A Generalized Zames-Falb Multiplier

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***Abstract*—Lur'e systems represent feedback interconnections of a Linear Time-Invariant system with a nonlinear operator. The Zames–Falb criterion is a powerful tool to determine the stability of a Lur'e system when the nonlinear operator is defined by a monotonic, odd and single-valued function. The article provides a generalization of the Zames–Falb criterion for the analysis of stability of a Lur'e system for nonlinear static operators that are "approximately" monotonic and odd. This result extends the standard Zames–Falb criterion with an additional notion of "robustness".**

***Index Terms*—Nonlinear control systems.**

NOTATION

$\mathbb{R}$ Set of real numbers.

C Set of complex numbers.

L_2 Set of square integrable functions on $\mathbb{R}$.

L_{2e} Set of functions on $\mathbb{R}$ that are square integrable on any compact set.

I. INTRODUCTION

In this work, we derive a generalized form of the Zames–Falb multiplier for the analysis of stability of Lur'e systems [1]. Lur'e systems are given by the feedback interconnection of a linear time-invariant system $\mathcal{G}$ with a nonlinear block Δ (see Fig. 1). A large number of real systems have this structure. Recent examples are provided by Atomic Force Microscopes [2], [3] and other Microelectromechanical systems [4]–[6]. Many studies have targeted the problem of "absolute stability," where the global asymptotic stability of the origin is sought with respect to a class of nonlinear operators, and thus provide a robust notion of stability. Classical results include the Popov (see [7]) and the circle criteria (see [8] and [9]) that provide sufficient conditions for global asymptotic stability when the nonlinearity is restricted to be time-invariant and time varying respectively. The input/output approach based on Integral Quadratic Constraint (IQC) methods was pioneered by Yakubovich [10] and applied in different scenarios [11], [12]. More recently, the development of efficient algorithms to solve Linear Matrix Inequalities has made these methods computationally appealing and extremely versatile [13]. In this approach, a quadratic constraint is used to characterize the nonlinearity playing the role of the sector condition used in the other classical absolute stability criteria. Since the sector relations can be derived via special IQC conditions, such an approach provides a unifying theoretical framework [13]. Furthermore, it provides the powerful capability of seamlessly integrating different characterizations of the nonlinearity, as well. The main limitation to the general applicability of this technique is given by the necessity of identifying an IQC that properly characterizes the nonlinearity of interest. For example, the adoption of "standard" IQC's such as those defining the circle or the Popov criteria can lead to conservative results. Indeed, the sector conditions to which they are equivalent could include a class of nonlinearities that is too broad. On the other hand, an IQC such as the Zames-Falb one, is satisfied by nonlinearities with very strict characteristics (odd symmetry and monotonicity), but it does not provide any notion of "robustness" in the case of uncertainties in the knowledge of the nonlinearity. In this technical note we introduce a natural generalization of the Zames-Falb multiplier that bridges the gap between the Popov criterion condition and the standard Zames-Falb one. The technical note is structured as follows: in Section II we derive an IQC that is satisfied by a class of nonlinearities that are approximatively odd and monotonic and we use such a result in order to provide a stability criterion; in Section III we show the utility of our technique through numerical examples.

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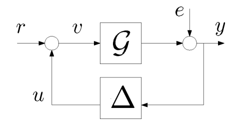


Fig. 1. Lur'e system.

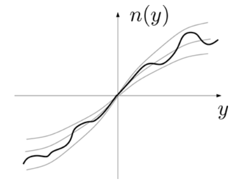


Fig. 2. Quasi-monotonic-and-odd function $n(y)$ is contained in an "envelope" of a monotonic and odd function $\overline{n}(y)$.

II. GENERALIZED ZAMES-FALB MULTIPLIER AND A STABILITY CRITERION VIA IQC'S

We introduce the class of quasi-monotonic-and-odd functions that will be used to formulate a stability criterion.

Definition 1: Let $n : \mathbb{R} \to \mathbb{R}$ be a single-valued function. We say that n is a quasi-monotonic-and-odd function with *spread* $D < 1$ and *skeleton* $\overline{n} : \mathbb{R} \to \mathbb{R}$ if there exists a function $\delta : \mathbb{R} \to \mathbb{R}$ such that

- $n(y) = \overline{n}(y)[1 + \delta(y)]$;
- $\overline{n}(y)$ is monotonic non-decreasing and odd;
- $|\delta(y)| \le D < 1$ for every $y \in \mathbb{R}$.

A graphical representation of a quasi-monotonic-and-odd function is given in Fig. 2.

We introduce the main technical result of the technical note.

Lemma 2.1 (Generalized Zames-Falb Multiplier): Let $n : \mathbb{R} \to \mathbb{R}$ be a quasi-monotonic-and-odd function with spread $D < 1$ and skeleton $\overline{n}$. Assume that

- $y(t) \in L_2$ implies $n(y(t)) \in L_2$;
- $h(t)$ is a absolutely integrable function such that

$$\|h(t)\|_1 \le \left(\frac{1-D}{1+D}\right)^2.$$

Then, for every $y(t) \in L_2$, we have that

$$\int_{-\infty}^{+\infty} n(y(t)) \left[y(t) - \int_{-\infty}^{+\infty} h(\tau)y(t+\tau)d\tau\right] dt \ge 0.$$

Proof: See the appendix. ∎

We provide the definition of well-posedness and (input-output) L_2 stability of a Lur'e interconnection.

Definition 2: Consider a Lur'e system where the linear part $\mathcal{G}$ is defined by a transfer function $G(s)$ and the nonlinear part is given by a nonlinear operator $\Delta : L_{2e} \to L_{2e}$. The *positive* feedback interconnection of $\mathcal{G}$ and Δ, as represented in Fig. 1, is described by the relations

$$\begin{cases} e = y - \mathcal{G}(u) \\ r = u - \Delta(y) \end{cases} \tag{1}$$

which define a map $\mathcal{M} : L_{2e} \times L_{2e} \to L_{2e} \times L_{2e}$ from the signals (u, y) to the L_{2e} signals (e, r). We say that the Lur'e interconnection is *well-posed* if $\mathcal{M}$ is causally invertible. We say that the Lur'e interconnection is L_2-*stable* if the restriction of $\mathcal{M}$ on L_2 is a bounded operator, that is, there exists a finite γ such that, for any $e, r \in L_2$

$$\|u\|^2 + \|y\|^2 < \gamma(\|e\|^2 + \|r\|^2).$$

The IQC framework for the stability analysis of Lur'e systems allows the expression of many absolute stability criteria in terms of a single unifying theory [13].

The fundamental result provided in [13] is reported for the sake of completeness.

Theorem 2.2 (Megretski-Rantzer Theorem): Consider a linear system $\mathcal{G}$ defined by a stable transfer function $G(s)$ and an operator $\Delta : L_{2e} \to L_{2e}$ bounded on its restriction on L_2. Let $\Pi(i\omega) : i\mathbb{R} \to C^{2\times 2}$ be a measurable and hermitian function. Let $v := \Delta(y)$ for any $y \in L_2$ and let $\hat{y}(i\omega)$ and $\hat{v}(i\omega)$ be the Fourier transform of $y(t)$ and $v(t)$ respectively. Assume that

- for every $\tau \in [0, 1]$ the *positive* feedback interconnection of $\mathcal{G}$ and $\tau\Delta$, as defined in (1), is well-posed;
- for every $\tau \in [0, 1]$ and for every $y \in L_2$

$$\int \begin{pmatrix} \hat{y}(i\omega) \\ \tau\hat{v}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \hat{y}(i\omega) \\ \tau\hat{v}(i\omega) \end{pmatrix} d\omega \geq 0. \tag{2}$$

- there exists $\epsilon > 0$ such that

$$\begin{pmatrix} G(i\omega) \\ 1 \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} G(i\omega) \\ 1 \end{pmatrix} \leq -\epsilon G(i\omega)^* G(i\omega). \tag{3}$$

Then, the feedback interconnection of $\mathcal{G}$ and Δ is L_2-stable.

We formulate the following criterion

Theorem 2.3 (Generalized Zames-Falb): Consider a Lur'e system given by the *positive* feedback interconnection of a LTI system $\mathcal{G}$, described by a stable transfer function $G(s)$, and a nonlinear operator $\Delta : L_{2e} \to L_{2e}$ defined as $\Delta(y) = v$, where $v(t) := n(y(t))$ for every $t \in \mathbb{R}$ and n is quasi-monotonic-and-odd function. Assume that

- the feedback inteconnection of $\mathcal{G}$ and $\tau\Delta$ is well-posed for every $\tau \in [0, 1]$;
- $yn(y) \geq 0$;
- $y(t) \in L_2$ implies $n(y(t)) \in L_2$;;
- Δ is bounded in its restriction on L_2
- there exists $\epsilon > 0$ and a summable function $h(t)$ such that

$$\|h\|_1 \leq \left(\frac{1 - D}{1 + D}\right)^2 \tag{4}$$

$$\mathrm{Real}\{G(i\omega)[1 + H(i\omega)]\} \leq -\epsilon G^*(i\omega)G(i\omega) \tag{5}$$

where $H(i\omega)$ is the Fourier transform of $h(t)$.

Then, the Lur'e system is stable.

Proof: Denote the Fourier transform of $y(t)$ and $v(t)$ as $\hat{y}(i\omega)$ and $\hat{v}(i\omega)$, respectively. Define

$$\Pi(i\omega) := \frac{1}{2} \begin{pmatrix} 0 & 1 - H(i\omega) \\ 1 - H(i\omega)^* & 0 \end{pmatrix}.$$

Then, the condition defined in (2) is equivalent to

$$\frac{\tau}{2} \int_{-\infty}^{+\infty} [\hat{y}(i\omega)^*(1 - H(i\omega))\hat{v}(i\omega) + \tag{6}$$

$$+\hat{y}(i\omega)(1 - H(i\omega)^*)\hat{v}(i\omega)^*] d\omega \geq 0. \tag{7}$$

Using the Parseval theorem, (6) becomes

$$\tau \int_{-\infty}^{+\infty} n(y(t)) \left[y(t) - \int_{-\infty}^{+\infty} h(\sigma)y(t + \sigma)d\sigma \right] dt \geq 0$$

that is satisfied because of Lemma 2.1. Finally, (3) is met by hypothesis. Thus, we can apply Theorem 2.2 and conclude the stability of the interconnection. ■

When $D = 0$, we find as a special case the standard Zames-Falb multiplier. Also, observe how the circle criterion (but in a formulation limited to time-invariant nonlinearities) can be obtained by using $H(i\omega) = 0$.

III. Numerical Examples

Condition (5) can be checked by solving a linear matrix inequality as proved in [14], however, we make use of an analytically tractable case, just for the purpose of showing the use of the generalized Zames-Falb stability criterion.

A. Saturation-Like Nonlinearity

Consider a Lur'e system defined as the *negative* feedback interconnection of a linear time-invariant operator described by a transfer function [Nyquist plot in Fig. 3(a)]

$$\overline{G}(s) = \frac{10(s + 0.25)s}{s^3 + 2s^2 + 2s + 1} \tag{8}$$

and the nonlinear function represented in (Fig. 3)

$$n(y) := K \arctan(y)[1 + D \sin(5y + \phi)]$$

with $K \geq 0$, $D \in [0, 0.05]$ and $\phi \in [-\pi, \pi]$. Nonlinearities of this kind are common in many applications, such as when interference fringes disturb the measurement obtained using photodiodes ([15] and [16]).

We stress that we have formulated our criterion assuming a *positive* feedback interconnection (according to the convention in [13]), but the example assumes a *negative* feedback. This can be taken into account simply by checking condition (5) for $G(s) = -\overline{G}(s)$. To this aim, consider $H(s) = 0.75/(s + 1)$ and observe that its impulse response $h(t)$ satisfies $\|h(t)\|_1 \leq 0.75$. Observe that

$$\mathrm{Real}\{[1 - H^*(i\omega)]G(i\omega)\} = \tag{9}$$

$$= -|1 - H^*(i\omega)|^2 \mathrm{Real}\left\{\frac{i\omega}{-\omega^2 + i\omega + 1}\right\} = \tag{10}$$

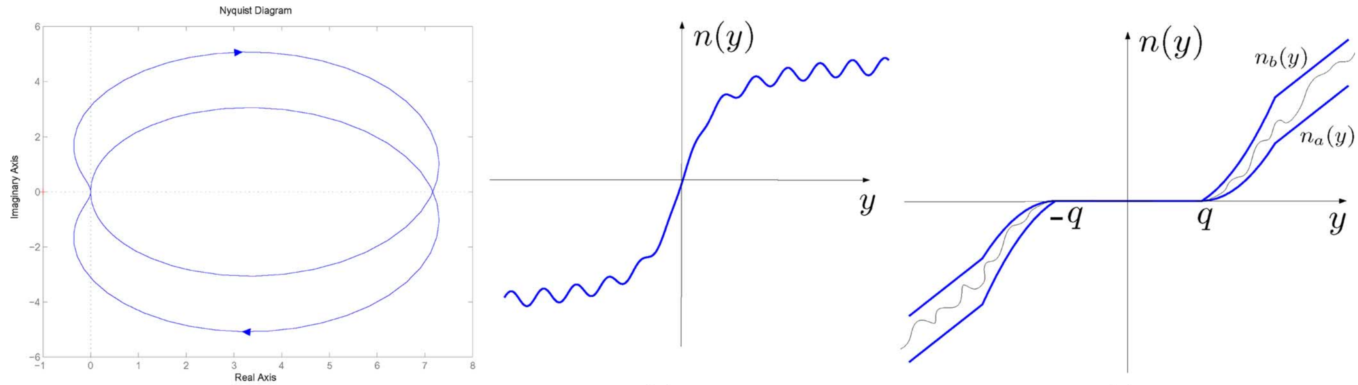
$$= -\frac{\omega^2|1 - H^*(i\omega)|^2}{\omega^4 - \omega^2 + 1} \leq -\epsilon\frac{\omega^2[0.25^2 + \omega^2]}{\omega^6 + 1} = -\epsilon|G(i\omega)|^2 \tag{11}$$

that is satisfied for some $\epsilon > 0$ sufficiently small. The generalized Zames-Falb condition holds for

$$0.75 < \left(\frac{1 - D}{1 + D}\right)^2 \Rightarrow D \leq \frac{1 - \sqrt{0.75}}{1 + \sqrt{0.75}} \simeq 0.0718$$

that defines the maximum deviation D of $n(y)$ from the monotonic function $\overline{n}(y) = K \arctan(y)$ that can be tolerated keeping the interconnection stable. Thus, the generalized Zames–Falb criterion guarantees stability for any $K > 0$ with a relative tolerance on the nonlinearity $\overline{n}$ of about 7%. The quasi-monotonic-and-odd nonlinearity has a spread of 5%, thus L_2-stability is ensured (implying global asymptotic stability of the origin).

The circle criterion guarantees stability for $K < \overline{K}_c \simeq 2.858$, while the Popov criterion guarantees stability for $K < \overline{K}_p \simeq 2.97$. The standard Zames–Falb criterion guarantees stability for any $K > 0$, but it can not be applied for $D \neq 0$.



(a) (b) (c)

Fig. 3. (a) Nyquist plot of the transfer function $\overline{G}(s) = 10(s+0.25)s/(s^3+2s^2+2s+1)$; (b) representation of the saturation nonlinearity in the first example; (c) representation of the dead zone nonlinearity in the second example.

B. Dead Zone Nonlinearity

Define the following class of nonlinearities with dead zone $[-q, q]$, with $q \geq 0$

$$n_\sigma(y) := \begin{cases} 0 & \text{if } 0 \leq y \leq q \\ (y-q)(y-\sigma) & \text{if } q < y \leq q+10 \\ 10(y-\sigma) & \text{if } q+10 < y \\ -n(-y) & \text{if } y < 0. \end{cases}$$

Consider the *negative* feedback interconnection of the linear system described by the transfer function $\overline{G}(s)$ as in (8) and a nonlinearity $n(y)$ such that $yn_a(y) \leq yn(y) \leq yn_b(y)$ with $b \leq a \leq q$ (to make the inequalities meaningful). This is represented schematically in Fig. 3(c). For different values of the parameters a and b, we ask what is the constraint on q that needs to be satisfied to guarantee stability. The circle and Popov criteria can not be applied since the nonlinearity $n(y)$ belongs to the sector $[0, 10]$ that is out of the sector for which they can guarantee stability.

For both the standard and the generalized Zames-Falb criteria we have to check condition (5). Since we are considering a *negative* feedback interconnection, the condition (5) is to be checked for $G(s) = -\overline{G}(s)$. This has already been done in (9) with the transfer function $H(s) = 0.75/(s+1)$ that has an impulse response $h(t)$ satisfying $\|h(t)\|_1 \leq 0.75$. The standard Zames-Falb criterion is not robust: it guarantees stability for any $q \geq a$, but it requires $a = b$ in order to make $n(y)$ odd and monotonic. For the generalized Zames–Falb criterion, the condition $\|h(t)\|_1 \leq 0.75$ leads to a maximum deviation from a monotonic odd nonlinearity $D \simeq 0.0718$. Define $\overline{n}(y) := n_{(a+b)/2}(y)$ and observe that, for $q < y < 10+q$

$$\frac{(y-b)}{\left(y-\frac{a+b}{2}\right)} \leq 1+\delta(y) = \frac{n(y)}{(y-q)\left(y-\frac{a+b}{2}\right)} \leq \frac{(y-a)}{\left(y-\frac{a+b}{2}\right)}.$$

This implies, for $q < y < 10+q$

$$|\delta(y)| \leq \frac{1}{2}\frac{a-b}{y-\frac{a+b}{2}} \leq \frac{1}{2}\frac{a-b}{q-\frac{a+b}{2}}. \tag{12}$$

The same relation can be shown for $q+10 \geq y$, while, for $0 < y \leq q$, the relation is trivially satisfied since $\delta(y) = 0$. By symmetry (12) holds for $y < 0$, as well. By imposing the inequality on the spread of $n(y)$, we find

$$\frac{1}{2}\frac{a-b}{q-\frac{a+b}{2}} \leq D \Rightarrow q \geq \frac{a+b}{2} + \frac{1}{D}\frac{a-b}{2}.$$

Thus, for $a = b$, stability is again guaranteed for $q \geq a$ as in the standard Zames–Falb criterion. However, when $b < a$, the generalized criterion finds sufficient conditions for stability, too: if the dead zone is large enough, stability is always guaranteed. The width of the required dead zone is interestingly proportional to the difference of the two parameters a, b, and proportional to the inverse of the maximum spread tolerable by the linear system.

IV. Conclusion

In this technical note, we have derived a generalization of the Zames–Falb multiplier for the stability analysis of Lur'e systems. The standard Zames–Falb multiplier allows for a formulation of a stability criterion when the feedback nonlinearity is strictly odd and monotonic. The new formulation takes into account possible deviations from the odd and monotonic behavior introducing a notion of robustness in the criterion.

Appendix

Proof of Lemma 2.1: The derivation follows the line of [17] for exactly monotonic and odd nonlinearities, but the additional degree of freedom given by the presence of the uncertainty D introduces technical complications that need to be taken into account.

Since n is quasi-monotonic-and odd it follows that $n(y)y \geq 0$ for every y. Define the potential function

$$P(y) := \int_0^y n(y')dy'.$$

Observe that

$$\int_a^b n(y')dy' = P(b) - P(a) \leq \tag{13}$$

$$\leq \begin{cases} (b-a)\sup_{a\leq y'\leq b} n(y') & \text{if } a \leq b \\ (a-b)\sup_{b\leq y'\leq a} -n(y') & \text{if } a > b. \end{cases} \tag{14}$$

Since $n(y)y \geq 0$, we can bound $n(y)$ as follows:

$$\overline{n}(y)(1 - D\text{sgn}(y)) \leq n(y) \leq \overline{n}(y)(1 + D\text{sgn}(y)).$$

Since $\overline{n}(y)$ is monotonic non-decreasing and odd, we have that

$$\begin{aligned} \sup_{a\leq y'\leq b} n(y') &\leq \overline{n}(b)[1 + D\text{sgn}(b)] && \text{if } a \leq b \\ \sup_{b\leq y'\leq a} -n(y') &\leq -\overline{n}(b)[1 - D\text{sgn}(b)] && \text{if } a > b. \\ \sup_{a\leq y'\leq b} n(y') &\leq -\overline{n}(-b)[1 + D\text{sgn}(b)] && \text{if } a \leq b \\ \sup_{b\leq y'\leq a} -n(y') &\leq \overline{n}(-b)[1 - D\text{sgn}(b)] && \text{if } a > b. \end{aligned}$$

Using the last four relations, the fact that $\overline{n}(b)b \geq 0$ and $0 \leq D < 1$, we get the following bounds in terms of $n(b)$ and $n(-b)$:

$$\sup_{a \leq y' \leq b} n(y') \leq n(b)\frac{1+D\text{sgn}(b)}{1-D\text{sgn}(b)} \tag{15}$$

$$\sup_{b \leq y' \leq a} -n(y') \leq -n(b)\frac{1-D\text{sgn}(b)}{1+D\text{sgn}(b)} \tag{16}$$

$$\sup_{a \leq y' \leq b} n(y') \leq -n(-b)\frac{1+D\text{sgn}(b)}{1-D\text{sgn}(b)} \tag{17}$$

$$\sup_{b \leq y' \leq a} -n(y') \leq n(-b)\frac{1-D\text{sgn}(b)}{1+D\text{sgn}(b)}. \tag{18}$$

Plugging the two relations (15) and (16) in (13) we get

$$P(b) - P(a) \leq n(b)(b-a)\frac{1+D\text{sgn}(b)}{1-D\text{sgn}(b)} \quad \text{if } a \leq b$$
$$P(b) - P(a) \leq n(b)(b-a)\frac{1-D\text{sgn}(b)}{1+D\text{sgn}(b)} \quad \text{if } a > b$$

or, more succintly

$$P(b) - P(a) \leq n(b)(b-a)\frac{1+D\text{sgn}[b(b-a)]}{1-D\text{sgn}[b(b-a)]}.$$

Using the other two relations (17) and (18) we find

$$P(b) - P(a) \leq -n(-b)(b-a)\frac{1+D\text{sgn}[b(b-a)]}{1-D\text{sgn}[b(b-a)]}.$$

Fix $y(t) \in L_2(-\infty, +\infty)$. Now, let $b = y(t)$ and $a = y(t+\tau)$. We obtain

$$\begin{aligned} &P(y(t)) - P(y(t+\tau)) \\ &\quad \leq n(y(t))[y(t) - y(t+\tau)]\frac{1}{q(t)} \end{aligned}$$

$$\begin{aligned} \text{where } 0 < \frac{1-D}{1+D} &\leq q(t) := \\ &:= \frac{1-D\text{sgn}[y(t)(y(t)-y(t+\tau))]}{1+D\text{sgn}[y(t)(y(t)-y(t+\tau))]} \leq \frac{1+D}{1-D}. \end{aligned}$$

Observe that

$$q(t)[P(y(t)) - P(y(t+\tau))] \leq n(y(t))[y(t) - y(t+\tau)].$$

Integrating both sides of the above inequality, we obtain

$$\int_{-\infty}^{\infty} q(t)P(y(t))dt - \int_{-\infty}^{\infty} q(t)P(y(t+\tau))dt \leq \int_{-\infty}^{\infty} n(t)[y(t) - y(t+\tau)]dt.$$

Observe that

$$\frac{1-D}{1+D}\int_{-\infty}^{\infty} P(y(t))dt \leq \int_{-\infty}^{\infty} q(t)P(y(t))dt$$
$$-\frac{1+D}{1-D}\int_{-\infty}^{\infty} P(y(t))dt \leq -\int_{-\infty}^{\infty} q(t)P(y(t+\tau))dt.$$

Then, we can write

$$\begin{aligned} &-\frac{4D}{(1-D)^2}\int_{-\infty}^{\infty} n(y(t))y(t)dt \\ &= -\left(\frac{1+D}{1-D} - \frac{1-D}{1+D}\right)\int_{-\infty}^{\infty}\left(\frac{1+D}{1-D}\right)n(y(t))y(t)dt \\ &\leq -\left(\frac{1+D}{1-D} - \frac{1-D}{1+D}\right)\int_{-\infty}^{\infty} P(y(t))dt \leq \\ &\leq \frac{1-D}{1+D}\int_{-\infty}^{\infty} P(y(t))dt - \frac{1+D}{1-D}\int_{-\infty}^{\infty} P(y(t+\tau))dt \leq \\ &\leq \int_{-\infty}^{\infty} q(t)[P(y(t)) - P(y(t+\tau))]dt \leq \\ &\leq \int_{-\infty}^{\infty} n(t)[y(t) - y(t+\tau)]dt. \end{aligned}$$

Thus, we get

$$\int_{-\infty}^{\infty} n(y(t))y(t+\tau)dt \leq \frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty} n(y(t))y(t)dt.$$

Proceeding in a similar way, for $b = -y(t)$ and $a = y(t+\tau)$, we find

$$-\int_{-\infty}^{\infty} n(y(t))y(t+\tau)dt \leq \frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty} n(y(t))y(t)dt.$$

Thus, we have an inequality involving the absolute value

$$\left|\int_{-\infty}^{\infty} n(y(t))y(t+\tau)dt\right| \leq \frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty} n(y(t))y(t)dt.$$

Let us evaluate

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau h(\tau)y(t+\tau)n(y(t)) \\ &\quad = \int_{-\infty}^{\infty} d\tau h(\tau)\int_{-\infty}^{\infty} dt y(t+\tau)n(y(t)) \\ &\quad \leq \int_{-\infty}^{\infty} d\tau |h(\tau)|\frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty} dt y(t)n(y(t)) \\ &\quad \leq \int_{-\infty}^{\infty} n(y(t))y(t)dt. \end{aligned}$$

This proves the assertion.

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Consensus in Directed Networks of Agents With Nonlinear Dynamics

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***Abstract*—This technical note studies the consensus problem for cooperative agents with nonlinear dynamics in a directed network. Both local and global consensus are defined and investigated. Techniques for studying the synchronization in such complex networks are exploited to establish various sufficient conditions for reaching consensus. The local consensus problem is first studied via a combination of the tools of complex analysis, local consensus manifold approach, and Lyapunov methods. A generalized algebraic connectivity is then proposed to study the global consensus problem in strongly connected networks and also in a broad class of networks containing spanning trees, for which ideas from algebraic graph theory, matrix theory, and Lyapunov methods are utilized.**

***Index Terms*—Algebraic graph theory, complex network, consensus, Lyapunov function, synchronization.**

I. Introduction

Cooperative collective behavior in networks of autonomous agents has received considerable attention in recent years due to the growing interest in understanding intriguing animal group behaviors, such as flocking and swarming, and also due to their emerging broad applications in sensor networks [1], unmanned air vehicles (UAV) formations, robotic teams, to name a few. To coordinate with other agents in a network, agents need to share information with their adjacent peers and agree on a certain value of interest. In this context, the consensus problem usually refers to the problem of how to reach an agreement among a group of autonomous agents in a dynamically changing environment [2]. One of the main challenges of solving such a consensus problem is that an agreement has to be reached by all the agents in the whole dynamic network while the information of each agent is shared only locally.

Various models have been used to study the consensus problem and some of the theoretical results obtained recently are closely related to what is presented in this technical note. In [3], Vicsek *et al.* studied a discrete-time system that models a group of autonomous agents moving in the plane with the same speed but different headings, which in essence is a simplified version of the model proposed earlier by Reynolds [4]. Analysis on Vicsek's model or its continuous-time version [5]–[9] shows that the connectivity of the time-varying graph that describes the neighbor relationships within the group is key in reaching consensus. In particular, in [6], Olfati-Saber and Murray established the relationship between the algebraic connectivity [10] (also called the Fiedler eigenvalue) and the speed of convergence when the directed graph is balanced. A broader class of directed graphs that may lead to consensus are those that contain spanning trees [8], [11], which are also called rooted graphs [9], [12]. Second-order and higher-order consensus in linear multi-agent systems was studied in [13], [33].

Vicsek's model is similar to a class of models discussed in studying the synchronization of complex networks [14]–[20], [26], [27]. In 1998, Pecora and Carroll made use of a master stability function to study the synchronization of coupled complex networks [17]. Thereafter, stability and synchronization of small-world and scale-free networks have been investigated extensively using this master stability function method. In [14], [15], local synchronization was studied using the transverse stability to the synchronization manifold, where synchronization was discussed with respect to small-world and scale-free networks. In [18], a distance from the collective states to the synchronization manifold was defined and then utilized to obtain conditions for global synchronization of coupled systems [19], [20]. It is clear that most of the real-world complex networks, e.g., World Wide Web and mobile communication networks, are directed networks. However, many existing tools developed for the study of synchronization in complex networks can only be applied to undirected networks. This is partly due to the fact that algebraic graph theory especially the algebraic connectivity has not been well developed for directed graphs. For example, there are no standard definitions for the algebraic connectivity and consensus convergence rate for directed graphs, while its counterparts for undirected graphs have been extensively used to study the synchronization problem.

Very recently, the consensus problem in directed networks with nonlinear dynamics has been discussed [21]–[25]. In [22], a class of feedback rules was used and a passivity-based design framework was developed to reach the velocity consensus among agents. Under the assumption that the vector fields satisfy a subtangentiality condition, it was proved in [23] that agents can reach consensus if and only if the network is connected sufficiently frequently over time. In [24], distributed algorithms for reaching network consensus was proposed based on nonsmooth analysis and many results assumed that the network is weight-balanced. In contrast, in this technical note, we consider the multi-agent system in which the dynamics of each agent consist of two terms:

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