Structured Optimal and Robust Control With Multiple Criteria: A Convex Solution

Xin Qi, Murti V. Salapaka, Petros G. Voulgaris, and Mustafa Khammash

***Abstract*—In this paper, the design of controllers that incorporate structural and multiobjective performance requirements is considered. The control structures under study cover nested, chained, hierarchical, delayed interaction and communications, and symmetric systems. Such structures are strongly related to several modern-day and future applications including integrated flight propulsion systems, platoons of vehicles, micro-electro-mechanical systems, networked control, control of networks, production lines and chemical processes. It is shown that the system classes presented have the common feature that all stabilizing controllers can be characterized by convex constraints on the Youla–Kucera parameter. Using this feature, a solution to a general optimal performance problem that incorporates time domain and frequency domain constraints is obtained. A synthesis procedure is provided which at every step yields a feasible controller together with a measure of its performance with respect to the optimal. Convergence to the optimal performance is established. An example of a multinode network congestion control problem is provided that illustrates the effectiveness of the developed methodology.**

***Index Terms*—Input–output, multiobjective, structure.**

NOMENCLATURE

c_0	Subspace of ℓ_∞ whose element x satisfies $\lim_{k\to\infty} x(k) = 0$.
ℓ_1	Banach space of right-sided absolutely summable real sequences with the norm given by $\|x\|_1 := \sum_{k=0}^{\infty} |x(k)|$.
$\ell_1^{m\times n}$	Banach space of matrix-valued right-sided real sequences with the norm given by $\|x\|_1 := \max_{1\le i\le m} \sum_{j=1}^{n} \|x_{ij}\|_1$, where $x \in \ell_1^{m\times n}$ is the matrix (x_{ij}) and each x_{ij} is in ℓ_1.
ℓ_2	Banach space of right-sided square summable real sequences with the norm given by $\|x\|_2 := \left(\sum_{k=0}^{\infty} x^2(k)\right)^{1/2}$.
ℓ_∞	Banach space of bounded real sequences with the norm given by $\|x\|_\infty := \sup_k |x(k)|$.
$\hat{x}(\lambda)$	λ transform of a right-sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{\infty} x(k)\lambda^k$.
$\mathcal{H}_2$	Isometrically isomorphic image of ℓ_2 under the λ transform with the norm given by $\|\hat{x}(\lambda)\|_{\mathcal{H}_2} = \|x\|_2$.
$\mathcal{H}_2^{m\times n}$	Hilbert space of matrix-valued right-sided real sequences with the norm given by $\|x\|_2 := (\sum_{i=1}^{m}\sum_{j=1}^{n} \|x_{ij}\|_2^2)^{1/2}$, where $x \in \ell_1^{m\times n}$ is the matrix (x_{ij}) and each x_{ij} is in $\mathcal{H}_2$.
$\mathcal{H}_\infty^{m\times n}$	Space of complex-valued matrix functions that are analytic in the open unit disc and bounded on the unit circle with the norm defined by $\|\hat{x}\|_{\mathcal{H}_\infty} := \text{ess}\ \sup_\theta \sigma_{\max}[\hat{x}(e^{i\theta})]$.
$\mathcal{N}$	Set of all the natural numbers.
$\mathcal{R}$	Real number system.
$\mathcal{R}^n$	n-dimensional Euclidean space.
$\mathcal{R}^{m\times n}$	Set of m-by-n dimensional matrices.
P_n	Truncation operator on the space of sequences defined as $P_n(x(0)\ x(1)\ \ldots) = (x(0)\, x(1)\ \ldots\ x(n)\, 0\, 0\ \ldots)$.
$W(X^*, X)$	Weak star topology on X^* induced by X.
X^*	Dual space of a normed linear vector space X. For $x^* \in X^*$, $< x, x^* >$ denotes the value of the bounded linear functional x^* at $x \in X$.

Throughout this paper we assume, unless stated otherwise, all systems to be finite dimensional linear, time-invariant, and discrete-time.

I. INTRODUCTION

IN LARGE, complex, and distributed systems, there is often the need of considering a specific structure on the overall control scheme (e.g., [1]). In this paper, we consider the general framework of Fig. 1 where G may represent a complex system consisting of subsystems interacting with each other. The overall controller for G is K. Both G and K are assumed to be linear, discrete-time systems. The controller K has to respect a specific structure that may be imposed by interaction and communication constraints. A typical example of structure that has been studied extensively in the literature is when K is decentralized. However, the optimal performance problem when structural constraints are present still remains a challenge, notably the lack of a convex characterization of the problem (see, for example, [2]–[4] and the references therein). Taking an input–output point of view and parameterizing all stabilizing

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K via the Youla–Kucera [5] parameter Q one can see as a major source of difficulty, the fact that structural constraints on K may lead to nonconvex constraints on Q. A main theme in this paper is the identification of specific classes of problems for which the constraints on Q are convex with the appropriate choice of the coprime factors of G. The various classes of systems identified include what are herein referred to as nested, chained, hierarchical, delayed interaction and communication, and symmetric systems. They are associated with several practical applications such as integrated flight propulsion systems, platoons of vehicles, networked control, production lines, and chemical processes. Common to all these problems is that G_{22}, the part of the system that relates controls u to measurements y, has a specific structure. It is the structure of G_{22} that matters for convexity; the remaining part of G can be unstructured.

In the classes we consider, G_{22} has the same structure as the one imposed on the controller K. This by itself is not in general a necessary or sufficient condition for the problem to be convex. As indicated in [6] where these structures were initially reported, there is an algebraic property between the K's and G_{22}'s under consideration. That is, with the exception of the symmetric case, they form a ring as the structure is preserved in products, additions and in $(I - G_{22}K)^{-1}$ whenever the inverse exists, as it should, for well-posedness. A very interesting relaxation of this ring property is possible as reported in [7] where it is shown that structures that are so-called quadratically invariant, lead to convex closed-loop maps in the term $K(I - G_{22}K)^{-1}$, which is the Youla–Kucera parameter in the case of stable G. However, by itself convexity in $K(I - G_{22}K)^{-1}$ does not provide a method to effectively synthesize controllers that incorporate structure and specifications on the closed-loop performance in terms of various measures that may reflect various frequency and time domain concerns. This is precisely what this paper achieves. In particular, in this article we explicitly identify the coprime factors to be employed in the Youla–Kucera parameterization that result in the parameter Q inheriting the structure of the controller K. The results obtained hold for stable as well as unstable plants G_{22}. The constructive nature of these results allows one to pose the problem of optimal performance as an optimization problem in the parameter Q. For various different specifications and performance measures on the desired closed-loop behavior the resulting optimization problem takes a convex form, which we solve within any prespecified tolerance from the optimal via tractable finite-dimensional convex optimization problems.

The need to design control systems that perform well with respect to a variety of performance criteria is important in its own right and has led to the development of a number of multiobjective techniques over the last decade (see, for example, [8], [9], and the references therein). The majority of these methods concentrate on optimizing a combination of two-criteria, selected among standard measures such as $\mathcal{H}_\infty$, $\mathcal{H}_2$, ℓ_1 closed-loop norms. Moreover, these results do not address the concerns of controllers that need to satisfy structure constraints. Here, we consider the general multiobjective (GMO) control synthesis problem that can simultaneously incorporate multiple measures on the closed-loop in terms of the ℓ_1, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ norms and can incorporate time-domain template constraints on

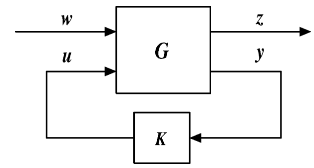


Fig. 1. Standard framework.

the response of the closed-loop due to specific inputs while respecting the structure constraints on the controller. As stated before, the results on translating the structure constraints on the controller K to structure constraints on the Youla–Kucera parameter Q makes it possible to use Q as the optimizing variable. An important advantage of incorporating Q as an optimization variable is that the controller can be easily retrieved from the optimal solution. This is in contrast to methods where interpolation constraints are imposed on the closed-loop map to characterize their achievability. This method has similarity to the basic Q-des method introduced in [10]. However, in this article emphasis is placed on obtaining a sequence of tractable finite dimensional convex optimization problems which provide converging upper and lower bounds to the optimal cost. Thus in this methodology, every step furnishes a feasible controller that meets the specifications together with the distance from the optimal performance. The development presented here is inspired in part by some of the ideas presented in [11] for the special case of the unstructured $\mathcal{H}_2/\ell_1$ problem. Our approach, apart from providing solutions to hitherto unaddressed multiobjective problems, relies only on the primal formulation of the problem. Thus, there is no need for constructing the dual and to perform the ensuing analysis of the dual and its relationship to the primal. This approach makes it straightforward to add new performance objectives into the problem formulation and its solution. Finally, it needs to be noted that problems with structured constraints have only recently been considered in [12]–[16], where the Youla–Kucera parameterization is used. These works all fall within the general classes of problems considered here.

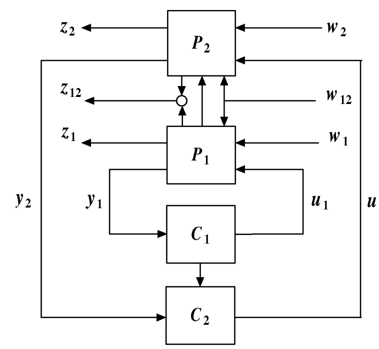
The paper is organized as follows. Previously, we introduced the notation to be used in this paper. In Section II, we present the various control structures of interest. In Sections III–V, we introduce the controller parameterization and describe the solution procedure to the structural multiobjective optimal control synthesis problem. In Section VI, we illustrate the effectiveness of the proposed framework with a network congestion control example. In Section VII, we conclude this paper.

II. Classes of System Structures

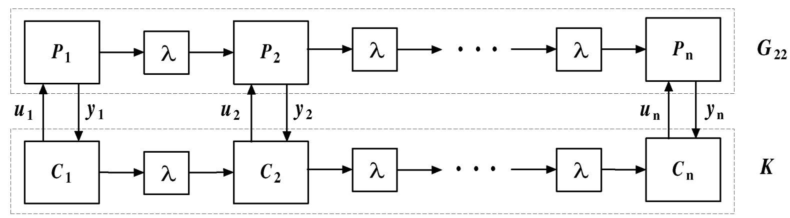
In this section, we present the basic classes of structured control systems that are of interest to this work. In what follows it is understood that all the subsystems in the various structures are multiple–input–multiple-output (MIMO), and the resulting G and K structural representations, whenever given, are in terms of MIMO subblocks.

A. Triangular Structures

In this class of systems, G_{22} and K are triangular transfer function matrices. This class includes what we term as nested structures, chains (or strings) with leader and followers, and



(a)



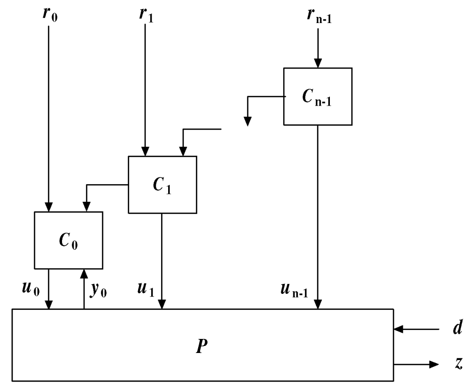
(b)

Fig. 2. (a) Nested structure. (b) Chain structure.

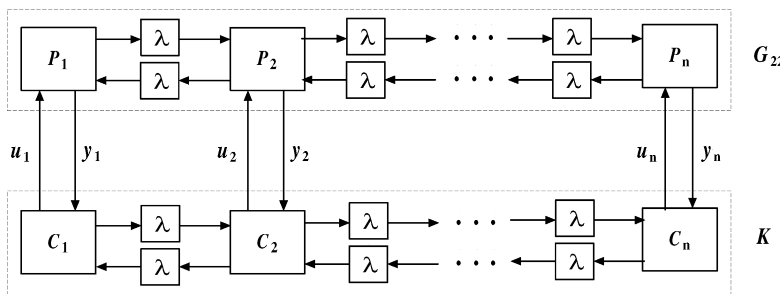
hierarchical type of schemes. In the sequel, we elaborate more on these.

Nested Systems: Simple Triangular Structures: This class represents the case where a subsystem is inside another and there is only one-way interaction, from inside to outside, or, the reverse. A practical application among several others that have this feature is the integrated flight propulsion control (IFPC) (see [17]). To illustrate the nested problem in simple terms we consider only two nests. The generalization to n nests is straightforward. Thus, we consider the case of Fig. 2(a), where there is a system comprised of two nests (subsystems). The internal subsystem consists of a plant P_1 together with its controller C_1 whereas the external subsystem consists of the plant P_2 together with the controller C_2. The internal and external subsystems have control inputs u_1 and u_2 and measured outputs y_1 and y_2 respectively. Due to the nested structure depicted in the figure, the control input u_1 depends only on the measurement y_1, whereas u_2 can depend on both y_1 and y_2. Moreover, y_1 is affected only by u_1 while y_2 is affected by both u_1 and u_2. The overall system is subjected to exogenous inputs (e.g., commands, disturbances, and sensor noise) and there are outputs to be regulated. In particular, we allow for inputs w_1 affecting directly the internal subsystem, inputs w_2 that affect the external subsystem only, and inputs w_{12} that affect both subsystems. Similarly, the outputs of interest z_1, z_2 and z_{12} are related, respectively, directly to the internal, directly to the external and to combination of both subsystems. A necessary assumption for the existence of a stabilizing overall controller K is that each subsystem P_i is stabilizable by each subcontroller C_i (this becomes clear for example by setting $w_{12} = w_1 = 0$ whereby C_2 is the only part of the controller that effects the plant component P_2). Thus, if the exogenously unforced (with no external disturbances) state–space descriptions for P_1 and P_2 are given by $x_1(k+1) = A_1x_1(k)+B_1u_1(k), y_1(k) = C_1x_1(k)+D_1u_1(k)$, and $x_2(k+1) = A_2x_2(k) + B_{21}u_1(k) + B_2u_2(k)$, $y_2(k) = C_2x_2(k) + D_{21}u_1(k) + D_2u_2(k)$, respectively, then we have that there exist observer and feedback gains, L_i and F_i respectively such that $A_i + L_iC_i$ and $A_i + B_iF_i$ have eigenvalues in the unit disc. We note here that, for simplicity, the particular state space descriptions used for P_is were state decoupled (i.e., u_1 only affects P_2). This is of no consequence in the developments to follow. Bringing the nested system to the standard $G - K$ control design framework of Fig. 1 we have the following signal identifications: $y := [y_1' \ \ y_2']'$, $u := [u_1' \ \ u_2']'$, $z := [z_1' \ \ z_{12}' \ \ z_2']'$, $w := [w_1' \ \ w_{12}' \ \ w_2']'$. The plant G_{22} has the following lower (block) triangular (l.b.t.) structure: $G_{22} := \begin{bmatrix} g_1 & 0 \\ g_{21} & g_2 \end{bmatrix}$ where $g_1 = C_1\lambda(I-\lambda A_1)^{-1}B_1+D_1, g_{21} = C_2\lambda(I-\lambda A_2)^{-1}B_{21}+D_{21}$, $g_2 = C_2\lambda(I-\lambda A_2)^{-1}B_2 + D_2$. Moreover, for the controller K to correspond to the nested structure of Fig. 2(a) it should be of the form $K = \begin{bmatrix} k_1 & 0 \\ k_{12} & k_2 \end{bmatrix}$, i.e., it should be a lower (block) triangular system.

Nested Systems: Chains: In the chain (or string) system of Fig. 2(b), there are n subsystems P_i with their corresponding subcontrollers C_i. Platoons of vehicles where there is a leader and followers that are obtaining information form their leading vehicles is a good example, among several others, to associate with this structure. The control action u_i in the subsystem P_i affects its follower P_{i+1} by a 1-step delay (λ) while the control action u_{i+1} in P_{i+1} does not affect its leader P_i. Also, subcontroller C_i passes information to its follower C_{i+1} with a 1-step delay while C_{i+1} does not transmit any information to C_i. Exogenous inputs w and regulated outputs z are admitted that may couple the dynamics but are not shown in the picture for clarity. Bringing in the general $G - K$ framework we have that the



(a)



(b)

Fig. 3. (a) Hierarchical structure. (b) Delayed interaction and communication structure.

structure of G_{22} and K is captured by $G_{22} = \Lambda\tilde{G}_{22}\Lambda^{-1}$ and $K = \Lambda\tilde{K}\Lambda^{-1}$ with $\tilde{G}_{22}$ and $\tilde{K}$ given by

$$\begin{bmatrix} g_1 & & & \\ g_{21} & g_2 & & \\ \vdots & & \ddots & \\ g_{n1} & \cdots & \cdots & g_n \end{bmatrix} \quad \begin{bmatrix} k_1 & & & \\ k_{21} & k_2 & & \\ \vdots & & \ddots & \\ k_{n1} & \cdots & \cdots & k_n \end{bmatrix}$$

respectively with $\Lambda = \operatorname{diag}(1, \lambda, \ldots, \lambda^{n-1})$. Thus, it follows that the chain problem can be addressed using an equivalent simple triangular structure.

Hierarchical Structures: Another interesting structure in this category is that of open hierarchies as depicted in Fig. 3(a). This is a multiple-input (or multiple-agent) system that needs to be regulated to follow certain external commands as well as to reject disturbances. Each control input is authorized by a single decision maker at a specified level in a decision making hierarchy. The decision maker receives signals from upper levels, possibly direct external commands, and, is allowed to pass information only to lower levels. The lowest level is the only level that receives feedback from the system. In particular, P is a linear time-invariant, discrete-time, system, the plant, that needs to be regulated, z represents the variables to be controlled, d some direct external disturbances and y_0 variables that are measured and can be used for feedback. The control inputs to P can be grouped to n (possibly vector) variables $u_0, \ldots, u_{n-1}$. Each variable u_i is authorized by a corresponding decision maker C_i at the ith level of a n-level hierarchical structure. In this structure, there is only one-way flow of signals, from upper to lower levels, and not vice-versa. Decision maker C_0 is the only one that processes the measurements y_0 from the plant P. There are also external direct commands to any level i denoted by r_i. For the setup to make sense C_0 should be able to stabilize P. The previously defined system can be put in a standard $G-K$ control design framework of Fig. 1 with the following signal identifications: $y := [\psi_0' \ r_1' \ \ldots \ r_{n-1}']'$, $u := [u_0' \ u_1' \ \ldots \ u_{n-1}']'$, $w := [d' \ r_0' \ \ldots \ r_{n-1}']'$ where $\psi_0 := [y_0' \ r_0']'$. For the controller K to be generated by the hierarchical structure described previously and *vice versa*, the constraint is that K should be an upper block triangular (u.b.t.) system. Considering the structure of G_{22} we have that it is u.b.t. and only its first row is nonzero.

As a 2-decision maker example of this abstract structure, one can imagine a high-priority traffic r_1 that needs to go through a node which also lets lower priority traffic r_0 to enter. In this case C_1 can be thought of as a high priority gate that does not "care" much for the queuing state y_0 of the node, but does communicate its decision to the lower priority gate C_0 which in turn needs to regulate how much of r_0 is to enter the node based on the available information y_0 and r_1. As an alternate example of this structure, one can think of situations when there is no flow of information from C_0 to decision makers $C_1, C_2, \ldots$ due to failures in the (one-way) communication, or, due to security reasons, and the system is required to operate on this mode for a significant amount of time.

An additional possible feature that is of interest in the overall scheme of Fig. 3(a) is that there can be different, yet commensurate, communication rates from layer to layer. For example, if C_0 samples y_0 and produces u_0 every T time units, C_1 can be sending signals to C_0 every n_1T time units, C_2 can be sending signals to C_1 every n_2n_1T time units, etc., where the n_i's are integers. By stacking the inputs and outputs to integer multiples of $n_{n-1}\ldots n_2n_1$ time steps, the so obtained lifted system representations (e.g., [18]) will have the same structure as before where now the subblocks themselves will have an extra block lower triangular structure in their feedthrough terms. The details are omitted for the sake of brevity.

Toeplitz Triangular Structures: An additional triangular structure that is interest that requires fewer building blocks is that of Toeplitz, i.e.,

$$G_{22} = \begin{bmatrix} g_1 & & & \\ \lambda g_2 & g_1 & & \\ \vdots & \ddots & \ddots & \\ \lambda^{n-1} g_n & \dots & \lambda g_2 & g_1 \end{bmatrix}$$

with K as in G_{22} by replacing g's with k's. Such structures appear for example in networking architectures and vehicular platoons where subsystems repeat (see the example at the end of the section.)

B. Delayed Interaction and Communication Networks: Band Structure

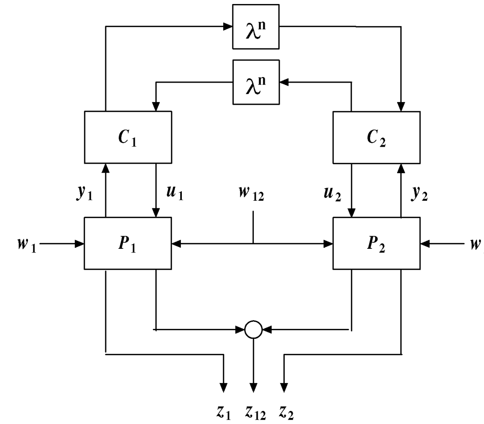
The network in this case is as in Fig. 3(b). In this figure, subsystem P_i and its subcontroller C_i interact with their respective neighbors with a 1-step delay in the transmission and reception of signals. Exogenous inputs w and regulated outputs z are admitted that may couple the dynamics but are not shown in the figure for clarity. Note that this is a generalized version of the chain structure of the previous subsection. A relevant example to associate in this case is the control of a large networked system over a network where, as an aggregate model, the neighbor-to-neighbor interaction and communication is subject to a unit delay, or, controlling an array of embedded micro sensors and actuators together with their corresponding micro controller. In the $G - K$ framework, the structure is reflected in G_{22} and K as

$$G_{22} = \begin{bmatrix} g_{11} & \lambda g_{12} & \cdots & \lambda^{n-1} g_{1n} \\ \lambda g_{21} & g_{22} & \cdots & \lambda^{n-2} g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{n-1} g_{n1} & \cdots & \cdots & g_{nn} \end{bmatrix} \tag{1}$$

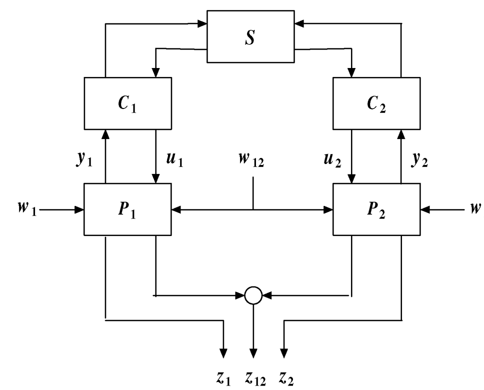
and similarly for K. We refer to the aforementioned structure in G_{22} and K as the band structure. For a distributed control system of a similar type in the case of spatially invariant systems, we refer to [19].

C. Other Structures

Delayed Observation Sharing: Two examples of communication patterns are given. Both are shown in the Fig. 4 and can be associated to the problem of controlling a production line of parallel operations to achieve a global objective, or controlling a power grid with decentralized stations. In Fig. 4(a), the measurement information from a local control station C_i is passed to the other with the delay of n time-steps. In Fig. 4(b), information exchanges between stations C_i are performed periodically every n time-steps through a data recording and supervising unit S. In both of these scenarios there is no interaction between the local plants P_i. There is however a coupling through the disturbances w and the variables z to be regulated. In the $G - K$ framework,



(a)



(b)

Fig. 4. (a) n-Step delayed information exchange structure. (b) n-Step information sharing structure.

the structure of the required controller K for the first case is $K = \begin{bmatrix} k_1 & \lambda^n k_{12} \\ \lambda^n k_{21} & k_2 \end{bmatrix}$. For the second case, assuming periodic control is used, one can lift [18] the system by stacking the inputs and outputs to integer multiples of n time steps to realize that the lifted controller K should have a feedthrough D-term of the form

$$D_K = \begin{bmatrix} f_1 & & & \\ f_2 & d_1 & & \\ \vdots & \vdots & \ddots & \\ f_n & d_{n-1} & \cdots & d_1 \end{bmatrix}$$

where f_i are full (block) 2×2 matrices and d_i are diagonal. In this case, the C_i's are n-time periodic controllers. Generalizations to tree-clusters of this structure are shown as in Fig. 5(a) where S_1 and S_2 communicate with a higher level unit Σ every $m \times n$ time-steps.

Symmetric Structures: This is the case where $G_{22} = G_{22}'$ and $K = K'$. Fig. 5(b) shows the case of a two-control input and two-measured output symmetric feedback system with g and k representing the coupling dynamics in the plant and controller, respectively. If $g_1 = g_2$, then we have a circulant symmetry in G_{22} [20]. Applications to associate with this class are cross-directional control of paper machines, multizone crystal growth furnaces, dyes for plastic films, etc.

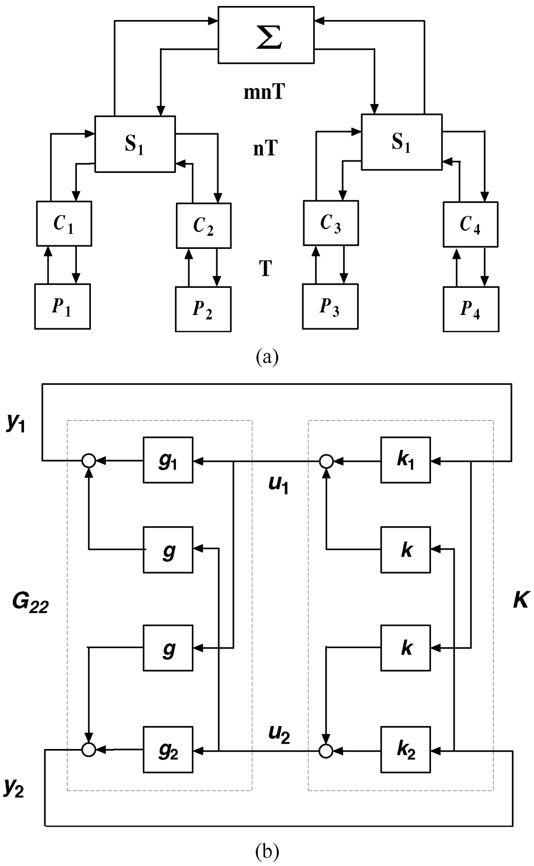


Fig. 5. (a) Tree structure. (b) Symmetric structure.

D. Platoon Structure Example

In the previous subsections, we presented various system structures (e.g., Figs. 2–5) that, when viewed in an overall G-K framework, impose structural constraints on K. The form of these constraints on K follows from the system architecture in a rather straightforward manner. That is, given the subcontroller blocks C_i there is a unique K in the indicated structure that is generated. The reverse direction, from a K to its identification and realization in terms of C_i's, is highly nonunique and depends entirely on what is selected as the information that the C_i's pass to each other. Herein, we provide an application example to illustrate in more detail how certain of the structures provided can appear and exhibit the connection between K and C_i's. Consider a platoon of n vehicles moving along a straight line [21]. The (simplified) dynamics for the ith vehicle are given as $x_i(k+1) = Ax_i(k) + B_{i-1}u_{i-1}(k) + B_iu_i(k)$ where $x_i(k) = \begin{pmatrix} p_i(k) \\ v_i(k) \end{pmatrix}$, $u_i(k) = a_i(k)$ with $p_i(k)$ and $v_i(k)$ the relative position and velocity, respectively, of the leading $(i-1)$th vehicle with respect to the ith vehicle at time k; $a_i(k)$ is the acceleration of the ith vehicle at time k; $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B_i = \begin{pmatrix} 0 \\ 1/m_i \end{pmatrix}$, with m_i the mass of the ith vehicle. For the sake of the example, assume that the "local" measurement y_i is $y_i(k) = x_i(k)$. For the leading vehicle $i = 1$, this can be interpreted with respect to a known frame (also, it is understood that in the above equations $B_{-1} = 0$ and $u_{-1} = 0$.) We can see that, in this case, we have

$$G_{22} = \begin{pmatrix} g_1 & & & \\ \lambda g_{21} & g_2 & & \\ & \ddots & \ddots & \\ & & \lambda g_{n(n-1)} & g_n \end{pmatrix}$$

with $g_i(\lambda) = (\lambda^{-1}I - A)^{-1}B_i$, $i = 1, \ldots, n$, $g_{i(i-1)}(\lambda) = (I - \lambda A)^{-1}B_{i-1}$, $i = 2, \ldots, n$.

We can identify this model with the nested plant G_{22} in Fig. 2(b) or, more generally, with G_{22} in Fig. 3(b). If the m_i's are identical then G_{22} is Toeplitz lower triangular. If we constrain the flow of information in the overall control scheme to be unidirectional as in Fig. 2(b), we are looking at generating the control signal in the functional form $u_i = \sum_{j=1}^{i-1} k_{ij}\lambda^{i-j}y_i + k_iy_i$ where the k_{ij} and k_i are controller transfer functions (it is also understood that when the upper limit in the summation term is smaller than the lower limit the term is zero). If G_{22} is Toeplitz we may impose a Toeplitz controller structure where $k_{ij} = k_{(i+1)(j+1)}$ and $k_i = k_1$. In the case of bidirectional flow as in Fig. 3(b), we have that u_i is of the functional form $u_i = \sum_{j=1}^{i-1} k_{ij}\lambda^{i-j}y_j + k_iy_i + \sum_{j=i+1}^{n} k_{ij}\lambda^{j-i}y_j$. Define the variables π_i^-, π_i^+ to represent information coming to station i from the leading and following stream, respectively, while ϖ_i^-, ϖ_i^+ represent information that station i sends to the following and leading stream, respectively. These variables are highly nonunique. For instance, if the overall controller structure K is stable, a particular selection for the information that is passed from subcontroller to subcontroller can be as $\pi_i^- = [\lambda^{i-1}y_1' \ \ \ldots \ \ \lambda y_{i-1}']'$, $\pi_i^+ = [\lambda y_{i+1}' \ \ \ldots \ \ \lambda^{n-i}y_n']'$, $\varpi_i^- = [\pi_i^{-'} \ \ y_i']'$, $\varpi_i^+ = [y_i' \ \ \pi_i^{+'}]'$. We can then rewrite $u_i = K_i^-\pi_i^- + k_iy_i + K_i^+\pi_i^+$ with $K_i^- = (k_{i1}\ldots k_{i(i-1)})$, $K_i^+ = (k_{i(i+1)}\ldots k_{in})$. Denoting $K_i = (K_i^- \ \ k_i \ \ K_i^+)$ a possible functional identification of the C_i in Fig. 3(b) is

$$\begin{pmatrix} \varpi_i^- \\ u_i \\ \varpi_i^+ \end{pmatrix} = C_i \begin{pmatrix} \pi_i^- \\ y_i \\ \pi_i^+ \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} I & & \\ & I & \\ & & 0 \end{pmatrix} \\ K_i \\ \begin{pmatrix} 0 & & \\ & I & \\ & & I \end{pmatrix} \end{pmatrix} \begin{pmatrix} \pi_i^- \\ y_i \\ \pi_i^+ \end{pmatrix}.$$

As mentioned earlier, this is not the only identification of K in terms of the C_i's. To illustrate the point, consider only the case of two vehicles with unidirectional communication and, for the sake of argument, let the resulting K be stable. Then, one possibility is $C_1 = [k_1' \ \ k_{21}']'$ and $C_2 = (\,I \quad k_2\,)$. An alternative, among the many, is $C_1 = [k_1' \ \ I]'$ and $C_2 = (\,k_{21} \quad k_2\,)$. Which identification, i.e., what is the information to be exchanged, and which implementation is better is not studied in this paper and represents current work by the authors [22].

III. Controller Parameterization

In this section, we show that for the classes of structures of the previous subsection, there exist factorizations that lead to a parameterization of all structured controllers in terms of a convexly constrained Youla parameter Q. More specifically, employing the Youla-Kucera parameterization, all stabilizing K, not necessarily with the structure required, are given by the parameterization [5]

$$\begin{aligned} K &= (Y_r - M_rQ)(X_r - N_rQ)^{-1} \\ &= (X_\ell - QN_\ell)^{-1}(Y_\ell - QM_\ell) \end{aligned} \tag{2}$$

where Q is a stable free parameter and Y_r, M_r, X_r, N_r, X_l, N_l, Y_l, and M_l can be obtained from a coprime factorization (e.g., [23] and [24]) of G_{22}. The coprime factors in (2) are highly nonunique. However, there is a particular choice of these factors such that the structural constraints on K transform to constraints on Youla parameter Q. In fact, these constraints are the same as in the required structure for K as developed here.

A. Triangular Structures

We first establish the result for simple triangular structures.

Lemma 4.1: Let the plant P that maps the control inputs $u = (u_1', \ldots, u_n')'$ to the measured outputs $y = (y_1', \ldots, y_n')'$ be lower triangular. Assume that the subsystem identified by the i^{th} row of P denoted by $P_i = \{P_{i1} \ \ldots \ P_{ii} \ 0 \ \ldots, 0\}$ admits a realization

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$$

with $B_i = \{B_{i1} \ \ldots \ B_{ii} \ 0 \ \ldots 0\}$, $D_i = (D_{i1} \, D_{i2} \, \ldots D_{ii} \, 0 \, \ldots 0)$ and

$$\left[\begin{array}{c|c} A_i & B_{ii} \\ \hline C_i & D_{ii} \end{array}\right]$$

the inherited state space realization of P_{ii} is stabilizable and detectable. Then there exist stable parameters Y_r, M_r, X_r, N_r, X_ℓ, N_ℓ, Y_ℓ, and M_ℓ such that the following statements are equivalent.

- K is lower triangular and it internally stabilizes the interconnection depicted in Fig. 6.
- There exists a stable Q that is lower triangular such that $K = (Y_r - M_rQ)(X_r - N_rQ)^{-1} = (X_\ell - QN_\ell)^{-1}(Y_\ell - QM_\ell)$.

Proof: Note that P admits a realization

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with $A = \mathrm{diag}\{A_1, \ldots, A_n\}$, $B = (B)_{ij}$, $C = \mathrm{diag}\{C_1, \ldots, C_n\}$, and $D = (D)_{ij}$. Note also that from the stabilizability and detectability condition we have that there exists matrices F_i and L_i such that the matrices $A_i + B_{ii}F_i$ and $A_i + L_iC_i$ are Hurwitz. Let $F := \mathrm{diag}\{F_1, \ldots, F_n\}$ and $L := \mathrm{diag}\{L_1, \ldots, L_n\}$. Then, $A + BF$ is lower triangular with its diagonal equal to $\mathrm{diag}\{A_1 + B_{11}F_1, \ldots, A_n + B_{nn}F_n\}$

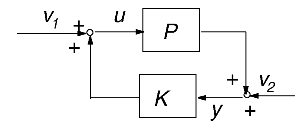


Fig. 6. Feedback interconnection of $P = G_{22}$ and K.

and $A + LC = \mathrm{diag}\{A_1 + L_1C_1, \ldots, A_n + L_nC_n\}$. Thus, $A + BF$ and $A + LC$ are Hurwitz. Let

$$N_r = \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \end{array}\right]$$
$$M_r = \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \end{array}\right]$$
$$Y_r = \left[\begin{array}{c|c} A + BF & -L \\ \hline F & 0 \end{array}\right]$$
$$X_r = \left[\begin{array}{c|c} A + BF & -L \\ \hline C + DF & I \end{array}\right].$$

Then, we have that K is internally stabilizing if and only if there exists a stable Q such that $K = (Y_r - M_rQ)(X_r - N_rQ)^{-1}$ (indeed the coprime factors are the standard factors obtained via the Luenberger observer; see [23]). Note that as $N_r = (C+DF)[\lambda^{-1}I - (A+BF)]^{-1}B + D$, where $(C+DF)$, $[\lambda^{-1}I - (A+BF)]$, B, and D are lower triangular and the lower triangular property is preserved under the operations of multiplication, inverse, and in addition, it follows that N_r transfer matrix is lower triangular. Similarly it can be shown that M_r, X_r and Y_r are lower triangular transfer matrices. Using the same arguments on the preservation of the triangular structure it follows that $K = (Y_r - M_rQ)(X_r - N_rQ)^{-1}$ is lower triangular if and only if Q is lower triangular. This proves the lemma. ∎

We note that in our setup the subsystems are assumed state-decoupled. In the case where there is coupling the same proof goes through. That is, if the matrices A and C in the realization of P in the proof are, instead of diagonal, triangular (coming for example from a minimal realization,) the stabilizability and detectability conditions guarantee the existence of diagonal F and L. Hence, the proof remains the same.

Toeplitz Triangular Structures: For Toeplitz triangular systems a similar result as proven for simple triangular structure can be established. The detailed construction of the coprime factors and the proof are provided in the Appendix .

Lemma 4.2: Let the plant P that maps the control inputs $u = (u_1', \ldots, u_n')'$ to the measured outputs $y = (y_1', \ldots, y_n')'$ be lower triangular and Toeplitz with

$$P = \begin{bmatrix} P_1 & & & & \\ P_2 & P_1 & & & \\ P_3 & P_2 & P_1 & & \\ & \ddots & \ddots & \ddots & \\ P_n & P_{n-1} & P_3 & P_2 & P_1 \end{bmatrix}.$$

Let T be the last row $T = (P_n, \ldots, P_1)$. Assume that T admits a stabilizable and a detectable realization

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with $B = (B_1 \ \ldots \ B_n)$ and $D = (D_1 \ D_2 \ \ldots D_n)$. Furthermore, assume that

$$\left[\begin{array}{c|c} A & B_1 \\ \hline C & D_1 \end{array}\right]$$

is a stabilizable and detectable realization of P_1. Then, there exist stable parameters Y_r, M_r, X_r, N_r, X_ℓ, N_ℓ, Y_ℓ and M_ℓ such that the following statements are equivalent.

- K is lower triangular Toeplitz and it internally stabilizes the interconnection depicted in Fig. 6.
- There exists a stable Q that is lower triangular Toeplitz such that $K = (Y_r - M_r Q)(X_r - N_r Q)^{-1} = (X_\ell - Q N_\ell)^{-1}(Y_\ell - Q M_\ell)$.

B. Delayed Interaction and Communication Structures

For the case of delayed interaction and communication structures (which have the band structure) the assumptions and the arguments to establish the result differ from that of triangular structure. Nevertheless, it can be established that the coprime factors can be constructed for the plant such that structure constraints can be equivalently posed on the Youla–Kucera parameter as shown later.

We consider band structures described by (1) where the dynamics for each subsystem P_i is of the form $x_i^+(t) := x_i(t+1) = A_{ii}x_i(t) + \sum_{j=1}^n B_{ij}u_j(t - |i-j|)$, $y_i(t) = C_{ii}x_i(t) + \sum_{j=1}^n D_{ij}u_j(t - |i-j|)$ where x_i is a local state, u_i is the local control, and y_i is the local measurement variables. We assume that the overall system is stabilizable by a controller that respects the information exchange pattern indicated by the band structure. Using a λ-transform representation and with some abuse of notation we can rewrite the ith subsystem dynamics as $x_i^+ = A_{ii}x_i + \sum_{j=1}^n B_{ij}\lambda^{|i-j|}u_j, \ y_i = C_{ii}x_i + \sum_{j=1}^n D_{ij}\lambda^{|i-j|}u_j$ The overall system dynamics can be compactly written as $x^+ = Ax + B(\lambda)u$, $y = Cx + D(\lambda)u$ where $x = (x_1', \ldots, x_n')'$, $u = (u_1', \ldots, u_n')'$, $y = (y_1', \ldots, y_n')'$. We note that in the previous representation the matrices $B(\lambda)$, $D(\lambda)$ have band structure; also A and C are diagonal, hence these trivially correspond to band structures. This system can be brought into a standard state-space form by introducing the state variable $\chi = (x' \ \lambda u' \ \lambda^2 u' \cdots \lambda^{n-1}u')'$ to obtain $\chi^+ = \overline{A}\chi + \overline{B}u, \ \ y = \overline{C}\chi + \overline{D}u$ where $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$ are constant matrices independent of λ. We will assume now that this is a stabilizable system in the following sense: There exists a constant gain $\overline{F}$ such that $u = \overline{F}\chi$. stabilizes the system (i.e., $\overline{A} + \overline{BF}$ Hurwitz) while it respects the subcontroller to subcontroller information exchange pattern. That is to say that u_i should depend only on a subset of χ given by x_i and $\left\{\lambda^{|i-j|}u_j\right\}_{j=1}^n$ i.e., $u = \overline{F}_{0,x}x + \overline{F}_{1,u}\lambda u + \cdots + \overline{F}_{n-1,u}\lambda^{n-1}u$ with $F_{0,x}$-diagonal and $F_{i,u}$ $(2i+1)$-diagonal matrices. More compactly, one can represent the above as $u = \overline{F}_{0,x}x + \overline{F}_u(\lambda)u$ where $\overline{F}_{0,x}$ and $\overline{F}_u(\lambda)$ have band structure. Since addition, multiplication, and inversion preserve the band structure, solving for u leads to $u = F(\lambda)x$ with $F(\lambda)$ being also a banded map of the form

$$F(\lambda) = \begin{pmatrix} F_{11}(\lambda) & \cdots & \lambda^{n-1}F_{1n}(\lambda) \\ \vdots & & \vdots \\ \lambda^{n-1}F_{n1}(\lambda) & \cdots & F_{nn}(\lambda) \end{pmatrix}.$$

It should be clear that for a given $\bar{F}$ there is a unique corresponding $F(\lambda)$.

As with stabilizability, we assume that the system is detectable in the following sense: There exists a constant gain $\overline{L}$ such that an observer $\widehat{\chi}^+ = \overline{A}\widehat{\chi} - \overline{L}(y - \hat{y})$, $\hat{y} = \overline{C}\widehat{\chi}$ has stable state error estimate dynamics (i.e., $\overline{A} + \overline{LC}$ is Hurwitz) with $\hat{x}_i^+$ depending only on a subset of available measurements that is consistent with the information pattern, i.e., on $\left\{\lambda^{|i-j|}y_j\right\}_{j=1}^n$. Thus, if $\overline{L} = (L_{0,\hat{x}}' \ \ L_{\hat{x}/\widehat{\chi}}')'$ the constraint is that $L_{0,\hat{x}}$ is diagonal which in turn leads, after simple manipulations, to a compact representation for the aforementioned observer as $\hat{x}^+ = A\hat{x} - L(\lambda)(y - \hat{y}), \ \ \hat{y} = C\hat{x}$ where $L(\lambda)$ is a band system

$$L(\lambda) = \begin{pmatrix} L_{11}(\lambda) & \cdots & \lambda^{n-1}L_{1n}(\lambda) \\ \vdots & & \vdots \\ \lambda^{n-1}L_{n1}(\lambda) & \cdots & L_{nn}(\lambda) \end{pmatrix}.$$

Again, there is a unique correspondence between $\overline{L}$ and $L(\lambda)$. Thus, to generate banded coprime factors of G_{22} and all controllers, an observer with band structure can be used which compactly written is as

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + B(\lambda)u - L(\lambda)(y - \hat{y}) \\ \hat{y} &= C\hat{x} + D(\lambda)u, \ \ u = F(\lambda)\hat{x} + v \end{aligned}$$

where $v = Q(\lambda)e$, $e = y - \hat{y}$. That Q is necessary and sufficient to be banded follows as A, $B(\lambda)$, $L(\lambda)$, C, D, and $F(\lambda)$ are banded. The aforementioned discussion leads to the following lemma.

Lemma 4.3: Let the plant $P = G_{22}$ that maps the control inputs $u = (u_1', \ldots, u_n')'$ to the measured outputs $y = (y_1', \ldots, y_n')'$ be represented by a realization as before. Let there be $\bar{F}$ and $\bar{L}$ with the properties as given above. Then there exist stable parameters Y_r, M_r, X_r, N_r, X_ℓ, N_ℓ, Y_ℓ, and M_ℓ such that the following statements are equivalent.

- K has the band structure and it internally stabilizes the interconnection shown in Fig. 6.
- There exists a stable Q with the band structure such that $K = (Y_r - M_r Q)(X_r - N_r Q)^{-1} = (X_\ell - Q N_\ell)^{-1}(Y_\ell - Q M_\ell)$.

We would like to note that in our setup the subsystems are assumed state-decoupled. In the case where there is coupling the same approach as in the proof goes through, i.e., if we consider subsystems dynamics for P_i of the form

$$\begin{aligned} x_i^+(t) &= \sum_{j=1}^n A_{ij}x_j(t - |i-j|) \\ &\quad + \sum_{j=1}^n B_{ij}u_j(t - |i-j|) \\ y_i(t) &= \sum_{j=1}^n C_{ij}x_j(t - |i-j|) + \sum_{j=1}^n D_{ij}u_j(t - |i-j|) \end{aligned}$$

under the same stabilizability condition, banded coprime factors can be obtained from a banded observer based controller.

C. Other Structures

The case of delayed observation sharing structures of Fig. 4 is similar to the previous subsection and thus it will not be discussed further. In fact, since G_{22} is diagonal, all the coprime factors will be diagonal and can be obtained by factorizing each block separately. The result is that Q has to have the same structure as K. For the symmetric structures mentioned we consider the case where G_{22} is stable and symmetric, i.e., $G'_{22} = G_{22}$. Then, all stabilizing K, possibly nonsymmetric, are given as $K = (I + QG_{22})^{-1}Q$. If K is to be symmetric, then

$$\begin{aligned} Q &= K(I - G_{22}K)^{-1} = K'(I - G'_{22}K')^{-1} \\ &= (I - K'G'_{22})^{-1}K' = ((I - G_{22}K)')^{-1}K' = Q' \end{aligned}$$

i.e., Q is symmetric. Similar argument shows that if Q is symmetric then $K = (I+QG_{22})^{-1}Q$ is symmetric. More generally, if G_{22} is unstable, one can choose the coprime factors appropriately as in [16] to have the same result, i.e., K is symmetric if and only if Q is symmetric.

Remark: It is to be noted that the realizations of G_{22} used in some proofs are not required to be minimal. This does not pose any problem in obtaining observer based controller parameterizations as long as the state-space descriptions for G_{22} are stabilizable and detectable.

Remark: It should be clear from our exposition so far that, with the exception of the symmetric structures, a ring-type property between the structures of G_{22} and K is present. For example, in the triangular and band structures, products, additions, and inversions preserve the structure. This also leads to coprime factors, and consequently Q, to have the same structure. For the symmetric structure, the ring-type property fails. Nonetheless, quadratic invariance [7], a relaxation of the aforementioned ring property, still holds as the product $KG_{22}K$ remains symmetric when K, G_{22} are symmetric (while KG_{22} is not symmetric in general). It is this property that leads to a symmetric $K(I - G_{22}K)^{-1}$ (which is equal to the Youla parameter Q in the stable case). The classes of structures presented here by no means constitute a complete characterization of structured control problems that lead to convex constraints in the Youla parameter Q (consider for example any similarity transformation TGT^{-1} and TKT^{-1} of these structures.) They represent a basic, diverse enough set that relates, as indicated, to current and emerging control application problems. It is for these specific structures that we explicitly construct a solution for the optimal performance problem of the next section.

IV. Optimal Performance With Multiple Objectives in the Presence of Structure Constraints

All the classes discussed in Section III require, when viewed in the G-K framework, that K be stabilizing and have a specific structure (triangular, band, symmetric, etc). From the discussion in the previous section, this is equivalent to requiring that Q has the same structure as K. For a multiobjective formulation of these classes of structured problems, we consider the system shown in Fig. 7, a suitable refinement of Fig. 1,

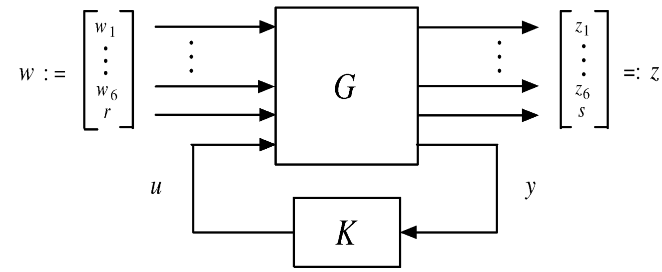


Fig. 7. Closed-loop system.

where $G : [w; u] \to [z; y]$ is the generalized discrete-time linear time-invariant plant and K is the structured controller. w, z, u, and y are the exogenous input, regulated output, control input, and measured output, respectively. r is a given scalar reference input (such as a step) and s is the corresponding time response. Let $\hat{R}$ denote the closed-loop transfer matrix from w to z. The set of all the achievable closed-loop maps is given by [25]

$$\{\hat{R} = G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} | K \text{ stabilizing and structured}\}$$

where $G = [G_{zw}\ G_{zu}; G_{yw}\ G_{yu}]$ is the open-loop transfer matrix from $[w; u]$ to $[z; y]$. To simplify the notations in the sequel, we use $\hat{R}^i$ $(i = 1, \ldots, 6)$ to denote the closed-loop transfer matrix from w_i to z_i and $\hat{R}^7$ the transfer function from r to s. The general multiobjective (GMO) problem studied in this paper can be stated as follows: *Given the plant G, constants* $c_i > 0$, $i = 1, \ldots, 6$, *and two sequences* $\{a_{\rm temp}(k)\}_{k=0}^{\infty}$ *and* $\{b_{\rm temp}(k)\}_{k=0}^{\infty}$, *solve the following problem*:

$$\begin{aligned} &\inf_{K \text{ stabilizing, structured}} f(K) \\ &\text{subject to} \\ &\|R^4(K)\|_1 \le c_4 \\ &\|\hat{R}^5(K)\|_{\mathcal{H}_2}^2 \le c_5 \\ &\|\hat{R}^6(K)\|_{\mathcal{H}_\infty} \le c_6 \\ &a_{\rm temp}(k) \le s(k) \le b_{\rm temp}(k) \end{aligned} \tag{3}$$

where $f(K) = c_1\|R^1(K)\|_1 + c_2\|\hat{R}^2(K)\|_{\mathcal{H}_2}^2 + c_3\|\hat{R}^3(K)\|_{\mathcal{H}_\infty}$ and $\{s(k)\}_{k=0}^{\infty}$ denotes the time response of the closed-loop system due to the exogenous reference input r with $w_i = 0$, $i = 1, \ldots, 6$. Let μ denote the optimal value of the aforementioned problem. From now on, we will always assume that problem (3) has a nonempty feasible set. The GMO problem defined previously represents a large class of multiobjective control problems. Many extensively studied (unstructured) multiobjective problems are special cases of the GMO setup, e.g., $\mathcal{H}_2/\ell_1$ [11], ℓ_1/TDC [26], [27]. Furthermore, for the first time, the $\mathcal{H}_\infty/\ell_1$ problem and $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ problem are addressed. The problem formulation in (3) also provides a uniform framework for the performance tradeoff study involving the ℓ_1, $\mathcal{H}_2$, $\mathcal{H}_\infty$, and TDC. By solving the GMO problem for various combinations of the parameters $c_i (i = 1, \ldots, 6)$ and template sequences $\{a_{\rm temp}(k)\}$ and $\{b_{\rm temp}(k)\}$, important information on the limits of system performance can be obtained both qualitatively and quantitatively. As indicated, for the classes of structural problems described in the previous sections, a suitable choice of the Youla parameterization leads to subspace type of restrictions on Q. We denote by $\mathcal{S}$ the closed subspace of stable systems $Q \in \ell_1^{n_u \times n_y}$ that have the required structure. Then, a characterization of all the achievable closed-loop maps can be given as follows: $\{R \in \ell_1^{n_z \times n_w} | R = H - U * Q * V \text{ with } Q \in \mathcal{S}\}$ where $H \in \ell_1^{n_z \times n_w}, U \in \ell_1^{n_z \times n_u}, V \in \ell_1^{n_y \times n_w}$, Q is a free parameter in $\mathcal{S}$ and '$*$' denotes the convolution operation. In the sequel, without any loss of generality, we shall always assume that $\hat{U}$ and $\hat{V}$ have full-column and row ranks, respectively (see [25]).

Also, it can be assumed $\hat{H}$, $\hat{U}$ and $\hat{V}$ are polynomial matrices in λ. This assumption is justified by the fact that H, U and V are operators in the $\ell_1^{n_z \times n_w}$ space. Another reason for considering U and V to polynomial is that the denominator polynomials of U and V can be absorbed by the Q parameter (without destroying the imposed structure). Let H, U and V in the Youla parameterization be partitioned into submatrices of compatible dimensions with the exogenous input component w_i and regulated output component z_i. Then, the closed-loop transfer matrix sequences from w_i to z_i can be expressed as $R^i(Q) = H^{ii} - U^i * Q * V^i, i = 1, \ldots, 7$. For the sake of simplicity, and without loss of generality, we will consider the case when r is a step sequence. Let A_{temp} be defined as

$$A_{\text{temp}} := \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Then, the time response of the closed-loop system due to the reference input r is given by $s = R^7 * r = A_{\text{temp}} R^7$ where with some abuse of notation, R^7 is being interpreted as the impulse response stacked into a column, s and r are the output $[s(0)\, s(1)\, \ldots]'$ and the input $[r(0)\, r(1)\, \ldots]'$ respectively. Based on the Youla parameterization and the previous discussion, the problem defined in (3) has the following equivalent formulation:

$$\begin{aligned} \mu = \inf_{Q \in \mathcal{S}} & f(Q) \\ \text{subject to } & \|R^4(Q)\|_1 \le c_4 \\ & \|R^5(Q)\|_2^2 \le c_5 \\ & \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \le c_6 \\ a_{\text{temp}}(k) \le & [A_{\text{temp}} R^7(Q)](k) \le b_{\text{temp}}(k) \end{aligned} \tag{4}$$

where $f(Q) := c_1\|R^1(Q)\|_1 + c_2\|R^2(Q)\|_2^2 + c_3\|\hat{R}^3(Q)\|_{\mathcal{H}_\infty}$, $R^i(Q) = H^{ii} - U^i * Q * V^i, i = 1, \ldots, 7$.

A. Auxiliary Problem

In the general case, (4) is a difficult problem to solve. To facilitate the solution of this problem, we define an auxiliary problem closely related to it. The auxiliary GMO problem statement is

$$\begin{aligned} \nu = \inf_{Q \in \mathcal{S}} & f(Q) \\ \text{subject to } & \|Q\|_1 \le \gamma \\ & \|R^4(Q)\|_1 \le c_4 \\ & \|R^5(Q)\|_2^2 \le c_5 \\ & \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \le c_6 \\ a_{\text{temp}}(k) \le & [A_{\text{temp}} R^7(Q)](k) \le b_{\text{temp}}(k). \end{aligned} \tag{5}$$

Note there is an extra one norm bound on the Youla parameter Q in the auxiliary problem compared with the original GMO problem (4). As will be seen later, this extra constraint $\|Q\|_1 \le \gamma$ plays an essential role in obtaining the solution to (4). Also, introducing Q as an optimization variable facilitates the computation of the optimal controller that avoids the numerical difficulties involved with zero interpolation methods. Another significant advantage (as will be seen later) over previous methods is that the optimal solution can be approached via the primal alone; there is no need to formulate or analyze the dual problem. This adds considerable flexibility to the approach that enables it to accommodate diverse control objectives in a seamless manner.

B. Relationship Between the GMO Problem and the Auxiliary Problem

In the problem formulation of (4), Q needs to satisfy the constraint $\|R^4(Q)\|_1 = \|H^{44} - U^4 * Q * V^4\|_1 \le c_4$. Let us suppose $\hat{U}^4$ and $\hat{V}^4$ have full normal column and row rank, respectively. Further suppose $\hat{U}^4$ and $\hat{V}^4$ have no zeros on the unit circle. Then, U^4 and V^4 are left- and right-invertible in ℓ_1 and it follows that $\|Q\|_1 \le \|(U^4)^{-l}\|_1(\|H^{44}\|_1 + c_4)\|(V^4)^{-r}\|_1 := \beta$, where $(U^4)^{-l}$ and $(V^4)^{-r}$ denote the left and right inverse of U^4 and V^4, respectively. Consequently, if we choose $\gamma \ge \beta$ in the auxiliary problem, the constraint $\|Q\|_1 \le \gamma$ is redundant in GMO problem and we get $\nu = \mu$. We now consider the case where $\hat{U}^4$ or $\hat{V}^4$ has zeros on the unit circle. Under this circumstance, there is a possibility that the original GMO problem does not admit an optimal solution and the one norm of the optimization variable Q cannot be restricted to any bounded set. Thus, from a computational point of view, it would be desirable to impose a reasonable bound on $\|Q\|_1$ in the optimization for this case as well.

V. Problem Solution

For the remainder of this paper, we shall focus our attention on the auxiliary problem (5). Note that (5) is an infinite-dimensional optimization problem and the existence of an optimal solution is not clear. We make the following assumption on the TDCs: *For all k, $a_{\text{temp}}(k) < b_{\text{temp}}(k)$. Furthermore, there exists N_1, N_2 so that $a_{\text{temp}}(k) = a_{\text{temp}}(N_1)$ for all $k \ge N_1$ and $b_{\text{temp}}(k) = b_{\text{temp}}(N_2)$ for all $k \ge N_2$.* This assumption is easily met in most cases of practical interest.

A. Existence of an Optimal Solution and Converging Lower Bounds

In this section, we develop a sequence of finite dimensional convex optimization problems whose objective values converges to ν from below. We will also prove the existence of an optimal solution to (5). For most of the functional analytic results utilized [28] is an excellent source for detailed exposition to the concepts used. Define

$$T_i(Q) := \begin{bmatrix} R^i(0) & 0 & 0 & \cdots & \cdots \\ R^i(1) & R^i(0) & 0 & \cdots & \cdots \\ R^i(2) & R^i(1) & R^i(0) & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix}.$$

$T_{i,k}(Q)$ will denote the matrix consisting of the first k rows of $T_i(Q)$. It is a standard result [25], [29] that

$$\|\hat{R}^i(Q)\|_{\mathcal{H}_\infty} = \|T_i(Q)\| := \sup_k \sigma_{\max}(T_{i,k}(Q)) = \sup_k \|T_{i,k}(Q)\|$$

where $\|\cdot\|$ denotes the matrix spectral norm. Furthermore, from standard results in linear algebra (e.g., [29, Th. 4.3.8] or [30, Ch. 2]), we have $\|T_{i,k}(Q)\| \le \|T_{i,k+1}(Q)\| \le \|T_i(Q)\|$, for all k. Based on the above discussion, we define a candidate lower bound of ν as

$$\begin{aligned} \nu_n &:= \inf_{Q\in\mathcal{S}} f_n(Q) \\ &\text{subject to } \|Q\|_1 \le \gamma \\ &\|P_n(R^4(Q))\|_1 \le c_4 \\ &\|P_n(R^5(Q))\|_2^2 \le c_5 \\ &\|T_{6,n}(Q)\| \le c_6 \\ &a_{\text{temp}}(k) \le [A_{\text{temp}} R^7(Q)](k) \le b_{\text{temp}}(k) \end{aligned} \tag{6}$$

where

$$f_n(Q) := c_1\|P_n(R^1(Q))\|_1 + c_2\|P_n(R^2(Q))\|_2^2 + c_3\|T_{3,n}(Q)\|$$

and $k = 0, 1, \ldots, n$. Since only the parameters of $Q(0), \ldots, Q(n)$ enter into the optimization, problem (6) is a finite dimensional convex programming problem. Thus, it always admits an FIR optimal solution on the nonempty compact feasible set. The following lemma is an immediate consequence of the previous definition.

Lemma 6.1: For all n, $\nu_n \le \nu_{n+1} \le \nu$.

Now, we present the main result of this section.

Theorem 6.1: There is an optimal solution Q^0 in $\ell_1^{n_u\times n_v}$ to problem (5) that satisfies the structural constraints described by $Q \in \mathcal{S}$. Moreover, $\nu_n \nearrow \nu$.

Proof: Suppose $Q_n \in \mathcal{S}$ is a finitely supported optimal solution to (6). Note that $\|Q_n\|_1 \le \gamma$ for any positive integer n and Banach–Alaoglu theorem (see [28]) implies that $B_\gamma := \{Q \in \ell_1^{n_u\times n_v} : \|Q\|_1 \le \gamma\}$ is weak-star compact. Thus, there exists a subsequence $\{Q_{n_m}\}$ of $\{Q_n\}$ and Q^0 in $\ell_1^{n_u\times n_v}$ such that $(Q_{n_m})_{ij} \to (Q^0)_{ij}$ ($i = 1, \ldots, n_n,\ j = 1, \ldots, n_y$) in the $W(c_0^*, c_0)$ topology. It follows that for all t, $Q_{n_m}(t) \to Q^0(t)$ and for all n, $P_n(R(Q_{n_m})) \to P_n(R(Q^0))$ and $T_{i,n}(Q_{n_m}) \to T_{i,n}(Q^0)$ ($i = 3, 6$) as $m \to \infty$. Moreover, suppose without loss of generality that, Q_{n_m} in $\mathcal{S}$ is required to be such that $(Q_{n_m})_{ij} = 0$. Then, this is equivalent to require that $(Q_{n_m})_{ij}(t) = 0, \forall t$ and so it follows from the aforementioned arguments that for all t, $Q^0_{ij}(t) = 0$, that is, $Q^0 \in \mathcal{S}$ (note that the same argument holds in case of other structures on Q like, for example, the Toeplitz structure). For any $n > 0$ and for any $n_m > n$, $f_n(Q_{n_m}) \le f_{n_m}(Q_{n_m}) = \nu_{n_m} \le \nu$. By letting $m \to \infty$, we get $f_n(Q^0) \le \nu$, $\forall n$. Since n is arbitrary, it follows that $f(Q^0) \le \nu$. Similar arguments show that $\|Q^0\|_1 \le \gamma$, $\|R^4(Q^0)\|_1 \le c_4$, $\|R^5(Q^0)\|_2^2 \le c_5$. Furthermore, for any given $k > 0$ and for any $n_m \ge k$, $\|T_{6,k}(Q_{n_m})\| \le \|T_{6,n_m}(Q_{n_m})\|$. Recall the fact that $T_{6,k}(Q_{n_m})$ is a function of $Q_{n_m}(0), \ldots, Q_{n_m}(k)$ only. By letting $m \to \infty$, we have $\|T_{6,k}(Q^0)\| \le c_6$, $\forall k$. Since k is arbitrary, it follows that $\|T_6(Q^0)\| := \sup_k \|T_{6,k}(Q^0)\| \le c_6$. Finally, for any given $k > 0$, there exists some $n_m > k$ so that $a_{\text{temp}}(k) \le [A_{\text{temp}} R^7(Q_{n_m})](k) \le b_{\text{temp}}(k)$. Then, for all $l \ge m$, we have $a_{\text{temp}}(k) \le [A_{\text{temp}} R^7(Q_{n_l})](k) \le b_{\text{temp}}(k)$. By letting l tend to infinity, it follows that $a_{\text{temp}}(k) \le [A_{\text{temp}} R^7(Q^0)](k) \le b_{\text{temp}}(k)$, $\forall k$. Thus, Q^0 is an optimal solution to problem (5). To prove that $\nu_n \nearrow \nu$, we note that for all $n > 0$ and $n_m > n$, $f_n(Q_{n_m}) \le f_{n_m}(Q_{n_m}) = \nu_{n_m}$. Taking the limit as m tends to infinity we have $f_n(Q^0) \le \lim_{m\to\infty} \nu_{n_m}$, for all $n > 0$. It follows that $f(Q^0) \le \lim_{m\to\infty} \nu_{n_m}$. Thus, we have shown that $\lim_{m\to\infty} \nu_{n_m} = \nu$. Since ν_n is a monotonically increasing sequence bounded above by ν, it follows that $\nu_n \nearrow \nu$. ■

B. Converging Upper Bounds

It is clear that ν_n itself does not provide any information on its distance to the optimal cost ν. This motivates the computation of an upper bound of ν. Let ν^n be defined by

$$\begin{aligned} \nu^n &:= \inf_{Q\in\mathcal{S}} f(Q) \\ &\|Q\|_1 \le \gamma \\ &\|R^4(Q)\|_1 \le c_4 \\ &\|R^5(Q)\|_2^2 \le c_5 \\ &\|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \le c_6 \\ &a_{\text{temp}}(k) \le [A_{\text{temp}} R^7(Q)](k) \le b_{\text{temp}}(k) \\ &Q(k) = 0 \text{ if } k > n \end{aligned} \tag{7}$$

The numerical solution of this problem amounts to solving a convex programming problem involving only $Q(0), \ldots, Q(n)$. It is clear that since H, U and V are all finitely supported, the time response $y = A_{\text{temp}} R^7(Q)$ would be a constant after some finite time instant $N \ge n$ and (7) is a finite-dimensional optimization problem.

Lemma 6.2: Given any Q in $\ell_1^{n_u\times n_v}$ and positive real number δ, there exists some N so that $n \ge N$ implies

$$\begin{aligned} &|\,\|R^4(P_n(Q))\|_1 - \|R^4(Q)\|_1\,| \\ &< \delta |\,\|R^5(P_n(Q))\|_2^2 - \|R^5(Q)\|_2^2\,| \\ &< \delta \;|\,\|\hat{R}^6(P_n(Q))\|_{\mathcal{H}_\infty} - \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty}\,| \\ &< \delta \;\left|\,[A_{\text{temp}} R^7(P_n(Q))](k) - [A_{\text{temp}} R^7(Q)](k)\,\right| < \delta \qquad \forall k. \end{aligned}$$

Proof: See the Appendix. ■

Define

$$\begin{aligned} C := \{\, &(\gamma, c_4, c_5, c_6, a_{\text{temp}}(0), \ldots, a_{\text{temp}}(N_1) \\ &b_{\text{temp}}(0), \ldots, b_{\text{temp}}(N_2)) \\ &\in \mathcal{R}^{6+N_1+N_2} |\ \text{there exists } Q \in \mathcal{S} \text{ so that} \\ &\|Q\|_1 \le \gamma,\ \|R^4(Q)\|_1 \le c_4.\ \|R^5(Q)\|_2^2 \le c_5,\ \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \\ &\le c_6\ a_{\text{temp}}(k) \le [A_{\text{temp}} R^7(Q)](k) \le b_{\text{temp}}(k) \qquad \forall k\}. \end{aligned}$$

Lemma 6.3: C is a convex set.

Lemma 6.4: ν is a continuous function with respect to $(\gamma, c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}})$ in the interior of C.

Proof: The conclusion follows immediately from Lemma 6.3. ■

In what follows, we will assume $(\gamma, c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}})$ lies in the interior of C.

Theorem 6.2: $\{\nu^n\}$ forms a monotonically decreasing sequence of upper bounds of ν. Furthermore, $\nu^n \searrow \nu$, as $n \to \infty$.

Proof: Clearly, $\nu^n \geq \nu^{n+1}$ since any Q in $\mathcal{S}$ which belongs to the feasible set of ν^n will also be feasible for problem ν^{n+1}. For same reason, we have $\nu^n \geq \nu$ for all n. Thus, $\{\nu^n\}$ is a decreasing sequence of real numbers bounded below by ν. For notational simplicity, in what follows, we will omit the symbol γ in ν^n and ν. For any given $\epsilon > 0$, the continuity of ν implies that there exists $\delta > 0$ such that

$$\nu(c_4-\delta, c_5-\delta, c_6-\delta, a_{\text{temp}}+\delta, b_{\text{temp}}-\delta) \\ -\nu(c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}}) < \frac{\epsilon}{4}$$

where $a_{\text{temp}}+\delta := \{a_{\text{temp}}(k)+\delta\}_{k=0}^{\infty}$ and $b_{\text{temp}}-\delta := \{b_{\text{temp}}(k)-\delta\}_{k=0}^{\infty}$. Also, there exists some Q^δ such that

$$\begin{aligned} &f(Q^\delta) - \nu(c_4-\delta, c_5-\delta, c_6-\delta, a_{\text{temp}}+\delta, b_{\text{temp}}-\delta) \\ &< \frac{\epsilon}{4} \quad \|R^4(Q^\delta)\|_1 \leq c_4 - \delta \quad \|R^5(Q^\delta)\|_2^2 \\ &\leq c_5 - \delta \quad \|\hat{R}^6(Q^\delta)\|_{\mathcal{H}_\infty} \leq c_6 - \delta \quad a_{\text{temp}}(k) + \delta \\ &\leq [A_{\text{temp}} R^7(Q^\delta)](k) \leq b_{\text{temp}}(k) - \delta \qquad \forall k \quad \|Q^\delta\|_1 \leq \gamma. \end{aligned}$$

By Lemma 6.2, there exists some positive integer N large enough so that $n \geq N$ implies

$$\begin{aligned} &f(P_n(Q^\delta)) - f(Q^\delta) \\ &< \frac{\epsilon}{2} \quad \|R^4(P_n(Q^\delta))\|_1 - \|R^4(Q^\delta)\|_1 \\ &< \frac{\delta}{2} \quad \|R^5(P_n(Q^\delta))\|_2^2 - \|R^5(Q^\delta)\|_2^2 \\ &< \frac{\delta}{2} \quad \|\hat{R}^6(P_n(Q^\delta))\|_{\mathcal{H}_\infty} - \|\hat{R}^6(Q^\delta)\|_{\mathcal{H}_\infty} \\ &< \frac{\delta}{2} \quad \left| [A_{\text{temp}} R^7(P_n(Q^\delta))](k) - [A_{\text{temp}} R^7(Q^\delta)](k) \right| \\ &< \frac{\delta}{2} \qquad \forall k. \end{aligned}$$

It follows from this that for all $n \geq N$

$$\begin{aligned} f(P_n(Q^\delta)) &< \nu(c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}}) + \epsilon \quad \|R^4(P_n(Q^\delta))\|_1 \\ &< c_4 \quad \|R^5(P_n(Q^\delta))\|_2^2 < c_5 \quad \|\hat{R}^6(P_n(Q^\delta))\|_{\mathcal{H}_\infty} \\ &< c_6 \quad a_{\text{temp}}(k) \leq [A_{\text{temp}} R^7(P_n(Q^\delta))](k) \\ &\leq b_{\text{temp}}(k) \qquad \forall k \quad \|P_n(Q^\delta)\|_1 \leq \gamma. \end{aligned}$$

Thus, $P_n(Q^\delta)$ satisfies all the constraints of problem $\nu^n(c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}})$ and it follows that for all $n \geq N$:

$$\nu^n(c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}}) - \nu(c_4, c_5, c_6, a_{\text{temp}}, b_{\text{temp}}) < \epsilon.$$

This proves the theorem. ∎

After establishing the convergence of ν_n and ν^n to ν, we now briefly address the issue of constructing suboptimal controllers from the optimizing Youla parameter Q. For any prescribed performance tolerance $\delta > 0$, the optimizing process can be stopped once for some n, $|\nu_n - \nu^n| \leq \delta$. The minimizing variable Q^n to the upper bound ν^n can then be used to recover a suboptimal controller which achieves the objective value ν^n.

We should also mention here that for a specific subclass of structured problems, recent very interesting work [31] has provided an LMI based framework that provides controllers with a degree bound which depends on the plant order only. This subclass satisfies a one-block hypothesis for some subsystem in the formulation of the performance objective, and considers LMI-type of performance criteria such as $\mathcal{H}_\infty$ and $\mathcal{H}_2$.

C. Uniqueness and Convergence of the Optimal Solution

Having established the existence of an optimal solution Q^0 to problem (5) and the convergence of ν^n to ν, we now present results addressing the uniqueness and convergence properties of the suboptimal and optimal solutions. The proofs for these results are given in the Appendix .

Theorem 6.3: Suppose $\hat{U}^2$ and $\hat{V}^2$ have full-column and row rank on the unit circle, respectively. Let Q^n denote an optimal solution to ν^n. Let Q^0 denote an optimal solution to ν. Let $R^n := H - U * Q^n * V, n = 0, 1, \ldots$. Then, R^n $(n = 0, 1, \ldots)$ is unique, and Q^n $(n = 0, 1, \ldots)$ is unique.

One direct consequence of Theorem 6.3 is that Q^0 is the weak-star convergent limit of a subsequence of $\{Q^n\}$.

Lemma 6.5: Suppose $\hat{U}^2$ and $\hat{V}^2$ have full-column and row rank on the unit circle, respectively. Then, there exists a subsequence $\{Q^{n_m}\}$ of $\{Q^n\}$ such that $(Q^{n_m})_{ij} \to (Q^0)_{ij}$ $(i = 1, \ldots, n_u,\ j = 1, \ldots, n_y)$ in the $W(c_0^*, c_0)$ topology.

If no $\mathcal{H}_\infty$ term is present in the objective function of the GMO problem, the conclusion of Theorem 6.3 can be made stronger. More explicitly, suppose $c_3 = 0$ in the GMO problem setup (5), i.e, $f(Q) = c_1\|R^1(Q)\|_1 + c_2\|R^2(Q)\|_2^2$. It can be easily seen that the conclusions established in Theorems 6.1 and 6.2 hold.

Lemma 6.6: Let $f : (R^1, R^2) \to \mathcal{R}$ (where R^1, R^2 are matrices consisting of elements in ℓ_1) be defined by: $f(R^1, R^2) := c_1\|R^1\|_1 + c_2\|R^2\|_2^2$. Let $\{(R^{1,k}, R^{2,k})\}$ be a sequence such that $(R^{1,k}(t), R^{2,k}(t)) \to (R^{1,o}(t), R^{2,o}(t))$ for all t and

$$f(R^{1,k}, R^{2,k}) \leq f(R^{1,o}, R^{2,o}), \qquad \text{for all } k. \tag{8}$$

Let $\|R^{1,o}\|_1 = \|(R^{1,o})_p\|_1$ where $(R^{1,o})_p$ represents the p^{th} row of $R^{1,o}$. Then, $c_1\|(R^{1,k})_p - (R^{1,o})_p\|_1 + c_2\|R^{2,k} - R^{2,o}\|_2^2 \to 0$ as $k \to \infty$. The same conclusion holds if (8) is replaced with the following condition:

$$f(R^{1,k}, R^{2,k}) \to f(R^{1,o}, R^{2,o}). \tag{9}$$

Theorem 6.4: Suppose $\hat{U}^2$ and $\hat{V}^2$ have full-column and row rank on the unit circle, respectively. Let Q^n denote an optimal solution to $\nu^n(c_3 = 0)$ and let $R^n := H - U * Q^n * V$, $R^{i,n} := H^{ii} - U^i * Q^n * V^i$, $i = 1, \ldots, 7$, $n = 0, 1, \ldots$. Then R^n $(n = 0, 1, \ldots)$ is unique, Q^n $(n = 0, 1, \ldots)$ is unique, and $\|R^{2,n} - R^{2,0}\|_2 \to 0$, as $n \to \infty$.

It should be remarked that the lower bound version of Theorem 6.4 also holds and the proof can be carried out in exactly the same manner as that for Theorem 6.4. As a concluding remark for this section, we want to point out that the GMO control design framework we have developed here is flexible. Given any finite numbers of $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ norm objectives and TDCs, they can be directly stacked into the GMO problem formalism and the theoretical and numerical schemes established in this and the previous section can be extended in a straightforward manner to obtain the solution.

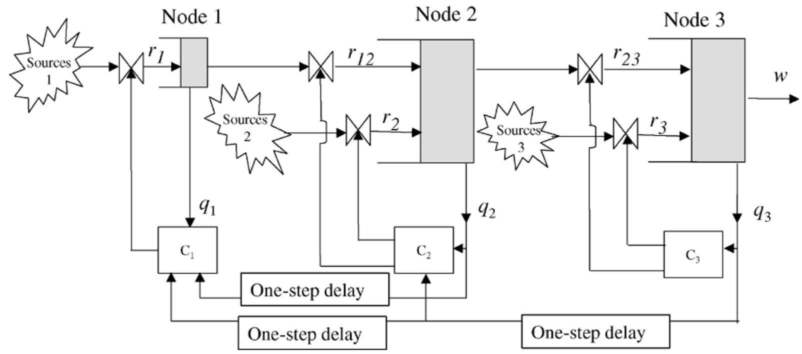


Fig. 8. 3-nodal ABR network.

VI. Example of Optimal Control Design for a 3-Nodal ABR Network

Consider the schematic in Fig. 8, that depicts a network of three nodes. The purpose of the model is to study various aspects of coordination control between various nodes and its relation to the information structure. An associated application is congestion control in the case of an available bit rate (ABR) communication network [33]. To keep the exposition relatively simple, we chose three nodes that captures the essential features of the related problem. The development is the same for any number of nodes. In Fig. 8, r_1, r_2 and r_3 denote the flow rates from data sources into network nodes 1, 2, and 3, respectively. r_{12} denotes the rate of flow from node 1 to node 2 and r_{23} denotes the rate of flow from node 2 to node 3. w represents the total capacity available for the three data sources. q_i denotes the buffer length at the i^{th} node. The network exerts control over the network traffic by assigning the rate for each data source. In particular, there are three (nodal) subcontrollers C_1, C_2, C_3 that dictate respectively r_1, (r_{12}, r_2), and (r_{21}, r_3). In other words each C_i regulates traffic entering node i. Moreover, there is a one-step delay in passing nodal information (q_i) from one nodal subcontroller C_i to its preceding one C_{i-1}, while each C_i does not receive information from any of the preceding nodes. A feature of such a structure is that even if some node i is congested due to for example a failure, the operation of the nodes following i is not affected. The goals are to prevent the node buffers from overflowing so as to avoid possible data loss ("stabilization goal"), and to optimally utilize the available transfer capacity w such that the sum of the data rates $r_i (i = 1, 2, 3)$ matches w as closely as possible ("optimality goal"). For this system, the exogenous input signal is identified as the available capacity w. The control input, and measured output signals are identified, respectively as: $u = [\ r_1\ r_{12}\ r_2\ r_{23}\ r_3\ ]'$, and $y = [\ q_1\ q_2\ q_3\ ]'$. The goal of the congestion control for the above network can be captured by adopting the following signal identification for the regulated output: $z = [\ q_1\ \ q_2\ \ q_3\ \ r_1 - wa_1\ \ r_2 - wa_2\ \ r_3 - wa_3\ ]'$, where a_i is a prescribed constant representing the ratio of available resource assigned to i_{th} source. Suppose also that steps are the typical exogenous input signals w we would like to optimally track. Then, we can impose TDCs on $z_i (i = 4, 5, 6)$ such that the step response of $z_i (i = 4, 5, 6)$ is forced to stay within a prescribed envelope. In the sequel, we consider the coordination of the network operation around a desired equilibrium point where the queues at the nodes and the traffic rates are at a desired nonzero, positive level. The linearized fluid model nodal dynamics that we adopt are given by

- node 1: $q_1(k+1) = q_1(k) + r_1(k) - r_{12}(k)$;
- node 2: $q_2(k+1) = q_2(k) + r_2(k) + r_{12}(k) - r_{23}(k)$;
- node 3: $q_3(k+1) = q_3(k) + r_3(k) + r_{23}(k) - w(k)$.

There are three local controllers corresponding to the three nodes such that the controllers are required to satisfy the following structural constraints: $C_1 : r_1 = f_1(q_1, \lambda q_2, \lambda^2 q_3)$

$$C_2 : \begin{cases} r_{12} & = & f_{12}(q_2, \lambda q_3) \\ r_2 & = & f_2(q_2, \lambda q_3) \end{cases} \quad C_3 : \begin{cases} r_{23} & = & f_{23}(q_3) \\ r_3 & = & f_3(q_3) \end{cases}$$

where the various f_i and f_{ij} are (causal) linear operators and λ is interpreted as the one step delay operator. Clearly, the plant G_{22} and the controller K are upper triangular operators of the following form:

$$\left[\begin{array}{c:cc:cc} * & \lambda * & \lambda * & \lambda^2 * & \lambda^2 * \\ \hdashline 0 & * & * & \lambda * & \lambda * \\ \hdashline 0 & 0 & 0 & * & * \end{array}\right], \quad \left[\begin{array}{c:c:c} * & \lambda * & \lambda^2 * \\ \hdashline 0 & * & \lambda * \\ 0 & * & \lambda * \\ \hdashline 0 & 0 & * \\ 0 & 0 & * \end{array}\right] \tag{10}$$

respectively. In this example, we provide a tradeoff study between ℓ_1 and $\mathcal{H}_2$ performance of the closed-loop system by solving the following multiobjective problem:

$$\nu := \inf c_1 \|R(K)\|_1 + c_2 \|R(K)\|_2^2$$
subject to
K is stabilizing
K satisfies (10)
$z_i (i = 4, 5, 6)$ satisfies prescribed TDCs

where c_1 and c_2 are prescribed weighting constants. Following the framework established in Section IV, we now detail the procedure of how the upper block triangular structural constraints

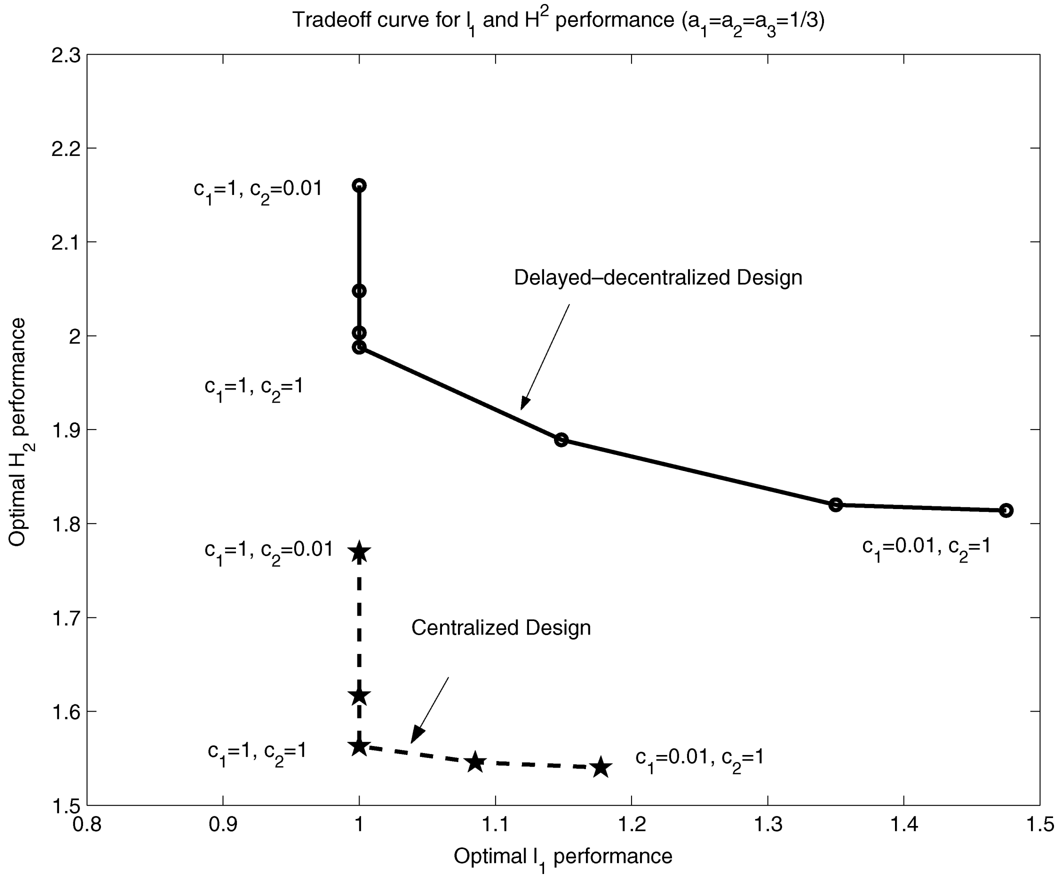


Fig. 9. Tradeoff curve between ℓ_1 and $\mathcal{H}_2$ performance.

on K as specified in (10) are transformed to the same structural constraints on Q. The state-space description of G_{22} is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|ccc} A_1 & 0 & 0 & B_1 & B_{12} & 0 \\ 0 & A_2 & 0 & 0 & B_2 & B_{23} \\ 0 & 0 & A_3 & 0 & 0 & B_3 \\ \hline C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 & 0 \end{array}\right]$$

where $A_1 = A_2 = A_3 = 1$, $C_1 = C_2 = C_3 = 1$, $B_1 = 1$, $B_{12} = [-1\ 0]$, $B_2 = [1\ 1]$, $B_{23} = [-1\ 0]$, and $B_3 = [1\ 1]$. The state feedback and observer gain matrices for G_{22} are chosen to be $F = \mathrm{diag}(F_1, F_2, F_3)$ and $L = \mathrm{diag}(L_1, L_2, L_3)$ $F_1 = -0.90$, $F_2 = F_3 = [0\ -0.9]'$, and $L_i = -0.90$. This choice of F and L guarantees that $A+BF$ and $A+LC$ are stable matrices. Using the F and L described and the resulting co-prime factors (see (2)) the structural constraints on the controller K transform to the same constraints on Q. Hence, we equivalently formulate problem ν as

$$\begin{aligned} \nu := &\inf c_1\|R(Q)\|_1 + c_2\|R(Q)\|_2^2 \\ &\text{subject to} \\ &Q \text{ is stable} \\ &Q \in \mathcal{S} \\ &z_i (i = 4, 5, 6) \text{ satisfies prescribed TDCs} \end{aligned}$$

where $\mathcal{S}$ characterizes the structure constraints and TDC are the time domain constraints that the three error signals z_4, z_5 and z_6 due to step inputs lie within the upper and lower templates b_{temp} and a_{temp} given by $b_{\mathrm{temp}} = [0.5\,0.5\,0.5\,0.5\,0.5\,0.5\,0.01\,0.01\ldots]$ and

$$a_{\mathrm{temp}} = [-0.5 - 0.5 - 0.5 - 0.5 - 0.5 - 0.5 - 0.01 - 0.01 \ldots].$$

The fairness index a_i is taken to be $a_1 = a_2 = a_3 = 1/3$ and the upper bounds of $\|Q\|_1$ are chosen to be $\gamma = 100$. For a given increasing sequence of nonnegative ratios of c_2/c_1 (seven points), the auxiliary problem of ν was solved by using GMO 1.0 package and the optimal Youla parameters and the values of $\|R(Q)\|_1$ and $\|R(Q)\|_2^2$ were obtained. For all pairs of c_1 and c_2, the ℓ_1 norms of the optimal Q's are far less than γ (typically $\|Q\|_1 \leq 1$). This shows that the extra ℓ_1 norm constraint on Q

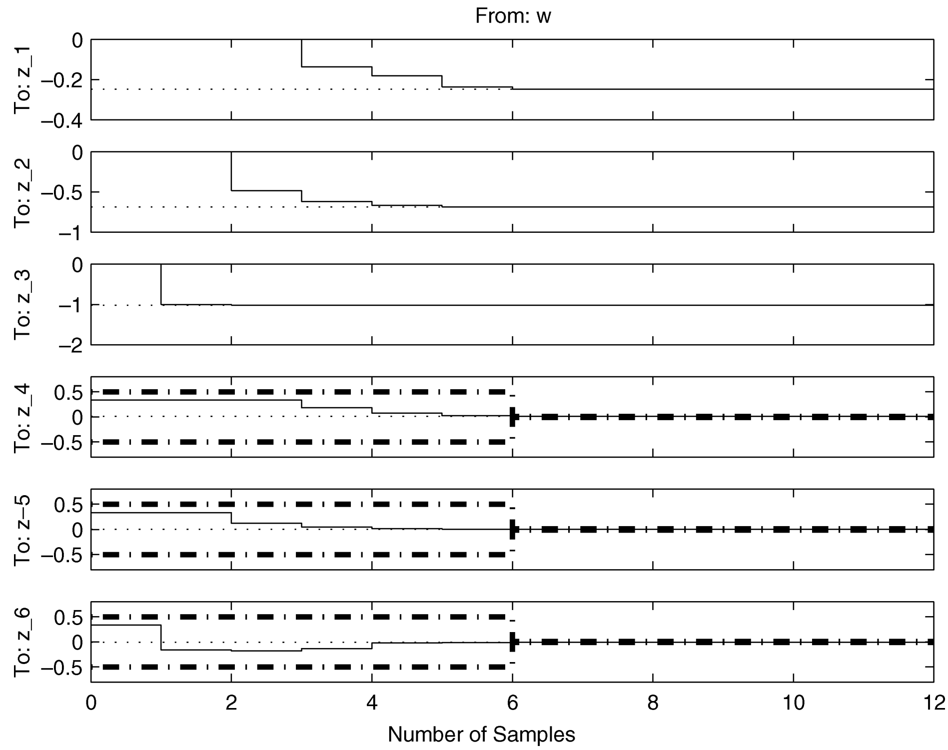


Fig. 10. Step response of closed-loop system with decentralized delayed controller.

is inactive and problem ν and its auxiliary problem admit the same optimal cost.

The plots of $\|R(Q)\|_1$ versus $\|R(Q)\|_2^2$ are shown in Fig. 9, where the dashed curve denotes the cases of centralized design with no information transfer delay while the solid curve denotes the cases where there exists transfer delay in the feedback path, as illustrated in Fig. 8. From these two curves, important information on the tradeoff among system performance specifications are obtained. It can also be concluded that for this example, the structure constraints imposed on the stabilizing controllers as specified in (10) induce a significant loss of the closed-loop system performance.

The impulse responses of the centralized sub-optimal controller (case $c_1 = c_2 = 1$, performance tolerance $\delta = 0.01$), and the decentralized, delayed sub-optimal controller (case $c_1 = c_2 = 1$, performance tolerance $\delta = 0.01$) show that the structural constraints imposed on the stabilizing controller are satisfied. That is, the controller admits the upper block triangular structure specified in (10) while the centralized controller does not admit such a structure. The order of the Youla parameter Q is 3 and the order of the corresponding decentralized, delayed suboptimal controller is 6. In Fig. 10, the step response of the closed-loop system with decentralized, delayed controller is plotted, where the dash-dotted lines denote the TDC envelops imposed on the step responses of $z_i (i = 4, 5, 6)$. It is clear from the response plots that the time response of $z_i (i = 4, 5, 6)$ satisfies the requirement of zero steady value, which implies that the optimality goal of the congestion control mechanism is achieved.

VII. Conclusion and Discussion

In this paper, we presented a convex optimization approach for optimal synthesis in systems in which the overall control scheme is required to have certain structure. These classes can be associated with several practical applications in integrated flight propulsion systems, platoons of vehicles, networked control, production lines, chemical processes, etc. The common thread in all of these classes is that by taking an input-output point of view we can characterize all stabilizing controllers in terms of convex constraints in the Youla–Kucera parameter. We showed that the regularized GMO problem admits a minimizing solution in ℓ_1 and, more importantly, from an engineering viewpoint, we showed how to obtain arbitrarily close to optimal controllers within any prespecified accuracy. The use of this method for simultaneous parameter and control design (see [34] and [35]) is under investigation.

Appendix

A. *Proof of Lemma 4.2*

Note that a particular realization of P is given by $\tilde{A} = \mathrm{diag}(A, \ldots, A)$, $\tilde{C} = \mathrm{diag}(C, \ldots, C)$ and $\tilde{B}$ and $\tilde{D}$ given by lower triangular toeplitz matrices As (A, C) is detectable one can show that $(\tilde{A}, \tilde{B})$ is stabilizable and $(\tilde{A}, \tilde{C})$ is detectable. Indeed, if F and L are such that $A + B_1 F$ and $A + LC$ are stable then it follows that $\tilde{A} + \tilde{B}\tilde{F}$ and

$\tilde{A} + \tilde{L}\tilde{C}$ are Hurwitz where $\tilde{F} = \operatorname{diag}\{F, F, \ldots F\}$ and $\tilde{L} = \operatorname{diag}\{L, L, \ldots, L\}$. Let

$$N_r = \left[\begin{array}{c|c} \tilde{A}+\tilde{B}\tilde{F} & \tilde{B} \\ \hline \tilde{C}+\tilde{D}\tilde{F} & \tilde{D} \end{array}\right]$$
$$M_r = \left[\begin{array}{c|c} \tilde{A}+\tilde{B}\tilde{F} & \tilde{B} \\ \hline \tilde{F} & I \end{array}\right]$$
$$Y_r = \left[\begin{array}{c|c} \tilde{A}+\tilde{B}\tilde{F} & -\tilde{L} \\ \hline \tilde{F} & 0 \end{array}\right]$$
$$X_r = \left[\begin{array}{c|c} \tilde{A}+\tilde{B}\tilde{F} & -\tilde{L} \\ \hline \tilde{C}+\tilde{D}\tilde{F} & I \end{array}\right].$$

Then, we have that K is internally stabilizing if and only if there exists a stable Q such that $K = (Y_r - M_r Q)(X_r - N_r Q)^{-1}$ (see [23]). Note that as $N_r = (\tilde{C}+\tilde{D}\tilde{F})[\lambda^{-1}I - (\tilde{A}+\tilde{B}\tilde{F})]^{-1}\tilde{B}+\tilde{D}$, where $(\tilde{C}+\tilde{D}\tilde{F})$, $[\lambda^{-1}I - (\tilde{A}+\tilde{B}\tilde{F})]$, $(\tilde{B}+\tilde{D})$ are lower triangular Toeplitz and the lower triangular Toeplitz property is preserved under the operations of multiplication, inverse and addition it follows that N_r transfer matrix is lower triangular Toeplitz. Similarly it can be shown that M_r, X_r and Y_r are lower triangular Toeplitz transfer matrices. Using the same arguments on the preservation of the triangular structure it follows that $K = (Y_r - M_r Q)(X_r - N_r Q)^{-1}$ is lower triangular Toeplitz if and only if Q is lower triangular Toeplitz. This proves the lemma. ∎

B. Proof of Lemma 6.2

It is clear that

$$\begin{aligned} \|R^4((I-P_n)(Q))\|_1 &= \|U^4 * ((I-P_n)(Q)) * V^4)\|_1 \\ &\le \|U^4\|_1 \|(I-P_n)(Q)\|_1 \|V^4\|_1 \\ \|A_{\text{temp}} R^7((I-P_n)(Q))\|_\infty &= \|R^7((I-P_n)(Q)) * r\|_\infty \\ &\le \|R^7((I-P_n)(Q))\|_1 \\ &\le \|U^7\|_1 \|(I-P_n)(Q)\|_1 \|V^7\|_1. \end{aligned}$$

Since ℓ_1 is a proper subspace of ℓ_2, we infer by Hölder inequality that $\|R^5((I-P_n)(Q))\|_2 \le \|U^5\|_1^1 \|(I-P_n)(Q)\|_1 \|V^5\|_1$. Moreover, for any given $x \in \ell_1^{n_u \times n_v}$, $\|\hat{x}\|_{\mathcal{H}_\infty} \le \sqrt{n_u}\|x\|_1$ and it follows that $\|\hat{R}^6((I-P_n)(Q))\|_{\mathcal{H}_\infty} \le \sqrt{n_u}\|U^6\|_1 \|(I-P_n)(Q)\|_1 \|V^6\|_1$. Since $Q \in \ell_1^{n_u \times n_v}$, $\|(I-P_n)(Q)\|_1$ can be made arbitrarily small by letting n large enough and the conclusion follows immediately from the previous four inequalities. ∎

C. Proof of Theorem 6.3

1) Define $A^0_{feas} := \{R \in \ell_1^{n_z \times n_w} :$ | there exists $Q \in \ell_1^{n_u \times n_v}$ so that

$$\begin{aligned} &R = H - U * Q * V \ \ \|Q\|_1 \le \gamma \ \ \|R^4(Q)\|_1 \\ &\le c_4 \ \ \|R^5(Q)\|_2^2 \le c_5 \ \ \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \le c_6 \ \ a_{\text{temp}}(k) \\ &\le [A_{\text{temp}} R^7(Q)](k) \le b_{\text{temp}}(k) \qquad \forall k\}. \end{aligned}$$

Then, A^0_{feas} is a convex set. Problem (5) is equivalent to $\nu = \inf_{R \in A^0_{feas}} c_1\|R^1\|_1 + c_2\|R^2\|_2^2 + c_3\|\hat{R}^3\|_{\mathcal{H}_\infty}$. $\hat{R}^i$ and $\hat{R}$ can be expressed as $\hat{R}^i = \hat{E}_i \hat{R} \hat{F}_i$ where $\hat{E}_i = [0_{n_{z_i} \times n_{z_1}} \cdots I_{n_{z_i}} \cdots 0_{n_{z_i} \times n_{z_7}}] \in \mathcal{R}^{n_{z_i} \times n_z}$, $\hat{F}_i = [0_{n_{w_1} \times n_{w_i}} \cdots I_{n_{w_i}} \cdots 0_{n_{w_7} \times n_{w_i}}]' \in \mathcal{R}^{n_w \times n_{w_i}}$, $i = 1, \ldots, 7$. Also, the λ transforms $\hat{E}_i = E_i$ and $\hat{F}_i = F_i$. Thus, the problem can be reformulated as $\nu = \inf_{R \in A^0_{feas}} g(R)$ where

$$\begin{aligned} g(R) &:= c_1\|E_1 * R * F_1\|_1 + c_2\|E_2 * R * F_2\|_2^2 \\ &+ c_3\|\hat{E}_3 \hat{R} \hat{F}_3\|_{\mathcal{H}_\infty} = c_1\|R^1\|_1 + c_2\|R^2\|_2^2 + c_3\|\hat{R}^3\|_{\mathcal{H}_\infty}. \end{aligned}$$

We claim that $g(R)$ is a strict convex function of R given the assumption $c_2 > 0$. To see this, choose $R, S \in A^0_{feas}$ such that $R \ne S$. Then, it follows from the invertibility of $\hat{U}^2$ and $\hat{V}^2$ that $R^2 \ne S^2$. Then, for any $\alpha \in (0,1)$

$$\begin{aligned} &g(\alpha R + (1-\alpha)S) \\ &= c_1\|E_1 * (\alpha R) * F_1 + E_1 * ((1-\alpha)S) * F_1\|_1 \\ &\quad + c_2\|E_2 * (\alpha R) * F_2 + E_2 * ((1-\alpha)S) * F_2\|_2^2 \\ &\quad + c_3\|\hat{E}_3(\alpha \hat{R}^3)\hat{F}_3 + \hat{E}_3((1-\alpha)\hat{S})\hat{F}_3\|_{\mathcal{H}_\infty} \\ &= c_1\|\alpha R^1 + (1-\alpha)S^1\|_1 + c_2\|\alpha R^2 + (1-\alpha)S^2\|_2^2 \\ &\quad + c_3\|\alpha \hat{R}^3 + (1-\alpha)\hat{S}^3\|_{\mathcal{H}_\infty} < c_1\alpha\|R^1\| + c_1(1-\alpha)\|S^1\|_1 \\ &\quad + c_2\alpha\|R^2\|_2^2 + c_2(1-\alpha)\|S^2\|_2^2 + c_3\alpha\|\hat{R}^3\|_{\mathcal{H}_\infty} \\ &\quad + c_3(1-\alpha)\|\hat{S}^3\|_{\mathcal{H}_\infty} = \alpha g(R) + (1-\alpha)g(S) \end{aligned}$$

where the strict convexity of $\|\cdot\|_2^2$ and the convexity of $\|\cdot\|_1$ and $\|\cdot\|_{\mathcal{H}_\infty}$ are employed to justify the strict inequality sign. This proves that $g(\cdot)$ is strictly convex on A^0_{feas}. Thus, R^0, the minimizing closed-loop map to problem is unique. Similar arguments as show before that R^n $(n = 1, 2, \ldots)$ is unique. 2) The uniqueness of $Q^n (n = 0, 1, \ldots)$ follows immediately from (1) and the invertibility of $\hat{U}^2$ and $\hat{V}^2$. ∎

D. Proof of Lemma 6.5

Since $\{Q^n\}$ is uniformly bounded by γ in ℓ_1 norm sense, the Banach–Alaoglu Theorem implies that there is a subsequence $\{Q^{n_m}\}$ of $\{Q^n\}$ and some $\bar{Q}^0 \in \ell_1^{n_u \times n_v}$ such that $(Q^{n_m})_{ij} \to (\bar{Q}^0)_{ij}$ $(i = 1, \ldots, n_u, j = 1, \ldots, n_y)$ in the $W(c_0^*, c_0)$ topology. So, it follows that $Q^{n_m}(t) \to \bar{Q}^0(t)$ for all t and for all n, $P_n(R(Q^{n_m})) \to P_n(R(Q^0))$ and $T_{i,n}(Q^{n_m}) \to T_{i,n}(Q^0)$ $(i = 3, 6)$ as $m \to \infty$. Now, it is clear that for any $n > 0$ and any $n_m > n$,

$$\begin{aligned} &c_1\|P_n(R^1(Q^{n_m}))\|_1 + c_2\|P_n(R^2(Q^{n_m}))\|_2^2 \\ &\quad + c_3\|T_{3,n}(Q^{n_m})\| c_1\|R^1(Q^{n_m})\|_1 \\ &\quad + c_2\|R^2(Q^{n_m})\|_2^2 + c_3\|\hat{R}^3(Q^{n_m})\|_{\mathcal{H}_\infty} =: \nu^{n_m} \end{aligned}$$

By letting $m \to \infty$, we have

$$\begin{aligned} &c_1\|P_n(R^1(\bar{Q}^0))\|_1 + c_2\|P_n(R^2(\bar{Q}^0))\|_2^2 + c_3\|T_{3,n}(\bar{Q}^0)\| \\ &\qquad \le \lim_{m\to\infty} \nu^{n_m} = \nu \qquad \forall n. \end{aligned}$$

So, it follows that $c_1\|R^1(\bar{Q}^0)\|_1 + c_2\|R^2(\bar{Q}^0)\|_2^2 + c_3\|\hat{R}^3((\bar{Q}^0)\|_{\mathcal{H}_\infty} \le \nu$. Furthermore, as in the proof of Theorem 6.1, we can verify that $\bar{Q}^0$ satisfies all the constraints of problem (5). Thus, $\bar{Q}^0$ is an optimal solution to ν and by the uniqueness of the optimal solution of ν, we have $\bar{Q}^0 = Q^0$. ∎

E. Proof of Lemma 6.6

We prove for the case when $c_2 > 0$ and when condition (8) is true. We leave the rest of the proof to the reader.

For notational convenience we will denote (R^1, R^2) by R. Also, we define $g((R^1, R^2)) := c_1\|(R^1)_p\|_1 + c_2\|R^2\|_2^2$. It is clear that $g((R^{1,k}, R^{2,k})) \leq g((R^{1,o}, R^{2,o}))$ for all k. We claim that $g(R^k) \rightarrow g(R^o)$ as $k \rightarrow \infty$. Suppose not, then there exists a subsequence $\{R^{k_s}\}$ of $\{R^k\}$ and an $\epsilon_1 > 0$ such that $g(R^o) - g(R^{k_s}) > \epsilon_1$ for all s. Choose m such that $g((I - P_m)R^o) \leq \epsilon_1/2$. Thus $g(P_mR^o) + \epsilon_1/2 - g(R^{k_s}) > g(R^o) - g(R^{k_s}) > \epsilon_1$, which implies that $g(P_mR^o) - g(P_mR^{k_s}) \geq g(P_mR^o) - g(R^{k_s}) > \epsilon_1/2$. But we know that R^{k_s} converges to R^o pointwise and therefore $g(P_mR^{k_s}) \rightarrow g(P_mR^o)$. Thus we have reached a contradiction to our supposition which proves our claim. Given $\epsilon > 0$ choose n such that $\sum_{(p,q)} \|(I - P_n)R_{pq}^{2,o}\|_2 \leq \epsilon/(8Mc_2)$ and $g((I - P_n)R^o) \leq \epsilon/8$. where M is an upper bound on $\sum_{(p,q)} \|R_{pq}^{22,k}\|_2$, which exists because $g(R^k) \leq f(R^o)$. As $g(R^k)$ converges to $g(R^o)$ and $R^k(t)$ converges to $R^o(t)$ for all t it follows that $g((I - P_n)R^k)$ converges to $g((I - P_n)R^o)$. Thus, there exists an integer K_1 such that $k > K_1$ implies that $g((I - P_n)R^k)) \leq g((I - P_n)R^o) + \epsilon/4$. As $R^k(t)$ converges to $R^o(t)$ for all t it also follows that $g(P_n(R^k - R^o))$ converges to zero. Thus, we can choose an integer K_2 such that if $k > K_2$ then $g(P_n(R^k - R^o)) \leq \epsilon/4$. Thus, for any $k > \max\{K_1, K_2\}$ we have

$$\begin{aligned}
&g(R^k - R^o)\\
&= g(P_n(R^k - R^o))\\
&\quad + g((I - P_n)(R^k - R^o)) \leq g(P_n(R^k - R^o))\\
&\quad + g((I - P_n)R^k) + g((I - P_n)R^o)\\
&\quad + 2c_2 \sum_{(p,q)} \sum_{t=n+1}^{\infty} |R_{pq}^{22,k}(t)||R_{pq}^{22,o}(t)|\\
&\leq g(P_n(R^k - R^o)) + g((I - P_n)R^k) + g((I - P_n)R^o)\\
&\quad + 2c_2 \sum_{(p,q)} \| (I - P_n)R_{pq}^k \|_2 \| (I - P_n)R_{pq}^0 \|_2\\
&\leq \frac{\epsilon}{4} + 2g((I - P_n)R^o) + \frac{\epsilon}{4} 2c_2M \sum_{(p,q)} \|(I - P_n)R_{pq}^0 \|_2 \leq \epsilon.
\end{aligned}$$

This proves the lemma. ■

F. Proof of Theorem 6.4

The proof for 1) and 2) can be carried out in the exactly the same way as the proof for 1) and 2) in Theorem 6.3 and will not be repeated here. We prove 3) by using contradiction. Suppose the sequence $\{\|R^{2,n} - R^{2,0}\|_2\}_{n=1}^{\infty}$ doesn't converge to zero. Then there exists a subsequence $\{R^{n_m}\}$ of R^n and an $\epsilon > 0$ such that $\|R^{2,n_m} - R^{2,0}\|_2 \geq \epsilon, \forall m$. Then by using the same argument as in the proof of Lemma 6.5, we can prove that a subsequence $\{Q^{n_{m_k}}\}$ of $\{Q^{n_m}\}$ converges pointwise to the optimal solution Q^0 of problem ν(with $c_3 = 0$). Furthermore, since $Q^{n_{m_k}}$ is the optimal solution to problem $\nu^{n_{m_k}}$, whose limit converges to ν as $k \rightarrow \infty$, we have

$$\begin{aligned}
\nu^{n_{m_k}} &= c_1\|R^{1,n_{m_k}}\|_1 + c_2\|R^{2,n_{m_k}}\|_2^2\\
&\rightarrow c_1\|R^{1,0}\|_1 + c_2\|R^{2,0}\|_2^2 = \nu
\end{aligned}$$

as $k \rightarrow \infty$. Thus, the assumptions of Lemma 6.6 are satisfied and it follows that $c_1\|R^{1,n_{m_k}} - R^{1,0}\|_1 + c_2\|R^{2,n_{m_k}} - R^{2,0}\|_2^2 \rightarrow 0$ as $k \rightarrow \infty$, which is a contradiction. ■

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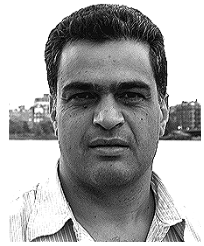

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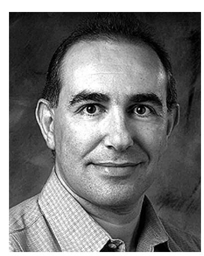

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