Z-Transforms Properties

Z-Transform has following properties:

Linearity Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and $y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \xleftrightarrow{\text{Z.T}} a\,X(Z) + b\,Y(Z)$$

Time Shifting Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then Time shifting property states that

$$x(n-m) \xleftrightarrow{\text{Z.T}} z^{-m} X(Z)$$

Multiplication by Exponential Sequence Property

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n\,.\,x(n) \xleftrightarrow{\text{Z.T}} X(Z/a)$$

Time Reversal Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{\text{Z.T}}{\longleftrightarrow} X(1/Z)$$

Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \overset{\text{Z.T}}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

Convolution Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$$

Correlation Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z^{-1})$$

Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z \to 1} [z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z-transform.

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.

- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

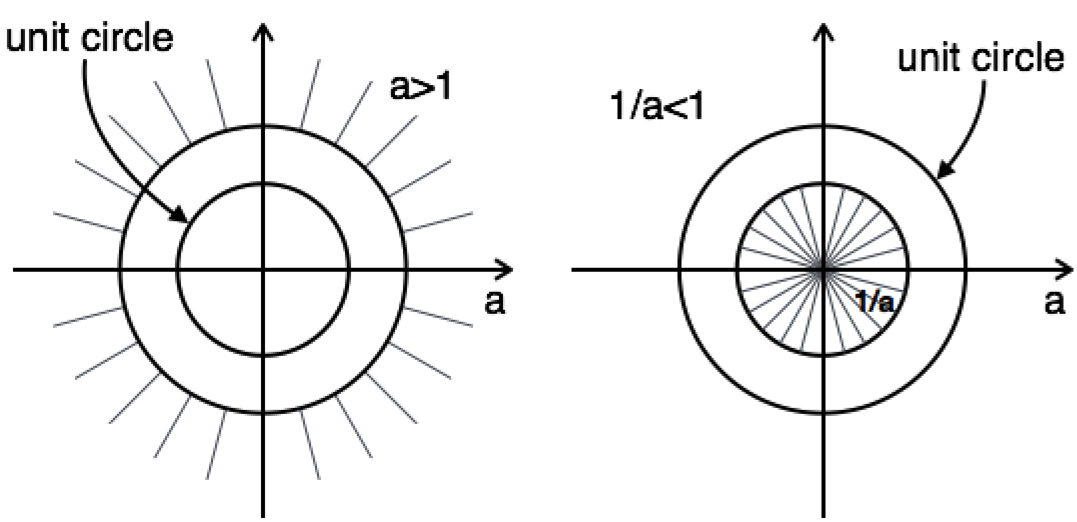
The concept of ROC can be explained by the following example:

Example 1: Find z-transform and ROC of $a^n u[n] + a^- nu[-n-1]$

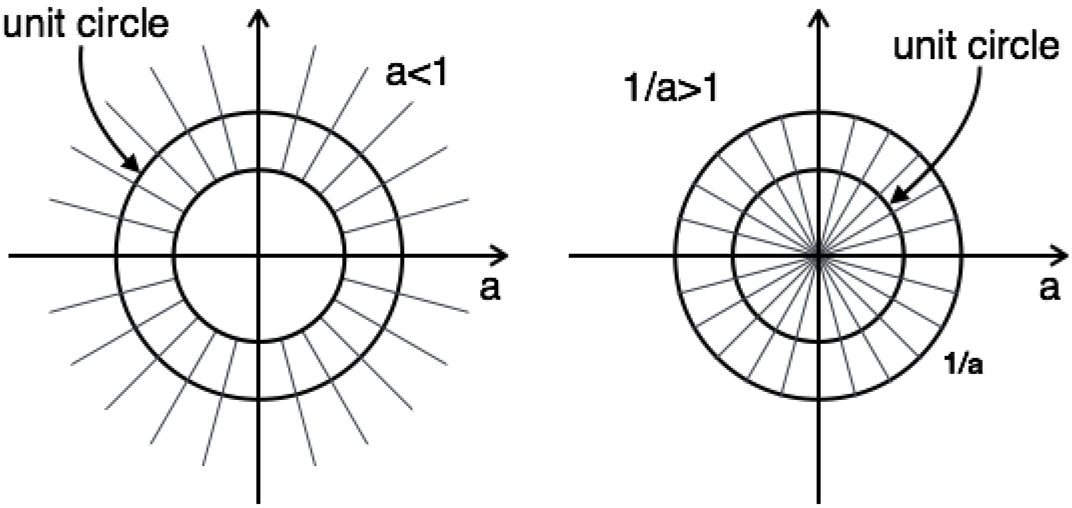
$$Z.T[a^n u[n]] + Z.T[a^{-n} u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

$$ROC: |z| > a \qquad ROC: |z| < \frac{1}{a}$$

The plot of ROC has two conditions as a > 1 and a < 1, as you do not know a.



In this case, there is no combination ROC.



Here, the combination of ROC is from $a < |z| < \frac{1}{a}$

Hence for this problem, z-transform is possible when a < 1.

Causality and Stability

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

- ROC is outside the outermost pole.
- In The transfer function H[Z], the order of numerator cannot be grater than the order of denominator.

Stability Condition for Discrete Time LTI Systems

A discrete time LTI system is stable when

- its system function H[Z] include unit circle |z|=1.
- all poles of the transfer function lay inside the unit circle |z|=1.

Z-Transform of Basic Signals

x(t)	X[Z]
δ	1
$u(n)$	$\frac{Z}{Z-1}$
$u(-n-1)$	$-\frac{Z}{Z-1}$
$\delta(n-m)$	z^{-m}
$a^n u[n]$	$\frac{Z}{Z-a}$
$a^n u[-n-1]$	$-\frac{Z}{Z-a}$
$n\,a^n u[n]$	$\frac{aZ}{\|Z-a\|^2}$
$n\,a^n u[-n-1]$	$-\frac{aZ}{\|Z-a\|^2}$
$a^n \cos \omega n u[n]$	$\frac{Z^2-aZ\cos\omega}{Z^2-2aZ\cos\omega+a^2}$
$a^n \sin \omega n u[n]$	$\frac{aZ\sin\omega}{Z^2-2aZ\cos\omega+a^2}$