

BUSINESS MATHEMATICS

HIGHER SECONDARY - SECOND YEAR
Volume-1

Untouchability is a sin
Untouchability is a crime
Untouchability is inhuman



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Text Book Committee

Chairperson

Dr.S.ANTONY RAJ

Reader in Mathematics
Presidency College
Chennai 600 005.

Reviewer

Dr. M.R. SRINIVASAN

Reader in Statistics
University of Madras,
Chennai - 600 005.

Reviewers - cum - Authors

Thiru. **N. RAMESH**

Selection Grade Lecturer
Department of Mathematics
Govt. Arts College (Men)
Nandanam, Chennai - 600 035.

Thiru. **R.MURTHY**

Selection Grade Lecturer
Department of Mathematics
Presidency College
Chennai 600005.

Authors

Thiru. **V. PRAKASH**

Lecturer (S.S.), Department of Statistics
Presidency College
Chennai - 600 005.

Thiru. **S. RAMACHANDRAN**

Post Graduate Teacher
The Chintadripet Hr. Sec. School
Chintadripet, Chennai - 600 002.

Thiru. **S.T. PADMANABHAN**

Post Graduate Teacher
The Hindu Hr. Sec. School
Triplicane, Chennai - 600 005.

Thiru. **S. RAMAN**

Post Graduate Teacher
Jaigopal Garodia National Hr. Sec. School
East Tambaram, Chennai - 600 059.

Tmt. **AMALI RAJA**

Post Graduate Teacher
Good Shepherd Matriculation
Hr. Sec. School, Chennai 600006.

Tmt. **M.MALINI**

Post Graduate Teacher
P.S. Hr. Sec. School (Main)
Mylapore, Chennai 600004.

Price : Rs.

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Preface

‘The most distinct and beautiful statement of any truth must atlast take the Mathematical form’ -Thoreau.

Among the Nobel Laureates in Economics more than 60% were Economists who have done pioneering work in Mathematical Economics. These Economists not only learnt Higher Mathematics with perfection but also applied it successfully in their higher pursuits of both Macroeconomics and Econometrics.

A Mathematical formula (involving stochastic differential equations) was discovered in 1970 by Stanford University Professor of Finance Dr.Scholes and Economist Dr.Merton. This achievement led to their winning Nobel Prize for Economics in 1997. This formula takes four input variables-duration of the option, prices, interest rates and market volatility-and produces a price that should be charged for the option. Not only did the formula work, it transformed American Stock Market.

Economics was considered as a deductive science using verbal logic grounded on a few basic axioms. But today the transformation of Economics is complete. Extensive use of graphs, equations and Statistics replaced the verbal deductive method. Mathematics is used in Economics by beginning with a few variables, gradually introducing other variables and then deriving the inter relations and the internal logic of an economic model. Thus Economic knowledge can be discovered and extended by means of mathematical formulations.

Modern Risk Management including Insurance, Stock Trading and Investment depend on Mathematics and it is a fact that one can use Mathematics advantageously to predict the future with more precision! Not with 100% accuracy, of course. But well enough so that one can make a wise decision as to where to invest money. The idea of using Mathematics to predict the future goes back to two 17th Century French Mathematicians Pascal and Fermat. They worked out probabilities of the various outcomes in a game where two dice are thrown a fixed number of times.

In view of the increasing complexity of modern economic problems, the need to learn and explore the possibilities of the new methods is becoming ever more pressing. If methods based on Mathematics and Statistics are used suitably according to the needs of Social Sciences they can prove to be compact, consistent and powerful tools especially in the fields of Economics, Commerce and Industry. Further these methods not only guarantee a deeper insight into the subject but also lead us towards exact and analytical solutions to problems treated.

This text book has been designed in conformity with the revised syllabus of Business Mathematics(XII) (to come into force from 2005 - 2006)-<http://www.tn.gov.in/schoolsyllabus/>. Each topic is developed systematically rigorously treated from first principles and many worked out examples are provided at every stage to enable the students grasp the concepts and terminology and equip themselves to encounter problems. Questions compiled in the Exercises will provide students sufficient practice and self confidence.

Students are advised to read and simultaneously adopt pen and paper for carrying out actual mathematical calculations step by step. As the Statistics component of this Text Book involves problems based on numerical calculations, Business Mathematics students are advised to use calculators. Those students who succeed in solving the problems on their own efforts will surely find a phenomenal increase in their knowledge, understanding capacity and problem solving ability. They will find it effortless to reproduce the solutions in the Public Examination.

We thank the Almighty God for blessing our endeavour and we do hope that the academic community will find this textbook triggering their interests on the subject!

“The direct application of Mathematical reasoning to the discovery of economic truth has recently rendered great services in the hands of master Mathematicians” – Alfred Marshall.

Malini	Amali Raja	Raman	Padmanabhan	Ramachandran
Prakash	Murthy	Ramesh	Srinivasan	Antony Raj

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APPLICATIONS OF MATRICES AND DETERMINANTS 1

The concept of matrices and determinants has extensive applications in many fields such as Economics, Commerce and Industry. In this chapter we shall develop some new techniques based on matrices and determinants and discuss their applications.

1.1 INVERSE OF A MATRIX

1.1.1 Minors and Cofactors of the elements of a determinant.

The minor of an element a_{ij} of a determinant A is denoted by M_{ij} and is the determinant obtained from A by deleting the row and the column where a_{ij} occurs.

The cofactor of an element a_{ij} with minor M_{ij} is denoted by C_{ij} and is defined as

$$C_{ij} = \begin{cases} M_{ij}, & \text{if } i + j \text{ is even} \\ -M_{ij}, & \text{if } i + j \text{ is odd.} \end{cases}$$

Thus, *cofactors are signed minors*.

In the case of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, we have

$$M_{11} = a_{22}, \quad M_{12} = a_{21}, \quad M_{21} = a_{12}, \quad M_{22} = a_{11}$$

Also $C_{11} = a_{22}, \quad C_{12} = -a_{21}, \quad C_{21} = -a_{12}, \quad C_{22} = a_{11}$

In the case of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we have

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix};$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix};$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix};$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \text{ and so on.}$$

1.1.2 Adjoint of a square matrix.

The transpose of the matrix got by replacing all the elements of a square matrix A by their corresponding cofactors in $|A|$ is called the **Adjoint** of A or **Adjugate** of A and is denoted by $\text{Adj } A$.

Thus, $\text{Adj } A = A_c^t$

Note

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A_c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$
 $\therefore \text{Adj } A = A_c^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Thus the Adjoint of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

can be written instantly as $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(ii) $\text{Adj } I = I$, where I is the unit matrix.

(iii) $A(\text{Adj } A) = (\text{Adj } A) A = |A| I$

(iv) $\text{Adj } (AB) = (\text{Adj } B) (\text{Adj } A)$

(v) If A is a square matrix of order 2, then $|\text{Adj } A| = |A|$

If A is a square matrix of order 3, then $|\text{Adj } A| = |A|^2$

Example 1

Write the Adjoint of the matrix $A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$

Solution :

$$\text{Adj } A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

Example 2

Find the Adjoint of the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

Solution :

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \quad \text{Adj } A = A_c^t$$

Now,

$$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8, \quad C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5,$$

$$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6, \quad C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1, \quad C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$\therefore A_c = \begin{pmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}$$

Hence

$$\text{Adj } A = \begin{pmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}^t = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$$

1.1.3 Inverse of a non singular matrix.

The **inverse of a non singular matrix** A is the matrix B such that $AB = BA = I$. B is then called the inverse of A and denoted by A^{-1} .

Note

- (i) A non square matrix has no inverse.
- (ii) The inverse of a square matrix A exists only when $|A| \neq 0$ that is, if A is a singular matrix then A^{-1} does not exist.
- (iii) If B is the inverse of A then A is the inverse of B. That is $B = A^{-1} \Rightarrow A = B^{-1}$.
- (iv) $A A^{-1} = I = A^{-1} A$
- (v) The inverse of a matrix, if it exists, is unique. That is, no matrix can have more than one inverse.
- (vi) The order of the matrix A^{-1} will be the same as that of A.

- (vii) $I^{-1} = I$
 (viii) $(AB)^{-1} = B^{-1} A^{-1}$, provided the inverses exist.
 (ix) $A^2 = I$ implies $A^{-1} = A$
 (x) If $AB = C$ then
 (a) $A = CB^{-1}$ (b) $B = A^{-1}C$, provided the inverses exist.
 (xi) We have seen that
 $A(\text{Adj}A) = (\text{Adj}A)A = |A| I$
 $\therefore A \frac{1}{|A|} (\text{Adj}A) = \frac{1}{|A|} (\text{Adj}A)A = I \quad (\text{if } |A| \neq 0)$

This suggests that

$$A^{-1} = \frac{1}{|A|} (\text{Adj}A). \text{ That is, } A^{-1} = \frac{1}{|A|} A_c^t$$

- (xii) $(A^{-1})^{-1} = A$, provided the inverse exists.

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } |A| = ad - bc \neq 0$$

$$\text{Now } A_c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad A_c^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\sim A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus the inverse of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written instantly as $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided $ad - bc \neq 0$.

Example 3

Find the inverse of $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$, if it exists.

Solution :

$$|A| = \begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} = -2 \neq 0 \quad \therefore A^{-1} \text{ exists.}$$

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$$

Example 4

Show that the inverses of the following do not exist :

$$(i) \ A = \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix} \quad (ii) \ A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 6 & 2 & -4 \end{pmatrix}$$

Solution :

$$(i) \quad |A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = 0 \quad \therefore A^{-1} \text{ does not exist.}$$

$$(ii) \quad |A| = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 6 & 2 & -4 \end{vmatrix} = 0 \quad \therefore A^{-1} \text{ does not exist.}$$

Example 5

Find the inverse of $A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, if it exists.

Solution :

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 15 \neq 0 \quad \therefore A^{-1} \text{ exists.}$$

We have, $A^{-1} = \frac{1}{|A|} A_c^t$

Now, the cofactors are

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5, \quad C_{12} = -\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = 7, \quad C_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$

$$C_{21} = -\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = 10, \quad C_{22} = \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -8, \quad C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1,$$

$$C_{31} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5, \quad C_{32} = -\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 10, \quad C_{33} = \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -5,$$

Hence

$$A_c = \begin{pmatrix} -5 & 7 & 1 \\ 10 & -8 & 1 \\ -5 & 10 & -5 \end{pmatrix}, \quad A_c^t = \begin{pmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{pmatrix} \therefore A^{-1} = \frac{1}{15} \begin{pmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{pmatrix}$$

Example 6

Show that $A = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{8}{17} & \frac{6}{17} & -\frac{9}{17} \\ \frac{10}{17} & -\frac{1}{17} & -\frac{7}{17} \end{pmatrix}$ are inverse of each other.

$$\begin{aligned}
 AB &= \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{8}{17} & \frac{6}{17} & -\frac{9}{17} \\ \frac{10}{17} & -\frac{1}{17} & -\frac{7}{17} \end{pmatrix} \\
 &= \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix} \frac{1}{17} \begin{pmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I
 \end{aligned}$$

Since A and B are square matrices and $AB = I$, A and B are inverse of each other.

EXERCISE 1.1

- 1) Find the Adjoint of the matrix $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$
- 2) Find the Adjoint of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$
- 3) Show that the Adjoint of the matrix $A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$ is A itself.
- 4) If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, verify that $A(\text{Adj } A) = (\text{Adj } A)A = |A|I$.
- 5) Given $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, verify that $\text{Adj}(AB) = (\text{Adj } B)(\text{Adj } A)$

- 6) In the second order matrix $A = (a_{ij})$, given that $a_{ij} = i+j$, write out the matrix A and verify that $|\text{Adj } A| = |A|$
- 7) Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$, verify that $|\text{Adj } A| = |A|^2$
- 8) Write the inverse of $A = \begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}$
- 9) Find the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$
- 10) Find the inverse of $A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and verify that $AA^{-1} = I$.
- 11) If $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ and none of the a 's are zero, find A^{-1} .
- 12) If $A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{pmatrix}$, show that the inverse of A is itself.
- 13) If $A^{-1} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, find A .
- 14) Show that $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{pmatrix}$ are inverse of each other
- 15) If $A = \begin{pmatrix} 2 & -3 \\ -4 & 8 \end{pmatrix}$, compute A^{-1} and show that $4A^{-1} = 10I - A$
- 16) If $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$ verify that $(A^{-1})^{-1} = A$
- 17) Verify $(AB)^{-1} = B^{-1} A^{-1}$, when $A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -6 & 0 \\ 0 & 9 \end{pmatrix}$
- 18) Find λ if the matrix $\begin{pmatrix} 6 & 7 & -1 \\ 3 & \ddot{e} & 5 \\ 9 & 11 & \ddot{e} \end{pmatrix}$ has no inverse.

19) If $X = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & p & q \end{pmatrix}$

find p, q such that $Y = X^{-1}$.

20) If $\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix} X = \begin{pmatrix} 14 \\ 29 \end{pmatrix}$, find the matrix X.

1.2 SYSTEMS OF LINEAR EQUATIONS

1.2.1 Submatrices and minors of a matrix.

Matrices obtained from a given matrix A by omitting some of its rows and columns are called **sub matrices** of A.

e.g. If $A = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 \\ 2 & 0 & 1 & -1 & 4 \\ 2 & 1 & 0 & 4 & 2 \\ 3 & 1 & 4 & 1 & 2 \end{pmatrix}$, some of the submatrices of A are :

$$\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & -1 \\ 4 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 & 4 & 5 \\ 2 & 0 & 1 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

The determinants of the square submatrices are called **minors** of the matrix.

Some of the minors of A are :

$$\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 5 \\ 3 & 2 \end{vmatrix}, \begin{vmatrix} 3 & 4 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 4 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & -1 & 4 \\ 0 & 4 & 2 \\ 4 & 1 & 2 \end{vmatrix}$$

1.2.2 Rank of a matrix.

A positive integer 'r' is said to be the **rank** of a non zero matrix A, denoted by $\rho(A)$, if

- (i) there is at least one minor of A of order 'r' which is not zero and
- (ii) every minor of A of order greater than 'r' is zero.

Note

- (i) The rank of a matrix A is the order of the largest non zero minor of A .
- (ii) If A is a matrix of order $m \times n$ then $\rho(A) \leq \text{minimum}(m, n)$
- (iii) The rank of a zero matrix is taken to be 0.
- (iv) For non zero matrices, the least value of the rank is 1.
- (v) The rank of a non singular matrix of order $n \times n$ is n .
- (vi) $\rho(A) = \rho(A^t)$
- (vii) $\rho(I_2) = 2, \rho(I_3) = 3$

Example 7

Find the rank of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & 5 \end{pmatrix}$

Solution :

Order of A is 3×3 . $\therefore \rho(A) \leq 3$

Consider the only third order minor

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 1 & 5 \end{vmatrix} = -2 \neq 0.$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$

Example 8

Find the rank of the matrix $A = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}$

Solution :

Order of A is 3×3 . $\therefore \rho(A) \leq 3$

Consider the only third order minor

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

The only minor of order 3 is zero. $\therefore \rho(A) \leq 2$

Consider the second order minors.

We find, $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3 \neq 0$

There is a minor of order 2 which is non zero. $\therefore \rho(A) = 2$.

Example 9

Find the rank of the matrix $A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{pmatrix}$

Solution :

Order of A is 3×3 . $\therefore \rho(A) \leq 3$
Consider the only third order minor

$$\begin{vmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{vmatrix} = 0 \quad (R_1 \propto R_2)$$

The only minor of order 3 is zero. $\therefore \rho(A) \leq 2$

Consider the second order minors. Obviously they are all zero.

$\therefore \rho(A) \leq 1$ Since A is a non zero matrix, $\rho(A) = 1$

Example 10

Find the rank of the matrix $A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$

Solution :

Order of A is 2×4 . $\therefore \rho(A) \leq 2$

Consider the second order minors.

We find,

$$\begin{vmatrix} 1 & -3 \\ 9 & 1 \end{vmatrix} = 28 \neq 0$$

There is a minor of order 2 which is not zero.

$\therefore \rho(A) = 2$

Example 11

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{pmatrix}$

Solution :

Order of A is 3×4 . $\therefore \rho(A) \leq 3$.

Consider the third order minors.

We find,

$$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & 3 \\ 8 & 1 & 9 \end{vmatrix} = -40 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$.

1.2.3 Elementary operations and equivalent matrices.

The process of finding the values of a number of minors in our endeavour to find the rank of a matrix becomes laborious unless by a stroke of luck we get a non zero minor at an early stage. To get over this difficulty, we introduce many zeros in the matrix by what are called **elementary operations** so that the evaluation of the minors is rendered easier. It can be proved that the elementary operations do not alter the rank of a matrix.

The following are the elementary operations :

- (i) *The interchange of two rows.*
- (ii) *The multiplication of a row by a non zero number.*
- (iii) *The addition of a multiple of one row to another row.*

If a matrix B is obtained from a matrix A by a finite number of elementary operations then we say that the matrices A and B are **equivalent matrices** and we write $A \sim B$.

Also, while introducing many zeros in the given matrix, it would be desirable (but not necessary) to reduce it to a **triangular form**.

A matrix $A = (a_{ij})$ is said to be in a triangular form if $a_{ij} = 0$ whenever $i > j$.

e.g., The matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 3 & 0 \\ 0 & 0 & 2 & 9 \end{pmatrix}$ is in a triangular form.

Example 12

Find the rank of the matrix $A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$

Solution :

Order of A is 3×4 . $\therefore \rho(A) \leq 3$.

Let us reduce the matrix A to a triangular form.

$$A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$$

Applying $R_1 \leftrightarrow R_3$

$$A \sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - 5R_1$

$$A \sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - 8R_2$

$$A \sim \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{pmatrix}$$

This is now in a triangular form.

We find,

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = -12 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$.

Example 13

Find the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \end{pmatrix}$

Solution :

Order of A is 3×4 . $\therefore \rho(A) \leq 3$

Let us reduce the matrix A to a triangular form.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \end{pmatrix}$$

This is now in a triangular form.

We find,

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & -8 \end{vmatrix} = -16 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$.

Example 14

Find the rank of the matrix $A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$

Solution :

Order of A is 3×4 . $\therefore \rho(A) \leq 3$.

$$A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$$

Applying $R_3 \rightarrow \frac{R_3}{4}$

$$A \sim \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

Applying $R_1 \leftrightarrow R_3$

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 6 \\ 4 & 5 & 2 & 2 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 4R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 1 & -6 & 2 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 0 & -11 & 8 \end{pmatrix}$$

This is in a triangular form.

We find,

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & -11 \end{vmatrix} = 11 \neq 0$$

There is a minor of order 3 which is not zero. $\therefore \rho(A) = 3$

1.2.4 Systems of linear equations.

A system of (simultaneous) equations in which the variables (ie. the unknowns) occur only in the first degree is said to be **linear**.

A system of linear equations can be represented in the form $AX = B$. For example, the equations $x - 3y + z = -1$, $2x + y - 4z = -1$, $6x - 7y + 8z = 7$ can be written in the matrix form as

$$\begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$$

$A \quad X = B$

A is called the **coefficient matrix**. If the matrix A is augmented with the column matrix B, at the end, we get the **augmented matrix**,

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 2 & 1 & -4 & -1 \\ 6 & -7 & 8 & 7 \end{array} \right) \text{ denoted by } (A, B)$$

A system of (simultaneous) linear equations is said to be **homogeneous** if the constant term in each of the equations is zero. A system of linear homogeneous equations can be represented in the form $AX = O$. For example, the equations $3x+4y-2z=0$, $5x+2y=0$, $3x-y+z=0$ can be written in the matrix form as

$$\begin{array}{ccc} \begin{pmatrix} 3 & 4 & -2 \\ 5 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ A & X & = O \end{array}$$

1.2.5 Consistency of equations

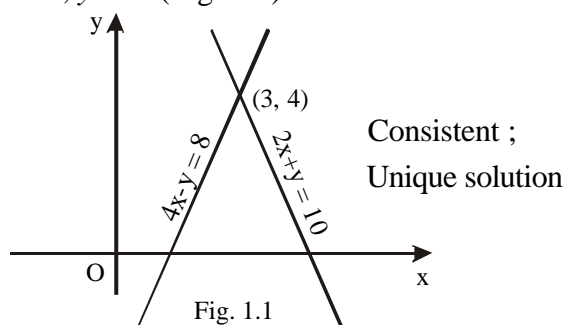
A system of equations is said to be **consistent** if it has at least one set of solution. Otherwise it is said to be **inconsistent**.

Consistent equations may have

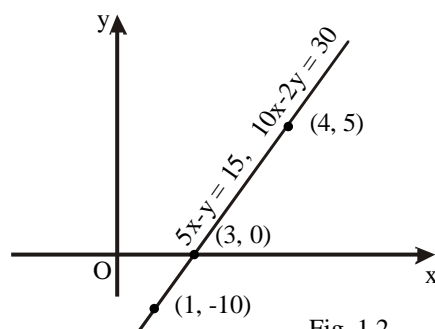
- (i) unique solution (that is, only one set of solution) or
- (ii) infinite sets of solution.

By way of illustration, consider first the case of linear equations in two variables.

The equations $4x-y=8$, $2x+y=10$ represent two straight lines intersecting at (3, 4). They are consistent and have the unique solution $x=3$, $y=4$. (Fig. 1.1)



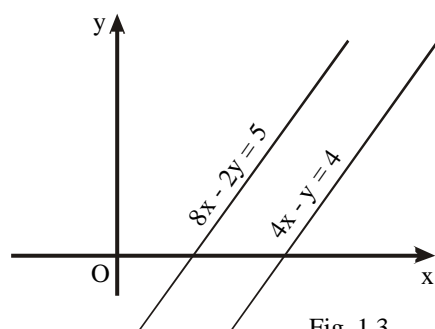
The equations $5x - y = 15$, $10x - 2y = 30$ represent two coincident lines. We find that any point on the line is a solution. The equations are consistent and have infinite sets of solution such as $x = 1, y = -10$; $x = 3, y = 0$; $x = 4, y = 5$ and so on (Fig. 1.2) Such equations are called dependent equations.



Consistent ;
Infinite sets of solution.

Fig. 1.2

The equations $4x - y = 4$, $8x - 2y = 5$ represent two parallel straight lines. The equations are inconsistent and have no solution. (Fig. 1.3)



Inconsistent ;
No solution

Fig. 1.3

Now consider the case of linear equations in three variables. The equations $2x + 4y + z = 5$, $x + y + z = 6$, $2x + 3y + z = 6$ are consistent and have only one set of unique solution viz. $x = 2, y = -1, z = 5$. On the other hand, the equations $x + y + z = 1$, $x + 2y + 4z = 1$, $x + 4y + 10z = 1$ are consistent and have infinite sets of solution such as $x = 1, y = 0, z = 0$; $x = 3, y = -3, z = 1$; and so on. All these solutions are included in $x = 1 + 2k, y = -3k, z = k$ where k is a parameter.

The equations $x + y + z = -3$, $3x + y - 2z = -2$, $2x + 4y + 7z = 7$ do not have even a single set of solution. They are inconsistent.

All homogeneous equations do have the **trivial solution** $x = 0, y = 0, z = 0$. Hence the homogeneous equations are all consistent and the question of their being consistent or otherwise does not arise at all.

The homogeneous equations may or may not have solutions other than the trivial solution. For example, the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y - z = 0$ have only the trivial solution viz., $x = 0, y = 0, z = 0$. On the other hand the equations $x + y - z = 0$, $x - 2y + z = 0$, $3x + 6y - 5z = 0$ have infinite sets of solution such as $x = 1, y = 2, z = 3$; $x = 3, y = 6, z = 9$ and so on. All these non trivial solutions are included in $x = t, y = 2t, z = 3t$ where t is a parameter.

1.2.6 Testing the consistency of equations by rank method.

Consider the equations $AX = B$ in ' n ' unknowns

- 1) If $\rho(A, B) = \rho(A)$, then the equations are consistent.
- 2) If $\rho(A, B) \neq \rho(A)$, then the equations are inconsistent.
- 3) If $\rho(A, B) = \rho(A) = n$, then the equations are consistent and have unique solution.
- 4) If $\rho(A, B) = \rho(A) < n$, then the equations are consistent and have infinite sets of solution.

Consider the equations $AX = 0$ in ' n ' unknowns

- 1) If $\rho(A) = n$ then equations have the trivial solution only.
- 2) If $\rho(A) < n$ then equations have the non trivial solutions also.

Example 15

Show that the equations $2x - y + z = 7$, $3x + y - 5z = 13$, $x + y + z = 5$ are consistent and have unique solution.

Solution :

The equations take the matrix form as

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 5 \end{pmatrix}$$

$$\underset{\text{A}}{\quad} \quad \quad \underset{\text{X}}{\quad} = \underset{\text{B}}{\quad}$$

$$\text{Now (A, B)} = \begin{pmatrix} 2 & -1 & 1 & \vdots & 7 \\ 3 & 1 & -5 & \vdots & 13 \\ 1 & 1 & 1 & \vdots & 5 \end{pmatrix}$$

Applying $R_1 \leftrightarrow R_3$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 3 & 1 & -5 & \vdots & 13 \\ 2 & -1 & 1 & \vdots & 7 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 2R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & -2 & -8 & \vdots & -2 \\ 0 & -3 & -1 & \vdots & -3 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - \frac{3}{2} R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & -2 & -8 & \vdots & -2 \\ 0 & 0 & 11 & \vdots & 0 \end{pmatrix}$$

Obviously,

$$\rho(A, B) = 3, \rho(A) = 3$$

The number of unknowns is 3.

Hence $\rho(A, B) = \rho(A) =$ the number of unknowns.

\therefore The equations are consistent and have unique solution.

Example 16

Show that the equations $x + 2y = 3, y - z = 2, x + y + z = 1$ are consistent and have infinite sets of solution.

Solution :

The equations take the matrix form as

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\text{A}}{\quad} \quad \quad \underset{\text{X}}{\quad} = \underset{\text{B}}{\quad}$$

$$\text{Now, (A, B)} = \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 1 & 1 & 1 & \vdots & 1 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -2 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

Obviously,

$$\rho(A, B) = 2, \quad \rho(A) = 2.$$

The number of unknowns is 3.

Hence $\rho(A, B) = \rho(A) < \text{the number of unknowns}$.

\therefore The equations are consistent and have infinite sets of solution.

Example 17

Show that the equations $x - 3y + 4z = 3$, $2x - 5y + 7z = 6$, $3x - 8y + 11z = 1$ are inconsistent.

Solution :

The equations take the matrix form as

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$$

$$\underset{\text{A}}{\quad} \quad \quad \underset{\text{X}}{\quad} = \underset{\text{B}}{\quad}$$

Now,

$$(A, B) = \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 2 & -5 & 7 & \vdots & 6 \\ 3 & -8 & 11 & \vdots & 1 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$(A, B) \sim \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & -8 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$(A, B) \sim \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & -8 \end{pmatrix}$$

Obviously,

$$\rho(A, B) = 3, \quad \rho(A) = 2$$

Hence $\rho(A, B) \neq \rho(A)$

\therefore The equations are inconsistent.

Exampe 18

Show that the equations $x + y + z = 0$, $2x + y - z = 0$, $x - 2y + z = 0$ have only the trivial solution.

Solution :

The matrix form of the equations is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$A \quad X = O$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & -3 & 0 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - 3R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$

Obviously,

$$\rho(A) = 3$$

The number of unknowns is 3.

Hence $\rho(A) =$ the number of unknowns.

\therefore The equations have only the trivial solution.

Example 19

Show that the equations $3x + y + 9z = 0$, $3x + 2y + 12z = 0$, $2x + y + 7z = 0$ have non trivial solutions also.

Solution :

The matrix form of the equations is

$$\begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \quad X = O$$

$$A = \begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{vmatrix} = 0, \quad \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \neq 0$$

$$\therefore \rho(A) = 2$$

The number of unknowns is 3.

Hence $\rho(A) <$ the number of unknowns.

\therefore The equations have non trivial solutions also.

Example 20

Find k if the equations $2x + 3y - z = 5$, $3x - y + 4z = 2$, $x + 7y - 6z = k$ are consistent.

Solution :

$$(A, B) = \begin{pmatrix} 2 & 3 & -1 & \vdots & 5 \\ 3 & -1 & 4 & \vdots & 2 \\ 1 & 7 & -6 & \vdots & k \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & -1 & 4 \\ 1 & 7 & -6 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & 4 \\ 1 & 7 & -6 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11 \neq 0$$

Obviously $\rho(A) = 2$.

For the equations to be consistent, $\rho(A, B)$ should also be 2.

Hence every minor of (A, B) of order 3 should be zero.

$$\therefore \begin{vmatrix} 3 & -1 & 5 \\ -1 & 4 & 2 \\ 7 & -6 & k \end{vmatrix} = 0$$

Expanding and simplifying, we get $k = 8$.

Example 21

Find k if the equations $x + y + z = 3$, $x + 3y + 2z = 6$, $x + 5y + 3z = k$ are inconsistent.

Solution :

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 3 & 2 & \vdots & 6 \\ 1 & 5 & 3 & \vdots & k \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 5 & 3 \end{pmatrix}$$

We find,

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$$

Obviously $\rho(A) = 2$.

For the equations to be inconsistent, $\rho(A, B)$ should not be 2.

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 3 & 2 & \vdots & 6 \\ 1 & 5 & 3 & \vdots & k \end{pmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 3 \\ 0 & 4 & 2 & \vdots & k-3 \end{pmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & k-9 \end{pmatrix}$$

$\rho(A, B) \neq 2$ only when $k \neq 9$

\therefore The equations are inconsistent when k assumes any real value other than 9.

Example 22

Find the value of k for the equations $kx + 3y + z = 0$, $3x - 4y + 4z = 0$, $kx - 2y + 3z = 0$ to have non trivial solution.

Solution :

$$A = \begin{pmatrix} k & 3 & 1 \\ 3 & -4 & 4 \\ k & -2 & 3 \end{pmatrix}$$

For the homogeneous equations to have non trivial solution, $\rho(A)$ should be less than the number of unknowns viz., 3.

$\therefore \rho(A) \neq 3$.

$$\text{Hence } \begin{vmatrix} k & 3 & 1 \\ 3 & -4 & 4 \\ k & -2 & 3 \end{vmatrix} = 0$$

Expanding and simplifying, we get $k = \frac{11}{4}$

Example 23

Find k if the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y + kz = 0$ have only the trivial solution.

Solution :

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{pmatrix}$$

For the homogeneous equations to have only the trivial solution, $\rho(A)$ should be equal to the number of unknowns viz., 3.

$$\therefore \begin{vmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{vmatrix} \neq 0, \quad k \neq 1.$$

The equations have only the trivial solution when k assumes any real value other than 1.

EXERCISE 1.2

- 1) Find the rank of each of the following matrices

$$(i) \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 2 & 5 \end{pmatrix} \quad (ii) \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 3 & 6 & 6 \end{pmatrix} \quad (iii) \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

$$(iv) \begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix} \quad (v) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -2 & -4 \end{pmatrix} \quad (vi) \begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ -1 & -3 & -4 & -7 \end{pmatrix}$$

$$(vii) \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 3 \end{pmatrix} \quad (viii) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad (ix) \begin{pmatrix} 9 & 6 \\ -6 & 4 \end{pmatrix}$$

- 2) Find the ranks of $A+B$ and AB where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$$

- 3) Prove that the points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are collinear

$$\text{if the rank of the matrix } \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \text{ is less than 3.}$$

- 4) Show that the equations $2x + 8y + 5z = 5$, $x + y + z = -2$, $x + 2y - z = 2$ are consistent and have unique solution.

- 5) Show that the equations $x-3y-8z = -10$, $3x+y-4z = 0$, $2x+5y+6z = 13$ are consistent and have infinite sets of solution.
- 6) Test the system of equations $4x-5y-2z = 2$, $5x-4y+2z = -2$, $2x+2y+8z = -1$ for consistency.
- 7) Show that the equations $4x-2y = 3$, $6x-3y = 5$ are inconsistent.
- 8) Show that the equations $x+y+z = -3$, $3x+y-2z = -2$, $2x+4y+7z = 7$ are not consistent.
- 9) Show that the equations $x+2y+2z = 0$, $x-3y-3z = 0$, $2x+y-z = 0$ have no other solution other than $x = 0$, $y = 0$ and $z = 0$.
- 10) Show that the equations $x+y-z = 0$, $x-2y+z = 0$, $3x+6y-5z = 0$ have non trivial solutions also.
- 11) Find k if the equations $x+2y-3z = -2$, $3x-y-2z = 1$, $2x+3y-5z = k$ are consistent.
- 12) Find k if the equations $x+y+z = 1$, $3x-y-z = 4$, $x+5y+5z = k$ are inconsistent.
- 13) Find the value of k for the equations $2x-3y+z = 0$, $x+2y-3z = 0$, $4x-y+kz = 0$ to have non trivial solutions.
- 14) Find k for which the equations $x+2y+3z = 0$, $2x+3y+4z = 0$ and $7x+ky+9z = 0$ have no non trivial solutions.

1.3 SOLUTION OF LINEAR EQUATIONS

1.3.1 Solution by Matrix method.

When $|A| \neq 0$, the equations $AX = B$ have the unique solution given by $X = A^{-1}B$.

Example 2:

Solve using matrices the equations $2x-y = 3$, $5x+y = 4$.

Solution :

The equations can be written in matrix form as

$$\begin{pmatrix} 2 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\underset{A}{\quad} \underset{X}{\quad} = \underset{B}{\quad}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = 7 \neq 0$$

\therefore The unique solution is given by

$$X = A^{-1}B$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \therefore x = 1, y = -1$$

Example 25

Solve the equations $2x + 8y + 5z = 5$, $x + y + z = -2$, $x + 2y - z = 2$ by using matrix method.

Solution :

The equations can be written in matrix form as

$$\begin{pmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix} \quad \begin{array}{l} \text{cofactors} \\ +(-1-2), \quad -(-1-1), \quad +(2-1) \\ -(-8-10), \quad +(-2-5), \quad -(4-8) \\ +(8-5), \quad -(2-5), \quad +(2-8) \end{array}$$

$$|A| = \begin{vmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 15 \neq 0$$

\therefore The unique solution is given by

$$X = A^{-1}B.$$

We now find A^{-1} .

$$A_c = \begin{pmatrix} -3 & 2 & 1 \\ 18 & -7 & 4 \\ 3 & 3 & -6 \end{pmatrix}$$

$$A_c^t = \begin{pmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} A^t_c = \frac{1}{15} \begin{pmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{pmatrix}$$

Now

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{15} \begin{pmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{15} \begin{pmatrix} -45 \\ 30 \\ -15 \end{pmatrix} \text{ ie., } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow x = -3, y = 2, z = -1.$$

Example 26

A woman invested different amounts at 8%, $8\frac{3}{4}\%$ and 9%, all at simple interest. Altogether she invested Rs. 40,000 and earns Rs. 3,455 per year. How much does she have invested at each rate if she has Rs. 4,000 more invested at 9% than at 8%? Solve by using matrices.

Solution :

Let x, y, z be the amounts in Rs. invested at 8%, $8\frac{3}{4}\%$ and 9% respectively.

According to the problem,

$$x + y + z = 40,000$$

$$\frac{x \times 8 \times 1}{100} + \frac{35 \times y \times 1}{400} + \frac{9 \times z \times 1}{100} = 3,455 \text{ and}$$

$$z - x = 4,000$$

$$\Rightarrow x + y + z = 40,000$$

$$32x + 35y + 36z = 13,82,000$$

$$x - z = -4,000$$

The equations can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 32 & 35 & 36 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 40,000 \\ 13,82,000 \\ -4,000 \end{pmatrix}$$

$$\underset{\text{A}}{\quad} \quad \quad \underset{\text{X}}{\quad} = \underset{\text{B}}{\quad}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 32 & 35 & 36 \\ 1 & 0 & -1 \end{vmatrix} = -2 \neq 0$$

∴ The unique solution is given by $X = A^{-1}B$

We now find A^{-1} .

$$A_c = \begin{pmatrix} -35 & 68 & -35 \\ 1 & -2 & 1 \\ 1 & -4 & 3 \end{pmatrix} \quad \begin{array}{l} \text{cofactors} \\ +(-35-0), \quad -(-32-36), \quad +(0-35) \\ -(-1-0), \quad +(-1-1), \quad -(0-1) \\ +(36-35), \quad -(36-32), \quad +(35-32) \end{array}$$

$$A_c^t = \begin{pmatrix} -35 & 1 & 1 \\ 68 & -2 & -4 \\ -35 & 1 & 3 \end{pmatrix}$$

Now,

$$A^{-1} = \frac{1}{|A|} A_c^t = \frac{1}{-2} \begin{pmatrix} -35 & 1 & 1 \\ 68 & -2 & -4 \\ -35 & 1 & 3 \end{pmatrix}$$

Now

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -35 & 1 & 1 \\ 68 & -2 & -4 \\ -35 & 1 & 3 \end{pmatrix} \begin{pmatrix} 40,000 \\ 13,82,000 \\ -4,000 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -22,000 \\ -28,000 \\ -30,000 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11,000 \\ 14,000 \\ 15,000 \end{pmatrix}$$

Hence the amounts invested at 8%, $8\frac{3}{4}\%$ and 9% are Rs. 11,000, Rs. 14,000 and Rs. 15,000 respectively.

1.3.2 Solution by Determinant method (Cramer's rule)

Let the equations be

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3.$$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$
$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

When $\Delta \neq 0$, the unique solution is given by

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}.$$

Example 27

Solve the equations $x + 2y + 5z = 23$,
 $3x + y + 4z = 26$, $6x + y + 7z = 47$ by determinant method.

Solution :

The equations are

$$x + 2y + 5z = 23$$

$$3x + y + 4z = 26$$

$$6x + y + 7z = 47$$

$$\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 6 & 1 & 7 \end{vmatrix} = -6 \neq 0; \quad \Delta_x = \begin{vmatrix} 23 & 2 & 5 \\ 26 & 1 & 4 \\ 47 & 1 & 7 \end{vmatrix} = -24$$
$$\Delta_y = \begin{vmatrix} 1 & 23 & 5 \\ 3 & 26 & 4 \\ 6 & 47 & 7 \end{vmatrix} = -12; \quad \Delta_z = \begin{vmatrix} 1 & 2 & 23 \\ 3 & 1 & 26 \\ 6 & 1 & 47 \end{vmatrix} = -18$$

By Cramer's rule

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-24}{-6} = 4 \quad ; \quad y = \frac{\Delta_y}{\Delta} = \frac{-12}{-6} = 2$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-18}{-6} = 3 \quad ; \quad \Rightarrow x = 4, y = 2, z = 3.$$

Example 28

Solve the equations $2x - 3y - 1 = 0$, $5x + 2y - 12 = 0$ by Cramer's rule.

Solution :

The equations are $2x - 3y = 1$, $5x + 2y = 12$

$$\Delta = \begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix} = 19 \neq 0 \quad ; \quad \Delta_x = \begin{vmatrix} 1 & -3 \\ 12 & 2 \end{vmatrix} = 38$$

$$\Delta_y = \begin{vmatrix} 2 & 1 \\ 5 & 12 \end{vmatrix} = 19 \quad ;$$

By Cramer's rule

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{38}{19} = 2, \quad y = \frac{\Delta_y}{\Delta} = \frac{19}{19} = 1$$

$$\Rightarrow x = 2, y = 1.$$

Example 29

A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission.

Months	Sales of units			Total commission drawn (in Rs.)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rates of commission on the items A, B and C. Solve by Cramer's rule.

Solution :

Let x , y and z be the rates of commission in Rs. per unit for A, B and C items respectively.

According to the problem,

$$90x + 100y + 20z = 800$$

$$130x + 50y + 40z = 900$$

$$60x + 100y + 30z = 850$$

Dividing each of the equations by 10 throughout,

$$9x + 10y + 2z = 80$$

$$13x + 5y + 4z = 90$$

$$6x + 10y + 3z = 85$$

Now,

$$\Delta = \begin{vmatrix} 9 & 10 & 2 \\ 13 & 5 & 4 \\ 6 & 10 & 3 \end{vmatrix} = -175 \neq 0 ; \quad \Delta_x = \begin{vmatrix} 80 & 10 & 2 \\ 90 & 5 & 4 \\ 85 & 10 & 3 \end{vmatrix} = -350$$

$$\Delta_y = \begin{vmatrix} 9 & 80 & 2 \\ 13 & 90 & 4 \\ 6 & 85 & 3 \end{vmatrix} = -700 ; \quad \Delta_z = \begin{vmatrix} 9 & 10 & 80 \\ 13 & 5 & 90 \\ 6 & 10 & 85 \end{vmatrix} = -1925$$

By Cramer's rule

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-350}{-175} = 2 ; \quad y = \frac{\Delta_y}{\Delta} = \frac{-700}{-175} = 4$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-1925}{-175} = 11$$

Hence the rates of commission for A, B and C are Rs. 2, Rs. 4 and Rs. 11 respectively.

EXERCISE 1.3

- 1) Solve by matrix method the equations $2x + 3y = 7$, $2x + y = 5$.
- 2) Solve by matrix method the equations
 $x - 2y + 3z = 1$, $3x - y + 4z = 3$, $2x + y - 2z = -1$
- 3) Solve by Cramer's rule the equations $6x - 7y = 16$, $9x - 5y = 35$.
- 4) Solve by determinant method the equations
 $2x + 2y - z - 1 = 0$, $x + y - z = 0$, $3x + 2y - 3z = 1$.
- 5) Solve by Cramer's rule : $x + y = 2$, $y + z = 6$, $z + x = 4$.

- 6) Two types of radio valves A, B are available and two types of radios P and Q are assembled in a small factory. The factory uses 2 valves of type A and 3 valves of type B for the type of radio P, and for the radio Q it uses 3 valves of type A and 4 valves of type B. If the number of valves of type A and B used by the factory are 130 and 180 respectively, find out the number of radios assembled. Use matrix method.
- 7) The cost of 2kg. of wheat and 1kg. of sugar is Rs. 7. The cost of 1kg. wheat and 1kg. of rice is Rs. 7. The cost of 3kg. of wheat, 2kg. of sugar and 1kg. of rice is Rs. 17. Find the cost of each per kg., using matrix method.
- 8) There are three commodities X, Y and Z which are bought and sold by three dealers A, B and C. Dealer A purchases 2 units of X and 5 units of Z and sells 3 units of Y, dealer B purchases 5 units of X, 2 units of Y and sells 7 units of Z and dealer C purchases 3 units of Y, 1 unit of Z and sells 4 units of X. In the process A earns Rs. 11 and C Rs. 5 but B loses Rs. 12. Find the price of each of the commodities X, Y and Z, by using determinants.
- 9) A company produces three products everyday. The total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total production of the first and third product is twice the production of second product. Determine the production level of each product by using Cramer's rule.

1.4 STORING INFORMATION

We know that a matrix provides a convenient and compact notation for representation of data which is capable of horizontal and vertical divisions. Now we shall study the applications of matrices in the study of (i) Relations on sets (ii) Directed routes and (iii) Cryptography.

Let us first recall the concept of relations on sets studied in earlier classes.

Relation :

A relation R from a set A to a set B is a subset of the cartesian product $A \times B$. Thus R is a set of ordered pairs where the first element comes from A and the second element comes from B . If $(a, b) \in R$ we say that ' a ' is related to ' b ' and write $a R b$. If $(a, b) \notin R$, we say that ' a ' is not related to ' b ' and write $a \not R b$. If R is a relation from a set A to itself then we say that R is a relation on A .

For example,

$$\text{Let } A = \{2, 3, 4, 6\} \text{ and } B = \{4, 6, 9\}$$

Let R be the relation from A to B defined by $x R y$ if x divides y exactly. Then

$$R = \{(2, 4), (2, 6), (3, 6), (3, 9), (4, 4), (6, 6)\}$$

Inverse relation.

Let R be any relation from a set A to a set B . Then the inverse of R , denoted by R^{-1} is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R . For example, the inverse of the relation $R = \{(1, y), (1, z), (3, y)\}$ from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$ is $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$ from B to A .

Composition of relations.

Let A, B and C be sets and let R be a relation from A to B and let S be a relation from B to C . i.e. R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by

$$R \circ S = \{(a, c) / \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}.$$

For example,

$$\text{Let } A = \{1, 2, 3, 4\}, B = \{a, b, c, d\} \text{ and } C = \{x, y, z\} \text{ and}$$

$$\text{let } R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$$

$$\text{and } S = \{(b, x), (b, z), (c, y), (d, z)\}$$

$$\text{Then } R \circ S = \{(2, z), (3, x), (3, z)\}$$

Types of relations.

A relation R on a set A is **reflexive** if aRa for every $a \in A$. that is $(a, a) \in R$ for every $a \in A$.

A relation R on a set A is **symmetric** if whenever aRb then bRa that is, whenever $(a, b) \in R$ then $(b, a) \in R$.

A relation R on a set A is **transitive** if whenever aRb and bRc then aRc that is, whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$.

A relation R is an **equivalence relation** if R is reflexive, symmetric and transitive.

For example, consider the following three relations on $A = \{1, 2, 3\}$.

$$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$$

$$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$$

$$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$$

R is not reflexive, S is reflexive and T is not reflexive.

R is not symmetric, S is symmetric and T is not symmetric.

R is transitive, S is transitive and T is not transitive.

1.4.1 Relation matrices.

A matrix is a convenient way to represent a relation R from X to Y . Such a relation can be analysed by using a computer.

We label the rows with the elements of X (in some arbitrary order) and we label the columns with the elements of Y (again in some arbitrary order). We then set the entry in row x and column y to 1 if xRy and to 0 otherwise. The matrix so obtained is called the **relation matrix** for R .

Example 30

Find the relation matrix for the relation R from $\{2, 3, 4\}$ to $\{5, 6, 7, 8\}$ where R is defined by xRy if x divides y exactly.

Solution :

$$R = \{(2, 6), (2, 8), (3, 6), (4, 8)\}$$

The relation matrix for R is

$$R = \begin{matrix} & \begin{matrix} 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Example 31

Let $S = \{1, 2, 3, 4\}$. Let R be the relation on S defined by mRn if $m < n$. Write out the relation matrix for R .

Solution :

$$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

The relation matrix for R is

$$R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Example 32

Given a relation matrix $R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{matrix}$ Write down the relation R in the form of a set of ordered pairs.

Solution :

$$R = \{(x, 1), (y, 1), (y, 2)\}$$

Matrix for inverse relation

If R is relation matrix, then its transpose R^t represents the inverse relation R^{-1} .

Example 33

Let $A = \{0, 1, 2\}$ Define a relation R on A by mRn if $mn = m$. Find the relation matrix for R . Use it to find the relation matrix for the inverse relation R^{-1} .

Solution :

$$R = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 1)\}$$

Relation matrix for R is

$$R = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Relation matrix for R^{-1} is

$$R^{-1} = R^t = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Relation matrix for composition of relations.

The relation matrix for $R_1 \circ R_2$ is obtained by replacing each non zero element in the matrix product $R_1 R_2$ by 1.

Example 34

Let R_1 be a relation from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ defined by $R_1 = \{(1, a), (2, b), (3, a), (3, b)\}$ and let R_2 be the relation from Y to $Z = \{x, y, z\}$ defined by $R_2 = \{(a, x), (a, y), (b, y), (b, z)\}$.

Find the relation matrices for R_1 and R_2 and using them find the relation matrix for $R_1 \circ R_2$.

Solution :

$$\text{The relation matrix for } R_1 \text{ is } R_1 = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$\text{The relation matrix for } R_2 \text{ is } R_2 = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

The matrix product $R_1 R_2$ is

$$R_1 R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Replacing each non zero element in $R_1 R_2$ by 1 we get,

$$R_1 \circ R_2 = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

Type of a relation as revealed by its matrix.

A relation R is reflexive if its matrix has only 1's on the main diagonal.

A relation R is symmetric if its matrix is symmetric

That is, $a_{ij} = a_{ji}$ for all i, j .

A relation R is transitive if whenever the entry i, j in the matrix product R^2 is non zero, the entry i, j in the relation matrix R is also non zero.

Example 35

Given a relation $R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$ on $A = \{a, b, c, d\}$. Find the relation matrix for R and using it identify the type of the relation.

Solution :

The relation matrix for R is

$$R = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

The matrix has only 1's on the main diagonal. Hence the relation is reflexive.

The matrix is symmetric. Hence the relation is symmetric.
The matrix product.

$$R^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Whenever an entry in R^2 is non zero, the corresponding entry in R is also non zero. Hence the relation is transitive.

Thus the relation R is reflexive, symmetric and transitive and hence an equivalence relation.

Example 36

Let R be the relation on $S = \{1, 2, 3, 4\}$ defined by mRn if $|m-n| \leq 1$. Find the relation matrix for R and using it, identify the type of the relation.

Solution :

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$$

The relation matrix for R is

$$R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

The matrix has only 1's on the main diagonal. Hence the relation is reflexive.

The matrix is symmetric. Hence the relation is symmetric.

$$R^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}$$

Now a_{13} element in R^2 is non zero but a_{13} element in R is zero.

Hence the relation is not transitive.

1.4.2 Route Matrices.

A **directed route** is a set of points P_1, P_2, \dots, P_n called vertices together with a finite set of directed edges each of which joins an ordered pair of distinct vertices. Thus the directed edge P_{ij} is different from the directed edge P_{ji} . There may be no directed edge from a vertex P_i to any other vertex and may not be any directed edge from any vertex to the vertex P_i . Also there can be no loops and multiple directed edges joining any two vertices.

Each edge of a directed route is called a **stage** of length 1. A **path** from the vertex P_i to the vertex P_j is a sequence of directed edges from P_i to P_j . It should start from P_i and end with P_j , repetition of any of the vertices including P_i and P_j being allowed.

If there is a path from P_i to P_j we say that P_j is **accessible** from P_i or that P_i has **access** to P_j .

A directed route is said to be **strongly connected** if for any pair of vertices P_i and P_j , there is a path from P_i to P_j and a path from P_j to P_i . Otherwise the route is said to be not strongly connected.

A directed route is represented by its **route matrix**. If G is a directed route with ' n ' vertices then the $n \times n$ matrix A where the (i, j) th element is 1 if there is a directed edge from P_i to P_j and zero otherwise is called the route matrix for the directed route G .

It is to be noted that the number of 1's in the route matrix will be equal to the number of edges in its route. Route matrices are also relation matrices. The route matrices are square matrices whereas the relation matrices need not be square matrices.

Path matrix of a directed route is the matrix $P = \{P_{ij}\}$ such that

$$P_{ij} = \begin{cases} 1, & \text{if there is a path from } P_i \text{ to } P_j \\ 0, & \text{otherwise.} \end{cases}$$

Example 37

Find the route matrix for each directed route given below:

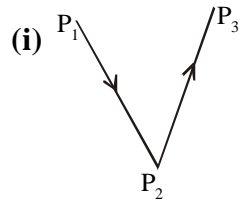


Fig. 1.4

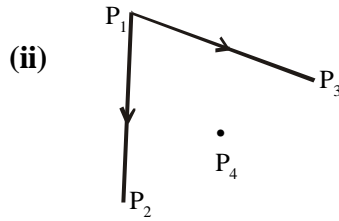


Fig. 1.5

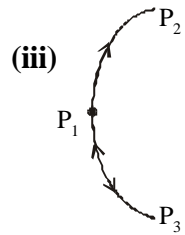


Fig. 1.6

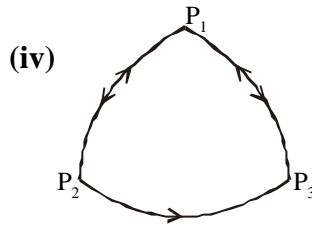


Fig. 1.7

Solution :

(i)

$$\begin{matrix} & P_1 & P_2 & P_3 \\ P_1 & \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\ P_2 & \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ P_3 & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

(ii)

$$\begin{matrix} & P_1 & P_2 & P_3 & P_4 \\ P_1 & \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \\ P_2 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \\ P_3 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \\ P_4 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

(iii)

$$\begin{matrix} & P_1 & P_2 & P_3 \\ P_1 & \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \\ P_2 & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\ P_3 & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

(iv)

$$\begin{matrix} & P_1 & P_2 & P_3 \\ P_1 & \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \\ P_2 & \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \\ P_3 & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Example 38

Draw the directed route for each route matrix given below :

(i)

$$\begin{matrix} & P_1 & P_2 & P_3 & P_4 \\ P_1 & \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \\ P_2 & \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \\ P_3 & \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \\ P_4 & \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

(ii)

$$\begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ P_1 & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \\ P_2 & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ P_3 & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ P_4 & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \\ P_5 & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ P_6 & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Solution :

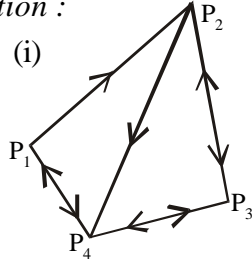


Fig. 1.8

(ii)

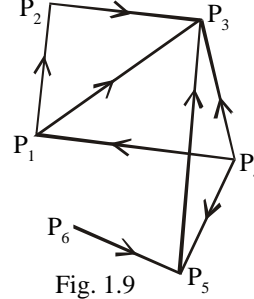


Fig. 1.9

Theorems on directed routes.

Six theorems (without proof) are stated below.

Theorem 1

If A is the route matrix then the i, j th element in A^r is the number of ways in which P_i has access to P_j in r stages

Theorem 2

If A is the route matrix then the sum of all the elements in the j th column of A^r is the number of ways in which P_j is accessible by all individuals in r stages.

Theorem 3

If A is the route matrix then the i, j th element in $A + A^2 + A^3 + \dots + A^r$ is the number of ways in which P_i has access to P_j in one, two, ... or r stages.

Theorem 4

If A is the route matrix then the sum of all the elements in the j th column in $A + A^2 + A^3 + \dots + A^r$ is the number of ways in which P_j is accessible by all individuals in one, two, ... or r stages.

Theorem 5

A directed route with n vertices and having the route matrix A is strongly connected if $A + A^2 + A^3 + \dots + A^n$ has no zero entries.

Theorem 6

If there are n vertices in a route and A is its route matrix then its path matrix is got by replacing each non zero element in $A + A^2 + A^3 + \dots + A^n$ by 1.

Now we shall have a few examples illustrating the application of these theorems.

Example 39

Consider the following directed route G .

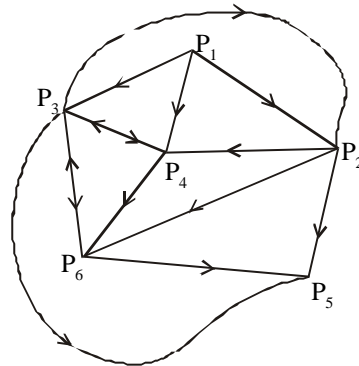


Fig. 1.10

- (i) Find the route matrix of G .
- (ii) Find the number of ways in which P_1 has access to P_3 in 3 stages. Indicate the paths.
- (iii) Find the number of paths from P_1 to P_5 in 3 stages. Indicate the paths.
- (iv) Find the number of ways in which P_6 can be accessed by others in 3 stages.
- (v) Find the number of ways in which P_1 has access to P_5 in one, two or three stages.
- (vi) Find the number of ways in which P_6 can be accessed by others in 3 or less stages.

Solution :

- (i) Route matrix of G is

$$A = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$(ii) \quad A^2 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix} \quad A^3 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{matrix} & \begin{pmatrix} 0 & 1 & 5 & 2 & 5 & 4 \\ 0 & 2 & 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & 2 & 4 & 3 \\ 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & 2 \end{pmatrix} \end{matrix}$$

P_1 has access to P_3 in 3 stages in 5 ways.

The paths are,

$P_1 P_2 P_4 P_3$, $P_1 P_2 P_6 P_3$, $P_1 P_4 P_6 P_3$, $P_1 P_3 P_4 P_3$ and $P_1 P_3 P_6 P_3$.

(iii) The number of paths from P_1 to P_5 in 3 stages is 5.

The paths are,

$P_1 P_2 P_6 P_5$, $P_1 P_3 P_2 P_5$, $P_1 P_3 P_6 P_5$, $P_1 P_4 P_6 P_5$ and $P_1 P_4 P_3 P_5$

(iv) The number of ways in which P_6 can be accessed by others in 3 stages = $4 + 2 + 3 + 3 + 0 = 12$

$$(v) \quad A + A^2 + A^3 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{matrix} & \begin{pmatrix} 0 & 3 & 7 & 5 & 7 & 7 \\ 0 & 2 & 3 & 3 & 5 & 4 \\ 0 & 3 & 5 & 4 & 7 & 6 \\ 0 & 2 & 4 & 3 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 4 & 3 \end{pmatrix} \end{matrix}$$

P_1 has access to P_5 in one, two, or three stages in 7 ways.

- (vi) The number of ways in which P_6 can be accessed by others in 3 or less stages = $7 + 4 + 6 + 5 + 0 = 22$

Example 40

Show, by using the route matrix and its powers, that the directed route G given below is strongly connected.

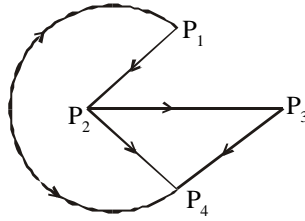


Fig. 1.11

Solution :

The route matrix of G is

$$A = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Since there are 4 vertices, let us find $A + A^2 + A^3 + A^4$.

$$A^2 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{matrix},$$

$$A^3 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

$$A^4 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \end{matrix},$$

$$A + A^2 + A^3 + A^4 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{pmatrix} 4 & 3 & 2 & 6 \\ 3 & 2 & 2 & 5 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix} \end{matrix}$$

There is no zero entry in this matrix

$\therefore G$ is strongly connected.

Example 41

Given a directed route :

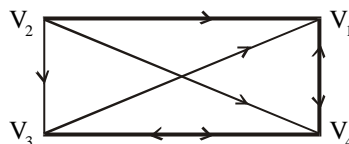


Fig. 1.12

Find the route matrix of G and using its powers examine whether G is strongly connected.

Solution :

Route matrix of G is

$$A = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Since there are 4 vertices, let us find $A + A^2 + A^3 + A^4$.

$$A^2 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix} \end{matrix} \quad A^3 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 2 \\ 3 & 0 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix} \end{matrix}$$

$$A^4 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 2 & 0 & 2 & 1 \\ 5 & 0 & 3 & 5 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 1 & 4 \end{pmatrix} \end{matrix}$$

$$\therefore A + A^2 + A^3 + A^4 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 4 & 0 & 3 & 4 \\ 11 & 0 & 7 & 11 \\ 7 & 0 & 4 & 7 \\ 7 & 0 & 4 & 7 \end{pmatrix} \end{matrix}$$

There is zero entry in this matrix. Hence G is not strongly connected.

Example 42

Show that the directed route G with route matrix

$$A = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{matrix} \text{ is strongly connected.}$$

Solution:

Since there are 3 vertices, we find $A + A^2 + A^3$.

$$A^2 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{matrix}, \quad A^3 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{matrix}.$$

$$\therefore A + A^2 + A^3 = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix} \end{matrix}$$

There is no zero entry in this matrix. $\therefore G$ is strongly connected.

Example 43

Find the path matrix of the directed route given below by using the powers of its route matrix.

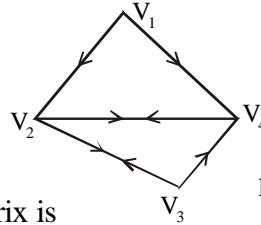


Fig. 1.13

Solution :

The route matrix is

$$A = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Since there are 4 vertices, let us find $A + A^2 + A^3 + A^4$.

$$A^2 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{matrix}, \quad A^3 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{pmatrix} \end{matrix},$$

$$A^4 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 3 & 2 & 3 \\ 0 & 4 & 1 & 3 \\ 0 & 3 & 2 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix} \end{matrix} \therefore A + A^2 + A^3 + A^4 = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 7 & 4 & 7 \\ 0 & 7 & 4 & 7 \\ 0 & 7 & 4 & 7 \\ 0 & 4 & 3 & 4 \end{pmatrix} \end{matrix}$$

Replacing each non zero entry by 1, we get the path matrix

$$P = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

Example 44

The route matrix of a directed route G is

$$A = \begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Find the path matrix of G without using the powers of A.

Solution :

The directed route G is

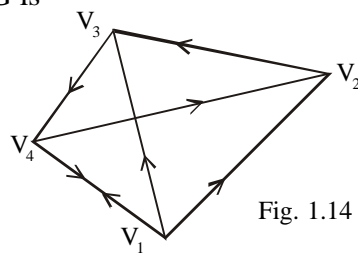


Fig. 1.14

The path matrix is written directly from G.

$$P = \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

1.4.3 Cryptography

Cryptography is the study of coding and decoding secret messages. A non singular matrix can be effectively used for this. The following example illustrates the method.

Example 45

Using the substitution scheme,

1	2	3	4	5	6	7	8	9	10	11	12	13
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
A	B	C	D	E	F	G	H	I	J	K	L	M
14	15	16	17	18	19	20	21	22	23	24	25	26
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

and the matrix $A = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$

(i) Code the message : **HARD WORK** and

(ii) Decode the message : **98, 39, 125, 49, 80, 31**

Solution :

(i) Using the substitution scheme,

H A R D W O R K
8 1 18 4 23 15 18 11

Grouping them,

$$\begin{pmatrix} 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 18 \\ 4 \end{pmatrix}, \begin{pmatrix} 23 \\ 15 \end{pmatrix}, \begin{pmatrix} 18 \\ 11 \end{pmatrix}$$

Applying the transformation $AX = B$,

$$\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 43 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} 102 \\ 40 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 23 \\ 15 \end{pmatrix} = \begin{pmatrix} 160 \\ 61 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 18 \\ 11 \end{pmatrix} = \begin{pmatrix} 123 \\ 47 \end{pmatrix}$$

The coded message is

43, 17, 102, 40, 160, 61, 123, 47

(ii) 98, 39, 125, 49, 80, 31

Grouping them,

$$\begin{pmatrix} 98 \\ 39 \end{pmatrix}, \begin{pmatrix} 125 \\ 49 \end{pmatrix}, \begin{pmatrix} 80 \\ 31 \end{pmatrix}$$

Now solve for $AX = B$.

$$\therefore X = A^{-1}B, \text{ where } A^{-1} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 98 \\ 39 \end{pmatrix} = \begin{pmatrix} 19 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 125 \\ 49 \end{pmatrix} = \begin{pmatrix} 22 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 80 \\ 31 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

Hence we have

19, 1, 22, 5, 13, 5

Using the substitution scheme, the message decoded is

S A V E M E

Note

When a 2×2 non singular matrix is used, we group the numbers in twos in order. If a number is left out without being paired, we can include one extraneous number of our own as the last number and ignore its decoding, when the process is over. When a 3×3 matrix is used we group the numbers in threes in order. If need be one or two extraneous numbers may be included and dispensed with when the process is over.

EXERCISE 1.4

- 1) Find the relation matrix for the relation R from $\{2, 5, 8, 9\}$ to $\{6, 8, 9, 12\}$ defined by $x R y$ if x divides y exactly.
- 2) Let $S = \{2, 4, 6, 9\}$ and R be the relation on S defined by $m R n$ if $m > n$. Write out the relation matrix for R .

3) Given the relation matrix $R = \begin{matrix} & \begin{matrix} l & m \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix}$

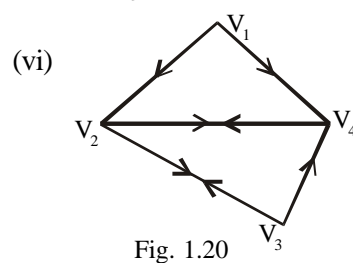
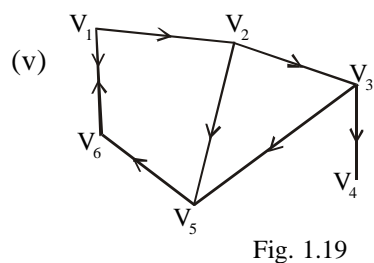
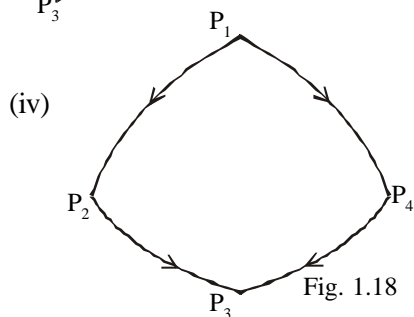
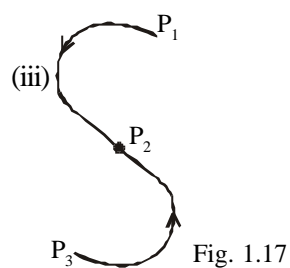
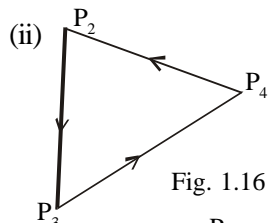
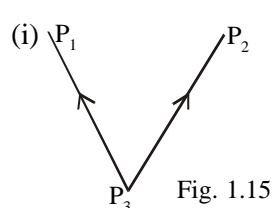
Write the relation R in the form of a set of ordered pairs.

4) Given the relation matrix $R = \begin{matrix} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \end{matrix}$

Write down the matrix for the inverse relation R^{-1} .

- 5) Let R be the relation from $X = \{3, 5, 9\}$ to $Y = \{4, 3, 8\}$ defined by $x R y$ if $x + y > 10$. Let S be the relation from Y to $Z = \{1, 2, 5\}$ defined by $y S z$ if $y < z$. Find the relation matrices for R , S and $R \circ S$.
- 6) Find the relation matrix for the relation.
 $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ on $\{1, 2, 3, 4\}$.
Use it to identify the type of the relation.
- 7) Find the relation matrix for the relation
 $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$ on $\{1, 2, 3, 4\}$.
Using it, decide the type of the relation.

- 8) Find the relation matrix for the relation
 $R = \{(2, 2), (3, 3), (4, 4), (1, 2)\}$ on $\{1, 2, 3, 4\}$.
 Using it, decide the nature of the relation.
- 9) Find the relation matrix for the relation
 $R = \{(1, 2), (2, 3)\}$ on $\{1, 2, 3, 4\}$.
 Using it, identify the type of the relation.
- 10) Find the route matrix for each of the directed routes :



- 11) Draw the directed route for each of the following route matrices

(i) $A = \begin{matrix} & \begin{matrix} P_1 & P_2 & P_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}$

(ii) $\begin{matrix} & \begin{matrix} V_1 & V_2 & V_3 & V_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$

12) Given the route matrix $M = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$

for a directed route G . Using the powers of M , find the number of paths from C to A with at most three stages. Indicate the paths.

13) Given the directed route G :

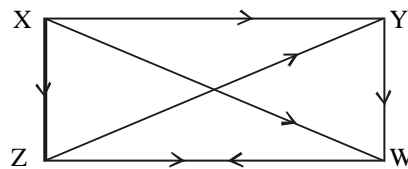


Fig. 1.21

- (i) Find the route matrix of G .
- (ii) Find whether G is strongly connected, by using the powers of the route matrix.
- (iii) Find the path matrix of G .

14) Given the following directed route G :

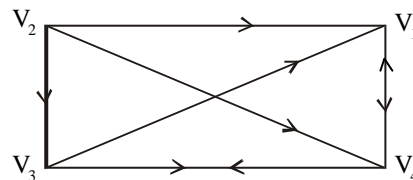


Fig. 1.22

- (i) Find the route matrix of G .
- (ii) Find the number of paths of length 3 from V_2 to V_3 . Indicate them.
- (iii) Find the number of paths of length 4 from V_2 to V_4 . Indicate them.
- (iv) Find the number of paths from V_4 to V_1 of length 3 or less. Indicate them.

- (v) Find the number of ways in which V_4 can be accessed by others in one, two or three stages.
- (vi) Is G strongly connected?
- (vii) Find the path matrix of G .
- 15) Given a directed route G :

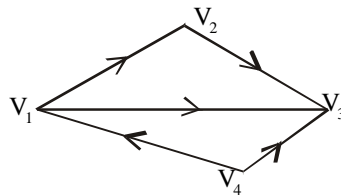


Fig. 1.23

Find the route matrix of G and using its powers show that G is not strongly connected.

- 16) Given the route matrix for a directed route G :

$$\begin{matrix} & P_1 & P_2 & P_3 & P_4 \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix} \quad \text{Show that } G \text{ is strongly connected.}$$

- 17) Given the directed route G :

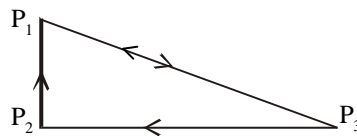


Fig. 1.24

Find the route matrix and using its powers, find the path matrix.

- 18) The route matrix of a directed route G is

$$A = \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Find its path matrix without using the powers of A .

- 19) Given a directed route G :

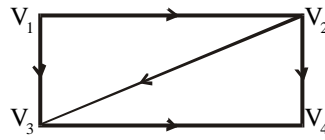


Fig. 1.25

Find its route and path matrices.

- 20) Using the substitution scheme,

1	2	3	4	5	6	7	8	9	10	11	12	13
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
A	B	C	D	E	F	G	H	I	J	K	L	M

14	15	16	17	18	19	20	21	22	23	24	25	26
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

and the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$

- (i) Code the message : CONSUMER and
(ii) Decode the message : 68, 48, 81, 60, 61, 42, 28, 27.

1.5 INPUT - OUTPUT ANALYSIS

Consider a simple economic model consisting of two industries A_1 and A_2 , where each produces only one type of product. Assume that each industry consumes part of its own output and rest from the other industry for its operation. The industries are thus interdependent. Further assume that whatever is produced is consumed. That is the total output of each industry must be such as to meet its own demand, the demand of the other industry and the external demand that is the final demand.

Our aim is to determine the output levels of each of the two industries in order to meet a change in final demand, based on a knowledge of the current outputs of the two industries, of course under the assumption that the structure of the economy does not change.

Let a_{ij} be the rupee value of the output of A_i consumed by A_j , $i, j = 1, 2$

Let x_1 and x_2 be the rupee value of the current outputs of A_1 and A_2 respectively.

Let d_1 and d_2 be the rupee value of the final demands for the outputs of A_1 and A_2 respectively.

These assumptions lead us to frame the two equations

$$\left. \begin{aligned} a_{11} + a_{12} + d_1 &= x_1 \\ a_{21} + a_{22} + d_2 &= x_2 \end{aligned} \right\} \text{-----(1)}$$

$$\text{Let } b_{ij} = \frac{a_{ij}}{x_j}, i, j = 1, 2$$

That is

$$b_{11} = \frac{a_{11}}{x_1}, b_{12} = \frac{a_{12}}{x_2}, b_{21} = \frac{a_{21}}{x_1}, b_{22} = \frac{a_{22}}{x_2},$$

Then equations (1) take the form

$$b_{11} x_1 + b_{12} x_2 + d_1 = x_1$$

$$b_{21} x_1 + b_{22} x_2 + d_2 = x_2$$

These can be rearranged as

$$(1 - b_{11}) x_1 - b_{12} x_2 = d_1$$

$$-b_{21} x_1 + (1 - b_{22}) x_2 = d_2$$

This takes the matrix form

$$\begin{pmatrix} 1 - b_{11} & -b_{12} \\ -b_{21} & 1 - b_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

That is $(I - B) X = D$

$$\text{Where } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Solving this

$$X = (I - B)^{-1} D.$$

The matrix B is known as the **technology matrix**.

Hawkins - Simon conditions ensure the viability of the system.

If B is the technology matrix then Hawkins – Simon conditions are

- (i) the main diagonal elements in $I - B$ must be positive and
- (ii) $|I - B|$ must be positive.

Example 46

The data below are about an economy of two industries P and Q. The values are in lakhs of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	16	12	12	40
Q	12	8	4	24

Find the technology matrix and test whether the system is viable as per Hawkins - Simon conditions.

Solution :

With the usual notation we have,

$$a_{11}=16, \quad a_{12}=12, \quad x_1=40$$

$$a_{21}=12, \quad a_{22}=8, \quad x_2=24$$

Now

$$b_{11} = \frac{a_{11}}{x_1} = \frac{16}{40} = \frac{2}{5}, \quad b_{12} = \frac{a_{12}}{x_2} = \frac{12}{24} = \frac{1}{2},$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{12}{40} = \frac{3}{10}, \quad b_{22} = \frac{a_{22}}{x_2} = \frac{8}{24} = \frac{1}{3}.$$

\therefore The technology matrix is

$$B = \begin{pmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{3} \end{pmatrix}$$

$$I - B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{5} & \frac{1}{2} \\ \frac{3}{10} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{2} \\ -\frac{3}{10} & \frac{2}{3} \end{pmatrix}$$

The main diagonal elements in $I - B$ viz., $\frac{3}{5}$ and $\frac{2}{3}$ are positive. Also

$$|I - B| = \begin{vmatrix} \frac{3}{5} & -\frac{1}{2} \\ -\frac{3}{10} & \frac{2}{3} \end{vmatrix} = \frac{1}{4} . \quad |I - B| \text{ is positive.}$$

\therefore The two Hawkins - Simon conditions are satisfied. Hence the system is viable.

Example 47

In an economy there are two industries P and Q and the following table gives the supply and demand positions in crores of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	10	25	15	50
Q	20	30	10	60

Determine the outputs when the final demand changes to 35 for P and 42 for Q.

Solution :

With the usual notation we have,

$$\begin{aligned} a_{11} = 10, a_{12} = 25 & \quad x_1 = 50 \\ a_{21} = 20, a_{22} = 30 & \quad x_2 = 60 \end{aligned}$$

Now,

$$\begin{aligned} b_{11} &= \frac{a_{11}}{x_1} = \frac{10}{50} = \frac{1}{5}, \quad b_{12} = \frac{a_{12}}{x_2} = \frac{25}{60} = \frac{5}{12}, \\ b_{21} &= \frac{a_{21}}{x_1} = \frac{20}{50} = \frac{2}{5}, \quad b_{22} = \frac{a_{22}}{x_2} = \frac{30}{60} = \frac{1}{2}. \end{aligned}$$

\therefore The technology matrix is

$$\begin{aligned} B &= \begin{pmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix} \\ I - B &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{pmatrix} \\ |I - B| &= \begin{vmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{vmatrix} = \frac{7}{30} \end{aligned}$$

$$(I - B)^{-1} = \frac{1}{\frac{7}{30}} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{30}{7} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix}$$

Now,

$$X = (I - B)^{-1} D$$

$$= \frac{1}{7} \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 35 \\ 42 \end{pmatrix} = \begin{pmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 150 \\ 204 \end{pmatrix}$$

The output of the industry P should be Rs.150 crores and that of Q should be Rs. 204 crores.

EXERCISE 1.5

- 1) The technology matrix of an economic system of two industries is $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{2}{3} \end{pmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.
- 2) The technology matrix of an economic system of two industries is $\begin{pmatrix} \frac{3}{5} & \frac{9}{10} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.
- 3) The technology matrix of an economic system of two industries is $\begin{pmatrix} \frac{2}{5} & \frac{1}{10} \\ \frac{7}{10} & \frac{3}{5} \end{pmatrix}$. Find the output levels when the final demand changes to 34 and 51 units.
- 4) The data below are about an economy of two industries P and Q. The values are in millions of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	14	6	8	28
Q	7	18	11	36

Determine the outputs if the final demand changes to 20 for P and 30 for Q.

- 5) Suppose the inter-relationship between the production of two industries P and Q in a year (in lakhs of rupees) is

Producer	User		Final Demand	Total Output
	P	Q		
P	15	10	10	35
Q	20	30	15	65

Find the outputs when the final demand changes to

(i) 12 for P and 18 for Q (ii) 8 for P and 12 for Q.

- 6) In an economy of two industries P and Q the following table gives the supply and demand positions in millions of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	16	20	4	40
Q	8	40	32	80

Find the outputs when the final demand changes to 18 for P and 44 for Q.

- 7) The data below are about an economy of two industries P and Q. The values are in crores of rupees.

Producer	User		Final Demand	Total Output
	P	Q		
P	50	75	75	200
Q	100	50	50	200

Find the outputs when the final demand changes to 300 for P and 600 for Q.

- 8) The inter - relationship between the production of two industries P and Q in crores of rupees is given below.

Producer	User		Total Output
	P	Q	
P	300	800	2,400
Q	600	200	4,000

If the level of final demand for the output of the two industries is 5,000 for P and 4,000 for Q, at what level of output should the two industries operate?

1.6 TRANSITION PROBABILITY MATRICES

These are matrices in which the individual elements are the probabilities of transition from one state to another of an event. The probabilities of the various changes applied to the initial state by matrix multiplication gives a forecast of the succeeding state. The following examples illustrate the method.

Example 48

Two products A and B currently share the market with shares 60% and 40 % each respectively. Each week some brand switching takes place. Of those who bought A the previous week, 70% buy it again whereas 30% switch over to B. Of those who bought B the previous week, 80% buy it again whereas 20% switch over to A. Find their shares after one week and after two weeks. If the price war continues, when is the equilibrium reached?

Solution :

Transition Probability matrix

$$T = \begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

Shares after one week

$$\begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A & B \\ 0.6 & 0.4 \end{matrix} & \begin{matrix} \begin{matrix} A & B \\ \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \end{matrix} \end{matrix} = \begin{matrix} A & B \\ 0.5 & 0.5 \end{matrix}$$

$$A = 50\%, \quad B = 50\%$$

Shares after two weeks

$$\begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A & B \\ 0.5 & 0.5 \end{matrix} & \begin{matrix} \begin{matrix} A & B \\ \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \end{matrix} \end{matrix} = \begin{matrix} A & B \\ 0.45 & 0.55 \end{matrix}$$

$$A = 45\%, \quad B = 55\%$$

Equilibrium

At equilibrium we must have

$$(A \ B) T = (A \ B) \text{ where } A + B = 1$$

$$\Rightarrow (A \ B) \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = (A \ B)$$

$$\Rightarrow 0.7 A + 0.2 B = A$$

$$\Rightarrow 0.7 A + 0.2 (1-A) = A$$

$$\text{Simplifying, we get } A = 0.4$$

\therefore Equilibrium is reached when A's share is 40% and B's share is 60%

Example 49

A new transit system has just gone into operation in a city. Of those who use the transit system this year, 10% will switch over to using their own car next year and 90% will continue to use the transit system. Of those who use their cars this year, 80% will continue to use their cars next year and 20% will switch over to the transit system. Suppose the population of the city remains constant and that 50% of the commuters use the transit system and 50% of the commuters use their own car this year,

- (i) what percent of commuters will be using the transit system after one year?**
- (ii) what percent of commuters will be using the transit system in the long run?**

Solution :

Transition Probability Matrix

$$T = \begin{matrix} & \begin{matrix} S & C \end{matrix} \\ \begin{matrix} S \\ C \end{matrix} & \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

Percentage after one year

$$\begin{matrix} & \begin{matrix} S & C \end{matrix} \\ \begin{matrix} S & C \end{matrix} & \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \end{matrix} \begin{matrix} & \begin{matrix} S & C \end{matrix} \\ \begin{matrix} S \\ C \end{matrix} & \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} S \\ C \end{matrix} & \begin{pmatrix} 0.55 & 0.45 \end{pmatrix} \end{matrix}$$

$$S = 55\% , \quad C = 45\%$$

Equilibrium will be reached in the long run.

At equilibrium we must have

$$(S \ C) T = (S \ C) \quad \text{where } S + C = 1$$

$$\Rightarrow (S \ C) \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (S \ C)$$

$$\Rightarrow 0.9S + 0.2C = S$$

$$\Rightarrow 0.9S + 0.2(1-S) = S$$

Simplifying, we get $S = 0.67$

$\therefore 67\%$ of the commuters will be using the transit system in the long run.

EXERCISE 1.6

- 1) Two products P and Q share the market currently with shares 70% and 30% each respectively. Each week some brand switching takes place. Of those who bought P the previous week, 80% buy it again whereas 20% switch over to Q. Of those who bought Q the previous week, 40% buy it again whereas 60% switch over to P. Find their shares after two weeks. If the price war continues, when is the equilibrium reached?
- 2) The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 60% of those who already subscribe will subscribe again while 25% of those who do not now subscribe will subscribe. On the last letter it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?
- 3) Two newspapers A and B are published in a city. Their present market shares are 15% for A and 85% for B. Of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after two years.

EXERCISE 1.7

Choose the correct answer

- 1) If the minor of a_{23} equals the cofactor of a_{23} in $|a_{ij}|$ then the minor of a_{23} is
(a) 1 (b) 2 (c) 0 (d) 3
- 2) The Adjoint of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is
(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$
- 3) The Adjoint of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is
(a) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
- 4) If $AB = BA = |A| I$ then the matrix B is
(a) the inverse of A (b) the transpose of A
(c) the Adjoint of A (d) 2A
- 5) If A is a square matrix of order 3 then $|\text{Adj}A|$ is
(a) $|A|^2$ (b) $|A|$ (c) $|A|^3$ (d) $|A|^4$
- 6) If $|A| = 0$ then $|\text{Adj}A|$ is
(a) 0 (b) 1 (c) -1 (d) ± 1
- 7) The inverse of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is
(a) $\begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$
(c) $\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

- 8) If $A = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$ then A^{-1} is
- (a) $\begin{pmatrix} -0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$ (b) $\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$ (c) $\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$ (d) $\begin{pmatrix} 0.2 & 0.4 \\ -0.4 & 0.2 \end{pmatrix}$
- 9) For what value of k the matrix A ,
where $A = \begin{pmatrix} 2 & k \\ 3 & 5 \end{pmatrix}$ has no inverse?
- (a) $\frac{3}{10}$ (b) $\frac{10}{3}$ (c) 3 (d) 10
- 10) If $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$ then $A^{-1}A$ is
- (a) 0 (b) A (c) I (d) A^2 .
- 11) The rank of an $n \times n$ matrix each of whose elements is 1 is
- (a) 1 (b) 2 (c) n (d) n^2
- 12) The rank of an $n \times n$ matrix each of whose elements is 2 is
- (a) 1 (b) 2 (c) n (d) n^2
- 13) The rank of a zero matrix is
- (a) 0 (b) 1 (c) -1 (d) ∞
- 14) The rank of a non singular matrix of order $n \times n$ is
- (a) n (b) n^2 (c) 0 (d) 1
- 15) A system of linear homogeneous equations has at least
- (a) one solution (b) two solutions
(c) three solutions (d) four solutions
- 16) The equations $AX = B$ can be solved by Cramer's rule only when
- (a) $|A| = 0$ (b) $|A| \neq 0$ (c) $A = B$ (d) $A \neq B$
- 17) The inverse of the relation $\begin{matrix} x & y \\ a & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ b & \end{matrix}$ is
- (a) $\begin{matrix} a & b \\ x & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ y & \end{matrix}$ (b) $\begin{matrix} x & y \\ a & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ b & \end{matrix}$ (c) $\begin{matrix} a & b \\ x & \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ y & \end{matrix}$ (d) $\begin{matrix} x & y \\ a & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ b & \end{matrix}$

- 18) The relation $R = \begin{matrix} & a & b \\ a & \begin{pmatrix} 0 & 1 \end{pmatrix} \\ b & \begin{pmatrix} 1 & 0 \end{pmatrix} \end{matrix}$ is
- (a) Reflexive (b) Symmetric
(c) Transitive (d) Reflexive and symmetric
- 19) The number of Hawkins - Simon conditions for the viability of an input - output model is
- (a) 1 (b) 3 (c) 4 (d) 2
- 20) If $T = \begin{matrix} & A & B \\ A & \begin{pmatrix} 0.7 & 0.3 \end{pmatrix} \\ B & \begin{pmatrix} x & 0.8 \end{pmatrix} \end{matrix}$ is a transition probability matrix, then the value of x is
- (a) 0.3 (b) 0.2 (c) 0.3 (d) 0.7

2.1 CONICS

Intersections of cone by a plane

The parabola, ellipse and hyperbola are all members of a class of curves called conics.

The above three curves can be obtained by cutting a cone with a plane and so they are called conics (Fig. 2.1).

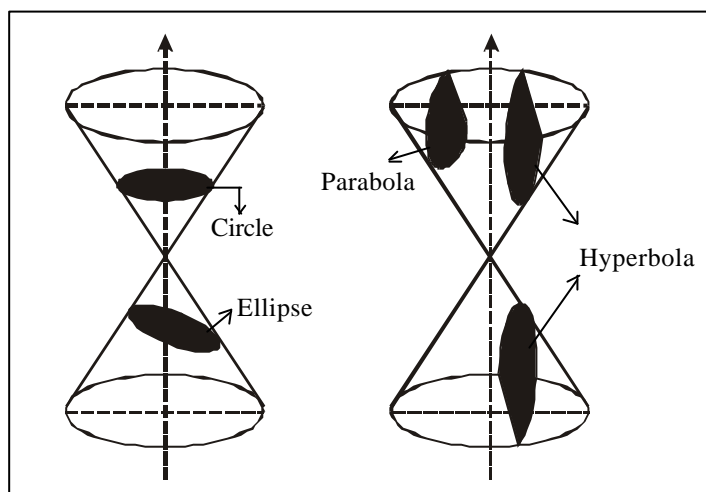


Fig.2.1

A **conic** is the locus of a point which moves in a plane such that its distance from a fixed point in the plane bears a constant ratio to its distance from a fixed straight line in that plane.

Focus, Directrix, Eccentricity: In the above definition of a conic, the fixed point is called the **focus**, the fixed line the **directrix** and the constant ratio, the **eccentricity** of the conic.

The eccentricity is usually denoted by the letter ' e '.

In the Fig. 2.2, S is the focus, the line LM is the directrix and

$$\frac{SP}{PM} = e$$

The conic is

a parabola if $e = 1$,
 an ellipse if $e < 1$ and
 a hyperbola if $e > 1$.

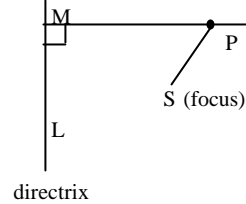


Fig.2.2

2.1.1 The general equation of a conic

We know that a conic is the locus of a point moving such that its distance from the focus bears a constant ratio to its distance from the directrix.

Let the focus be $S(x_1, y_1)$ and the directrix be $Ax + By + C = 0$

Let the eccentricity of the conic be e and $P(x, y)$ be any point on it.

$$\text{Then } SP = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

Perpendicular distance of P from $Ax + By + C = 0$ is

$$PM = \pm \frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

$$\frac{SP}{PM} = e$$

$$\Rightarrow \frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{\pm \frac{Ax + By + C}{\sqrt{A^2 + B^2}}} = e$$

$$\text{or } (x - x_1)^2 + (y - y_1)^2 = e^2 \left[\frac{(Ax + By + C)^2}{(A^2 + B^2)} \right]$$

Simplifying, we get an equation of the second degree in x and y of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

This is the general equation of a conic.

Remarks :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

- (i) a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

- (ii) a circle if $a = b$ and $h = 0$
If the above two conditions are not satisfied, then
 $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,
- (iii) a parabola if $h^2 - ab = 0$
- (iv) an ellipse if $h^2 - ab < 0$
- (v) a hyperbola if $h^2 - ab > 0$

Example 1

The equation $4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$ represents a conic. Identify the conic.

Solution :

Comparing the given equation,

$4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$ with the general second degree equation in x and y

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get

$$a = 4, 2h = 4, b = 1$$

$$\therefore h^2 - ab = (2)^2 - 4(1) = 0$$

\therefore The given conic is a parabola.

Example 2

**Identify the conic represented by
 $16x^2 + 25y^2 - 118x - 150y - 534 = 0$.**

Solution:

Here, $a = 16, 2h = 0, b = 25$

$$\therefore h^2 - ab = 0 - 16 \times 25 = -400 < 0$$

\therefore The conic is an ellipse.

EXERCISE 2.1

Identify the conics represented by the following equations:

1) $x^2 - 6xy + 9y^2 + 26x - 38y + 49 = 0$

$$2) \quad 7x^2 + 12xy - 2y^2 + 22x + 16y - 7 = 0$$

$$3) \quad 7x^2 + 2xy + 7y^2 - 60x - 4y + 44 = 0$$

2.2 PARABOLA

2.2.1 Standard Equation of parabola

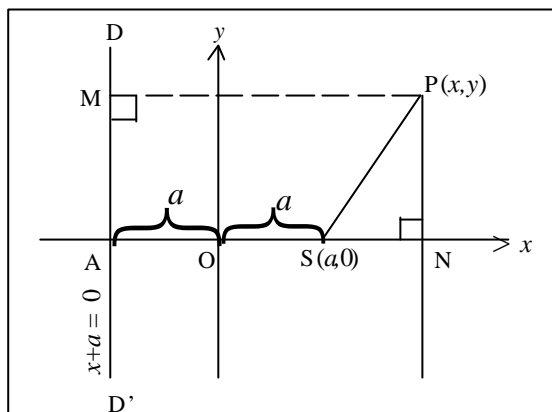


Fig. 2.5

Let S be the focus and the line DD' be the directrix. Draw SA perpendicular to DD' cutting DD' at A. Let $SA = 2a$. Take AS as the x axis and Oy perpendicular to AS through the middle point O of AS as the y axis.

Then S is $(a, 0)$ and the directrix DD' is the line $x + a = 0$.

Let $P(x, y)$ be any point on the parabola. Draw $PM \perp DD'$ and $PN \perp Ox$

$$PM = NA = NO + OA = x + a.$$

$$SP^2 = (x - a)^2 + y^2$$

$$\text{Then } \frac{SP}{PM} = e \quad [P \text{ is a point on the parabola}]$$

$$\text{or, } SP^2 = e^2 (PM)^2$$

$$\text{or, } (x - a)^2 + y^2 = (x + a)^2 \quad (\because e = 1)$$

$$\text{or, } y^2 = 4ax$$

This is the standard equation of the parabola.

Note

- (i) In any parabola the line which passes through the focus and is perpendicular to the directrix is called the **axis** of the parabola, and the point of intersection of the curve and its axis is called the **vertex**.
- (ii) The chord which passes through the focus and is perpendicular to the axis is called the **latus rectum**.

2.2.2 Tracing of the parabola $y^2 = 4ax$

- 1) (a) Putting $y = 0$, the only value of x we get is zero.
 \therefore The curve cuts the x -axis at $(0,0)$ only.
- (b) If $x < 0$, y is imaginary. Hence the curve does not exist for negative values of x .
- (c) The equation of the parabola is unaltered if y is replaced by $-y$. Hence the curve is symmetrical about x -axis.
- (d) As x increases, y also increases. As $x \rightarrow \infty$, $y \rightarrow \pm \infty$. Hence the curve diverges and assumes the form as shown in Fig. 2.4.

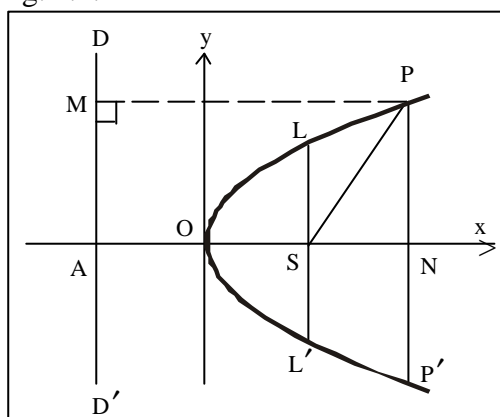


Fig. 2.4

- 2) **Directrix** : The directrix is a line parallel to the y -axis and its equation is $x = -a$ or, $x + a = 0$.
- 3) The x -axis is the **axis** of the parabola and the y -axis is the **tangent at the vertex**.

- 4) **Latus rectum:** Through S, LSL' be drawn \perp to AS.
 Corresponding to $x = a$, $y^2 = 4a^2$ or, $y = \pm 2a$
 $\therefore SL = SL' = 2a$.
 So, $LL' = 4a$. LL' is called **Latus rectum** of the parabola.
 SL (or SL') is the Semi - latus rectum.
 $OS = \frac{1}{4} LL' = a$

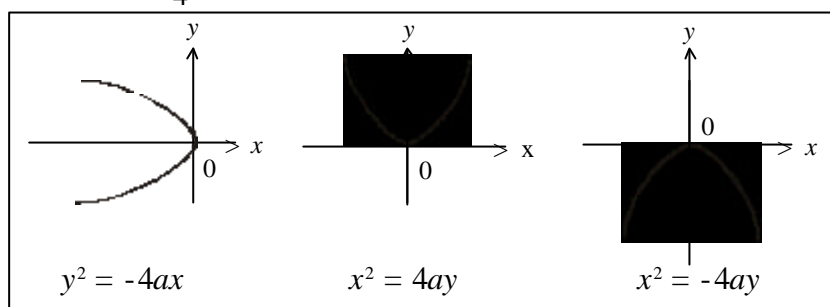


Fig. 2.5

Note

In Fig 2.5, $y^2 = -4ax$ is a parabola which lies only on the negative side of the x -axis. $x^2 = 4ay$ is a parabola whose axis of symmetry is the y -axis and it lies on the positive side of y -axis. $x^2 = -4ay$ lies on the negative side of y -axis.

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a,0)$	$(-a, 0)$	$(0,a)$	$(0,-a)$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Latus rectum	$4a$	$4a$	$4a$	$4a$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$

Example 3

Find the equation of the parabola whose focus is the point $(2, 1)$ and whose directrix is the straight line $2x + y + 1 = 0$

Solution :

Let P (x, y) be a point on the parabola.
 If PM is drawn perpendicular to the directrix,

$\frac{SP}{PM} = 1$ where S is the focus of the parabola.

$$\therefore SP^2 = PM^2$$

$$\text{or, } (x-2)^2 + (y-1)^2 = \left(\frac{2x+y+1}{\sqrt{2^2+1^2}} \right)^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = \frac{(2x+y+1)^2}{5}$$

$$5x^2 + 5y^2 - 20x - 10y + 25 = 4x^2 + y^2 + 1 + 4xy + 2y + 4x$$

$$x^2 - 4xy + 4y^2 - 24x - 12y + 24 = 0.$$

This is the required equation.

Example 4

Find the focus, latus rectum, vertex and directrix of the parabola $y^2 - 8x - 2y + 17 = 0$.

Solution :

The given equation can be written as

$$y^2 - 2y = 8x - 17$$

$$y^2 - 2y + 1 = 8x - 16 \quad (\text{completing the square})$$

$$(y-1)^2 = 8(x-2)$$

Changing the origin to the point (2,1) by putting $y-1 = Y$,
 $x-2 = X$, the equation of the parabola is $Y^2 = 8X$.

\therefore The vertex is the new origin and latus rectum is 8. So focus is the point (2,0) in the new coordinates.

So, with respect to the original axes the focus is the point (4,1) and the directrix is the line $X + 2 = 0$

$$\text{or, } x - 2 + 2 = 0 \quad \text{or} \quad x = 0$$

The vertex is $(X = 0, Y = 0) = (x = 2, y = 1) = (2, 1)$

Example 5

Find the vertex, focus, axis, directrix and length of semi-latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$

Solution :

$$4y^2 + 12x - 20y + 67 = 0$$

$$\text{or } 4y^2 - 20y = -12x - 67$$

which can be rearranged and written as $4(y^2 - 5y) = -12x - 67$

$$4 \left\{ y^2 - 5y + \frac{25}{4} - \frac{25}{4} \right\} = -12x - 67$$

$$4 \left[\left(y - \frac{5}{2} \right)^2 - \frac{25}{4} \right] = -12x - 67$$

$$4 \left(y - \frac{5}{2} \right)^2 = 25 - 12x - 67$$

$$= -12 \left(x + \frac{7}{2} \right)$$

$$\Rightarrow \left(y - \frac{5}{2} \right)^2 = 3 \left(-x - \frac{7}{2} \right)$$

To bring it to the form $Y^2 = 4aX$,

$$\text{set } X = -x - \frac{7}{2} \text{ and } Y = y - \frac{5}{2}$$

We now get $Y^2 = 3X$. Here $4a = 3$ and so $a = \frac{3}{4}$

We now tabulate the results.

Referred to (X, Y)		Referred to (x, y) $x = -X - \frac{7}{2}, y = Y + \frac{5}{2}$
Vertex	(0,0)	$\left(0 - \frac{7}{2}, 0 + \frac{5}{2} \right) = \left(-\frac{7}{2}, \frac{5}{2} \right)$
Axis	$Y = 0$ (X -axis)	$y - \frac{5}{2} = 0$ or $y = \frac{5}{2}$
Focus	$(a, 0) = \left(\frac{3}{4}, 0 \right)$	$\left(-\frac{3}{4} - \frac{7}{2}, 0 + \frac{5}{2} \right) = \left(-\frac{17}{4}, \frac{5}{2} \right)$
Directrix	$X = -a$ i.e. $X = -\frac{3}{4}$	$-x - \frac{7}{2} = -\frac{3}{4}$ or $x = -\frac{11}{4}$
Semi latus rectum	$2a = 2 \times \frac{3}{4} = \frac{3}{2}$	$\frac{3}{2}$

Note

We could also have taken a transformation $X = x + \frac{7}{2}$ and $Y = y - \frac{5}{2}$ so that we would have got $Y^2 = -3X$ and compared it with $y^2 = -4ax$

Example 6

The average cost y of a monthly output x kgs. of a firm producing a metal is Rs. $(\frac{1}{10}x^2 - 3x + 50)$. Show that the average variable cost curve is a parabola. Find the output and average cost at the vertex of the parabola.

Solution:

The average variable cost curve is

$$y = \frac{1}{10}x^2 - 3x + 50$$

$$10y = x^2 - 30x + 500$$

$$10y = (x - 15)^2 + 275$$

$$(x-15)^2 = 10y - 275$$

$$(x-15)^2 = 10(y - 27.5)$$

$$X^2 = 10Y \text{ where}$$

$$X = x-15, Y = y - 27.5 \text{ and}$$

$$4a = 10 \Rightarrow a = 2.5$$

Thus we get the average variable cost curve as a parabola whose vertex is $(X = 0, Y = 0)$

$$\text{or } (x = 15, y = 27.5)$$

$$\text{or } (15, 27.5)$$

Hence the output and average cost at the vertex are 15 kgs. and Rs. 27.50 respectively.

Example 7

The supply of a commodity is related to the price by the relation $x = 5\sqrt{2p - 10}$. Show that the supply curve is a parabola. Find its vertex and the price below which supply is 0?

Solution:

The supply price relation is given by,

$$x^2 = 25(2p - 10) \Rightarrow x^2 = 50(p - 5)$$

$$\Rightarrow X^2 = 4aP \text{ where } X = x$$

$$\text{and } P = p - 5$$

\therefore the supply curve is a parabola whose vertex is $(X = 0, P = 0)$

$$\Rightarrow (x = 0, p = 5) \Rightarrow (0, 5)$$

and supply is zero below $p = 5$.

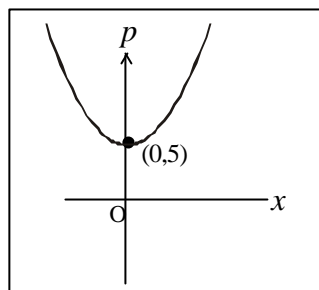


Fig.2.6

Example 8

The girder of railway bridge is a parabola with its vertex at the highest point, which is 15 metres above the span of length 150 metres. Find its height 30 metres from the mid point.

Solution :

Let the parabola be $y^2 = 4ax$.

This passes through $(15, 75)$

$$\Rightarrow (75)^2 = 4a(15)$$

$$4a = \frac{(75)^2}{15} = 375$$

Hence the parabola is $y^2 = 375x$

Now $B(x, 30)$ lies on the parabola

$$\Rightarrow 375x = 30^2$$

$$\text{or } x = \frac{900}{375} = \frac{12}{5} = 2.4$$

Required height = 2.4m

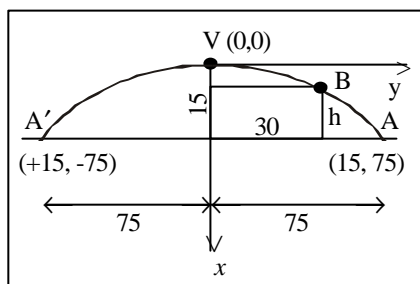


Fig.2.7

Example 9

The profit Rs. y accumulated in lakhs in x months is given by $y = -4x^2 + 28x - 40$. Find the best time to end the project.

Solution :

$$4x^2 - 28x = -40 - y$$

$$4(x^2 - 7x) = -40 - y$$

$$4(x^2 - 7x + \frac{49}{4}) = -40 - y + 49$$

$$(x - \frac{7}{2})^2 = \frac{1}{4} (9 - y)$$

$$(x - \frac{7}{2})^2 = -\frac{1}{4} (y - 9)$$

$$\text{Required time} = \frac{7}{2} = 3\frac{1}{2} \text{ months}$$

EXERCISE 2.2

- Find the equations of the parabolas with the following foci and directrices:
 - (1, 2) ; $x + y - 2 = 0$
 - (1, -1) ; $x - y = 0$
 - (0, 0) ; $x - 2y + 2 = 0$
 - (3, 4) ; $x - y + 5 = 0$
- Find the vertex, axis, focus and directrix of the following parabola:
 - $x^2 = 100y$
 - $y^2 = 20x$
 - $y^2 = -28x$
 - $x^2 = -60y$
- Find the foci, latus recta, vertices and directrices of the following parabolas:
 - $y^2 + 4x - 2y + 3 = 0$
 - $y^2 - 4x + 2y - 3 = 0$
 - $y^2 - 8x - 9 = 0$
 - $x^2 - 3y + 3 = 0$
- The average variable cost of a monthly output of x tonnes of a firm producing a valuable metal is Rs. $\frac{1}{10}x^2 - 3x + 62.5$. Show that the average variable cost curve is a parabola. Find also the output and the average cost at the vertex of the parabola.

Note

The point (x_1, y_1) lies outside, on or inside the parabola according as $y_1^2 - 4ax_1$ is greater than, equal to or less than zero.

2.3 ELLIPSE

2.3.1 Standard Equation of ellipse:

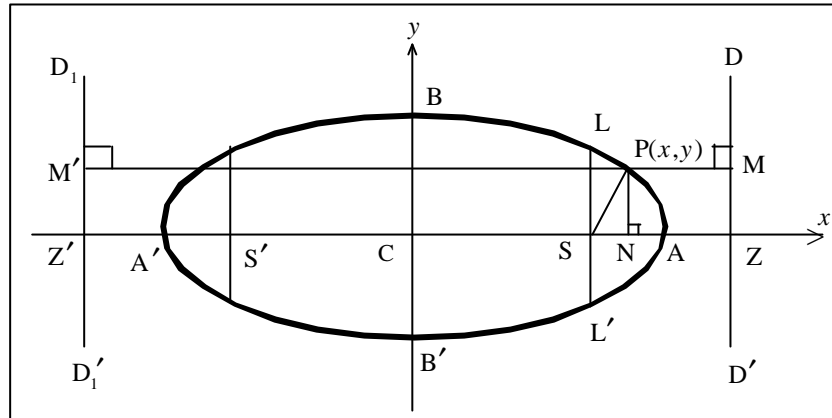


Fig.2.8

Let S be the focus and DD' be the directrix.

Draw SZ perpendicular to DD'. Let A, A' divide SZ internally and externally respectively in the ratio $e:1$ where e is the eccentricity. Then A, A' are points on the ellipse.

Let C be the mid-point of AA' and let $AA' = 2a$. Take CA to be the x axis and Cy which is \perp to CA to be the y axis. C is the origin.

$$\begin{aligned} \frac{SA}{AZ} &= e, \quad \frac{SA'}{A'Z} = e \\ \therefore SA &= e(AZ) & \text{----- (1)} \\ A'S &= e(A'Z) & \text{----- (2)} \\ (1) + (2) &\Rightarrow SA + A'S = e(AZ + A'Z) \\ AA' &= e(CZ - CA + A'C + CZ) \\ 2a &= e(2CZ) \quad (\text{since } CA = CA') \\ \Rightarrow CZ &= \frac{a}{e} \\ (2) - (1) &\Rightarrow A'S - SA = e(A'Z - AZ) \\ A'C + CS - (CA - CS) &= e(AA') \end{aligned}$$

$$\text{or} \quad 2CS = e \cdot 2a$$

$$\Rightarrow \quad CS = ae$$

So, S is the point $(ae, 0)$.

Let $P(x, y)$ be any point on the ellipse, draw $PM \perp DD'$ and $PN \perp CZ$.

$$\Rightarrow \quad PM = NZ = CZ - CN$$

$$= \frac{a}{e} - x$$

$$\frac{SP}{PM} = e \quad (\text{since the point } P \text{ lies on the ellipse})$$

$$SP^2 = e^2 PM^2$$

$$(x - ae)^2 + y^2 = e^2 \left(\frac{a}{e} - x\right)^2 = (a - ex)^2$$

$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$$\text{Put} \quad b^2 = a^2(1 - e^2)$$

$$\text{Hence} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$$

This is the equation of ellipse in the standard form.

2.3.2 Tracing of the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- (i) The curve does not pass through the origin. When $y = 0$, $x = \pm a$. Therefore it meets the x axis at the points $(\pm a, 0)$. Similarly it meets the y axis at the points $(0, \pm b)$.
- (ii) The curve is symmetrical about both the axes of co-ordinates since both the powers of x and y are even. If (x, y) be a point on the curve, so also are $(-x, y)$, $(x, -y)$ and $(-x, -y)$.
- (iii) We may put the equation of the ellipse in the form

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

If $|x| > a$. i.e. if either $x > a$ or $x < -a$, $a^2 - x^2$ becomes negative so that $\sqrt{a^2 - x^2}$ is imaginary. Hence there is no point of the curve lying either to the right of the line $x = a$ or to the left of the line $x = -a$.

If $|x| \leq a$ i.e. if $-a \leq x \leq a$, the expression under the radical sign is positive and we get two equal and opposite values of y .

The curve lies entirely between these two lines. Note that $x = a$ is a tangent at $A(a, 0)$ and $x = -a$ is a tangent at $A'(-a, 0)$

- (iv) Similarly writing the equation in the form

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$$

we find that the curve does not extend above the line $y = b$ and below the line $y = -b$.

In fact the curve lies entirely between the lines $y = b$, and $y = -b$ which are respectively tangents to the curve at B and B'.

- (v) If x increases from 0 to a , y decreases from b to 0. Similarly if y increases from 0 to b , x decreases from a to 0.
- (vi) **Latus rectum :** Through S, LSL' be drawn perpendicular to AS. Corresponding to $x = ae$, we have

$$\frac{a^2 e^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } y^2 = b^2 (1 - e^2) = b^2 \frac{b^2}{a^2} = \frac{b^4}{a^2}$$

or $y = \pm \frac{b^2}{a} \Rightarrow SL = SL' = \frac{b^2}{a}$. So, $LL' = \frac{2b^2}{a}$ is the latus rectum of the ellipse.

These information about the curve are sufficient to enable us to find the shape of the curve as given in the Fig. 2.9. Unlike the parabola, the ellipse is a closed curve.

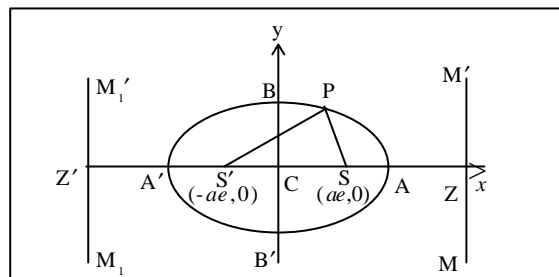


Fig. 2.9

An important property

If P is any point on the ellipse whose foci are S and S', then $SP + S'P = 2a$ where $2a$ is the length of the major axis.

2.3.3 Centre, vertices, foci, axes and directrices for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$)

(i) Centre

We have seen that if (x, y) be a point on the curve, then $(-x, -y)$ is also a point on the curve. Again if $(x, -y)$ is a point on the curve, $(-x, y)$ is also a point on the curve. This shows that every line through C meets the curve at two points equidistant from C. Thus every chord through C is bisected at C. The point C is therefore called the centre of the ellipse. $C(0,0)$ is the middle point of AA' .

(ii) Vertices

The points A and A' where the line joining the foci S and S' meets the curve are called the vertices of the ellipse. A is the point $(a, 0)$ and A' is the point $(-a, 0)$.

(iii) Foci

The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the ellipse.

(iv) Axes

The two lines AA' and BB' with respect to which the curve is symmetrical are called respectively the major axis and the minor axis of the ellipse.

Since $e < 1$, $1 - e^2$ is also less than unity.

Therefore, $b^2 = a^2 (1 - e^2)$ is less than a^2 so that $b < a$.

Thus $BB' < AA'$. AA' is called the **major axis** and BB' the **minor axis**. The segment $CA = a$ is called the semi-major axis and the segment $CB = b$ the semi minor axis.

(v) Directrices

Equation to the directrix MZ in the Fig. 2.9 is $x = \frac{a}{e}$

Equation to the directrix $M_1'Z'$ in the Fig. 2.9 is $x = -\frac{a}{e}$

Due to symmetry, there are two directrices.

$$(vi) \quad b^2 = a^2 (1 - e^2) \quad \backslash \quad e = \sqrt{1 - \frac{b^2}{a^2}}$$

Example 10

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(-1, 1)$ and the corresponding directrix is $x - y + 3 = 0$.

Solution :

Given the focus is $S(-1, 1)$, directrix is $x - y + 3 = 0$ and

$$e = \frac{1}{2}$$

Let $P(x_1, y_1)$ be any point on the ellipse. Then

$SP^2 = e^2 PM^2$ where PM is the perpendicular distance of $x - y + 3 = 0$ from P.

$$(x_1 + 1)^2 + (y_1 - 1)^2 = \frac{1}{4} \left(\frac{x_1 - y_1 + 3}{\sqrt{1+1}} \right)^2$$

$$8(x_1 + 1)^2 + 8(y_1 - 1)^2 = (x_1 - y_1 + 3)^2$$

$$7x_1^2 + 2x_1y_1 + 7y_1^2 + 10x_1 - 10y_1 + 7 = 0$$

Locus of (x_1, y_1) , i.e., the equation of the ellipse is

$$7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$$

Example 11

Find the equation of the ellipse whose foci are (2,0) and (-2, 0) and eccentricity is $\frac{1}{2}$.

Solution:

We know that S (ae , 0) and S' ($-ae$, 0) are the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($a > b$)

Now the foci are (2,0) and (-2, 0) and $e = \frac{1}{2}$

$$\begin{aligned} \Rightarrow ae &= 2 & \text{and} & \quad e = \frac{1}{2} \\ \Rightarrow a &= 4 & \text{or} & \quad a^2 = 16 \end{aligned}$$

The centre C is the mid-point of SS' and hence C is (0,0). S and S' are on the x axis. So the equation of the ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$b^2 = a^2 (1 - e^2) = 16 \left(1 - \frac{1}{4}\right) = 12$$

\therefore the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Example 12

Find the eccentricity, foci and latus rectum of the ellipse $9x^2 + 16y^2 = 144$.

Solution :

The equation of the ellipse is

$$9x^2 + 16y^2 = 144 \quad \therefore \frac{x^2}{16} + \frac{y^2}{9} = 1$$

The given equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then $a = 4$ and $b = 3$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

The foci are S ($ae, 0$) and S' ($-ae, 0$).

or, S($\sqrt{7}, 0$) and S'($-\sqrt{7}, 0$)

$$\text{The latus rectum} = \frac{2b^2}{a} = \frac{2(3^2)}{4} = \frac{9}{2}$$

Example 13

Find the centre, eccentricity, foci and directrices of the ellipse $3x^2 + 4y^2 - 6x + 8y - 5 = 0$

Solution :

The given equation can be written as

$$(3x^2 - 6x) + (4y^2 + 8y) = 5$$

$$3(x^2 - 2x) + 4(y^2 + 2y) = 5$$

$$\Rightarrow 3(x^2 - 2x + 1 - 1) + 4(y^2 + 2y + 1 - 1) = 5$$

$$\Rightarrow 3(x-1)^2 + 4(y+1)^2 = 5 + 3 + 4 = 12$$

$$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y+1)^2}{3} = 1$$

If we put $X = x - 1$ and $Y = y + 1$ in the above equation

$$\text{we get, } \frac{X^2}{4} + \frac{Y^2}{3} = 1$$

which can be compared with the standard equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$b^2 = a^2 (1 - e^2) \text{ gives } 3 = 4(1 - e^2) \text{ or } e^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$$

Now let us tabulate the results:

	Referred to (X, Y)	Referred to (x, y) $x = X+1, y = Y-1$
Centre	(0,0)	(0+1, 0-1) = (1, -1)
Foci	($\pm ae, 0$) = (1,0) and (-1, 0)	(2, -1) and (0, -1)
Directrices	$X = \pm \frac{a}{e}$ or $X = \pm 4$	$x - 1 = \pm 4$ or $x = 5$ and $x = -3$

EXERCISES 2.3

- 1) Find the equation of the ellipse whose
 - (i) focus is (1, 2) directrix is $2x - 3y + 6 = 0$ and eccentricity is $\frac{2}{3}$
 - (ii) focus is (0, 0) directrix is $3x + 4y - 1 = 0$ and eccentricity is $\frac{5}{6}$
 - (iii) focus is (1, -2) directrix is $3x - 2y + 1 = 0$ and $e = \frac{1}{\sqrt{2}}$
- 2) Find the equation of the ellipse whose
 - (i) foci are (4, 0) and (-4, 0) and $e = \frac{1}{3}$
 - (ii) foci are (3, 0) and (-3, 0) and $e = \sqrt{\frac{3}{8}}$
 - (iii) the vertices are $(0, \pm 5)$ and foci are $(0, \pm 4)$.
- 3) Find the centre, vertices, eccentricity, foci and latus rectum and directrices of the ellipse.
 - (i) $9x^2 + 4y^2 = 36$
 - (ii) $7x^2 + 4y^2 - 14x + 40y + 79 = 0$
 - (iii) $9x^2 + 16y^2 + 36x - 32y - 92 = 0$

2.4 HYPERBOLA

2.4.1 Standard Equation of Hyperbola

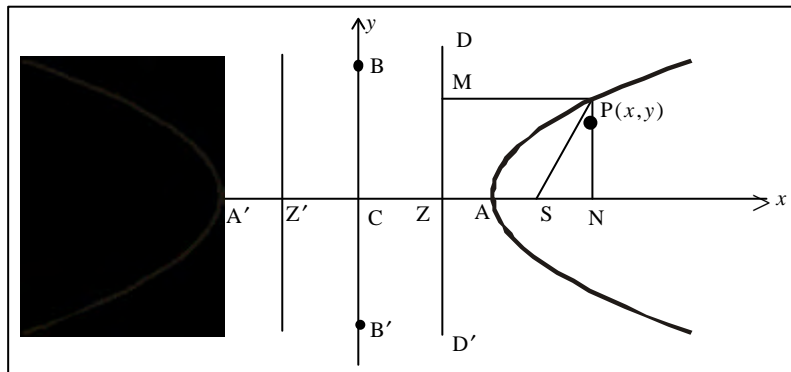


Fig. 2.10

Let S be the focus and DD' be the directrix. Draw $SZ \perp DD'$. Let A, A' divide SZ internally and externally respectively

in the ratio $e:1$ where e is the eccentricity; then A, A' are points on the hyperbola. Take C , the mid-point of AA' as origin, CZ as x-axis and Cy perpendicular to CZ as y-axis.

Let $AA' = 2a$. Now $\frac{SA}{AZ} = e$, $\frac{SA'}{A'Z} = e$.

$$\text{So } SA = e(AZ) \quad \text{-----(1)}$$

$$\text{and } SA' = e(A'Z) \quad \text{-----(2)}$$

$$(1) + (2) \Rightarrow SA + SA' = e(AZ + A'Z)$$

$$\text{or, } CS - CA + CS + CA' = e \cdot AA'$$

$$2CS = e \cdot 2a$$

$$CS = ae$$

$$(2) - (1) \Rightarrow SA' - SA = e(A'Z - AZ)$$

$$\text{or, } AA' = e(CZ + CA' - CA + CZ)$$

$$2a = e \cdot 2CZ$$

$$\text{So, } CZ = \frac{a}{e}$$

Let $P(x, y)$ be any point on the hyperbola, and let $PM \perp DD'$ and $PN \perp CA$.

$$\text{Then } \frac{SP}{PM} = e \quad \text{or, } SP^2 = e^2 PM^2$$

$$\begin{aligned} \Rightarrow (x - ae)^2 + (y - 0)^2 &= e^2 [CN - CZ]^2 \\ &= e^2 \left(x - \frac{a}{e}\right)^2 = (xe - a)^2 \end{aligned}$$

$$\Rightarrow x^2(e^2 - 1) - y^2 = a^2e^2 - a^2$$

$$\Rightarrow x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

$$\text{Put } b^2 = a^2(e^2 - 1)$$

$$\text{Hence } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the standard equation of hyperbola.

The line AA' is the **transverse axis** and the line through C which is perpendicular to AA' is the **conjugate axis** of the hyperbola.

2.4.2 Tracing of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- (i) The curve does not pass through the origin. When $y = 0$, $x = \pm a$. Therefore it meets the x axis at the points $(\pm a, 0)$. Thus the points A and A' in which the curve cuts the x axis are equidistant from the centre.

We have $CA = CA' = a$ and $AA' = 2a$.

Similarly when $x = 0$, y becomes imaginary. Therefore the curve does not cut the y -axis.

Let us take two points B and B' on the y -axis such that $CB = CB' = b$. Then $BB' = 2b$.

- (ii) The curve is symmetrical about both x axis and y axis since both the powers of x and y are even. If (x, y) be a point on the curve, so also are the points $(-x, y)$, $(x, -y)$ and $(-x, -y)$.

- (iii) We write the equation of the hyperbola in the form

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$$

If $|x| \geq a$ i.e. if either $x \geq a$ or $x \leq -a$, $x^2 - a^2 \geq 0$ and we get two equal and opposite values of y . In this case, as the value of x numerically increases, the corresponding two values of y increase numerically.

The curve therefore consists of two branches each extending to infinity in two directions as shown in the Fig. 2.11.

If $|x| < a$ or, $-a < x < a$, then $x^2 - a^2$ is a negative quantity. Therefore y imaginary and there is no point of the curve between the lines $x = -a$ and $x = a$. The curve lies to the left of the line $x = -a$ and to the right of $x = a$.

- (iv) Similarly by writing the equation in the form

$$x = \pm \frac{a}{b} \sqrt{y^2 + b^2}$$

we find that y can have any real value without limitation and that for each value of y we get two equal and opposite values of x .

These information about the curve are sufficient to enable us to find the shape of the curve. The curve drawn is therefore as shown in the Fig. 2.11.

- (v) **Latus rectum :** Through S , LSL' be drawn perpendicular to AS . Corresponding to $x = ae$, we have

$$\frac{a^2 e^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } y^2 = b^2 (e^2 - 1) = b^2 \frac{b^2}{a^2} = \frac{b^4}{a^2}$$

or $y = \pm \frac{b^2}{a} \Rightarrow SL = SL' = \frac{b^2}{a}$. So, $LL' = \frac{2b^2}{a}$ is the latus rectum of the hyperbola.

Note that the hyperbola is not a closed curve. The curve consists of two parts detached from each other.

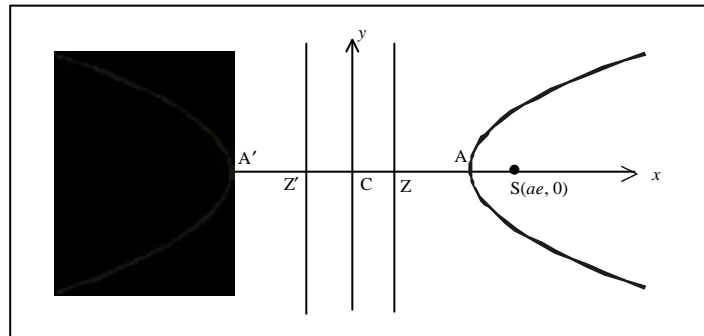


Fig.2.11

An important property:

The difference between the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis of the hyperbola. i.e., $SP - S'P = 2a$.

2.4.3 Asymptote of a curve

A straight line which touches a curve at infinity but does not lie altogether at infinity is called an **asymptote** of that curve.

Note

$ax^2 + bx + c = 0$ has both roots equal to zero if $b = c = 0$ and both roots infinite if $a = b = 0$.

The asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The coordinates of the point of intersection of the line $y = mx + c$ and the hyperbola are given by $\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$

$$x^2 \left(\frac{1}{a^2} - \frac{m^2}{b^2} \right) - \frac{2mc}{b^2}x - \frac{c^2}{b^2} - 1 = 0$$

$$x^2 (b^2 - a^2m^2) - 2ma^2cx - a^2c^2 - a^2b^2 = 0$$

Now if $y = mx + c$ is an asymptote, both the roots of this equation are infinite.

\therefore Coefficient of $x = 0$ and coefficient of $x^2 = 0$.

$$\Rightarrow -2ma^2c = 0 \text{ and } b^2 - a^2m^2 = 0. \quad \therefore c = 0, m = \pm \frac{b}{a}$$

Hence there are two asymptotes, namely

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

$$\text{or, } \frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0$$

Their combined equation is

$$\left(\frac{x}{a} - \frac{y}{b} \right) \left(\frac{x}{a} + \frac{y}{b} \right) = 0 \quad \text{or, } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Note

- (i) The asymptotes evidently pass through the centre $C(0,0)$ of the hyperbola (Fig 2.12).
- (ii) The slopes of the asymptotes are $\frac{b}{a}$ and $-\frac{b}{a}$. Hence the asymptotes are equally inclined to the transverse axis. That is, the transverse and conjugate axes bisect the angles between the asymptotes (Fig 2.12).

- (iii) If 2α is the angle between the asymptotes then $\tan \alpha = \frac{b}{a}$
 \therefore Angle between the asymptotes $= 2 \tan^{-1} \left(\frac{b}{a} \right)$
- (iv) The combined equation of the asymptotes differs from that of the hyperbola by a constant only.

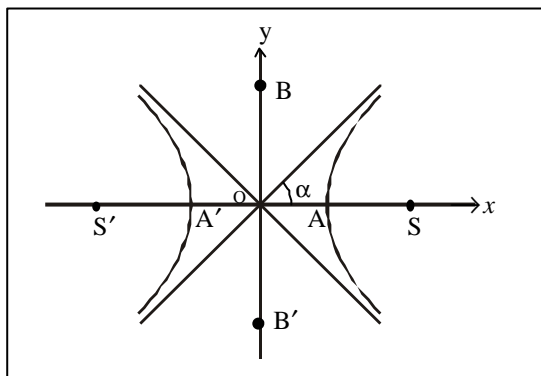


Fig.2.12.

Example 14

Find the equation of the hyperbola in standard form whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16

Solution :

$$\text{Given } e = \sqrt{2}$$

Let S and S' be the foci, then $S'S = 16$

$$\text{But } S'S = 2ae \quad \therefore 2ae = 16$$

$$\text{Thus we have } (2a)(\sqrt{2}) = 16, \quad \Rightarrow a = 4\sqrt{2}$$

$$\begin{aligned} \text{Also } b^2 &= a^2 (e^2 - 1) \\ &= (4\sqrt{2})^2 (2 - 1) = 32 \end{aligned}$$

The equation of the hyperbola is

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{32} - \frac{y^2}{32} = 1$$

$$\Rightarrow x^2 - y^2 = 32$$

2.4.4 Rectangular Hyperbola (R.H.)

A hyperbola is said to be rectangular if the asymptotes are at right angles.

If 2α is the angle between the asymptotes, then $\tan \alpha = \frac{b}{a}$.

If the hyperbola is R.H., then $2\alpha = 90^\circ$

$$\therefore \alpha = 45^\circ \Rightarrow a = b.$$

$$\therefore \text{Equation of the rectangular hyperbola is } x^2 - y^2 = a^2.$$

\Rightarrow A hyperbola is also said to be rectangular when its transverse and conjugate axes are equal in length. i.e. $a = b$

$$\therefore b^2 = a^2(e^2 - 1) \text{ gives } a^2 = a^2(e^2 - 1). \text{ or } e = \sqrt{2}$$

2.4.5 Standard equation of rectangular hyperbola.

Let the asymptotes of a rectangular hyperbola be taken as the coordinate axes.

The equations of the asymptotes are $x = 0$ and $y = 0$. Their combined equation is $xy = 0$.

Since the equation of the hyperbola differs from the equation of asymptotes by a constant, the equation of the hyperbola is

$$xy = k \quad \text{----- (1)}$$

where k is a constant.

Let the transverse axis, $AA' = 2a$.

Draw AM perpendicular to one asymptote, the x -axis.

$\angle ACM = 45^\circ$ where C is the centre

$$\text{So, } CM = CA \cos 45^\circ = \frac{a}{\sqrt{2}}$$

$$\text{and } MA = CA \sin 45^\circ = \frac{a}{\sqrt{2}}$$

$$\therefore \text{The coordinates of A are } \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}} \right)$$

It lies on the rectangular hyperbola. Therefore,

$$k = \frac{a}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}}$$

or, $k = \frac{a^2}{2}$

The equation for the rectangular hyperbola is

$$xy = \frac{a^2}{2}$$

Let $c^2 = \frac{a^2}{2}$

Hence $xy = c^2$

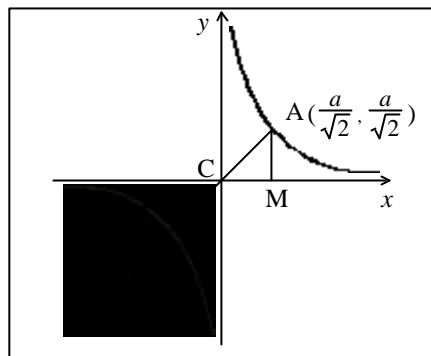


Fig.2.13.

This is the standard form of the equation of the rectangular hyperbola.

Example 15

Find the equation of the hyperbola whose eccentricity is $\sqrt{3}$, focus is $(1, 2)$ and the corresponding directrix is $2x + y = 1$.

Solution :

Focus S is $(1, 2)$, directrix is $2x + y = 1$ and $e = \sqrt{3}$

If $P(x_1, y_1)$ is any point on the hyperbola, then $SP^2 = e^2 PM^2$, where PM is the perpendicular to the directrix $2x + y = 1$.

$$(x_1 - 1)^2 + (y_1 - 2)^2 = 3 \frac{(2x_1 + y_1 - 1)^2}{5}$$

$$5(x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4) = 3(2x_1 + y_1 - 1)^2$$

$$7x_1^2 + 12x_1y_1 - 2y_1^2 - 2x_1 + 14y_1 - 22 = 0$$

\therefore Locus of (x_1, y_1) is

$$7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$$

Example 16

Find the equations of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$

Solution :

The combined equation of the asymptotes differs from the equation of the hyperbola only by a constant.

So, the combined equation of the asymptotes is

$$2x^2 + 5xy + 2y^2 - 11x - 7y + k = 0 \dots\dots\dots(1)$$

which is a pair of straight lines satisfying the condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \dots\dots\dots(2)$$

In the given equation

$$a = 2, h = \frac{5}{2}, b = 2, f = \frac{-7}{2}, g = -\frac{11}{2}, c = k$$

Substituting in (2) we get $k = 5$.

So the combined equation of the asymptotes is

$$2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$$

$$\Rightarrow (2x^2 + 5xy + 2y^2) - 11x - 7y + 5 = 0$$

$$\Rightarrow (2x + y)(x + 2y) - 11x - 7y + 5 = 0$$

$$\Rightarrow (2x + y + l)(x + 2y + m) = 0$$

$$\Rightarrow l + 2m = -11 \quad (\text{comparing the coefficients of } x)$$

$$\text{and } 2l + m = -7 \quad (\text{comparing the coefficients of } y)$$

$$\Rightarrow l = -1, m = -5$$

The equations of the asymptotes are

$$2x + y - 1 = 0 \text{ and } x + 2y - 5 = 0.$$

Example 17

Find the centre, eccentricity, foci and latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$

Solution :

The given equation can be written as

$$9(x^2 - 2x) - 16(y^2 + 4y) = 199$$

Completing squares in x and y , we get

$$9(x - 1)^2 - 16(y + 2)^2 = 199 + 9 - 64 = 144$$

$$\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$$

To bring this to the standard form, set $X = x - 1$ and $Y = y + 2$;

we get,

$$\frac{X^2}{16} - \frac{Y^2}{9} = 1$$

$$b^2 = a^2 (e^2 - 1)$$

$$\Rightarrow e^2 = \frac{9}{16} + 1 = \frac{25}{16} \quad \text{or, } e = \frac{5}{4}$$

Now we can tabulate the results:

	Referred to (X, Y)	Referred to (x, y) $x = X+1, y = Y-2$
Centre	(0,0)	$(0 + 1, 0 - 2) = (1, -2)$
Foci	$(\pm ae, 0)$ $= (5, 0) \text{ and } (-5, 0)$	$(5+1, 0-2) \text{ and } (-5+1, 0-2)$ $(6, -2) \text{ and } (-4, -2)$
Latus rectum	$= \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$	$\frac{9}{2}$

Example 18

Find the equation to the hyperbola which has the lines $x + 4y - 5 = 0$ and $2x - 3y + 1 = 0$ for its asymptotes and which passes through the point (1,2).

Solution :

The combined equation of the asymptotes is

$$(x + 4y - 5) (2x - 3y + 1) = 0$$

The equation of the hyperbola differs from this combined equation of asymptotes only by a constant. Thus the hyperbola is

$$(x + 4y - 5) (2x - 3y + 1) = k, \quad \text{where } k \text{ is a constant}$$

Since the hyperbola passes through (1, 2)

$$[1 + 4(2) - 5] [2(1) - 3(2) + 1] = k \quad \text{i.e. } k = -12$$

\Rightarrow The equation of the hyperbola is $(x + 4y - 5) (2x - 3y + 1) = -12$

$$\text{or} \quad 2x^2 + 5xy - 12y^2 - 9x + 19y + 7 = 0.$$

Example 19

The cost of production of a commodity is Rs.12 less per unit at a place A than it is at a place B and distance between A and B is 100km. Assuming that the route of delivery of the commodity is along a straight line and that the delivery cost is 20 paise per unit per km, find the curve, at any point of which the commodity can be supplied from either A or B at the same total cost.

Solution:

Choose the midpoint of AB as the origin O(0,0).

Let P be a point on the required curve so that the commodity supplied from either A or B at the same total cost.

Let the cost per unit at B = C

\therefore the cost per unit at A = $C - 12$

Delivery cost per unit from A to P = $\frac{20}{100}$ AP

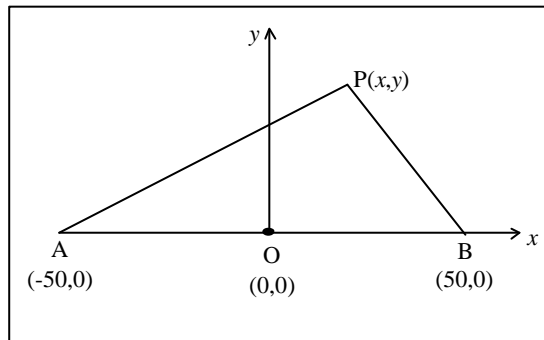


Fig.2.14.

Delivery cost per unit from B to P = $\frac{20}{100}$ BP

Total cost is same whether the commodity is delivered from either A or B.

$$\therefore (C-12) + \frac{20}{100}(AP) = C + \frac{20}{100}(BP)$$

$$\begin{aligned}\therefore \quad \frac{AP}{5} - \frac{BP}{5} &= 12 \quad \text{i.e. } AP - BP = 60 \\ \sqrt{(x+50)^2 + y^2} - \sqrt{(x-50)^2 + y^2} &= 60 \\ \sqrt{x^2 + y^2 + 100x + 2500} - \sqrt{x^2 + y^2 - 100x + 2500} &= 60\end{aligned}$$

Simplifying we get,

$$\begin{aligned}6400x^2 - 3600y^2 &= 5760000 \\ \therefore \quad 16x^2 - 9y^2 &= 14400 \\ \frac{x^2}{900} - \frac{y^2}{1600} &= 1 \quad \therefore \quad \frac{x^2}{(30)^2} - \frac{y^2}{(40)^2} = 1\end{aligned}$$

Thus we get the required curve as hyperbola.

Example 20

A machine sells at Rs. p and the demand, x (in hundreds) machines per year is given by $x = \frac{90}{p+5} - 6$. What type of demand curve corresponds to the above demand's law? At what price does the demand tend to vanish?

Solution:

The demand curve is

$$\begin{aligned}x + 6 &= \frac{90}{p+5} \Rightarrow (x+6)(p+5) = 90 \\ \Rightarrow \quad XP &= 90\end{aligned}$$

where $X = x+6$, $P = p+5$

\Rightarrow The demand curve is a rectangular hyperbola.

$$\text{Demand} = 0 \Rightarrow 6(p+5) = 90$$

$\Rightarrow p = \text{Rs. } 10$.

EXERCISE 2.4

- 1) Find the equation of the hyperbola with
 - (a) focus $(2, 2)$, eccentricity $\frac{3}{2}$ and directrix $3x - 4y = 1$.
 - (b) focus $(0, 0)$, eccentricity $\frac{5}{4}$ and directrix $x \cos \alpha + y \sin \alpha = p$.

- 2) Find the equation of the hyperbola whose foci are (6, 4) and (-4, 4) and eccentricity 2.
- 3) Find the equation of the hyperbola whose
 - (a) centre is (1, 0), one focus is (6, 0) and transverse axis 6.
 - (b) centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2).
 - (c) centre is (6, 2), one focus is (4, 2) and $e = 2$.
- 4) Find the centre, eccentricity, foci and directrices for the following hyperbolas:
 - (a) $9x^2 - 16y^2 = 144$
 - (b) $\frac{(x+2)^2}{9} - \frac{(y+4)^2}{7} = 1$
 - (c) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$
- 5) Find the equation to the asymptotes of the hyperbola
 - (a) $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$
 - (b) $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$
- 6) Find the equation to the hyperbola which passes through (2,3) and has for its asymptotes the lines $4x + 3y - 7 = 0$ and $x - 2y = 1$.
- 7) Find the equation to the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ for asymptotes and which passes through the origin.

EXERCISE 2.5

Choose the correct answer

- 1) The eccentricity of a parabola is
 - (a) 1
 - (b) 0
 - (c) 2
 - (d) -1
- 2) The eccentricity of a conic is $\frac{1}{\sqrt{2}}$. The conic is
 - (a) a parabola
 - (b) an ellipse
 - (c) a circle
 - (d) a hyperbola
- 3) Latus rectum of $y^2 = 4ax$ is
 - (a) $2a$
 - (b) $3a$
 - (c) $4a$
 - (d) a
- 4) Focus of $y^2 = -4ax$ is
 - (a) $(a, 0)$
 - (b) $(0, a)$
 - (c) $(0, -a)$
 - (d) $(-a, 0)$

- 5) Equation of the directrix of $x^2 = 4ay$ is
 (a) $x + a = 0$ (b) $x - a = 0$ (c) $y + a = 0$ (d) $y - a = 0$
- 6) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an ellipse ($a > b$) if
 (a) $b^2 = a^2(1 - e^2)$ (b) $b^2 = -a^2(1 - e^2)$
 (c) $b^2 = \frac{a^2}{1 - e^2}$ (d) $b^2 = \frac{1 - e^2}{a^2}$
- 7) Latus rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) is
 (a) $\frac{2a^2}{b}$ (b) $\frac{a^2}{2b}$ (c) $\frac{2b^2}{a}$ (d) $\frac{b^2}{2a}$
- 8) Focus of $y^2 = 16x$ is
 (a) (2, 0) (b) (4, 0) (c) (8, 0) (d) (2, 4)
- 9) Equation of the directrix of $y^2 = -8x$ is
 (a) $x + 2 = 0$ (b) $x - 2 = 0$ (c) $y + 2 = 0$ (d) $y - 2 = 0$
- 10) The length of the latus rectum of $3x^2 + 8y = 0$, is
 (a) $\frac{8}{3}$ (b) $\frac{2}{3}$ (c) 8 (d) $\frac{3}{8}$
- 11) The parabola $x^2 + 16y = 0$ is completely
 (a) above x -axis (b) below x -axis
 (c) left of y -axis (d) right of y -axis
- 12) The semi major and semi minor axes of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is
 (a) (4, 5) (b) (8, 10) (c) (5, 4) (d) (10, 8)
- 13) The length of latus rectum of $4x^2 + 9y^2 = 36$ is
 (a) $\frac{4}{3}$ (b) $\frac{8}{3}$ (c) $\frac{4}{9}$ (d) $\frac{8}{9}$
- 14) In an ellipse $e = \frac{3}{5}$, the length of semi minor axis is 2. The length of major axis is
 (a) 4 (b) 5 (c) 8 (d) 10
- 15) Eccentricity of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ is
 (a) $\frac{3}{2}$ (b) $\frac{9}{4}$ (c) $\frac{5}{4}$ (d) 4

- 16) The sum of focal distances of any point on the ellipse is equal to length of its
 (a) minor axis (b) semi minor axis
 (c) major axis (d) semi major axis
- 17) The difference between the focal distances of any point on the hyperbola is equal to length of its
 (a) transverse axis (b) semi transverse axis
 (c) conjugate axis (d) semi conjugate axis.
- 18) Asymptotes of a hyperbola pass through
 (a) one of the foci (b) one of the vertices
 (c) the centre of the hyperbola (d) one end of its latus rectum.
- 19) Eccentricity of the rectangular hyperbola is
 (a) 2 (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$
- 20) If a is the length of the semi transverse axis of rectangular hyperbola $xy = c^2$ then the value of c^2 is
 (a) a^2 (b) $2a^2$ (c) $\frac{a^2}{2}$ (d) $\frac{a^2}{4}$

APPLICATIONS OF DIFFERENTIATION - I

3

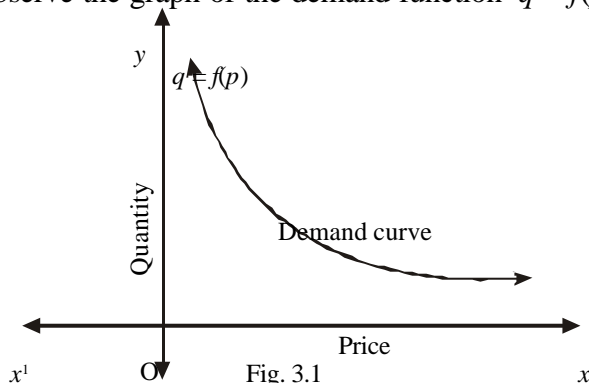
Differentiation plays a vital role in Economics and Commerce. Before we embark on demonstrating applications of differentiation in these fields, we introduce a few Economic terminologies with usual notations.

3.1 FUNCTIONS IN ECONOMICS AND COMMERCE

3.1.1 Demand Function

Let q be the demand (quantity) of a commodity and p the price of that commodity. The demand function is defined as $q = f(p)$ where p and q are positive. Generally, p and q are inversely related.

Observe the graph of the demand function $q = f(p)$



Following observations can be made from the graph (Fig 3.1)

- (i) only the first quadrant portion of the graph of the demand function is shown since p and q are positive.
- (ii) slope of the demand curve is negative.

3.1.2 Supply Function

Let x denotes amount of a particular commodity that sellers offer in the market at various price p , then the supply function is given by $x = f(p)$ where x and p are positive.

Generally x and p are directly related

Observe the graph of the supply function, $x = f(p)$

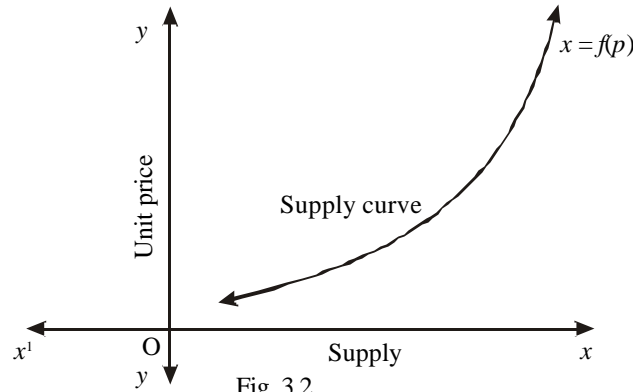


Fig. 3.2

Following observations can be made from the graph (Fig 3.2)

- (i) only the first quadrant portion of the graph of the supply function is shown since the function has meaning only for non-negative values of q and p .
- (ii) slope of the supply function is positive.

3.1.3 Cost Function

Normally total cost consists of two parts.

(i) Variable cost and (ii) fixed cost. Variable cost is a single-valued function of output, but fixed cost is independent of the level of output.

Let $f(x)$ be the variable cost and k be the fixed cost when the output is x units. The total cost function is defined as $C(x) = f(x) + k$, where x is positive.

Note that $f(x)$ does not contain constant term.

We define Average Cost (AC), Average Variable Cost (AVC), Average Fixed Cost (AFC), Marginal Cost (MC), and Marginal Average Cost (MAC) as follows.

- (i) Average Cost (AC) = $\frac{f(x) + k}{x} = \frac{\text{Total Cost}}{\text{Output}}$

- (ii) Average Variable Cost (AVC) = $\frac{f(x)}{x} = \frac{\text{Variable Cost}}{\text{Output}}$
- (iii) Average Fixed Cost (AFC) = $\frac{k}{x} = \frac{\text{Fixed Cost}}{\text{Output}}$
- (iv) Marginal Cost (MC) = $\frac{d}{dx} C(x) = C'(x)$
- (v) Marginal Average Cost (MAC) = $\frac{d}{dx} (AC)$

Note

If $C(x)$ is the total cost of producing x units of some product then its derivative $C'(x)$ is the marginal cost which is the approximate cost of producing 1 more unit when the production level is x units. The graphical representation is shown here (Fig 3.3).

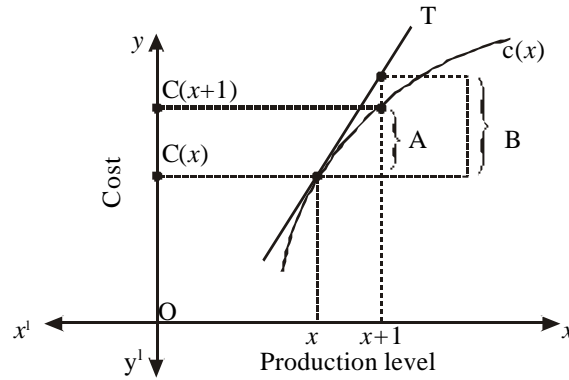


Fig. 3.3

$$A = C(x+1) - C(x)$$

$$B = C'(x) = \text{Marginal Cost}$$

3.1.4 Revenue Function

Let x units be sold at Rs. p per unit. Then the total revenue $R(x)$ is defined as $R(x) = px$, where p and x are positive.

$$\text{Average revenue (AR)} = \frac{\text{Total revenue}}{\text{quantity sold}} = \frac{px}{x} = p.$$

(i.e. Average revenue and price are the same)

$$\text{Marginal revenue (MR)} = \frac{d}{dx} (R) = R'(x)$$

Note

If $R(x)$ be the total revenue gained from selling x units of some product, then its derivative, $R'(x)$ is the marginal revenue, which is approximate revenue gained from selling 1 more unit when the sales level is x units. The graphical representation is shown here (Fig 3.4)

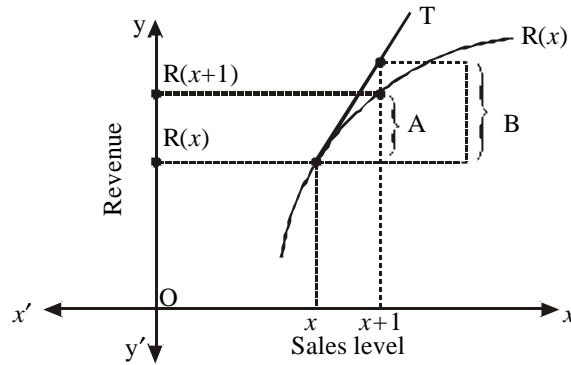


Fig. 3.4

$$A = R(x+1) - R(x)$$

$$B = R'(x) = \text{marginal revenue}$$

3.1.5 Profit Function

The profit function $P(x)$ is defined as the difference between the total revenue and the total cost. i.e. $P(x) = R(x) - C(x)$.

3.1.6. Elasticity

The elasticity of a function $y = f(x)$, with respect to x , is defined as

$$\begin{aligned} \eta &= \frac{E_y}{E_x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} \\ &= \frac{x}{y} \frac{dy}{dx} \end{aligned}$$

Thus the elasticity of y with respect to x is the limit of the ratio of the relative increment in y to the relative increment in x , as the increment in x tends to zero. The elasticity is a pure number, independent of the units in x and y .

3.1.7 Elasticity of Demand

Let $q = f(p)$ be the demand function, where q is the demand and p is the price. Then the elasticity of demand is $\eta_d = \frac{p}{q} \frac{dq}{dp}$

(Fig 3.5)

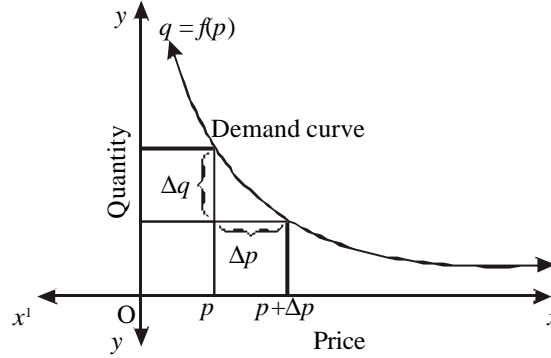


Fig. 3.5

$$\text{Elasticity of demand} = \lim_{\Delta p \rightarrow 0} \frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} = \frac{p}{q} \frac{dq}{dp}$$

Since the slope of the demand curve is negative and elasticity is a positive quantity the elasticity of demand is given by

$$\eta_d = - \frac{p}{q} \frac{dq}{dp}$$

3.1.8. Elasticity of Supply

Let $x = f(p)$ be the supply function, where x is the supply and p is the price. The elasticity of supply is defined as

$$\eta_s = \frac{p}{x} \frac{dx}{dp}$$

3.1.9 Equilibrium Price

The price at which quantity demanded is equal to quantity supplied is called equilibrium price.

3.1.10 Equilibrium Quantity

The quantity obtained by substituting the value of equilibrium price in any one of the given demand or supply functions is called equilibrium quantity.

3.1.11 Relation between Marginal Revenue and Elasticity of Demand

Let q units be demanded at unit price p so that $p = f(q)$ where f is differentiable. The revenue is given by

$$R(q) = qp$$

$$R(q) = q f(q) \quad [p = f(q)]$$

Marginal revenue is obtained by differentiating $R(q)$ with respect to q .

$$\begin{aligned} \therefore R'(q) &= q f'(q) + f(q) \\ &= q \frac{dp}{dq} + p \quad \left[\frac{dp}{dq} = f'(q) \right] \\ R'(q) &= p \left(1 + \frac{q}{p} \frac{dp}{dq} \right) \end{aligned}$$

$$\begin{aligned} &= p \left[1 + \frac{1}{\frac{p}{q} \frac{dq}{dp}} \right] \\ &= p \left[1 + \left\{ \frac{-1}{-\frac{p}{q} \frac{dq}{dp}} \right\} \right] \end{aligned}$$

Since $\eta_d = - \frac{p}{q} \frac{dq}{dp}$,

$$\text{Marginal Revenue} = R'(q) = p \left[1 - \frac{1}{\eta_d} \right]$$

Example 1

A firm produces x tonnes of output at a total cost

$$C(x) = \frac{1}{10} x^3 - 4x^2 + 20x + 5$$

Find (i) Average cost (ii) Average Variable Cost
(iii) Average Fixed Cost (iv) Marginal Cost and
(v) Marginal Average Cost.

Solution :

$$C(x) = \frac{1}{10} x^3 - 4x^2 + 20x + 5$$

$$\begin{aligned}
\text{(i) Average Cost} &= \frac{\text{Total cost}}{\text{output}} \\
&= \left(\frac{1}{10} x^2 - 4x + 20 + \frac{5}{x} \right) \\
\text{(ii) Average Variable Cost} &= \frac{\text{Variable cost}}{\text{output}} \\
&= \frac{1}{10} x^2 - 4x + 20 \\
\text{(iii) Average Fixed Cost} &= \frac{\text{Fixed cost}}{\text{output}} = \frac{5}{x} \\
\text{(iv) Marginal Cost} &= \frac{d}{dx} C(x) \\
&= \frac{d}{dx} \left(\frac{1}{10} x^3 - 4x^2 + 20x + 5 \right) \\
&= \left(\frac{3}{10} x^2 - 8x + 20 \right) \\
\text{(v) Marginal Average Cost} &= \frac{d}{dx} (AC) \\
&= \frac{d}{dx} \left(\frac{1}{10} x^2 - 4x + 20 + \frac{5}{x} \right) \\
&= \left(\frac{1}{5} x - 4 - \frac{5}{x^2} \right)
\end{aligned}$$

Example 2

The total cost C of making x units of product is $C = 0.00005x^3 - 0.06x^2 + 10x + 20,000$. Find the marginal cost at 1000 units of output.

Solution :

$$C = 0.00005x^3 - 0.06x^2 + 10x + 20,000$$

$$\begin{aligned}
\text{Marginal Cost } \frac{dC}{dx} &= (0.00005) (3x^2) - (0.06) 2x + 10 \\
&= 0.00015 x^2 - 0.12x + 10
\end{aligned}$$

$$\text{when } x = 1000$$

$$\begin{aligned}
\frac{dC}{dx} &= (0.00015)(1000)^2 - (0.12)(1000) + 10 \\
&= 150 - 120 + 10 = 40
\end{aligned}$$

At $x = 1000$ units, Marginal Cost is Rs. 40

Example 3

Find the elasticity of demand for the function $x = 100 - p - p^2$ when $p = 5$.

Solution :

$$x = 100 - p - p^2$$

$$\frac{dx}{dp} = -1 - 2p.$$

$$\begin{aligned} \text{Elasticity of demand } \eta_d &= -\frac{p}{x} \frac{dx}{dp} \\ &= -\frac{p(-1-2p)}{100-p-p^2} = \frac{p+2p^2}{100-p-p^2} \end{aligned}$$

$$\begin{aligned} \text{When } p = 5, \eta_d &= \frac{5+50}{100-5-25} \\ &= \frac{55}{70} = \frac{11}{14} \end{aligned}$$

Example 4

Find the elasticity of supply for the supply function $x = 2p^2+8p+10$

Solution :

$$x = 2p^2+8p+10$$

$$\frac{dx}{dp} = 4p+8$$

$$\begin{aligned} \text{Elasticity of supply } \eta_s &= \frac{p}{x} \frac{dx}{dp} \\ &= \frac{4p^2+8p}{2p^2+8p+10} = \frac{2p^2+4p}{p^2+4p+5} \end{aligned}$$

Example 5

For the function $y = 4x-8$ find the elasticity and also obtain the value when $x = 6$.

Solution :

$$y = 4x-8$$

$$\frac{dy}{dx} = 4$$

$$\text{Elasticity } \eta = \frac{x}{y} \frac{dy}{dx}$$

$$\eta = \frac{x}{4x-8} (4) = \frac{x}{x-2}$$

$$\text{when } x = 6, \eta = \frac{6}{6-2} = \frac{3}{2}$$

Example 6

If $y = \frac{1-2x}{2+3x}$ find $\frac{E_y}{E_x}$. Obtain the values of η when $x = 0$ and $x = 2$.

Solution :

$$\text{We have } y = \frac{1-2x}{2+3x}$$

Differentiating with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2+3x)(-2) - (1-2x)(3)}{(2+3x)^2} \\ &= \frac{-4 - 6x - 3 + 6x}{(2+3x)^2} = \frac{-7}{(2+3x)^2} \end{aligned}$$

$$\begin{aligned} \eta &= \frac{E_y}{E_x} = \frac{x}{y} \frac{dy}{dx} \\ &= \frac{x(2+3x)}{(1-2x)} \times \frac{-7}{(2+3x)^2} \\ \eta &= \frac{-7x}{(1-2x)(2+3x)} \end{aligned}$$

$$\text{when } x = 0, \eta = 0$$

$$\text{when } x = 2, \eta = \frac{7}{12}$$

Example 7

A demand function is given by $x p^n = k$, where n and k are constants. Calculate price elasticity of demand.

Solution :

$$\text{Given } x p^n = k$$

$$\Rightarrow x = k p^{-n}$$

$$\frac{dx}{dp} = -nk p^{-n-1}$$

$$\begin{aligned}\text{Elasticity of demand } \eta_d &= -\frac{p}{x} \frac{dx}{dp} \\ &= -\frac{p}{kp^{-n}} (-nk p^{-n-1}) \\ &= n, \quad \text{which is a constant.}\end{aligned}$$

Example 8

Show that the elasticity of demand at all points on the curve $xy^2 = c$ (c is constant), where y represents price will be numerically equal to 2.

Solution :

$$\text{We have } xy^2 = c$$

$$x = \frac{c}{y^2}$$

Differentiating with respect to y ,

$$\frac{dx}{dy} = -\frac{2c}{y^3}$$

$$\text{Elasticity of demand } \eta_d = -\frac{y}{x} \frac{dx}{dy} = \frac{-y}{\frac{c}{y^2}} \left(\frac{-2c}{y^3} \right) = 2$$

Example 9

The demand curve for a monopolist is given by $x = 100 - 4p$

- (i) Find the total revenue, average revenue and marginal revenue.
- (ii) At what value of x , the marginal revenue is equal to zero?

Solution :

$$\text{We have } x = 100 - 4p$$

$$p = \frac{100-x}{4}$$

$$\text{Total revenue } R = px$$

$$= \left(\frac{100-x}{4} \right) x = \frac{100x-x^2}{4}$$

$$\begin{aligned}
\text{Average revenue} = p &= \frac{100-x}{4} \\
\text{Marginal revenue} &= \frac{d}{dx} (R) \\
&= \frac{d}{dx} \left(\frac{100x - x^2}{4} \right) \\
&= \frac{1}{4} [100 - 2x] = \frac{50-x}{2}
\end{aligned}$$

(ii) Marginal revenue is zero implies

$$\begin{aligned}
\frac{50-x}{2} = 0 &\Rightarrow x = 50 \\
\therefore \text{Marginal revenue is zero when } x &= 50.
\end{aligned}$$

Example 10

If AR and MR denote the average and marginal revenue at any output level, show that elasticity of demand is equal to $\frac{AR}{AR - MR}$. Verify this for the linear demand law $p = a + bx$ where p is price and x is the quantity.

Solution :

$$\begin{aligned}
\text{Total Revenue } R &= px \\
\text{Average Revenue } AR &= p \\
\text{Marginal Revenue } MR &= \frac{d}{dx} (R) = \frac{d}{dx} (px) \\
&= p + x \frac{dp}{dx}
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{AR}{(AR-MR)} &= \frac{p}{p - (p + x \frac{dp}{dx})} \\
&= - \frac{p}{x} \frac{dx}{dp} \\
&= \text{Elasticity of demand } \eta_d \\
\therefore \frac{AR}{(AR-MR)} &= \eta_d
\end{aligned}$$

Given $p = a + bx$

Differentiating with respect to x ,

$$\frac{dp}{dx} = b$$

$$R = px = ax + bx^2$$

$$AR = a + bx \quad (\text{AR} = \text{price})$$

$$\begin{aligned} MR &= \frac{d}{dx}(ax + bx^2) \\ &= a + 2bx. \end{aligned}$$

$$\therefore \frac{AR}{(AR-MR)} = \frac{a+bx}{a+bx-a-2bx} = -\frac{(a+bx)}{bx} \text{ -----(1)}$$

$$\begin{aligned} \eta_d &= -\frac{p}{x} \frac{dx}{dp} \\ &= -\frac{(a+bx)}{x} \frac{1}{b} = -\frac{(a+bx)}{bx} \text{ -----(2)} \end{aligned}$$

from (1) and (2) we get that $\frac{AR}{(AR-MR)} = \eta_d$

Example 11

Find the equilibrium price and equilibrium quantity for the following demand and supply functions, $Q_d = 4 - 0.06p$ and $Q_s = 0.6 + 0.11p$

Solution :

At the equilibrium price

$$Q_d = Q_s$$

$$\Rightarrow 4 - 0.06p = 0.6 + 0.11p$$

$$\Rightarrow 0.17p = 3.4$$

$$\begin{aligned} \Rightarrow p &= \frac{3.4}{0.17} \\ p &= 20 \end{aligned}$$

$$\begin{aligned} \text{when } p &= 20, \quad Q_d = 4 - (0.06)(20) \\ &= 4 - 1.2 = 2.8 \end{aligned}$$

\therefore Equilibrium price = 20 and Equilibrium quantity = 2.8

Example 12

The demand for a given commodity is given by $q = \frac{p}{p-5}$ ($p > 5$), where p is the unit price. Find the elasticity of demand when $p = 7$. Interpret the result.

Solution :

$$\text{Demand function } q = \frac{p}{p-5}$$

Differentiating with respect to p , we get

$$\frac{dq}{dp} = \frac{(p-5)(1) - p(1)}{(p-5)^2} = \frac{-5}{(p-5)^2}$$

$$\begin{aligned} \text{Elasticity of demand } \eta_d &= -\frac{p}{q} \frac{dq}{dp} = \frac{-p(p-5)}{p} \left\{ -\frac{5}{(p-5)^2} \right\} \\ &= \frac{5}{p-5} \end{aligned}$$

$$\text{when } p = 7, \quad \eta_d = \frac{5}{7-5} = 2.5$$

This means that if the price increases by 1% when $p = 7$, the quantity demanded will decrease by approximately 2.5%. Also if the price decreases by 1% when $p = 7$, the quantity demanded will increase by approximately 2.5%.

Example 13

The demand for a given commodity is $q = -60p + 480$, ($0 < p < 7$) where p is the price. Find the elasticity of demand and marginal revenue when $p = 6$.

Solution :

$$\text{Demand function } q = -60p + 480$$

Differentiating with respect to p , we get

$$\frac{dq}{dp} = -60$$

$$\text{Elasticity of demand } \eta_d = -\frac{p}{q} \frac{dq}{dp} = \frac{-p}{-60p+480} (-60) = -\frac{p}{p-8}$$

$$\text{when } p = 6, \quad \eta_d = \frac{-6}{6-8} = 3$$

$$\text{Marginal revenue} = p \left(1 - \frac{1}{\eta_d}\right) = 6\left(1 - \frac{1}{3}\right) = 4$$

\therefore Marginal revenue = Rs. 4

EXERCISE 3.1

- 1) A firm produces x tonnes of output at a total cost $C(x) = \text{Rs. } \left(\frac{1}{2}x^3 - 4x^2 + 25x + 8\right)$. Find (i) Average Cost (ii) Average Variable Cost and (iii) Average Fixed Cost. Also find the value of each of the above when the output level is 10 tonnes.
- 2) The total cost C of making x units of product is $C(x) = 25 + 3x^2 + \sqrt{x}$. Find the marginal cost at output level of 100 units.
- 3) The total cost of making x units is given by $C(x) = 50 + 5x + 2\sqrt{x}$. What is the marginal cost at 100 units of output?
- 4) If the cost of making x units is $C = \frac{1}{2}x + 26\sqrt{x+4}$. Find the marginal cost at output of 96 units.
- 5) If the total cost C of making x tonnes of a product is $C = 10 + 30\sqrt{x}$. Find the marginal cost at 100 tonnes output and find the level of output at which the marginal cost is Rs. 0.40 per ton.
- 6) The cost function for the production of x units of an item is given by $C = \frac{1}{10}x^3 - 4x^2 + 8x + 4$. Find (i) the average cost (ii) the marginal cost and (iii) the marginal average cost.
- 7) If the total cost C of making x units is $C = 50 + 10x + 5x^2$. Find the average cost and marginal cost when $x = 1.3$.
- 8) The total cost C of producing x units is $C = 0.00004x^3 - 0.002x^2 + 3x + 10,000$. Find the marginal cost of 1000 units output.
- 9) Show that the elasticity of demand at all points on the curve $xy = c^2$ (y being price, and c is the constant) will be numerically equal to one.

- 10) Find the elasticity of demand when the demand is $q = \frac{20}{p+1}$ and $p = 3$. Interpret the result.
- 11) Given the demand function $q = 165 - 3p - 2p^2$, find the elasticity of demand at the price $p = 5$. Interpret the result.
- 12) Show that the elasticity of demand function $p = \frac{100}{q}$ is unity for every value of q .
- 13) Find the elasticity of demand with respect to the price for the following demand functions.
 (i) $p = \sqrt{a - bx}$, a and b are constants (ii) $x = \frac{8}{p^{3/2}}$
- 14) A demand curve is $xp^m = b$ where m and b are constants. Calculate the price elasticity of demand.
- 15) Find the elasticity of supply for the supply function $x = 2p^2 + 5$.
- 16) The supply of certain items is given by the supply function $x = a\sqrt{p - b}$, where p is the price, a and b are positive constants. ($p > b$). Find an expression for elasticity of supply η_s . Show that it becomes unity when the price is $2b$.
- 17) For the demand function $p = 550 - 3x - 6x^2$ where x is the quantity demanded and p is the price per unit, find the average revenue and marginal revenue.
- 18) The sales S , for the product with price x is given by $S = 20,000 e^{-0.6x}$.
 Find (i) total sales revenue R , where $R = xS$
 (ii) Marginal revenue
- 19) The price and quantity x of a commodity are related by the equation $x = 30 - 4p - p^2$. Find the elasticity of demand and marginal revenue.
- 20) Find the equilibrium price and equilibrium quantity for the following demand and supply functions.
 $q_d = 4 - 0.05p$ and $q_s = 0.8 + 0.11p$
- 21) Find the marginal revenue for the revenue function
 $R(x) = 100x + \frac{x^2}{2}$, where $x = 10$.

- 22) The price and quantity q of a commodity are related by the equation $q = 32 - 4p - p^2$. Find the elasticity of demand and marginal revenue when $p = 3$.

3.2 DERIVATIVE AS A RATE OF CHANGE

Let a relation between two variables 'x' and 'y' be denoted by $y = f(x)$. Let Δx be the small change in x and Δy be the corresponding change in y . Then we define average rate of change of y with respect to x is $\frac{\Delta y}{\Delta x}$, where $\Delta y = f(x + \Delta x) - f(x)$ and

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

= Instantaneous rate of change of y with respect to x .

3.2.1 Rate of change of a quantity

Let the two quantities x and y be connected by the relation $y = f(x)$. Then $f'(x_0)$ represents the rate of change of y with respect to x at $x = x_0$.

3.2.2 Related rates of change

We will find the solution to the problems which involve equations with two or more variables that are implicit functions of time. Since such variables will not usually be defined explicitly in terms of time, we will have to differentiate implicitly with respect to time to determine the relation between their time-rates of change.

Example 14

If $y = \frac{300}{x}$, find the average rate of change of y with respect to x when x increases from 10 to 10.5. Find also the instantaneous rate of change of y at $x = 10$.

Solution :

- (i) Average rate of change of y with respect to x at $x = x_0$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Here $f(x) = \frac{300}{x}$, $x = 10$ and $\Delta x = 0.5$

∴ Average rate of change of y with respect to x is

$$\frac{f(10.5) - f(10)}{0.5} = \frac{28.57 - 30}{0.5} = \frac{-1.43}{0.5}$$

$$= -2.86 \text{ units per unit change in } x.$$

The negative sign indicates that y decreases per unit increase in x .

(ii) The instantaneous rate of change of y is $\frac{dy}{dx}$

$$y = \frac{300}{x}$$

$$\therefore \frac{dy}{dx} = \frac{-300}{x^2}$$

$$\text{At } x = 10, \frac{dy}{dx} = \frac{-300}{(10)^2} = -3$$

⇒ The instantaneous rate of change at $x = 10$ is -3 units per unit change in x . The negative sign indicates the decrease rate of change with respect to x .

Example 15

A point moves along the graph of $xy = 35$ in such a way that its abscissa is increasing at the rate of 3 units per second when the point is $(5, 7)$. Find the rate of change of y - coordinate at that moment.

Solution :

Here x and y are functions of time ' t '.

$$xy = 35$$

Differentiating with respect to ' t ' we get

$$\frac{d}{dt}(xy) = \frac{d}{dt}(35)$$

$$\Rightarrow x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{y}{x} \frac{dx}{dt}$$

We need the value of $\frac{dy}{dt}$ when $x = 5$, $y = 7$ and $\frac{dx}{dt} = 3$

$$\therefore \frac{dy}{dt} = -\frac{7}{5} \times 3$$

$$= -4.2 \text{ units per second.}$$

i.e. y co-ordinate is decreasing at the rate of 4.2 units per second.

Example 16

The unit price, p of a product is related to the number of units sold, x , by the demand equation $p = 400 - \frac{x}{1000}$. The cost of producing x units is given by $C(x) = 50x + 16,000$. The number of units produced and sold, x is increasing at a rate of 200 units per week. When the number of units produced and sold is 10,000, determine the instantaneous rate of change with respect to time, t (in weeks) of (i) Revenue (ii) Cost (iii) Profit.

Solution :

$$\begin{aligned} \text{(i) Revenue } R &= px \\ &= \left(400 - \frac{x}{1000}\right)x \\ R &= 400x - \frac{x^2}{1000} \\ \frac{d}{dt}(R) &= \frac{d}{dt}(400x) - \frac{d}{dt}\left(\frac{x^2}{1000}\right) \\ \frac{dR}{dt} &= \left(400 - \frac{x}{500}\right)\frac{dx}{dt} \\ \text{when } x &= 10,000, \text{ and } \frac{dx}{dt} = 200 \\ \frac{dR}{dt} &= \left(400 - \frac{10,000}{500}\right)(200) \\ &= \text{Rs. } 76,000 \text{ per week.} \end{aligned}$$

i.e. Revenue is increasing at a rate of Rs.76,000 per week.

$$\begin{aligned} \text{(ii) } C(x) &= 50x + 16,000. \\ \frac{d}{dt}(C) &= \frac{d}{dt}(50x) + \frac{d}{dt}(16,000) \end{aligned}$$

$$\begin{aligned}\frac{dC}{dt} &= 50 \frac{dx}{dt} + 0 \\ &= 50 \frac{dx}{dt}\end{aligned}$$

when $\frac{dx}{dt} = 200$, $\frac{dC}{dt} = 50 \times 200$
 $= \text{Rs.}10,000$ per week.

i.e. Cost is increasing at a rate of Rs.10,000 per week.

(iii) Profit $P = R - C$

$$\begin{aligned}\frac{dP}{dt} &= \frac{dR}{dt} - \frac{dC}{dt} \\ &= 76,000 - 10,000 \\ &= \text{Rs.}66,000 \text{ per week.}\end{aligned}$$

i.e. Profit is increasing at a rate of Rs.66,000 per week.

Example 17

If the perimeter of a circle increases at a constant rate, prove that the rate of increase of the area varies as the radius of the circle.

Solution :

Let P be the perimeter and A be the area of the circle of radius r .

$$\text{Then } P = 2\pi r \text{ and } A = \pi r^2$$

$$\frac{dP}{dt} = 2\pi \frac{dr}{dt} \quad \text{-----(1)}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \text{-----(2)}$$

using (1) and (2) we get,

$$\frac{dA}{dt} = r \frac{dP}{dt}$$

Since perimeter P increases at a constant rate $\frac{dP}{dt}$ is constant.

$\therefore \frac{dA}{dt} \propto r$ i.e. the rate of increase of A is proportional to the radius.

Example 18

A metal cylinder is heated and expands so that its radius increases at a rate of 0.4 cm per minute and its height increases at a rate of 0.3 cm per minute retaining its shape. Determine the rate of change of the surface area of the cylinder when its radius is 20 cms. and height is 40 cms.

Solution :

The surface area of the cylinder is

$$A = 2\pi rh.$$

Differentiating both sides with respect to 't'

$$\frac{dA}{dt} = 2\pi \left[r \frac{dh}{dt} + h \frac{dr}{dt} \right]$$

$$\text{Given } r = 20, \quad h = 40, \quad \frac{dr}{dt} = 0.4, \quad \frac{dh}{dt} = 0.3$$

$$\begin{aligned} \therefore \frac{dA}{dt} &= 2\pi [20 \times 0.30 + 40 \times 0.40] \\ &= 2\pi [6 + 16] \\ &= 44\pi \text{ cm}^2 / \text{minute.} \end{aligned}$$

Example 19

For the function $y = x^3 + 21$, what are the values of x , when y increases 75 times as fast as x ?

Solution :

$$y = x^3 + 21$$

Differentiating both sides with respect to 't'

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt} + 0$$

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

$$\text{Given } \frac{dy}{dt} = 75 \frac{dx}{dt}$$

$$\therefore 3x^2 \frac{dx}{dt} = 75 \frac{dx}{dt}$$

$$\begin{aligned}\Rightarrow 3x^2 &= 75 \\ x^2 &= 25 \\ x &= \pm 5.\end{aligned}$$

Example 20

The demand y for a commodity is $y = \frac{12}{x}$, where x is the price. Find the rate at which the demand changes when the price is Rs. 4.

Solution :

The rate of change of the demand y with respect to the price is $\frac{dy}{dx}$.

$$\text{We have } y = \frac{12}{x}$$

Differentiating with respect to x , we get $\frac{dy}{dx} = -\frac{12}{x^2}$

\therefore The rate of change of demand with respect to price x is $-\frac{12}{x^2}$

When the price is Rs. 4 the rate of change of demand is $\frac{-12}{16} = -\frac{3}{4}$

This means that when the price is Rs. 4, an increase in price by 1% will result in the fall of demand by 0.75%.

EXERCISE 3.2

- 1) If $y = \frac{500}{x}$ find the average rate of change of y with respect to x , when x increases from 20 to 20.5 units. Find also the instantaneous rate of change of y at $x = 20$.
- 2) A point moves on the graph of $xy = 8$ in such a manner that its y - coordinate is increasing at a rate of 2 units per second, when the point is at (2, 4). Find the rate of change of the x - coordinate at that moment.
- 3) A point moves on the curve $4x^2 + 2y^2 = 18$ in such a way that its x co-ordinate is decreasing at a rate of 3 units per second when the point is at (2,1). Find the rate of change of the y - coordinate at that moment.

- 4) A point moves along the curve $y^2 = 12x$ in such a way that its x - coordinate is increasing at the rate of $5\sqrt{2}$ units per second when the point is at (3, 6). Show that the y - coordinate increases at the same rate as that of x - coordinate.
- 5) Given are the following revenue, cost and profit equations $R = 800x - \frac{x^2}{10}$, $C = 40x + 5,000$, $P = R - C$, where x denotes the number of units produced and sold (per month). When the production is at 2000 units and increasing at the rate of 100 units per month, determine the instantaneous rate of change with respect to time, t (in months), of (i) Revenue (ii) Cost and (iii) Profit
- 6) The unit price, p of some product is related to the number of units sold, x , by the demand function $p = 200 - \frac{x}{1000}$. The cost of producing x units of this product is given by $C = 40x + 12,000$. The number of units produced and sold x is increasing at the rate of 300 units per week. When the number of units produced and sold is 20,000 determine the instantaneous rate of change with respect to time, t (in weeks) of (i) Revenue (ii) Cost (iii) Profit.
- 7) Using derivative as a rate measure prove the following statement : "If the area of a circle increases at a uniform rate, then the rate of increase of the perimeter varies inversely as the radius of the circle".
- 8) The radius of a circular plate is increasing at the rate of 0.2 cm per second. At what rate is the area increasing when the radius of the plate is 25 cm.?
- 9) A metal cylinder when heated, expands in such a way that its radius r , increases at the rate of 0.2 cm. per minute and its height h increases at a rate of 0.15 cm per minute. retaining its shape. Determine the rate of change of the volume of the cylinder when its radius is 10 cms and its height is 25 cms.
- 10) For what values of x , is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

3.3 DERIVATIVE AS MEASURE OF SLOPE

3.3.1 Slope of the Tangent Line

Geometrically $\frac{dy}{dx}$ represents the slope or gradient of the tangent line to the curve $y = f(x)$ at the point $P(x, y)$. If θ is the inclination of the tangent line with the positive direction of x -axis, then slope of the line (Fig. 3.6).

$$m = \tan \theta = \frac{dy}{dx} \text{ at } P(x, y).$$

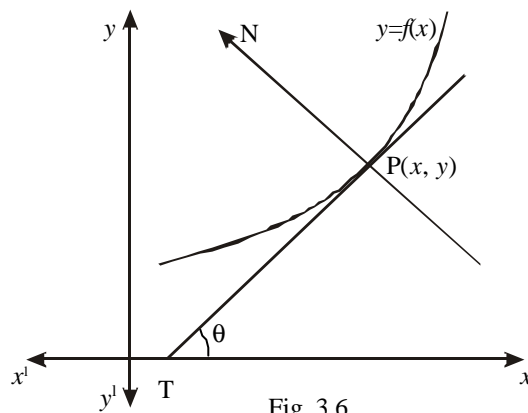


Fig. 3.6

Note

- (i) If the tangent to the curve is parallel to the x -axis, then $\theta = 0$
which implies $\tan \theta = 0 \therefore \frac{dy}{dx} = 0$ at that point
- (ii) If the tangent to the curve is parallel to the y -axis, then $\theta = 90^\circ$
which implies $\tan \theta = \infty$
 $\therefore \frac{dy}{dx} = \infty$ or $\frac{dx}{dy} = 0$ at that point.

3.3.2 Equation of the Tangent

From Analytical Geometry, the equation of the tangent to the curve $y = f(x)$ at $P(x_1, y_1)$ is

$$y - y_1 = \frac{dy}{dx} (x - x_1) \text{ where } \frac{dy}{dx} \text{ is the slope of the tangent at } P.$$

or $y - y_1 = m(x - x_1)$ where $m = \frac{dy}{dx}$ at P.

The point P is called the Point of Contact.

Note

Two tangents to the curve $y = f(x)$ will be

- (i) parallel if the slopes are equal and
- (ii) perpendicular to each other if the product of their slopes is -1 .

3.3.3 Equation of the Normal

The line which is perpendicular to the tangent at the point of contact $P(x, y)$ is called normal.

\therefore The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1), \text{ provided } \frac{dy}{dx} \neq 0 \text{ at } (x_1, y_1)$$

$$\text{or } y - y_1 = -\frac{1}{m}(x - x_1) \text{ where } m = \left(\frac{dy}{dx}\right) \text{ at } (x_1, y_1).$$

Example 21

Find the slope of the curve $y = \frac{x^2 - 12}{x - 4}$, $(x \neq 4)$ at the point $(0, 3)$ and determine the points where the tangent is parallel to the axis of x .

Solution :

$$\text{We have } y = \frac{x^2 - 12}{x - 4}$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{(x-4)(2x) - (x^2-12)(1)}{(x-4)^2}$$

$$= \frac{x^2 - 8x + 12}{(x-4)^2}$$

$$\begin{aligned} \therefore \text{ The slope of the curve at } (0, 3) &= \frac{dy}{dx} \text{ at } (0, 3) \\ &= \frac{3}{4} \end{aligned}$$

The points at which tangents are parallel to the x -axis are given by $\frac{dy}{dx} = 0$

$$\Rightarrow x^2 - 8x + 12 = 0$$

$$\Rightarrow (x - 2)(x - 6) = 0 \quad \therefore x = 2, 6$$

$$\text{when } x = 2, y = 4$$

$$\text{when } x = 6, y = 12$$

\therefore The points at which the tangents are parallel to the x - axis are (2, 4) and (6, 12).

Example 22

Determine the values of l and m so that the curve, $y = lx^2 + 3x + m$ may pass through the point (0, 1) and have its tangent parallel to the x -axis at $x = 0.75$.

Solution :

$$\text{We have } y = lx^2 + 3x + m.$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = 2lx + 3.$$

$$\begin{aligned} \text{At } x = 0.75, \frac{dy}{dx} &= 2l(0.75) + 3 \\ &= 1.5l + 3. \end{aligned}$$

The tangent at $x = 0.75$ is parallel to the x -axis

$$\therefore \text{ At } x = 0.75, \frac{dy}{dx} = 0$$

$$\Rightarrow 1.5l + 3 = 0$$

$$\Rightarrow l = -\frac{3}{1.5} = -2.$$

Since the curve is passing through the point (0, 1) we get that

$$1 = l(0)^2 + 3(0) + m \Rightarrow m = 1.$$

$$\therefore l = -2 \text{ and } m = 1.$$

Example 23

For the cost function $y = 2x \left(\frac{x+4}{x+3} \right) + 3$, prove that the marginal cost falls continuously as the output x increases.

Solution :

We have $y = 2x \left(\frac{x+4}{x+3} \right) + 3$

$$y = \frac{2x^2+8x}{x+3} + 3 \quad \text{----- (1)}$$

Marginal cost is $\frac{dy}{dx}$

\therefore Differentiating (1) with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x+3)(4x+8) - (2x^2+8x)(1)}{(x+3)^2} + 0 \\ &= \frac{2(x^2+6x+12)}{(x+3)^2} = \frac{2(x^2+6x+9+3)}{(x+3)^2} \\ &= 2 \left(\frac{(x+3)^2+3}{(x+3)^2} \right) = 2 \left(1 + \frac{3}{(x+3)^2} \right) \end{aligned}$$

This shows that as x increases, the marginal cost $\frac{dy}{dx}$ decreases.

Example 24

Prove that for the cost function $C = 100 + x + 2x^2$, where x is the output, the slope of AC curve $= \frac{1}{x}(\text{MC}-\text{AC})$.

(MC is the marginal cost and AC is the average cost)

Solution :

Cost function is $C = 100 + x + 2x^2$

$$\begin{aligned} \text{Average cost (AC)} &= \frac{100 + x + 2x^2}{x} \\ &= \frac{100}{x} + 1 + 2x \end{aligned}$$

$$\begin{aligned}\text{Slope of AC} &= \frac{d}{dx} (\text{AC}) \\ &= \frac{d}{dx} \left(\frac{100}{x} + 1 + 2x \right) = -\frac{100}{x^2} + 2 \text{ -----(1)}\end{aligned}$$

$$\begin{aligned}\text{Marginal cost MC} &= \frac{d}{dx} (\text{C}) \\ &= \frac{d}{dx} (100 + x + 2x^2) = 1 + 4x\end{aligned}$$

$$\begin{aligned}\text{MC} - \text{AC} &= (1 + 4x) - \left(\frac{100}{x} + 1 + 2x \right) \\ &= -\frac{100}{x} + 2x\end{aligned}$$

$$\begin{aligned}\frac{1}{x} (\text{MC} - \text{AC}) &= \frac{1}{x} \left(-\frac{100}{x} + 2x \right) \\ &= -\frac{100}{x^2} + 2 \text{ -----(2)}\end{aligned}$$

From (1) and (2) we get

$$\text{Slope of AC} = \frac{1}{x} (\text{MC} - \text{AC})$$

Example 25

Find the equations of the tangent and normal at the point

(a cos θ, b sin θ) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution :

$$\text{We have } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating with respect to x , we get

$$\begin{aligned}\frac{1}{a^2} (2x) + \frac{1}{b^2} 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{b^2 x}{a^2 y}\end{aligned}$$

$$\text{At } (a \cos \theta, b \sin \theta) \quad \frac{dy}{dx} = -\frac{b \cos \theta}{a \sin \theta} = m.$$

Equation of the tangent is

$$y - y_1 = m (x - x_1)$$

$$\begin{aligned}
\Rightarrow y - b \sin\theta &= -\frac{b \cos\theta}{a \sin\theta} (x - a \cos\theta) \\
\Rightarrow ay \sin\theta - ab \sin^2\theta &= -bx \cos\theta + ab \cos^2\theta \\
\Rightarrow bx \cos\theta + ay \sin\theta &= ab (\sin^2\theta + \cos^2\theta) = ab
\end{aligned}$$

Dividing both sides by 'ab', we get

$$\frac{x}{a} \cos\theta + \frac{y}{b} \sin\theta = 1$$

\therefore Equation of the tangent is $\frac{x}{a} \cos\theta + \frac{y}{b} \sin\theta = 1$

Equation of the normal is

$$y - y_1 = -\frac{1}{m} (x - x_1)$$

$$\Rightarrow y - b \sin\theta = \frac{a \sin\theta}{b \cos\theta} (x - a \cos\theta)$$

$$\Rightarrow by \cos\theta - b^2 \sin\theta \cos\theta = ax \sin\theta - a^2 \sin\theta \cos\theta$$

$$\Rightarrow ax \sin\theta + by \cos\theta = \sin\theta \cos\theta (a^2 - b^2)$$

When $\sin\theta \cos\theta \neq 0$, dividing by ' $\sin\theta \cos\theta$ ' we get

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

\therefore The equation of the normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

Example 26

Find the equation of the tangent and normal to the demand curve $y = 10 - 3x^2$ at (1, 7).

Solution :

Demand curve $y = 10 - 3x^2$

Differentiating both sides with respect x , we have

$$\frac{dy}{dx} = -6x$$

$$\text{At (1, 7)} \quad \frac{dy}{dx} = -6 = m.$$

Equation of the tangent is

$$y - y_1 = m (x - x_1)$$

$$\Rightarrow y - 7 = -6(x - 1)$$

$$\Rightarrow 6x + y - 13 = 0.$$

Equation of the normal is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 7 = -\frac{1}{-6}(x - 1)$$

$$y - 7 = \frac{1}{6}(x - 1)$$

$$6y - 42 = x - 1$$

$$\Rightarrow x - 6y + 41 = 0.$$

Example 27

Find the points on the curve $y = (x-1)(x-2)$ at which the tangent makes an angle 135° with the positive direction of the x -axis.

Solution :

We have $y = (x-1)(x-2)$

Differentiating with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= (x-1)(1) + (x-2)(1) \\ &= 2x - 3 \quad \text{-----(1)} \end{aligned}$$

Also the tangent is making 135° with the x -axis.

$$\begin{aligned} \therefore m = \frac{dy}{dx} &= \tan \theta & \left| \begin{array}{l} \tan 135^\circ = \tan (180^\circ - 45^\circ) \\ = -\tan 45^\circ \\ = -1 \end{array} \right. \\ &= \tan 135^\circ \\ &= -1 \quad \text{-----(2)} \end{aligned}$$

Equating (1) and (2) we get

$$\begin{aligned} 2x - 3 &= -1 \quad \text{or} \quad 2x = 2 \\ x &= 1 \end{aligned}$$

When $x = 1$, $y = (1-1)(1-2) = 0$.

\therefore The required point is $(1, 0)$.

EXERCISE 3.3

- 1) Find the slope of the tangent line at the point (0, 5) of the curve $y = \frac{1}{3}(x^2 + 10x - 15)$. At what point of the curve the slope of the tangent line is $\frac{8}{5}$?
- 2) Determine the coefficients a and b so that the curve $y = ax^2 - 6x + b$ may pass through the point (0, 2) and have its tangent parallel to the x -axis at $x = 1.5$.
- 3) For the cost function $y = 3x\left(\frac{x+7}{x+5}\right) + 5$, prove that the marginal cost falls continuously as the output x increases.
- 4) Find the equations of the tangents and normals to the following curves
 - (i) $y^2 = 4x$ at (1, 2)
 - (ii) $y = \sin 2x$ at $x = \frac{\pi}{6}$
 - (iii) $x^2 + y^2 = 13$ at (-3, -2)
 - (iv) $xy = 9$ at $x = 4$.
 - (v) $y = x^2 \log x$ at $x = e$
 - (vi) $x = a \cos \theta, y = b \sin \theta$ at $\theta = \frac{\pi}{4}$
- 5) Find the equation of the tangent and normal to the supply curve $y = x^2 + x + 2$ when $x = 6$.
- 6) Find the equation of the tangent and normal to the demand curve $y = 36 - x^2$ when $y = 11$.
- 7) At what points on the curve $3y = x^3$ the tangents are inclined at 45° to the x -axis.
- 8) Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = b e^{-x/a}$ at the point where the curve cuts the y -axis.
- 9) Find the equation of the tangent and normal to the curve $y(x-2)(x-3) - x + 7 = 0$ at the point where it cuts the x axis.
- 10) Prove that the curves $y = x^2 - 3x + 1$ and $x(y+3) = 4$ intersect at right angles at the point (2, -1).
- 11) Find the equation of the tangent and normal at the point $(a \sec \theta, b \tan \theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

- 12) At what points on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangent is parallel to (i) x -axis (ii) y -axis.

EXERCISE 3.4

Choose the correct answer

- 1) The average fixed cost of the function $C = 2x^3 - 3x^2 + 4x + 8$ is
 (a) $\frac{2}{x}$ (b) $\frac{4}{x}$ (c) $\frac{-3}{x}$ (d) $\frac{8}{x}$
- 2) If 60 units of some product cost Rs. 1400 and 40 units cost Rs. 1200 to manufacture, then the variable cost per unit is
 (a) Rs. 100 (b) Rs. 2600 (c) Rs. 10 (d) Rs. 5
- 3) If 20 units of some product cost Rs. 2500 and 50 units cost Rs. 3400 to produce, the linear cost function is
 (a) $y = 30x + 1900$ (b) $y = 20x + 5900$
 (c) $y = 50x + 3400$ (d) $y = 10x + 900$
- 4) Variable cost per unit is Rs. 40, fixed cost is Rs. 900 and unit selling price is Rs. 70. Then the profit equation is
 (a) $P = 30x - 900$ (b) $P = 15x - 70$
 (c) $P = 40x - 900$ (d) $P = 70x + 3600$
- 5) For the cost function $c = \frac{1}{10} e^{2x}$, the marginal cost is
 (a) $\frac{1}{10}$ (b) $\frac{1}{5} e^{2x}$ (c) $\frac{1}{10} e^{2x}$ (d) $\frac{1}{10} e^x$
- 6) Given the demand equation $p = -x + 10$; ($0 \leq x \leq 10$) where p denotes the unit selling price and x denotes the number of units demanded of some product. Then the marginal revenue at $x = 3$ units is
 (a) Rs. 5 (b) Rs. 10 (c) Rs. 4 (d) Rs. 30
- 7) The demand for some commodity is given by $q = -3p + 15$ ($0 < p < 5$) where p is the unit price. The elasticity of demand is
 (a) $\frac{9p^2 + 15}{p}$ (b) $\frac{9p - 45}{p}$ (c) $\frac{15p - 9}{p}$ (d) $\frac{p}{-p + 5}$

- 8) For the function $y = 3x + 2$ the average rate of change of y when x increases from 1.5 to 1.6 is
 (a) 1 (b) 0.5 (c) 0.6 (d) 3.
- 9) If $y = 2x^2 + 3x$, the instantaneous rate of change of y at $x = 4$ is
 (a) 16 (b) 19 (c) 30 (d) 4.
- 10) If the rate of change of y with respect to x is 6 and x is changing at 4 units/sec, then the rate of change of y per sec is
 (a) 24 units/sec (b) 10 units/sec
 (c) 2 units/sec (d) 22 units/sec
- 11) The weekly profit P , in rupees of a corporation is determined by the number x of shirts produced per week according to the formula $P = 2000x - 0.03x^2 - 1000$. Find the rate at which the profit is changing when the production level x is 1000 shirts per week.
 (a) Rs.140 (b) Rs. 2000
 (c) Rs.1500 (d) Rs. 1940
- 12) The bottom of a rectangular swimming tank is $25m$ by $40m$. Water is pumped into the tank at the rate of $500m^3/\text{min}$. Find the rate at which the level of the water in the tank is rising?
 (a) $0.5m/\text{min}$ (b) $0.2m/\text{min}$
 (c) $0.05m/\text{min}$ (d) $0.1m/\text{min}$
- 13) The slope of the tangent at $(2, 8)$ on the curve $y = x^3$ is
 (a) 3 (b) 12 (c) 6 (d) 8
- 14) The slope of the normal to the curve $\sqrt{x} + \sqrt{y} = 5$ at $(9, 4)$ is
 (a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$
- 15) For the curve $y = 1 + ax - x^2$ the tangent at $(1, -2)$ is parallel to x-axis. The value of ' a ' is
 (a) -2 (b) 2 (c) 1 (d) -1

- 16) The slope of the tangent to the curve $y = \cos t$, $x = \sin t$ at $t = \frac{\pi}{4}$ is
 (a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) -1
- 17) The point at which the tangent to the curve $y^2 = x$ makes an angle $\frac{\pi}{4}$ with the x -axis is
 (a) $(\frac{1}{2}, \frac{1}{4})$ (b) $(\frac{1}{2}, \frac{1}{2})$ (c) $(\frac{1}{4}, \frac{1}{2})$ (d) (1, -1)
- 18) The tangent to the curve $y = 2x^2 - x + 1$ at (1, 2) is parallel to the line
 (a) $y = 3x$ (b) $y = 2x + 4$
 (c) $2x + y + 7 = 0$ (d) $y = 5x - 7$
- 19) The slope of the tangent to the curve $y = x^2 - \log x$ at $x = 2$ is
 (a) $\frac{7}{2}$ (b) $\frac{2}{7}$ (c) $-\frac{7}{2}$ (d) $-\frac{2}{7}$
- 20) The slope of the curve $x = y^2 - 6y$ at the point where it crosses the y axis is
 (a) 5 (b) -5 (c) $\frac{1}{6}$ (d) $-\frac{1}{16}$

APPLICATIONS OF DIFFERENTIATION-II

4

The concept of maxima and minima is applied in Economics to study profit maximisation, inventory control and economic order quantity.

We also learn what a partial derivative is and how to calculate it. Application of partial derivatives are also discussed with the production function, marginal productivities of labour and capital and with partial elasticities of demand.

4.1 MAXIMA AND MINIMA

4.1.1 Increasing and Decreasing Functions

A function $y = f(x)$ is said to be an increasing function of x in an interval, say $a \leq x \leq b$, if y increases as x increases. i.e. if $a \leq x_1 < x_2 \leq b$, then $f(x_1) \leq f(x_2)$.

A function $y = f(x)$ is said to be a decreasing function of x in an interval, say $a \leq x \leq b$, if y decreases as x increases. i.e. if $a \leq x_1 < x_2 \leq b$, then $f(x_1) \geq f(x_2)$.

4.1.2. Sign of the derivative

Let f be an increasing function defined in a closed interval $[a, b]$. Then for any two values x_1 and x_2 in $[a, b]$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$.

$$\therefore f(x_1) \leq f(x_2) \text{ and } x_2 - x_1 > 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$$

$$\Rightarrow \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0, \text{ if this limit exists.}$$

$$\Rightarrow f'(x) \geq 0 \text{ for all } x \in [a, b].$$

Similarly, if f is decreasing on $[a, b]$ then $f'(x) \leq 0$, if the derivative exists.

The converse holds with the additional condition, that f is continuous on $[a, b]$.

Note

Let f be continuous on $[a, b]$ and has derivative at each point of the open interval (a, b) , then

- (i) If $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing on $[a, b]$
- (ii) If $f'(x) < 0$ for every $x \in (a, b)$, then f is strictly decreasing on $[a, b]$
- (iii) If $f'(x) = 0$ for every $x \in (a, b)$, then f is a constant function on $[a, b]$
- (iv) If $f'(x) \geq 0$ for every $x \in (a, b)$, then f is increasing on $[a, b]$
- (v) If $f'(x) \leq 0$ for every $x \in (a, b)$, then f is decreasing on $[a, b]$

The above results are used to test whether a given function is increasing or decreasing.

4.1.3 Stationary Value of a Function

A function $y = f(x)$ may neither be an increasing function nor be a decreasing function of x at some point of the interval $[a, b]$. In such a case, $y = f(x)$ is called stationary at that point. At a stationary point $f'(x) = 0$ and the tangent is parallel to the x - axis.

Example 1

If $y = x - \frac{1}{x}$, prove that y is a strictly increasing function for all real values of x . ($x \neq 0$)

Solution :

We have $y = x - \frac{1}{x}$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = 1 + \frac{1}{x^2} > 0 \text{ for all values of } x, \text{ except } x = 0$$

$\therefore y$ is a strictly increasing function for all real values of x . ($x \neq 0$)

Example 2

If $y = 1 + \frac{1}{x}$, show that y is a strictly decreasing function for all real values of x . ($x \neq 0$)

Solution :

$$\text{We have } y = 1 + \frac{1}{x}$$

$$\frac{dy}{dx} = 0 - \frac{1}{x^2} < 0 \text{ for all values of } x. (x \neq 0)$$

$\therefore y$ is a strictly decreasing function for all real values of x . ($x \neq 0$)

Example 3

Find the ranges of values of x in which $2x^3 - 9x^2 + 12x + 4$ is strictly increasing and strictly decreasing.

Solution :

$$\text{Let } y = 2x^3 - 9x^2 + 12x + 4$$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

$$= 6(x^2 - 3x + 2)$$

$$= 6(x - 2)(x - 1)$$

$$\frac{dy}{dx} > 0 \text{ when } x < 1 \text{ or } x > 2$$

x lies outside the interval $(1, 2)$.

$$\frac{dy}{dx} < 0 \text{ when } 1 < x < 2$$

\therefore The function is strictly increasing outside the interval $[1, 2]$ and strictly decreasing in the interval $(1, 2)$

Example 4

Find the stationary points and the stationary values of the function $f(x) = x^3 - 3x^2 - 9x + 5$.

Solution :

$$\text{Let } y = x^3 - 3x^2 - 9x + 5$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

At stationary points, $\frac{dy}{dx} = 0$

$$\therefore 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x + 1)(x - 3) = 0$$

The stationary points are obtained when $x = -1$ and $x = 3$

$$\text{when } x = -1, \quad y = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$$

$$\text{when } x = 3, \quad y = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$$

\therefore The stationary values are 10 and -22

The stationary points are $(-1, 10)$ and $(3, -22)$

Example 5

For the cost function $C = 2000 + 1800x - 75x^2 + x^3$ find when the total cost (C) is increasing and when it is decreasing. Also discuss the behaviour of the marginal cost (MC)

Solution :

$$\text{Cost function } C = 2000 + 1800x - 75x^2 + x^3$$

$$\frac{dC}{dx} = 1800 - 150x + 3x^2$$

$$\frac{dC}{dx} = 0 \Rightarrow 1800 - 150x + 3x^2 = 0$$

$$\Rightarrow 3x^2 - 150x + 1800 = 0$$

$$\Rightarrow x^2 - 50x + 600 = 0$$

$$\Rightarrow (x - 20)(x - 30) = 0$$

$$\Rightarrow x = 20 \text{ or } x = 30$$



For ,

$$(i) \quad 0 < x < 20, \quad \frac{dC}{dx} > 0 \quad \left| \quad (i) \quad x = 10 \text{ then } \frac{dC}{dx} = 600 > 0 \right.$$

$$\begin{array}{l|l}
 \text{(ii)} & 20 < x < 30, \frac{dC}{dx} < 0 & \text{(ii)} \ x = 25 \text{ then } \frac{dC}{dx} = -75 < 0 \\
 \text{(iii)} & x > 30 ; \frac{dC}{dx} > 0 & \text{(iii)} \ x = 40 \text{ then } \frac{dC}{dx} = 600 > 0
 \end{array}$$

\therefore C is increasing for $0 < x < 20$ and for $x > 30$.

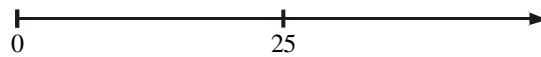
C is decreasing for $20 < x < 30$

$$MC = \frac{d}{dx} (C)$$

$$\therefore MC = 1800 - 150x + 3x^2$$

$$\frac{d}{dx}(MC) = -150 + 6x$$

$$\begin{aligned}
 \frac{d}{dx}(MC) = 0 &\Rightarrow 6x = 150 \\
 &\Rightarrow x = 25.
 \end{aligned}$$



$$\begin{array}{l|l}
 \text{(i)} \ 0 < x < 25, \frac{d}{dx}(MC) < 0 & \text{For,} \\
 \text{(ii)} \ x > 25, \frac{d}{dx}(MC) > 0 & \text{(i)} \ x = 10 \text{ then } \frac{d}{dx}(MC) = -90 < 0 \\
 & \text{(ii)} \ x = 30 \text{ then } \frac{d}{dx}(MC) = 30 > 0
 \end{array}$$

\therefore MC is decreasing for $x < 25$ and increasing for $x > 25$.

4.1.4 Maximum and Minimum Values

Let f be a function defined on $[a, b]$ and c an interior point of $[a, b]$ (i.e.) c is in the open interval (a, b) . Then

- (i) $f(c)$ is said to be a maximum or relative maximum of the function f at $x = c$ if there is a neighbourhood $(c - \delta, c + \delta)$ of c such that for all $x \in (c - \delta, c + \delta)$ other than c , $f(c) > f(x)$
- (ii) $f(c)$ is said to be a minimum or relative minimum of the function f at $x = c$ if there is a neighbourhood $(c - \delta, c + \delta)$ of c such that for all $x \in (c - \delta, c + \delta)$ other than c , $f(c) < f(x)$.

- (iii) $f(c)$ is said to be an extreme value of f or extremum at c if it is either a maximum or minimum.

4.1.5 Local and Global Maxima and Minima

Consider the graph (Fig. 4.1) of the function $y = f(x)$.

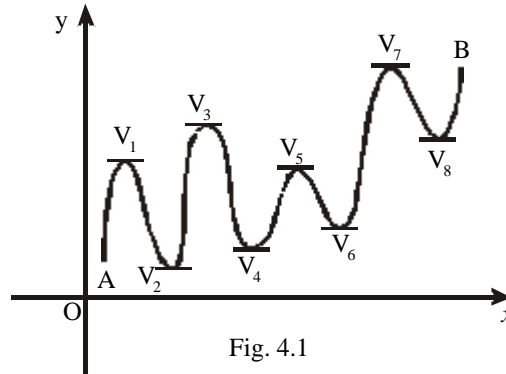


Fig. 4.1

The function $y = f(x)$ has several maximum and minimum points. At the points V_1, V_2, \dots, V_8 , $\frac{dy}{dx} = 0$. In fact the function has maxima at V_1, V_3, V_5, V_7 and minima at V_2, V_4, V_6, V_8 . Note that maximum value at V_5 is less than the minimum value at V_8 . These maxima and minima are called local or relative maxima and minima. If we consider the part of the curve between A and B then the function has absolute maximum or global maximum at V_7 and absolute minimum or global minimum at V_2 .

Note

By the terminology maximum or minimum we mean local maximum or local minimum respectively.

4.1.6. Criteria for Maxima and Minima.

	Maximum	Minimum
Necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient condition	$\frac{dy}{dx} = 0 ; \frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0 ; \frac{d^2y}{dx^2} > 0$

4.1.7 Concavity and Convexity

Consider the graph (Fig. 4.2) of the function $y = f(x)$.

Let PT be the tangent to the curve $y = f(x)$ at the point P . The curve (or an arc of the curve) which lies above the tangent line PT is said to be concave upward or convex downward.

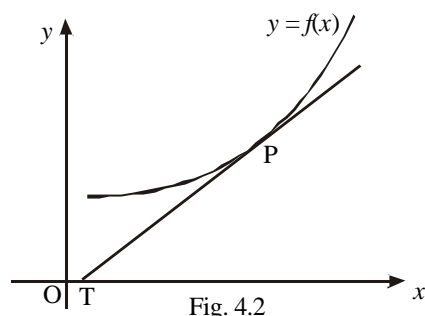


Fig. 4.2

The curve (or an arc of the curve) which lies below the tangent line PT (Fig. 4.3) is said to be convex upward or concave downward.

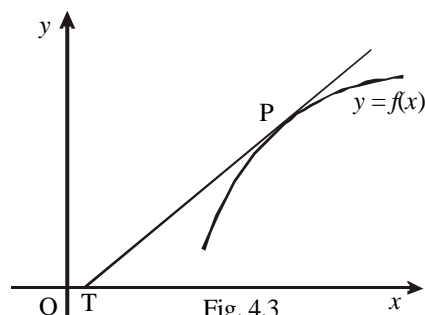


Fig. 4.3

4.1.8 Conditions for Concavity and Convexity.

Let $f(x)$ be twice differentiable. Then the curve $y = f(x)$ is

- (i) concave upward on any interval if $f''(x) > 0$
- (ii) convex upward on any interval if $f''(x) < 0$

4.1.9 Point of Inflection

A point on a curve $y = f(x)$, where the concavity changes from up to down or vice versa is called a Point of Inflection.

For example, in $y = x^{\frac{1}{3}}$ (Fig. 4.4) has a point of inflection at $x = 0$

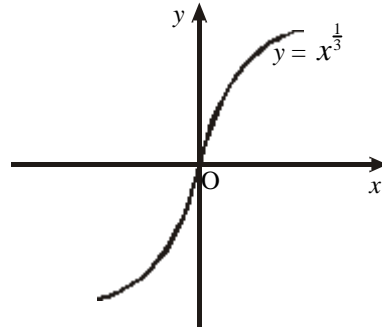


Fig. 4.4

4.1.10 Conditions for point of inflection

A point $(c, f(c))$ on a curve $y = f(x)$ is a point of inflection
 (i) if $f''(c) = 0$ or $f''(c)$ is not defined and (ii) if $f''(x)$ changes sign as x increases through c i.e. $f'''(c) \neq 0$ when $f'''(x)$ exists

Example 6

Investigate the maxima and minima of the function $2x^3 + 3x^2 - 36x + 10$.

Solution :

$$\text{Let } y = 2x^3 + 3x^2 - 36x + 10$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = 6x^2 + 6x - 36 \quad \text{-----(1)}$$

$$\begin{aligned} \frac{dy}{dx} = 0 &\Rightarrow 6x^2 + 6x - 36 = 0 \\ &\Rightarrow x^2 + x - 6 = 0 \\ &\Rightarrow (x + 3)(x - 2) = 0 \\ &\Rightarrow x = -3, 2 \end{aligned}$$

Again differentiating (1) with respect to x , we get

$$\frac{d^2y}{dx^2} = 12x + 6$$

$$\text{when } x = -3, \frac{d^2y}{dx^2} = 12(-3) + 6 = -30 < 0$$

- \therefore It attains maximum at $x = -3$
 \therefore Maximum value is $y = 2(-3)^3 + 3(-3)^2 - 36(-3) + 10 = 91$
 when $x = 2$, $\frac{d^2 y}{dx^2} = 12(2) + 6 = 30 > 0$
 \therefore It attains minimum at $x = 2$
 \therefore Minimum value is $y = 2(2)^3 + 3(2)^2 - 36(2) + 10 = -34$

Example 7

Find the absolute (global) maximum and minimum values of the function $f(x) = 3x^5 - 25x^3 + 60x + 1$ in the interval $[-2, 1]$

Solution :

$$\text{Given } f(x) = 3x^5 - 25x^3 + 60x + 1$$

$$f'(x) = 15x^4 - 75x^2 + 60$$

The necessary condition for maximum and minimum is

$$f'(x) = 0$$

$$\Rightarrow 15x^4 - 75x^2 + 60 = 0$$

$$\Rightarrow x^4 - 5x^2 + 4 = 0$$

$$\Rightarrow x^4 - 4x^2 - x^2 + 4 = 0$$

$$\Rightarrow (x^2 - 1)(x^2 - 4) = 0$$

$$\therefore x = \pm 1, -2, \quad (2 \notin [-2, 1])$$

$$f''(x) = 60x^3 - 150x$$

$$f''(-2) = 60(-2)^3 - 150(-2) = -180 < 0$$

$\therefore f(x)$ is maximum.

$$f''(-1) = 60(-1)^3 - 150(-1) = 90 > 0$$

$\therefore f(x)$ is minimum.

$$f''(1) = 60(1)^3 - 150(1) = -90 < 0$$

$\therefore f(x)$ is maximum.

The maximum value when $x = -2$ is

$$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1 = -15$$

The minimum value when $x = -1$ is

$$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1 = -37$$

The maximum value when $x = 1$ is

$$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1 = 39$$

\therefore Absolute maximum value = 39.

and Absolute minimum value = -37

Example 8

What is the maximum slope of the tangent to the curve $y = -x^3 + 3x^2 + 9x - 27$ and at what point is it?

Solution :

$$\text{We have } y = -x^3 + 3x^2 + 9x - 27$$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

\therefore Slope of the tangent is $-3x^2 + 6x + 9$

$$\text{Let } M = -3x^2 + 6x + 9$$

Differentiating with respect to x , we get

$$\frac{dM}{dx} = -6x + 6 \text{ -----(1)}$$

Slope is maximum when $\frac{dM}{dx} = 0$ and $\frac{d^2M}{dx^2} < 0$

$$\begin{aligned} \frac{dM}{dx} = 0 &\Rightarrow -6x + 6 = 0 \\ &\Rightarrow x = 1 \end{aligned}$$

Again differentiating (1) with respect to x , we get

$$\frac{d^2M}{dx^2} = -6 < 0, \therefore M \text{ is maximum at } x = 1$$

\therefore Maximum value of M when $x = 1$ is

$$M = -3(1)^2 + 6(1) + 9 = 12$$

When $x = 1$; $y = -(1)^3 + 3(1)^2 + 9(1) - 27 = -16$

\therefore Maximum slope = 12

The required point is (1, -16)

Example 9**Find the points of inflection of the curve**

$$y = 2x^4 - 4x^3 + 3.$$

*Solution :*We have $y = 2x^4 - 4x^3 + 3$ Differentiate with respect to x , we get

$$\frac{dy}{dx} = 8x^3 - 12x^2$$

$$\frac{d^2y}{dx^2} = 24x^2 - 24x$$

$$\begin{aligned} \frac{d^2y}{dx^2} = 0 &\Rightarrow 24x(x-1) = 0 \\ &\Rightarrow x = 0, 1 \end{aligned}$$

$$\frac{d^3y}{dx^3} = 48x - 24$$

$$\text{when } x = 0, 1 \quad \frac{d^3y}{dx^3} \neq 0.$$

 \therefore points of inflection exist.

$$\text{when } x = 0, \quad y = 2(0)^4 - 4(0)^3 + 3 = 3$$

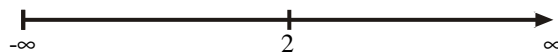
$$\text{when } x = 1, \quad y = 2(1)^4 - 4(1)^3 + 3 = 1$$

 \therefore The points of inflection are $(0, 3)$ and $(1, 1)$ **Example 10****Find the intervals on which the curve $f(x) = x^3 - 6x^2 + 9x - 8$ is convex upward and convex downward.***Solution :*We have $f(x) = x^3 - 6x^2 + 9x - 8$ Differentiating with respect to x ,

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12$$

$$f''(x) = 0 \Rightarrow 6(x-2) = 0 \quad \therefore x = 2$$



	For
(i) $-\infty < x < 2, f''(x) < 0$	(i) $x = 0$ then $f''(x) = -12 < 0$
(ii) $2 < x < \infty, f''(x) > 0$	(ii) $x = 3$ then $f''(x) = 6 > 0$

\therefore The curve is convex upward in the interval $(-\infty, 2)$

The curve is convex downward in the interval $(2, \infty)$

Exercise 4.1

- 1) Show that the function $x^3 + 3x^2 + 3x + 7$ is an increasing function for all real values of x .
- 2) Prove that $75 - 12x + 6x^2 - x^3$ always decreases as x increases.
- 3) Separate the intervals in which the function $x^3 + 8x^2 + 5x - 2$ is increasing or decreasing.
- 4) Find the stationary points and the stationary values of the function $f(x) = 2x^3 + 3x^2 - 12x + 7$.
- 5) For the following total revenue functions, find when the total revenue (R) is increasing and when it is decreasing. Also discuss the behaviour of marginal revenue (MR).
(i) $R = -90 + 6x^2 - x^3$ (ii) $R = -105x + 60x^2 - 5x^3$
- 6) For the following cost functions, find when the total cost (C) is increasing and when it is decreasing. Also discuss the behaviour of marginal cost (MC).
(i) $C = 2000 + 600x - 45x^2 + x^3$ (ii) $C = 200 + 40x - \frac{1}{2}x^2$.
- 7) Find the maximum and minimum values of the function
(i) $x^3 - 6x^2 + 7$ (ii) $2x^3 - 15x^2 + 24x - 15$
(iii) $x^2 + \frac{16}{x}$ (iv) $x^3 - 6x^2 + 9x + 15$
- 8) Find the absolute (global) maximum and minimum values of the function $f(x) = 3x^5 - 25x^3 + 60x + 15$ in the interval $[-\frac{3}{2}, 3]$.
- 9) Find the points of inflection of the curve $y = x^4 - 4x^3 + 2x + 3$.
- 10) Show that the maximum value of the function $f(x) = x^3 - 27x + 108$ is 108 more than the minimum value.

- 11) Find the intervals in which the curve $y = x^4 - 3x^3 + 3x^2 + 5x + 1$ is convex upward and convex downward.
- 12) Determine the value of output q at which the cost function $C = q^2 - 6q + 120$ is minimum.
- 13) Find the maximum and minimum values of the function $x^5 - 5x^4 + 5x^3 - 1$. Discuss its nature at $x = 0$.
- 14) Show that the function $f(x) = x^2 + \frac{250}{x}$ has a minimum value at $x = 5$.
- 15) The total revenue (TR) for commodity x is $TR = 12x + \frac{x^2}{2} - \frac{x^3}{3}$. Show that at the highest point of average revenue (AR), $AR = MR$ (where $MR = \text{Marginal Revenue}$).

4.2 APPLICATION OF MAXIMA AND MINIMA

The concept of zero slope helps us to determine the maximum value of profit functions and the minimum value of cost functions. In this section we will analyse the practical application of Maxima and Minima in commerce.

Example 11

A firm produces x tonnes of output at a total cost $C = (\frac{1}{10}x^3 - 5x^2 + 10x + 5)$. At what level of output will the marginal cost and the average variable cost attain their respective minimum?

Solution :

$$\text{Cost } C(x) = \text{Rs.}(\frac{1}{10}x^3 - 5x^2 + 10x + 5)$$

$$\text{Marginal Cost} = \frac{d}{dx}(C)$$

$$MC = \frac{3}{10}x^2 - 10x + 10$$

$$\text{Average variable cost} = \frac{\text{Variable cost}}{x}$$

$$AVC = (\frac{1}{10}x^2 - 5x + 10)$$

(i) Let $y = MC = \frac{3}{10}x^2 - 10x + 10$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{3}{5}x - 10$$

Marginal cost is minimum when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{3}{5}x - 10 = 0 \text{ or } x = \frac{50}{3}$$

$$\text{when } x = \frac{50}{3}, \frac{d^2y}{dx^2} = \frac{3}{5} > 0 \therefore \text{MC is minimum.}$$

\therefore Marginal cost attains its minimum at $x = \frac{50}{3}$ units.

(ii) Let $z = AVC = \frac{1}{10}x^2 - 5x + 10$

Differentiating with respect to x , we get

$$\frac{dz}{dx} = \frac{1}{5}x - 5$$

AVC is minimum when $\frac{dz}{dx} = 0$, and $\frac{d^2z}{dx^2} > 0$

$$\frac{dz}{dx} = 0 \Rightarrow \frac{1}{5}x - 5 = 0 \Rightarrow x = 25.$$

$$\text{when } x = 25, \frac{d^2z}{dx^2} = \frac{1}{5} > 0 \therefore \text{AVC is minimum at } x = 25 \text{ units.}$$

\therefore Average variable cost attains minimum at $x = 25$ units.

Example 12

A certain manufacturing concern has total cost function $C = 15 + 9x - 6x^2 + x^3$. Find x , when the total cost is minimum.

Solution :

$$\text{Cost } C = 15 + 9x - 6x^2 + x^3$$

Differentiating with respect to x , we get

$$\frac{dC}{dx} = 9 - 12x + 3x^2 \quad \text{-----(1)}$$

Cost is minimum when $\frac{dC}{dx} = 0$ and $\frac{d^2C}{dx^2} > 0$

$$\begin{aligned}\frac{dC}{dx} = 0 &\Rightarrow 3x^2 - 12x + 9 = 0 \\ &\quad x^2 - 4x + 3 = 0 \\ &\Rightarrow x = 3, x = 1\end{aligned}$$

Differentiating (1) with respect to x we get

$$\frac{d^2C}{dx^2} = -12 + 6x$$

when $x = 1$; $\frac{d^2C}{dx^2} = -12 + 6 = -6 < 0 \therefore C$ is maximum

when $x = 3$, $\frac{d^2C}{dx^2} = -12 + 18 = 6 > 0 \therefore C$ is minimum

\therefore when $x = 3$, the total cost is minimum

Example 13

The relationship between profit P and advertising cost x is given by $P = \frac{4000x}{500+x} - x$. Find x which maximises P .

Solution :

$$\text{Profit } P = \frac{4000x}{500+x} - x$$

Differentiating with respect to x we get

$$\begin{aligned}\frac{dP}{dx} &= \frac{(500+x)4000 - (4000x)(1)}{(500+x)^2} - 1 \\ &= \frac{2000000}{(500+x)^2} - 1 \quad \text{-----(1)}\end{aligned}$$

Profit is maximum when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$$\begin{aligned}\frac{dP}{dx} = 0 &\Rightarrow \frac{2000000}{(500+x)^2} - 1 = 0 \\ &\Rightarrow 2000000 = (500+x)^2 \\ &\Rightarrow 1000 \times \sqrt{2} = 500+x\end{aligned}$$

$$1000 \times 1.414 = 500 + x$$

$$x = 914.$$

Differentiating (1) with respect to x we get

$$\frac{d^2P}{dx^2} = -\frac{4000000}{(500+x)^3}$$

$$\therefore \text{ when } x = 914 ; \frac{d^2P}{dx^2} < 0 \quad \therefore \text{ Profit is maximum.}$$

Example 14

The total cost and total revenue of a firm are given by $C = x^3 - 12x^2 + 48x + 11$ and $R = 83x - 4x^2 - 21$. Find the output (i) when the revenue is maximum (ii) when profit is maximum.

Solution :

(i) Revenue $R = 83x - 4x^2 - 21$

Differentiating with respect to x ,

$$\frac{dR}{dx} = 83 - 8x$$

$$\frac{d^2R}{dx^2} = -8$$

Revenue is maximum when $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2} < 0$

$$\frac{dR}{dx} = 0 \Rightarrow 83 - 8x = 0 \quad \therefore x = \frac{83}{8}$$

Also $\frac{d^2R}{dx^2} = -8 < 0$. $\therefore R$ is maximum

\therefore When the output $x = \frac{83}{8}$ units, revenue is maximum

(ii) Profit $P = R - C$

$$= (83x - 4x^2 - 21) - (x^3 - 12x^2 + 48x + 11)$$

$$= -x^3 + 8x^2 + 35x - 32$$

Differentiating with respect to x ,

$$\frac{dP}{dx} = -3x^2 + 16x + 35$$

$$\frac{d^2P}{dx^2} = -6x + 16$$

Profit is maximum when $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$

$$\therefore \frac{dP}{dx} = 0 \Rightarrow -3x^2 + 16x + 35 = 0$$

$$\Rightarrow 3x^2 - 16x - 35 = 0$$

$$\Rightarrow (3x + 5)(x - 7) = 0$$

$$\Rightarrow x = \frac{-5}{3} \text{ or } x = 7$$

$$\text{when } x = \frac{-5}{3}, \frac{d^2P}{dx^2} = -6\left(\frac{-5}{3}\right) + 16 = 26 > 0 \therefore P \text{ is minimum}$$

$$\text{when } x = 7, \frac{d^2P}{dx^2} = -6(7) + 16 = -26 < 0 \therefore P \text{ is maximum}$$

\therefore when $x = 7$ units, profit is maximum.

Example 15

A telephone company has a profit of Rs. 2 per telephone when the number of telephones in the exchange is not over 10,000. The profit per telephone decreases by 0.01 paisa for each telephone over 10,000. What is the maximum profit?

Solution :

Let x be the number of telephones.

The decrease in the profit per telephone

$$= (x - 10,000)(0.01), \quad x > 10,000.$$

$$= (0.01x - 100)$$

The profit per telephone

$$= 200 - (0.01x - 100)$$

$$= (300 - 0.01x)$$

The total profit for x telephones

$$= x(300 - 0.01x)$$

$$= 300x - 0.01x^2$$

Let the total profit $P = 300x - 0.01x^2$

Differentiating with respect to x , we get

$$\frac{dP}{dx} = 300 - 0.02x \quad \text{-----(1)}$$

Conditions for the maximum profit are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2P}{dx^2} < 0$$

$$\frac{dP}{dx} = 0 \Rightarrow 300 - 0.02x = 0$$

$$\Rightarrow x = \frac{300}{0.02} = 15,000.$$

Differentiating (1) with respect to x we get

$$\frac{d^2P}{dx^2} = -0.02 < 0 \quad \therefore P \text{ is maximum}$$

\therefore when $x = 15,000$, the maximum profit

$$\begin{aligned} P &= (300 \times 15,000) - (0.01) \times (15,000)^2 \text{ paise} \\ &= \text{Rs. } (45,000 - 22,500) = \text{Rs. } 22,500 \end{aligned}$$

\therefore Maximum profit is Rs. 22,500.

Example 16

The total cost function of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ where x is the output. A tax at Rs. 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $p = 2530 - 5x$, where Rs. p is the price per unit of output, find the profit maximising output and price.

Solution :

$$\begin{aligned} \text{Total Revenue (R)} &= px \\ &= (2530 - 5x)x = 2530x - 5x^2 \end{aligned}$$

Total cost after the imposition of tax is

$$\begin{aligned} C + 2x &= \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x \\ &= \frac{1}{3}x^3 - 5x^2 + 30x + 10 \end{aligned}$$

Profit = Revenue – Cost

$$= (2530x - 5x^2) - \left(\frac{1}{3}x^3 - 5x^2 + 30x + 10\right)$$

$$P = -\frac{1}{3}x^3 + 2500x - 10$$

Differentiating P with respect to x,

$$\frac{dP}{dx} = -x^2 + 2500 \quad \text{-----(1)}$$

Conditions for maximum profit are

$$\frac{dP}{dx} = 0 \text{ and } \frac{d^2P}{dx^2} < 0$$

$$\begin{aligned} \frac{dP}{dx} = 0 &\Rightarrow 2500 - x^2 = 0 \\ &\Rightarrow x^2 = 2500 \text{ or } x = 500 \end{aligned}$$

Differentiating (1) with respect to x

$$\frac{d^2P}{dx^2} = -2x$$

When $x = 50$, $\frac{d^2P}{dx^2} = -50 < 0 \therefore P$ is maximum

\therefore Profit maximising output is 50 units

When $x = 50$, price $p = 2530 - (5 \times 50)$
 $= 2530 - 250 = \text{Rs. } 2280$

4.2.1 Inventory Control

Inventory is defined as the stock of goods. In practice raw materials are stored upto a capacity for smooth and efficient running of business.

4.2.2 Costs Involved in Inventory Problems

(i) **Holding cost or storage cost or inventory carrying cost. (C_1)**

The cost associated with carrying or holding the goods in stock is known as holding cost per unit per unit time.

(ii) **Shortage cost (C_2)**

The penalty costs that are incurred as a result of running out of stock are known as shortage cost.

(iii) Set up cost or ordering cost or procurement cost : (C_3)

This is the cost incurred with the placement of order or with the initial preparation of production facility such as resetting the equipment for production.

4.2.3 Economic Order Quantity (EOQ)

Economic order quantity is that size of order which minimises total annual cost of carrying inventory and the cost of ordering under the assumed conditions of certainty with the annual demands known. Economic order quantity is also called Economic lot size formula.

4.2.4 Wilson's Economic Order Quantity Formula

The formula is to determine the optimum quantity ordered (or produced) and the optimum interval between successive orders, if the demand is known and uniform with no shortages.

Let us have the following assumptions.

- (i) Let R be the uniform demand per unit time.
- (ii) Supply or production of items to the inventory is instantaneous.
- (iii) Holding cost is Rs. C_1 per unit per unit time.
- (iv) Let there be n orders (cycles) per year, each time q units are ordered (produced).
- (v) Let Rs C_3 be the ordering (set up) cost per order (cycle). Let t be the time taken between each order.

Diagrammatic representation of this model is given below :

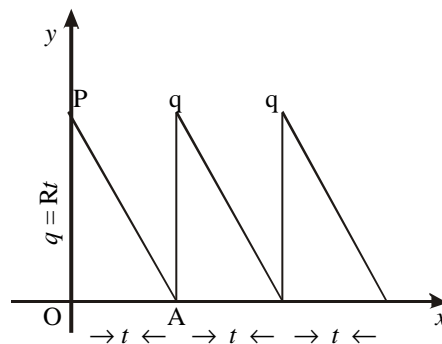


Fig. 4.5

If a production run is made at intervals t , a quantity $q = Rt$ must be produced in each run. Since the stock in small time dt is $Rt dt$, the stock in period t is

$$\begin{aligned}\int_0^t Rt dt &= \frac{1}{2} Rt^2 \\ &= \frac{1}{2} qt \quad (\text{as } Rt = q)\end{aligned}$$

= Area of the inventory triangle OAP (Fig. 4.5).

$$\text{Cost of holding inventory per production run} = \frac{1}{2} C_1 Rt^2.$$

$$\text{Set up cost per production run} = C_3.$$

$$\therefore \text{Total cost per production run} = \frac{1}{2} C_1 Rt^2 + C_3$$

Average total cost per unit time

$$C(t) = \frac{1}{2} C_1 Rt + \frac{C_3}{t} \text{ -----(1)}$$

$$C(t) \text{ is minimum if } \frac{d}{dt} C(t) = 0 \text{ and } \frac{d^2}{dt^2} C(t) > 0$$

Differentiating (1) with respect to t we get

$$\frac{d}{dt} C(t) = \frac{1}{2} C_1 R - \frac{C_3}{t^2} \text{ -----(2)}$$

$$\frac{d}{dt} C(t) = 0 \Rightarrow \frac{1}{2} C_1 R - \frac{C_3}{t^2} = 0$$

$$\Rightarrow t = \sqrt{\frac{2C_3}{C_1 R}}$$

Differentiating (2) with respect to t , we get

$$\frac{d^2}{dt^2} C(t) = \frac{2C_3}{t^3} > 0, \text{ when } t = \sqrt{\frac{2C_3}{C_1 R}}$$

Thus $C(t)$ is minimum for optimum time interval

$$t_o = \sqrt{\frac{2C_3}{C_1 R}}$$

Optimum quantity q_o to be produced during each production run,

$$EOQ = q_0 = Rt_0 = \sqrt{\frac{2C_3R}{C_1}}$$

This is known as the Optimal Lot - size formula due to Wilson.

Note : (i) Optimum number of orders per year

$$n_0 = \frac{\text{demand}}{EOQ} = R\sqrt{\frac{C_1}{2C_3R}} = \sqrt{\frac{RC_1}{2C_3}} = \frac{1}{t_0}$$

(ii) Minimum average cost per unit time, $C_0 = \sqrt{2C_1C_3R}$

(iii) Carrying cost = $\frac{q_0}{2} \times C_1$, Ordering cost = $\frac{R}{q_0} \times C_3$

(iv) At EOQ, Ordering cost = Carrying cost.

Example 17

A manufacturer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and no shortages are allowed. The inventory holding cost is 20 paise per unit per month and the set up cost per run is Rs.350. Determine (i) the optimum run size q_0 (ii) optimum scheduling period t_0 (iii) minimum total variable yearly cost.

Solution :

$$\text{Supply rate } R = \frac{12,000}{12} = 1,000 \text{ units / month.}$$

$$C_1 = 20 \text{ paise per unit per month}$$

$$C_3 = \text{Rs. 350 per run.}$$

$$(i) \quad q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 350 \times 1000}{0.20}} \\ = 1,870 \text{ units / run.}$$

$$t_0 = \sqrt{\frac{2C_3}{C_1R}} = \sqrt{\frac{2 \times 350}{0.20 \times 1000}} = 56 \text{ days}$$

$$(iii) \quad C_0 = \sqrt{2C_1C_3R} = \sqrt{2 \times 0.20 \times 12 \times 350 \times (1000 \times 12)} \\ = \text{Rs.4,490 per year.}$$

Example 18

A company uses annually 24,000 units of raw materials which costs Rs. 1.25 per unit, placing each order costs Rs. 22.50 and the holding cost is 5.4% per year of the average inventory. Find the EOQ, time between each order, total number of orders per year. Also verify that at EOQ carrying cost is equal to ordering cost

Solution :

$$\text{Requirement} = 24,000 \text{ units / year}$$

$$\text{Ordering Cost (C}_3\text{)} = \text{Rs.}22.50$$

$$\text{Holding cost (C}_1\text{)} = 5.4\% \text{ of the value of each unit.}$$

$$= \frac{5.4}{100} \times 1.25$$

$$= \text{Rs.}0.0675 \text{ per unit per year.}$$

$$\text{EOQ} = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 24000 \times 22.5}{0.0675}} = 4000 \text{ units.}$$

$$\text{Time between each order} = t_o = \frac{q_o}{R} = \frac{4000}{24000} = \frac{1}{6} \text{ year}$$

$$\text{Number of order per year} = \frac{R}{q_o} = \frac{24000}{4000} = 6$$

$$\text{At EOQ carrying cost} = \frac{q_o}{2} \times C_1 = \frac{4000}{2} \times 0.0675 = \text{Rs.}135$$

$$\text{Ordering cost} = \frac{R}{q_o} \times C_3 = \frac{24000}{4000} \times 22.50 = \text{Rs.}135$$

Example 19

A manufacturing company purchases 9000 parts of a machine for its annual requirements. Each part costs Rs.20. The ordering cost per order is Rs.15 and carrying charges are 15% of the average inventory per year.

Find (i) economic order quantity

(ii) time between each order

(iii) minimum average cost

Solution :

Requirement $R = 9000$ parts per year

$C_1 = 15\%$ unit cost

$$= \frac{15}{100} \times 20 = \text{Rs.}3 \text{ each part per year.}$$

$C_3 = \text{Rs.}15$ per order

$$\begin{aligned} \text{EOQ} &= \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 15 \times 9000}{3}} \\ &= 300 \text{ units.} \end{aligned}$$

$$\begin{aligned} t_0 &= \frac{q_0}{R} = \frac{300}{9000} = \frac{1}{30} \text{ year} \\ &= \frac{365}{30} = 12 \text{ days (approximately).} \end{aligned}$$

$$\begin{aligned} \text{Minimum Average cost} &= \sqrt{2C_1C_3R} \\ &= \sqrt{2 \times 3 \times 15 \times 9000} = \text{Rs.}900 \end{aligned}$$

EXERCISE 4.2

- 1) A certain manufacturing concern has the total cost function $C = \frac{1}{5}x^2 - 6x + 100$. Find when the total cost is minimum.
- 2) A firm produces an output of x tons of a certain product at a total cost given by $C = 300x - 10x^2 + \frac{1}{3}x^3$. Find the output at which the average cost is least and the corresponding value of the average cost.
- 3) The cost function, when the output is x , is given by $C = x(2e^x + e^{-x})$. Show that the minimum average cost is $2\sqrt{2}$.
- 4) A firm produces x tons of a valuable metal per month at a total cost C given by $C = \text{Rs.}(\frac{1}{3}x^3 - 5x^2 + 75x + 10)$. Find at what level of output, the marginal cost attains its minimum.

- 5) A firm produces x units of output per week at a total cost of Rs. $(\frac{1}{3}x^3 - x^2 + 5x + 3)$. Find the level at which the marginal cost and the average variable cost attain their respective minimum.
- 6) It is known that in a mill the number of labourers x and the total cost C are related by $C = \frac{3}{2(x-4)} + \frac{3}{32}x$. What value of x will minimise the cost?
- 7) $R = 21x - x^2$ and $C = \frac{x^3}{3} - 3x^2 + 9x + 16$ are respectively the sales revenue and cost function of x units sold.
Find (i) At what output the revenue is maximum? What is the total revenue at this point?
(ii) What is the marginal cost at a minimum?
(iii) What output will maximise the profit?
- 8) A firm has revenue function $R = 8x$ and a production cost function $C = 150000 + 60\left(\frac{x^2}{900}\right)$. Find the total profit function and the number of units to be sold to get the maximum profit.
- 9) A radio manufacturer finds that he can sell x radios per week at Rs p each, where $p = 2(100 - \frac{x}{4})$. His cost of production of x radios per week is Rs. $(120x + \frac{x^2}{2})$. Show that his profit is maximum when the production is 40 radios per week. Find also his maximum profit per week.
- 10) A manufacturer can sell x items per week at a price of $p = 600 - 4x$ rupees. Production cost of x items works out to Rs. C where $C = 40x + 2000$. How much production will yield maximum profit?
- 11) Find the optimum output of a firm whose total revenue and total cost functions are given by $R = 30x - x^2$ and $C = 20 + 4x$, x being the output of the firm.
- 12) Find EOQ for the data given below. Also verify that carrying costs is equal to ordering costs at EOQ.

Item	Monthly Requirements	Ordering cost per order	Carrying cost Per unit.
A	9000	Rs. 200	Rs. 3.60
B	25000	Rs. 648	Rs. 10.00
C	8000	Rs. 100	Rs. 0.60

- 13) Calculate the EOQ in units and total variable cost for the following items, assuming an ordering cost of Rs.5 and a holding cost of 10%

Item	Annual demand	Unit price (Rs.)
A	460 Units	1.00
B	392 Units	8.60
C	800 Units	0.02
D	1500 Units	0.52

- 14) A manufacturer has to supply his customer with 600 units of his products per year. Shortages are not allowed and storage cost amounts to 60 paise per unit per year. When the set up cost is Rs. 80 find,
- (i) the economic order quantity.
 - (ii) the minimum average yearly cost
 - (iii) the optimum number of orders per year
 - (iv) the optimum period of supply per optimum order.
- 15) The annual demand for an item is 3200 units. The unit cost is Rs.6 and inventory carrying charges 25% per annum. If the cost of one procurement is Rs.150, determine (i) Economic order quantity. (ii) Time between two consecutive orders (iii) Number of orders per year (iv) minimum average yearly cost.

4.3 PARTIAL DERIVATIVES

In differential calculus, so far we have discussed functions of one variable of the form $y = f(x)$. Further one variable may be expressed as a function of several variables. For example, production may be treated as a function of labour and capital and

price may be a function of supply and demand. In general, the cost or profit depends upon a number of independent variables, for example, prices of raw materials, wages on labour, market conditions and so on. Thus a dependent variable y depends on a number of independent variables $x_1, x_2, x_3 \dots x_n$. It is denoted by $y = f(x_1, x_2, x_3 \dots x_n)$ and is called a function of n variables. In this section, we will restrict the study to functions of two or three variables and their derivatives only.

4.3.1. Definition

Let $u = f(x, y)$ be a function of two independent variables x and y . The derivative of $f(x, y)$ with respect to x , keeping y constant, is called partial derivative of u with respect to x and is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or u_x . Similarly we can define partial derivative of f with respect to y .

Thus we have

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

provided the limit exists.

(Here y is fixed and Δx is the increment of x)

$$\text{Also } \frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limit exists.

(Here x is fixed and Δy is the increment of y).

4.3.2 Successive Partial Derivatives.

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are in general functions of x and y . So we can differentiate functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ partially with respect to x and y . These derivatives are called second order partial derivatives of $f(x, y)$. Second order partial derivatives are denoted

$$\begin{aligned}\text{by } \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) &= \frac{\partial^2 f}{\partial x^2} = f_{xx} \\ \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) &= \frac{\partial^2 f}{\partial y^2} = f_{yy} \\ \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \\ \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) &= \frac{\partial^2 f}{\partial y \partial x} = f_{yx}\end{aligned}$$

Note

If f , f_x , f_y are continuous then $f_{xy} = f_{yx}$

4.3.3 Homogeneous Function

A function $f(x, y)$ of two independent variable x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for $t > 0$.

4.3.4 Euler's Theorem on Homogeneous Function

Theorem : Let f be a homogeneous function in x and y of degree n , then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f.$$

Corollary : In general if $f(x_1, x_2, x_3, \dots, x_m)$ is a homogeneous function of degree n in variables $x_1, x_2, x_3, \dots, x_m$, then,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + \dots + x_m \frac{\partial f}{\partial x_m} = n f.$$

Example 20

It $u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$, find each of the following.

$$\text{(i) } \frac{\partial u}{\partial x} \quad \text{(ii) } \frac{\partial u}{\partial y} \quad \text{(iii) } \frac{\partial^2 u}{\partial x^2} \quad \text{(iv) } \frac{\partial^2 u}{\partial y^2} \quad \text{(v) } \frac{\partial^2 u}{\partial x \partial y} \quad \text{(vi) } \frac{\partial^2 u}{\partial y \partial x}$$

Solution :

$$u(x, y) = 1000 - x^3 - y^2 + 4x^3y^6 + 8y$$

$$\begin{aligned}
\text{(i)} \quad \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} (1000 - x^3 - y^2 + 4x^3y^6 + 8y) \\
&= 0 - 3x^2 - 0 + 4(3x^2)y^6 + 0 \\
&= -3x^2 + 12x^2y^6.
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} (1000 - x^3 - y^2 + 4x^3y^6 + 8y) \\
&= 0 - 0 - 2y + 4x^3(6y^5) + 8 \\
&= -2y + 24x^3y^5 + 8
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \\
&= \frac{\partial}{\partial x} (-3x^2 + 12x^2y^6) \\
&= -6x + 12(2x)y^6 \\
&= -6x + 24xy^6.
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \\
&= \frac{\partial}{\partial y} (-2y + 24x^3y^5 + 8) \\
&= -2 + 24x^3(5y^4) + 0 \\
&= -2 + 120x^3y^4
\end{aligned}$$

$$\begin{aligned}
\text{(v)} \quad \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) \\
&= \frac{\partial}{\partial x} (-2y + 24x^3y^5 + 8) \\
&= 0 + 24(3x^2)y^5 + 0 \\
&= 72x^2y^5.
\end{aligned}$$

$$\begin{aligned}
\text{(vi)} \quad \frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) \\
&= \frac{\partial}{\partial y} (-3x^2 + 12x^2y^6) \\
&= 0 + 12x^2(6y^5) = 72x^2y^5
\end{aligned}$$

Example 21

If $f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 5$ find (i) $f_x(1, -1)$
(ii) $f_{yy}(1, 1)$ (iii) $f_{xy}(2, 1)$

Solution :

$$(i) \quad f(x, y) = 3x^2 + 4y^3 + 6xy - x^2y^3 + 5$$

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}(f) = \frac{\partial}{\partial x}(3x^2 + 4y^3 + 6xy - x^2y^3 + 5) \\ &= 6x + 0 + 6(1)y - (2x)y^3 + 0 \\ &= 6x + 6y - 2xy^3. \end{aligned}$$

$$f_x(1, -1) = 6(1) + 6(-1) - 2(1)(-1)^3 = 2$$

$$(ii) \quad f_y = \frac{\partial}{\partial y}(f) = \frac{\partial}{\partial y}(3x^2 + 4y^3 + 6xy - x^2y^3 + 5)$$

$$= 12y^2 + 6x - 3x^2y^2$$

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \\ &= \frac{\partial}{\partial y}(12y^2 + 6x - 3x^2y^2) \\ &= 24y - 6x^2y \end{aligned}$$

$$\therefore f_{yy}(1, 1) = 18$$

$$(iii) \quad f_{xy} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x}(12y^2 + 6x - 3x^2y^2)$$

$$= 6 - 6xy^2$$

$$\therefore f_{xy}(2, 1) = -6$$

Example 22

If $u = \log \sqrt{x^2 + y^2 + z^2}$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{x^2 + y^2 + z^2}$$

Solution :

$$\text{We have } u = \frac{1}{2} \log (x^2 + y^2 + z^2) \quad \text{-----}(1)$$

Differentiating (1) partially with respect to x ,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1}{2} \frac{2x}{x^2 + y^2 + z^2} = \frac{x}{x^2 + y^2 + z^2} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2 + z^2} \right) = \frac{(x^2 + y^2 + z^2)(1) - x(2x)}{(x^2 + y^2 + z^2)^2} \\ &= \frac{x^2 + y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2}\end{aligned}$$

Differentiating (1) partially with respect to y we get,

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{y}{x^2 + y^2 + z^2} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{(x^2 + y^2 + z^2)(1) - y(2y)}{(x^2 + y^2 + z^2)^2} = \frac{-y^2 + z^2 + x^2}{(x^2 + y^2 + z^2)^2}\end{aligned}$$

Differentiating (1) partially with respect to z we get,

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{z}{x^2 + y^2 + z^2} \\ \frac{\partial^2 u}{\partial z^2} &= \frac{(x^2 + y^2 + z^2)(1) - z(2z)}{(x^2 + y^2 + z^2)^2} = \frac{-z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^2} \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{-x^2 + y^2 + z^2 - y^2 + z^2 + x^2 - z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^2} \\ &= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}\end{aligned}$$

Example 23

Verify Euler's theorem for the function

$$u(x, y) = x^3 + y^3 + x^2y.$$

Solution :

We have $u(x, y) = x^3 + y^3 + x^2y$ -----(1)

$$u(tx, ty) = t^3x^3 + t^3y^3 + t^2x^2(ty)$$

$$= t^3 (x^3 + y^3 + x^2y) = t^3 u(x, y)$$

$\therefore u$ is a homogeneous function of degree 3 in x and y .

We have to verify that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u$.

Differentiating (1) partially with respect to x , we get

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy$$

$$\therefore x \frac{\partial u}{\partial x} = 3x^3 + 2x^2y$$

Differentiating (1) partially with respect to y , we get

$$\frac{\partial u}{\partial y} = 3y^2 + x^2$$

$$\therefore y \frac{\partial u}{\partial y} = 3y^3 + x^2y$$

$$\begin{aligned} \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= 3x^3 + 2x^2y + 3y^3 + x^2y \\ &= 3(x^3 + x^2y + y^3) = 3u \end{aligned}$$

Thus Euler's Theorem is verified, for the given function.

Example 24

Using Euler's theorem if $u = \log \frac{x^4 + y^4}{x - y}$

show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$.

Solution :

$$u = \log \frac{x^4 + y^4}{x - y}$$

$$\Rightarrow e^u = \frac{x^4 + y^4}{x - y}$$

This is a homogeneous function of degree 3 in x and y

\therefore By Euler's theorem,

$$x \frac{\partial}{\partial x} (e^u) + y \frac{\partial}{\partial y} (e^u) = 3e^u$$

$$x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 3e^u$$

dividing by e^u we get $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$

Example 25

Without using Euler's theorem prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 4u, \text{ if } u = 3x^2yz + 4xy^2z + 5y^4$$

Solution :

$$\text{We have } u = 3x^2yz + 4xy^2z + 5y^4 \quad \text{-----(1)}$$

Differentiating partially with respect to x , we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3(2x)yz + 4(1)y^2z + 0 \\ &= 6xyz + 4y^2z \end{aligned}$$

Differentiating (1) partially with respect to y , we get

$$\begin{aligned} \frac{\partial u}{\partial y} &= 3x^2(1)z + 4x(2y)z + 20y^3 \\ &= 3x^2z + 8xyz + 20y^3 \end{aligned}$$

Differentiating (1) partially with respect to z , we get

$$\begin{aligned} \frac{\partial u}{\partial z} &= 3x^2y(1) + 4xy^2(1) + 0 \\ &= 3x^2y + 4xy^2 \end{aligned}$$

$$\begin{aligned} \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} &= 6x^2yz + 4xy^2z + 3x^2yz + 8xy^2z + 20y^4 + 3x^2yz + 4xy^2z \\ &= 12x^2yz + 16xy^2z + 20y^4 \\ &= 4(3x^2yz + 4xy^2z + 5y^4) = 4u. \end{aligned}$$

Example 26

The revenue derived from selling x calculators and y adding machines is given by $R(x, y) = -x^2 + 8x - 2y^2 + 6y + 2xy + 50$. If 4 calculators and 3 adding machines are sold, find the marginal revenue of selling (i) one more calculator (ii) one more adding machine.

Solution :

- (i) The marginal revenue of selling one more calculator is R_x .

$$\begin{aligned} R_x &= \frac{\partial}{\partial x}(R) = \frac{\partial}{\partial x}(-x^2 + 8x - 2y^2 + 6y + 2xy + 50) \\ &= -2x + 8 - 0 + 0 + 2(1)(y) \end{aligned}$$

$$R_x(4, 3) = -2(4) + 8 + 2(3) = 6$$

\therefore At (4, 3), revenue is increasing at the rate of Rs.6 per calculator sold.

\therefore Marginal revenue is Rs. 6.

- (ii) Marginal Revenue of selling one more adding machine is R_y

$$\begin{aligned} R_y &= \frac{\partial}{\partial y}(R) = \frac{\partial}{\partial y}(-x^2 + 8x - 2y^2 + 6y + 2xy + 50) \\ &= 0 + 0 - 4y + 6 + 2x(1) \\ &= -4y + 6 + 2x \end{aligned}$$

$$R_y(4, 3) = -4(3) + 6 + 2(4) = 2$$

Thus at (4, 3) revenue is increasing at the rate of approximately Rs.2 per adding machine.

Hence Marginal revenue is Rs.2.

EXERCISE 4.3

- 1) If $u = 4x^2 - 3y^2 + 6xy$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
- 2) If $u = x^3 + y^3 + z^3 - 3xyz$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 3u$
- 3) If $z = 4x^6 - 8x^3 - 7x + 6xy + 8y + x^3y^5$, find each of the following
 - (i) $\frac{\partial u}{\partial x}$ (ii) $\frac{\partial u}{\partial y}$ (iii) $\frac{\partial^2 z}{\partial x^2}$ (iv) $\frac{\partial^2 z}{\partial y^2}$ (v) $\frac{\partial^2 z}{\partial x \partial y}$ (vi) $\frac{\partial^2 z}{\partial y \partial x}$
- 4) If $f(x, y) = 4x^2 - 8y^3 + 6x^5y^2 + 4x + 6y + 9$, evaluate the following.

(i) f_x	(ii) $f_x(2, 1)$	(iii) f_y	(iv) $f_y(0, 2)$
(v) f_{xx}	(vi) $f_{xx}(2, 1)$	(vii) f_{yy}	(viii) $f_{yy}(1, 0)$
(ix) f_{xy}	(x) $f_{xy}(2, 3)$	(xi) $f_{yx}(2, 3)$	

- (ii) If $u = \log \frac{x^2 + y^2}{x + y}$, then prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$
- 14) The cost of producing x washers and y dryers is given by $C(x, y) = 40x + 200y + 10xy + 500$. Presently, 50 washers and 90 dryers are being produced. Find the marginal cost of producing (i) one more washer (ii) one more dryer.
- 15) The revenue derived from selling x pens and y note books is given by $R(x, y) = 2x^2 + y^2 + 4x + 5y + 800$
At present, the retailer is selling 30 pens and 50 notebooks. Which of these two product lines should be expanded in order to yield the greater increase in revenue?
- 16) The annual profit of a certain hotel is given by $P(x, y) = 100x^2 + 4y^2 + 2x + 5y + 10000$. Where x is the number of rooms available for rent and y is the monthly advertising expenditures. Presently, the hotel has 90 rooms available and is spending Rs.1000 per month on advertising.
- If an additional room is constructed, how will this affect annual profit?
 - If an additional rupee is spent on monthly advertising expenditures, how will this affect annual profit?

4.4 APPLICATIONS OF PARTIAL DERIVATIVES

In this section we learn how the concept of partial derivatives are used in the field of Commerce and Economics.

4.4.1 Production Function

Production P of a firm depends upon several economic factors like investment or capital (K), labour (L), raw material (R), etc. Thus $P = f(K, L, R, \dots)$. If P depends only on labour (L) and capital (K), then we write $P = f(L, K)$.

4.4.2 Marginal Productivities

Let $P = f(L, K)$ represent a production function of two variables L and K .

$\frac{\partial P}{\partial L}$ is called the 'Marginal Productivity of Labour' and $\frac{\partial P}{\partial K}$ is the 'Marginal Productivity of Capital'.

4.4.3 Partial Elasticities of Demand

Let $q_1 = f(p_1, p_2)$ be the demand for commodity A which depends upon the prices p_1 and p_2 of commodities A and B respectively.

The partial elasticity of demand q_1 with respect to p_1 is defined as

$$-\frac{p_1}{q_1} \cdot \frac{\partial q_1}{\partial p_1} = \frac{Eq_1}{Ep_1}$$

Similarly the partial elasticity of demand of q_1 with respect to price p_2 is $-\frac{p_2}{q_1} \cdot \frac{\partial q_1}{\partial p_2} = \frac{Eq_1}{Ep_2}$

Example 27

Find the marginal productivities of capital (K) and labour (L), if $P = 10K - K^2 + KL$, when $K = 2$ and $L = 6$

Solution :

$$\text{We have } P = 10K - K^2 + KL \text{ -----(1)}$$

The marginal productivity of capital is $\frac{\partial P}{\partial K}$

\therefore Differentiating (1) partially with respect to K we get

$$\begin{aligned} \frac{\partial P}{\partial K} &= 10 - 2K + (1) L \\ &= 10 - 2K + L \end{aligned}$$

$$\text{when } K = 2, \text{ and } L = 6, \frac{\partial P}{\partial K} = 10 - 2(2) + 6 = 12$$

The marginal productivity of labour is $\frac{\partial P}{\partial L}$

\therefore Differentiating (1) partially with respect to L we get

$$\begin{aligned} \frac{\partial P}{\partial L} &= K \\ \text{when } K &= 2, \text{ and } L = 6 \quad \frac{\partial P}{\partial L} = 2. \end{aligned}$$

\therefore Marginal productivity of capital = 12 units

\therefore Marginal productivity of labour = 2 units

Example 28

For some firm, the number of units produced when using x units of labour and y units of capital is given by the production function $f(x, y) = 80 x^{\frac{1}{4}} y^{\frac{3}{4}}$. Find (i) the equations for both marginal productivities. (ii) Evaluate and interpret the results when 625 units of labour and 81 units of capital are used.

Solution :

$$\text{Given } f(x, y) = 80 x^{\frac{1}{4}} y^{\frac{3}{4}} \quad \text{-----(1)}$$

Marginal productivity of labour is $f_x(x, y)$.

\therefore Differentiating (1) partially with respect to x , we get

$$f_x = 80 \cdot \frac{1}{4} x^{-\frac{3}{4}} y^{\frac{3}{4}} = 20 x^{-\frac{3}{4}} y^{\frac{3}{4}}$$

Marginal productivity of capital is $f_y(x, y)$

\therefore Differentiating (1) partially with respect to y we get

$$f_y = 80 x^{\frac{1}{4}} \left(\frac{3}{4} \right) y^{-\frac{1}{4}} = 60 x^{\frac{1}{4}} y^{-\frac{1}{4}}$$

$$(ii) \quad f_x(625, 81) = 20(625)^{-\frac{3}{4}}(81)^{\frac{3}{4}}$$

$$= 20 \left(\frac{1}{125} \right) (27) = 4.32$$

i.e. when 625 units of labour and 81 units of capital are used, one more unit of labour results in 4.32 more units of production.

$$\begin{aligned} f_y(625, 81) &= 60 (625)^{\frac{1}{4}}(81)^{-\frac{1}{4}} \\ &= 60(5) \left(\frac{1}{3} \right) = 100 \end{aligned}$$

(i.e.) when 625 units of labour and 81 units of capital are used, one more unit of capital results in 100 more units of production.

Example 29

The demand for a commodity A is $q_1 = 240 - p_1^2 + 6p_2 - p_1 p_2$.
Find the partial Elasticities $\frac{Eq_1}{Ep_1}$ and $\frac{Eq_1}{Ep_2}$ when $p_1 = 5$ and $p_2 = 4$.

Solution :

$$\text{Given } q_1 = 240 - p_1^2 + 6p_2 - p_1 p_2$$

$$\frac{\partial q_1}{\partial p_1} = -2p_1 - p_2$$

$$\frac{\partial q_1}{\partial p_2} = 6 - p_1$$

$$\begin{aligned} \text{(i)} \quad \frac{Eq_1}{Ep_1} &= - \frac{p_1}{q_1} \frac{\partial q_1}{\partial p_1} \\ &= \frac{-p_1}{240 - p_1^2 + 6p_2 - p_1 p_2} (-2p_1 - p_2) \end{aligned}$$

$$\text{when } p_1 = 5 \text{ and } p_2 = 4$$

$$\left(\frac{Eq_1}{Ep_1} \right) = \frac{-(5)(-10-4)}{240 - 25 + 24 - 20} = \frac{70}{219}$$

$$\begin{aligned} \text{(ii)} \quad \frac{Eq_1}{Ep_2} &= - \frac{p_2}{q_1} \frac{\partial q_1}{\partial p_2} \\ &= \frac{-p_2(6-p_1)}{240 - p_1^2 + 6p_2 - p_1 p_2} \end{aligned}$$

$$\text{when } p_1 = 5 \text{ and } p_2 = 4$$

$$\left(\frac{Eq_1}{Ep_2} \right) = \frac{-4(6-5)}{240 - 25 + 24 - 20} = \frac{-4}{219}$$

EXERCISE 4.4

- 1) The production function of a commodity is
 $P = 10L + 5K - L^2 - 2K^2 + 3KL$.

Find (i) the marginal productivity of labour

(ii) the marginal productivity of capital

(iii) the two marginal productivities when $L = 1$ and $K = 2$.

- 2) If the production of a firm is given by $P = 3K^2L^2 - 2L^4 - K^4$, prove that $L \frac{\partial P}{\partial L} + K \frac{\partial P}{\partial K} = 4P$.
- 3) If the production function is $Z = y^2 - xy + x^2$ where x is the labour and y is the capital find the marginal productivities of x and y when $x = 2$ and $y = 3$.
- 4) For some firm, the number of units produced when using x units of labour and y units of capital is given by the production function $f(x, y) = 100x^{\frac{1}{5}}y^{\frac{4}{5}}$. Find
 (i) both marginal productivities.
 (ii) interpret the results when 243 units of labour and 32 units of capital are used.
- 5) For the production function $p = 5(L)^{0.7} (K)^{0.3}$ find the marginal productivities of labour (L) and capital (K) when $L = 10$ and $K = 3$.
- 6) For the production function $P = C(L)^\alpha (K)^\beta$ where C is a positive constant and if $\alpha + \beta = 1$ show that
- $$K \frac{\partial P}{\partial K} + L \frac{\partial P}{\partial L} = P.$$
- 7) The demand for a quantity A is $q_1 = 16 - 3p_1 - 2p_2^2$. Find
 (i) the partial elasticities $\frac{Eq_1}{Ep_1}, \frac{Eq_1}{Ep_2}$ (ii) the partial elasticities for $p_1 = 2$ and $p_2 = 1$.
- 8) The demand for a commodity A is $q_1 = 10 - 3p_1 - 2p_2$. Find the partial elasticities when $p_1 = p_2 = 1$.
- 9) The demand for a commodity X is $q_1 = 15 - p_1^2 - 3p_2$. Find the partial elasticities when $p_1 = 3$ and $p_2 = 1$.
- 10) The demand function for a commodity Y is $q_1 = 12 - p_1^2 + p_1p_2$. Find the partial elasticities when $p_1 = 10$ and $p_2 = 4$.

EXERCISE 4.5

Choose the correct answer

- 1) The stationary value of x for $f(x) = 3(x-1)(x-2)$ is
(a) 3 (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{-3}{2}$
- 2) The maximum value of $f(x) = \cos x$ is
(a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) 1
- 3) $y = x^3$ is always
(a) an increasing function of x (b) decreasing function of x
(c) a constant function (d) none of these.
- 4) The curve $y = 4 - 2x - x^2$ is
(a) concave upward (b) concave downward
(c) straight line (d) none of these.
- 5) If $u = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to
(a) $y^2 u$ (b) $x^2 u$ (c) $2xu$ (d) $2yu$
- 6) If $u = \log(e^x + e^y)$ then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ is equal to
(a) $\frac{1}{e^x + e^y}$ (b) $\frac{e^x}{e^x + e^y}$ (c) 1 (d) $e^x + e^y$
- 7) If $u = x^y$ ($x > 0$) then $\frac{\partial u}{\partial y}$ is equal to
(a) $x^y \log x$ (b) $\log x$ (c) $y^x \log x$ (d) $\log y^x$
- 8) $f(x, y) = \frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}$ is a homogeneous function of degree
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$
- 9) If $f(x, y) = 2x + ye^{-x}$, then $f_y(1, 0)$ is equal to
(a) e (b) $\frac{1}{e}$ (c) e^2 (d) $\frac{1}{e^2}$
- 10) If $f(x, y) = x^3 + y^3 + 3xy$ then f_{xy} is
(a) $6x$ (b) $6y$ (c) 2 (d) 3

- 11) If marginal revenue is Rs.25 and the elasticity of demand with respect to price is 2, then average revenue is
 (a) Rs.50 (b) Rs.25 (c) Rs.27 (d) Rs.12.50
- 12) The elasticity of demand when marginal revenue is zero, is
 (a) 1 (b) 2 (c) -5 (d) 0
- 13) The marginal revenue is Rs.40 and the average revenue is Rs.60. The elasticity of demand with respect to price is
 (a) 1 (b) 0 (c) 2 (d) 3
- 14) If $u = x^2 - 4xy + y^2$ then $\frac{\partial^2 u}{\partial y^2}$ is
 (a) 2 (b) $2xy$ (c) $2x^2y$ (d) $2xy^2$
- 15) If $z = x^3 + 3xy^2 + y^3$ then the marginal productivity of x is
 (a) $x^2 + y^2$ (b) $6xy + 3y^2$ (c) $3(x^2 + y^2)$ (d) $(x^2 + y^2)^2$
- 16) If $q_1 = 2000 + 8p_1 - p_2$ then $\frac{\partial q_1}{\partial p_1}$ is
 (a) 8 (b) -1 (c) 2000 (d) 0
- 17) The marginal productivity of labour (L) for the production function $P = 15K - L^2 + 2KL$ when $L = 3$ and $K = 4$ is
 (a) 21 (b) 12 (c) 2 (d) 3
- 18) The production function for a firm is $P = 3L^2 - 5KL + 2k^2$. The marginal productivity of capital (K) when $L = 2$ and $K = 3$ is
 (a) 5 (b) 3 (c) 6 (d) 2
- 19) The cost function $y = 40 - 4x + x^2$ is minimum when x
 (a) $x = 2$ (b) $x = -2$ (c) $x = 4$ (d) $x = -4$
- 20) If $R = 5000$ units / year, $C_1 = 20$ paise, $C_3 = \text{Rs.}20$ then EOQ is
 (a) 1000 (b) 5000 (c) 200 (d) 100

APPLICATIONS OF INTEGRATION

5

In the present chapter we give some properties of definite integral, geometrical interpretation of definite integral and applications of integration in finding total and average functions from the given marginal functions. We further find demand function when the price and elasticity of demand are known. Finally we discuss a few problems under consumers' surplus and producers' surplus.

5.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

We state below an important theorem which enables us to evaluate definite integrals by making use of antiderivative.

Theorem :

Let f be a continuous function defined on the closed interval $[a, b]$. Let F be an antiderivative of f . Then

$$\int_a^b f(x) dx = F(b) - F(a)$$

5.1.1 Properties of definite integrals

$$1) \quad \int_a^b f(x) dx = - \int_b^a f(x) dx$$

Proof :

Let $F(x)$ be the antiderivative of $f(x)$. Then we have,

$$\begin{aligned} \int_a^b f(x) dx &= [F(x)]_a^b = F(b) - F(a) = -[F(a) - F(b)] \\ &= - \int_b^a f(x) dx \end{aligned}$$

$$2) \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad \text{for } a < c < b.$$

Proof:

Let a, b, c be three real numbers such that $a < c < b$.

$$\text{L.H.S.} = \int_a^b f(x) dx = F(b) - F(a) \quad \text{-----(1)}$$

$$\begin{aligned} \text{R.H.S.} &= \int_a^c f(x) dx + \int_c^b f(x) dx = F(c) - F(a) + F(b) - F(c) \\ &= F(b) - F(a) \quad \text{-----(2)} \end{aligned}$$

$$\text{From (1) and (2), } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$3) \quad \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

Proof:

$$\text{Let } a + b - x = t \quad \therefore -dx = dt$$

$$x = a \Rightarrow t = b$$

$$x = b \Rightarrow t = a$$

Thus when x varies from a to b ,

t varies from b to a .

$$\begin{aligned} \therefore \int_a^b f(x) dx &= - \int_b^a f(a+b-t) dt \\ &= \int_a^b f(a+b-t) dt \quad [\text{by property (1)}] \\ &= \int_a^b f(a+b-x) dx \quad [\text{since } \int_a^b f(x) dx = \int_a^b f(t) dt] \end{aligned}$$

$$4) \quad \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

Proof:

$$\text{Let } a - x = t \quad \therefore -dx = dt$$

$$x = 0 \Rightarrow t = a$$

$$x = a \Rightarrow t = 0$$

$$\begin{aligned}\therefore \int_0^a f(x) dx &= \int_a^0 f(a-t) (-dt) = \int_0^a f(a-t) dt \\ &= \int_0^a f(a-x) dx\end{aligned}$$

$$5) \text{ (i) } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \text{if } f(x) \text{ is an even function.}$$

$$\text{(ii) } \int_{-a}^a f(x) dx = 0 \quad \text{if } f(x) \text{ is an odd function.}$$

Proof:

(i) If $f(x)$ is an even function, then $f(-x) = f(x)$.

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad [\text{by property (2)}]$$

Put $t = -x$ in the first integral then, $dt = -dx$

$$x = -a \Rightarrow t = a$$

$$x = 0 \Rightarrow t = 0$$

$$\begin{aligned}\therefore \int_{-a}^a f(x) dx &= - \int_a^0 f(-t) dt + \int_0^a f(x) dx \\ &= \int_0^a f(-x) dx + \int_0^a f(x) dx \\ &= \int_0^a f(x) dx + \int_0^a f(x) dx \quad (f(x) \text{ is an even function}) \\ &= 2 \int_0^a f(x) dx\end{aligned}$$

(ii) If $f(x)$ is an odd function then

$$f(-x) = -f(x)$$

$$\therefore \int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$$

Put $t = -x$ in the first integral. Then $dt = -dx$

$$x = -a \Rightarrow t = a$$

$$x = 0 \Rightarrow t = 0$$

$$\begin{aligned}
\therefore \int_{-a}^a f(x) dx &= -\int_a^0 f(-t) dt + \int_0^a f(x) dx \\
&= \int_0^a f(-x) dx + \int_0^a f(x) dx \\
&= -\int_0^a f(x) dx + \int_0^a f(x) dx \quad (\text{since } f(x) \text{ is an odd function}) \\
&= 0
\end{aligned}$$

Example 1

Evaluate $\int_{-1}^1 (x^3 + x) dx$

Solution :

$f(x) = x^3 + x$ is an odd function.

$$\Rightarrow \int_{-1}^1 (x^3 + x) dx = 0 \quad [\text{by property 5(ii)}]$$

Example 2

Evaluate $\int_{-2}^2 (x^4 + x^2) dx$

Solution :

$f(x) = x^4 + x^2$ is an even function

$$\begin{aligned}
\Rightarrow \int_{-2}^2 (x^4 + x^2) dx &= 2 \int_0^2 (x^4 + x^2) dx \quad [\text{by property 5(i)}] \\
&= 2 \left[\frac{x^5}{5} + \frac{x^3}{3} \right]_0^2 \\
&= 2 \left[\frac{2^5}{5} + \frac{2^3}{3} \right] = \frac{272}{15}
\end{aligned}$$

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x} + \sqrt{\cos^3 x}} dx$

Solution :

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x + \cos^3 x}} dx \quad \text{-----(1)}$$

By property (4), $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\begin{aligned} \text{Again } I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin^3(\frac{\pi}{2}-x)}}{\sqrt{\sin^3(\frac{\pi}{2}-x) + \cos^3(\frac{\pi}{2}-x)}} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos^3 x}}{\sqrt{\cos^3 x + \sin^3 x}} dx \quad \text{-----(2)} \end{aligned}$$

Adding (1) and (2) we get,

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin^3 x} + \sqrt{\cos^3 x}}{\sqrt{\sin^3 x + \cos^3 x}} dx \\ &= \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \end{aligned}$$

$$\therefore I = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x + \cos^3 x}} dx = \frac{\pi}{4}$$

Example 4

Evaluate $\int_0^1 x(1-x)^5 dx$

Solution :

By property (4), $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\therefore \int_0^1 x(1-x)^5 dx = \int_0^1 (1-x)(1-1+x)^5 dx = \int_0^1 (1-x)x^5 dx$$

$$= \int_0^1 (x^5 - x^6) dx = \left[\frac{x^6}{6} - \frac{x^7}{7} \right]_0^1 = \frac{1}{42}$$

$$\therefore \int_0^1 x(1-x)^5 dx = \frac{1}{42}$$

Example 5

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

Solution :

$$\begin{aligned} \text{Let } I &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x} dx}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \text{-----(1)} \end{aligned}$$

By property (3), $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\begin{aligned} \therefore I &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos(\frac{\pi}{6} + \frac{\pi}{3} - x)}}{\sqrt{\sin(\frac{\pi}{6} + \frac{\pi}{3} - x)} + \sqrt{\cos(\frac{\pi}{6} + \frac{\pi}{3} - x)}} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \text{-----(2)} \end{aligned}$$

Adding (1) and (2) we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{6}$$

$$\therefore I = \frac{\frac{\pi}{12}}{\frac{\frac{\pi}{6}}{\frac{\pi}{3}}} \quad \therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \frac{\frac{\pi}{12}}{\frac{\pi}{6}}$$

EXERCISE 5.1

Evaluate the following using the properties of definite integral:

- 1) $\int_{-10}^{10} (4x^5 + 6x^3 + \frac{2}{3}x) dx$
- 2) $\int_{-2}^2 (3x^2 + 5x^4) dx$
- 3) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$
- 4) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$
- 5) $\int_0^2 x\sqrt{2-x} dx$
- 6) $\int_0^1 x(1-x)^3 dx$
- 7) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\cot x}}$
- 8) $\int_0^2 \frac{\sqrt{x} dx}{\sqrt{x} + \sqrt{2-x}}$
- 9) $\int_0^{\frac{\pi}{2}} x \sin^2 x dx$
- 10) $\int_0^{\frac{\pi}{2}} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$

5.2 GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL AS AREA UNDER A CURVE

The area A of the region bounded by the curve $y = f(x)$, the x -axis and the ordinates at $x = a$ and $x = b$ is given by,

$$\begin{aligned} \text{Area, } A &= \int_a^b y dx \\ &= \int_a^b f(x) dx \end{aligned}$$

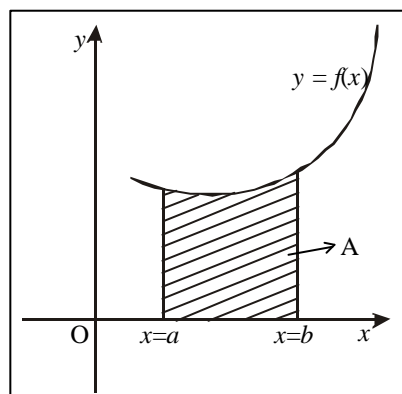


Fig 5.1

Note

The graph of $y = f(x)$ must not cross the x -axis between $x = a$ and $x = b$.

Similarly the area A of the region bounded by the curve $x = g(y)$, the y -axis and the abscissae $y = c$ and $y = d$ is given by

$$\begin{aligned}\text{Area, } A &= \int_c^d x \, dy \\ &= \int_c^d g(y) \, dy\end{aligned}$$

Note

The graph of $x = g(y)$ must not cross the axis of y between $y = c$ and $y = d$.

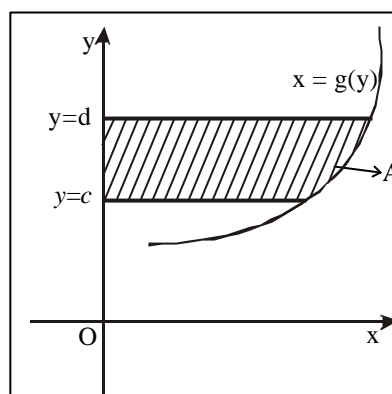


Fig. 5.2

Example 6

Find the area enclosed by the parabola $y^2 = 4x$, $x = 1$, $x = 4$ and the x -axis.

Solution :

The area under the curve is

$$\begin{aligned}A &= \int_1^4 y \, dx \\ &= \int_1^4 \sqrt{4x} \, dx \\ &= 2 \int_1^4 \sqrt{x} \, dx = 2 \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\ &= 2 \times \frac{2}{3} (4^{\frac{3}{2}} - 1^{\frac{3}{2}}) \\ &= \frac{28}{3} \text{ sq. units.}\end{aligned}$$

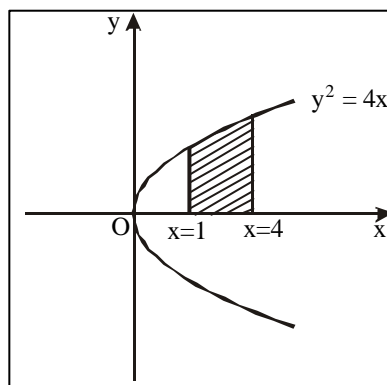


Fig. 5.3

Example 7

Find the area of the region bounded by the parabola $x^2 = 4y$, $y = 2$, $y = 4$ and the y - axis

Solution :

The area under the curve is,

$$\begin{aligned}
 A &= \int_c^d x \, dy \\
 &= \int_2^4 \sqrt{4y} \, dy \\
 &= 2 \int_2^4 \sqrt{y} \, dy = 2 \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right]_2^4 \\
 &= 2 \times \frac{2}{3} (4^{\frac{3}{2}} - 2^{\frac{3}{2}}) = \frac{32-8\sqrt{2}}{3} \text{ sq. units.}
 \end{aligned}$$

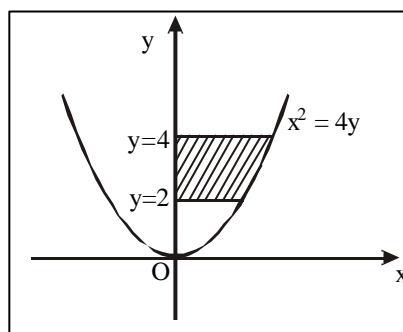


Fig. 5.4

Example 8

Find the area under the curve $y = 4x^2 - 8x + 6$ bounded by the y - axis x - axis and the ordinate at $x = 2$.

Solution :

The y - axis is the ordinate at $x = 0$. The area bounded by the ordinates at $x = 0$, $x = 2$ and the given curve is

$$\begin{aligned}
 A &= \int_a^b y \, dx \\
 &= \int_0^2 (4x^2 - 8x + 6) \, dx \\
 &= \left[4 \frac{x^3}{3} - \frac{8x^2}{2} + 6x \right]_0^2 \\
 &= \frac{4}{3} (2)^3 - 4(2)^2 + 6(2) - 0 \\
 &= \frac{20}{3} \text{ sq. units.}
 \end{aligned}$$

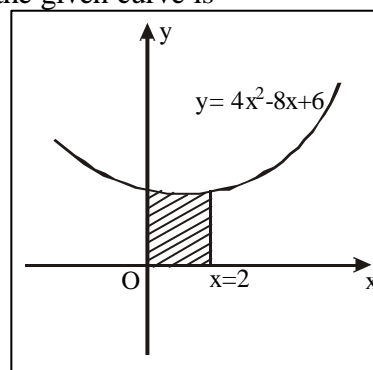


Fig. 5.5

Example 9

Find the area bounded by the semi cubical parabola $y^2 = x^3$ and the lines $x = 0$, $y = 1$ and $y = 2$.

Solution :

$$\begin{aligned}
 \text{Area, } A &= \int_c^d x \, dy \\
 &= \int_1^2 y^{\frac{2}{3}} \, dy = \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}} \right]_1^2 \\
 &= \frac{3}{5} \left[2^{\frac{5}{3}} - 1 \right] \text{ sq. units.}
 \end{aligned}$$

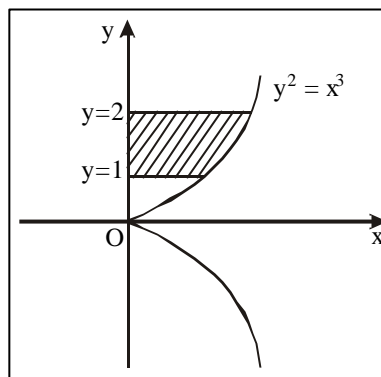


fig 5.6

Example 10

Find the area bounded by one arch of the curve $y = \sin ax$ and the x - axis.

Solution :

The limits for one arch of the curve $y = \sin ax$ are $x = 0$ and $x = \frac{\pi}{a}$

$$\begin{aligned}
 \text{Area, } A &= \int_a^b y \, dx \\
 &= \int_0^{\frac{\pi}{a}} \sin ax \, dx \\
 &= \left[-\frac{\cos ax}{a} \right]_0^{\frac{\pi}{a}} \\
 &= -\frac{1}{a} [\cos \pi - \cos 0] \\
 &= \frac{2}{a} \text{ sq. units}
 \end{aligned}$$

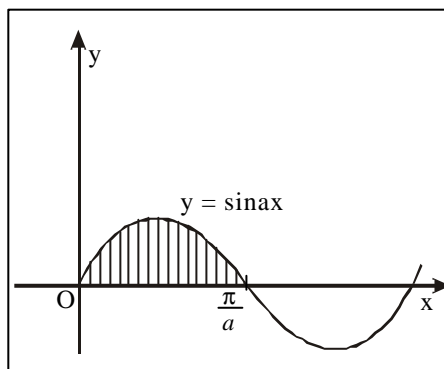


fig 5.7

Example 11

Find the area of one loop of the curve $y^2 = x^2 (4 - x^2)$ between $x = 0$ and $x = 2$.

Solution :

Equation of the curve is

$$y^2 = x^2 (4 - x^2)$$

$$\therefore y = \pm x \sqrt{4 - x^2}$$

$$\text{Area, } A = \int_a^b y \, dx$$

$= 2 \times \text{Area in the I quadrant}$

$$= 2 \int_0^2 x \sqrt{4 - x^2} \, dx$$

$$\therefore A = 2 \int_4^0 \sqrt{t} \frac{(-dt)}{2} = \int_0^4 \sqrt{t} \, dt$$

$$= \left[\frac{\frac{3}{2}}{\frac{3}{2}} \right]_0^4$$

$$= \frac{16}{3} \text{ sq. units.}$$

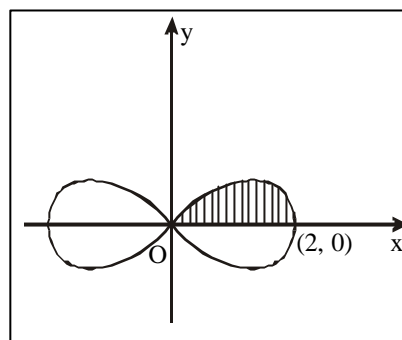


fig 5.8

($\because y > 0$ in the I quadrant)

$$\text{Put } t = 4 - x^2$$

$$dt = -2x \, dx$$

$$-\frac{dt}{2} = x \, dx.$$

$$\text{when } x = 0, \quad t = 4$$

$$\text{when } x = 2, \quad t = 0$$

EXERCISE 5.2

- 1) Find the area under the curve $y = 4x - x^2$ included between $x = 0$, $x = 3$ and the x -axis.
- 2) Find the area of the region bounded by the curve $y = 3x^2 - 4x + 5$, the x -axis and the lines $x = 1$ and $x = 2$.
- 3) Find the area under the curve $y = \frac{1}{1+x^2}$, x -axis, $x = -1$, and $x = 1$.
- 4) Find the area contained between the x -axis and one arch of the curve $y = \cos x$ bounded between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

- 5) Find the area of one loop of the curve $y^2 = x^2 (1-x^2)$ between $x = 0$ and $x = 1$.
- 6) Find the area under the demand curve $xy = 1$ bounded by the ordinates $x = 3$, $x = 9$ and x -axis.
- 7) Find the area cut off from the parabola $y^2 = 4ax$ by its latus rectum.
- 8) Find the area bounded by the curve $x = 3y^2 - 9$ and the lines $x = 0$, $y = 0$ and $y = 1$.
- 9) Find the area above the axis of x bounded by $y = \frac{4}{x}$, $x = 1$ and $x = 4$.
- 10) Find the area of the circle of radius ' a ' using integration.
- 11) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5.3 APPLICATIONS OF INTEGRATION IN ECONOMICS AND COMMERCE

We learnt already that the marginal function is obtained by differentiating the total function. We were given the total cost, total revenue or demand function and we obtained the marginal cost, marginal revenue or elasticity of demand.

Now we shall obtain the total function when marginal function is given.

5.3.1 The cost function and average cost function from marginal cost function :

If C is the cost of producing an output x then marginal cost function, $MC = \frac{dC}{dx}$. Using integration as reverse process of differentiation we obtain,

$$\text{Cost function, } C = \int (MC) dx + k$$

where k is the constant of integration which can be evaluated if the fixed cost is known. If the fixed cost is not known, then $k = 0$.

$$\text{Average cost function, } AC = \frac{C}{x}, x \neq 0$$

Example 12

The marginal cost function of manufacturing x units of a commodity is $6 + 10x - 6x^2$. Find the total cost and average cost, given that the total cost of producing 1 unit is 15.

Solution :

Given that,

$$\begin{aligned} MC &= 6 + 10x - 6x^2 \\ C &= \int (MC) dx + k \\ &= \int (6 + 10x - 6x^2) dx + k \\ &= 6x + \frac{10x^2}{2} - \frac{6x^3}{3} + k \\ &= 6x + 5x^2 - 2x^3 + k \quad \text{-----(1)} \end{aligned}$$

Given, when $x = 1$, $C = 15$

$$\therefore (1) \Rightarrow 15 = 6 + 5 - 2 + k$$

$$\Rightarrow k = 6$$

$$\therefore \text{Total Cost function, } C = 6x + 5x^2 - 2x^3 + 6$$

$$\begin{aligned} \text{Average Cost function, } AC &= \frac{C}{x}, x \neq 0 \\ &= 6 + 5x - 2x^2 + \frac{6}{x} \end{aligned}$$

Example 13

The marginal cost function of manufacturing x units of a commodity is $3x^2 - 2x + 8$. If there is no fixed cost find the total cost and average cost functions.

Solution :

Given that,

$$\begin{aligned} MC &= 3x^2 - 2x + 8 \\ C &= \int (MC) dx + k \\ &= \int (3x^2 - 2x + 8) dx + k \\ &= x^3 - x^2 + 8x + k \end{aligned}$$

No fixed cost $\Rightarrow k = 0$

\therefore Total cost, $C = x^3 - x^2 + 8x$

$$\begin{aligned}\text{Average Cost, } AC &= \frac{C}{x}, \quad x \neq 0 \\ &= x^2 - x + 8.\end{aligned}$$

Example 14

The marginal cost function of manufacturing x units of a commodity is $3 - 2x - x^2$. If the fixed cost is 200, find the total cost and average cost functions.

Solution :

Given that,

$$MC = 3 - 2x - x^2$$

$$\begin{aligned}C &= \int (MC) dx + k \\ &= \int (3 - 2x - x^2) dx + k \\ &= 3x - x^2 - \frac{x^3}{3} + k \quad \text{-----(1)}\end{aligned}$$

Given that fixed cost $C = 200$

$$\therefore (1) \Rightarrow k = 200$$

$$\therefore C = 3x - x^2 - \frac{x^3}{3} + 200$$

$$\begin{aligned}AC &= \frac{C}{x}, \quad x \neq 0 \\ &= 3 - x - \frac{x^2}{3} + \frac{200}{x}\end{aligned}$$

5.3.2 The revenue function and demand function from marginal revenue function

If R is the total revenue function when the output is x , then marginal revenue

$$MR = \frac{dR}{dx} . \quad \text{Integrating with respect to 'x' we get}$$

$$\text{Revenue function, } R = \int (MR) dx + k$$

where 'k' is the constant of integration which can be evaluated under given conditions.

If the total revenue $R = 0$, when $x = 0$,

$$\text{Demand function, } p = \frac{R}{x}, \quad x \neq 0$$

Example 15

If the marginal revenue for a commodity is

MR = 9 - 6x² + 2x, find the total revenue and demand function.

Solution :

Given that, $MR = 9 - 6x^2 + 2x$

$$\begin{aligned} R &= \int (MR) dx + k \\ &= \int (9 - 6x^2 + 2x) dx + k \\ &= 9x - 2x^3 + x^2 + k \end{aligned}$$

Since $R = 0$ when $x = 0$, $k = 0$

$$\therefore R = 9x - 2x^3 + x^2$$

$$p = \frac{R}{x}, \quad x \neq 0 \Rightarrow p = 9 - 2x^2 + x$$

Example 16

For the marginal revenue function $MR = 3 - 2x - x^2$, find the revenue function and demand function.

Solution :

Given that

$$\begin{aligned} MR &= 3 - 2x - x^2 \\ R &= \int (MR) dx + k \\ &= \int (3 - 2x - x^2) dx + k \end{aligned}$$

$$= 3x - x^2 - \frac{x^3}{3} + k$$

since $R = 0$, when $x = 0$, $k = 0$

$$\therefore R = 3x - x^2 - \frac{x^3}{3}$$

$$p = \frac{R}{x}, \quad x \neq 0$$

$$p = 3 - x - \frac{x^2}{3}$$

Example 17

If the marginal revenue for a commodity is

$MR = \frac{e^x}{100} + x + x^2$, find the revenue function.

Solution :

Given that,

$$MR = \frac{e^x}{100} + x + x^2$$

$$\therefore R = \int (MR) dx + k$$

$$= \int \left(\frac{e^x}{100} + x + x^2 \right) dx + k$$

$$= \frac{e^x}{100} + \frac{x^2}{2} + \frac{x^3}{3} + k$$

when no product is sold, revenue is zero.

when $x = 0$, $R = 0$.

$$\therefore 0 = \frac{e^0}{100} + 0 + 0 + k \quad \therefore k = -\frac{1}{100}$$

$$\therefore \text{Revenue, } R = \frac{e^x}{100} + \frac{x^2}{2} + \frac{x^3}{3} - \frac{1}{100}$$

5.3.3 The demand function when the elasticity of demand is given

We know that,

$$\text{Elasticity of demand } \eta_d = \frac{-p}{x} \frac{dx}{dp}$$

$$\Rightarrow \frac{-dp}{p} = \frac{dx}{x} \frac{1}{\eta_d}$$

Integrating both sides

$$-\int \frac{dp}{p} = \frac{1}{\eta_d} \int \frac{dx}{x}$$

This equation yields the demand function 'p' as a function of 'x'.

The revenue function can be found out by using the relation, $R = px$.

Example 18

The elasticity of demand with respect to price p for a commodity is $\frac{x-5}{x}$, $x > 5$ when the demand is 'x'. Find the demand function if the price is 2 when demand is 7. Also find the revenue function.

Solution :

Given that,

$$\text{Elasticity of demand, } \eta_d = \frac{x-5}{x}$$

$$\text{i.e. } -\frac{p}{x} \frac{dx}{dp} = \frac{x-5}{x}$$

$$\Rightarrow \frac{dx}{x-5} = -\frac{dp}{p}$$

Integrating both sides,

$$\int \frac{dx}{x-5} = -\int \frac{dp}{p} + \log k$$

$$\Rightarrow \log (x - 5) = -\log p + \log k$$

$$\Rightarrow \log (x - 5) + \log p = \log k$$

$$\Rightarrow \log p (x - 5) = \log k$$

$$\Rightarrow p (x - 5) = k \quad \text{-----(1)}$$

when $p = 2$, $x = 7$,

$$k = 4$$

\therefore The demand function is,

$$p = \frac{4}{x-5}$$

$$\text{Revenue, } R = px \quad \text{or} \quad R = \frac{4x}{x-5}, \quad x > 5$$

Example 19

The elasticity of demand with respect to price for a commodity is a constant and is equal to 2. Find the demand function and hence the total revenue function, given that when the price is 1, the demand is 4.

Solution :

Given that,

$$\text{Elasticity of demand, } \eta_d = 2$$

$$\Rightarrow -\frac{p}{x} \frac{dx}{dp} = 2$$

$$\Rightarrow \frac{dx}{x} = -2 \frac{dp}{p}$$

Integrating both sides,

$$\Rightarrow \int \frac{dx}{x} = -2 \int \frac{dp}{p} + \log k$$

$$\log x = -2 \log p + \log k$$

$$\log x + \log p^2 = \log k$$

$$p^2 x = k \quad \text{-----(1)}$$

$$\text{Given, when } x = 4, \quad p = 1$$

$$\text{From (1) we get } k = 4$$

$$\therefore (1) \Rightarrow xp^2 = 4 \quad \text{or} \quad p^2 = \frac{4}{x}$$

$$\text{Demand function } p = \frac{2}{\sqrt{x}} \quad ; \quad \text{Revenue } R = px = 2\sqrt{x}$$

Example 20

The marginal cost and marginal revenue with respect to a commodity of a firm are given by $C'(x) = 4 + 0.08x$ and $R'(x) = 12$. Find the total profit, given that the total cost at zero output is zero.

Solution :

Given that,

$$\begin{aligned} MC &= 4 + 0.08x \\ C(x) &= \int (MC) dx + k_1 \\ &= \int (4 + 0.08x) dx + k_1 \\ &= 4x + 0.08 \frac{x^2}{2} + k_1 \\ &= 4x + 0.04x^2 + k_1 \quad \text{----- (1)} \end{aligned}$$

But given when $x = 0, C = 0$

$$\begin{aligned} \therefore (1) &\Rightarrow 0 = 0 + 0 + k_1 \\ \therefore k_1 &= 0 \\ \therefore C(x) &= 4x + 0.04x^2 \quad \text{----- (2)} \end{aligned}$$

Given that,

$$\begin{aligned} MR &= 12. \\ \therefore R(x) &= \int MR dx + k_2 \\ &= \int 12 dx + k_2 \\ &= 12x + k_2 \\ \text{Revenue} &= 0 \text{ when } x = 0. \\ \therefore k_2 &= 0 \\ \therefore R(x) &= 12x \quad \text{----- (3)} \end{aligned}$$

$$\begin{aligned} \text{Total profit function, } P(x) &= R(x) - C(x) \\ &= 12x - 4x - 0.04x^2 \\ &= 8x - 0.04x^2. \end{aligned}$$

Example 21

The marginal revenue function (in thousands of rupees) of a commodity is $7 + e^{-0.05x}$ where x is the number of units sold. Find the total revenue from the sale of 100 units ($e^{-5} = 0.0067$)

Solution :

Given that,

Marginal revenue, $R'(x) = 7 + e^{-0.05x}$

\therefore Total revenue from sale of 100 units is

$$\begin{aligned} R &= \int_0^{100} (7 + e^{-0.05x}) dx \\ &= \left[7x + \frac{e^{-0.05x}}{-0.05} \right]_0^{100} \\ &= 700 - \frac{100}{5} (e^{-5} - 1) \\ &= 700 - 20 (0.0067 - 1) \\ &= 700 + 20 - 0.134 \\ &= (720 - 0.134) \quad \text{thousands} \\ &= 719.866 \times 1000 \end{aligned}$$

Revenue, $R = \text{Rs. } 7,19,866$.

Example 22

The marginal cost $C'(x)$ and marginal revenue $R'(x)$ are given by $C'(x) = 20 + \frac{x}{20}$ and $R'(x) = 30$. The fixed cost is Rs. 200. Determine the maximum profit.

Solution :

$$\begin{aligned} \text{Given } C'(x) &= 20 + \frac{x}{20} \\ \therefore C(x) &= \int C'(x) dx + k_1 \\ &= \int (20 + \frac{x}{20}) dx + k_1 \\ &= 20x + \frac{x^2}{40} + k_1 \quad \text{-----(1)} \end{aligned}$$

When quantity produced is zero, the fixed cost is Rs. 200.

i.e. when $x = 0$, $C = 200$,

$$\Rightarrow k_1 = 200$$

Cost function is $C(x) = 20x + \frac{x^2}{40} + 200$

The revenue, $R'(x) = 30$

$$\begin{aligned}\therefore R(x) &= \int R'(x) dx + k_2 \\ &= \int 30 dx + k_2 \\ &= 30x + k_2\end{aligned}$$

When no product is sold, revenue = 0 -----(2)

i.e., when $x = 0$, $R = 0$

$$\therefore \text{Revenue, } R(x) = 30x$$

Profit, $P = \text{Total revenue} - \text{Total cost}$

$$= 30x - 20x - \frac{x^2}{40} - 200 = 10x - \frac{x^2}{40} - 200$$

$$\frac{dP}{dx} = 10 - \frac{x}{20} ; \frac{dP}{dx} = 0 \Rightarrow x = 200$$

$$\frac{d^2P}{dx^2} = \frac{-1}{20} < 0$$

\therefore Profit is maximum when $x = 200$

$$\therefore \text{Maximum profit is } P = 2000 - \frac{40000}{40} - 200$$

Profit = Rs. 800.

Example 23

A company determines that the marginal cost of producing x units is $C'(x) = 10.6x$. The fixed cost is Rs. 50. The selling price per unit is Rs. 5. Find (i) Total cost function (ii) Total revenue function (iii) Profit function.

Solution :

Given,

$$C'(x) = 10.6x$$

$$\begin{aligned}\therefore C(x) &= \int C'(x) dx + k = \int 10.6x dx + k \\ &= 10.6 \frac{x^2}{2} + k\end{aligned}$$

$$= 5.3x^2 + k \quad \text{-----}(1)$$

Given fixed cost = Rs. 50

(i.e.) when $x = 0$, $C = 50 \therefore k = 50$

Hence Cost function, $C = 5.3x^2 + 50$

(ii) Total revenue = number of units sold \times price per unit

Let x be the number of units sold. Given that selling price per unit is Rs. 5.

$$\therefore \text{Revenue } R(x) = 5x.$$

(iii) Profit, $P = \text{Total revenue} - \text{Total cost}$

$$= 5x - (5.3x^2 + 50)$$

$$= 5x - 5.3x^2 - 50.$$

Example 24

Determine the cost of producing 3000 units of commodity if the marginal cost in rupees per unit is $C'(x) = \frac{x}{3000} + 2.50$

Solution :

$$\text{Given, Marginal cost, } C'(x) = \frac{x}{3000} + 2.50$$

$$\begin{aligned} \therefore C(x) &= \int C'(x) dx + k = \int \left(\frac{x}{3000} + 2.50 \right) dx + k \\ &= \frac{x^2}{6000} + 2.50x + k. \end{aligned}$$

$$\text{When } x = 0, C = 0 \therefore k = 0. \Rightarrow C(x) = \frac{x^2}{6000} + 2.50x$$

When $x = 3000$,

$$\text{Cost of production, } C(x) = \text{Rs. } 9000$$

Example 25

The marginal cost at a production level of x units is given by $C'(x) = 85 + \frac{375}{x^2}$. Find the cost of producing 10 incremental units after 15 units have been produced.

Solution :

$$\text{Given, } C'(x) = 85 + \frac{375}{x^2} \quad \therefore C(x) = \int C'(x) dx + k$$

The cost of producing 10 incremental units after 15 units have been produced

$$\begin{aligned} &= \int_{15}^{25} C'(x) dx = \int_{15}^{25} \left(85 + \frac{375}{x^2} \right) dx \\ &= \left[85x - \frac{375}{x} \right]_{15}^{25} = \text{Rs. } 860. \end{aligned}$$

$$\therefore \text{Required cost} = \text{Rs. } 860$$

EXERCISE 5.3

- 1) The marginal cost function of production x units, is $MC = 10 + 24x - 3x^2$ and the total cost of producing one unit is Rs.25. Find the total cost function and the average cost function.
- 2) The marginal cost function is $MC = \frac{100}{x}$. Find the cost function $C(x)$ if $C(16) = 100$. Also find the average cost function.
- 3) The marginal cost of manufacturing x units of product is $MC = 3x^2 - 10x + 3$. The total cost of producing one unit of the product is Rs.7. Find the total cost and average cost function.
- 4) For the marginal cost function $MC = 5 - 6x + 3x^2$, x is the output. If the cost of producing 10 items is Rs.850, find the total cost and average cost function.
- 5) The marginal cost function is $MC = 20 - 0.04x + 0.003x^2$ where x is the number of units produced. The fixed cost of production is Rs. 7,000. Find the total cost and the average cost.
- 6) If the marginal revenue function is $R'(x) = 15 - 9x - 3x^2$, find the revenue function and average revenue function.
- 7) If the marginal revenue of a commodity is given by $MR = 9 - 2x + 4x^2$, find the demand function and revenue function.

- 8) Find the total revenue function and the demand function for the marginal revenue function $MR = 100 - 9x^2$.
- 9) Find the revenue function and the demand function if the marginal revenue for x units is $MR = 2 + 4x - x^2$.
- 10) The marginal revenue of a commodity is given by $MR = 4 - 3x$. Find the revenue function and the demand function.
- 11) The elasticity of demand with respect to price ' p ' is $\frac{3-x}{x}$, $x < 3$. Find the demand function and the revenue function when the price is 2 and the demand is 1.
- 12) The elasticity of demand with respect to price p for a commodity is $\frac{p}{x^2}$, when the demand is x . Find the demand function and revenue function if the demand is 2 when the price is 3.
- 13) Find the demand function for which the elasticity of demand is 1.
- 14) The marginal cost function of a commodity in a firm is $2 + 3e^{3x}$ where x is the output. Find the total cost and average cost function if the fixed cost is Rs.500.
- 15) The marginal revenue function is given by $R'(x) = \frac{3}{x^2} - \frac{2}{x}$. Find the revenue function and demand function if $R(1) = 6$.
- 16) The marginal revenue is $R'(x) = 16 - x^2$. Find the revenue and demand function.
- 17) The marginal cost of production of a firm is given by $C'(x) = 5 + 0.13x$. The marginal revenue is given by $R'(x) = 18$. The fixed cost is Rs.120. Find the profit function.
- 18) The marginal revenue (in thousands of rupees) of a commodity is $R'(x) = 4 + e^{-0.03x}$ where x denotes the number of units sold. Determine the total revenue from the sale of 100 units of the commodity ($e^{-3} = 0.05$)

5.4 CONSUMERS' SURPLUS

A demand curve for a commodity shows the amount of the commodity that will be bought by people at any given price p .

Suppose that the prevailing market price is p_0 . At this price an amount x_0 of the commodity determined by the demand curve will be sold. However there are buyers who would be willing to pay a price higher than p_0 . All such buyers will gain from the fact that the prevailing market price is only p_0 . This gain is called **Consumers' Surplus**. It is represented by the area below the demand curve $p = f(x)$ and above the line $p = p_0$.

Thus Consumers' Surplus,

CS = [Total area under the demand function bounded by $x = 0$, $x = x_0$ and x -axis – Area of the rectangle OAPB]

$$\therefore \text{CS} = \int_0^{x_0} f(x) dx - p_0 x_0$$

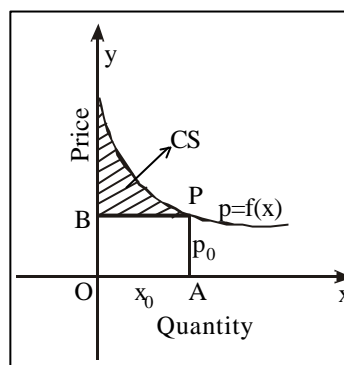


Fig. 5.9

Example 26

Find the consumers' surplus for the demand function $p = 25 - x - x^2$ when $p_0 = 19$.

Solution :

Given that,

The demand function is $p = 25 - x - x^2$

$$p_0 = 19$$

$$\therefore 19 = 25 - x - x^2$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x + 3)(x - 2) = 0$$

$$\Rightarrow x = 2 \text{ (or) } x = -3$$

$$\therefore x_0 = 2$$

[demand cannot be negative]

$$\therefore p_0 x_0 = 19 \times 2 = 38$$

$$\text{CS} = \int_0^{x_0} f(x) dx - p_0 x_0$$

$$\begin{aligned}
&= \int_0^2 (25 - x - x^2) dx - 38 \\
&= \left[25x - \frac{x^2}{2} - \frac{x^3}{3} \right]_0^2 - 38 \\
&= \left[25(2) - 2 - \frac{8}{3} \right] - 38 = \frac{22}{3} \text{ units}
\end{aligned}$$

Example 27

The demand of a commodity is $p = 28 - x^2$. Find the consumers' surplus when demand $x_0 = 5$

Solution :

Given that,

The demand function, $p = 28 - x^2$

when $x_0 = 5$

$$p_0 = 28 - 25$$

$$= 3$$

$$\therefore p_0 x_0 = 15$$

$$\begin{aligned}
\text{CS} &= \int_0^{x_0} f(x) dx - p_0 x_0 \\
&= \int_0^5 (28 - x^2) dx - 15 \\
&= \left[28x - \frac{x^3}{3} \right]_0^5 - 15 \\
&= \left[28 \times 5 - \frac{125}{3} \right] - 15 = \frac{250}{3} \text{ units}
\end{aligned}$$

Example 28

The demand function for a commodity is $p = \frac{12}{x+3}$. Find the consumers' surplus when the prevailing market price is 2.

Solution :

Given that, Demand function, $p = \frac{12}{x+3}$

$$p_0 = 2 \Rightarrow 2 = \frac{12}{x+3}$$

$$\text{or } 2x + 6 = 12 \quad \text{or } x = 3 \quad \therefore x_0 = 3 \Rightarrow p_0 x_0 = 6$$

$$\text{CS} = \int_0^{x_0} f(x) dx - p_0 x_0 = \int_0^3 \frac{12}{x+3} dx - 6$$

$$= 12 [\log(x+3)]_0^3 - 6.$$

$$= 12[\log 6 - \log 3] - 6 = 12 \log \frac{6}{3} - 6 = 12 \log 2 - 6$$

5.5 PRODUCERS' SURPLUS

A supply curve for a commodity shows the amount of the commodity that will be brought into the market at any given price p . Suppose the prevailing market price is p_0 . At this price an amount x_0 of the commodity, determined by the supply curve, will be offered to buyers. However, there are producers who are willing to supply the commodity at a price lower than p_0 . All such producers will gain from the fact that the prevailing market price is only p_0 . This gain is called 'Producers' Surplus'. It is represented by the area above the supply curve $p = g(x)$ and below the line $p = p_0$.

Thus Producers' Surplus,

PS = [Area of the whole rectangle OAPB – Area under the supply curve bounded by $x = 0$, $x = x_0$ and x - axis]

$$\therefore \text{PS} = p_0 x_0 - \int_0^{x_0} g(x) dx$$

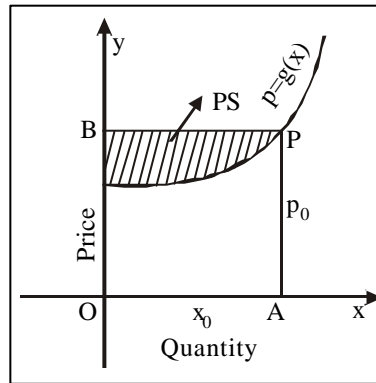


Fig 5.10

Example 29

The supply function for a commodity is $p = x^2 + 4x + 5$ where x denotes supply. Find the producers' surplus when the price is 10.

Solution :

Given that,

Supply function, $p = x^2 + 4x + 5$

For $p_0 = 10$,

$$10 = x^2 + 4x + 5 \Rightarrow x^2 + 4x - 5 = 0$$

$$\Rightarrow (x + 5)(x - 1) = 0 \Rightarrow x = -5 \quad \text{or} \quad x = 1$$

Since supply cannot be negative, $x = -5$ is not possible.

$$\therefore x = 1$$

$$\therefore p_0 = 10 \text{ and } x_0 = 1 \Rightarrow p_0 x_0 = 10$$

Producers' Surplus,

$$\begin{aligned} \text{PS} &= p_0 x_0 - \int_0^{x_0} g(x) dx \\ &= 10 - \int_0^1 (x^2 + 4x + 5) dx \\ &= 10 - \left[\frac{x^3}{3} + \frac{4x^2}{2} + 5x \right]_0^1 \\ &= 10 - \left[\frac{1}{3} + 2 + 5 \right] = \frac{8}{3} \text{ units.} \end{aligned}$$

Example 30

Find the producers' surplus for the supply function $p = x^2 + x + 3$ when $x_0 = 4$.

Solution :

Given that,

supply function $p = x^2 + x + 3$

when $x_0 = 4$, $p_0 = 4^2 + 4 + 3 = 23$

$$\therefore p_0 x_0 = 92.$$

Producers' Surplus

$$\text{PS} = p_0 x_0 - \int_0^{x_0} g(x) dx = 92 - \int_0^4 (x^2 + x + 3) dx$$

$$\begin{aligned}
&= 92 - \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x \right]_0^4 \\
&= 92 - \left[\frac{64}{3} + \frac{16}{2} + 12 \right] = \frac{152}{3} \text{ units.}
\end{aligned}$$

Example 31

Find the producers' surplus for the supply function $p = 3 + x^2$ when the price is 12.

Solution :

Given that,

supply function, $p = 3 + x^2$. When $p_0 = 12$,

$$12 = 3 + x^2 \text{ or } x^2 = 9 \text{ or } x = \pm 3$$

Since supply cannot be negative,

$$x = 3. \quad \text{i.e.} \quad x_0 = 3,$$

$$\therefore p_0 x_0 = 36.$$

Producers' Surplus,

$$\begin{aligned}
\text{PS} &= p_0 x_0 - \int_0^{x_0} g(x) dx \\
&= 36 - \int_0^3 (3 + x^2) dx = 36 - \left[3x + \frac{x^3}{3} \right]_0^3 \\
&= 36 - \left[9 + \frac{27}{3} - 0 \right] = 18 \text{ units.}
\end{aligned}$$

Example 32

The demand and supply functions under pure competition are $p_d = 16 - x^2$ and $p_s = 2x^2 + 4$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

Solution :

For market equilibrium,

Quantity demanded = Quantity supplied

$$\Rightarrow 16 - x^2 = 2x^2 + 4 \Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4 \quad \Rightarrow x = \pm 2 \quad \text{But } x = -2 \text{ is inadmissible.}$$

$$\therefore x = 2 \quad (\text{i.e.}) \quad x_0 = 2$$

$$\therefore p_0 = 16 - (2)^2 = 12$$

$$\therefore p_0 x_0 = 12 \times 2 = 24.$$

Consumers' Surplus,

$$\begin{aligned} \text{CS} &= \int_0^{x_0} f(x) dx - p_0 x_0 \\ &= \int_0^2 (16 - x^2) dx - 24 \\ &= \left[16x - \frac{x^3}{3} \right]_0^2 - 24 = 32 - \frac{8}{3} - 24 = \frac{16}{3} \text{ units.} \end{aligned}$$

Producers' Surplus

$$\begin{aligned} \text{PS} &= p_0 x_0 - \int_0^{x_0} g(x) dx \\ &= 24 - \int_0^2 (2x^2 + 4) dx = 24 - \left[\frac{2x^3}{3} + 4x \right]_0^2 \\ &= 24 - \frac{2 \times 8}{3} - 8 = \frac{32}{3} \text{ units.} \end{aligned}$$

EXERCISE 5.4

- 1) If the demand function is $p = 35 - 2x - x^2$ and the demand x_0 is 3, find the consumers' surplus.
- 2.) If the demand function for a commodity is $p = 36 - x^2$ find the consumers' surplus for $p_0 = 11$.
- 3) The demand function for a commodity is $p = 10 - 2x$. Find the consumers' surplus for (i) $p = 2$ (ii) $p = 6$.
- 4) The demand function for a commodity is $p = 80 - 4x - x^2$. Find the consumers' surplus for $p = 20$.
- 5) If the supply function is $p = 3x^2 + 10$ and $x_0 = 4$, find the producers' surplus.
- 6) If the supply law is $p = 4 - x + x^2$, find the producers' surplus when the price is 6.
- 7) The supply function for a commodity is $p = 3 + x$. Find the producers' surplus when (i) $x_0 = 3$. (ii) $x_0 = 6$.

- 8) For a commodity, the supply law is $p = \frac{x^2}{2} + 3$. Find the producers' surplus when the price is 5.
- 9) The demand and supply function for a commodity are $p_d = 16 - 2x$ and $p_s = x^2 + 1$. Find the consumers' surplus and producers' surplus at the market equilibrium price.
- 10) The demand and supply law under a pure competition are given by $p_d = 23 - x^2$ and $p_s = 2x^2 - 4$. Find the consumers' surplus and producers' surplus at the market equilibrium price.
- 11) Under pure competition the demand and supply laws for commodity are $p_d = 56 - x^2$ and $p_s = 8 + \frac{x^2}{3}$. Find the consumers' surplus and producers' surplus at the equilibrium price.
- 12) Find the consumers' surplus and the producers' surplus under market equilibrium if the demand function is $p_d = 20 - 3x - x^2$ and the supply function is $p_s = x - 1$.
- 13) In a perfect competition the demand and supply curves of a commodity are given by $p_d = 40 - x^2$ and $p_s = 3x^2 + 8x + 8$. Find the consumers' surplus and producers' surplus at the market equilibrium price.
- 14) The demand and supply function for a commodity are given by $p_d = 15 - x$ and $p_s = 0.3x + 2$. Find the consumers' surplus and producers' surplus at the market equilibrium price.
- 15) The demand and supply curves are given by $p_d = \frac{16}{x+4}$ and $p_s = \frac{x}{2}$. Find the consumers' surplus and producers' surplus at the market equilibrium price.

EXERCISE 5.5

Choose the correct answer

- 1) If $f(x)$ is an odd function then $\int_{-a}^a f(x) dx$ is

(a) 1	(b) 2a	(c) 0	(d) a
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- 2) If $f(x)$ is an even function then $\int_{-a}^a f(x) dx$ is
 (a) $2 \int_0^a f(x) dx$ (b) $\int_0^a f(x) dx$ (c) $-2a$ (d) $2a$
- 3) $\int_{-3}^3 x dx$ is
 (a) 0 (b) 2 (c) 1 (d) -1
- 4) $\int_{-2}^2 x^4 dx$ is
 (a) $\frac{32}{5}$ (b) $\frac{64}{5}$ (c) $\frac{16}{5}$ (d) $\frac{8}{5}$
- 5) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx$ is
 (a) 0 (b) -1 (c) 1 (d) $\frac{\pi}{2}$
- 6) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx$ is
 (a) 2 (b) -2 (c) -1 (d) 1
- 7) The area under the curve $y = f(x)$, the x -axis and the ordinates at $x = a$ and $x = b$ is
 (a) $\int_a^b y dx$ (b) $\int_a^b y dy$ (c) $\int_a^b x dy$ (d) $\int_a^b x dx$
- 8) The area under the curve $x = g(y)$, the y - axis and the lines $y = c$ and $y = d$ is
 (a) $\int_c^d y dy$ (b) $\int_c^d x dy$ (c) $\int_c^d y dx$ (d) $\int_c^d x dx$
- 9) The area bounded by the curve $y = e^x$, the x - axis and the lines $x = 0$ and $x = 2$ is
 (a) $e^2 - 1$ (b) $e^2 + 1$ (c) e^2 (d) $e^2 - 2$
- 10) The area bounded by $y = x$, y - axis and $y = 1$ is
 (a) 1 (b) $\frac{1}{2}$ (c) $\log 2$ (d) 2

- 11) The area of the region bounded by $y = x + 1$ the x - axis and the lines $x = 0$ and $x = 1$ is
 (a) $\frac{1}{2}$ (b) 2 (c) $\frac{3}{2}$ (d) 1
- 12) The area bounded by the demand curve $xy = 1$, the x - axis, $x = 1$ and $x = 2$ is
 (a) $\log 2$ (b) $\log \frac{1}{2}$ (c) $2 \log 2$ (d) $\frac{1}{2} \log 2$
- 13) If the marginal cost function $MC = 3e^{3x}$, then the cost function is
 (a) $\frac{e^{3x}}{3}$ (b) $e^{3x}+k$ (c) $9e^{3x}$ (d) $3e^{3x}$
- 14) If the marginal cost function $MC = 2 - 4x$, then the cost function is
 (a) $2x-2x^2+k$ (b) $2-4x^2$ (c) $\frac{2}{x} - 4$ (d) $2x - 4x^2$
- 15) The marginal revenue of a firm is $MR = 15 - 8x$. Then the revenue function is
 (a) $15x-4x^2+k$ (b) $\frac{15}{x} - 8$ (c) -8 (d) $15x - 8$
- 16) The marginal revenue $R'(x) = \frac{1}{x+1}$ then the revenue function is
 (a) $\log |x+1| + k$ (b) $-\frac{1}{(x+1)}$ (c) $\frac{1}{(x+1)^2}$ (d) $\log \frac{1}{x+1}$
- 17) The consumers' surplus for the demand function $p = f(x)$ for the quantity x_0 and price p_0 is
 (a) $\int_0^{x_0} f(x) dx - p_0 x_0$ (b) $\int_0^{x_0} f(x) dx$
 (c) $p_0 x_0 - \int_0^{x_0} f(x) dx$ (d) $\int_0^{p_0} f(x) dx$
- 18) The producers' surplus for the supply function $p = g(x)$ for the quantity x_0 and price p_0 is
 (a) $\int_0^{x_0} g(x) dx - p_0 x_0$ (b) $p_0 x_0 - \int_0^{x_0} g(x) dx$
 (c) $\int_0^{x_0} g(x) dx$ (d) $\int_0^{p_0} g(x) dx$

ANSWERS

1. APPLICATIONS OF MATRICES AND DETERMINANTS

Exercise 1.1

$$\begin{aligned}
 &1) \begin{pmatrix} 1 & -3 \\ -2 & -1 \end{pmatrix} \quad 2) \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \quad 8) \frac{1}{16} \begin{pmatrix} 2 & -4 \\ 3 & 2 \end{pmatrix} \\
 &9) \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ -4 & -2 & 5 \\ 1 & -1 & 1 \end{pmatrix} \quad 10) \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\
 &11) \begin{pmatrix} \frac{1}{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & \frac{1}{a_3} \end{pmatrix} \quad 13) \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 10 \\ 1 & 2 & -7 \end{pmatrix} \quad 18) 4, -2 \quad 19) -1, 0 \\
 &20) \begin{pmatrix} 5 \\ 2 \end{pmatrix}
 \end{aligned}$$

Exercise 1.2

- 1) (i) 3 (ii) 2 (iii) 1 (iv) 3 (v) 2 (vi) 3 (vii) 1 (viii) 2 (ix) 2
 2) 2, 0. 6) inconsistent
 11) $k = -3$ 12) k assumes any real value other than 0
 13) $k = -3$ 14) k assumes any real value other than 8

Exercise 1.3

- 1) 2, 1. 2) 0, 1, 1. 3) 5, 2. 4) 2, -1, 1.
 5) 0, 2, 4. 6) 20, 30. 7) Rs.2, Rs.3, Rs.5.
 8) Re.1, Rs.2, Rs.3. 9) 11 tons, 15 tons, 19 tons.

Exercise 1.4

$$\begin{aligned}
 &1) \begin{matrix} 6 & 8 & 9 & 12 \\ 2 & \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \\ 5 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \\ 8 & \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \\ 9 & \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix} \quad 2) \begin{matrix} 2 & 4 & 6 & 9 \\ 2 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \\ 4 & \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \\ 6 & \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \\ 9 & \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}
 \end{aligned}$$

$$3) \{(a, l), (b, m), (c, m)\} \quad 4) \begin{matrix} a & b & c & d \\ a \begin{pmatrix} 1 & 0 & 0 & 1 \\ b \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \end{matrix}$$

$$5) \begin{matrix} 4 & 3 & 8 \\ 3 \begin{pmatrix} 0 & 0 & 1 \\ 5 \begin{pmatrix} 0 & 0 & 1 \\ 9 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{matrix} \end{matrix}, \begin{matrix} 1 & 2 & 5 \\ 4 \begin{pmatrix} 0 & 0 & 1 \\ 3 \begin{pmatrix} 0 & 0 & 1 \\ 8 \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{matrix} \end{matrix}, \begin{matrix} 1 & 2 & 5 \\ 3 \begin{pmatrix} 0 & 0 & 0 \\ 5 \begin{pmatrix} 0 & 0 & 0 \\ 9 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{matrix} \end{matrix}$$

$$6) \begin{matrix} 1 & 2 & 3 & 4 \\ 1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 4 \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix} \end{matrix} \end{matrix} ; \text{Equivalence relation.}$$

$$7) \begin{matrix} 1 & 2 & 3 & 4 \\ 1 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 4 \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix} \end{matrix} \end{matrix} ; \text{Reflexive, Not symmetric, Transitive.}$$

$$8) \begin{matrix} 1 & 2 & 3 & 4 \\ 1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 4 \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix} \end{matrix} \end{matrix} ; \text{Not reflexive, Not symmetric, Transitive.}$$

$$9) \begin{matrix} 1 & 2 & 3 & 4 \\ 1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix} \end{matrix} \end{matrix} ; \text{Not reflexive, Not symmetric, Not transitive.}$$

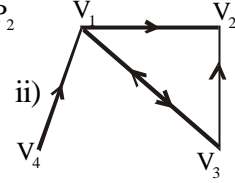
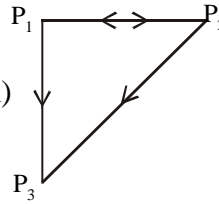
$$10) \begin{matrix} P_1 & P_2 & P_3 \\ (i) \begin{pmatrix} P_1 \begin{pmatrix} 0 & 0 & 0 \\ P_2 \begin{pmatrix} 0 & 0 & 0 \\ P_3 \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \end{pmatrix} \end{matrix} \end{matrix} \quad (ii) \begin{matrix} P_1 & P_2 & P_3 & P_4 \\ \begin{pmatrix} P_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ P_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ P_3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ P_4 \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix} \end{matrix} \end{matrix}$$

$$(iii) \begin{matrix} P_1 & P_2 & P_3 \\ \begin{pmatrix} P_1 \begin{pmatrix} 0 & 1 & 0 \\ P_2 \begin{pmatrix} 0 & 0 & 0 \\ P_3 \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \end{matrix} \quad (iv) \begin{matrix} P_1 & P_2 & P_3 & P_4 \\ \begin{pmatrix} P_1 \begin{pmatrix} 0 & 1 & 0 & 1 \\ P_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ P_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ P_4 \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \end{matrix}$$

$$(v) \begin{matrix} & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

$$(vi) \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

11) i)



12) 3, CBA, CDA, CDBA

$$13) (i) \begin{matrix} & X & Y & Z & W \\ \begin{matrix} X \\ Y \\ Z \\ W \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix} \quad (ii) \text{ Not strongly connected.}$$

$$(iii) \begin{matrix} & X & Y & Z & W \\ \begin{matrix} X \\ Y \\ Z \\ W \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

14) (i)

$$\begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

(ii) 2, $V_2 V_1 V_4 V_3$, $V_2 V_3 V_4 V_3$.

(iii) 5,

$$\begin{aligned} & V_2 V_1 V_4 V_1 V_4, \\ & V_2 V_1 V_4 V_3 V_4, \\ & V_2 V_4 V_3 V_1 V_4, \\ & V_2 V_3 V_4 V_1 V_4, \\ & V_2 V_3 V_4 V_3 V_4. \end{aligned}$$

(iv) 4, $V_4 V_1$, $V_4 V_3 V_1$, $V_4 V_3 V_4 V_1$, $V_4 V_1 V_4 V_1$.

$$(v) \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

(vi) Not strongly connected. (vii)

$$15) \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ V_1 & \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \\ V_2 & \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\ V_3 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \\ V_4 & \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix} \quad 17) \begin{matrix} & P_1 & P_2 & P_3 \\ P_1 & \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ P_2 & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ P_3 & \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \end{matrix}; \begin{matrix} & P_1 & P_2 & P_3 \\ P_1 & \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\ P_2 & \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\ P_3 & \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

$$18) \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ V_1 & \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \\ V_2 & \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \\ V_3 & \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \\ V_4 & \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

$$19) \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ V_1 & \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \\ V_2 & \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \\ V_3 & \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \\ V_4 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}; \begin{matrix} & V_1 & V_2 & V_3 & V_4 \\ V_1 & \begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix} \\ V_2 & \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \\ V_3 & \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \\ V_4 & \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

20) (i) 24, 21, 61, 47, 76, 55, 33, 28 (ii) THURSDAY

Exercise 1.5

- 1) The system is viable. 2) The system is not viable.
- 3) 110 units, 320 units.
- 4) Rs.72 millions, Rs.96 millions. 5) (i) Rs 42 lakhs, Rs. 78 lakhs
(ii) Rs.28 lakhs, Rs.52 lakhs. 6) Rs. 80 millions, Rs. 120 millions.
- 7) Rs. 1200 crores, Rs. 1600 crores.
- 8) Rs. 7104 crores, Rs. 6080 crores.

Exercise 1.6

- 1) 74.8%, 25.2% ; 75%, 25% 2) 39% 3) 54.6%, 45.4%

Exercise 1.7

- 1) c 2) b 3) c 4) c 5) a 6) a 7) b 8) b
- 9) b 10) c 11) a 12) a 13) a 14) a 15) a 16) b
- 17) a 18) b 19) d 20) b

ANALYTICAL GEOMETRY

Exercise 2.1

- 1) a parabola 2) a hyperbola 3) an ellipse

Exercise 2.2

- 1) (a) $x^2 + y^2 - 2xy - 4y + 6 = 0$
 (b) $x^2 + y^2 + 2xy - 4x + 4y + 4 = 0$
 (c) $4x^2 + 4xy + y^2 - 4x + 8y - 4 = 0$
 (d) $x^2 + 2xy + y^2 - 22x - 6y + 25 = 0$
- 2) (a) $(0, 0), (0, 25), x = 0, y + 25 = 0$
 (b) $(0, 0), (5, 0), y = 0, x + 5 = 0$
 (c) $(0, 0), (-7, 0), y = 0, x - 7 = 0$
 (d) $(0, 0), (0, -15), x = 0, y - 15 = 0$
- 3) (a) $(-\frac{1}{2}, 1), (-\frac{3}{2}, 1), 2x - 1 = 0, 4$
 (b) $(-1, -1), (0, -1), x + 2 = 0, 4$
 (c) $(-\frac{9}{8}, 0), (\frac{7}{8}, 0), 8x + 25 = 0, 8$
 (d) $(0, 1), (0, \frac{7}{4}), 4y - 1 = 0, 3$
- 4) Output = 15 tons. and cost = Rs.40

Exercise 2.3

- 1) (i) $101x^2 + 48xy + 81y^2 - 330x - 324y + 441 = 0$
 (ii) $27x^2 + 20y^2 - 24xy + 6x + 8y - 1 = 0$
 (iii) $17x^2 + 22y^2 + 12xy - 58x + 108y + 129 = 0$
- 2) (i) $\frac{x^2}{144} + \frac{y^2}{128} = 1$ (ii) $\frac{x^2}{24} + \frac{y^2}{15} = 1$
 (iii) $\frac{x^2}{9} + \frac{y^2}{25} = 1$
- 3) (i) $(0, 0), (0, \pm 3); \frac{\sqrt{5}}{3}; (0, \pm\sqrt{5}); \frac{8}{3}$
 (ii) $(1, -5), (1, \pm\sqrt{7}-5); \frac{\sqrt{3}}{\sqrt{7}}; (1, \pm\sqrt{3}-5); \frac{8}{\sqrt{7}}$
 $y = \frac{7}{\sqrt{3}} - 5, y = \frac{-7}{\sqrt{3}} - 5,$
 (iii) $(-2, 1), (2, 1) (-6, 1); \frac{\sqrt{7}}{4}; (\pm\sqrt{7}-2, 1); \frac{9}{2}$
 $x = \frac{16}{\sqrt{7}} - 2, x = \frac{-16}{\sqrt{7}} - 2,$

Exercise 2.4

- 1) (a) $19x^2 + 216xy - 44y^2 - 346x - 472y + 791 = 0$
 (b) $16(x^2 + y^2) = 25(x \cos \alpha + y \sin \alpha - p)^2$
- 2) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$
- 3) (a) $16x^2 - 9y^2 - 32x - 128 = 0$
 (b) $3x^2 - y^2 - 18x + 4y + 20 = 0$
 (c) $3x^2 - y^2 - 36x + 4y + 101 = 0$
- 4) (a) $(0, 0); \frac{5}{4}; (\pm 5, 0); 5x \pm 16 = 0$
 (b) $(-2, -4); \frac{4}{3}; (2, -4) (-6, -4); 4x - 1 = 0, 4x + 17 = 0$
 (c) $(1, 4); 2; (6, 4) (-4, 4); 4x - 9 = 0, 4x + 1 = 0$
- 5) (a) $3x + y + 2 = 0$ and $x - 2x + 5 = 0$;
 (b) $4x - y + 1 = 0$ and $2x + 3y - 1 = 0$
- 6) $4x^2 - 5xy - 6y^2 - 11x + 11y + 57 = 0$
- 7) $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$

Exercise 2.5

- 1) a 2) b 3) c 4) d 5) c 6) a 7) c 8) b
- 9) b 10) a 11) b 12) c 13) b 14) b 15) a 16) c
- 17) a 18) c 19) c 20) c

APPLICATIONS OF DIFFERENTIATION-I**Exercise 3.1**

- 1) (i) $\frac{1}{2}x^2 - 4x + 25 + \frac{8}{x}$ (ii) $\frac{1}{2}x^2 - 4x + 25$
 (iii) $\frac{8}{x}$. AC = Rs.35.80, AVC = Rs.35, AFC = Rs.0.80
- 2) Rs.600.05 3) Rs.5.10 4) Rs.1.80 5) Rs. 1.50, Rs.1406.25
- 6) (i) $\frac{1}{10}x^2 - 4x + 8 + \frac{4}{x}$ (ii) $\frac{3}{10}x^2 - 8x + 8$ (iii) $\frac{1}{5}x - 4 - \frac{4}{x^2}$
- 7) Rs. 55, Rs.23 8) Rs.119 10) 0.75 11) 1.15

$$\begin{aligned}
13) & \text{ (i) } \frac{2(a-bx)}{bx} \quad \text{(ii) } \frac{3}{2} \quad 14) m \quad 15) \frac{4p^2}{2p^2+5} \quad 16) \frac{p}{2(p-b)} \\
17) & AR = p, \quad MR = 550 - 6x - 18x^2 \quad 18) \text{ (i) } R = 20,000 x e^{-0.6x} \\
& \text{(ii) } MR = 20,000 x e^{-0.6x} [1 - 0.6x] \quad 19) \frac{4p+2p^2}{30-4p-p^2}, \\
& \frac{3p^2+8p-30}{2(p+2)} \quad 20) 20, 3 \quad 21) \text{ Rs.110} \quad 22) \frac{30}{11}, \text{ Rs.1.90}
\end{aligned}$$

Exercise 3.2

- 1) -1.22, -1.25 2) -1 unit / sec 3) 12 units / sec.
- 5) (i) revenue is increasing at the rate of Rs.40,000 per month
(ii) cost is increasing at the rate of Rs.4,000 per month
(iii) profit is increasing at the rate of Rs.36,000 per month
- 6) (i) revenue is increasing at the rate of Rs.48,000 per week
(ii) cost is increasing at the rate of Rs.12,000 per week
(iii) profit is increasing at the rate of Rs.36,000 per week.
- 8) $10\pi \text{ cm}^2 / \text{sec}$ 9) $115\pi \text{ cm}^3 / \text{minute}$ 10) $x = \frac{1}{3}, 3$.

Exercise 3.3

- 1) $\frac{10}{3}, \frac{-13}{5}$ 2) $a = 2, b = 2$
- 4) (i) $x - y + 1 = 0, x + y - 3 = 0$
(ii) $2x - 2y + \sqrt{3} - \frac{\pi}{3} = 0; 2x + 2y - \sqrt{3} - \frac{\pi}{3} = 0$
(iii) $3x + 2y + 13 = 0; 2x - 3y = 0$
(iv) $9x + 16y - 72 = 0; 64x - 36y - 175 = 0$
(v) $3ex - y - 2e^2 = 0; x + 3ey - 3e^3 - e = 0$
(vi) $\sqrt{2}bx + \sqrt{2}ay - 2ab = 0; \sqrt{2}ax - \sqrt{2}by - a^2 + b^2 = 0$
- 5) $13x - y - 34 = 0; x + 13y - 578 = 0$
- 6) $10x + y - 61 = 0; x - 10y + 105 = 0$

- 7) $(1, \frac{1}{3})$, $(-1, -\frac{1}{3})$ 9) $x - 20y - 7 = 0$; $20x + y - 140 = 0$
 11) $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$; $\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$.
 12) (i) $(1, 0)$ and $(1, 4)$ (ii) $(3, 2)$ and $(-1, 2)$

Exercise 3.4

- 1) d 2) c 3) a 4) a 5) b 6) c 7) d
 8) d 9) b 10) a 11) d 12) a 13) b 14) c
 15) b 16) d 17) c 18) a 19) a 20) c

APPLICATIONS OF DIFFERENTIATION-II

Exercise 4.1

- 3) increasing in $(-\infty, -5)$ and $(-\frac{1}{3}, \infty)$ decreasing in $(-5, -\frac{1}{3})$
 4) $(-2, 27)$, $(1, 0)$
 5) (i) R is increasing for $0 < x < 4$ decreasing for $x > 4$, MR is increasing for $0 < x < 2$ and decreasing for $x > 2$.
 (ii) R is increasing for $1 < x < 7$, decreasing for $0 < x < 1$ and $x > 7$. MR is increasing for $0 < x < 4$ and decreasing for $x > 4$.
 6) (i) TC is increasing for $0 < x < 10$ and for $x > 20$ and decreasing for $10 < x < 20$. MC is decreasing for $0 < x < 15$ and increasing for $x > 15$.
 (ii) TC is increasing for $0 < x < 40$ and decreasing for $x > 40$. MC is always decreasing.
 7) (i) $x = 0$ max. value = 7, $x = 4$ min. value = -25
 (ii) $x = 1$ max. value = -4, $x = 4$ mini. value = -31
 (iii) $x = 2$ min. value = 12
 (iv) $x = 1$ max. value = 19, $x = 3$ mini. value = 15
 8) $x = 1$ max. value = 53, $x = -1$ min. value = -23.
 9) $(0, 3)$, $(2, -9)$
 11) convex up for $\frac{1}{2} < x < 1$
 convex down for $-\infty < x < \frac{1}{2}$ and $1 < x < \infty$.

- 12) $q = 3$.
 13) $x = 1$ max. value = 0
 $x = 3$ mini. value = -28
 $x = 0$ point of inflexion exists.

Exercise 4.2

- 1) $x = 15$ 2) 15, 225 4) $x = 5$ 5) $1, \frac{3}{2}$
 6) $x = 8$ 7) (i) 10.5, Rs.110.25 (ii) 3, 0 (iii) $x = 6$
 8) $x = 60$ 9) Rs.1600 10) $x = 70$ 11) $x = 13$
 12) A : 1000, B : 1800, C : 1633
 13) A : 214.476, Rs.21.44 B : 67.51 Rs.58.06 C : 2000, Rs.4,
 D : 537.08, Rs.27.93
 14) (i) 400 (ii) Rs.240 (iii) $\frac{3}{2}$ orders / year (iv) $\frac{2}{3}$ of a year
 15) (i) 800 (ii) $\frac{1}{4}$ of a year (iii) 4 (iv) Rs.1200.

Exercise 4.3

- 1) $8x + 6y; 6x - 6y$
 3) (i) $24x^5 + 3x^2y^5 - 24x^2 + 6y - 7$
 (ii) $5x^3y^4 + 6x + 8$
 (iii) $120x^4 - 48x + 6xy^5$
 (iv) $20x^3y^3$
 (v) $15x^2y^4 + 6$
 (vi) $15x^2y^4 + 6$
 4) (i) $30x^4y^2 + 8x + 4$ (ii) 500
 (iii) $12x^5y - 24y^2 + 6$ (iv) -90
 (v) $120x^3y^2 + 8$ (vi) 968
 (vii) $12x^5 - 48y$ (viii) 12
 (ix) $60x^4y$ (x) 2880 (xi) 2880

14) (i) 940 (ii) 700

15) Note Book (16) (i) Rs.18,002 (ii) Rs.8005

Exercise 4.4

1) (i) $10 - 2L + 3K$, (ii) $5 - 4K + 3L$ (iii) 14, 0

3) 1, 4 4) 3.95, 120 5) 2.438, 3.481

7) (i) $\frac{3}{4}$ (ii) $\frac{1}{2}$ 8) $\frac{3}{5}, \frac{2}{5}$ 9) 6, 1 10) $-\frac{10}{3}, \frac{5}{6}$

Exercise 4.5

1) b 2) d 3) a 4) b 5) c 6) c 7) a
8) c 9) b 10) d 11) a 12) a 13) d 14) a
15) c 16) a 17) c 18) d 19) a 20) a

APPLICATIONS OF INTEGRATION

Exercise 5.1

1) 0 2) 80 3) $\frac{\pi}{2}$ 4) 2 5) $\frac{16}{15} \sqrt{2}$ 6) $\frac{1}{20}$ 7) $\frac{\pi}{12}$
8) 1 9) $\frac{\pi^2}{4}$ 10) $(a + b) \frac{\pi}{4}$

Exercise 5.2

Answers are in square units.

1) 9 2) 6 3) $\frac{\pi}{2}$ 4) 2 5) $\frac{2}{3}$ 6) $\log 3$ 7) $\frac{8a^2}{3}$ 8) 8
9) $4\log 4$ 10) πa^2 11) πab

Exercise 5.3

1) $C = 10x + 12x^2 - x^3 + 4$, $AC = 10 + 12x - x^2 + \frac{4}{x}$

2) $C = 100 (\log \frac{x}{16} + 1)$, $AC = \frac{100}{x} (\log \frac{x}{16} + 1)$

3) $C = x^3 - 5x^2 + 3x + 8$, $AC = x^2 - 5x + 3 + \frac{8}{x}$

4) $C = 5x - 3x^2 + x^3 + 100$, $AC = 5 - 3x + x^2 + \frac{100}{x}$

- 5) $C = 20x - 0.02x^2 + 0.001x^3 + 7000$
 $AC = 20 - 0.02x + 0.001x^2 + \frac{7000}{x}$
- 6) $R = 15x - \frac{9x^2}{2} - x^3$, $AR = 15 - \frac{9x}{2} - x^2$
- 7) $R = 9x - x^2 + \frac{4x^3}{3}$, $p = 9 - x + \frac{4x^2}{3}$
- 8) $R = 100x - 3x^3$, $p = 100 - 3x^2$
- 9) $R = 2x + 2x^2 - \frac{x^3}{3}$, $p = 2 + 2x - \frac{x^2}{3}$
- 10) $R = 4x - \frac{3x^2}{2}$, $p = 4 - \frac{3x}{2}$
- 11) $p = 3 - x$, $R = 3x - x^2$
- 12) $p = 5 - \frac{x^2}{2}$, $R = 5x - \frac{x^3}{2}$ 13) $p = \frac{k}{x}$, k is constant.
- 14) $C = 2x + e^{3x} + 500$, $AC = 2 + \frac{e^{3x}}{x} + \frac{500}{x}$
- 15) $R = -\frac{3}{x} - \log x^2 + 9$, $p = -\frac{3}{x^2} - \frac{\log x^2}{x} + \frac{9}{x}$
- 16) $R = 16x - \frac{x^3}{3}$, $p = 16 - \frac{x^2}{3}$ 17) $13x - 0.065x^2 - 120$
- 18) $R = \text{Rs. } 4,31,667$

Exercise 5.4

- 1) $CS = 27$ units 2) $CS = \frac{250}{3}$ units
- 3) (i) $CS = 16$ units (ii) $CS = 4$ units
- 4) $CS = 216$ units 5) $PS = 128$ units 6) $PS = \frac{10}{3}$ units
- 7) (i) $PS = \frac{9}{2}$ units (ii) $PS = 18$ units 8) $PS = \frac{8}{3}$ units
- 9) $CS = 9$ units ; $PS = 18$ units 10) $CS = 18$ units ; $PS = 36$ units
- 11) $CS = 144$ units $PS = 48$ units
- 12) $CS = \frac{63}{2}$ units $PS = \frac{9}{2}$ units
- 13) $CS = \frac{16}{3}$ units $PS = 32$ units

14) CS = 50 units PS = 15 units

15) CS = $16 \log_2 -8$ units PS = 4 units

Exercise 5.5

- | | | | | |
|---------|---------|---------|---------|---------|
| 1) (c) | 2) (a) | 3) (a) | 4) (b) | 5) (a) |
| 6) (a) | 7) (a) | 8) (b) | 9) (a) | 10) (b) |
| 11) (c) | 12) (a) | 13) (b) | 14) (a) | 15) (a) |
| 16) (a) | 17) (a) | 18) (b) | | |