M

SCALAR TRIPLE PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. If we introduce dot and cross between $\vec{a}, \vec{b}$ and $\vec{c}$ in the same alphabetical order , we have the following products:

$$(\vec{a}.\ \vec{b}).\ \vec{c}\ ,\ (\vec{a}.\ \vec{b}) \times \vec{c}\ ,\ (\vec{a} \times \vec{b}).\ \vec{c} \text{ and } (\vec{a} \times \vec{b}) \times \vec{c}.$$

We know that dot and cross products are defined between two vector quantities.

Now $\vec{a}.\ \vec{b}$ is a scalar quantity. So $(\vec{a}.\ \vec{b}).\ \vec{c}$ and $(\vec{a}.\ \vec{b}) \times \vec{c}$ are not meaningful quantities. Again $(\vec{a} \times \vec{b})$ is a vector quantity. So its dot and cross product with other vector quantity $\vec{c}$ is meaningful.

DEFINITION :

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then the scalar $(\vec{a} \times \vec{b}).\ \vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}, \vec{b}, \vec{c}]$ or $[\vec{a}\ \ \vec{b}\ \vec{c}]$.

$[\vec{a}, \vec{b}, \vec{c}]$ or $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}).\ \vec{c}$.

NOTE:

1) If $(\vec{a} \times \vec{b}).\ \vec{c} = (\vec{a}\ .\vec{b}) \times \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}] = 0$

Then the vectors are coplanar.

2) Geometrically $[\vec{a}\ \vec{b}\ \vec{c}]$ represent volume of a parallelepiped with edges $\vec{a}, \vec{b}$ and $\vec{c}$.

Coplanar vectors are the vectors which lie on the same plane, in a three-dimensional space. These are vectors which are parallel to the same plane. We can always find in a plane any two random vectors, which are coplanar.

Conditions for Coplanar vectors

- If there are three vectors in a 3d-space and their scalar triple product is zero, then these three vectors are coplanar.
- If there are three vectors in a 3d-space and they are linearly independent, then these three vectors are coplanar.
- In case of n vectors, if no more than two vectors are linearly independent, then all vectors are coplanar.

A linear combination of vectors $v_1, \ldots, v_n$ with coefficients $a_1, \ldots, a_n$ is a vector, such that;

$a_1v_1 + \ldots + a_nv_n$

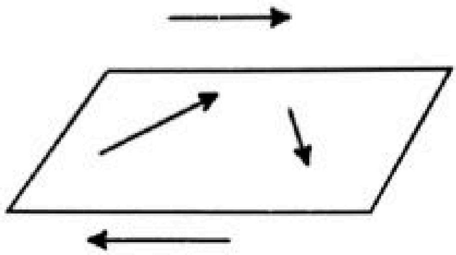
A linear combination $a_1v_1 + ... + a_nv_n$ is called trivial if all the coefficients $a_1, ..., a_n$ is zero and if at least one of the coefficients is not zero, then it is known as non-trivial.

What are Linearly independent vectors?

The vectors, $v_1, ...v_n$ are linearly independent if no non-trivial combination of these vectors is equal to the zero vector. That means $a_1v_1 + ... + a_nv_n = 0$ and the coefficients $a_1= 0 ..., a_n=0$.

What are Linearly dependent vectors?

The vectors, $v_1, ...v_n$ are linearly dependent if there exist at least one non-trivial combination of these vectors equal to zero vector.



Coplanar vectors

Examples

1. Find is Coplanar (A, B, C)?
***A* = (1, 2, 3), *B* = (2, 4, 6), *C* = (3, 4, 5)**

Solution:
Here, $A = (1, 2, 3)$, $B = (2, 4, 6)$, $C = (3, 4, 5)$

The 3 vectors are coplanar, if their scalar triple product is zero
1. Calculate scalar triple product
$A\cdot(B\times C)$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 1(4 \times 5 - 6 \times 4) - 2(2 \times 5 - 6 \times 3) + 3(2 \times 4 - 4 \times 3)$$
$$= 1(20 - 24) - 2(10 - 18) + 3(8 - 12)$$
$$= 1(-4) - 2(-8) + 3(-4)$$
$$= -4 - 16 - 12$$
$$= 0$$

Here scalar triple product is zero, so vectors are coplanar.

2. Find is Coplanar (A, B, C)?
***A* = (5, -1, 1), *B* = (-2, 3, 4), *C* = (3, 4, 5)**

Solution:
Here, $A = (5, -1, 1)$, $B = (-2, 3, 4)$, $C = (3, 4, 5)$

The 3 vectors are coplanar, if their scalar triple product is zero

1. Calculate scalar triple product

$A \cdot (B \times C)$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} 5 & -1 & 1 \\ -2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 5(3 \times 5 - 4 \times 4) - -1((-2) \times 5 - 4 \times 3) + 1((-2) \times 4 - 3 \times 3)$$

$$= 5(15 - 16) - -1(-10 - 12) + 1(-8 - 9)$$

$$= 5(-1) - -1(-22) + 1(-17)$$

$$= -5 - 22 - 17$$

$$= -44 \neq 0$$

Here scalar triple product is not zero, so vectors are not coplanar.

PROBLEMS:-

1) Find the volume of parallelepiped whose edges are represented by

$$\vec{A} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\vec{B} = \hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{C} = 3\hat{i} - \hat{j} + 2\hat{k}$$

sol: Volume of a parallelepiped $= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= 2(4-1)+3(2+3)+4(-1-6)$$

$= 7$ **cubic unit.**

2) Show that points (4,5,1) , (0,-1,1) , (3,9,4) , (-4,4,4) are coplanar.

Sol: *let*

$$\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\vec{AB} = 0\vec{B} - 0\vec{A}$$

$$= 0\hat{i} - \hat{j} + \hat{k} - 4\hat{i} - 5\hat{j} - \hat{k}$$

$$= -4\hat{i} - 6\hat{j}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$

$$= 3\hat{i} + 9\hat{j} + 4\hat{k} - 4\hat{i} - 5\hat{j} - \hat{k}$$

$$= -\hat{i} - 9\hat{j} + 3\hat{k}$$

$$\vec{AD} = \vec{OD} - \vec{OA}$$

$$= -4\hat{i} + 4\hat{j} + 4\hat{k} - 4\hat{i} - 5\hat{j} - \hat{k}$$

$$= -8\hat{i} - \hat{j} + 3\hat{k}$$

This can be represented as = $\begin{vmatrix} -4 & -6 & 0 \\ -1 & -9 & 3 \\ -8 & -1 & 3 \end{vmatrix}$

= -4 (-27+3) + 6 (-3+24) + 0
= -4 (-24) + 6 (21)
= 96 + 126
= 222 $\neq 0$

as the result is not equal to zero
so these given points are **not coplanar.**

3)Find 'P' such that

$$\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$$
$$\vec{B} = \hat{i} + p\hat{j} - 3\hat{k}$$
$$\vec{C} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

are coplanar.

Sol: If three vectors are coplanar then

$(\vec{a} \times \vec{b}).\ \vec{c} = 0$

So $\begin{vmatrix} 2 & -1 & 1 \\ 1 & p & -3 \\ 3 & 4 & 5 \end{vmatrix} = 0$

= 2 (5p+12) + 1 (5+9) + (4-3p) = 0
= 10p + 24 + 14 + 4 – 3p = 0
= 7p = -42
= p = -6

VECTOR TRIPLE PRODUCT:

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then the cross product of $\vec{a}$ and $\vec{b} \times \vec{c}$ or cross product of $\vec{a} \times \vec{b}$ and $\vec{c}$ is called vector triple product of $\vec{a}, \vec{b}, \vec{c}$ and is written as $\vec{a} \times (\vec{b} \times \vec{c})$ or $(\vec{a} \times \vec{b}) \times \vec{c}$ respectively.

NOTE: 1 $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

2 $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which lies in plane of $\vec{b}\ and\ \vec{c}$ and perpendicular to $\vec{a}$.

3 $(\vec{a} \times \vec{b}) \times \vec{c}$ is a vector which lies in plane of $\vec{a}\ and\ \vec{b}$ and perpendicular to $\vec{c}$.

QUESTIONS RELATED TO VECTOR TRIPLE PRODUCT:

Q1. find $\vec{a} \times (\vec{b} \times \vec{c})$, where $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$.

Sol: here $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{c} = \hat{i} + \hat{j} - \hat{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= (-2 - 3)\hat{i} - (-1 - 3)\hat{j} + (1 - 2)\hat{k}$$

$$\therefore \vec{b} \times \vec{c} = -5\hat{i} + 4\hat{j} - \hat{k}$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ -5 & 4 & -1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 4 & -5 \\ 4 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & -5 \\ -5 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 4 \\ -5 & 4 \end{vmatrix}$$

$$= (-4 + 20)\hat{i} - (-2 - 25)\hat{j} + (8 + 20)\hat{k}$$

$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = 16\hat{i} + 27\hat{j} + 28\hat{k}$$

Q2. PROVE THAT:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

sol: L.H.S = $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{a}.\vec{b})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{c})\vec{b}$

$[\therefore \vec{a}.\vec{b} = \vec{b}.\vec{a}]$

= 0.

Q3. PROVE THAT:

$$\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$$

sol: L.H.S = $\hat{\imath} \times (\vec{a} \times \hat{\imath}) + \hat{\jmath} \times (\vec{a} \times \hat{\jmath}) + \hat{k} \times (\vec{a} \times \hat{k})$

$= (\hat{\imath}.\hat{\imath})\vec{a} - (\hat{\imath}.\vec{a})\hat{\imath} + (\hat{\jmath}.\hat{\jmath})\vec{a} - (\hat{\jmath}.\vec{a})\hat{\jmath} + (\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}$

$= \vec{a} - (\vec{a}.\hat{\imath})\hat{\imath} + \vec{a} - (\vec{a}.\hat{\jmath})\hat{\jmath} + \vec{a} - (\vec{a}.\hat{k})\hat{k}$

$= 3\vec{a} - \{(\vec{a}.\hat{\imath})\hat{\imath} + (\vec{a}.\hat{\jmath})\hat{\jmath} + (\vec{a}.\hat{k})\hat{k}\}$

$= 3\vec{a} - \vec{a}$

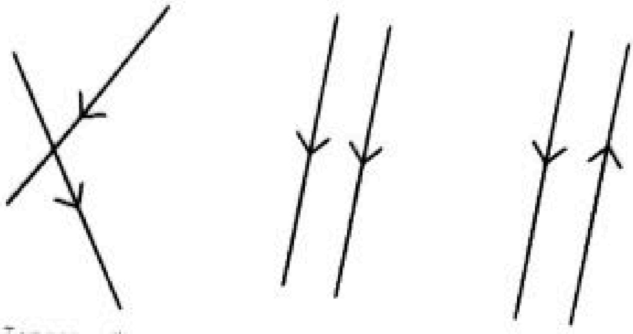
$= 2\vec{a}$

= R.H.S.

Collinear Vectors

Any two given vectors can be considered as collinear vectors if these vectors are parallel to the same given line. Thus, we can consider any two vectors as collinear vectors if and only if these two vectors are either along the same line or these vectors are parallel to each other. For any two vectors to be parallel to one another, the condition is that one of the vectors should be a scalar multiple of another vector.

Collinear Vectors



Q4. show that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ if and only if $\vec{a}$ *and* $\vec{c}$ are collinear.

Sol: (I) assume that $\vec{a}$ *and* $\vec{c}$ *are* collinear.
$\therefore \vec{c} = \lambda\vec{a}$ where λ is a scalar.

Now $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times (\vec{b} \times \lambda\vec{a}) = (\vec{a}.\lambda\vec{a})\vec{b} - (\vec{a}.\vec{b})\lambda\vec{a}$
$=\lambda[(\vec{a}.\vec{a})\vec{b} - (\vec{a}.\vec{b})\vec{a}]$

also $(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) = -\lambda\vec{a} \times (\vec{a} \times \vec{b})$
$= -\lambda[(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}] = \lambda[(\vec{a}.\vec{a})\vec{b} - (\vec{a}.\vec{b})\vec{a}]$

$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

assume that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times b) \times \vec{c}$

$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = -\vec{c} \times (\vec{a} \times \vec{b})$

$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}]$

$(\vec{c}.\vec{a})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}$

$\Rightarrow -(\vec{a}.\vec{b})\vec{c} = -(\vec{c}.\vec{b})\vec{a} \Rightarrow (\vec{a}.\vec{b})\vec{c} = (\vec{c}.\vec{b})\vec{a}$
$\Rightarrow \vec{a}$ *and* $\vec{c}$ are collinear

Q5. show that

$$[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = (\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$$

$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$$

$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{c}\times\vec{a})$$

$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{c}) = 2[\vec{a} \quad \vec{b} \quad \vec{c}]$$

Examples

1. Find is Collinear (A,B)
$A = (3, 4, 5)$, $B = (6, 8, 10)$

Solution:
Here, $A = (3, 4, 5)$, $B = (6, 8, 10)$

Two vectors are collinear if relations of their coordinates are equal.

$$\frac{A_1}{B_1} = \frac{3}{6} = \frac{1}{2}, \quad \frac{A_2}{B_2} = \frac{4}{8} = \frac{1}{2}, \quad \frac{A_3}{B_3} = \frac{5}{10} = \frac{1}{2}$$

Since $\frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3}$ So vectors are collinear

3. Find is Collinear(A, B)?
$A = (3, 4, 0)$, $B = (2, 2, 1)$

Solution:
Here, $A = (3, 4, 0)$, $B = (2, 2, 1)$

Two vectors $\vec{A}$ and $\vec{B}$ are collinear if there exists a number n such that $\vec{B} = n \cdot \vec{A}$

Find the first nonzero coefficient of vector $\vec{A}$

$A_1 = 3$

$n = \frac{B_1}{A_1} = \frac{2}{3} = 0.6667$

$0.6667 \cdot \vec{A}$

$= \left(0.6667 \cdot A_1, 0.6667 \cdot A_2, 0.6667 \cdot A_3\right)$

$= (0.6667 \cdot 3, 0.6667 \cdot 4, 0.6667 \cdot 0)$

$= (2, 2.6667, 0)$

$\neq \vec{B}$

Since $\vec{B} \neq 0.6667 \cdot \vec{A}$, so vectors are not collinear.

PRODUCT OF FOUR VECTORS

If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are any four vectors , then

1. *$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is scalar product and*
2. *$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is a vector product .*

a) Scalar product of four vectors :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ be any four vectors , then scalar product of these four vectors is given as :

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

QUES : Show that

$$(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{d}}) = \begin{vmatrix} \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}} & \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{c}} \\ \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{d}} & \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{d}} \end{vmatrix}$$

Proof We have L.H.S $= (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

$= (\vec{a} \times \vec{b}) \cdot \vec{A}$, where $\vec{A} = (\vec{c} \times \vec{d})$

$= \vec{a} \cdot (\vec{b} \times \vec{A})$ *[because of property of scalar triple product]*

$= \vec{a} \cdot [\vec{b} \times (\vec{c} \times \vec{d})]$

$= \vec{a} \cdot [(\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}]$ $[\because \vec{a} \times (\vec{b} \times \vec{c}) = [(\vec{b} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}]]$

$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}$

= **R.H.S** *hence proved.*

b) *Vector product of four vectors :*

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be any four vectors , then scalar product of these four vectors is given as :

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$$

or

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$$

QUES : Show that

I. $\boldsymbol{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}}$

Proof : We have L.H.S $= (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$= \vec{A} \times (\vec{c} \times \vec{d})$ *Here* $\vec{A} = (\vec{a} \times \vec{b})$

$= (\vec{A} \cdot \vec{d})\vec{c} - (\vec{A} \cdot \vec{c})\vec{d}$ $[\because \vec{a} \times (\vec{b} \times \vec{c}) = [(\vec{b} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}]]$

$= ((\vec{a} \times \vec{b}) \cdot \vec{d})\vec{c} - ((\vec{a} \times \vec{b}) \cdot \vec{c})\vec{d}$

$= [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$ $[\because \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]]$

= R.H.S ***Hence proved.***

QUES : Show that

$\boldsymbol{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}}$

Proof : We have L.H.S $= (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$= (\vec{a} \times \vec{b}) \times \vec{B}$ Here $\vec{B} = (\vec{c} \times \vec{d})$

$= -\vec{B} \times (\vec{a} \times \vec{b})$ $[\because \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})]$

$= -(\vec{B} \cdot \vec{b})\vec{a} - (\vec{B} \cdot \vec{a})\vec{b}$ $[\because \vec{a} \times (\vec{b} \times \vec{c}) = [(\vec{b} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{a})\vec{b}]]$

$= -((\vec{c} \times \vec{d}) \cdot \vec{b})\vec{a} + ((\vec{c} \times \vec{d}) \cdot \vec{a})\vec{b}$ $[\because \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]]$

$= [\vec{c}\ \vec{d}\ \vec{a}]\,\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a}$

Thus $\boldsymbol{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}}$

Hence proved.

QUES : Prove that
$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$

Proof : We have given L.H.S $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

Now by using the value of scalar product of four vectors

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) \quad ...(1)$$

Thus by using equation (1) we get

L.H.S $\Rightarrow (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d})-(\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})+(\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$
$+(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

Now As we know $(\vec{a}\cdot\vec{b})=(\vec{b}\cdot\vec{a})$, $(\vec{a}\cdot\vec{c})=(\vec{c}\cdot\vec{a})$, $(\vec{a}\cdot\vec{d})=(\vec{d}\cdot\vec{a})$

(similarly all other vectors obey commutative property.)

L.H.S $\Rightarrow (\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})-(\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})+(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$
$+(\vec{c}\cdot\vec{a})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$=0=$R.H.S ***hence proved.***

<u>RECIPROCAL SYSTEM OF VECTORS</u>

The set of vector $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}\,', \vec{b}', \vec{c}\,'$ are called reciprocal set if
$\vec{a}\,\vec{a}\,'=\vec{b}\vec{b}\,'=\vec{c}\,\vec{c}\,'=1$.
$\vec{a}\,'\,\vec{b}=\vec{a}\,'\,\vec{c}=\vec{b}\,'\vec{a}=\vec{b}\,'\,\vec{c}=\vec{c}\,'\,\vec{a}=\vec{c}\,'\,\vec{b}=0$.

OR

If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non collinear and non-coplanar vectors, then the three vectors $\vec{a}\,', \vec{b}', \vec{c}\,'$ given by

$$\vec{a}\,'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}\,'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c}\,'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

are the reciprocal system to the vectors $\vec{a}, \vec{b}, \vec{c}$.

<u>QUESTIONS</u>

Q: Prove that the orthogonal vector triad $\hat{i}, \hat{j}, \hat{k}$ is self reciprocal.

Sol: Let $\hat{i}', \hat{j}', \hat{k}'$ be reciprocal vectors of $\hat{i}, \hat{j}, \hat{k}$ respectively.

Now, $\hat{i}' = \frac{\hat{j}\times\hat{k}}{[\hat{i}, \hat{j}, \hat{k}]} = \frac{\hat{i}}{1} = \hat{i}$. $\quad$ (*because*, $[\hat{i}, \hat{j}, \hat{k}]=1$)

$\hat{j}' = \frac{\hat{k}\times\hat{i}}{[\hat{i}, \hat{j}, \hat{k}]} = \frac{\hat{j}}{1} = \hat{j}$.

$\hat{k}' = \frac{\hat{i}\times\hat{j}}{[\hat{i}, \hat{j}, \hat{k}]} = \frac{\hat{k}}{1} = \hat{k}$.

$\therefore$ orthogonal vector triad $\hat{i}, \hat{j}, \hat{k}$ is self reciprocal.

Q: If $\vec{a}.\vec{b}\times\vec{c} \neq 0$ and $\vec{a}' = \frac{\vec{b}\times\vec{c}}{\vec{a}.\vec{b}\times\vec{c}}$, $\vec{b}' = \frac{\vec{c}\times\vec{a}}{\vec{b}.\vec{c}\times\vec{a}}$, $\vec{c}' = \frac{\vec{a}\times\vec{b}}{\vec{c}.\vec{a}\times\vec{b}}$

show that $\vec{a}.\vec{a}' + \vec{b}.\vec{b}' + \vec{c}.\vec{c}' = 3$.

Sol: Here $\vec{a}' = \frac{\vec{b}\times\vec{c}}{\vec{a}.\vec{b}\times\vec{c}}$, $\vec{b}' = \frac{\vec{c}\times\vec{a}}{\vec{b}.\vec{c}\times\vec{a}}$, $\vec{c}' = \frac{\vec{a}\times\vec{b}}{\vec{c}.\vec{a}\times\vec{b}}$

L.H.S= $\vec{a}.\vec{a}' + \vec{b}.\vec{b}' + \vec{c}.\vec{c}'$

$= \vec{a}.\frac{\vec{b}\times\vec{c}}{\vec{a}.\vec{b}\times\vec{c}} + \vec{b}.\frac{\vec{c}\times\vec{a}}{\vec{b}.\vec{c}\times\vec{a}} + \vec{c}.\frac{\vec{a}\times\vec{b}}{\vec{c}.\vec{a}\times\vec{b}}$

$= \frac{\vec{a}.\vec{b}\times\vec{c}}{\vec{a}.\vec{b}\times\vec{c}} + \frac{\vec{b}.\vec{c}\times\vec{a}}{\vec{b}.\vec{c}\times\vec{a}} + \frac{\vec{c}.\vec{a}\times\vec{b}}{\vec{c}.\vec{a}\times\vec{b}}$

$=1+1+1=3$

=R.H.S.

Q: Find the reciprocal of vectors. And verify $[\vec{a}'\ \vec{b}'\ \vec{c}']\ [\vec{a}\ \vec{b}\ \vec{c}]=1$

given

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$

$\vec{b}=\hat{i}-\hat{j}-2\hat{k}$

$\vec{c}=-\hat{i}+2\hat{j}+2\hat{k}$.

Sol: Given $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}-2\hat{k}$, $\vec{c}=-\hat{i}+2\hat{j}+2\hat{k}$

We know that reciprocal system is given as

$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Now $[\vec{a}\vec{b}\ \vec{c}\,] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 3$

Now $\vec{b}\times\vec{c}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ -1 & 2 & 2 \end{vmatrix}=2\hat{\imath}+\hat{k}$

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}=2\hat{\imath}/3+\hat{k}/3$$

$$\vec{c}\times\vec{a}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix}=-8\hat{\imath}+3\hat{\jmath}-7\hat{k}$$

$$\vec{b}\,'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}=-8\hat{\imath}/3+\hat{\jmath}-7\hat{k}/3$$

$$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix}=7\hat{\imath}+3\hat{\jmath}-5\hat{k}$$

$$\vec{c}\,'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}=7\hat{\imath}/3+\hat{\jmath}-5\hat{k}/3$$

Therefore reciprocal system is given as,

$$[\vec{a}'\ \vec{b}\,'\vec{c}']=1/27\begin{vmatrix} 2 & 0 & 1 \\ -8 & 3 & -7 \\ -7 & 3 & -5 \end{vmatrix}=\frac{1}{3}$$

Further $[\vec{a}'\ \vec{b}\,'\vec{c}'][\vec{a}\ \vec{b}\ \vec{c}\,]=\frac{1}{3}\times 3=1$

Hence verified.

DERIVATIVE OF A VECTOR-VALUED FUNCTION

The derivative $r'(t)$ of the vector-valued function $r(t)$ is defined by

$$\frac{dr}{dt} = r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t)-r(t)}{\Delta t}$$

for any values of t for which the limit exists.

The vector $r'(t)$ is called the tangent vector to the curve defined by r.

If $r(t) = \langle f(t), g(t), h(t)\rangle$ where f,g,h are differentiable functions, then

$$r'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Thus we can differentiate vector-valued functions by differentiating their component functions.

PHYSICAL INTERPRETATION

If $r(t)$ represents the position of a particle , then the derivative is the velocity of the particle:

$$v(t) = \frac{dr}{dt} = r'(t)$$

In a similar way, the derivative of the velocity is acceleration:

$$a(t) = \frac{dv}{dt} = v'(t) = r''(t)$$

CONSTANT VECTOR

A vector having constant magnitude and constant direction is said to be a constant vector.

Note:- The derivative of a constant vector is zero.

Example : $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

here, $\vec{a}$ is a constant vector and the derivative of $\vec{a}$ is equal to

$$\frac{d\vec{a}}{dt} = \frac{d(1)}{dt}\hat{i} + \frac{d(2)}{dt}\hat{j} + \frac{d(3)}{dt}\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

Thus, it is proved that derivative of a constant vector is zero.

Scalar function: Let 'D' be any subset of set of real numbers, then a rule denoted by 'f' which associates to each scalar 't' belonging to 'D' a unique scalar that is, $f(t)$ is called scalar function of scalar variable 't'.

Vector function: Let 'D' be any subset of set of real numbers, then a rule denoted by 'f' which associates to each scalar 't' belonging to 'D' a unique vector that is, $\vec{f}(t)$ is called vector function of scalar variable 't'.

Limit of a vector function:- $\lim_{t \to t_0} \vec{f}(t) = \vec{l}$

OR $\quad |\vec{f}(t) - \vec{l}| < \epsilon, \qquad 0 < |t - t_0| < \delta$

where δ depends upon ϵ

Note :-

- $\frac{d}{dt}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$
- $\frac{d}{dt}(\vec{A}.\vec{B}) = \vec{B}.\frac{d\vec{A}}{dt} + \vec{A}.\frac{d\vec{B}}{dt}$
- $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$
- $\frac{d}{dt}(\phi\vec{A}) = \phi\frac{d\vec{A}}{dt} + \vec{A}\frac{d\phi}{dt}$ Where ϕ is some scalar function of 't'.

QUES-1: Prove that

$\vec{r}(t)$ ***is a constant vector if and only if*** $\frac{d\vec{r}}{dt} = 0$ ***and conversely.***

SOLUTION: Given $\vec{r}(t)$ = constant = $\vec{c}$ where $\vec{c}$ is a constant vector

$\therefore \vec{r}(t + \delta t) = constant \Rightarrow \vec{r}(t + \delta t) = \vec{r}(t)$ -----(1)

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t) - \vec{r}(t)}{\delta t} = 0$$

[from (1) $\quad \vec{r}(t + \delta t) = \vec{r}(t)$]

$\Rightarrow \frac{d\vec{r}}{dt} = 0$

Conversely: $\frac{d\vec{r}}{dt} = 0$ (given)

Let $\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$ ------(2)

Where r_1, r_2 and r_3 *are functions of 't'.*

Now $\frac{d\vec{r}}{dt} = 0 \Rightarrow \frac{d}{dt}(r_1\hat{i} + r_2\hat{j} + r_3\hat{k}) = 0$

$$\frac{dr_1}{dt}\hat{i} + \frac{dr_2}{dt}\hat{j} + \frac{dr_3}{dt}\hat{k} = \vec{0}$$

$\Rightarrow \frac{dr_1}{dt}\hat{i} + \frac{dr_2}{dt}\hat{j} + \frac{dr_3}{dt}\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow \frac{dr_1}{dt} = 0 \Rightarrow r_1$=constant

Similarly, r_2=constant and r_3=constant

Now by taking r_1, r_2 and r_3 is equal to constant in (2), we get

$\Rightarrow \vec{r}$ = *constant* OR $\vec{r}(t)$ = *constant*

QUES-2 : *If* $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$, *where,* $\vec{a}, \vec{b}$ *and* ω *are constants*

Then prove that i) $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$ ii) $\vec{r} \times \frac{d\vec{r}}{dt} = -\omega(\vec{a} \times \vec{b})$

SOLUTION :

Given $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$ -------(1)

i) So, $\frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{a}\sin\omega t + \vec{b}\cos\omega t) = \frac{d}{dt}(\vec{a}\sin\omega t) + \frac{d}{dt}(\vec{b}\cos\omega t)$

$\Rightarrow \frac{d\vec{r}}{dt} = \vec{a}\frac{d}{dt}(\sin\omega t) + \vec{b}\frac{d}{dt}(\cos\omega t) = \vec{a}\,\omega\cos\omega t + \vec{b}\,\omega(-\sin\omega t)$

$\Rightarrow \frac{d\vec{r}}{dt} = \vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t$ ----(2)

Now, $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(\vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t)$

$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}(\vec{a}\,\omega\cos\omega t) - \frac{d}{dt}(\vec{b}\,\omega\sin\omega t)$

$\Rightarrow \frac{d^2\vec{r}}{dt^2} = \vec{a}\,\omega^2(-\sin\omega t) - \vec{b}\,\omega^2\cos\omega t$

$\frac{d^2\vec{r}}{dt^2} = -\vec{a}\,\omega^2\sin\omega t - \vec{b}\,\omega^2\cos\omega t = -\omega^2(\vec{a}\sin\omega t + \vec{b}\cos\omega t)$

[from (1) we know $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$ *]*

$\therefore \frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$ ***Hence proved*** *(ii) By taking the values of* $\vec{r}$ *and* $\frac{d\vec{r}}{dt}$ *from (1) and (2), we have*

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{a}\sin\omega t & \vec{b}\cos\omega t \\ \vec{a}\omega\cos\omega t & -\vec{b}\omega\sin\omega t \end{vmatrix}$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = -\omega(\vec{a} \times \vec{b})\sin^2\omega t - \omega(\vec{a} \times \vec{b})\cos^2\omega t$$

$$\Rightarrow \vec{r} \times \frac{d\vec{r}}{dt} = -\omega(\vec{a} \times \vec{b})(\sin^2\omega t + \cos^2\omega t) = -\omega(\vec{a} \times \vec{b})$$ **Hence proved**

$$[as \ \sin^2\omega t + \cos^2\omega t = 1]$$

__QUES-3__ : *If* $\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ *and* $\vec{B} = \sin t\,\hat{i} + \cos t\,\hat{j}$ *then find* $\frac{d}{dt}(\vec{A}.\vec{B})$.

__SOLUTION__ :

As we know, $\frac{d}{dt}(\vec{A}.\vec{B}) = \vec{B}.\frac{dA}{dt} + \vec{A}.\frac{dB}{dt}$

$$\therefore \frac{d}{dt}(\vec{A}.\vec{B}) = (\sin t\,\hat{i} + \cos t\,\hat{j}).\frac{d}{dt}(5t^2\hat{i} + t\hat{j} - t^3\hat{k}) + (5t^2\hat{i} + t\hat{j} - t^3\hat{k}).\frac{d}{dt}(\sin t\,\hat{i} + \cos t\,\hat{j})$$

$$\Rightarrow \frac{d}{dt}(\vec{A}.\vec{B}) = (\sin t\,\hat{i} + \cos t\,\hat{j}).\left\{\frac{d}{dt}(5t^2\hat{i}) + \frac{d}{dt}(t\hat{j}) - \frac{d}{dt}(t^3\hat{k})\right\} + (5t^2\hat{i} + t\hat{j} - t^3\hat{k}).\left\{\frac{d}{dt}(\sin t\,\hat{i}) + \frac{d}{dt}(\cos t\,\hat{j})\right\}$$

$$\frac{d}{dt}(\vec{A}.\vec{B}) = (\sin t\,\hat{i} + \cos t\,\hat{j}).(10t\hat{i} + \hat{j} - 3t^2\hat{k}) + (5t^2\hat{i} + t\hat{j} - t^3\hat{k}).(\cos t\,\hat{i} - \sin t\,\hat{j})$$

[as we know $\hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1$ *and* $\hat{i}.\hat{j} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0$*]*

$$\therefore \frac{d}{dt}(\vec{A}.\vec{B}) = 10t\sin t + \cos t + 5t^2\cos t - t\sin t$$

$$\frac{d}{dt}(\vec{A}.\vec{B}) = 9t\sin t + \cos t + 5t^2\cos t$$

Or $\frac{d}{dt}(\vec{A}.\vec{B}) = 9t\sin t + \cos t(1 + 5t^2)$ __Answer__

Partial differentiation of a vector function

Firstly , A **partial derivative** is the rate of change of a multi-variable function when we allow only one of the variables to change. Specifically, we differentiate with respect to only one variable, regarding all others as constants (now we see the relation to partial functions!). Which essentially means if you know how to take a derivative, you know how to take a partial derivative.

A partial derivative of a function f with respect to a variable x, say $z=f(x,y_1,y_2,...y_n)$ (where the y_i' s are other independent variables) is commonly denoted in the following ways:

$\frac{\partial z}{\partial x}$ *(referred to as ``partial z, partial x'')*

$\frac{\partial f}{\partial x}$ *(referred to as ``partial f, partial x'')*

$\frac{\partial^2 f}{\partial x\,\partial y}$ *(referred to as ``partial f, partial x , partial z'')*

Vector function and its partial differentiation

Q. If $\vec{A} = x^2yz\hat{\imath} - 2xz^3\hat{\jmath} + xz^2\hat{k}$ and $\vec{B} = 2z\hat{\imath} + y\hat{\jmath} - x^2\hat{k}$, then find the value of $\frac{\partial^2 f}{\partial x\partial y}(\vec{A}\times\vec{B})$ at point (1,0,-2)

Sol. The cross product of 2 vectors $\vec{A}$ *and* $\vec{B}$ be given as :

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x^2yz & -2xz^3 & xz^3 \\ 2z & y & -x^2 \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \hat{\imath}\,(2x^3z^3 - xyz^2) - \hat{\jmath}(-x^4yz - 2xz^3) + (x^2y^2z + 4xz^4)\hat{k}$$

Now partially differentiating above term w.r.t. y, we get

$$\frac{\partial(\vec{A}\times\vec{B})}{\partial y} = -xz^2\hat{\imath} + x^4z\hat{\jmath} + 2x^2yz\hat{k}$$

Now partially differentiating above term w.r.t. x, we get

$$\frac{\partial^2(\vec{A}\times\vec{B})}{\partial x\partial y} = -z^2\hat{\imath} + 4x^3z\hat{\jmath} + 4xyz\hat{k}$$

Above equation at point (1,0,-2) is

$$\frac{\partial^2(\vec{A}\times\vec{B})}{\partial x\partial y} = -4\hat{\imath} + 4(-2)\hat{\jmath} = -4\hat{\imath} - 8\hat{\jmath}$$