Wavelet Transform

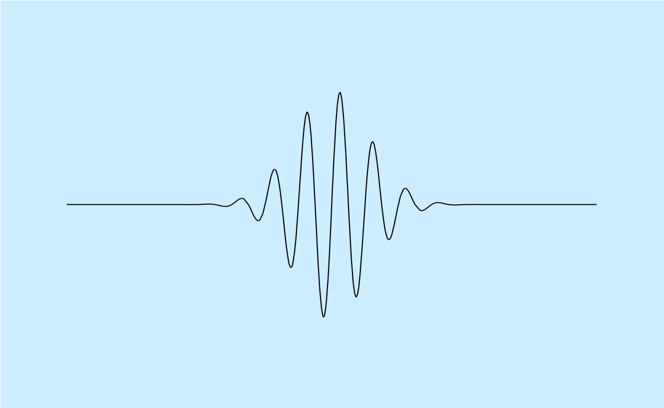
ICT4201

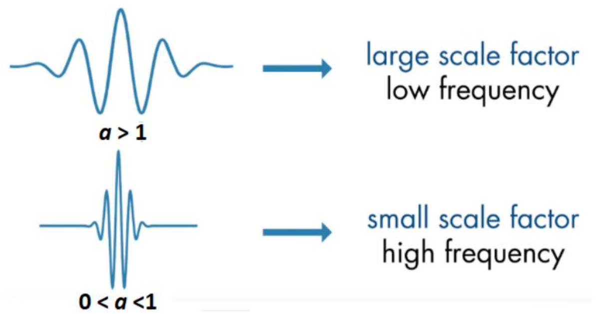
Preview

- Since 1950 Fourier Transform is used for image transformation from frequency to time domain.
- There are some limitation of Fourier Transform.
- Wavelet transform makes image easier to compress, transmit and analyzed.
  - In FT sinusoid functions of infinite durations are used for signal analysis
  - WT a small wave with varying frequency with limited duration.
  - WT was introduced in 1987 as a powerful new approach for signal processing.

What is wavelet?

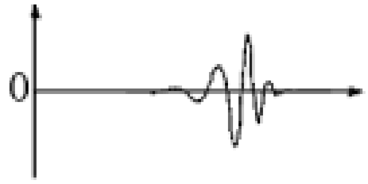
- Wavelet is a piece of wave/little wave.
- Wavelet means ripple or small oscillation.
- Wavelets can be time and amplitude varying using shifting and scaling functions.
- Scale
  - $a > 1$: dilate the signal
  - $a < 1$: compress the signal
- Shifting
  - Shifting of wavelet by τ





$$\Psi^{*}_{\tau,a}(t) = \frac{1}{\sqrt{a}}\Psi\left(\frac{t-\tau}{a}\right)$$

Wavelet function
$\psi(t)$



Shifted wavelet function
$\psi(t-k)$

Wavelet Transform

✓Wavelet is an oscillatory function of finite duration.

✓A sinusoidal wave $y(t) = A\cos\omega_c t$ is continuous over the interval $[\infty, -\infty]$ and the energy of the signal is infinite over the interval $[\infty, -\infty]$ known as energy signal. The energy of $y(t)$ is uniformly distributed over the entire interval.

✓If the wave $y(t)$ is modulated by a smooth Gaussian window function $g(t) = e^{-t^2}$, the modulated wave,

$$\psi(t) = g(t)y(t) = Ae^{-t^2}\cos(\omega_c t)$$

is also continuous over the interval $[\infty, -\infty]$ but almost all of its energy is confined within a small interval

Wavelet Transform

- Transform changes signals into another form, it is a mathematical mode for changing signal domain.
- Wavelet transform changes signal or image form time domain to time and frequency domain.
- We can use wavelet transform for multiresolution signal processing.
- Multiresolution processing provides a framework for extracting image of various levels of resolution.

FT vs WT

Fourier Transform	Wavelet Transform
1. The basis functions of Fourier Transform are sinusoids	1. The basis functions of WT are small waves of varying frequency.
2. FT is used to analyze signals by converting signals into a continuous series of sine and cosine functions each with a constant frequency and amplitude and of infinite duration.	2. WT converts a signal into a series of wavelets. WT basis function basis functions are obtained by scaling and shifting the mother wavelet.
3. Fourier Transform only provides the frequency information.	3. Wavelet Transform provides both time and frequency information.
4. During the transformation process, the temporal (time) information is lost.	4. During the transformation process, the temporal (time) information is not lost.
5. Fourier Transform only provides the notes or frequency information of musical score.	5. Wavelets are also provides an image with the equivalent of musical score revealing not only the appropriate times to play each note and frequency but also the notes of themselves.
6. Fourier Transform removes time information.	6. Wavelets do not remove information but move them it around, spreading out the noise and averaging signal.

FT vs WT

- Fourier transform breaks up a signal into sine waves of infinite number of frequencies and amplitudes. Wavelet analysis transform breaks up a signal into the shifted and scaled versions of the original (or mother) wavelet.

- Actually the continuous wavelet transform provides a time-frequency representation of a signal that offers very good time and frequency resolution.

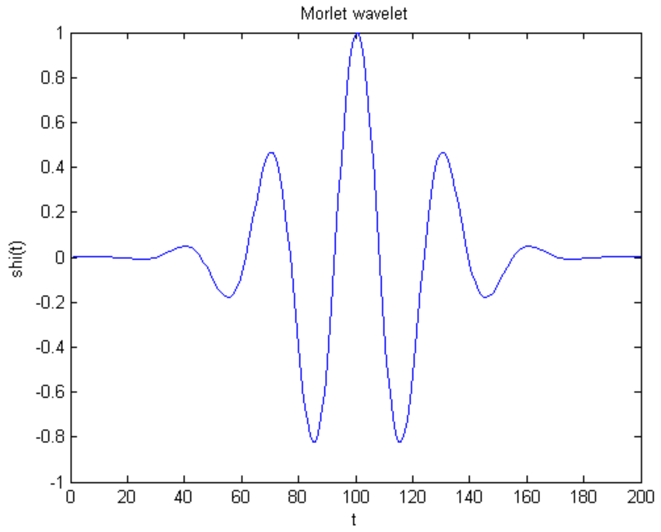
Morlet wavelet,

$$\psi(t) = e^{-t^2}\cos\left(\pi\sqrt{\frac{2}{\ln 2}}\,t\right) \text{ ;where } \omega_c = \pi\sqrt{\frac{2}{\ln 2}}$$

Here more than 99% of total energy of $\psi(t)$ lies in the interval $|t| \leq 2.5$

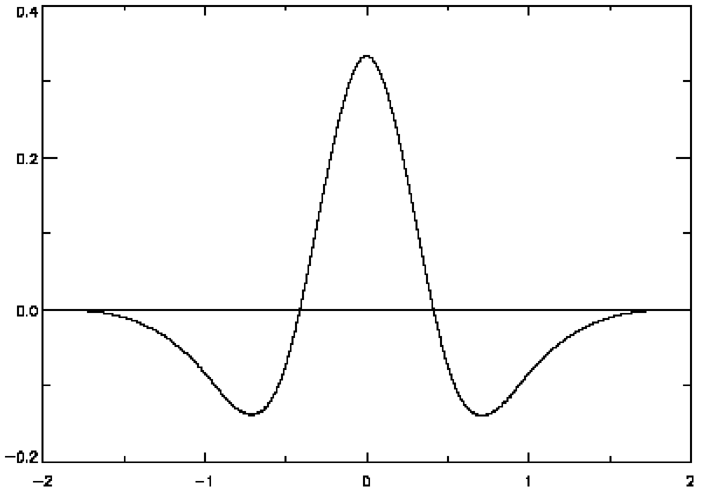
Mexican hat wavelet,

$$\psi(t) = (1-2t^2)e^{-t^2}$$

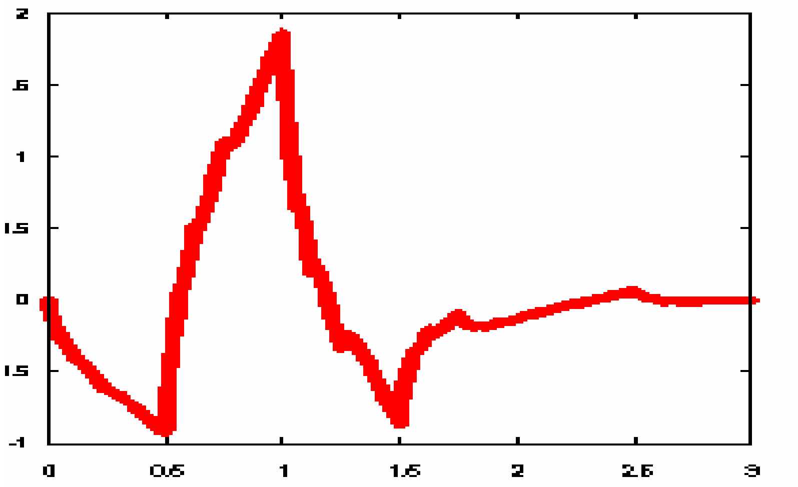


Morlet wavelet is shown in fig. above.

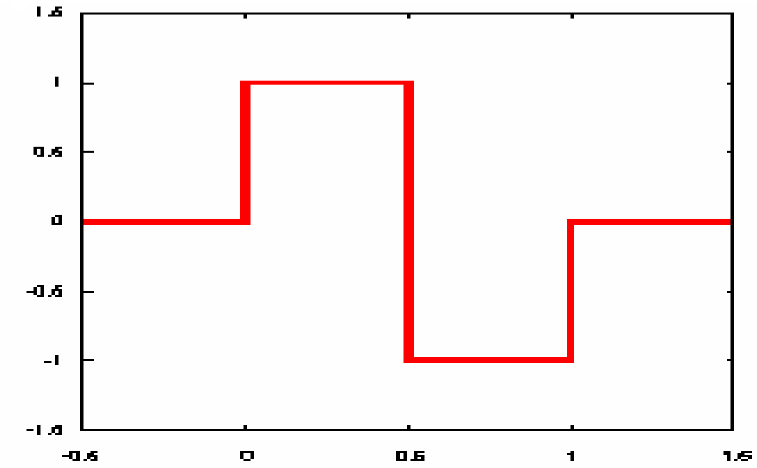
Example of wavelets



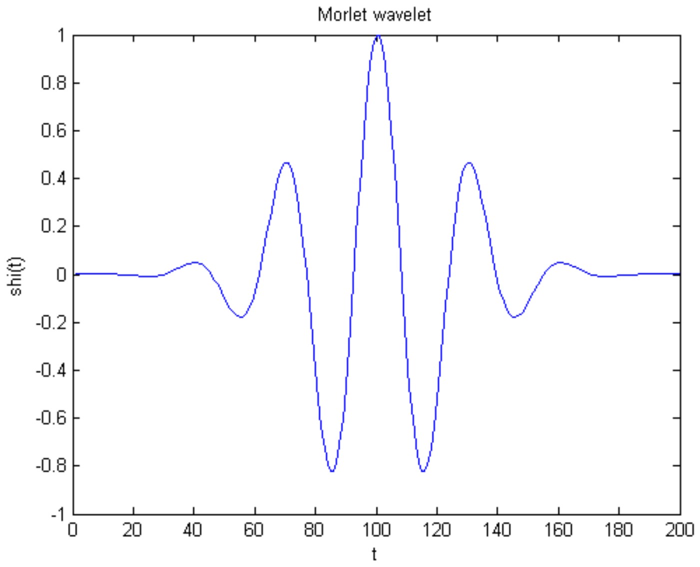
The mexican hat wavelet



Daubechhies Wavelet



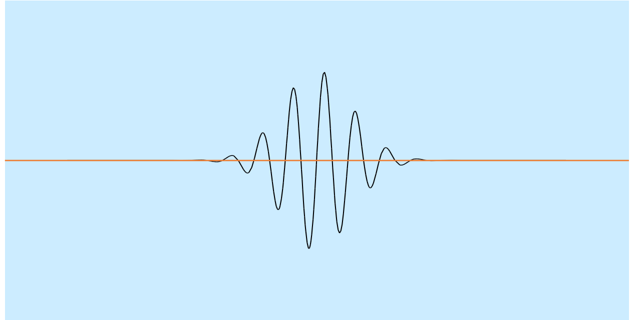
Haar Wavelet



Morlet wavelet

Wavelet function *Ψ(t)* has two main properties :

$$\int_{-\infty}^{\infty} \Psi(t)dt = 0;$$



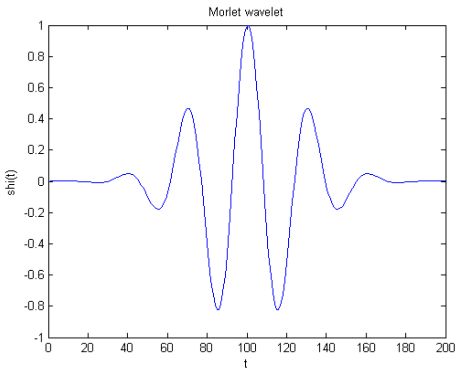
This means, the function is oscillatory or has wavy appearance.

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt < \infty;$$

This implies that most of the energy in *Ψ(t)* is confined to a finite duration.

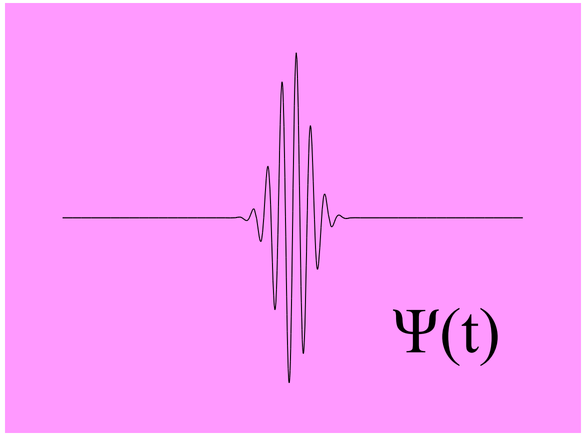
Wavelet is expressed in standard form like,

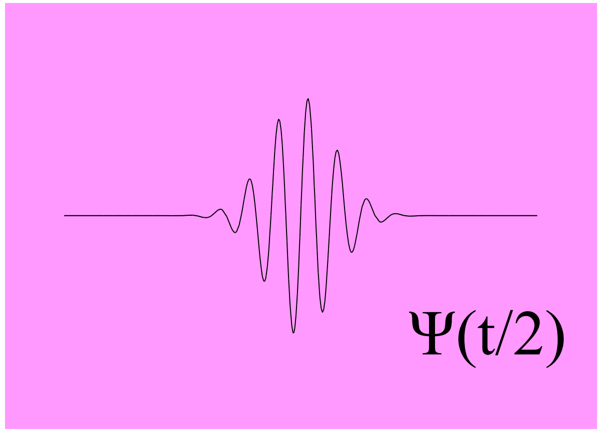
$$\psi_{a,b} = \psi\left(\frac{t-b}{a}\right)dt$$

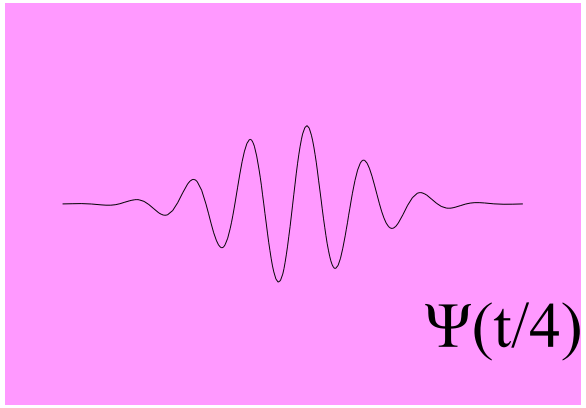


; where a and b are the scaling and shifting parameters. Scaling a wavelet simply means stretching (or compressing) it. The smaller the scale factor, the more "compressed" the wavelet. Shifting a wavelet simply means delaying (or hastening) its onset.

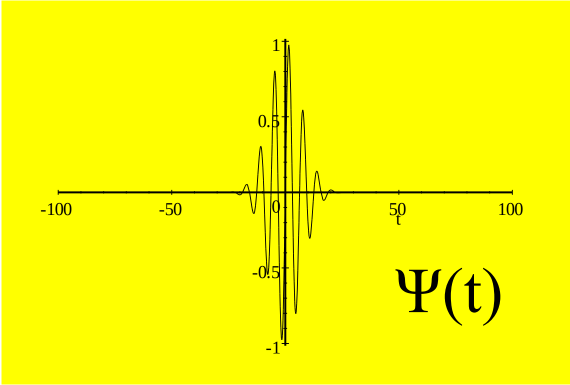
A family built by dilation

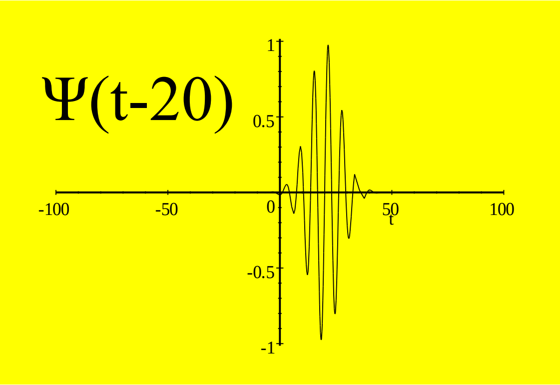


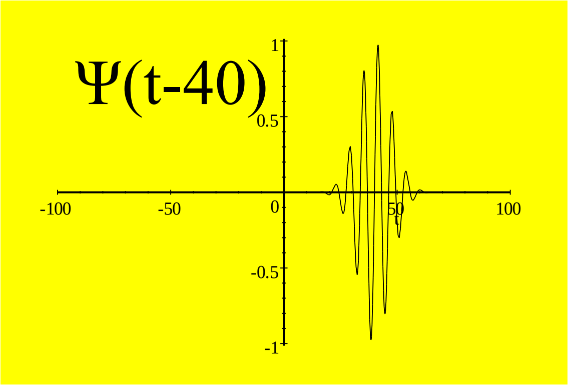




A family built by translation :







Example - 1

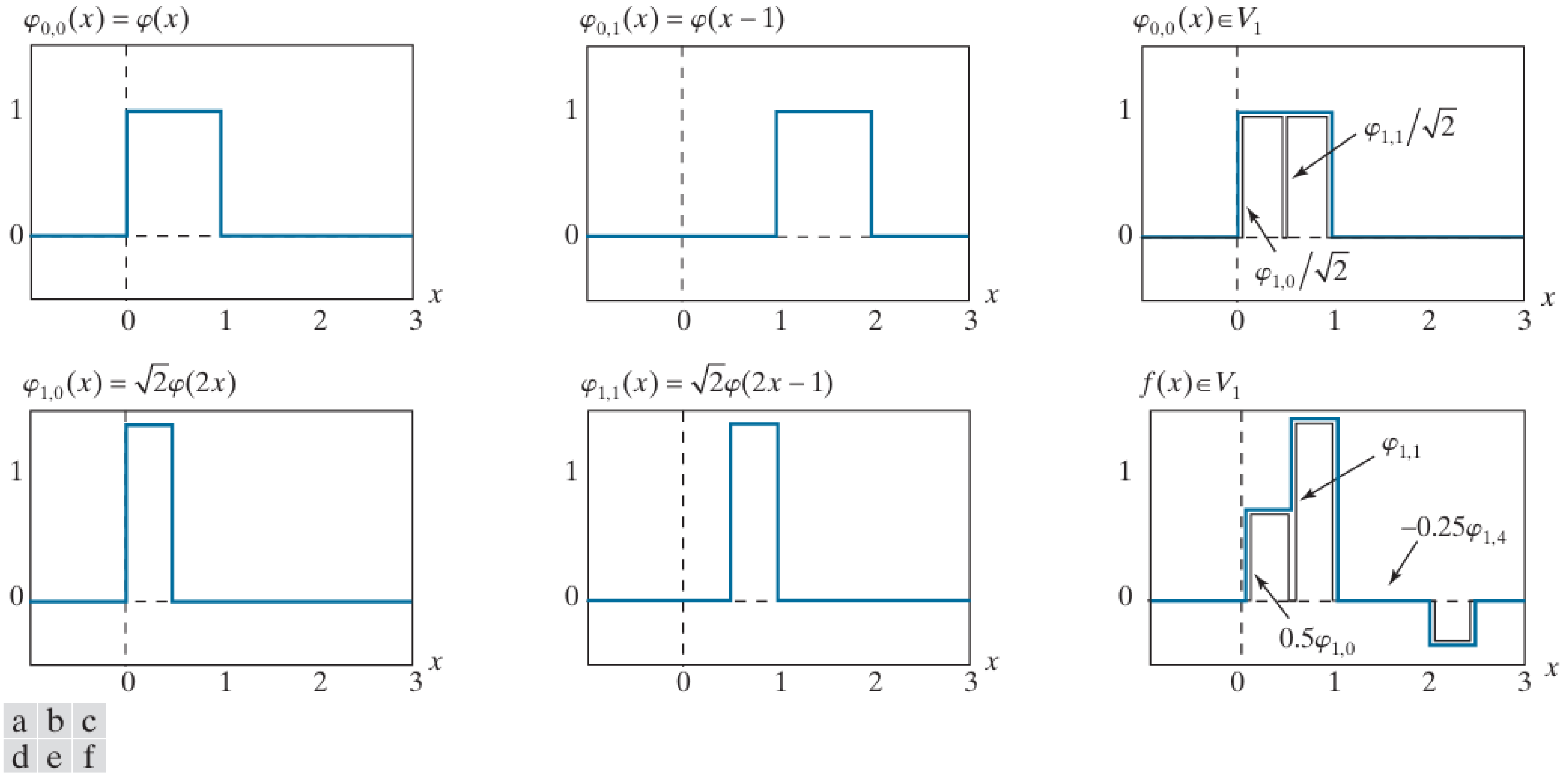


FIGURE 7.19 The Haar scaling function.

Example - 2

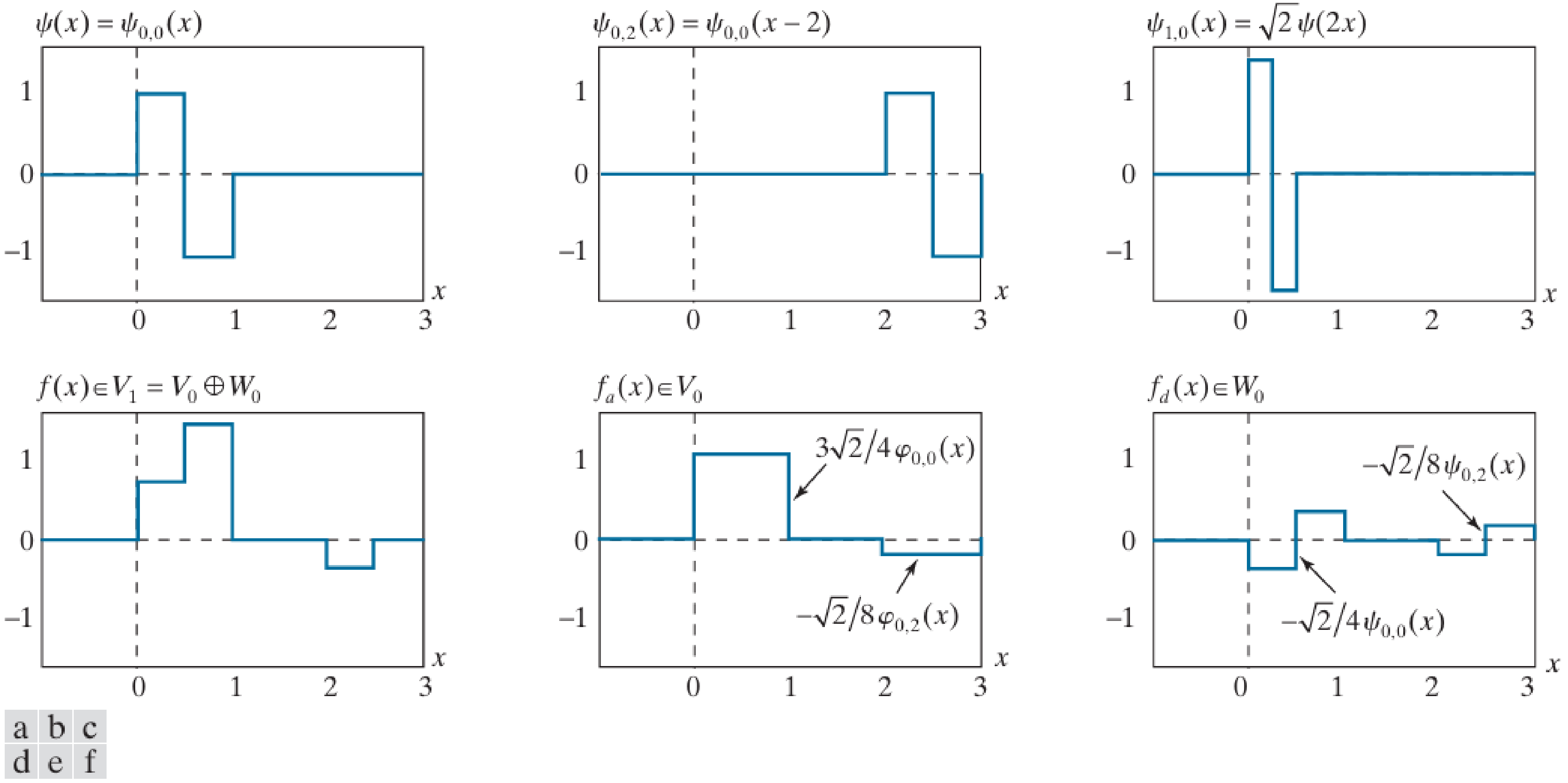


FIGURE 7.21 Haar wavelet functions.

The Continuous Wavelet Transform

Let $f(t)$ be any square integrable function. The CWT or continuous-time wavelet transform of f(t) with respect to a wavelet $\psi(t)$ is defined as

$$W(a,b) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{|a|}} \psi^{*}\left(\frac{t-b}{a}\right) dt \tag{1}$$

Where a and b are real and * denotes conjugation.

Inverse CWT operation can be expressed like,

$$f(t) = \frac{1}{C}\int_{a=-\infty}^{\infty}\int_{b=-\infty}^{\infty}\frac{1}{a^2}W(a,b)\psi_{a,b}(t)\,da\,db$$

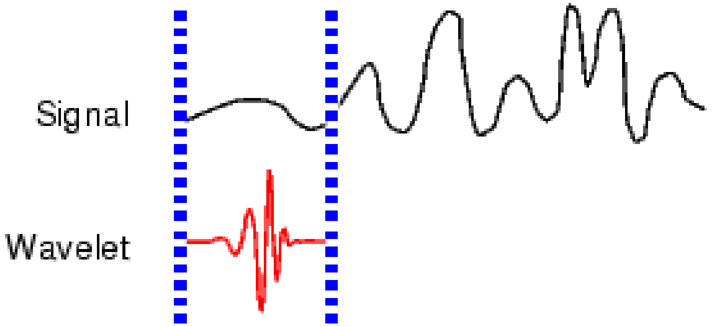
where $C = \int_{-\infty}^{\infty}\frac{|\psi(\omega)|^2}{|\omega|}d\omega$ and

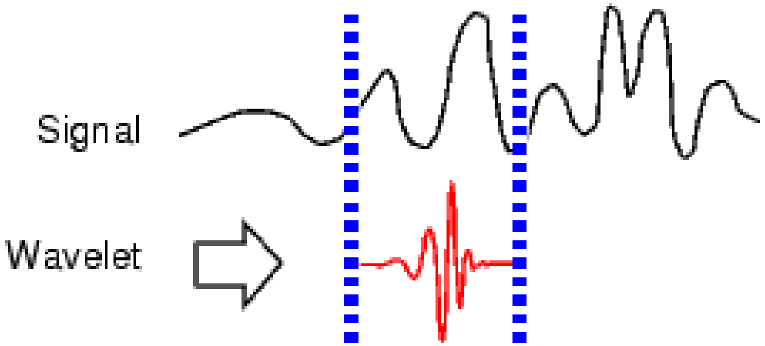
$$\psi(t) \leftrightarrow \psi(\omega)$$

COMPUTATION OF CWT

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^{*}\left(\frac{t-\tau}{s}\right) dt$$

- Step 1: The wavelet is placed at the beginning of the signal, and set s = 1 (the most compressed wavelet);
- Step 2: The wavelet function at scale “1” is multiplied by the signal, and integrated over all times; then multiplied by $1/\sqrt{s}$;
- Step 3: Shift the wavelet to $t = \tau$, and get the transform value at $t = \tau$ and $s = 1$;



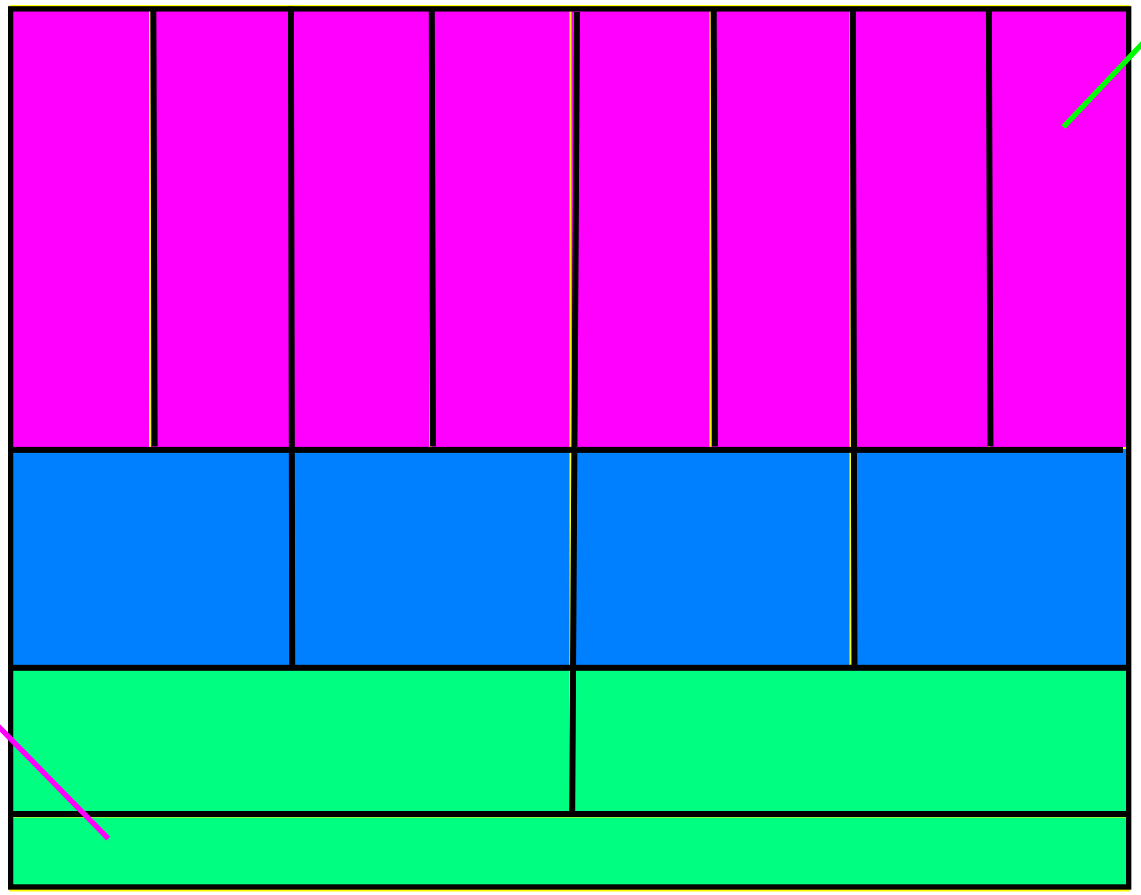


Step 4: Repeat the procedure until the wavelet reaches the end of the signal;
Step 5: Scale s is increased by a sufficiently small value, the above procedure is repeated for all *s;*
Step 6: Each computation for a given *s* fills the single row of the time-scale plane;
Step 7: CWT is obtained if all s are calculated.

RESOLUTION OF TIME & FREQUENCY

Better time resolution; Poor frequency resolution

Frequency

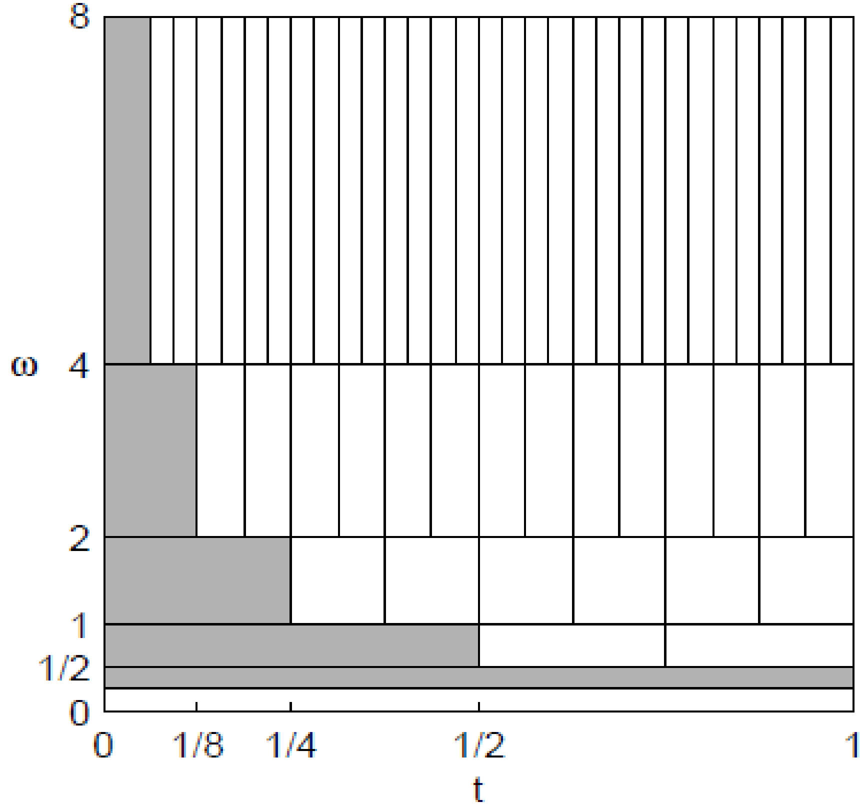


Time

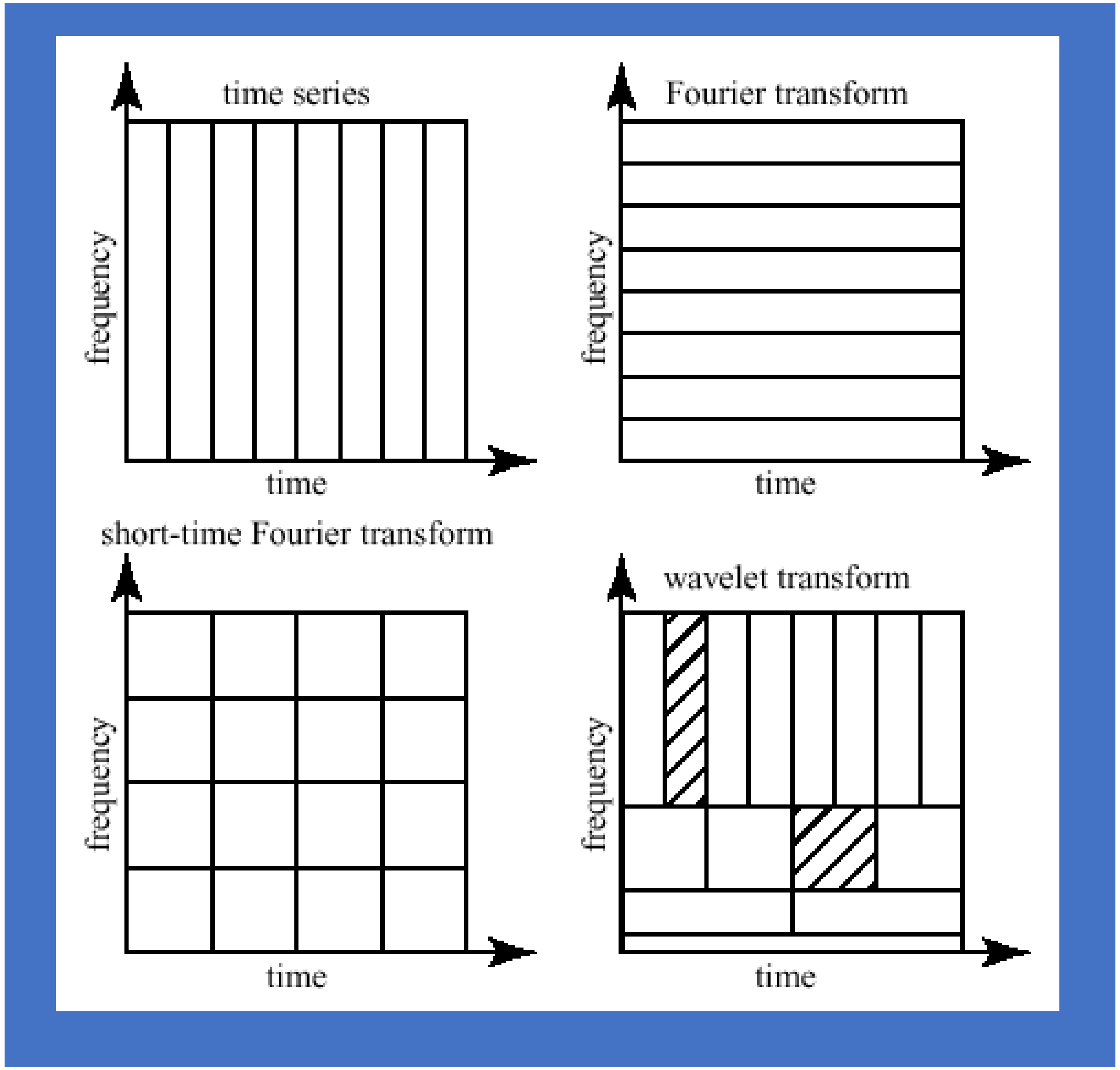
Better frequency resolution; Poor time resolution

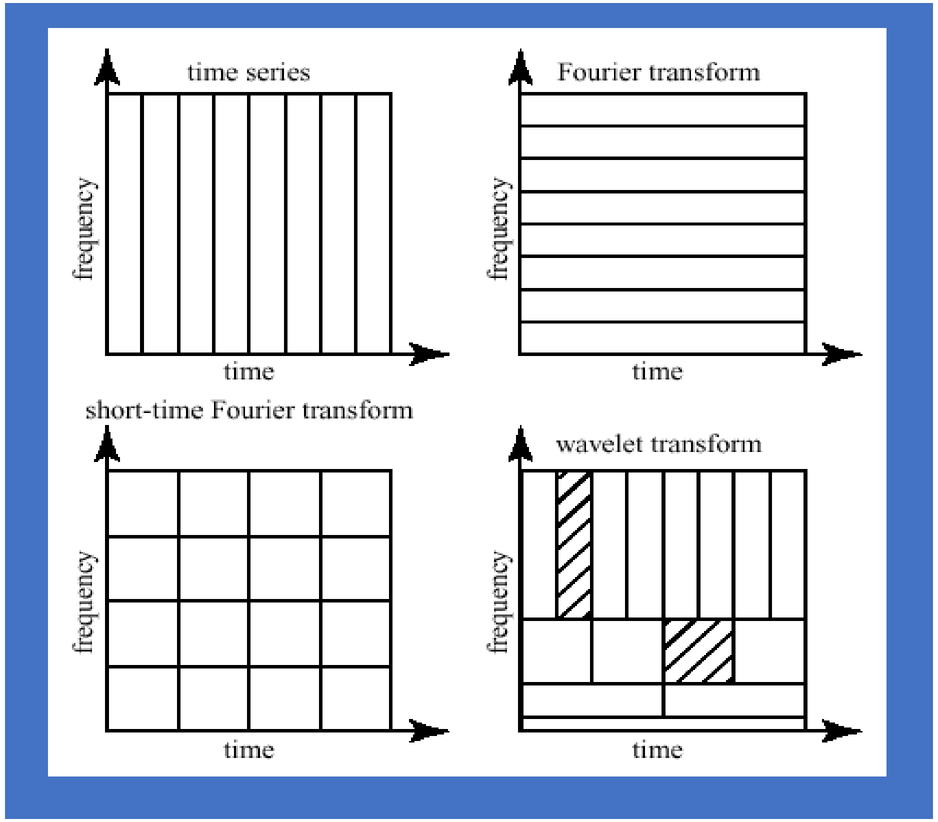
- Each box represents a equal portion
- Resolution in STFT is selected once for entire analysis

In signal analysis, analogously, this is stated as $\triangle\omega\ \triangle t = c$ for the frequency ω and the time t. If a pulse in the time space is much wider, which means that we are less definite in the time space (low resolution). It implies high resolution in frequency space. For example a wider sinc(t) pulse in time domain results in narrow rectangular pulse in frequency space. This is illustrated schematically in Figure below based on, for example, $\triangle\omega\ \triangle t = 1/4$.

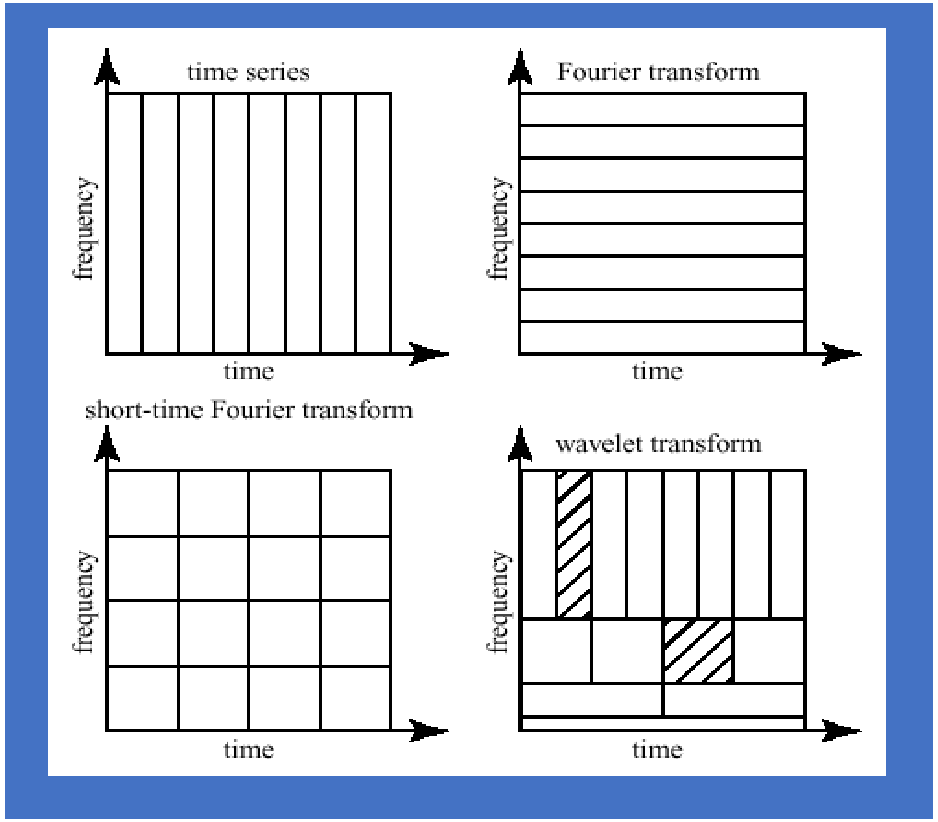


COMPARSION OF TRANSFORMATIONS





In fig. a. each strip can be considered as a narrow rectangular pulse or impulse. We know, $\delta(t) \rightarrow 1$ i.e. single frequency. Therefore y-axis indicates single frequency. Other shifted impulse, $|\delta(t-t_0)| \rightarrow 1$ also shows the same result.



In fig. b. a time domain signal of full duration has dc, fundamental and harmonics at equidistant. Actually the figure represents DFT.

Discrete Wavelet transform

One drawback of the CWT is that the representation of the signal is often redundant, since a and b are continuous over R (the real number). The original signal can be completely reconstructed by a sample version of *W(b, a)*. Typically, we sample *W(b, a)* in dyadic grid, i.e., $a = 2^m$ and $b = n2^m$

$$\because \Psi_{a,b}(t) = \Psi\left(\frac{t-b}{a}\right) = \Psi\left(\frac{t-n2^m}{2^m}\right) = \Psi(2^{-m}t-n) = \Psi_{m,n}(t)$$

is the dilated and translated version of the mother wavelet.

The DWT or continuous-time wavelet transform of *f(t)* with respect to a wavelet *Ψ(t)* is defined as,

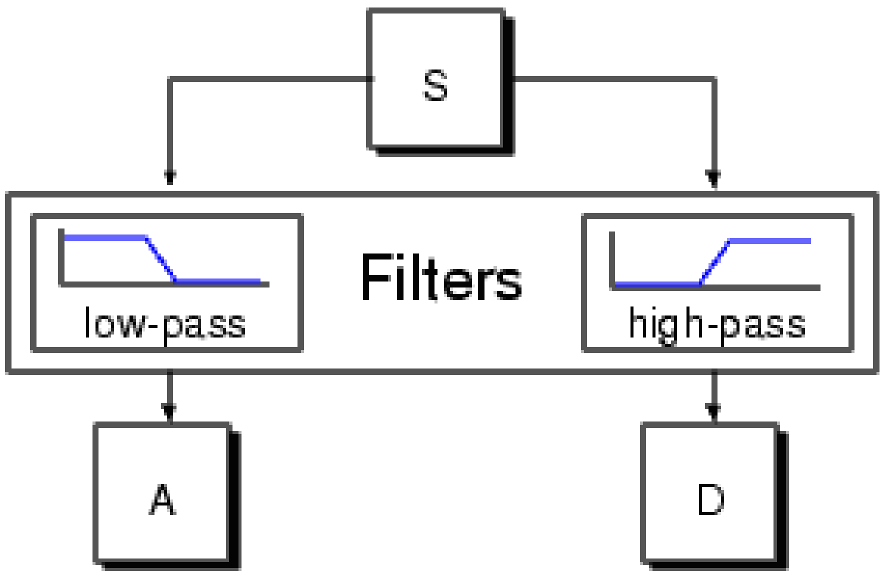
$$d(m,n) = \frac{1}{2^m}\int_{2^m n}^{2^m (n+1)} f(t)\psi(2^{-m}t - n)dt$$

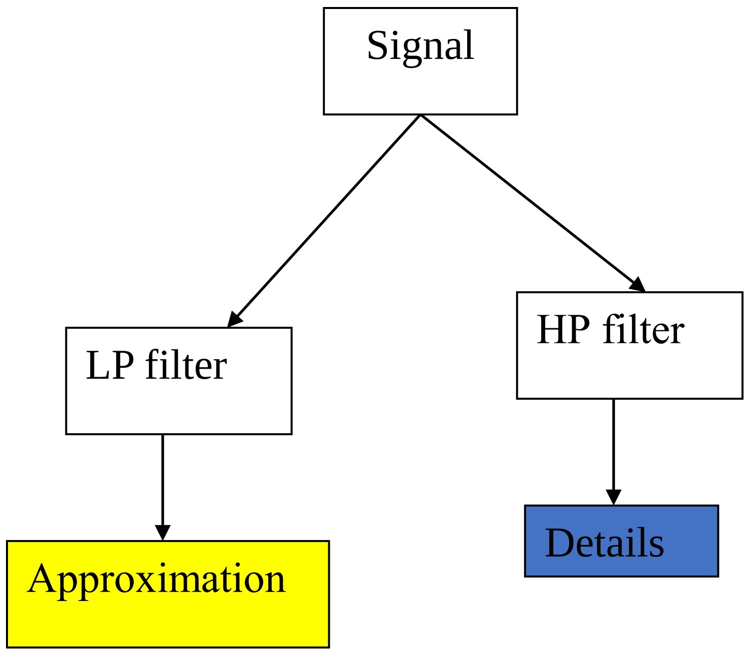
Here *d(m, n)* is equivalent to continuous wavelet transform *W(a, b)* when $a = 2^m$ and $b = n2^m$

Inverse operation i.e IDWT is expressed like,

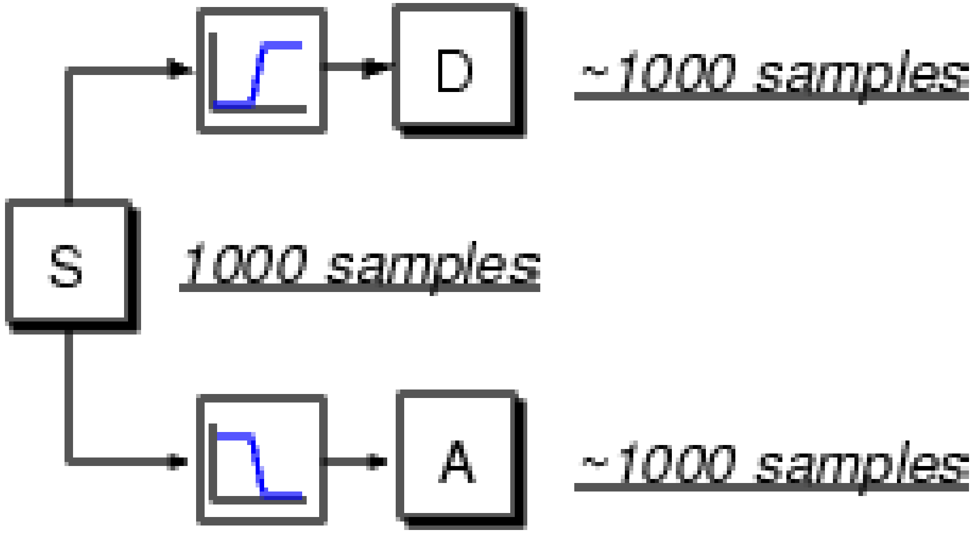
$$f(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} d(m,n)2^{-m/2}\psi(2^{-m}t - n)$$

The DWT is computed by successive lowpass and highpass filtering of the discrete time-domain signal





Under sampling to keep the number of samples constant:



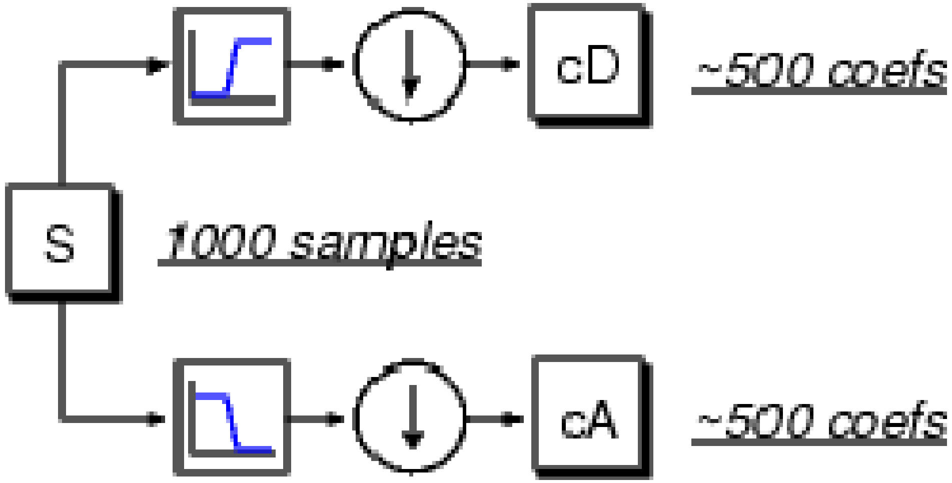
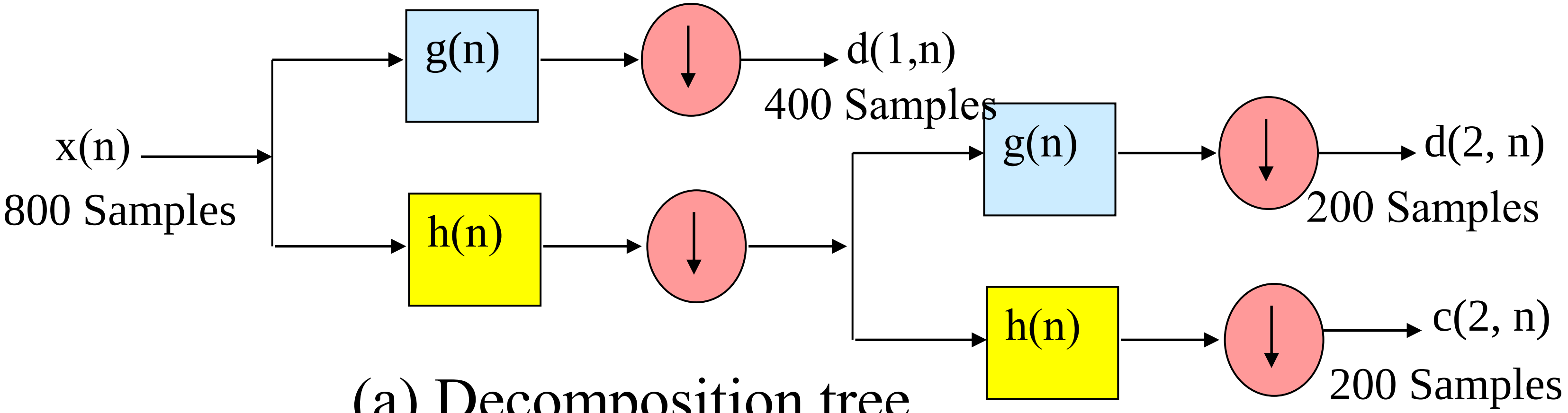
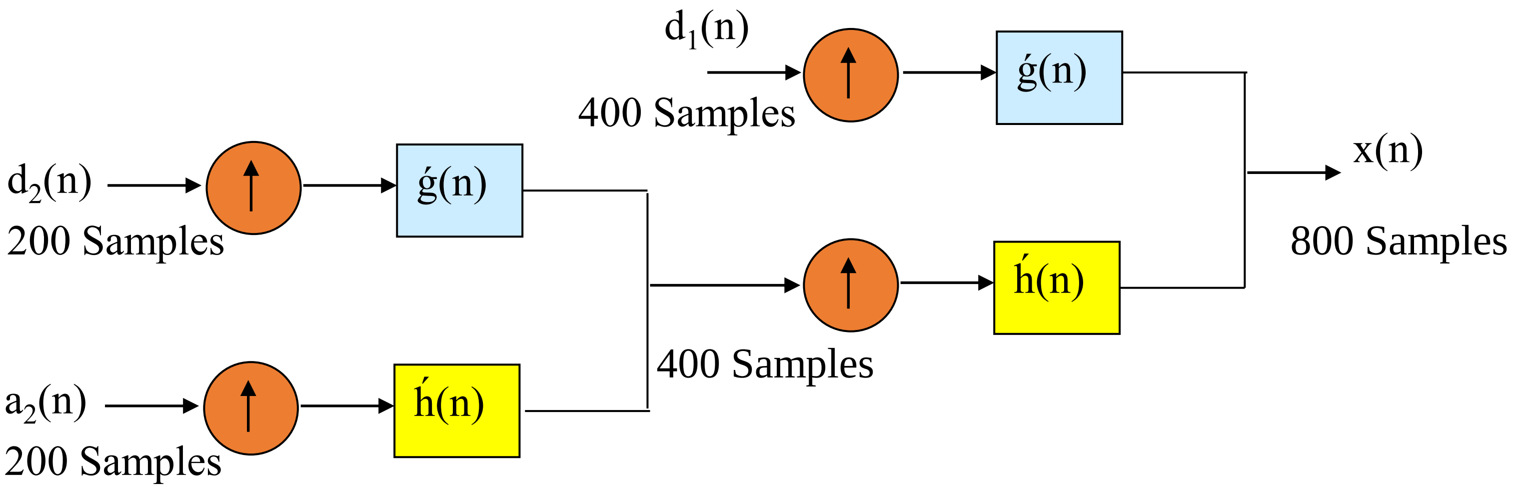


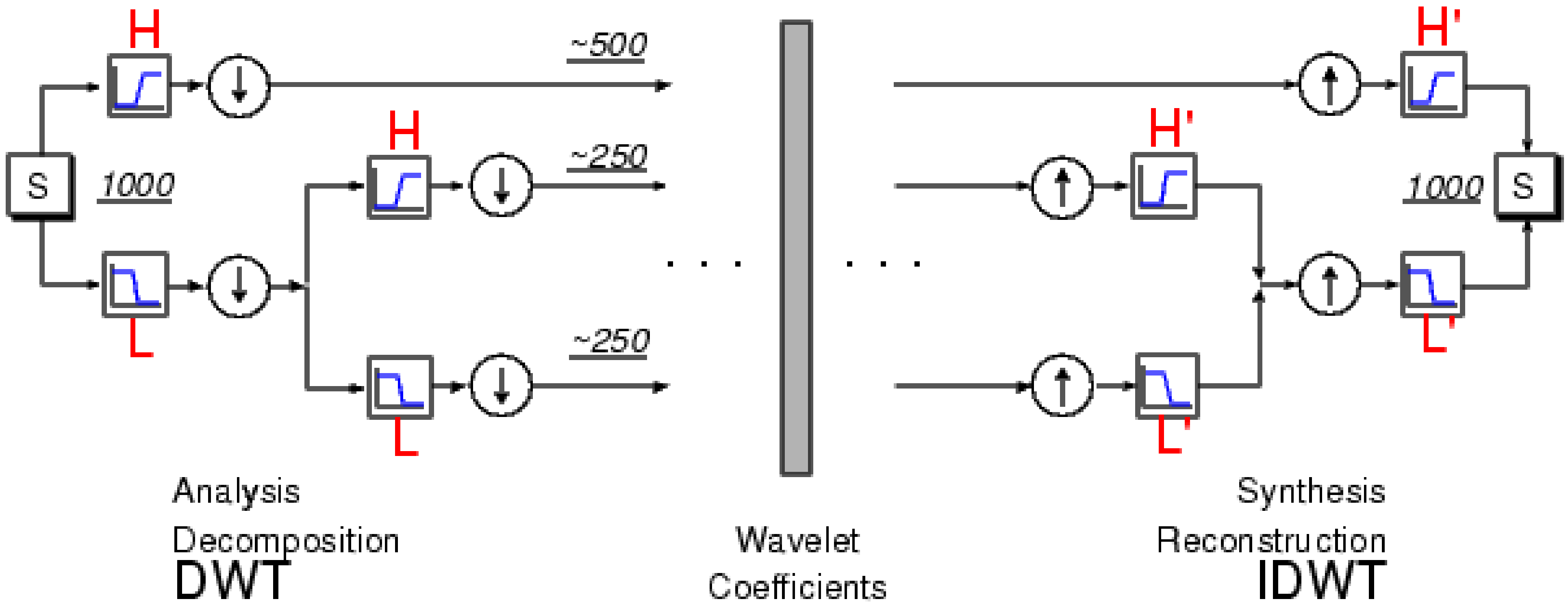
Fig. below shows two levels wavelet decomposition and reconstruction tree.



(a) Decomposition tree



(b) Reconstruction tree



Two levels filter bank of discrete wavelet transform

Applications

Figure 2.5 shows the general steps followed in a signal processing application. Processing may involve compression, encoding, denoising etc.

Figure 2.5 Signal processing application using Wavelet Transform.

Applications…

- One of the prominent applications is in the FBI fingerprint compression standard. Wavelet Transforms are used to compress the fingerprint pictures for storage in their data bank. The previously chosen Discrete Cosine Transform (DCT) did not perform well at high compression ratios.
- At present, the application of wavelets for image compression is one the hottest areas of research. Recently, the Wavelet Transforms have been chosen for the JPEG 2000 compression standard.

- Wavelets also find application in speech compression, which reduces transmission time in mobile applications.
- They are used in denoising, edge detection, feature extraction, speech recognition, echo cancellation and others.
- Wavelets also have numerous applications in digital communications. Orthogonal Frequency Division Multiplexing (OFDM) is one of them.
- Wavelets are used in biomedical imaging. For example, the ECG signals, measured from the heart, are analyzed using wavelets or compressed for storage.

Thank You