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Mixed Objective Control Synthesis: Optimal $\ell_1/\mathcal{H}_2$ Control [1]

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Abstract

In this paper we consider the problem of minimizing the ℓ_1 norm of the transfer function from the exogenous input to the regulated output over all internally stabilizing controllers while keeping its $\mathcal{H}_2$ norm under a specified level. The problem is analysed for the discrete-time, SISO, linear time invariant case. It is shown that an optimal solution always exists. Duality theory is employed to show that any optimal solution is a finite impulse response sequence and an *a priori* bound is given on its length. The problem is reduced to a finite dimensional convex optimization problem with an *a priori* determined dimension. Finally it is shown that, in the region of interest of the $\mathcal{H}_2$ constraint level the optimal is unique and continuous with respect to changes in the constraint level.

1. Notation

In this section we present the notation employed in this paper. $\hat{x}(\lambda)$ is the λ transform of a right sided real sequence $x = (x(k))_{k=0}^{\infty}$. $\ell_1, \ell_2, \ell_\infty$ are the well known banach spaces of sequences. X^* is the dual space of the banach space X. $< x, x^* >$ denotes the value of the bounded linear functional x^* at $x \in X$. T^* is the adjoint operator of $T : X \rightarrow Y$ which maps Y^* to X^*.

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2. Introduction

Consider the finite dimensional linear time invariant system depicted in Figure 1 where P denotes the plant and K denotes the controller. The signal w is the exogenous input and z is the regulated output. The signals u and y denote the control input and the measured output respectively. Let T_{zw} be the closed loop transfer function which maps w to z.

Many important control problems can be reduced to the above setup where the objective is to minimize a suitably defined measure of T_{zw}. In the standard ℓ_1 problem the design of an internally stabilizing controller which minimizes ℓ_∞ norm of the regulated output z due to the worst case magnitude bounded disturbance w, is addressed. It is shown in [3] that this problem reduces to solving a finite dimensional linear program. The analogous problem with the signal measures being the ℓ_2 norm is the standard $\mathcal{H}_\infty$ problem. The standard $\mathcal{H}_2$ problem is concerned with the minimization of the energy contained in the pulse response of the closed loop, T_{zw} . This can be viewed as minimizing the variance of the regulated output

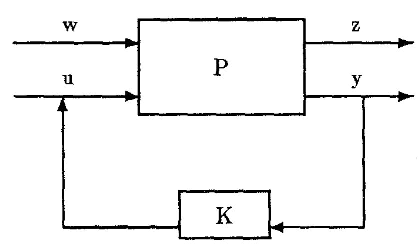

Figure 1: Plant Controller Configuration

z due to a white noise input w. Both problems are discussed in [7].

As a logical step, the design of controllers to satisfy mixed performance criteria has recently been the focus of researchers. Khargaonekar and Rotea in [6] address the problem of minimizing the two norm of a input output transfer function while keeping the two-induced norm ($\mathcal{H}_\infty$ norm) of another transfer function below a prescribed level. A state space approach was taken by the authors for this problem.

In [5] it is shown that a wide variety of control problems reduce to convex optimization problems and it is argued that the present technology makes it possible to deem the problem solved if it can be reduced to a convex optimization problem. In this light it is appropriate to exploit as much structure in the problem as possible so that the standard software available becomes computationally efficient. In [4] the problem of minimizing the ℓ_1 norm of the closed loop under linear inequality constraints is addressed. The problem of minimizing the ℓ_1 norm of the closed loop while keeping the $\mathcal{H}_\infty$ norm under a prescribed level falls under the above category.

In [10] the problem of minimizing the $\ell_1$norm of a single input single output transfer function while keeping the $\mathcal{H}_\infty$norm of the closed loop system under a specified value is reduced to solving a sequence of finite dimensional convex optimization problems and an unconstrained $\mathcal{H}_\infty$ problem. In [11] a similar problem in continuous time is considered. In [8] it is shown that the $\mathcal{H}_2/\ell_1$ problem; the problem of minimizing the $\mathcal{H}_2$ norm of the closed loop while maintaining the ℓ_1 norm below a prescribed value, reduces to a finite dimensional convex optimization problem.

In this paper the $\ell_1/\mathcal{H}_2$ problem which is the problem of minimizing the ℓ_1 norm of T_{zw} while keeping the $\mathcal{H}_2$ norm below a prescribed level is considered. In section 3 relevant duality theory results are given. In section 4 the problem statement is made precise. In section 5 results on the existence of optimal solutions and their properties are given. An *a priori* bound is given on the length of the impulse response of the optimal. In section 6 a simple example is given to demonstrate the theory developed. Finally in section 7 conclusions are given.

3. Mathematical Preliminaries

In this section we present a Lagrange duality theorem that applies to the minimization of a convex functional subject to both equality and inequality constraints. A sensitivity result which follows directly from the Lagrange duality theorem is presented. We employ the terminology used in [1] which is standard. First, we need the following definitions.

Definition 1 *Let P be a convex cone in a vector space X. We write $x \geq y$ if $x-y \in P$. We write $x > 0$ if $x \in int(P)$. Similarly $x \leq y$ if $x - y \in -P := N$ and $x < 0$ if $x \in int(N)$.*

Definition 2 *Let X be a vector space and Z be a vector space with positive cone P. A mapping $G : X \rightarrow Z$ is convex if $G(tx + (1-t)y) \leq tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with $0 \leq t \leq 1$. It is strictly convex if $G(tx + (1-t)y) < tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with $0 < t < 1$.*

The following is a lagrange duality theorem where we denote the interior of a set by *int*.

Theorem 1 *[1] Let X be a Banach space, Ω be a convex subset of X, Y be a finite dimensional space, Z be a normed space with positive cone P. Let $f : \Omega \rightarrow R$ be a real valued convex functional, $g : X \rightarrow Z$ be a convex mapping, $H : X \rightarrow Y$ be an affine linear map and $0 \in int[range(H)]$. Define*

$$\mu_0 := inf\{f(x) : \; g(x) \leq 0, \; H(x) = 0, \; x \in \Omega\}. \quad (1)$$

Suppose there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$ and suppose μ_0 is finite. Then,

$$\mu_0 = max\{\varphi(z^*, y) : z^* \geq 0, \; z^* \in Z^*, \; y \in Y\}, \quad (2)$$

where $\varphi(z^*, \; y) := inf \; \{ \; f(x) + < g(x), z^* > + < H(x), y >: \; x \in \Omega \; \}$ and the maximum is achieved for some $z_0^* \geq 0, \; z_0^* \in Z^*, \; y_0 \in Y$.

Furthermore if infimum in (1) is achieved by some $x_0 \in \Omega$ then

$$< g(x_0), z_0^* > + < H(x_0), y_0 >= 0, \quad (3)$$

and x_0 minimizes

$$f(x) + < g(x), z_0^* > + < H(x), y_0 >, \quad over \; all \; x \in \Omega. \quad (4)$$

We refer to (1) as the **Primal** problem and (2) as the **Dual** problem.

Corollary 1 *[1] Let X, Y, Z, f, H, g, Ω be as in Theorem 1. Let x_0 be the solution to the problem*

$$minimize \; f(x)$$
$$subject \; to \; x \in \Omega, \; H(x) = 0, \; g(x) \leq z_0$$

with (z_0^*, y_0) as the dual solution. Let x_1 be the solution to the problem

$$\begin{gathered} \textit{minimize } f(x) \\ \textit{subject to } x \in \Omega,\ H(x) = 0,\ g(x) \le z_1 \end{gathered}$$

with (z_1^*, y_1) as the dual solution. Then,

$$< z_1 - z_0, z_1^* > \le f(x_0) - f(x_1) \ \le < z_1 - z_0, z_0^* > . \tag{5}$$

4. Problem Formulation

Consider the standard feedback problem represented in Figure 1 where P and K are the plant and the controller respectively. Let w represent the exogenous input, z represent the output of interest, y is the measured output and u is the control input where z, w are assumed scalar. Let ϕ be the closed loop map which maps $w \longrightarrow z$. From Youla parametrization it is known that all achievable closed loop maps under stabilizing controllers are given by $\phi = h - u * q$ ($*$ denotes convolution), where $h, u, q \in \ell_1$; h, u depend only on the plant P and q is a free parameter in ℓ_1. Throughout the paper we make the following assumption.

Assumption 1 *All the zeros of $\hat{u}$ (the λ transform of u) inside the unit disc are real and distinct. Also, $\hat{u}$ has no zeros on the unit circle.*

The assumption that all zeros of $\hat{u}$ which are inside the open unit disc are real and distinct is not restrictive and is made to streamline the presentation of the paper. Let the zeros of u which are inside the unit disc be given by $z_1, z_2, \ldots, z_n$. Let

$$\Theta := \{\phi : \textit{there exists } q \in \ell_1 \textit{ with } \phi = h - u * q\}.$$

Θ is the set of all achievable closed loop maps under stabilizing controllers. Let $A : \ell_1 \longrightarrow R^n$ be given by

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \cdots \\ 1 & z_2 & z_2^2 & z_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \cdots \end{pmatrix},$$

and $b \in R^n$ be given by

$$b = \begin{pmatrix} \hat{h}(z_1) \\ \hat{h}(z_2) \\ \vdots \\ \hat{h}(z_n) \end{pmatrix}.$$

Theorem 2 *The following is true:*

$$\begin{aligned} \Theta &= \{\phi \in \ell_1 : \hat{\phi}(z_i) = \hat{h}(z_i) \textit{ for all } i = 1, \ldots, n\} \\ &= \{\phi \in \ell_1 : A\phi = b\}. \end{aligned}$$

Proof: Given in [2]. ■

The following problem

$$\begin{aligned} \nu_\infty &:= inf\{\| h - u * q \|_1 : q \in \ell_1\} \\ &= inf\{\| \phi \|_1 : \phi \in \ell_1 \textit{ and } A\phi = b\}, \end{aligned} \tag{6}$$

is the standard ℓ_1 problem. In [3] it is shown that this problem has a solution which is possibly non-unique. Optimal solutions are shown to be finite impulse response sequences. Let

$$\begin{aligned} \mu_\infty &:= inf\{\| h - u * q \|_2^2 : q \in \ell_1\}, \\ &= inf\{\| \phi \|_2^2 : \phi \in \ell_1 \textit{ and } A\phi = b\}, \end{aligned} \tag{7}$$

which is the standard $\mathcal{H}_2$ problem. The solution to this problem is unique but the solution is an infinite impulse response sequence. Define

$$m_1 := \inf_{A\phi = b, \| \phi \|_2^2 \le \mu_\infty} \| \phi \|_1, \tag{8}$$

which is the ℓ_1 norm of the unique optimal solution of the standard $\mathcal{H}_2$ problem. Let

$$m_2 := \inf_{A\phi = b, \| \phi \|_1 \le \nu_\infty} \| \phi \|_2^2, \tag{9}$$

which is the infimum over the ℓ_2 norms of the optimal solutions of the standard ℓ_1 problem.

The problem of interest is : *Given a positive constant $\gamma > \mu_\infty$ obtain a solution to the following mixed objective problem:*

$$\begin{aligned} \nu_\gamma := inf\{&\| h - u * q \|_1 : q \in \ell_1 \textit{ and} \\ &< h - u * q, h - u * q > \le \gamma\} \\ = inf\{&\| \phi \|_1 : \phi \in \ell_1, A\phi = b \textit{ and} \\ &< \phi, \phi > \le \gamma\}. \end{aligned} \tag{10}$$

Note that $< .,. >$ is the inner product associated with ℓ_2. In the following sections we will study this problem from the points of view of existence, structure and continuity of the optimal solution.

5. Existence and Properties of Optimal Solutions

First we present a theorem which states that (10) always has a solution. We then present results which show that any solution to (10) is of finite length and an *a priori* bound on the length exists.

Theorem 3 *There exists $\phi_0 \in \Phi$ such that*

$$\| \phi_0 \|_1 = \inf_{\phi \in \Phi} \{ \| \phi \|_1 \},$$

where $\Phi := \{\phi \in \ell_1 : A\phi = b$ and $< \phi, \phi > \leq \gamma\}$ with $\gamma > \mu_\infty$. Therefore the infimum in (10) is a minimum.

This theorem follows from the Banach-Alaoglu lemma on weak star compactness and the fact that the zero interpolation conditions lie in c_0.

Theorem 4
Define $\mathcal{T} := \{\phi \in \ell_1$: there exists L^* with $\phi(i) = 0$ if $i \geq L^*\}$. The dual of the problem is given by:

$$\max\{\varphi(y_1, y_2) : y_1 \geq 0, y_2 \in R^n\}, \qquad (11)$$

where

$$\varphi(y_1, y_2) := \inf_{\phi \in \mathcal{T}, \phi(i)v(i) \geq 0} \{ \| \phi \|_1 + y_1(< \phi, \phi > -\gamma) + < b, y_2 > - < \phi, v > \}.$$

$v(i)$ defined by $v(i) = A^* y_2(i)$. Also, any optimal solution ϕ_0 of (10) belongs to $\mathcal{T}$.

The above theorem is proved by using Theorem 1 to obtain the dual of (10) and then exploiting the structure of the dual problem. Now we give an *a priori* bound on the length of any solution to (10).

Theorem 5 *Let*

$$\alpha_\gamma = \frac{2m_1\sqrt{\gamma}}{\gamma - \mu_\infty} + 1,$$

and let $L \geq (n-1)$ be an integer. Define, $A_L^* : R^n \rightarrow R^{L+1}$ by:

$$A_L^* = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ z_1 & z_2 & z_3 & \dots & z_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^L & z_2^L & z_3^L & \dots & z_n^L \end{pmatrix}.$$

Let $(A_L^*)^{-l}$ denote the left inverse of A_L^* with its induced norm denoted by $\| (A_L^*)^{-l} \|_{\infty,1}$. Let L^* be such that

$$\max_{k=1,\dots,n} |z_k|^{L^*} \ \| (A_L^*)^{-l} \|_{\infty,1} \, \alpha_\gamma < 1.$$

Then every solution ϕ_0 of the primal (10) is such that $\phi(i) = 0$ if $i \geq L^*$.

This theorem can be proved by using Corollary 1 to establish an *a priori* bound on the dual variable y_1, which in turn can be used to bound the infinity norm of $v = A^* y_2$, by α_γ. The above theorem is the main theorem of this paper. It reduces the infinite dimensional convex optimization problem given by (10) to the following finite dimensional convex optimization with *a priori* known dimension:

$$\nu_\gamma = \inf \sum_{t=0}^{L^*-1} |\phi(t)|,$$

subject to

$$\phi \in R^{L^*}, \ A_{L^*}\phi = b, \ \sum_{t=0}^{L^*-1} |\phi(t)|^2 \leq \gamma.$$

It can also be shown that this problem reduces to an LMI. The following theorem is a result on the uniqueness of the optimal.

Theorem 6 *Define $S := \{\phi : A\phi = b$ and $\| \phi \|_1 = \nu_\infty\}$, $m_2 := \inf_{\phi \in S} < \phi, \phi >$. The following is true:*

1) If $\gamma \geq m_2$ then problem (10) is equivalent to the standard ℓ_1 problem whose solution is possibly nonunique.

2) If $\mu_\infty < \gamma < m_2$ then the solution to (10) is unique.

The following theorem establishes the continuity of the optimal with respect to changes in the constraint level in the region where the optimal is unique.

Theorem 7 *Let $\mu_\infty < \gamma < m_2$. Let ϕ_γ represent the solution of $\nu_\gamma = \min_{A\phi = b, < \phi, \phi > \leq \gamma} \| \phi \|_1$. Then $\phi_{\gamma_k} \rightarrow \phi_\gamma$ in the norm topology if $\gamma_k \rightarrow \gamma$.*

6. An Example

In this section we illustrate the theory developed in the previous sections with an example taken from [8]. Consider the SISO plant,

$$\hat{P}(\lambda) = \lambda - \frac{1}{2}, \qquad (12)$$

where we are interested in the sensitivity map $\phi := (I - PK)^{-1}$. Using Youla parametrization we get that all achievable transfer functions are given by $\hat{\phi} = (I - \hat{P}\hat{K})^{-1} = 1 - (\lambda - \frac{1}{2})\hat{q}$ where $\hat{q}$ is a stable map. The matrix A and b are given by

$$A = (1, \frac{1}{2}, \frac{1}{2^2}, \dots), \ \ b = 1.$$

It is easy to check that for this problem

$$\mu_\infty := inf\{\| \phi \|_2{}^2 : \phi \in \ell_1 \ and \ A\phi = b\} = 0.75$$

and

$$m_1 := \inf_{A\phi=b, \| \phi \|_2{}^2 \leq \mu_\infty} \| \phi \|_1 = 1.5.$$

Performing a standard ℓ_1 optimization [2] we obtain

$$\nu_\infty := inf\{\| \phi \|_1 : \phi \in \ell_1 \ and \ A\phi = b\} = 1$$

and

$$m_2 := \inf_{A\phi=b, \| \phi \|_1 \leq \nu_\infty} \| \phi \|_2{}^2 = 1.$$

with the optimal solution being $\phi_1 = 1$. We choose the constraint level to be 0.95. Therefore, $\alpha_\gamma = \frac{2m_1\sqrt{\gamma}}{\gamma - \mu_\infty} + 1 = 10.75$. For this example $n = 1$ and $z_1 = \frac{1}{2}$. L^* the *a priori* bound on the length of the optimal is chosen to satisfy

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \ \| (A_L^*)^{-l} \|_{\infty,1} \ \alpha_\gamma < 1. \tag{13}$$

where L is any positive integer such that $L \geq (n-1)$. We choose $L = 0$ and therefore $A_L = 1$ and $\| (A_L^*)^{-l} \|_{\infty,1} = 1$. We choose $L^* = 4$ which satisfies (13). Therefore, the optimal solution ϕ_0 satisfies $\phi_0(i) = 0$ if $i \geq 4$. The problem reduces to the following finite dimensional convex optimization problem:

$$\nu_\gamma = \min_{A_{L^*}\phi=1, \| \phi \|_2{}^2 \leq 0.95} \{\sum_{k=0}^{3} |\phi(k)| : \phi \in R^4\},$$

where $A_{L^*} = (1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3})$. We obtain (using Matlab Optimization Toolbox) the optimal solution ϕ_0 to be:

$$\hat{\phi}_0(\lambda) = 0.9732 + 0.0535\lambda.$$

Therefore we have $\| \phi_0 \|_1 = 1.02670$ and $\| \phi \|_2{}^2 \cong 0.95$.

7. Conclusions and Further Research

In this paper the mixed problem of $\ell_1/\mathcal{H}_2$ for the SISO discrete time case is solved. The problem was reduced to a finite dimensional convex optimization problem with an *a priori* determined dimension. Results on continuity and uniqueness of the optimal were given.

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