Lecture: 03

Fourier Analysis of Continuous-time

Signals & Systems

By
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Signal Representations

- Signals can be represented in a number of forms:

  1) Time domain representation
  2) S-domain representation (where s is a complex variable called Laplace operator and $s = \sigma + j\omega$. Here σ is real part and $j\omega$ is imaginary part of s).
  3) Z-domain representation (where z is a complex variable expressed in polar form as $z = re^{j\Omega}$ (where r is the magnitude of z and Ω is the angle of z).
  4) Frequency-domain representation

- Instead of time domain representation, signals converted into s, z, and f-domain are sometimes more convenient to analyze and process for many purposes. In addition, greater insights into the nature and properties of many signals and systems are provided by these transformations.

Signal Transformation

- A signal can be converted between the time and frequency domains with a pair of mathematical operators called a transform.
  - An example is the Fourier transform, which decomposes a function into the sum of a number of sine wave frequency components. The 'spectrum' of frequency components is the frequency domain representation of the signal.
- The inverse Fourier transform converts the *frequency domain function back to a time function.*

- To convert continuous time-domain signals into complex s-domain,
  - Laplace transform is used (which is also used for the solution of differential equations and the analysis of filters).
- To convert discrete time-domain signals into complex z-domain,
  - z transform is used.
- To convert time-domain signals into frequency domain,
  - Fourier series and Fourier transform is used.

Fourier Series Expansion is used for periodic signals to expand them in terms of their harmonics which are sinusoidal and orthogonal to one another

periodic Signal → CT → CTFS
periodic Signal → DT → DTFS

$\left.\begin{array}{l}\text{CTFS}\\ \text{DTFS}\end{array}\right\}$ PS

$\left.\begin{array}{l}\text{CTFT}\\ \text{DTFT}\end{array}\right\}$ Non PS

Ana.

CT → Laplace Tr. → Design.

DT → Z-Tr. → Design

$x(t) \longrightarrow$ periodic

$x(t \pm T) = x(t)$

time per. $= nT_0$

Difference: Time Domain Vs. Frequency Domain

- Time domain is the analysis of signal and system with respect to time, how the signal or system changes over time. But frequency-domain analysis shows how the signal's energy is distributed over a range of frequencies.
  - A time-domain graph shows how a signal changes with time, whereas a frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies.
  - In the time domain, the signal’s value is known for all real numbers, for the case of continuous time, or at various separate instants in the case of discrete time. An oscilloscope is a tool commonly used to visualize real-world signals in the time domain.
  - Frequency-domain analysis is widely used in such areas as communications, geology, remote sensing, and image processing.

- As an example, consider a typical electro cardiogram (ECG). If the doctor maps the heartbeat with time say the recording is done for 20 minutes, we call it a time domain signal.

- However, as in ECG, a number of peaks are there (of different types). Say in one heartbeat 4 types of peaks or variation in amplitude occurs. So in frequency domain, over the entire time period of recording, how many times each peak comes is recorded. Frequency is nothing but the number of times each event has occurred during total period of observation.

Fourier Series

- In mathematics, a Fourier series is a way to represent a continuous-time signal (or wave-like function) as the sum of simple sine waves.

- A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines.

- A Fourier series is a summation of harmonically related sinusoidal functions, also known as components or harmonics.

  - It decomposes any periodic continuous-time signal into the sum of a (possibly infinite) set of simple oscillating functions, namely ***sines*** and ***cosines*** (or, equivalently, complex exponentials).

  - The Discrete-time Fourier transform is a periodic function, often defined in terms of a Fourier series.

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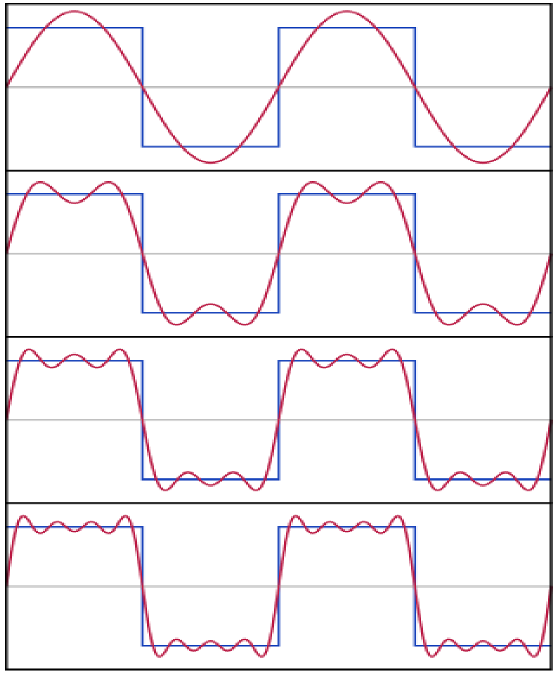
  - Figure shows the first four partial sums of the Fourier series for a square wave.



Fourier Series Representation of Signals

Fourier Series for Periodic Signals:

- We know that signals that repeat over and over are said to be periodic. A continuous-time signal x(t) is said to be periodic if it satisfies the condition

  $x(t) = x(t + T)$ for all t

  Where T is a positive constant called period of the signal.

- The fundamental period T_0 of x(t) is the smallest positive value of T for which the above relation is satisfied, and $f_0 = 1/T_0$ is referred to as the fundamental frequency.

- Two basic examples of periodic signals are:

  - real sinusoidal signal $x(t) = \cos(\omega_0 t + \phi)$

  - complex exponential signal $x(t) = e^{j\omega_0 t}$

  where $\omega_0 = 2\pi / T_0 = 2\pi f_0$ is called the fundamental angular frequency.

Fourier Series Representation of Signals

Fourier Series for Periodic Signals:

- The Fourier series representation of a complex exponential periodic signal x(t) with fundamental period T_0 is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \qquad (5.4)$$

- where c_k are known as the **complex Fourier coefficients** and are given by

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt \qquad (5.5)$$

- where $\int_{T0}$ denotes the integral over any one period and 0 to T_0 or $-T_0/2$ to $T_0/2$ is commonly used for the integration. Setting k= 0 in Eq. (5.5), we have

$$c_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt \qquad (5.6)$$

- which indicates that c_0 equals the average value of x(t) over a period.

Conditions to be Fourier Series

- It is known that a periodic signal x(t) has a Fourier series representation if it satisfies the following Dirichlet conditions:

    1. x(t) is absolutely integrable over any period, that is,

    $$\int_{T_0} |x(t)|\, dt < \infty$$

    2. x(t) has a finite number of maxima and minima within any finite interval of t.

    3. x(t) has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

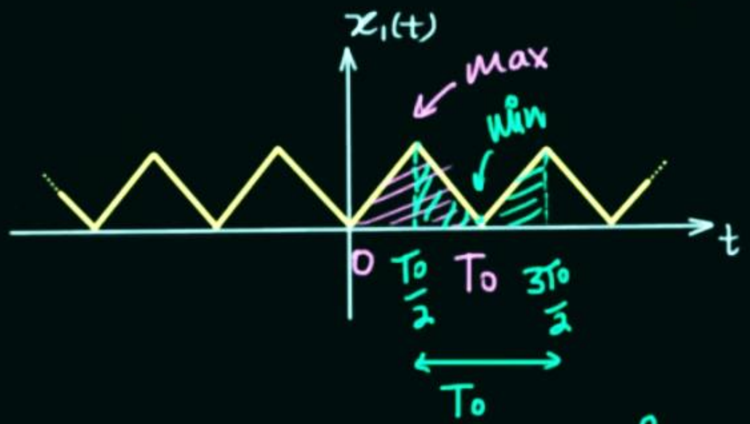
Conditions For Existance of Fourier Series (Dirichlet Conditions)

PS → FS ✓

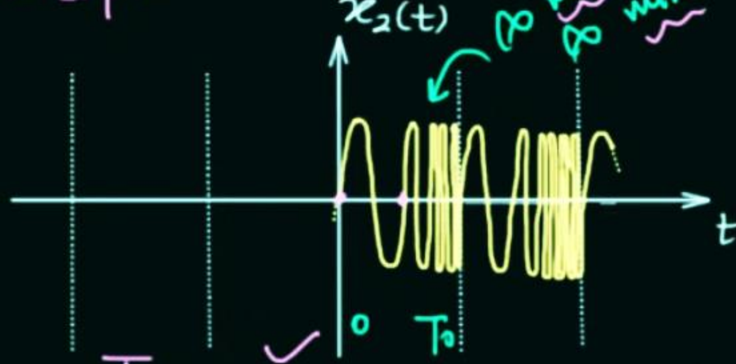
NPS → FT

Condition 1: Signal should have finite number of maxima and minima over the range of time period

FS Exp. ✓



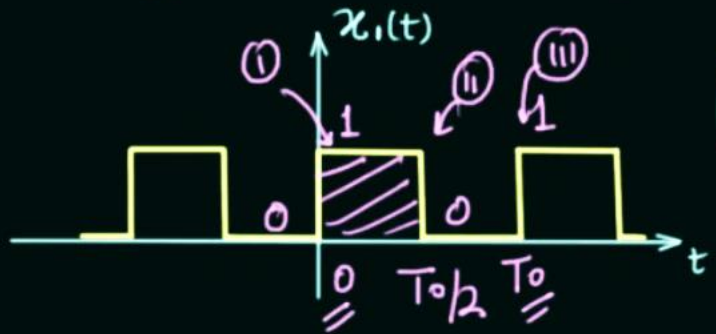
FS Exp ×



$T_{c1} \Rightarrow T_{c2} = T_{c1}/2 \quad T_{c3} = \frac{T_{c2}}{2}$

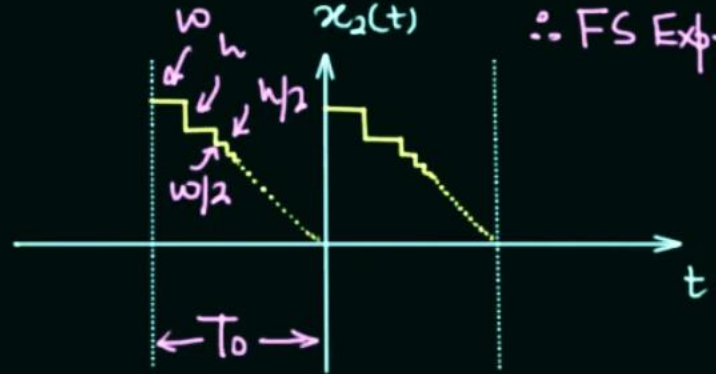
Condition 2: Signal should have finite number of discontinuities over the range of time period

FS Exp ✓



∞ discontinuities

∴ FS Exp. ×



PS → Power Signal, NENP

Condition 3: Signal should be absolutely integrable over the range of time period

Energy Signals

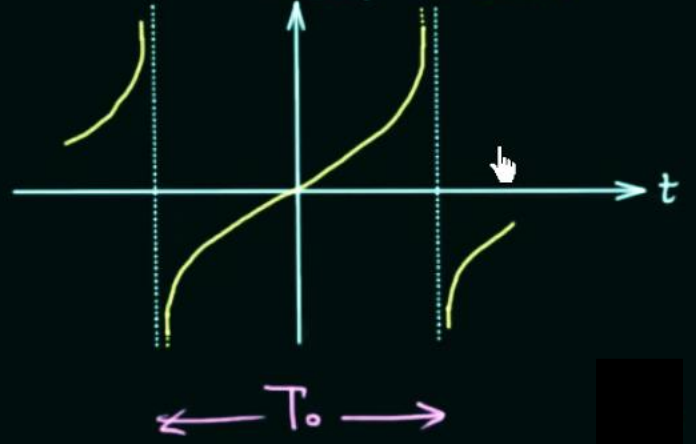
Stable LTI

$\int_{-\infty}^{\infty} |x_1(t)| dt < \infty$

$\int_{T_0} |x_1(t)| dt < \infty$ ← finite

$\text{area} = A \times \frac{T_0}{2}$

$x_2(t) = \tan t$



Fourier Series Representation of Signals

Trigonometric Fourier Series:

- The trigonometric Fourier series representation of a periodic signal x(t) with fundamental period T_0 is given by

$$x(t) = a_0 + \sum_{n=1}^{k} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \ldots (1)$$

where a_k, and b_k are the **Fourier coefficients** given by

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t \, dt \qquad (5.9a)$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t \, dt \qquad (5.9b)$$

- The **Fourier coefficients a_k, b_k** and the **complex Fourier coefficients c_k** are related by

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k}) \qquad (5.10)$$

- From Eq. (5.10) we obtain

$$c_k = \tfrac{1}{2}(a_k - jb_k) \qquad c_{-k} = \tfrac{1}{2}(a_k + jb_k) \quad (5.11)$$

Fourier Series Representation of Signals

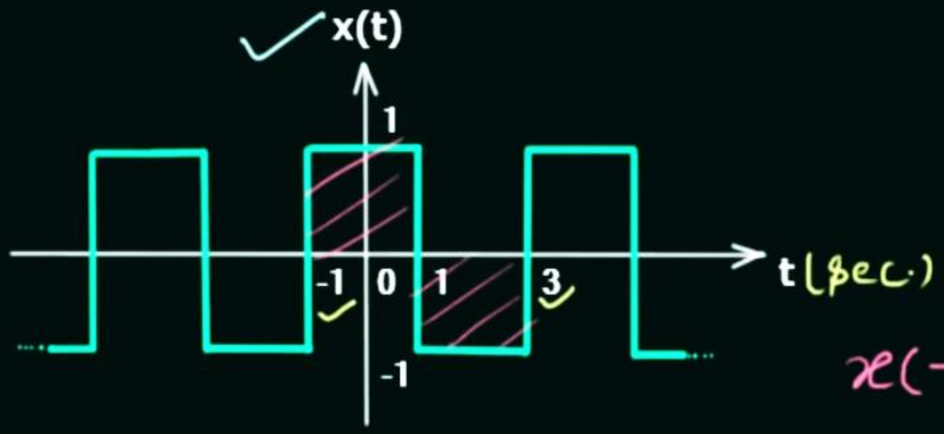
Fourier Series for Even and Odd Signals:

- If a continuous-time periodic signal x(t) is even, then $b_k=0$ and its Fourier series (5.8) contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0} \qquad (5.13)$$

- If x(t) is odd, then $a_k=0$ and its Fourier series contains only sine terms:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0} \qquad (5.14)$$



$$x(-t) = x(t)$$

$$x(t) = \cancelto{0}{a_0} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} \cancelto{0}{b_n} \sin n\omega_0 t$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cdot \cos n\omega_0 t$$

a_0 = dc component (Avg area of one time period)

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$T_0 = 4$ sec

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot \cos n\omega_0 t \, dt$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4}$$

$$\omega_0 = \frac{\pi}{2} \text{ rad/sec}$$

$$a_n = \frac{1}{2}\int_{-1}^{3} x(t)\cdot \cos \frac{n\pi}{2} t \, dt$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{1} (1)\cos \frac{n\pi}{2} t\, dt + \int_{1}^{3} (-1)\cdot \cos \frac{n\pi}{2} t\, dt\right]$$

$$\frac{n\pi}{2} t = \theta$$

$$\frac{n\pi}{2} dt = d\theta \Rightarrow dt = \frac{2}{n\pi} d\theta$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{1} \cos \frac{n\pi}{2} t\, dt - \int_{1}^{3} \cos \frac{n\pi}{2} t\, dt\right]$$

$$t = -1 \Rightarrow \theta = -\frac{n\pi}{2}$$

$$t = 1 \Rightarrow \theta = \frac{n\pi}{2}$$

$$t = 3 \Rightarrow \theta = \frac{3n\pi}{2}$$

$$a_n = \frac{1}{2}\left[\int_{-n\pi/2}^{n\pi/2} \cos\theta \frac{2}{n\pi} d\theta - \int_{n\pi/2}^{3n\pi/2} \cos\theta \frac{2}{n\pi} d\theta\right]$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{1} \cos\frac{n\pi}{2}t\,dt - \int_{1}^{3} \cos\frac{n\pi}{2}t\,dt\right]$$

$\frac{n\pi}{2}dt = d\theta \Rightarrow dt = \frac{2}{n\pi}d\theta$

$t = -1 \Rightarrow \theta = -\frac{n\pi}{2}$

$t = 1 \Rightarrow \theta = \frac{n\pi}{2}$

$t = 3 \Rightarrow \theta = \frac{3n\pi}{2}$

$$a_n = \frac{1}{2}\left[\int_{-n\pi/2}^{n\pi/2} \cos\theta \frac{2}{n\pi}d\theta - \int_{n\pi/2}^{3n\pi/2} \cos\theta \frac{2}{n\pi}d\theta\right]$$

$$a_n = \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{n\pi}\left[\int_{-n\pi/2}^{n\pi/2} \cos\theta\,d\theta - \int_{n\pi/2}^{3n\pi/2} \cos\theta\,d\theta\right]$$

$$a_n = \frac{1}{n\pi}\left[(\sin\theta)_{-n\pi/2}^{n\pi/2} - (\sin\theta)_{n\pi/2}^{3n\pi/2}\right]$$

$$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin(-n\pi/2) - \sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right]$$

$\rightarrow \sin\left(-\frac{n\pi}{2}\right)$

$\sin\left(\frac{\pi}{2} - \theta\right)$

$+\cos\theta$

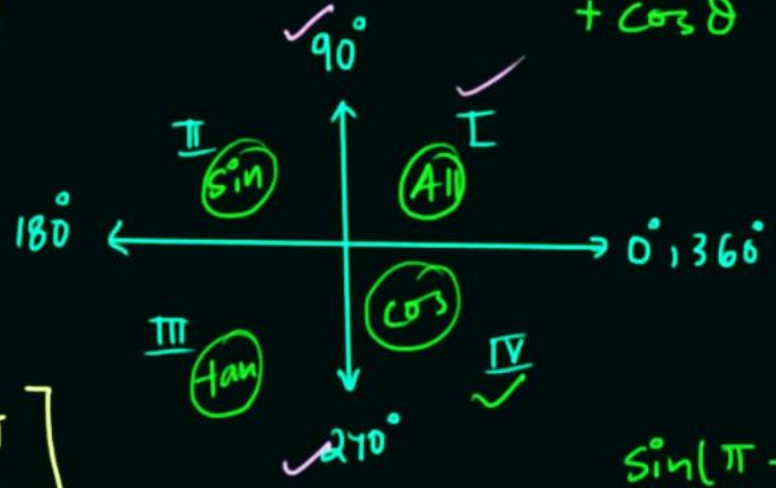
$\sin\left(n\pi + \frac{n\pi}{2}\right)$

C I: $n = \text{even}$

2π

$4\pi = 2 \times 360$

$\sin n\pi/2$



C II: $n = \text{odd}$

$n = 1$

$\sin(\pi + \pi$

$-\sin\frac{n\pi}{2}$

$\sin(\pi - \theta)$

$+\sin\theta$

$\sin\left(0 - \frac{n\pi}{2}\right)$

$-\sin\frac{n\pi}{2}$

Case I :- $n = \text{even}$

$$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \cancel{\sin\frac{n\pi}{2}} - \cancel{\sin\frac{n\pi}{2}} + \sin\frac{n\pi}{2}\right]$$

$$a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$\boxed{a_n = 0}$$

$n = 2$

$$a_n = \frac{2}{2\pi}\sin\frac{2\pi}{2} \Rightarrow a_n = \frac{1}{\pi}\sin\pi \qquad \boxed{\sin n\pi = 0}$$

Case II :- $n = \text{odd}$

$$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right] \Rightarrow a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

Case a: $n = 1, 5, 9, 13, \ldots$

$$a_n = \frac{4}{n\pi}$$

Case b: $n = 3, 7, 11, 15, \ldots$

$$a_n = -\frac{4}{n\pi}$$

$$a_3 = -\frac{4}{3\pi}$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{2}t$$

$$= \frac{4}{\pi}\cos\frac{\pi}{2}t - \frac{4}{3\pi}\cos\frac{3\pi}{2}t + \frac{4}{5\pi}\cos\frac{5\pi}{2}t - \frac{4}{7\pi}\cos\frac{7\pi}{2}t + \cdots$$

$$= \frac{4}{\pi}\left[\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - \frac{1}{7}\cos\frac{7\pi}{2}t + \cdots\right]$$

Exponential Fourier Series

- A periodic signal can be represented over a certain interval of time in terms of the linear combination of orthogonal functions. If these orthogonal functions are exponential functions, then it is called the exponential Fourier series

- The exponential Fourier series is the most widely used form of the Fourier series. In this representation, the periodic function *x(t)* is expressed as a weighted sum of the complex exponential functions. It is the convenient and compact form of the Fourier series and have extensive application in communication theory.

- For any periodic signal $x(t)$, the exponential form of Fourier series is given by,

$$X(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

- Where, $\omega_0 = 2\pi/T$ is the angular frequency of the periodic function.

Exponential Fourier Series

Therefore, *the Fourier coefficient of the exponential Fourier series* C_n is given by,

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt \quad \ldots.(2)$$

Equation (2) is also called as ***the analysis equation***.

Also, the *DC component* C_0 of *the exponential Fourier series* is given by,

$$C_0 = \frac{1}{T}\int_{t_0}^{(t_0+T)} x(t)dt \quad \ldots.(3)$$

Conjugate symmetric Signal

Conjugate symmetric Signal is a signal which satisfies the relation

$f(t) = f^{*}(-t)$. It is also known as even conjugate signal.

Example:

$f(t) = e^{j\omega_o t}$

$f(-t) = e^{j\omega_o (-t)}$

$f^{*}(-t) = e^{(-j)\omega_o (-t)} = e^{j\omega_o t} = f(t)$

Hence, $f(t) = f^{*}(-t)$

The **real part** is always **even signal** and the imaginary part is always odd signal for a conjugate symmetric signal.

$C_n \rightarrow$ complex Exp. Fourier Coeff

$$C_n = \frac{1}{T_0}\int_{T_0} x(t) \cdot e^{-jn\omega_0 t}\, dt \quad \text{—— (I)}$$

$\downarrow$ Reversal $(n = -n)$

$$C_{-n} = \frac{1}{T_0}\int_{T_0} x(t) \cdot e^{jn\omega_0 t}\, dt$$

$\downarrow *$

$$C^*_{-n} = \frac{1}{T_0}\int_{T_0} x^*(t) \cdot e^{-jn\omega_0 t}\, dt \quad \text{—— (II)}$$

if C_n is CS :-

$$C_n = C^*_{-n} \Rightarrow x(t) = x^*(t) \Rightarrow \underline{\text{Real}}$$

$$C_n = |C_n| e^{j\angle C_n} \quad \text{—— (III)}$$

$\downarrow n = -n$

$$C_{-n} = |C_{-n}| e^{j\angle C_{-n}}$$

$\downarrow *$

$$C^*_{-n} = |C_{-n}| e^{-j\angle C_{-n}} \quad \text{—— (IV)}$$

$$|C_n| e^{j\angle C_n} = |C_{-n}| e^{-j\angle C_{-n}}$$

$$|C_n| = |C_{-n}| \Rightarrow \text{Even}$$

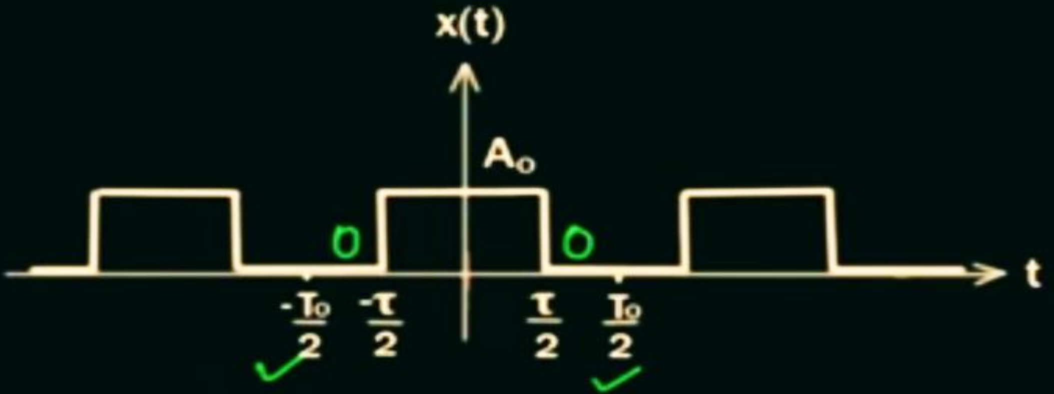
$$\angle C_n = -\angle C_{-n}$$

$$-\angle C_n = \angle C_{-n} \Rightarrow \text{Odd}$$

Complex Exponential Fourier Series (Example-1)

Question: Find C_n for signal $x(t)$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}$$



ઉકેલ:-

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\left[\int_{-T_0/2}^{-\tau/2} 0\cdot e^{-jn\omega_0 t}\,dt + \int_{-\tau/2}^{\tau/2} A_0\cdot e^{-jn\omega_0 t}\,dt + \int_{\tau/2}^{T_0/2} 0\cdot e^{-jn\omega_0 t}\,dt\right]$$

$$= \frac{A_0}{T_0}\int_{-\tau/2}^{\tau/2} e^{-jn\omega_0 t}\,dt$$

let $-jn\omega_0 t = \theta$

$-jn\omega_0 dt = d\theta \Rightarrow dt = \frac{-1}{jn\omega_0}d\theta$

when, $t = -\tau/2$

$\theta = jn\omega_0\tau/2$

$t = \tau/2$

$\theta = -jn\omega_0\tau/2$

$$= \frac{A_0}{T_0}\int_{jn\omega_0\tau/2}^{-jn\omega_0\tau/2} e^{\theta}\left(\frac{-1}{jn\omega_0}\right)d\theta \Rightarrow C_n = \frac{-A_0}{jn\omega_0 T_0}\int_{jn\omega_0\tau/2}^{-jn\omega_0\tau/2} e^{\theta}d\theta \Rightarrow C_n = \frac{-A_0}{jn\omega_0 T_0}\left[e^{\theta}\right]_{jn\omega_0\tau/2}^{-jn\omega_0\tau/2}$$

$$C_n = \frac{-A_0}{jn\omega_0 T_0}\left[e^{-j\underbrace{n\omega_0\tau/2}_{x}} - e^{j\underbrace{n\omega_0\tau/2}_{x}}\right]$$

$$e^{ix} = \cos x + i\sin x$$

$$C_n = \frac{-A_0}{jn\omega_0 T_0}\left[e^{-jx} - e^{jx}\right] \Rightarrow C_n = \frac{-A_0}{jn\omega_0 T_0}\left[\cos(-x) + j\sin(-x) - \{\cos x + j\sin x\}\right]$$

$$C_n = \frac{-A_0}{jn\omega_0 T_0}\left[\cancel{\cos x} - j\sin x - \cancel{\cos x} - j\sin x\right]$$

$$C_n = \frac{-A_0}{jn\omega_0 T_0}\left[-2j\sin\frac{n\omega_0\tau}{2}\right]$$

Fourier Transform

- Fourier transform is a mathematical process that converts a time domain signal into frequency domain.

- The Fourier transform of a function of time itself is a complex-valued function of frequency, whose absolute value represents the amount of that frequency present in the original function, and whose complex argument is the phase offset of the basic sinusoid in that frequency.

- The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.
  - For many functions of practical interest one can define an operation that reverses this: the inverse Fourier transformation, also called Fourier synthesis, of a frequency domain representation combines the contributions of all the different frequencies to recover the original function of time.

- Figure-A shows a rectangular pulse function f(t) in time domain. Figure-B shows its Fourier transform f`(ω), a function of frequency ω.

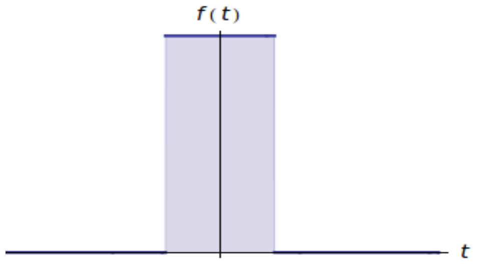


Figure: A

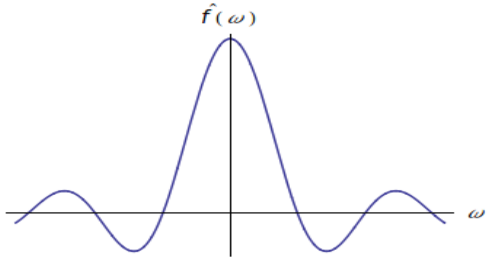


Figure: B

Fourier Transform

- Fourier Transform is a mathematical tool used for frequency analysis of signals.

- There are many Fourier transforms-
  - DFT (Discrete Fourier Transform)
  - DTFT (Discrete-time Fourier Transform)
  - FFT (Fast Fourier Transform)
  - IDFT (Discrete Inverse Fourier Transform)

Fourier Transform

- Let x(t) be an integrable continuous-time signal.
- The Fourier transform X(ω) of this signal is given by the following relation:

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Where ω is called angular frequency and ω = 2Πf.

- In the above relation, we see that the independent variable t (representing time) is transformed into ω (representing frequency).
- The functions x(t) and X(ω) are often referred to as a Fourier integral pair or Fourier transform pair.

Inverse Fourier Transform

- Let X(ω) be the Fourier transform of an integrable continuous-time signal x(t).
- Under suitable condition, x(t) can be reconstructed from X(ω) using inverse Fourier transform , as:

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

[redacted]

rad/sec

$\omega = 2\pi f$

>> Representation: $x(t) \rightleftharpoons \underset{\displaystyle X(\omega)}{\underline{\underline{X(j\omega)}}}$ or $X(f)$

Hz

$$X(j\omega) = |X(j\omega)| \angle X(j\omega)$$

>> Formulae:

FT pair

1) $x(t) \longrightarrow X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \longleftarrow \text{FT}$$

2) $X(j\omega) \longrightarrow x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \, d\omega \longleftarrow \text{IFT}$$

Properties of Fourier Transform: *Linearity*

Addition of two functions corresponding to the addition of the two frequency spectrum is called the linearity. If we multiply a function by a constant, the Fourier transform of the resultant function is multiplied by the same constant. The Fourier transform of sum of two or more functions is the sum of the Fourier transforms of the functions.

Case I.

If h(x) -> H(f) then ah(x) -> aH(f)

Case II.

If h(x) -> H(f) and g(x) -> G(f) then h(x)+g(x) -> H(f)+G(f)

1. Linearity: let, $\left\{\begin{array}{l} x_1(t) \rightleftharpoons C_{1n} \\ x_2(t) \rightleftharpoons C_{2n} \end{array}\right.$ "T_0"

$$\mathrm{LCM}[T_0, T_0] = T_0$$

$$\alpha x_1(t) + \beta x_2(t) = \underbrace{x(t)}_{\text{"}T_0\text{"}} \rightleftharpoons C_n$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} [\alpha x_1(t) + \beta x_2(t)]\cdot e^{-jn\omega_0 t}\,dt$$

$$= \alpha\cdot\frac{1}{T_0}\int_0^{T_0} x_1(t)\cdot e^{-jn\omega_0 t}\,dt + \beta\cdot\frac{1}{T_0}\int_0^{T_0} x_2(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$C_n = \alpha C_{1n} + \beta C_{2n}$$

** $\alpha x_1(t) + \beta x_2(t) \rightleftharpoons \alpha C_{1n} + \beta C_{2n}$

1. Linearity:

$$x_1(t) \rightleftharpoons X_1(j\omega)$$

$$x_2(t) \rightleftharpoons X_2(j\omega)$$

$$\alpha\, x_1(t) \rightleftharpoons \alpha\, X_1(j\omega)$$

$$\beta\, x_2(t) \rightleftharpoons \beta\, X_2(j\omega)$$

$$\underbrace{\alpha\, x_1(t) + \beta\, x_2(t)}_{x(t)} \rightleftharpoons \alpha X_1(j\omega) + \beta X_2(j\omega)$$

proof:- $x(t) \rightleftharpoons X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [\alpha\, x_1(t) + \beta\, x_2(t)]\cdot e^{-j\omega t}\, dt$$

$$= \alpha \underbrace{\int_{-\infty}^{\infty} x_1(t)\cdot e^{-j\omega t}\, dt}_{X_1(j\omega)} + \beta \underbrace{\int_{-\infty}^{\infty} x_2(t)\cdot e^{-j\omega t}\, dt}_{X_2(j\omega)}$$

$$X(j\omega) = \alpha X_1(j\omega) + \beta X_2(j\omega) \rightleftharpoons x(t) = \alpha x_1(t) + \beta x_2(t)$$

Properties of Fourier Transform: Conjugation

The conjugation property states that the conjugate of function x(t) in time domain results in conjugation of its Fourier transform in the frequency domain and ω is replaced by (−ω), i.e., if

$$\mathbf{x(t)} \overset{\mathbf{FT}}{\leftrightarrow} \mathbf{X}(\omega)$$

Then, according to conjugation property of Fourier transform,

$$\mathbf{x^*(t)} \overset{\mathbf{FT}}{\leftrightarrow} \mathbf{X}^*(-\omega)$$

Proof

From the definition of Fourier transform, we have

$$\mathbf{X}(\omega) = \int_{-\infty}^{\infty} \mathbf{x(t)e^{-j\omega t}dt}$$

Taking conjugate on both sides, we get

$$\mathbf{X}^*(\omega) = [\int_{-\infty}^{\infty} \mathbf{x(t)e^{-j\omega t}dt}]^*$$

$$\Rightarrow \mathbf{X}^*(\omega) = \int_{-\infty}^{\infty} \mathbf{x^*(t)e^{j\omega t}dt}$$

Now, by replacing (ω) by (−ω), we obtain,

$$\mathbf{X}^*(-\omega) = \int_{-\infty}^{\infty} \mathbf{x^*(t)e^{-j\omega t}dt} = \mathbf{F[x^*(t)]}$$

$$\therefore \mathbf{F[x^*(t)]} = \mathbf{X}^*(-\omega)$$

Or, it can also be represented as,

$$\mathbf{x^*(t)} \overset{\mathbf{FT}}{\leftrightarrow} \mathbf{X}^*(-\omega)$$

"T_0"

2. Conjugation:

$$x(t) \rightleftharpoons C_n$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

take conj. on both sides.

$$C_n^* = \frac{1}{T_0}\int_0^{T_0} x^*(t)\cdot e^{+jn\omega_0 t}\,dt$$

$n \longrightarrow -n$

$$C_{-n}^* = \frac{1}{T_0}\int_0^{T_0} x^*(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$x^*(t) \rightleftharpoons C_{-n}^*$$

Properties of Fourier Transform: *Time Reversal*

Statement – The time reversal property of Fourier transform states that if a function $x(t)$ is reversed in time domain, then its spectrum in frequency domain is also reversed, i.e., if

$$x(t) \leftrightarrow X(j\omega)$$

Then, according to the time-reversal property of Fourier transform,

$$x(-t) \leftrightarrow X(-j\omega)$$

Proof :- $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$

$t \rightarrow -t$

$x(t) \longrightarrow x(-t) \rightleftharpoons X'(j\omega) = ?$

$$X'(j\omega) = \int_{-\infty}^{\infty} x(-t) \cdot e^{-j\omega t}\, dt$$

$-t = \tau \Rightarrow t = -\tau \Rightarrow dt = -d\tau$

$$\left.\begin{aligned} t = -\infty &\Rightarrow \tau = \infty \\ t = \infty &\Rightarrow \tau = -\infty \end{aligned}\right\}$$

$$= -\int_{+\infty}^{-\infty} x(\tau) \cdot e^{+j\omega\tau}\, d\tau$$

$t \rightarrow \tau$

$\omega \rightarrow -\omega$

$$X'(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j(-\omega)\tau}\, d\tau = X(-j\omega) \rightleftharpoons x(-t)$$

Properties of Fourier Transform: *Time Scaling*

Statement – The time-scaling property of Fourier transform states that if a signal is expended in time by a quantity (a), then its Fourier transform is compressed in frequency by the same amount. Therefore, if

$$\mathbf{x}(\mathbf{t}) \overset{\mathbf{FT}}{\leftrightarrow} \mathbf{X}(\omega)$$

Then, according to the time-scaling property of Fourier transform

$$\mathbf{x}(\mathbf{at}) \overset{\mathbf{FT}}{\leftrightarrow} \frac{1}{|\mathbf{a}|}\mathbf{X}\left(\frac{\omega}{\mathbf{a}}\right)$$

- When $a > 1$, then $x(at)$ is the compressed version of $x(t)$, and
- When $a < 1$, the function $x(at)$ is the expanded version of $x(t)$.

Proof :— $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$

$x(t) \longrightarrow x(at) \rightleftharpoons X'(j\omega)$

any real const.

I) a is positive

$$X'(j\omega) = \int_{-\infty}^{\infty} x(at) \cdot e^{-j\omega t}\, dt$$

$$at = \tau \Rightarrow t = \frac{\tau}{a} \Rightarrow dt = \frac{d\tau}{a}$$

$$X'(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega \frac{\tau}{a}}\, \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$X'(j\omega) = \frac{1}{a} X\left(j\frac{\omega}{a}\right)$$

Properties of Fourier Transform: *Time-Shifting*

Statement – The time shifting property of Fourier transform states that if a signal $x(t)$ is shifted by t_0 in time domain, then the frequency spectrum is modified by a linear phase shift of slope $(-\omega t_0)$. Therefore, if,

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then, according to the time-shifting property of Fourier transform,

$$x(t - t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

proof :—

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$t \longrightarrow t - t_0$$

$$x(t) \longrightarrow x(t - t_0) \rightleftharpoons X'(j\omega)$$

$$X'(j\omega) = \int_{-\infty}^{\infty} x(t - t_0) \cdot e^{-j\omega t}\, dt$$

$$t - t_0 = \underline{\tau} \Rightarrow \quad t = \underline{\tau + t_0} \Rightarrow dt = d\tau$$

$$X'(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega(\tau + t_0)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau} \cdot e^{-j\omega t_0}\, d\tau$$

$$X'(j\omega) = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau}\, d\tau}$$ $t \to \tau$

$$X'(j\omega) = e^{\pm j\omega t_0} \cdot X(j\omega) \xrightleftharpoons{X(j\omega)} x(t \pm t_0)$$

Properties of Fourier Transform: *Frequency Shifting*

Statement – Frequency shifting property of Fourier transform states that the multiplication of a time domain signal x(t) by an exponential (ejω0t) causes the frequency spectrum to be shifted by ω0. Therefore, if

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

Then, according to the frequency shifting property,

$$e^{j\omega_0 t}\, x(t) \overset{FT}{\leftrightarrow} X(\omega - \omega_0)$$

Proof :—

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$\underbrace{e^{+j\omega_0 t} \cdot x(t)}\rightleftharpoons X'(j\omega)$$

$$X'(j\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) \cdot e^{-j\omega t} \, dt$$

$$X'(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j(\omega-\omega_0)t} \, dt$$

$$X'(j\omega) = X[j(\omega-\omega_0)] \rightleftharpoons e^{+j\omega_0 t} \cdot x(t)$$

Home Task

- Find the Fourier transformed value for the given signal:

X(t) = 2t – (Last digit of your class ID, if last digit is zero then use 7 instead)

Thank You