CS229, Fall 2018
Problem Set #0 Solutions: Linear Algebra and Multivariable Calculus

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1.

(a)

$$\nabla f(x) = \nabla \left(\frac{1}{2} x^T A x + b^T x \right) = Ax + b$$

(b)

$$\frac{\partial \left(g(h(x)) \right)}{\partial x_i} = \frac{\partial g(h(x))}{\partial h(x)} \cdot \frac{\partial h(x)}{\partial x_i} = g'(h(x)) \cdot \frac{\partial h(x)}{\partial x_i}$$
$$\nabla f(x) = \nabla g(h(x)) = g'(h(x)) \cdot \nabla h(x)$$

(c)

$$\nabla f(x) = \nabla \left(\frac{1}{2} x^T A x + b^T x \right) = Ax + b$$
$$\nabla (Ax + b) = A$$

(d)

$$\nabla f(x) = \nabla g(a^T x) = g'(a^T x) \cdot \nabla a^T x = g'(a^T x) \cdot a$$
$$\frac{\partial^2 \left(g(a^T x) \right)}{\partial x_i \partial x_j} = \frac{\partial^2 g(a^T x)}{\partial (a^T x)^2} \cdot \frac{\partial a^T x}{\partial x_i} \cdot \frac{\partial a^T x}{\partial x_j} = g''(a^T x) \cdot \frac{\partial a^T x}{\partial x_i} \cdot \frac{\partial a^T x}{\partial x_j}$$
$$g''(a^T x) \cdot a_i \cdot a_j = g''(a^T x) \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = g''(a^T x) \cdot (a^T a)$$

2.

(a)

$$A^T = (zz^T)^T = (z^T z) = zz^T = A$$
$$x^T A x = x^T z z^T x = (x^T z (x^T z)^T) = (x^T z)^2 \geq 0$$

(b)

$$Am = (zz^T)m = z(z^T m)$$
$$Am = 0 \iff z^T m = 0$$
$$N(A) = \{ m \in \mathbf{R}^n : m^T z = 0 \}$$

(c)

$$(BAB^T)^T = (B^T)^T A^T B^T = BA^T B^T = (BAB^T)$$
$$x^T BAB^T x = x^T BA (x^T B)^T \geq 0$$

3.

(a)

$$A = T\Lambda T^{-1}$$

$$AT = T\Lambda$$

$$A\begin{bmatrix} t^{(1)} & t^{(2)} & t^{(3)} & \dots & t^{(n)} \end{bmatrix} = \begin{bmatrix} t^{(1)} & t^{(2)} & t^{(3)} & \dots & t^{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} At^{(1)} & At^{(2)} & At^{(3)} & \dots & At^{(n)} \end{bmatrix} = \begin{bmatrix} \lambda_1 t^{(1)} & \lambda_2 t^{(2)} & \lambda_3 t^{(3)} & \dots & \lambda_n t^{(n)} \end{bmatrix}$$

$$At^i = \lambda_i t^{(i)}$$

b.

$$A = U\Lambda U^T$$

$$AU = U\Lambda U^T U = U\Lambda$$

$$A\begin{bmatrix} u_{(1)} & u_{(2)} & \dots u_{(n)} \end{bmatrix} = \begin{bmatrix} u_{(1)} & u_{(2)} & \dots u_{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} Au_{(1)} & Au_{(2)} & \dots Au_{(n)} \end{bmatrix} = \begin{bmatrix} \lambda_1 u_{(1)} & \lambda_2 u_{(2)} & \dots \lambda_n u_{(n)} \end{bmatrix}$$

$$Au^{(i)} = \lambda u^{(i)}$$

c.

$$At_{(i)} = \lambda_{(i)} t^{(i)}$$

$$(t^{(i)})^T At_{(i)} = \lambda_{(i)} ||t^{(i)}||^2 \geq 0$$