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MIXED OBJECTIVE CONTROL SYNTHESIS: OPTIMAL $\ell_1/\mathcal{H}_2$ CONTROL*

MURTI V. SALAPAKA†, MOHAMMED DAHLEH†, AND PETROS VOULGARIS‡

Abstract. In this paper we consider the problem of minimizing the ℓ_1 norm of the transfer function from the exogenous input to the regulated output over all internally stabilizing controllers while keeping its $\mathcal{H}_2$ norm under a specified level. The problem is analyzed for the discrete-time, single-input single-output (SISO), linear-time invariant case. It is shown that an optimal solution always exists. Duality theory is employed to show that any optimal solution is a finite impulse response sequence, and an a priori bound is given on its length. Thus, the problem can be reduced to a finite-dimensional convex optimization problem with an a priori determined dimension. Finally, it is shown that, in the region of interest of the $\mathcal{H}_2$ constraint level, the optimal is unique and continuous with respect to changes in the constraint level.

Key words. robust control, duality, ℓ_1 optimization, discrete time

AMS subject classifications. 49N05, 49N10, 49N15, 49N35, 93C55

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1. Notation.

The following notation is employed in this paper:

$\text{int}(X)$	The interior of a set X.
$\|x\|_1$	The 1-norm of the vector $x \in R^n$.
$\|x\|_2$	The 2-norm of the vector $x \in R^n$.
$\hat{x}(\lambda)$	The λ transform of a right-sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{\infty} x(k)\lambda^k$.
ℓ_1	The Banach space of right-sided absolutely summable real sequences with the norm given by $\parallel x \parallel_1 := \sum_{k=0}^{\infty} \|x(k)\|$.
ℓ_∞	The Banach space of right-sided, bounded sequences with the norm given by $\parallel x \parallel_\infty := \sup_k \|x(k)\|$.
c_0	The subspace of ℓ_∞ with elements x that satisfy $\lim_{k\to\infty} x(k) = 0$.
ℓ_2	The Banach space of right-sided square summable sequences with the norm given by $\parallel x \parallel_2 := \left[\sum_{k=0}^{\infty} x(k)^2\right]^{\frac{1}{2}}$.
$\mathcal{H}_2$	The isometric isomorphic space of l_2 under the λ transform $\hat{x}(\lambda)$ with the norm given by $\parallel \hat{x}(\lambda) \parallel_2 = \parallel x \parallel_2$.
X^*	The dual space of the Banach space X. $\langle x, x^* \rangle$ denotes the value of the bounded linear functional x^* at $x \in X$.
$W(X^*, X)$	The weak star topology on X^* induced by X. In this topology, $x_n^* \to x^*$ if and only if $\langle x, x_n^* \rangle \to \langle x, x^* \rangle$ for all $x \in X$.
T^*	The adjoint operator of $T : X \to Y$ which maps Y^* to X^*.

The following identities also hold (see, e.g., [7]): $(\ell_1)^* = \ell_\infty,\ (c_0)^* = \ell_1,\ (\ell_2)^* = \ell_2$.

2. Introduction.

Consider the finite-dimensional linear time invariant system depicted in Figure 2.1, where P denotes the plant and K denotes the controller. The

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†Mechanical and Environmental Engineering Department, University of California at Santa Barbara, Santa Barbara, CA 93106 (vasu@engineering.ucsb.edu, dahleh@engineering.ucsb.edu).

‡Coordinated Science Laboratory, University of Illinois at Urbana–Champaign, 1308 W. Main St., Urbana, IL 61801 (petros@ktisivios.csl.uiuc.edu).

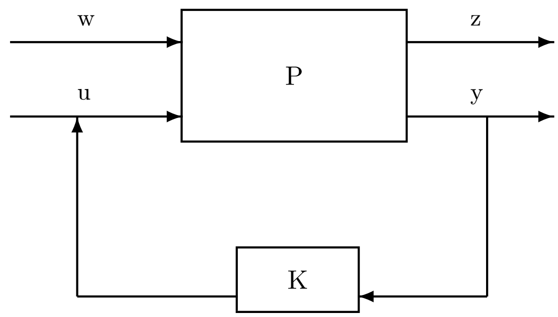

FIG. 2.1. *Plant controller configuration.*

signal w is the exogenous input and z is the regulated output. The signals u and y denote the control input and the measured output, respectively. Let T_{zw} be the closed-loop transfer function which maps w to z.

Many important control problems can be reduced to the above setup, where the objective is to minimize a suitably defined measure of T_{zw}. In the standard ℓ_1 problem the design of an internally stabilizing controller such that the ℓ_∞ norm of the regulated output z due to the worst-case magnitude bounded disturbance w is addressed. It is shown in [3] that this problem reduces to solving finite-dimensional linear programs. The analogous problem, with the signal measures being the ℓ_2 norm, is the standard $\mathcal{H}_\infty$ problem. The standard $\mathcal{H}_2$ problem is concerned with the minimization of the energy contained in the pulse response of the closed loop, T_{zw}. This can be viewed as minimizing the variance of the regulated output z due to a white noise input w. Both problems are discussed in [4].

It is well known (see, e.g., [2]) that optimization with respect to a particular norm may not necessarily yield good performance with respect to another. Thus, if enhanced performance is required with respect to multiple measures, then it is necessary to include all these measures directly into the design process. As a logical step, the design of controllers to satisfy mixed performance criteria has recently been the focus of researchers. Several state-space results on the interplay between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ are available (see, e.g., [5]). In [1] it is shown that a wide variety of mixed control problems reduce to convex optimization problems, and it is argued that the present technology makes it possible to deem the problem solved if it can be reduced to a convex optimization problem. In this light it is appropriate to exploit as much structure in the problem as possible, so that the standard software available becomes computationally efficient. In [6] the problem of minimizing the ℓ_1 norm of the closed loop under linear inequality constraints is addressed. Every such problem is equivalent to a linear programming problem which has a canonical dual problem associated with it. Contrary to the finite-dimensional case, it is not true that every infinite-dimensional linear program has the same optimal value as its dual. However, it was shown by the authors that under some conditions this "duality gap" does not exist between the primal and the dual, which is advantageous from a computational point of view. The problem of minimizing the ℓ_1 norm of the closed loop while keeping the $\mathcal{H}_\infty$ norm under a prescribed level falls under the above category. In [8] the problem of minimizing the ℓ_1 norm of a single-input single-output (SISO) transfer function while keeping the $\mathcal{H}_\infty$ norm of the closed-loop system under a specified value is reduced to solving a sequence of finite-dimensional convex optimization problems and an unconstrained $\mathcal{H}_\infty$ problem. In [9] it is shown that the $\mathcal{H}_2/\ell_1$ problem, the problem of minimizing the $\mathcal{H}_2$ norm of the closed loop while maintaining the ℓ_1 norm below a prescribed value, reduces to a finite-dimensional convex optimization problem. However, no a priori bound on its dimension is furnished, which substantially degrades the efficiency of the solution procedure since it may require a large number of iterations.

In this paper, the $\ell_1/\mathcal{H}_2$ problem, which is the problem of minimizing the ℓ_1 norm of T_{zw} while keeping the $\mathcal{H}_2$ norm below a prescribed level, is considered. This problem not only complements the one studied in [9], but it also turns out that much stronger results can be obtained which make this problem considerably more attractive to solve. In particular, it is shown that the problem reduces to a convex finite-dimensional one, and an a priori bound on its dimension is established. The latter feature is important in reducing the computational burden. Furthermore, the developments in this paper are substantially different than those in [9] and are more far reaching.

The paper is organized as follows. In section 3 relevant duality theory results are given. In section 4 the problem statement is made precise. In section 5 it is shown that an optimal solution always exists, and that it is a finite impulse response sequence. An a priori bound is given on its length. In section 6 the region of interest of the constraint level on the $\mathcal{H}_2$ norm is determined. It is shown that in this region the optimal is unique and is continuous with respect to changes in the constraint level. In section 7 an example is given to demonstrate the theory developed.

3. Mathematical preliminaries. In this section we present a Lagrange duality theorem that applies to the minimization of a convex functional subject to both equality and inequality constraints. A sensitivity result which follows directly from the Lagrange duality theorem is presented. We employ the terminology used in [7], which is standard. First, we need the following definitions.

DEFINITION 3.1. *Let P be a convex positive cone in a vector space X. We write $x \geq y$ if $x-y \in P$. We write $x > 0$ if $x \in \operatorname{int}(P)$. Similarly, $x \leq y$ if $x-y \in -P := N$ and $x < 0$ if $x \in \operatorname{int}(N)$. Given a vector space X with positive cone P the positive cone in X^*, $P^\oplus$ is defined as*

$$P^\oplus := \{x^* \in X^* : \langle x, x^* \rangle \geq 0 \text{ for all } x \in P\}.$$

DEFINITION 3.2. *Let X be a vector space and Z be a vector space with positive cone P. A mapping $G : X \to Z$ is convex if $G(tx+(1-t)y) \leq tG(x)+(1-t)G(y)$ for all x, y in X and t with $0 \leq t \leq 1$ and is strictly convex if $G(tx+(1-t)y) < tG(x)+(1-t)G(y)$ for all $x \neq y$ in X and t with $0 < t < 1$.*

The following is a Lagrange duality theorem.

THEOREM 3.3. *Let X be a Banach space, Ω be a convex subset of X, Y be a finite-dimensional normed space, and Z be a normed space with positive cone P. Let Z^* denote the dual space of Z with a positive cone $P^\oplus$. Let $f : \Omega \to R$ be a real valued convex functional, $g : X \to Z$ be a convex mapping, $H : X \to Y$ be an affine linear map, and $0 \in \operatorname{int}[\text{range(H)}]$. Define*

$$\mu_0 := \inf\{f(x) : \ g(x) \leq 0, \ H(x) = 0, \ x \in \Omega\}. \tag{3.1}$$

Suppose that there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$, and suppose that μ_0 is finite. Then

$$\mu_0 = \max\{\varphi(z^*, y) : z^* \geq 0, \ z^* \in Z^*, \ y \in Y\}, \tag{3.2}$$

where $\varphi(z^*,\ y):= \inf\{f(x) + \langle g(x), z^*\rangle + \langle H(x), y\rangle : x \in \Omega\ \}$, *and the maximum is achieved for some* $z_0^* \geq 0,\ z_0^* \in Z^*,\ y_0 \in Y$.

Furthermore, if the infimum in (3.1) *is achieved by some* $x_0 \in \Omega$, *then*

$$\langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle = 0, \tag{3.3}$$

and

$$x_0 \text{ minimizes } f(x) + \langle g(x), z_0^*\rangle + \langle H(x), y_0\rangle \text{ over all } x \in \Omega. \tag{3.4}$$

Proof. Given any $z^* \geq 0,\ y \in Y$, we have $\inf\{f(x)+\langle g(x), z^*\rangle+\langle H(x), y\rangle : x \in \Omega\} \leq \inf\{f(x) + \langle g(x), z^*\rangle + \langle H(x), y\rangle : x \in \Omega,\ g(x) \leq 0,\ H(x) = 0\} \leq \inf\{f(x) : x \in \Omega,\ g(x) \leq 0,\ H(x) = 0\}= \mu_0$. Therefore, it follows that $\max\{\varphi(z^*, y) : z^* \geq 0,\ y \in Y\}\leq \mu_0$. From Problem 7 of [7, Chap. 8] (see Lemma 9.1 in the Appendix for problem statement and proof), we know that there exists $z_0^* \in Z^*, z_0^* \geq 0,\ y_0 \in Y$ such that $\mu_0 = \varphi(z_0^*,\ y_0)$. This proves (3.2).

Suppose there exists $x_0 \in \Omega, H(x_0) = 0,\ g(x_0) \leq 0$, and $\mu_0 = f(x_0)$; then $\mu_0 = \varphi(z_0^*,\ y_0) \leq f(x_0) + \langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle \leq f(x_0) = \mu_0$. Therefore, we have $\langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle = 0$ and $\mu_0 = f(x_0) + \langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle$. This proves the theorem. ∎

We refer to (3.1) as the *primal* problem and (3.2) as the *dual* problem.

COROLLARY 3.4. *Let* X, Y, Z, f, H, g, Ω *be as in Theorem* 3.3. *Let* x_0 *be the solution to the problem*

$$\begin{gathered} \textit{minimize } f(x) \\ \textit{subject to } x \in \Omega,\ H(x) = 0,\ g(x) \leq z_0, \end{gathered}$$

with (z_0^*, y_0) *as the dual solution. Let* x_1 *be the solution to the problem*

$$\begin{gathered} \textit{minimize } f(x) \\ \textit{subject to } x \in \Omega,\ H(x) = 0,\ g(x) \leq z_1, \end{gathered}$$

with (z_1^*, y_1) *as the dual solution. Then*

$$\langle z_1 - z_0, z_1^*\rangle \ \leq f(x_0) - f(x_1) \ \leq \langle z_1 - z_0, z_0^*\rangle. \tag{3.5}$$

Proof. From Theorem 3.3 we know that for any $x \in \Omega$,

$$\begin{aligned} &f(x_0) + \langle g(x_0) - z_0, z_0^*\rangle + \langle H(x_0), y_0\rangle \\ \leq\ &f(x) + \langle g(x) - z_0, z_0^*\rangle + \langle H(x), y_0\rangle. \end{aligned}$$

In particular, we have

$$\begin{aligned} &f(x_0) + \langle g(x_0) - z_0, z_0^*\rangle + \langle H(x_0), y_0\rangle \\ \leq\ &f(x_1) + \langle g(x_1) - z_0, z_0^*\rangle + \langle H(x_1), y_0\rangle. \end{aligned}$$

From Theorem 3.3 we know that $\langle g(x_0) - z_0, z_0^*\rangle + \langle H(x_0), y_0\rangle = 0$ and $H(x_1) = 0$. This implies

$$f(x_0) - f(x_1) \ \leq \langle g(x_1) - z_0, z_0^*\rangle \ \leq \langle z_1 - z_0, z_0^*\rangle.$$

A similar argument gives the other inequality. This proves the corollary. ∎

4. Problem formulation. Consider the standard feedback problem represented in Figure 2.1, where P and K are the plant and the controller, respectively. Let w represent the exogenous input, z represent the output of interest, y be the measured output, and u be the control input where z and w are assumed scalar. Let ϕ be the closed-loop map which maps $w \to z$. From the Youla parametrization (see, e.g., [2]) it is known that all achievable closed-loop maps under stabilizing controllers are given by $\phi = h - u * q$ ($*$ denotes convolution), where $h, u, q \in \ell_1$, h, u depend only on the plant P, and q is a free parameter in ℓ_1. Throughout the paper we make the following assumption.

ASSUMPTION 1. All the zeros of $\hat{u}$ (the λ transform of u) inside the unit disc are real and distinct. Also, $\hat{u}$ has no zeros on the unit circle.

The assumption that all zeros of $\hat{u}$ which are inside the open unit disc are real and distinct is not restrictive and is made to streamline the presentation of the paper. Let the zeros of u which are inside the unit disc be given by $z_1, z_2, \ldots, z_n$. Let

$$\Theta := \{\phi : \text{ there exists } q \in \ell_1 \text{ with } \phi = h - u * q\}.$$

Θ is the set of all achievable closed-loop maps under stabilizing controllers. Let $A : \ell_1 \to R^n$ be given by

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \cdots \\ 1 & z_2 & z_2^2 & z_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \cdots \end{pmatrix},$$

and $b \in R^n$ be given by

$$b = \begin{pmatrix} \hat{h}(z_1) \\ \hat{h}(z_2) \\ \vdots \\ \hat{h}(z_n) \end{pmatrix}.$$

THEOREM 4.1. *The following is true:*

$$\begin{aligned} \Theta &= \{\phi \in \ell_1 : \hat{\phi}(z_i) = \hat{h}(z_i) \text{ for all } i = 1, \ldots, n\} \\ &= \{\phi \in \ell_1 : A\phi = b\}. \end{aligned}$$

Proof. The proof is given in [2, pp. 104–105]. □

The problem

$$\begin{aligned} \nu_\infty &:= \inf\{\| h - u * q \|_1 : q \in \ell_1\} \\ &= \inf\{\| \phi \|_1 : \phi \in \ell_1 \text{ and } A\phi = b\} \end{aligned} \tag{4.1}$$

is the standard ℓ_1 problem. In [3] it is shown that this problem has a solution which is possibly nonunique. Optimal solutions are shown to be finite impulse response sequences. Let

$$\begin{aligned} \mu_\infty &:= \inf\{\| h - u * q \|_2^2 : q \in \ell_1\}, \\ &= \inf\{\| \phi \|_2^2 : \phi \in \ell_1 \text{ and } A\phi = b\}, \end{aligned} \tag{4.2}$$

which is the standard $\mathcal{H}_2$ problem. The solution to this problem is unique, and the solution is an infinite impulse response sequence. Define

$$m_1 := \inf_{A\phi=b, \|\phi\|^2 \leq \mu_\infty} \| \phi \|_1, \tag{4.3}$$

which is the ℓ_1 norm of the unique optimal solution of the standard $\mathcal{H}_2$ problem. Let

$$m_2 := \inf_{A\phi=b, \|\phi\| \leq \nu_\infty} \| \phi \|_2{}^2, \tag{4.4}$$

which is the infimum over the ℓ_2 norms of the optimal solutions of the standard ℓ_1 problem.

The problem of interest is as follows: *given a positive constant $\gamma > \mu_\infty$ obtain a solution to the following mixed objective problem:*

$$\begin{aligned} \nu_\gamma &:= \inf\{\| h - u * q \|_1 : q \in \ell_1 \text{ and} \langle h - u * q, h - u * q\rangle \leq \gamma\} \\ &= \inf\{\| \phi \|_1 : \phi \in \ell_1 \ , A\phi = b \text{ and } \langle \phi, \phi\rangle \leq \gamma\}. \end{aligned} \tag{4.5}$$

Note that $\langle ., .\rangle$ is the inner product associated with ℓ_2. In the following sections we will study this problem from the point of view of existence, structure, continuity, and computation of the optimal solutions.

5. Analysis of optimal solutions and their properties. In the first part of this section we show that (4.5) always has a solution. In the second part we show that any solution to (4.5) is of finite length, and in the third, we give an a priori bound on the length.

5.1. Existence of a solution. Here we show that a solution to (4.5) always exists. We use the following lemma (see, e.g., [7]) to prove the main result of this subsection.

LEMMA 5.1 (Banach–Alaoglu). *Let X be a Banach space with X^* as its dual. Then the set $\{x^* : x^* \in X^*, \| x^* \| \leq M\}$ is $W(X^*, X)$ compact for any $M \in R$.*

THEOREM 5.2. *There exists $\phi_0 \in \Phi$ such that*

$$\| \phi_0 \|_1 = \inf_{\phi \in \Phi} \{\| \phi \|_1\},$$

where $\Phi := \{\phi \in \ell_1 : A\phi = b$ and$\langle \phi, \phi\rangle \leq \gamma\}$ with $\gamma > \mu_\infty$. Therefore, the infimum in (4.5) *is a minimum.*

Proof. We denote the feasible set of our problem by $\Phi := \{\phi \in \ell_1 : A\phi = b \text{ and} \langle \phi, \phi\rangle \leq \gamma\}$. $\nu_\gamma < \infty$ because $\gamma > \mu_\infty$, and therefore the feasible set is not empty. Let $B := \{\phi \in \ell_1 : \| \phi \|_1 \leq \nu_\gamma + 1\}$. It is clear that

$$\nu_\gamma = \inf_{\phi \in \Phi \cap B} \{\| \phi \|_1\}.$$

Therefore, given $i > 0$, there exists $\phi_i \in \Phi \cap B$ such that $\| \phi_i \|_1 \leq \nu_\gamma + \frac{1}{i}$. B is a bounded set in $\ell_1 = c_0^*$. It follows from the Banach–Alaoglu lemma that B is $W(c_0^*, c_0)$ compact. Using the fact that c_0 is separable, we know that there exists a subsequence $\{\phi_{i_k}\}$ of $\{\phi_i\}$ and $\phi_0 \in \Phi \cap B$ such that $\phi_{i_k} \to \phi_0$ in the $W(c_0^*, c_0)$ sense; that is, for all v in c_0,

$$\langle v, \phi_{i_k}\rangle \to \langle v, \phi_0\rangle \text{ as } k \to \infty. \tag{5.1}$$

Let the jth row of A be denoted by a_j and the jth element of b be given by b_j. Then, as $a_j \in c_0$ we have

$$\langle a_j, \phi_{i_k} \rangle \to \langle a_j, \phi_0 \rangle \text{ as } k \to \infty \text{ for all } j = 1, 2, \ldots, n. \tag{5.2}$$

As $A(\phi_{i_k}) = b$ we have $\langle a_j, \phi_{i_k} \rangle = b_j$ for all k and for all j which implies $\langle a_j, \phi_0 \rangle = b_j$ for all j. Therefore, we have $A(\phi_0) = b$. As $l_2 \subset c_0$ we have from (5.1) that for all v in l_2,

$$\langle v, \phi_{i_k} \rangle \to \langle v, \phi_0 \rangle \text{ as } k \to \infty, \tag{5.3}$$

which shows that $\phi_{i_k} \to \phi_0$ in $W(l_2^*, l_2)$. Also, from the construction of ϕ_{i_k}, we know that $\| \phi_{i_k} \|_2 \leq \sqrt{\gamma}$. From Lemma 5.1 we conclude that $\langle \phi_0, \phi_0 \rangle \leq \gamma$, and therefore we have shown that $\phi_0 \in \Phi$. Recall that ϕ_{i_k} were chosen so that $\| \phi_{i_k} \|_1 \leq \nu_\gamma + \frac{1}{i_k}$. From Lemma 5.1 we have that $\| \phi_0 \|_1 \leq \nu_\gamma + \frac{1}{i_k}$ for all k. Therefore $\| \phi_0 \|_1 \leq \nu_\gamma$. As $\phi_0 \in \Phi$ (which is the feasible set) we have $\| \phi_0 \|_1 = \nu_\gamma$. This proves the theorem. ∎

5.2. Structure of optimal solutions. In this subsection we use Lagrange duality results to show that every optimal solution is of finite length. The following two lemmas establish the dual problem.

LEMMA 5.3.

$$\nu_\gamma = \max\{\varphi(y_1, y_2) : y_1 \geq 0 \textit{ and } y_2 \in R^n\}, \tag{5.4}$$

where

$$\varphi(y_1, y_2) := \inf_{\phi \in \ell_1} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b - A\phi, y_2 \rangle\}.$$

Proof. We will apply Theorem 3.3 to get the result. Let X, Ω, Y, Z in Theorem 3.3 correspond to ℓ_1, ℓ_1, R^n, and R, respectively. Let $g(\phi) := \langle \phi, \phi \rangle - \gamma$, $H(\phi) := b - A\phi$. With this notation, we have $Z^* = R$.

A has full range, which implies $0 \in \text{int}[\text{range}(H)]$. $\gamma > \mu_\infty$, and therefore there exists ϕ_1 such that $\langle \phi_1, \phi_1 \rangle - \gamma < 0$ and $H(\phi_1) = 0$. Therefore, all the conditions of Theorem 3.3 are satisfied. From Theorem 3.3 we have

$$\nu_\gamma = \max_{y_1 \geq 0, y_2 \in R^n} \inf_{\phi \in \ell_1} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b - A\phi, y_2 \rangle\}.$$

This proves the lemma. ∎

The right-hand side of (5.4) is the *dual problem.*

LEMMA 5.4. *The dual problem is given by*

$$\max\{\varphi(y_1, y_2) : y_1 \geq 0 \textit{ and } y_2 \in R^n\}, \tag{5.5}$$

where

$$\varphi(y_1, y_2) := \inf_{\phi \in \ell_1, \phi(i)v(i) \geq 0} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b, y_2 \rangle - \langle \phi, v \rangle\}.$$

$v(i)$ *is defined by* $v(i) := A^* y_2(i)$.

Proof. Let $y_1 \geq 0$, $y_2 \in R^n$. It is clear that

$$\begin{aligned} &\inf_{\phi \in \ell_1} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b - A\phi, y_2 \rangle\} \\ &= \inf_{\phi \in \ell_1} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b, y_2 \rangle - \langle \phi, v \rangle\}. \end{aligned}$$

Suppose $\phi \in \ell_1$ and there exists i such that $\phi(i)v(i) < 0$; then define $\phi^1 \in \ell_1$ such that $\phi^1(j) = \phi(j)$ for all $j \neq i$ and $\phi^1(i) = 0$. Therefore, we have $\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b, y_2 \rangle - \langle \phi, v \rangle \geq \| \phi^1 \|_1 + y_1(\langle \phi^1, \phi^1 \rangle - \gamma) + \langle b, y_2 \rangle - \langle \phi^1, v \rangle$. This shows that we can restrict ϕ in the infimization to satisfy $\phi(i)v(i) \geq 0$. This proves the lemma. □

The following theorem is the main result of this subsection. It shows that any solution of (4.5) is a finite impulse response sequence.

THEOREM 5.5. *Define* $\mathcal{T}:=\{\phi \in \ell_1 : \textit{there exists } L^* \textit{ with } \phi(i) = 0 \textit{ if } i \geq L^*\}$. *The dual of the problem is given by*

$$\max\{\varphi(y_1, y_2) : y_1 \geq 0, y_2 \in R^n\}, \tag{5.6}$$

where

$$\varphi(y_1, y_2) := \inf_{\phi \in \mathcal{T}, \phi(i)v(i) \geq 0} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b, y_2 \rangle - \langle \phi, v \rangle\}.$$

$v(i)$ *defined by* $v(i) = A^* y_2(i)$. *Also, any optimal solution* ϕ_0 *of* (4.5) *belongs to* $\mathcal{T}$.

Proof. Let $y_1^\gamma \geq 0$, $y_2^\gamma \in R^n$ be the solution to

$$\max_{y_1 \geq 0, y_2 \in R^n} \inf_{\phi \in \ell_1, \phi(i)v(i) \geq 0} \{\| \phi \|_1 + y_1(\langle \phi, \phi \rangle - \gamma) + \langle b - A\phi, y_2 \rangle\}.$$

It is easy to show that there exists L^* such that $v^\gamma(i) := (A^* y_2^\gamma)(i)$ satisfies $|v^\gamma(i)| < 1$ if $i \geq L^*$. If $\phi(i)v^\gamma(i) \geq 0$ for all i, then,

$$\begin{aligned}
&\| \phi \|_1 + y_1^\gamma(\langle \phi, \phi \rangle - \gamma) + \langle b, y_2^\gamma \rangle - \langle \phi, v^\gamma \rangle \\
&= \sum_{i=0}^{\infty} \{|\phi(i)| + y_1^\gamma(\phi(i))^2 - \phi(i)v^\gamma(i)\} - y_1^\gamma \gamma + \langle y_2^\gamma, b \rangle \\
&= \sum_{i=0}^{\infty} \{\phi(i)(\operatorname{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi(i))^2\} - y_1^\gamma \gamma + \langle y_2^\gamma, b \rangle \\
&= \sum_{i=0}^{L^*} \{\phi(i)(\operatorname{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi(i))^2\} \\
&\quad + \sum_{i=L^*+1}^{\infty} \{\phi(i)(\operatorname{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi(i))^2\} - y_1^\gamma \gamma + \langle y_2^\gamma, b \rangle.
\end{aligned}$$

Suppose $|v^\gamma(i)| < 1$. Then we have

$$\phi(i)(\operatorname{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi(i))^2 \geq 0$$

and equal to zero only if $\phi(i) = 0$. Therefore, in the infimization, we can restrict $\phi(i) = 0$ whenever $|v^\gamma(i)| < 1$. As $|v^\gamma(i)| < 1$ for all $i \geq L^*$ it follows that we can restrict ϕ to $\mathcal{T}$ in the infimization. In Theorem 5.2 we showed that there exists a solution ϕ_0 to the primal. From Theorem 3.3 we have that ϕ_0 minimizes

$$\| \phi \|_1 + y_1^\gamma(\langle \phi, \phi \rangle - \gamma) + \langle b, y_2^\gamma \rangle - \langle \phi, v^\gamma \rangle,$$

over all ϕ in ℓ_1. From the previous discussion it follows that $\phi_0 \in \mathcal{T}$. This proves the theorem. □

5.3. An a priori bound on the length of any optimal solution. In this subsection we give an a priori bound on the length of any solution to (4.5). First we establish the following three lemmas.

LEMMA 5.6. *Let* $\gamma > \mu_\infty$, $m_1 := \inf_{A\phi=b,\langle\phi,\phi\rangle\leq\mu_\infty} \parallel \phi \parallel_1$, *and* $\nu_\gamma := \inf_{A\phi=b,\langle\phi,\phi\rangle\leq\gamma} \parallel \phi \parallel_1$. *Let* y_1^γ, y_2^γ *represent a dual solution as obtained in* (5.5). *Then* $y_1^\gamma \leq M_\gamma$ *where* $M_\gamma := \frac{m_1}{\gamma-\mu_\infty}$.

Proof. Let $\gamma > \gamma_1 > \mu_\infty$ and $\nu_{\gamma_1} := \inf_{A\phi=b,<\phi,\phi>\leq\gamma_1} \parallel \phi \parallel_1$. Let y_1^γ, y_2^γ represent a dual solution as obtained in (5.5). From Corollary 3.4 we have

$$\langle \gamma-\gamma_1, y_1^\gamma\rangle \leq \nu_{\gamma_1} - \nu_\gamma \leq \nu_{\gamma_1} \leq m_1,$$

which implies that $y_1^\gamma \leq \frac{m_1}{\gamma-\gamma_1}$. This holds for all $\gamma > \gamma_1 > \mu_\infty$. Therefore, $M_\gamma := \frac{m_1}{\gamma-\mu_\infty}$ is an a priori bound on y_1^γ. This proves the lemma. ☐

LEMMA 5.7. *Let* ϕ_0 *be a solution of the primal* (4.5). *Let* y_1^γ, y_2^γ *represent the corresponding dual solution as obtained in* (5.5). *Let* $v^\gamma := A^* y_2^\gamma$, *then*

$$\begin{aligned} y_1^\gamma \phi_0(i) &= \frac{v^\gamma(i)-1}{2} \; if \;\; v^\gamma(i) > 1 \\ &= \frac{v^\gamma(i)+1}{2} \; if \;\; v^\gamma(i) < -1 \\ &= \quad 0 \qquad if \;\; |v^\gamma(i)| \leq 1. \end{aligned}$$

Also, $\parallel v^\gamma \parallel_\infty \leq \alpha_\gamma$ *where* $\alpha_\gamma = \frac{2m_1\sqrt{\gamma}}{\gamma-\mu_\infty} + 1$.

Proof. Let

$$L(\phi) := \sum_{i=0}^{\infty} \{\phi(i)(\mathrm{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi(i))^2\} - \gamma y_1^\gamma + \langle b, y_2^\gamma\rangle.$$

Suppose $|v^\gamma(i)| = 1$. Now, if $y_1^\gamma = 0$, then it is clear that $y_1^\gamma\phi_0(i) = 0$. If $y_1^\gamma > 0$, then as ϕ_0 minimizes $L(\phi)$ we have $\phi_0(i) = 0$. We have already shown that if $|v^\gamma(i)| < 1$, then $\phi_0(i) = 0$. Therefore, $y_1^\gamma\phi_0(i) = 0$ if $|v^\gamma(i)| \leq 1$.

Suppose $v^\gamma(i) > 1$. Then it is easy to show that there exists $\phi(i)$ such that $\phi(i) \geq 0$ and $\phi(i)(\mathrm{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi(i))^2 < 0$. As any optimal minimizes $L(\phi)$, we know that $\phi_0(i)(\mathrm{sgn}(v^\gamma(i)) - v^\gamma(i)) + y_1^\gamma(\phi_0(i))^2 < 0$, which implies $\phi_0(i) > 0$ and therefore $1 - v^\gamma(i) + 2y_1^\gamma\phi_0(i) = 0$. This implies that $y_1^\gamma\phi_0(i) = \frac{v^\gamma(i)-1}{2}$. Similarly, the result follows when $v^\gamma(i) < -1$. Therefore, $\parallel v^\gamma \parallel_\infty \leq 2M_\gamma \parallel \phi_0 \parallel_\infty + 1 \leq \frac{2m_1}{\gamma-\mu_\infty} \parallel \phi_0 \parallel_\infty + 1 \leq \frac{2m_1\sqrt{\gamma}}{\gamma-\mu_\infty} + 1$. The last inequality follows from the fact that $\langle\phi_0,\phi_0\rangle \leq \gamma$. This implies that $\alpha_\gamma := \frac{2m_1\sqrt{\gamma}}{\gamma-\mu_\infty} + 1$ is an a priori upper bound on $\parallel v^\gamma \parallel_\infty$. This proves the lemma. ☐

LEMMA 5.8 (see [2]). *If* $y_2 \in R^n$ *is such that* $\parallel A^* y_2 \parallel_\infty \leq \alpha_\gamma$, *then there exists a positive integer* L^* *independent of* y_2 *such that* $|(A^* y_2)(i)| < 1$ *for all* $i \geq L^*$.

Proof. Define

$$A_L^* = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ z_1 & z_2 & z_3 & \dots & z_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^L & z_2^L & z_3^L & \dots & z_n^L \end{pmatrix},$$

$A_L^* : R^n \to R^{L+1}$. With this definition we have $A_\infty^* = A^*$. Let $y_2 \in R^n$ be such that $\| A^* y_2 \|_\infty \leq \alpha_\gamma$. Choose any L such that $L \geq (n-1)$. As $z_i, i = 1, \ldots, n$, are distinct, A_L^* has full column rank. A_L^* can be regarded as a linear map taking $(R^n, \| \,.\, \|_1) \to (R^{L+1}, \| \,.\, \|_\infty)$. As A_L^* has full column rank, we can define the left inverse of A_L^*, $(A_L^*)^{-l}$, which takes $(R^{L+1}, \| \,.\, \|_\infty) \to (R^n, \| \,.\, \|_1)$. Let the induced norm of $(A_L^*)^{-l}$ be given by $\| (A_L^*)^{-l} \|_{\infty,1}$. $y_2 \in R^n$ is such that $\| A^* y_2 \|_\infty \leq \alpha_\gamma$, and therefore $\| A_L^* y_2 \|_\infty \leq \alpha_\gamma$. It follows that

$$\| y_2 \|_1 \leq \| (A_L^*)^{-l} \|_{\infty,1} \| A_L^* y_2 \|_\infty \leq \| (A_L^*)^{-l} \|_{\infty,1} \alpha_\gamma. \tag{5.7}$$

Choose L^* such that

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \; \| (A_L^*)^{-l} \|_{\infty,1} \alpha_\gamma < 1. \tag{5.8}$$

There always exists such an L^* because $|z_k| < 1$ for all $k = 1, \ldots, n$. Note that L^* does not depend on y_2. For any $i \geq L^*$ we have

$$\begin{aligned} |(A^* y_2)(i)| = \left| \sum_{k=1}^{k=n} z_k^i y_2(k) \right| &\leq \max_{k=1,\ldots,n} |z_k|^i \| y_2 \|_1 \\ &\leq \max_{k=1,\ldots,n} |z_k|^i \; \| (A_L^*)^{-l} \|_{\infty,1} \alpha_\gamma \\ &\leq \max_{k=1,\ldots,n} |z_k|^{L^*} \; \| (A_L^*)^{-l} \|_{\infty,1} \alpha_\gamma. \end{aligned}$$

The second inequality follows from (5.7). From (5.8) we have $|(A^* y_2)(i)| < 1$ if $i \geq L^*$. This proves the lemma. □

The following theorem is the main result of the section.

THEOREM 5.9. *Every solution ϕ_0 of the primal* (4.5) *is such that $\phi(i) = 0$ if $i \geq L^*$, where L^* given in Lemma* 5.8 *can be determined a priori. Furthermore, the upper bound on lengths of the optimal solutions is nonincreasing as a function of γ.*

Proof. Using Lemma 5.7 we can bound on $\| v^\gamma \|_\infty$ by α_γ. Applying Lemma 5.8, we conclude that there exists L_γ^* (which can be determined a priori) such that $|v^\gamma(i)| < 1$ if $i \geq L_\gamma^*$. Using the fact that $\phi_0(i) = 0$ if $|v^\gamma(i)| < 1$, we conclude that $\phi_0 = 0$ if $i \geq L_\gamma^*$. L_γ^*; was chosen to satisfy

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \; \| (A_L^*)^{-l} \|_{\infty,1} \alpha_\gamma < 1.$$

α_γ is nonincreasing as a function of γ. Therefore L_γ^* is nonincreasing as a function of γ. This proves the theorem. □

Note that as $\alpha_\gamma = \frac{2m_1\sqrt{\gamma}}{\gamma - \mu_\infty} + 1$, we have that the upper bound on lengths of the solutions increases to infinity as γ decreases to μ_∞. This is commensurate with the fact that the optimal solution for the standard $\mathcal{H}_2$ problem (4.2) is an infinite impulse response sequence.

The above theorem shows that the problem at hand is a finite-dimensional convex problem of a priori determined dimension. In particular, in view of Theorem 5.9 the problem that needs to be solved is as follows:

$$\nu_\gamma = \min_{A_{L^*}\phi = b, \langle \phi, \phi \rangle \leq \gamma} \sum_{k=0}^{L^*} |\phi(k)|, \tag{5.9}$$

where

$$A_{L^*} = \begin{pmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{L^*} \\ 1 & z_2 & z_2^2 & \dots & z_2^{L^*} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & \dots & z_n^{L^*} \end{pmatrix},$$

and L^* is given in Lemma 5.8. An alternative representation can be given, as the following lemma suggests.

LEMMA 5.10. *The primal is given by*

$$\begin{aligned} & \textit{minimize} \sum_{k=0}^{L^*} \phi^+(k) + \phi^-(k) \\ \textit{subject to} \quad & A_{L^*}(\phi^+ - \phi^-) = b \\ & \langle \phi^+ - \phi^-, \phi^+ - \phi^- \rangle \le \gamma \\ & \phi^+, \phi^- \textit{ in } R^{L^*} \textit{ with } \phi^+, \phi^- \ge 0. \end{aligned} \tag{5.10}$$

Proof. Note that in the above theorem the ordering is componentwise for the inequalities. We will show that (5.9) is equivalent to (5.10). Let p_0 denote the value attained by the objective functional in (5.10). Suppose ϕ^+, ϕ^- satisfy the constraints of (5.10). Let $\phi := \phi^+ - \phi^-$. Then it is clear that ϕ satisfies the constraints of (5.9). Also, for each k, $|\phi(k)| = |\phi^+ - \phi^-| \le |\phi^+| + |\phi^-| = \phi^+(k) + \phi^-(k)$. This implies that $\nu_\gamma \le p_0$.

Suppose that ϕ satisfies the constraints of (5.9). Define ϕ^+ such that $\phi^+(k) = \phi(k)$ if $\phi(k) \ge 0$, and 0 otherwise. Similarly, define ϕ^- such that $\phi^-(k) = -\phi(k)$ if $\phi(k) \le 0$, and 0 otherwise. It is clear that $\phi = \phi^+ - \phi^-$ and that ϕ^+, ϕ^- satisfy the constraints of (5.10). Also, $|\phi(k)| = \phi^+(k) + \phi^-(k)$. This proves that $\nu_\gamma \ge p_0$. Therefore, $\nu_\gamma = p_0$. It is easy to show that if ϕ_0^+, ϕ_0^- is optimal for (5.10) then $\phi_0 := \phi_0^+ - \phi_0^-$ is optimal for (5.9). This proves the lemma. ∎

This type of convex problem can be solved efficiently using standard methods [1].

6. Uniqueness and continuity of the optimal solution. In this section we address the issue of uniqueness and continuity of solutions to the primal problem with respect to changes in the constraint level on the $\mathcal{H}_2$ norm of the closed-loop map. In the first part we address the issue of uniqueness, and in the second part, we show that the optimal solution is continuous in the region where it is unique.

6.1. Uniqueness of the optimal solution. The following three lemmas are established before the main result of this subsection.

LEMMA 6.1. *Let $y_1^\gamma \ge 0$, $y_2^\gamma \in R^n$ be a solution to* (5.5)*. If $y_1^\gamma = 0$, then $\nu_\gamma = \nu_\infty$. This implies that* (4.5) *reduces to solving a standard ℓ_1 problem.*

Proof. Let $v := A^* y_2$ and ϕ^1 be such that $A\phi^1 = b$. If $y_1^\gamma = 0$, then the dual (5.5) is given by

$$\begin{aligned} & \max_{y_2 \in R^n} \inf_{\phi(i)v(i) \ge 0} \{\| \phi \|_1 + \langle b - A\phi, y_2 \rangle\} \\ = & \max_{y_2 \in R^n} \inf_{\phi(i)v(i) \ge 0} \sum_{i=0}^{\infty} \{\phi(i)(\mathrm{sgn}(v(i)) - v(i))\} + \langle \phi^1, v \rangle \\ = & \max_{v \in Range(A^*)} \inf_{\phi(i)v(i) \ge 0} \sum_{i=0}^{\infty} \{\phi(i)(\mathrm{sgn}(v(i)) - v(i))\} + \langle \phi^1, v \rangle. \end{aligned}$$

Suppose $\| v \|_\infty > 1$; then there exists j such that $|v(j)| > 1$. Thus we can choose $\phi(j)$ with $\phi(j)v(j) \geq 0$ such that $\phi(j)(\mathrm{sgn}(v(j)) - v(j)) < M$ for any M. This implies that

$$\inf_{\phi(i)v(i)\geq 0} \sum_{i=0}^{\infty} \{\phi(i)(\mathrm{sgn}(v(i)) - v(i))\} + \langle \phi^1, v\rangle = -\infty.$$

Therefore, we can restrict v in the maximization to satisfy $\| v \|_\infty \leq 1$. From arguments similar to that of the proof of Theorem 5.9, $\phi(i) = 0$ whenever $|v(i)| < 1$. Therefore, the infimum term is zero whenever $\| v \|_\infty \leq 1$. This implies that the dual problem reduces to

$$\max_{v\in Range(A^*), \|v\|_\infty \leq 1} \langle \phi^1, v\rangle,$$

which is the same as the dual of the standard ℓ_1 problem as given in (4.1) [2]. This proves the lemma. $\square$

LEMMA 6.2. *Let Ω be a convex subset of a Banach space X and $f : \Omega \to R$ be strictly convex. If f achieves its minimum on Ω then the minimizer is unique.*

Proof. Let $m := \min_{x\in\Omega} f(x)$. Let $x_1, x_2 \in \Omega$ be such that $f(x_1) = f(x_2) = m$. Let $0 < \lambda < 1$. From convexity of Ω we have $\lambda x_1 + (1-\lambda)x_2 \in \Omega$. From strict convexity of f we have that if $x_1 \neq x_2$ then $f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2) = m$, which is a contradiction. Therefore $x_1 = x_2$. This proves the lemma. $\square$

LEMMA 6.3. *Let $y_1^\gamma \geq 0$, $y_2^\gamma \in R^n$ be a solution in* (5.5). *If $y_1^\gamma > 0$ then the solution ϕ_0 of* (4.5) *is unique.*

Proof. Let $L(\phi) := \| \phi \|_1 + y_1^\gamma(\langle \phi, \phi\rangle - \gamma) + \langle b - A\phi, y_2^\gamma\rangle$. From Theorem 3.3 we know that ϕ_0 minimizes $L(\phi)$, $\phi \in \ell_1$. If $y_1^\gamma > 0$ then it is easy to show that $L(\phi)$ is strictly convex in ℓ_1. From the previous lemma it follows that ϕ_0 is unique. This proves the lemma. $\square$

The main result of this subsection is now presented.

THEOREM 6.4. *Define $S := \{\phi : A\phi = b$ and $\| \phi \|_1 = \nu_\infty\}$, $m_2 := \inf_{\phi\in S}\langle \phi, \phi\rangle$. The following is true.*

(1) *If $\gamma \geq m_2$, then problem* (4.5) *is equivalent to the standard ℓ_1 problem whose solution is possibly nonunique.*

(2) *If $\mu_\infty < \gamma < m_2$, then the solution to* (4.5) *is unique.*

Proof. Suppose $m_2 < \gamma$. Then there exists $\phi_1 \in \ell_1$ such that $A\phi_1 = b$, $\| \phi_1 \|_1 = \nu_\infty$, and $\langle \phi_1, \phi_1\rangle \leq \gamma$. This implies that $\nu_\gamma = \inf_{A\phi=b,\langle\phi,\phi\rangle\leq\gamma} \| \phi \|_1 \leq \nu_\infty$. The other inequality is obvious. This proves (1).

Let $\mu_\infty < \gamma < m_2$ and suppose $y_1^\gamma = 0$; then we have shown in Lemma 6.1 that $\nu_\gamma = \nu_\infty$. Therefore, there exists $\phi_1 \in \ell_1$ such that $\| \phi_1 \|_1 = \nu_\infty$, $A\phi_1 = b$, and $\langle \phi_1, \phi_1\rangle \leq \gamma < m_2$. This implies that $\phi_1 \in S$ and $\langle \phi_1, \phi_1\rangle < m_2$, which is a contradiction. Therefore $y_1^\gamma > 0$. From Lemma 6.3 we know that ϕ_0 is unique. This proves (2). $\square$

The above theorem shows that in the region where the constraint level on the $\mathcal{H}_2$ is essentially of interest (i.e., active) the optimal solution is unique.

6.2. Continuity of the optimal solution. Following is a theorem which shows that the ℓ_1 norm of the optimal solution is continuous with respect to changes in the constraint level γ.

THEOREM 6.5. *Let $\nu_\gamma := \inf_{A\phi=b,\langle\phi,\phi\rangle\leq\gamma} \| \phi \|_1$. Then ν_γ is a continuous function of γ on (μ_∞, ∞).*

Proof. If $\gamma \in (\mu_\infty, \infty)$, then it is obvious that $\gamma \in \operatorname{int}\{\operatorname{dom}(\nu_\gamma)\}$, where $\operatorname{dom}(\nu_\gamma) := \{\gamma : -\infty < \nu_\gamma < \infty\}$ is the domain of ν_γ. From Proposition 1 of [7, §8.3] we know that ν_γ is a convex function of γ. The theorem follows from the fact that every convex function is continuous in the interior of its domain. ∎

Now we prove that the optimal solution is continuous with respect to changes in the constraint level in the region where the optimal is unique.

THEOREM 6.6. *Let $\mu_\infty < \gamma < m_2$. Let ϕ_γ represent the solution of $\nu_\gamma = \min_{A\phi=b,\langle\phi,\phi\rangle\leq\gamma} \| \phi \|_1$. Then $\phi_{\gamma_k} \to \phi_\gamma$ in the norm topology if $\gamma_k \to \gamma$.*

Proof. Let $m_1 := \min_{A\phi=b,\langle\phi,\phi\rangle\leq\mu_\infty} \| \phi \|_1$. Then it is obvious that $\| \phi_\gamma \|_1 = \nu_\gamma \leq m_1$. Without loss of generality, assume that $\gamma_k \geq \gamma/2$. Let L^* represent the upper bound on the length of $\phi_{\frac{\gamma}{2}}$. Then, as the upper bound is nonincreasing (see Theorem 5.9) we can assume that $\phi_{\gamma_k} \in R^{L^*}$. Let $B := \{x : x \in R^{L^*} : \| x \|_1 \leq m_1\}$; then we have $\phi_{\gamma_k} \in B$. Therefore, there exists a subsequence ϕ_{k_i} of ϕ_{γ_k} and ϕ_1 such that

$$\phi_{k_i} \to \phi_1 \text{ as } i \to \infty \text{ in } (R^{L^*}, \| \,.\, \|_1). \tag{6.1}$$

It is clear, as in the proof of Theorem 5.2, that $A\phi_1 = b$ as $A\phi_{k_i} = b$ for all i. Also,

$$\| \phi_1 \|_2{}^2 \leq \| \phi_1 - \phi_{k_i} \|_2{}^2 + \| \phi_{k_i} \|_2{}^2 \leq \| \phi_1 - \phi_{k_i} \|_2{}^2 + \gamma_{k_i}.$$

Taking limits on both sides as $i \to \infty$ we get $\langle\phi_1, \phi_1\rangle \leq \gamma$. This implies that ϕ_1 is a feasible element in the problem of ν_γ. From Theorem 6.5 we have $\| \phi_{k_i} \|_1 \to \nu_\gamma$. From (6.1) we have $\| \phi_1 \|_1 = \nu_\gamma$. From uniqueness of the optimal solution we have $\phi_1 = \phi_\gamma$. From uniqueness of the optimal solution, it also follows that $\phi_{\gamma_k} \to \phi_\gamma$. This proves the theorem. ∎

7. An example. In this section we illustrate the theory developed in the previous sections with an example. Consider the SISO plant,

$$\hat{P}(\lambda) = \lambda - \frac{1}{2}, \tag{7.1}$$

where we are interested in the sensitivity map $\phi := (I - PK)^{-1}$. Using Youla parametrization, we get that all achievable transfer functions are given by $\hat{\phi} = (I - \hat{P}\hat{K})^{-1} = 1 - (\lambda - \frac{1}{2})\hat{q}$ where $\hat{q}$ is a stable map. The matrices A and b are given by

$$A = \left(1, \frac{1}{2}, \frac{1}{2^2}, \dots\right), \quad b = 1.$$

It is easy to check that for this problem

$$\mu_\infty := \inf\{\| \phi \|_2{}^2 : \phi \in \ell_1 \text{ and } A\phi = b\} = 0.75$$

and

$$m_1 := \inf_{A\phi=b, \|\phi\|_2{}^2 \leq \mu_\infty} \| \phi \|_1 = 1.5,$$

with the optimal solution ϕ_2 given by

$$\hat{\phi}_2(\lambda) = \sum_{t=0}^{\infty} \frac{0.75}{2^t} \lambda^t.$$

Performing a standard ℓ_1 optimization [2] we obtain

$$\nu_\infty := \inf\{\| \phi \|_1 : \phi \in \ell_1 \text{ and } A\phi = b\} = 1$$

and

$$m_2 := \inf_{A\phi=b, \|\phi\|_1 \le \nu_\infty} \| \phi \|_2{}^2 = 1,$$

with the optimal solution $\phi_1 = 1$. We choose the constraint level to be 0.95. Therefore, $\alpha_\gamma = \frac{2m_1\sqrt{\gamma}}{\gamma-\mu_\infty} + 1 = 15.62$. For this example $n = 1$ and $z_1 = \frac{1}{2}$. L^*, the a priori bound on the length of the optimal, is chosen to satisfy

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \ \| (A_L^*)^{-l} \|_{\infty,1} \alpha_\gamma < 1, \tag{7.2}$$

where L is any positive integer such that $L \ge (n-1)$. We choose $L = 0$, and therefore $A_L = 1$ and $\| (A_L^*)^{-l} \|_{\infty,1}= 1$. We choose $L^* = 4$, which satisfies (7.2). Therefore, the optimal solution ϕ_0 satisfies $\phi_0(i) = 0$ if $i \ge 4$. The problem reduces to the following finite-dimensional convex optimization problem:

$$\nu_\gamma = \min_{A_{L^*}\phi=1, \|\phi\|_2{}^2 \le 0.95} \left\{ \sum_{k=0}^{3} |\phi(k)| : \phi \in R^4 \right\},$$

where $A_{L^*} = (1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3})$. We obtain (using Matlab Optimization Toolbox) the optimal solution ϕ_0 to be

$$\hat{\phi}_0(\lambda) = 0.9732 + 0.0535\lambda.$$

Therefore, we have $\| \phi_0 \|_1 = 1.02670$ and $\| \phi_0 \|_2{}^2 \cong 0.95$. The same computation was carried out for various values of the constraint level, $\gamma \in [0.75,\ 1]$. The tradeoff curve between the ℓ_1 and the $\mathcal{H}_2$ norms of the optimal solution is given in Figure 7.1. For all values of γ in the chosen range, the square of the $\mathcal{H}_2$ norm of the optimal closed loop was equal to the constraint level γ. Although, when the constraint level γ equals 0.75, the optimal closed-loop map is an infinite impulse response sequence, the optimal closed-loop map has very few nonzero terms in its impulse response even for values of γ very close to 0.75. For example, with $\gamma = 0.755$ the optimal closed-loop map is given by

$$\hat{\phi}_{0.755} = 0.7708 + 0.3632\lambda + 0.1596\lambda^2 + 0.0578\lambda^3 + 0.0065\lambda^4.$$

As a final remark, we can use the structure of this example to illustrate that the optimal unconstrained $\mathcal{H}_2$ solution can have $\mathcal{H}_2$ norm much smaller than the $\mathcal{H}_2$ norm of the optimal ℓ_1 (unconstrained) solution. Hence, minimizing only the ℓ_1 norm, which is an upper bound on the $\mathcal{H}_2$ norm, may require substantial sacrifices in terms of $\mathcal{H}_2$ performance. Indeed, instead of the P used in the example before, consider the plant $\hat{P}_a(\lambda) = \lambda - a$, where now a is a zero in the unit disk (i.e., $|a| < 1$) and very close to the unit circle (i.e., $|a| \cong 1$). Then the optimal unconstrained $\mathcal{H}_2$ norm given by

$$(b_a(A_aA_a^*)^{-1}b_a)^{1/2} = (1 - |a|^2)^{1/2},$$

where $A_a = (1, a, a^2, \ldots)$, $b_a = 1$ [2], is close to 0. On the other hand, for the optimal ℓ_1 unconstrained solution $\phi_{a,1}$, we have $\phi_{a,1} = 1$, which has $\mathcal{H}_2$ norm equal to 1. Therefore, minimizing only with respect to ℓ_1 may lead to undesirable $\mathcal{H}_2$ performance.

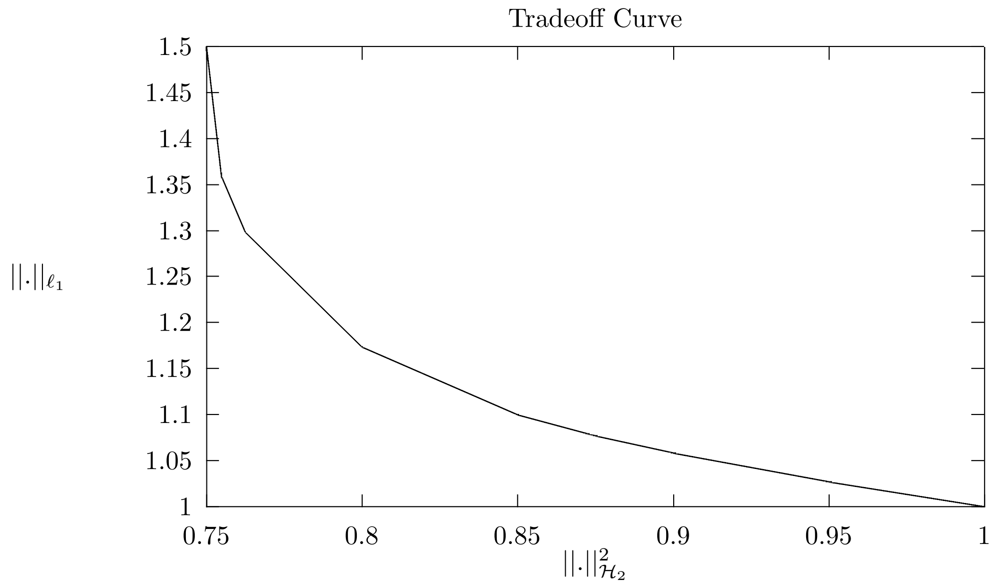

FIG. 7.1. *Here the ℓ_1 and the $\mathcal{H}_2$ norms of the optimal closed loop for various values of γ are plotted. The x axis can be read as the square of $\mathcal{H}_2$ norm or the value of γ.*

8. Conclusions. In this paper the mixed problem of $\ell_1/\mathcal{H}_2$ for the SISO discrete-time case is solved. The problem was reduced to a finite-dimensional convex optimization problem with an a priori determined dimension. The region of the constraint level in which the optimal is unique was determined, and it was shown that in this region the optimal solution is continuous with respect to changes in the constraint level of the $\mathcal{H}_2$ norm. A Lagrange duality theorem and a sensitivity result were used. The techniques used in obtaining the results of this paper are general enough to be adapted to analyze other mixed objective problems. Also, several of the results seem to have natural extensions for the multiple-input, multiple-output case, but a full investigation of this case is beyond the scope of this paper.

9. Appendix. Here we give the lemma needed in the proof of Theorem 3.3.

LEMMA 9.1. *Let X be a Banach space, Ω be a convex subset of X, Y be a finite-dimensional normed space, and Z be a normed space with positive cone P. Let Z^* denote the dual space of Z with a positive cone $P^{\oplus}$. Let $f : \Omega \to R$ be a real valued convex functional, $g : X \to Z$ be a convex mapping, $H : X \to Y$ be an affine linear map, and $0 \in \mathrm{int}[\{y \in Y : H(x) = y \text{ for some } x \in \Omega\}]$. Define*

$$\mu_0 := \inf\{f(x) : g(x) \leq 0,\ H(x) = 0,\ x \in \Omega\}. \tag{9.1}$$

Suppose there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$ and suppose μ_0 is finite. Then, there exist $z_0^* \geq 0$ and y_0^* such that

$$\mu_0 = \inf\{f(x) + \langle g(x), z_0^*\rangle + \langle H(x), y_0^*\rangle : x \in \Omega\}. \tag{9.2}$$

Proof. Let

$$\Omega_1 := \{x : x \in \Omega,\ H(x) = 0\}.$$

Applying Theorem 8.3.1 of [7, p. 217] to Ω_1 we know that there exists $z_0^* \in P^{\oplus}$ such that

$$\mu_0 = \inf\{f(x) + \langle g(x), z_0^*\rangle : x \in \Omega_1\}. \tag{9.3}$$

Consider the convex subset

$$H(\Omega) := \{y \in Y : \ y = H(x) \text{ for some } x \in \Omega\}$$

of Y. For $y \in H(\Omega)$, define

$$k(y) := \inf\{f(x) + \langle g(x), z_0^*\rangle : \ x \in \Omega, \ H(x) = y\}.$$

We now show that k is convex. Suppose $y, y' \in H(\Omega)$ and x, x' are such that $H(x) = y$ and $H(x') = y'$. Suppose $0 < \lambda < 1$. We have

$$\begin{aligned} \lambda(f(x) + \langle g(x), z_0^*\rangle) + (1-\lambda)(f(x') + \langle g(x'), z_0^*\rangle) \quad &\geq \quad f(\lambda x + (1-\lambda)x') \\ &\quad\ +\langle g(\lambda x + (1-\lambda)x'), z_0^*\rangle \\ &\geq \quad k(\lambda y + (1-\lambda)y'). \end{aligned}$$

(The first inequality follows from the convexity of f and g. The second inequality is true because $H(\lambda x + (1-\lambda)x') = (\lambda y + (1-\lambda)y')$. Taking the infimum on the left-hand side, we obtain $\lambda k(y) + (1-\lambda)k(y') \geq k(\lambda y + (1-\lambda)y')$. This proves that k is a convex function.

We now show that $k : H(\Omega) \to R$ (i.e., we show that $k(y) > -\infty$ for all $y \in H(\Omega)$). As $0 \in \text{int}[H(\Omega)]$, we know that there exists an $\epsilon > 0$ such that if $||y|| \leq \epsilon$, then $y \in H(\Omega)$. Take any $y \in H(\Omega)$ such that $y \neq 0$. Choose λ, y' such that

$$\lambda = \frac{\epsilon}{2||y||} \text{ and } y' = -\lambda y.$$

This implies that $y' \in H(\Omega)$. Let $\beta = \dfrac{\lambda}{\lambda + 1}$. We have

$$(1-\beta)y' + \beta y = 0.$$

Therefore, from convexity of the function k, we have

$$\beta k(y) + (1-\beta)k(y') \geq k(0) = \mu_0.$$

But $\mu_0 > -\infty$ by assumption. Therefore, $k(y) > -\infty$. Note that for all $y \in H(\Omega), \ k(y) < \infty$. This proves that k is a well-defined function.

Let $[k, H(\Omega)]$ be defined as given below:

$$[k, H(\Omega)] := \{(r, y) \in R \times Y : y \in H(\Omega), \ k(y) \leq r\}.$$

We first show that $[k, H(\Omega)]$ has a nonempty interior. As k is a well-defined convex function on the finite-dimensional convex set $H[\Omega]$ and $0 \in \text{int}[H(\Omega)]$, we have from Corollary 7.9.1 of [7, p. 194] that k is continuous at 0. Let $r_0 = k(0) + 2$ and choose ϵ' such that $0 < \epsilon' < 1$. As k is continuous at 0 we know that there exists $\delta > 0$ such that $y \in H(\Omega), $ and $||y|| \leq \delta$ implies that

$$|k(y) - k(0)| < \epsilon'.$$

This means that if $y \in H(\Omega)$ and $||y|| \leq \delta,$ then

$$k(y) < k(0) + \epsilon' < k(0) + 1 < r_0 - \frac{1}{2}.$$

Therefore, for all $y \in H(\Omega)$ with $||y|| \leq \delta$, we have $k(y) < r_0 - \frac{1}{2}$. This implies that for all $(r, y) \in R \times Y$ such that $|r - r_0| < \frac{1}{4}, y \in H(\Omega)$, and $||y|| \leq \delta$, we have $k(y) < r$. This proves that $(r_0, 0) \in \text{int}([k, H(\Omega)])$.

It is clear that $(k(0), 0) \in R \times Y$ is not in the interior of $[k, H(\Omega)]$. Using Theorem 5.12.2 of [7, p. 133] we know that there exists $(s, y^*) \neq (0, 0) \in R \times Y^*$ such that for all $(r, y) \in [k, H(\Omega)]$ the following is true:

$$\langle y, y^* \rangle + rs \geq \ \langle 0, y^* \rangle + k(0)s = s\mu_0. \tag{9.4}$$

In particular, $rs \geq s\mu_0$ for all $r \geq \mu_0$ (note that $(r, 0) \in [k, H(\Omega)]$ for all $r \geq \mu_0$). This means that $s \geq 0$.

Suppose $s = 0$. We have from (9.4) that $\langle y, y^* \rangle \geq 0$ for all $y \in H(\Omega)$. As $0 \in \text{int}[H(\Omega)]$, it follows that there exists an $\epsilon \in R$ such that $||y|| \leq \epsilon$ implies that $\langle y, y^* \rangle \geq 0$ and $\langle -y, y^* \rangle \geq 0$. This implies that if $||y|| \leq \epsilon$, then $\langle y, y^* \rangle = 0$. But then, for any $y \in Y$, one can choose a positive constant α such that $||\alpha y|| \leq \epsilon$, and therefore $\langle \alpha y, y^* \rangle = 0$. This implies that $(s, y^*) = (0, 0)$, which is not possible. Therefore, we conclude that $s > 0$.

Let $y_0^* = y^*/s$. From (9.4) we have

$$\langle y, y_0^* \rangle + r \geq \mu_0 \ \text{ for all } (r, y) \in [k, H(\Omega)]. \tag{9.5}$$

This implies that for all $y \in H(\Omega)$,

$$\langle y, y_0^* \rangle + k(y) \geq \mu_0. \tag{9.6}$$

(This is because $(k(y), y) \in [k, H(\Omega)]$.) Therefore, for all $x \in \Omega$,

$$\langle H(x), y_0^* \rangle + f(x) + \langle g(x), z_0^* \rangle \geq \mu_0, \tag{9.7}$$

which implies that

$$\inf\{f(x) + \langle g(x), z_0^* \rangle + \langle H(x), y_0^* \rangle : \ x \in \Omega\} \geq \mu_0. \tag{9.8}$$

But if $x \in \Omega$ is such that $H(x) = 0$, then

$$f(x) + \langle g(x), z_0^* \rangle = f(x) + \langle g(x), z_0^* \rangle + \langle H(x), y_0^* \rangle \tag{9.9}$$

$$\geq \inf\{f(x) + \langle g(x), z_0^* \rangle + \langle H(x), y_0^* \rangle : \ x \in \Omega\} \geq \mu_0. \tag{9.10}$$

Taking the infimum on the left-hand side of the above inequality over all $x \in \Omega$ which satisfy $H(x) = 0$ (that is infimum over all $x \in \Omega_1$), we have

$$\mu_0 = \inf\{f(x) + \langle g(x), z_0^* \rangle + \langle H(x), y_0^* \rangle : \ x \in \Omega\}. \tag{9.11}$$

This proves the lemma. $\square$

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