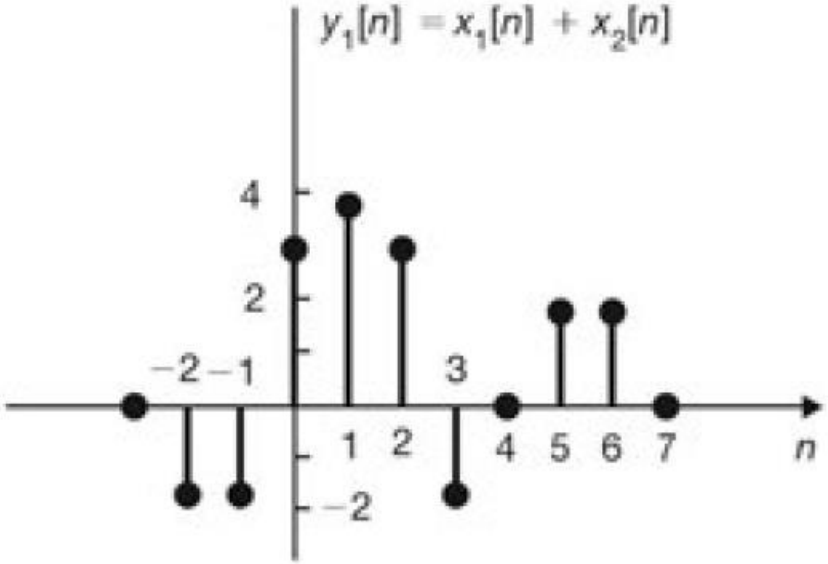
$y_1[4] = 0+0= 0$

$y_1[5] = 2+0= 2$

$y_1[6] = 2+0= 2$

$y_1[7] = 0+0= 0$

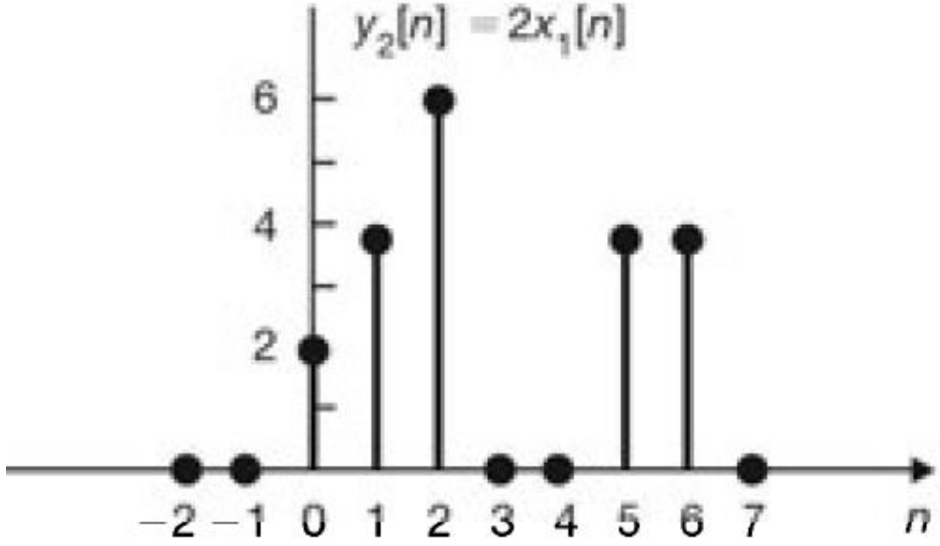
$$y_1[n] = \{\ldots, 0, -2, -2, \underset{\uparrow}{3}, 4, 3, -2, 0, 2, 2, 0, \ldots\}$$



Example Problem

$$x_1[n] = \{\ldots\ldots, 0, 0, \underset{\uparrow}{\underline{1}}, 2,3,0,0,2,2,0, \ldots\ldots.\}$$

(b) $y_2[n] = 2x_1[n] = \{\ldots\ldots, 0, 0, \underset{\uparrow}{\underline{2}}, 4,6,0,0,4,4,0, \ldots\ldots.\}$



(c) $y_3[n] = x_1[n] * x_2[n]$

$y_3[-3] = 0*0=0$

$y_3[-2] = 0*(-2)= 0$

$y_3[-1] = 0*(-2)= 0$

$y_3[0] = 1*2= 2$

$y_3[1] = 2*2= 4$

$y_3[2] = 3*0= 0$

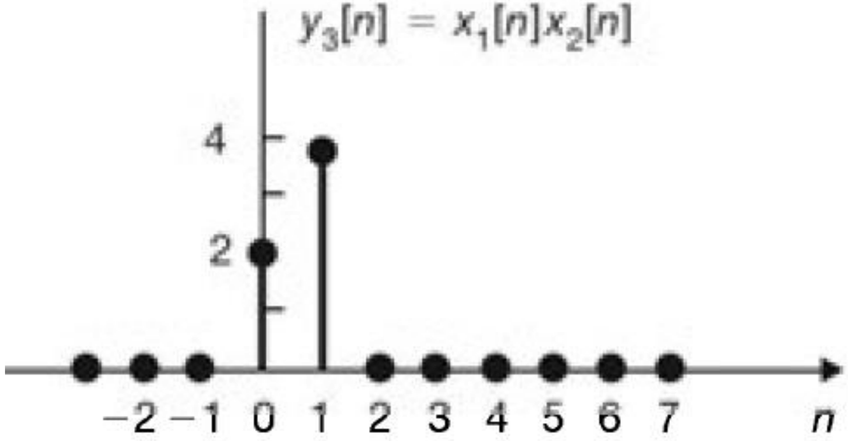
$y_3[3] = 0*(-2)= 0$

$y_3[4] = 0*0= 0$

$y_3[5] = 2*0= 0$

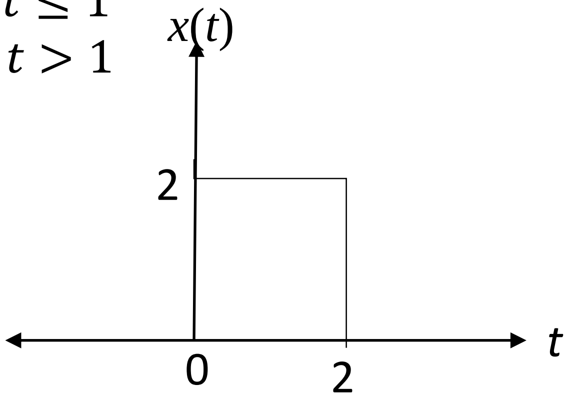
$y_3[6] = 2*0= 0$

$y_3[7] = 0*0= 0$



Time Scaling

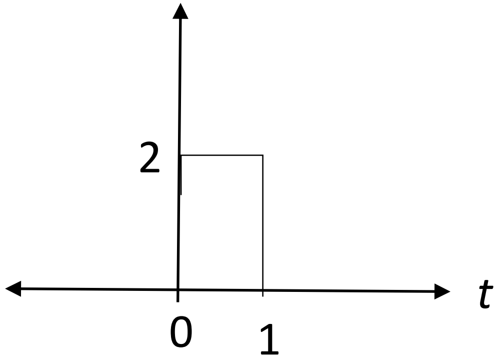
- Time scaling is an operation where there is either time compression of time expansion. Time scaling can be expressed as $x(t) \xrightarrow{Time\ Scaling} y(t) = x(at); a \neq 0$
- **Case I**: Where $|a| > 1; a \in (-\infty, -1) \cup (1, \infty)$ i.e. a is an integer. This is the case of compression.
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \leq t \leq 2 \\ 0; & t > 2 \end{cases}$; For a=2, $y(t) = x(2t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \leq t \leq 1 \\ 0; & t > 1 \end{cases}$
  - t= 0, x(0) = 0 t= 0; y(0)= x(2*0)= x(0) = 0
  - t = 2, x(2) = 2; t = 1; y(1)= x(2*1)= x(2) = 2



- Rules of time scaling is as follows:
  i. Amplitude remains same
  ii. Time, t is divided by the scaling factor, a. For example, if a=2 then $x\left(\frac{at}{a}\right)$;

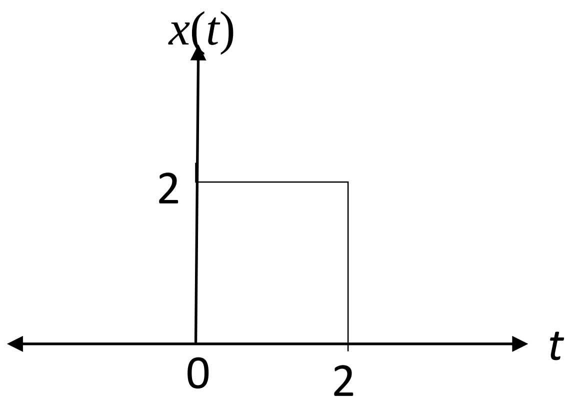
  i.e. $\frac{0}{2} \leq t \leq \frac{2}{2}; 0 \leq t \leq 1$

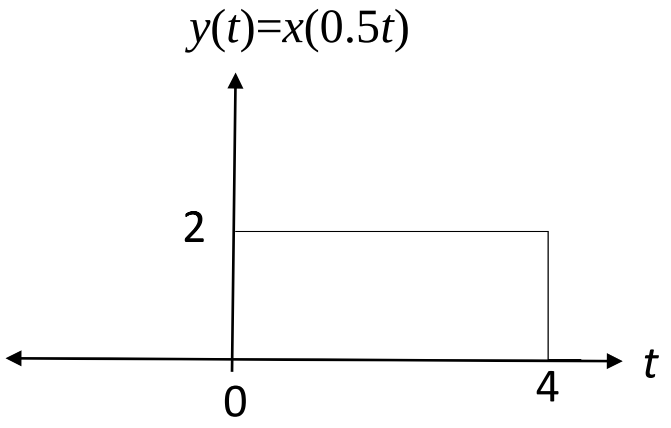
y(t)=x(2t)



Time Scaling

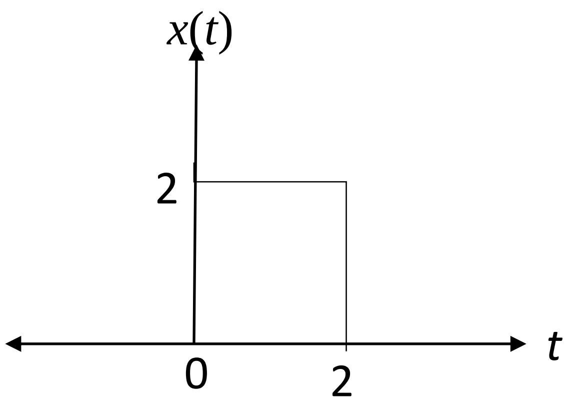
- Case II: Where $|a| < 1; a \in (-1, 0) \cup (0, -1)$ i.e. a is not an integer. This is the case of expansion.
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases}$; For a=0.5, $y(t) = x(0.5t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 4 \\ 0; & t > 4 \end{cases}$
  - If a = 0.5, $y(t) = x(0.5t)$ => $x\left(\frac{0.5t}{0.5}\right)$; i.e. $\frac{0}{0.5} \le t \le \frac{2}{0.5}; 0 \le t \le 4$

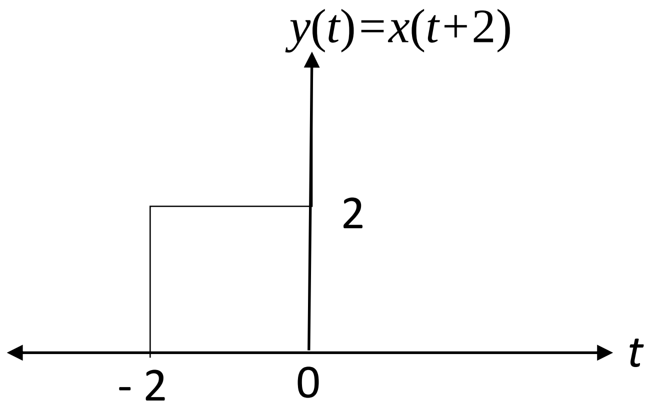




Time Shifting

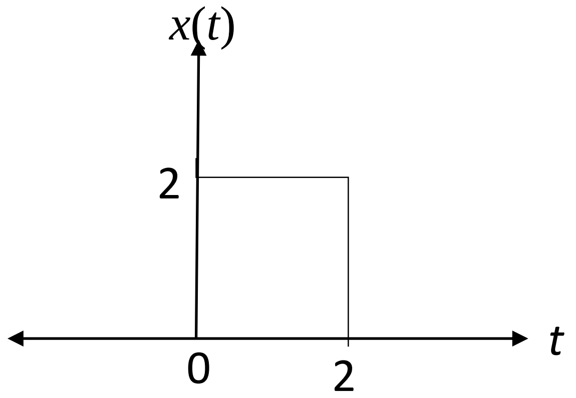
- Time shifting can be expressed for any signal $x(t)$ with a constant (calculate in sec) m as: $x(t) \xrightarrow{\text{Time Shifting}} y(t) = x(t \pm m)$
- **Case I**: Where $m > 0$; i.e. m is positive. This is called left shifting or advanced in time.
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases}$; For m=2, $y(t) = x(t+m) = \begin{cases} 0; & t < -2 \\ 2; & -2 \le t \le 0 \\ 0; & t > 0 \end{cases}$
  - $t=0$ then $x(0) = 2$
  - $t = 2$ then $x(2) = 2$
  - $t = -2$ then $y(-2) = x(-2+2) = x(0) = 2$
  - $t=0$ then $y(0) = x(0+2) = x(2) = 2$
  - $t = 1$ then $y(1) = x(1+2) = x(3) = 0$

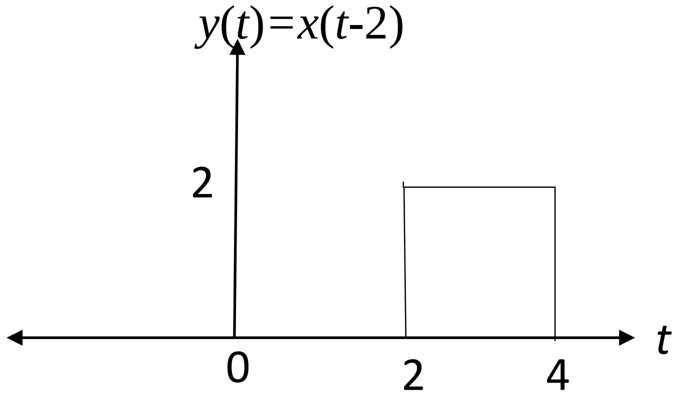




Time Shifting

- Case II: Where $m < 0$; i.e. m is negative. This is called right shifting or delay in time.
  - Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \leq t \leq 2 \\ 0; & t > 2 \end{cases}$; For $m = -2$, $y(t) = x(t-m) = \begin{cases} 0; & t < 2 \\ 2; & 2 \leq t \leq 4 \\ 0; & t > 4 \end{cases}$
  - $t=0$ then $x(0) = 2$
  - $t = 2$ then $x(2) = 2$
  - $t = -2$ then $y(-2) = x(-2-2) = x(-4) = 0$
  - $t=0$ then $y(0) = x(0 - 2) = x(-2) = 0$
  - $t = 1$ then $y(1) = x(1 - 2) = x(-1) = 0$
  - $t = 2$ then $y(2) = x(2 - 2) = x(0) = 2$
  - $t = 3$ then $y(3) = x(3 - 2) = x(1) = 2$
  - $t = 4$ then $y(4) = x(4 - 2) = x(2) = 2$
  - $t = 5$ then $y(5) = x(5 - 2) = x(3) = 0$





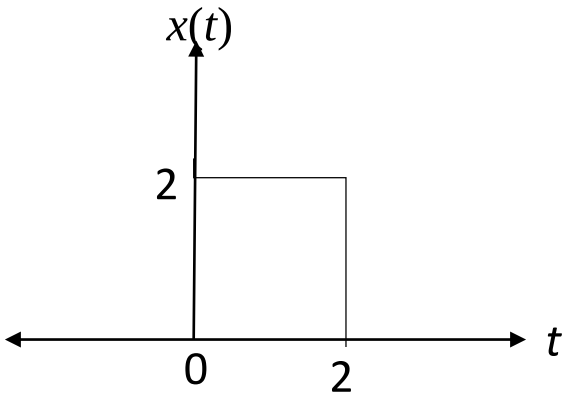
Reversal of Time

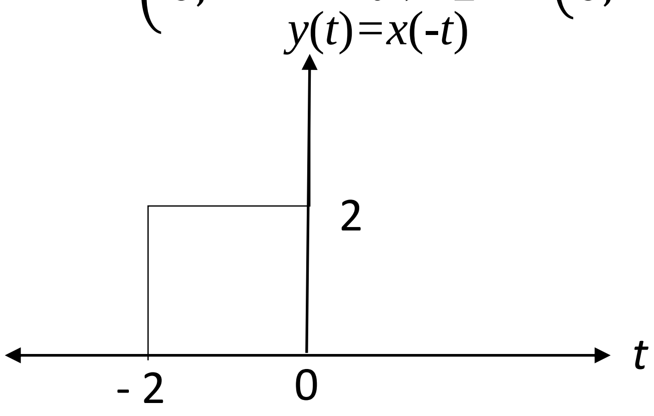
- For any signal, there are two types of reversal
    i. Time reversal
    ii. Amplitude reversal

- Time reversal is a special case time scaling with $a = -1$; It is also called refection/folding, where we get the mirror image of the actual signal.

$$x(t) \xrightarrow{T.R.} y(t) = x(at) = x(-t) \Rightarrow x(t) \xrightarrow{T.R.} x(-t)$$

- Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases} \xrightarrow{\text{T.R}} x(-\mathrm{t}) = \begin{cases} 0; & -t < 0 \\ 2; & 0 \le -t \le 2 \\ 0; & -t > 2 \end{cases} = \begin{cases} 0; & t > 0 \\ 2; & -2 \le t \le 0 \\ 0; & t < -2 \end{cases}$



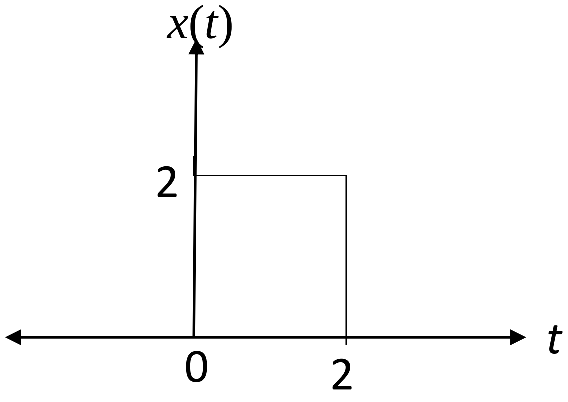


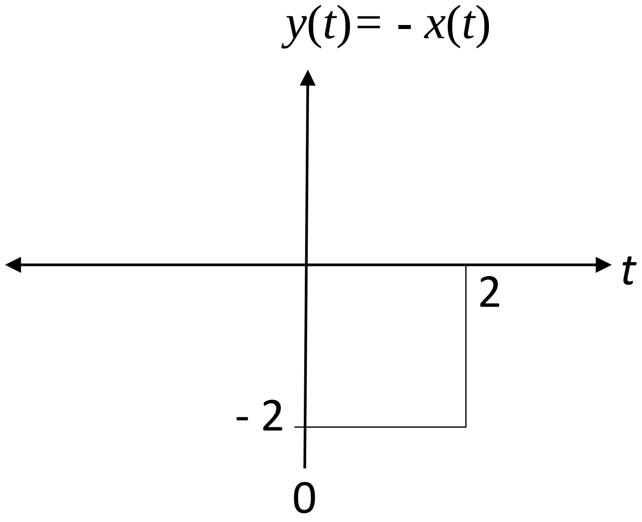
Reversal of Amplitude

- It is a special case of amplitude scaling with C = -1

$$x(t) \xrightarrow{A.R.} y(t) = x(Ct) = Cx(t) = -x(t)$$

- Given signal $x(t) = \begin{cases} 0; & t < 0 \\ 2; & 0 \le t \le 2 \\ 0; & t > 2 \end{cases} \xrightarrow{\text{A.R}} -x(\mathrm{t}) = \begin{cases} 0; & t < 0 \\ -2; & 0 \le -t \le 2 \\ 0; & t > 2 \end{cases}$





Reflection

Example 1.2 Consider the triangular pulse $x(t)$ shown in Fig. 1.21(a). Find the reflected version of $x(t)$ about the amplitude axis.

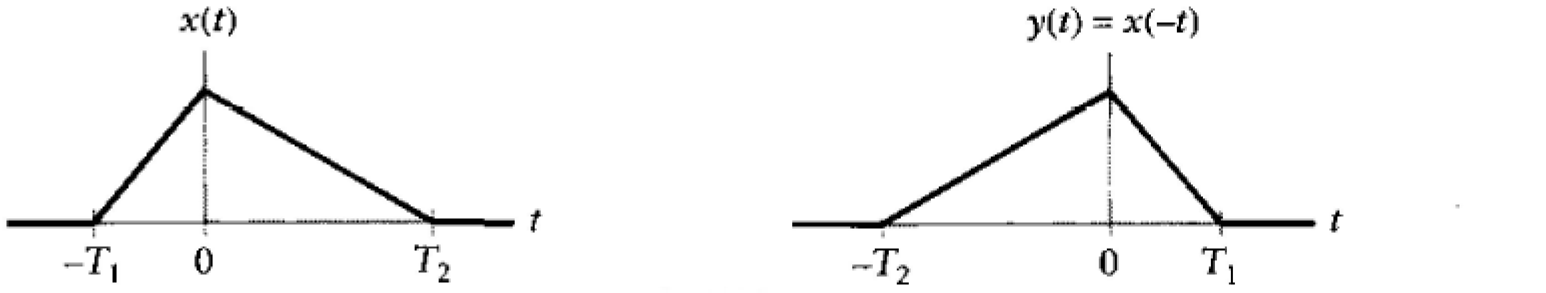
Solution: Replacing the independent variable t in $x(t)$ with $-t$, we get the result $y(t) = x(-t)$ shown in Fig. 1.21(b).

Note that for this example, we have

$$x(t) = 0 \quad \text{for } t < -T_1 \text{ and } t > T_2$$

Correspondingly, we find that

$$y(t) = 0 \quad \text{for } t > T_1 \text{ and } t < -T_2$$



(a) continuous-time signal $x(t)$

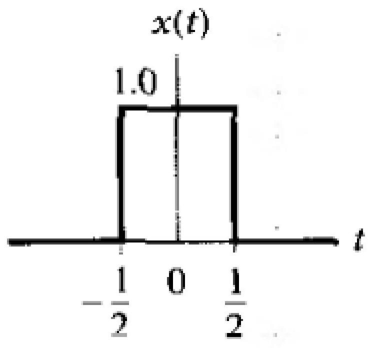
(b) reflected version of $x(t)$ about the origin

Figure 1.21 Operation of reflection

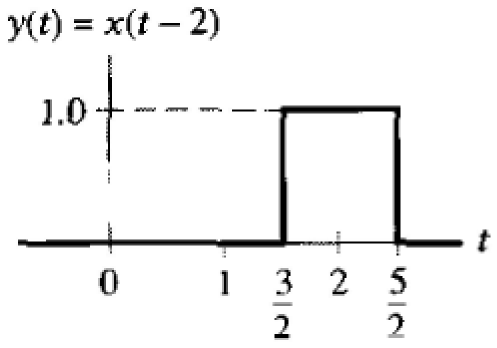
Time Shifting (cont…):

EXAMPLE 1.3 Figure 1.22(a) shows a rectangular pulse $x(t)$ of unit amplitude and unit duration. Find $y(t) = x(t - 2)$.

Solution: In this example, the time shift t_0 equals 2 time units. Hence, by shifting $x(t)$ to the right by 2 time units we get the rectangular pulse $y(t)$ shown in Fig. 1.22(b). The pulse $y(t)$ has exactly the same shape as the original pulse $x(t)$; it is merely shifted along the time axis.



(a) continuous-time signal in the form of a rectangular pulse of amplitude 1.0 and duration 1.0 symmetric about the origin;



(b) time-shifted version of $x(t)$ by 2 time units.

FIGURE 1.22 Time-shifting operation:

Precedence Rule for Time Shifting & Time Scaling

Let $y(t)$ denote a continuous-time signal that is derived from another continuous-time signal $x(t)$ through a combination of time shifting and time scaling, as described here:

$$y(t) = x(at - b) \qquad (1.25)$$

This relation between $y(t)$ and $x(t)$ satisfies the following conditions:

$$y(0) = x(-b) \qquad (1.26)$$

$$y\left(\frac{b}{a}\right) = x(0) \qquad (1.27)$$

which provide useful checks on $y(t)$ in terms of corresponding values of $x(t)$.

To correctly obtain $y(t)$ from $x(t)$, the time-shifting and time-scaling operations must be performed in the correct order.

Precedence Rule for Time Shifting & Time Scaling

The proper order is based on the fact that the scaling operation always replaces t by at, while the time-shifting operation always replaces t by $t - b$.

Hence the time-shifting operation is performed first on $x(t)$, resulting in an intermediate signal $v(t)$ defined by

$$v(t) = x(t - b)$$

The time shift has replaced t in $x(t)$ by $t - b$. Next, the time-scaling operation is performed on $v(t)$. This replaces t by at, resulting in the desired output

$$\begin{aligned} y(t) &= v(at) \\ &= x(at - b) \end{aligned}$$

To illustrate how the operation described in Eq. (1.25) can arise in a real-life situation, consider a voice signal recorded on a tape recorder. If the tape is played back at a rate faster than the original recording rate, we get compression (i.e., $a > 1$). If, on the other hand, the tape is played back at a rate slower than the original recording rate, we get expansion (i.e., $a < 1$). The constant b, assumed to be positive, accounts for a delay in playing back the tape.

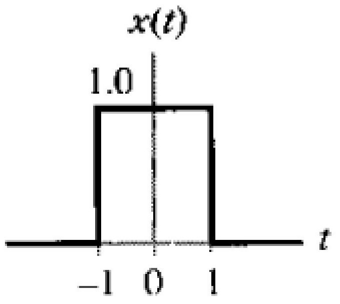
Precedence Rule for Time Shifting & Time Scaling

Example: Precedence rule for continuous-time signal

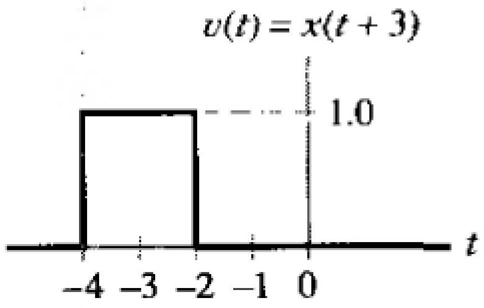
EXAMPLE 1.4 Consider the rectangular pulse $x(t)$ of unit amplitude and duration of 2 time units depicted in Fig. 1.23(a). Find $y(t) = x(2t + 3)$.

Solution: In this example, we have $a = 2$ and $b = -3$. Hence shifting the given pulse $x(t)$ to the left by 3 time units relative to the time axis gives the intermediate pulse $v(t)$ shown in Fig. 1.23(b). Finally, scaling the independent variable t in $v(t)$ by $a = 2$, we get the solution $y(t)$ shown in Fig. 1.23(c).

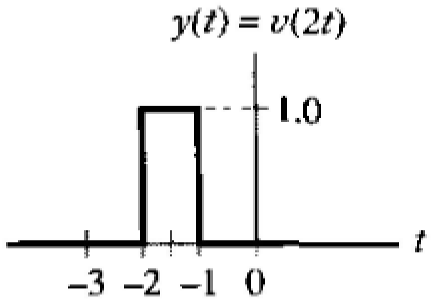
Note that the solution presented in Fig. 1.23(c) satisfies both of the conditions defined in Eqs. (1.26) and (1.27).



(a) Rectangular pulse $x(t)$ of amplitude 1.0 and duration 2.0, symmetric about the origin.



(b) Intermediate pulse $v(t)$, representing time-shifted version of $x(t)$.



(c) Desired signal $y(t)$, resulting from the compression of $v(t)$ by a factor of 2.

FIGURE 1.23 The proper order in which the operations of time scaling and time shifting should be applied for the case of a continuous-time signal.

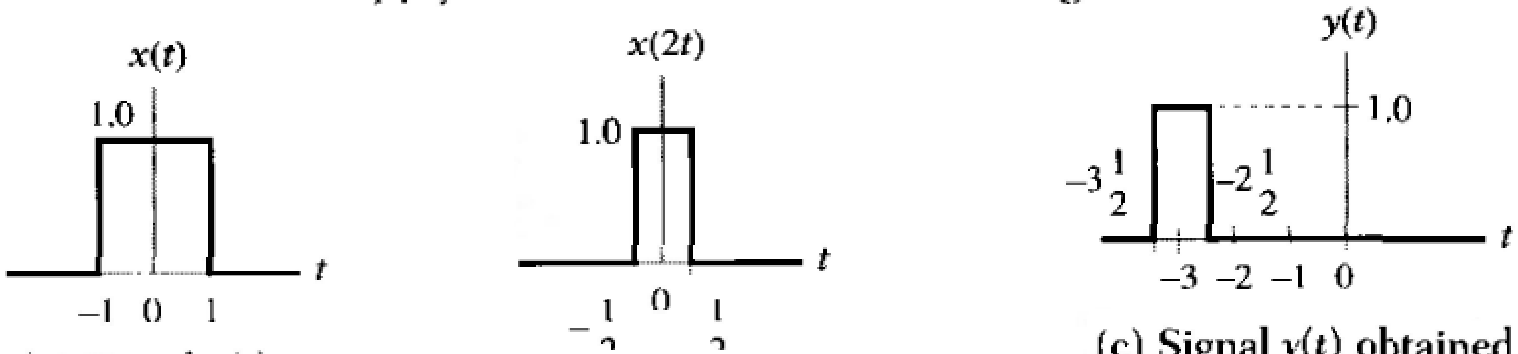
Precedence Rule for Time Shifting & Time Scaling

Example: Precedence rule for continuous-time signal

Suppose next that we purposely do not follow the precedence rule; that is, we first apply time scaling, followed by time shifting. For the given signal $x(t)$, shown in Fig. 1.24(a), the waveforms resulting from the application of these two operations are shown in Figs. 1.24(b) and (c), respectively. The signal $y(t)$ so obtained fails to satisfy the condition of Eq. (1.27).

This example clearly illustrates that if $y(t)$ is defined in terms of $x(t)$ by Eq. (1.25), then $y(t)$ can only be obtained from $x(t)$ correctly by adhering to the precedence rule for time shifting and time scaling.

Similar remarks apply to the case of discrete-time signals.



(a) Signal $x(t)$ (b) Time-scaled signal $x(2t)$. (c) Signal $y(t)$ obtained by shifting $x(2t)$ by 3 time units.

FIGURE 1.24 The incorrect way of applying the precedence rule.

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Similar remarks apply to the case of discrete-time signals.

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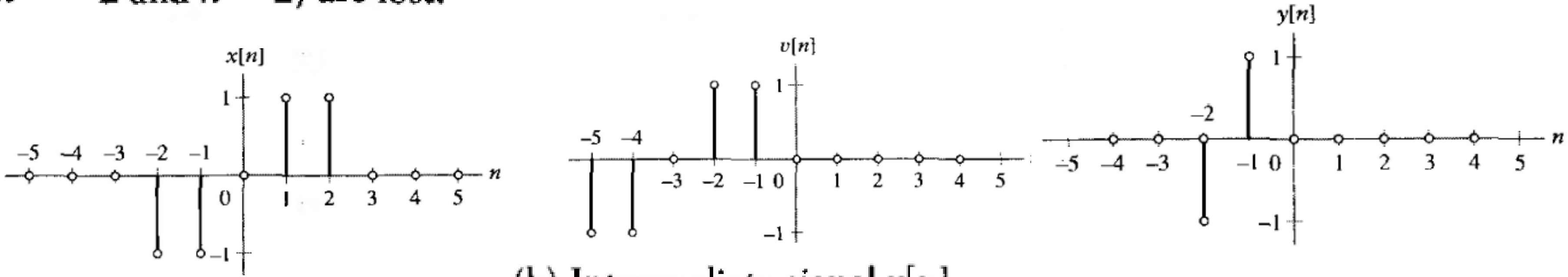
EXAMPLE 1.5 A discrete-time signal $x[n]$ is defined by

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}$$

Find $y[n] = x[2n + 3]$.

Solution: The signal $x[n]$ is displayed in Fig. 1.25(a). Time shifting $x[n]$ to the left by 3 yields the intermediate signal $v[n]$ shown in Fig. 1.25(b). Finally, scaling n in $v[n]$ by 2, we obtain the solution $y[n]$ shown in Fig. 1.25(c).

Note that as a result of the compression performed in going from $v[n]$ to $y[n] = v[2n]$, the samples of $v[n]$ at $n = -5$ and $n = -1$ (i.e., those contained in the original signal at $n = -2$ and $n = 2$) are lost.



(a) Discrete-time signal $x[n]$, antisymmetric about the origin.

(b) Intermediate signal $v[n]$ obtained by shifting $x[n]$ to the left by 3 samples.

(c) Discrete-time signal $y[n]$ resulting from the compression of $v[n]$ by a factor of 2, as a result of which two samples of the original $x[n]$ are lost.

FIGURE 1.25 The proper order of applying the operations of time scaling and time shifting for the case of a discrete-time signal.

Reference

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