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Multi-objective MIMO Optimal Control Design without Zero Interpolation

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Abstract

In this paper optimal controller design which address the concerns of the $\mathcal{H}_2$ and the ℓ_1 norms is studied. In the first problem a positive combination of the $\mathcal{H}_2$ and the ℓ_1 norms of the closed loop is minimized. In the second the ℓ_1 norm of a transfer function is minimized while restraining the another below a prespecified level. Converging upper and lower bounds are obtained. Relation to the pure ℓ_1 problem is established. The solution methodology does not involve zero interpolation to characterize the achievable closed-loop maps. Thus obtaining the controller from the optimal closed-loop map is very straightforward.

Keywords: duality, ℓ_1 optimization, $\mathcal{H}_2$ optimization, multiple objectives, robust control.

1 Introduction

It has been recognized that controllers which optimize a particular measure (particularly the $\mathcal{H}_2, \mathcal{H}_\infty$ or the ℓ_1 norm) might be unacceptable because their performance might be poor with respect to an alternate measure. Indeed it has been shown that in many cases optimal performance with respect to a certain measure can lead to a compromise with respect to some other measure. This has led to a number of results which address the design of controllers that incorporate the objectives of two or more measures. An important class of controllers which include the concerns of the $\mathcal{H}_2$ problem and other time domain criteria have been the recent focus of attention. In [6, 8, 10] it is shown that such problems can be solved via finite dimensional convex optimization. It is also shown that *a priori* estimates on the dimension can be obtained for the problems solved in [6, 9]. However, the referred work is limited to single-input single-output problems. In [4, 5] the interaction of the $\mathcal{H}_2$ and the ℓ_1 norms for the multi-input multi-output case was studied. It was shown that approximating solutions within any *a priori* given tolerance to the optimal can be obtained via quadratic programming problems. The approach was based on the Delay-Augmentation method which was first introduced for solving the pure ℓ_1 problem [1]. The difficulties that afflict the pure ℓ_1 solution methodology which employ the zero-interpolation scheme are also present in this approach. For such schemes the optimal controller needs to retrieved from the optimal closed loop map. This usually implies inversions of certain maps. Thus it becomes essential to determine exactly which zeros of the optimal solution get interpolated. This is particularly difficult to determine due to the inaccuracy introduced by the error present in any numerical scheme.

Researchers are focussing on alternate methods of solving the pure ℓ_1 problem which avoid the difficulties mentioned [2, 3]. A scheme for the pure ℓ_1 problem which does not use the zero-interpolation method was introduced in [3]. This method is particularly attractive because obtaining the controller from the optimal solution is very straightforward and the method can be generalized to solve multi-objective problems.

In this paper we present two problems. The first problem considers a positive combination of the ℓ_1 and the $\mathcal{H}_2$ norms. It is shown that approximating solutions which converge in the ℓ_2 norm to the optimal can be obtained. Conditions are given under which the optimal is unique. In the second problem the ℓ_1 norm of a transfer function is minimized while keeping the two norm of another transfer function below a prespecified level.

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2 Combination problem

Consider the system of Figure 1, where $w := (w_1\ w_2)'$ is the exogenous disturbance, $z := (z_1\ z_2)'$ is the regulated output, u is the control input and y is the measured output. In feedback control design the objective is to design a controller K such that with $u = Ky$ the resulting closed loop map Φ_{zw} from w to z is stable (see Figure 1) and satisfies certain performance criteria. In [11] a nice parameterization of all closed loop maps which are achievable via stabilizing controllers was first derived. A good treatment of the issues involved is presented in [1]. Following the notation used in [1] we denote by n_u, n_w, n_z and n_y the number of control inputs, exogenous inputs, regulated outputs and measured outputs respectively of the plant G. We represent by Θ, the set of closed loop maps of the plant G which are achievable through stabilizing controllers. $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$ and $V \in \ell_1^{n_y \times n_w}$ characterize the Youla parametrization of the plant [11]. The following theorem follows from Youla parameterization.

Theorem 1 $\Theta = \{\Phi \in \ell_1^{n_z \times n_w}$: there exists a $Q \in \ell_1^{n_u \times n_y}$ with $\hat{\Phi} = \hat{H} - \hat{U}\hat{Q}\hat{V}\}$, *where $\hat{f}$ denotes the λ transform of f (see [1]).*

If Φ is in Θ we say that Φ is an *achievable* closed loop map. We assume throughout the paper that $\hat{U}$ has normal rank n_u and $\hat{V}$ has normal rank n_y. There is no loss of generality in making this assumption [1]. Let $f : (\Phi^{11}, \Phi^{22}) \to R$ (where Φ^{11} and Φ^{22} are matrices consisting of elements in ℓ_1) be defined by:

$$f(\Phi^{11}, \Phi^{22}) = c_1||\Phi^{11}||_1 + c_2||\Phi^{22}||_2^2.$$

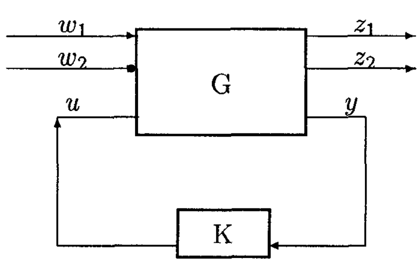

Figure 1: Closed Loop System.

We also define the truncation operator, P_r on the set of sequences by:

$$\begin{aligned} P_r(\Phi)(k) &= \Phi(k) \text{ if } k \leq r \\ &= 0 \text{ if } k > r. \end{aligned}$$

2.1 Problem Statement

Given a plant G, let the Youla parametrization be characterized by the transfer function matrices H, U, and V so that a closed loop map is achievable via a stabilizing controller if and only if it can be written as $H - U * Q * V$ for some $Q \in \ell_1^{n_u \times n_y}$. Further let H, U, and V be partitioned according to the following equation;

$H - U * Q * V = \begin{pmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{pmatrix} - \begin{pmatrix} U^1 \\ U^2 \end{pmatrix} * Q * \begin{pmatrix} V^1 & V^2 \end{pmatrix}$. The *combination* problem statement is: Given a plant G, positive constants c_1 and c_2 solve the following problem,

$$\inf_{Q \in \ell_1^{n_u \times n_y}} f(Q), \tag{1}$$

where $f(Q) := c_1||H^{11} - U^1 * Q * V^1||_1 + c_2||H^{22} - U^2 * Q * V^2||_2^2$. Now we define an associated problem which is more tractable. The *auxiliary combination* problem statement is: Given a plant G, positive constants c_1 and c_2 solve the following problem,

$$\nu = \inf_{||Q||_1 \leq \alpha} f(Q) \tag{2}$$

2.2 Relation between the auxiliary and the main problem

It is clear that the optimization in (1) can be restricted to the set $\{Q : Q \in \ell_1^{n_u \times n_y}$ with $c_1||H^{11} - U^1 * Q * V^1||_1 \leq c_1||H^{11}||_1 + c_2||H^{22}||_2^2\}$. This implies that for any relevant Q in the optimization in (1) $||U * Q * V||_1 \leq 2||H^{11}||_1 + \frac{c_2}{c_1}||H^{22}||_2^2$. Suppose, U^1 has more rows than columns and V^1 has more columns than rows and both have full normal rank. Thus the left inverse of U^1 exists (given by $(U^1)^{-l}$) and the right inverse of V^1 exists (given by $(V^1)^{-r}$). Further suppose that U^1 and V^1 have no zeros on the unit circle. Then it can be shown that there exists a β (which depends only on $(U^1)^{-l}$ and $(V^1)^{-r}$) such that $||Q||_1 \leq \beta$. Thus if in the auxiliary problem we choose $\alpha = \beta$ then the constraint $||Q||_1 \leq \alpha$ is redundant in the problem statement of ν and the solutions of (1) and (2) will be identical. The extra constraint in the problem statement of ν is useful because it regularizes the problem (as will be seen). Furthermore an *a priori* estimate on how large β has to be for the extra constraint to be redundant can be obtained. Now we state a result which shows that problem (2) does not have any anamolous behavior with respect to α.

Theorem 2 *Define* $\nu : [0,\infty) \to R$ *be defined by* $\nu(\alpha) := \inf_{||Q||_1 \leq \alpha} f(Q)$, *where* α *is a non-negative real number. Then* $\nu(\alpha)$ *is continuous with respect to* α *on* $(0,\infty)$.

Corollary 1 *Let*

$$\nu(\alpha^n) := \inf_{||Q||_1 \leq \alpha^n} f(Q), \tag{3}$$

where α *and the sequence* α^n *are in* $(0,1)$. *Let* Q^n *denote a solution of (3). Then there exists a subsequence* $\{Q^{n_m}_{i,j}\}$ *of* $\{Q^n_{i,j}\}$ *and* $Q^\alpha_{i,j}$ *such that* $Q^{n_m}_{i,j} \to Q^\alpha_{i,j}$ *where* Q^α *is a solution to* $\nu(\alpha)$. *Furthermore, if* $(\Phi^{11,n}, \Phi^{22,n}) := (H^{11} - U^1 * Q^n * V^1, H^{22} - U^2 * Q^n * V^2), (\Phi^{11,\alpha}, \Phi^{22,\alpha}) := (H^{11} - U^1 * Q^\alpha * V^1, H^{22} - U^2 * Q^\alpha * V^2)$ *and* $||(\Phi^{11,\alpha})_p||_1 = ||\Phi^{11,\alpha}||_1$, *where* $(\Phi^{11})_p$ *denotes the* p^{th} *row of* Φ^{11}, *then there exists a subsequence* $\{(\Phi^{11,n_m}, \Phi^{22,n_m})\}$ *such that*

$$c_1||(\Phi^{11,n_m})_p - (\Phi^{11,\alpha})_p||_1 + c_2||\Phi^{22,n_m} - \Phi^{22,\alpha}||_2^2 \to 0.$$

Also, if $H^{11} - U^1 * Q * V^1 = H^{22} - U^2 * Q * V^2$ *for all* $Q \in \ell_1^{n_u \times n_y}$ *then*

$$||\Phi^{22,n} - \Phi^{22,\alpha}||_2^2 \to 0.$$

In the rest of this section we will focus on the solution of the auxiliary combination problem given by (2).

2.3 Converging lower bounds

Let ν_n be defined by

$$\nu_n := \inf_{||Q||_1 \leq \alpha} f_n(Q) \tag{4}$$

where $f_n(Q) := c_1||P_n(H^{11} - U^1 * Q * V^1)||_1 + c_2||P_n(H^{22} - U^2 * Q * V^2)||_2^2$. It is clear that the above problem is finite dimensional and can be reduced to a finite dimensional quadratic programming problem. Thus every such approximate problem has a solution. It is also clear that $\nu_n < \nu_m < \nu$ for all $n < m$.

Theorem 3 *There exists a solution* Q^o *to problem (2). Also,* $\nu_n \nearrow \nu$. *Furthermore, if* $(\Phi^{11,n}, \Phi^{22,n}) := (P_n(H^{11} - U^1 * Q^n * V^1), P_n(H^{22} - U^2 * Q^n * V^2)), (\Phi^{11,o}, \Phi^{22,o}) := (H^{11} - U^1 * Q^o * V^1, H^{22} - U^2 * Q^o * V^2)$ *and* $||(\Phi^{11,o})_p||_1 = ||\Phi^{11,o}||_1$, *where* Q^n *is a solution to (4) and* $(\Phi^{11})_p$ *denotes the* p^{th} *row of* Φ^{11}. *then there exists a subsequence* $\{(\Phi^{11,n_m}, \Phi^{22,n_m})\}$ *such that* $c_1||(\Phi^{11,n_m})_p - (\Phi^{11,o})_p||_1 + c_2||\Phi^{22,n_m} - \Phi^{22,o}||_2^2 \to 0$.

Corollary 2 *Suppose in problem (2)* $H := H^{11} = H^{22}$, $U := U^1 = U^2$ *and* $V := V^1 = V^2$, *and* $c_2 > 0$. *Let* Q^o *be a solution of problem (2) and let* $\Phi^{11,o} = \Phi^{22,o} := H - U * Q^o * V$. *Then* Φ^o *is unique. If* Q^n *denotes the optimal solution to (4) then* $\Phi^n := P_n(H - U * Q^n * V)$ *is such that* $||\Phi^n - \Phi^o||_2 \to 0$.

Corollary 3 *There exists a solution* Q^o *to the following problem:* $\nu = \inf_{||Q||_1 \leq \alpha} ||H - U * Q * V||_1$.

Furthermore if Q^n *denotes the optimal solution to* $\nu_n = \inf_{||Q||_1 \leq \alpha} ||P_n(H - U * Q * V)||_1$, *then* $\nu_n \nearrow \nu$. *If* p *is such that* $||(H - U * Q^o * V)_p||_1 = ||H - U * Q^o * V||_1$ *then there exists a subsequence of* $\{H - U * Q^{n_m} * V\}$ *of* $\{H - U * Q^o * V\}$ *such that*

$$||(P_{n_m}(H - U * Q^{n_m} * V))_p - (H - U * Q^o * V)_p||_1 \to 0.$$

The results of this corollary relate very well with the ones obtained in [3] where the following problem was addressed: $\inf_{Q \in \ell_1^{n_u \times n_y}} \max\{||H - U * Q * V||_1, \alpha||Q||_1\}$. Adding the $\mathcal{H}_2$ norm of the closed loop makes the lower bounds behave nicer because then the original suboptimal solution sequence converges in the $\mathcal{H}_2$ norm to the optimal (see Corollary 2).

2.4 Converging upper bounds

Let ν^n be defined by

$$\begin{aligned} & \inf_{Q \in \ell_1^{n_u \times n_y}} f(Q) \\ \text{subject to} \quad & \\ & ||Q||_1 \leq \alpha \\ & Q(k) = 0 \text{ if } k > n. \end{aligned} \tag{5}$$

It is clear that $\nu^n \geq \nu^{n+1}$ because any $Q \in \ell_1^{n_u \times n_y}$ which satisfies the constraints in the problem definition of ν^n will satisfy the constraints in the problem definition of ν^{n+1}. For the same reason we also have $\nu^n \geq \nu$ for all relevant n.

Similar results as developed for the lower bounds can be proven for these upper bounds. The details are left to the reader.

3 $\ell_1/\mathcal{H}_2$ problem

3.1 Problem statement

The $\ell_1/\mathcal{H}_2$ problem statement is: Given a plant G, positive real number γ solve the following problem;

$$\begin{aligned} & \inf_{Q \in \ell_1^{n_u \times n_y}} \|H^{11} - U^1 * Q * V^1\|_1 \\ \text{subject to} \quad & \\ & \|H^{22} - U^2 * Q * V^2\|_1 \leq \gamma \\ & Q \in \ell_1^{n_u \times n_y}. \end{aligned} \tag{6}$$

The $\ell_1/\mathcal{H}_2$ *auxiliary* problem statement is: Given a plant G, positive real number γ solve the following problem;

$$\begin{aligned} \eta = & \inf_{Q \in \ell_1^{n_u \times n_y}} \|H^{11} - U^1 * Q * V^1\|_1 \\ \text{subject to} \quad & \\ & \|H^{22} - U^2 * Q * V^2\|_1 \leq \gamma \\ & \|Q\|_1 \leq \alpha. \end{aligned} \tag{7}$$

The parameter Q in the optimization stated in (6) can be restricted to the set $\{Q \in \ell_1^{n_u \times n_y}$ such that $\|H^{11} - U^1 * Q * V^1\|_1 \leq \|H\|_1\}$. Thus for all relevant Q for the optimization in the main $\ell_1/\mathcal{H}_2$ problem given by (6) $\|U^1 * Q * V^1\|_1 \leq \|H^{11}\|_1$. Once we have the above bound available on $R := U^1 * Q * V^1$ the rest of the discussion in Subsection 2.2 is applicable to the main and the auxiliary $\ell_1/\mathcal{H}_2$ problems given by (6) and (7) respectively. This establishes the relevance of the auxiliary $\ell_1/\mathcal{H}_2$ problem. Now we establish a result which shows that problem (7) does not have any anamolous behavior with respect to α and γ.

Theorem 4
Define $S := \{\gamma \in [0, \infty)$: there exists $Q \in \ell_1^{n_u \times n_y}$ with $\|H^{22} - U^2 * Q * V^2\|_2^2 \leq \gamma\}$. Let $\eta : [0, \infty) \times S \to R$ be defined by

$$\begin{aligned} \eta(\alpha, \gamma) := & \inf_{Q \in \ell_1^{n_u \times n_y}} \|H^{11} - U^1 * Q * V^1\|_1 \\ \textit{subject to} \quad & \\ & \|H^{22} - U^2 * Q * V^2\|_1 \leq \gamma \\ & \|Q\|_1 \leq \alpha. \end{aligned}$$

where α is any non-negative real number and $\gamma \in S$. Then $\eta(\alpha, \gamma)$ is continuous with respect to α and γ on $(0, \infty) \times int(S)$.

Results similar to the one established in Corollary 1 can be obtained for the $\ell_1/\mathcal{H}_2$ problem. Such a result will reflect on the norm convergence property for solutions to the problems with constraint on the one norm of Q being given by α^n and the constraint on $\|H^{22} - U^2 * Q * V^2\|_2^2$ being given by γ^n. This result is not presented for the sake of brevity.

Now we obtain converging upper and lower bounds to the auxiliary problem.

3.2 Converging lower and upper bounds

Let η_n be defined by

$$\begin{aligned} & \inf_{Q \in \ell_1^{n_u \times n_y}} \|P_n(H^{11} - U^2 * Q * V^2)\|_2^2 \\ \text{subject to} \quad & \\ & \|P_n(H^{22} - U^2 * Q * V^2)\|_1 \leq \gamma \\ & \|Q\|_1 \leq \alpha. \end{aligned} \tag{8}$$

It is clear that only the parameters of $Q(0), \ldots, Q(n)$ enter into the optimization problem and therefore (8) is a finite dimensional quadratic programming problem. Once optimal $Q(0), \ldots, Q(n)$ are found, $Q = \{Q(0), \ldots, Q(n), 0, \ldots\}$ will be an FIR optimal solution to (8).

Theorem 5 *Suppose the constraint set in problem (7) is nonempty. Then problem (7) always has an optimal solution $Q^0 \in \ell_1^{n_u \times n_y}$. Furthermore, $\eta_n \nearrow \eta$. Also, if $\Phi^{11,o} := H^{11} - U^1 * Q^0 * V^1$ and $\Phi^{11,n} := P_n(H^{11} - U^1 * Q^n * V^1)$ where Q^n is a solution to (8) then there exists a subsequence $\{\Phi^{11,n_m}\}$ of the sequence $\{\Phi^{11,n}\}$ such that $\|(\Phi^{11,n_m})_p - (\Phi^{11,o})_p\|_1 \to 0$ as $m \to \infty$, where p is any row such that*

$$\|(\Phi^{11,o})_p\|_1 = \|\Phi^{11,o}\|_1.$$

3.3 Converging upper bounds

Let η^n be defined by

$$\begin{array}{ll} & \inf_{Q\in\ell_1^{n_u\times n_y}} \|H^{11} - U^1 * Q * V^1\|_1 \\ \text{subject to} & \\ & \|H^{22} - U^2 * Q * V^2\|_2^2 \leq \gamma \\ & \|Q\|_1 \leq \alpha \\ & Q(k) = 0 \text{ if } k > n. \end{array} \tag{9}$$

It is clear that $\eta^n \geq \eta^{n+1}$ because any $Q \in \ell_1^{n_u\times n_y}$ which satisfies the constraints in the problem definition of η^n will satisfy the constraints in the problem definition of η^{n+1}. For the same reason we also have $\eta^n \geq \eta$ for all relevant n.

Similar results as developed for the lower bounds can be proven for these upper bounds. The details are left to the reader.

4 Conclusions

In this paper we have studied two problems. In the first termed the combination problem a positive combination of the one norm of a transfer function and the square of the two norm of another transfer function is minimized. Converging upper and lower bounds were obtained. It was established that approximating solutions to the optimal can be obtained which converge to the optimal in the two norm. The relation to the pure ℓ_1 problem was also discussed.

In the second problem the ℓ_1 norm of a transfer function was minimized while restraining the $\mathcal{H}_2$ norm of another below a prespecifed level. Results similar to the combination problem were obtained.

The machinery developed seems appropriate for other time domain norms like the ℓ_∞ norm of the response due to a given signal and possibly the $\mathcal{H}_\infty$ norm. A general framework in which the interplay of the $\mathcal{H}_2$, ℓ_1 $\mathcal{H}_\infty$ and other time domain measures is the topic of ongoing research. This paper lays down the foundations for such a framework.

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