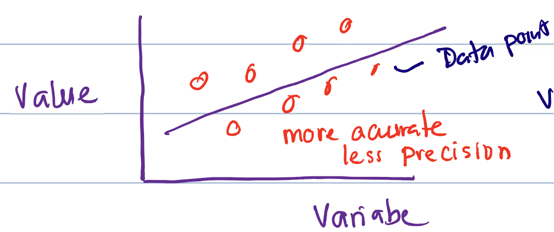
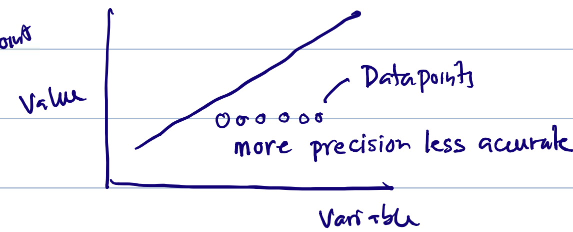
Quantification of Error

Error arises in processing problems of numerical methods during computation and pre-computations due to several reasons

Accuracy: How close the result value is to true value
the closer the result to the true value, more accurate it is

Precision How closely values agree to each other





Measurement of error:

True Error (E_t) = True value − Approximate value
↳ Can be +ve/−ve

If true value is 7.893 and approximate value is 7.975 then

$$E_t = 7.893 - 7.975 = -0.082$$

Absolute true error = $|E_t|$

$$\text{Relative error} = \frac{E_t}{\text{True Value}} = \frac{\overbrace{\text{True value} - \text{App value}}^{\text{True error}}}{\text{True value}}$$

It is usually represented as %

If true value be 20 m, true error is 1 cm

$$|e_r| = \left|\frac{1}{2000}\right| = 0.05\%$$

If true value is 1 cm and true error is 1 cm

$$|e_r| = \left|\frac{1}{1}\right| = 100\%$$

Approximate Errors :

Absolute approximate error is given by

- $|\text{Approximate error}| = |\text{current estimate} - \text{Previous estimate}|$

- Approximate relative error $(|e_a|) = \left|\dfrac{\text{Approximate error}}{\text{Current estimate}}\right|$

Approximate % Relative Error

a	$f(a)$	b	$f(b)$	c_k	$f(c_k)$
0	-1	1	1	0.5	-0.375
0.5	-0.375	1	0.17188	0.75	0.1719
0.5	-0.375	0.75	0.17188	0.625	-0.1309
0.625	-0.1309	0.75	0.01245	0.6875	0.0125
0.625	-0.1309	0.6875	0.01245	0.65625	-0.0611
0.65625	-0.0611	0.6875	0.01245	0.67188	-0.0248
0.67188	-0.0248	0.6875	0.01245	0.67969	-0.0063
0.67969	-0.0063	0.6875	0.01245	0.6836	0.0031
0.67969	-0.0063	0.6836	0.00305	0.68165	-0.0016
0.68165	-0.0016	0.6836	0.00305	0.68263	0.0007
0.68165	-0.0016	0.68263	0.00072	0.68214	-0.0005
0.68214	-0.0005	0.68263	0.00072	0.68239	0.0001
0.68214	-0.0005	0.68239	0.00015	0.68227	-0.0001
0.68227	-0.0001	0.68239	0.00015	0.68233	0

Iteration - I

$$e_a = \frac{0.75 - 0.5}{0.75} * 100 = 33\%$$

Iteration - II

$$= \frac{0.625 - 0.75}{0.625} * 100 = 20\%$$

Iteration – last but one

$$= \frac{0.68227 - 0.68239}{0.68227} * 100 = 0.01758\%$$

Iteration – last

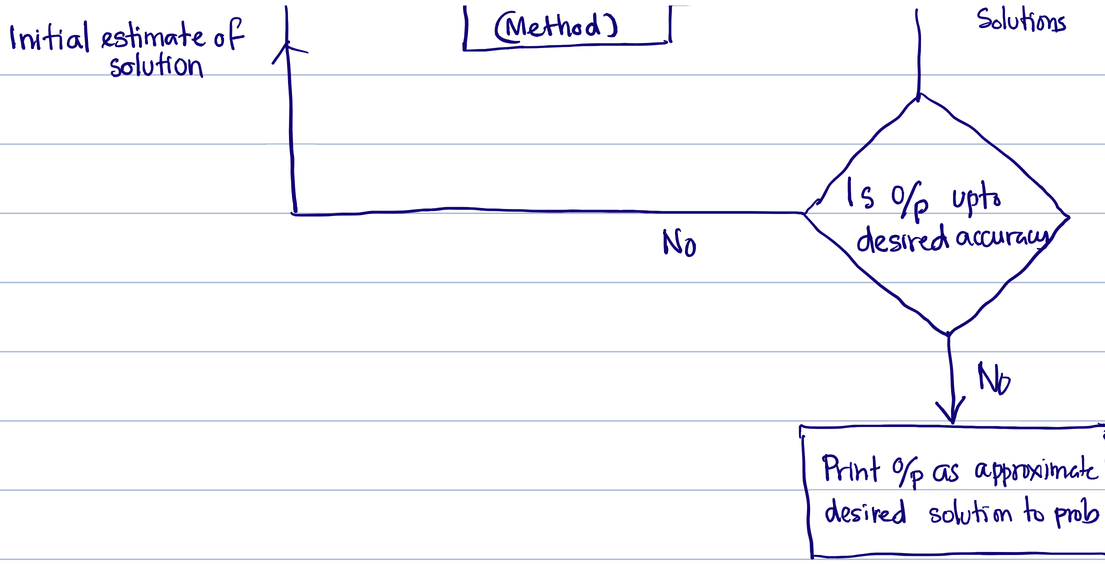
$$= \frac{0.68233 - 0.68227}{0.68233} * 100 = 0.00879\%$$

Importance of Numerical Methods

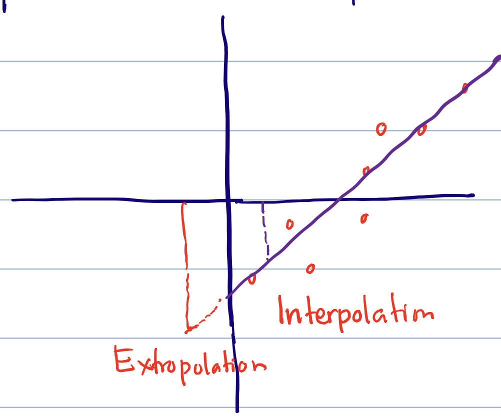
- Analytic methods do not exist
- Data available does not admit applicability of direct analytic method
- Analytic methods exists but are quite time consuming due to huge data / complex function involved.

Flow diagram

Input ——— $\boxed{\text{Process}}$ ——— O/p closer approximate to



Interpolation Vs Extrapolation



Error in a series approximation

The truncation error committed in a series approximation can be evaluated by Taylor's series. If x_i and x_{i+1} are two sucessive value of x then

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i) + \frac{(x_{i+1} - x_i)^n}{n!} f^n(x_i) + R_{n+1}(x_{i+1})$$

where $R_{n+1}(x_{i+1}) = \frac{(x_{i+1} - x_i)^{n+1}}{(n+1)!} f^{(n+1)}(\varepsilon) \quad x_i < \varepsilon < x_{i+1}$

Remainder term

If $f(x_{i+1})$ is approximated by first n terms then the maximum error committed by using n^{th} order approximation which is given by $R_{n+1}(x_{i+1})$

The interval length $x_{i+1} - x_i = h$

$$f(x_{i+1}) = f(x_i) + h f'(x) + \frac{h^2}{2!} f''(x_i) + \cdots + \frac{h^n}{n!} f^n(x_i) + O(h^{n+1})$$

the truncation error is of the order of h^{n+1}

If the series is approximated after 1st terms. This gives us

0^{th} approximation $\quad f(x_{i+1}) = f(x_i) + O(h)$

1^{st} order " $\quad f(x_{i+1}) = f(x_i) + hf'(x) + O(h^2)$

Example 1.10 Evaluate $f(1)$ using Taylor's series for

$f(x) = x^3 - 3x^2 + 5x - 10 \qquad f(1) = -7$

Let $h = 1 \quad x_i = 0 \quad x_{i+1} = 1$

$f'(x) = 3x^2 - 6x + 5 \qquad f''(x) = 6x - 6 \qquad f'''(x) = 6$

$f^{iv}(x)$ and higher order derivatives are zero Hence

$f'(x_i) = f'(0) = 5 \qquad f''(x_i) = f''(0) = -6 \qquad f'''(0) = 6$

Also $\qquad f(x_i) = f(0) = -10$

The Taylor Series gives

$$f(x_{i+1}) = f(x_i) + h f'(x_i) + \frac{h^2}{2} f''(x_i) + \frac{h^3}{6} f'''(x_i)$$

0^{th} order app. $\quad f(x_{i+1}) = f(x_i) + O(h)$

$f(1) = f(0) + O(h) = -10$

the error is $-7 + 10 = 3$

1^{st} order app $\quad f(x_{i+1}) = f(x_i) + h f'(x_i) + O(h^2)$

≈ -5

$f(1) = -10 + 5 + O(h^2) \approx -5$

the error is $-7 + 5 = -2$

2nd order app

$f(x_{i+1}) = f(x_i) + h f'(x_i) + \frac{h^2}{2} f''(x_i) + O(h^3)$

$f(1) = -10 + 5 + \frac{1}{2}(-6) + O(h^3) = -8$

the error is $-7 + 8 = 1$

3rd order app

$f(x_{i+1}) = f(x_i) + h f'(x_i) + \frac{h^2}{2} f''(x_i) + \frac{h^3}{3} f'''(x_i)$

$f(1) = f(0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3} f'''(x_0)$

$= -10 + 5 + \frac{1}{2}(-6) + \frac{1}{6} 6 = -7$

error $= 0$