JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 91, No. 1, pp. 91-113, OCTOBER 1996

Controller Design to Optimize a Composite Performance Measure[1]

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Communicated by J. B. Pearson

Abstract. This paper studies a mixed objective problem of minimizing a composite measure of the l_1, $\mathscr{H}_2$, and l_∞-norms together with the l_∞-norm of the step response of the closed loop. This performance index can be used to generate Pareto-optimal solutions with respect to the individual measures. The problem is analyzed for discrete-time, single-input single-output (SISO), linear time-invariant systems. It is shown via Lagrange duality theory that the problem can be reduced to a convex optimization problem with a priori known dimension. In addition, continuity of the unique optimal solution with respect to changes in the coefficients of the linear combination that defines the performance measure is established.

Key Words. Duality theory, l_1-optimization, multiobjective control.

1. Introduction

Consider the standard feedback configuration of Fig. 1, and let ϕ be the closed-loop transfer function which maps the exogenous input w to the regulated output z. The aim in feedback control is to design a controller K such that the system is internally stable and the closed-loop map between w and z meets certain specifications. In many cases, the objective is to determine a controller K which minimizes a relevant norm of the closed loop

[1]This research was supported by the National Science Foundation under Grants No. ECS-92-04309, ECS-92-16690 and ECS-93-08481.

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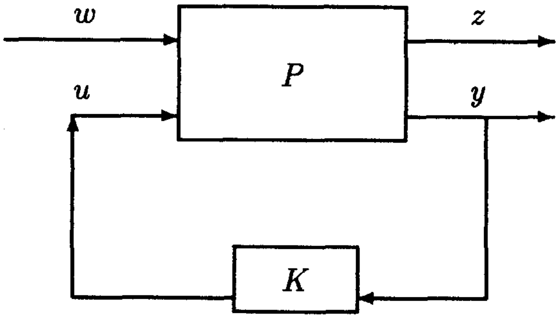

Fig. 1. Plant controller configuration.

over all stabilizing controllers. For example, in the standard $\mathcal{H}_2$-problem, the norm being minimized is the $\mathcal{H}_2$-norm of the map ϕ. This is a measure of the variance in the regulated output z for a white noise input w. This problem is studied in detail in Ref. 1. On the other hand, the standard l_1-problem addresses the design of an internally stabilizing controller such that the l_∞-norm of the regulated output z due to a worst-case l_∞-bounded disturbance w. It is shown in Ref. 2 that, for the 1-block case, the problem reduces to solving a finite-dimensional linear program. Another relevant measure can be the l_∞-norm of the closed-loop response to specific inputs w (e.g., steps, impulses, sinusoids, etc.), i.e., the maximum magnitude of the regulated output z to a specific w. Such types of problems have been considered in Ref. 3 where approximate solutions within any a priori established tolerance are obtained via linear programming.

In several situations, however, it becomes necessary to consider different measures of the regulated output together, since it is well known that a controller that gives good performance with respect to any particular measure may not guarantee good performance with respect to some other measure (e.g., Ref. 4). Thus, a mixed type of performance index is of importance. In recent years, results were obtained for problems incorporating mixed performance indices. Many state space results are available on the interplay of the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$-norms of the closed loop (e.g., Ref. 5 and Ref. 6). In Ref. 7 and Ref. 8, the problem of minimizing the l_1-norm of the closed loop, while keeping its $\mathcal{H}_\infty$-norm below a prespecified level, is considered. It is shown that approximate solutions can be obtained via finite-dimensional convex optimization. In Ref. 9, it is shown that approximate solutions to a continuous-time problem of minimizing the maximum magnitude of z due to a specified input w, while keeping the $\mathcal{H}_\infty$-norm of the closed loop below a given level, can be obtained by solving a finite-dimensional convex constrained optimization problem and a standard unconstrained $\mathcal{H}_\infty$-problem.

A wide variety of mixed objective optimization problems can be reduced to convex optimization problems (Ref. 10). However, these problems are in general infinite dimensional, and only approximate solutions can be obtained by using the methods given in Ref. 10. In addition, it may be hard to check how close to optimal the approximation is. Hence, it is only appropriate to exploit as much structure in the problem as possible so that certain properties of the problem at hand are revealed and computation of solutions becomes more efficient. In particular, it is important to identify classes of problems that, although initially cast as infinite dimensional, are in fact finite dimensional. In this case, it is also helpful to give, if possible, a priori bounds on the dimension of the problem. Within this context, several results on multiobjective mixed optimization involving the l_1-norm are becoming available. In Ref. 11, the problem of minimizing the $\mathscr{H}_2$-norm of the closed loop, while keeping its l_1-norm below a prespecified level, is reduced to a finite-dimensional quadratic optimization problem. In Ref. 12, the converse problem of minimizing the l_1-norm of the closed loop while constraining its l_2-norm to lie below a prespecified level is considered. It is shown that the problem reduces to a finite-dimensional convex optimization problem with an a priori known dimension.

In a similar vein, we consider in this paper the problem of minimizing a given linear combination of the l_1-norm, the square of the $\mathscr{H}_2$-norm, and the l_∞-norms of the step and pulse responses respectively of the closed loop over all stabilizing controllers. This performance index can be associated with the performance to inputs that are deterministic worst-case amplitude-bounded (hence, the l_1-norm), random (hence, the $\mathscr{H}_2$-norm), and fixed steps and pulses (hence, the l_∞-norms of the step and pulse responses). The problem is reduced to a finite-dimensional convex optimization with a priori known dimension, which can be readily solved via standard numerical algorithms. Its solution represents a Pareto-optimal point with respect to the individual measures involved. Continuity of the unique optimal solution with respect to changes in the coefficients of the linear combination is also established.

The paper is organized as follows. In Section 3, the statement of the problem is made precise and its relation to Pareto optimality is established. In Section 4, it is shown that the problem has a unique solution which has a finite impulse response. The problem is then reduced to a finite-dimensional convex optimization problem. In Section 5, an example is given to illustrate the theory developed. In Section 6, continuity properties of the optimal solution are studied. Finally, we conclude in Section 7.

1.1. Notation. The following notation is employed in this paper:

$|x|_1 = $ 1-norm of the vector $x \in R^n$;

$|x|_2 = $ 2-norm of the vector $x \in R^n$;

$\hat{x}(\lambda)$ = λ-transform of a right-sided real sequence $x=(x(k))_{k=0}^{\infty}$, defined as $\hat{x}(\lambda) := \sum_{k=0}^{k=\infty} x(k)\lambda^k$;

l_1 = Banach space of right-sided absolutely summable real sequences with the norm given by $\|x\|_1 := \sum_{k=0}^{\infty} |x(k)|$;

l_∞ = Banach space of right-sided, bounded sequences with the norm given by $\|x\|_\infty := \sup_k |x(k)|$;

c_0 = subspace of l_∞ with elements x that satisfy $\lim_{k\to\infty} x(k) = 0$;

l_2 = Banach space of right-sided square summable sequences with the norm given by $\|x\|_2 := [\sum_{k=0}^{k=\infty} x(k)^2]^{1/2}$;

$\mathscr{H}_2$ = isometric isomorphic image of l_2 under the λ-transform $\hat{x}(\lambda)$, with the norm given by $\|\hat{x}(\lambda)\|_2 = \|x\|_2$;

X^* = dual space of the Banach space X;

$\langle x, x^*\rangle$ = value of the bounded linear functional $x^* \in X^*$ at $x \in X$;

$W(X^*, X)$ = weak star topology on X^* induced by X;

T^* = adjoint operator of $T: X \to Y$ which maps Y^* to X^*.

We have from functional analysis that

$$(l_1)^* = l_\infty, \qquad (c_0)^* = l_1, \qquad (l_2)^* = l_2.$$

2. Problem Formulation

Consider Fig. 1, where P and K are the plant and the controller respectively. Let w represent the exogenous input, z the output of interest, y the measured output, and u the control input, where z and w are assumed scalar. Let ϕ be the closed-loop map which maps $w \to z$. From Youla parametrization (Ref. 13), it is known that all achievable closed-loop maps under stabilizing controllers are given by

$$\phi = h - u * q,$$

where $*$ denotes convolution, $h, u, q \in l_1$ with h, u depending only on the plant P and q a free parameter in l_1. Throughout the paper, we make the following assumption.

Assumption A1. All the zeros of $\hat{u}$ (the λ-transform of u) inside the unit disc are real and distinct. Also, $\hat{u}$ has no zeros on the unit circle.

The assumption that all zeros of $\hat{u}$ which are inside the open unit disc are real and distinct is not restrictive and is made to streamline the presentation of the paper. Let the zeros of $\hat{u}$ which are inside the unit disc be given by $z_1, z_2, \ldots, z_n$. Let

$$\Phi := \{\phi: \text{there exists } q \in l_1 \text{ with } \phi = h - u * q\}.$$

Φ is the set of all achievable closed-loop maps under stabilizing controllers. Let $A: l_1 \to R^n$ be given by

$$A = \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 & \cdots \\ 1 & z_2 & z_2^2 & z_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \cdots \end{bmatrix},$$

and let $b \in R^n$ be given by

$$b = \begin{bmatrix} \hat{h}(z_1) \\ \hat{h}(z_2) \\ \vdots \\ \hat{h}(z_n) \end{bmatrix}.$$

Theorem 2.1. The following is true:

$$\begin{aligned} \Phi &= \{\phi \in l_1 : \hat{\phi}(z_i) = \hat{h}(z_i), \text{ for all } i = 1, \ldots, n\} \\ &= \{\phi \in l_1 : A\phi = b\}. \end{aligned} \tag{1}$$

Proof. See Ref. 4. □

Let w_1 be the unit step input, i.e.,

$$w_1 = (1, 1, \ldots).$$

Let the corresponding output be denoted as z_1, i.e.,

$$z_1(i) = (\phi * w_1)(i) = \sum_{k=0}^{i} \phi(i-k) w_1(k) = \sum_{k=0}^{i} \phi(k).$$

Then, the problem of interest can be stated as follows: Given $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$, obtain a solution to the following mixed objective problem:

$$\begin{aligned} v &:= \inf_{\phi \text{ achievable}} \{c_1 \|\phi_1\|_1 + c_2 \|\phi\|_2^2 + c_3 \|\phi * w_1\|_\infty + c_4 \|\phi\|_\infty\} \\ &= \inf_{\phi \in l_1,\, A\phi = b} \{c_1 \|\phi\|_1 + c_2 \|\phi_2^2 + c_3 \|\phi * w_1\|_\infty + c_4 \|\phi\|_\infty\}. \end{aligned} \tag{2}$$

We define $f: l_1 \to R$ by

$$f(\phi) := c_1\|\phi\|_1 + c_2\|\phi\|_2^2 + c_3\|\phi * w_1\|_\infty + c_4\|\phi\|_\infty,$$

which is the objective functional in the optimization given by (2).

In the following sections, we will study the existence, structure, and computation of the optimal solution. Before we initiate our study toward these goals, it is worthwhile to point out certain connections between the cost under consideration and the notion of Pareto optimality.

2.1. Relation to Pareto Optimality. The notion of Pareto optimality can be stated as follows (see, for example, Ref. 10). Given a set of m nonnegative functionals $\bar{f}_i$, $i=1, \ldots, m$, on a normed linear space X, a point $x_0 \in X$ is Pareto optimal with respect to the vector-valued criterion $\bar{f} := (\bar{f}_1, \ldots, \bar{f}_m)$ if there does not exist any $x \in X$ such that

$$\bar{f}_i(x) \leq \bar{f}_i(x_0), \qquad \forall i \in \{1, \ldots, m\},$$
$$\bar{f}_i(x) < \bar{f}_i(x_0), \qquad \text{for some } i \in \{1, \ldots, m\}.$$

Under certain conditions, the set of all Pareto optimal solutions can be generated by solving a minimization of the weighted sum of the functionals, as the following theorem indicates.

Theorem 2.2. See Ref. 14. Let X be a normed linear space, and let each nonnegative functional $\bar{f}_i$ be convex. Also, let

$$S_m := \left\{c \in R^m : c_i \geq 0, \sum_{i=1}^{m} c_i = 1\right\},$$

and for each $c \in R^m$ consider the following scalar-valued optimization problem:

$$\inf_{x \in X} \sum_{i=1}^{m} c_i \bar{f}_i(x).$$

If $x_0 \in X$ is Pareto optimal with respect to the vector-valued criterion $\bar{f}(x)$, then there exists some $c \in S_m$ such that x_0 solves the above minimization problem. Conversely, given $c \in S_m$, if the above minimization problem has at most one solution x_0, then x_0 is Pareto optimal with respect to $\bar{f}(x)$.

In the next section, we show that there is a unique solution ϕ_0 to Problem (2). Furthermore, since u is assumed to be a scalar, there is a unique optimal $q \in l_1$. Hence, in view of the aforementioned theorem, we have that, if we restrict our attention to parameters c_1, c_2, c_3, c_4 such that

$$(c_1, c_2, c_3, c_4) \in \Sigma_4$$
$$:= \{(c_1, c_2, c_3, c_4) : c_1 + c_2 + c_3 + c_4 = 1, c_1, c_2, c_3, c_4 > 0\},$$

we will produce a set of Pareto-optimal solutions with respect to the vector-valued function

$$\begin{aligned}\bar{f}(q) &:= (\|h-u*q\|_1, \|h-u*q\|_2^2, \|(h-u*q)*w_1\|_\infty, \|h-u*q\|_\infty)\\ &=: (\bar{f}_1(q), \bar{f}_2(q), \bar{f}_3(q), \bar{f}_4(q)),\end{aligned}$$

where $q \in l_1$. Thus, if ϕ_0 is the optimal solution for Problem (2) with a corresponding q_0 for some given $(c_1, c_2, c_3, c_4) \in \Sigma_4$, then there does not exist a preferable alternative ϕ, with

$$\phi = h - u * q, \qquad \text{for some } q \in l_1,$$

such that

$$\begin{aligned}&\bar{f}_i(q) \leq \bar{f}_i(q_0), &&\forall i \in \{1, \ldots, 4\},\\ &\bar{f}_i(q) < \bar{f}_i(q_0), &&\text{for some } i \in \{1, \ldots, 4\}.\end{aligned}$$

As a final note, we mention that, if (c_1, c_2, c_3, c_4) do not satisfy

$$c_1 + c_2 + c_3 + c_4 = 1,$$

then we can define a new set of parameters $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ by

$$\begin{aligned}\bar{c}_1 &= c_1/(c_1 + c_2 + c_3 + c_4),\\ \bar{c}_2 &= c_2/(c_1 + c_2 + c_3 + c_4),\\ \bar{c}_3 &= c_3/(c_1 + c_2 + c_3 + c_4),\\ \bar{c}_4 &= c_4/(c_1 + c_2 + c_3 + c_4),\end{aligned}$$

with

$$\bar{c}_1 + \bar{c}_2 + \bar{c}_3 + \bar{c}_4 = 1.$$

These new parameters would yield the same optimal solution as with (c_1, c_2, c_3, c_4).

3. Existence, Uniqueness, and Properties of the Optimal Solution

In the first part of this section, we show that Problem (2) always has a solution. In the second part, we show that any solution to Problem (2) is a finite impulse response sequence; in the third part, we give an a priori bound on the length.

3.1. Existence of a Solution. Here, we show that a solution to (2) always exists. We use the following well-known lemma (see, for example, Ref. 15) to prove the main result of this subsection.

Lemma 3.1. Banach–Alaoglu Lemma. Let X be a separable Banach space with X^* as its dual. Then, the set $\{x^*: x^* \in X^*, \|x^*\| \leq M\}$ is $W(X^*, X)$ sequentially compact for any $M \in R$.

Theorem 3.1. There exists $\phi_0 \in \Phi$ such that

$$f(\phi_0) = \inf_{\phi \in \Phi} \{c_1\|\phi\|_1 + c_2\|\phi\|_2^2 + c_3\|\phi * w_1\|_\infty + c_4\|\phi\|_\infty\},$$

where $\Phi := \{\phi \in l_1 : A\phi = b\}$.

Therefore, the infimum in (2) is a minimum.

Proof. We denote the feasible set of our problem by

$$\Phi := \{\phi \in l_1 : A\phi = b\}.$$

Let

$$B := \{\phi \in l_1 : c_1\|\phi\|_1 + c_2\|\phi\|_2^2 + c_3\|\phi * w_1\|_\infty + c_4\|\phi\|_\infty \leq v + 1\}.$$

It is clear that

$$v = \inf_{\phi \in \Phi \cap B} \{c_1\|\phi\|_1 + c_2\|\phi\|_2^2 + c_3\|\phi * w_1\|_\infty + c_4\|\phi\|_\infty\}.$$

Therefore, given $i > 0$, there exists $\phi_i \in \Phi \cap B$ such that

$$c_1\|\phi_i\|_1 + c_2\|\phi_i\|_2^2 + c_3\|\phi_i * w_1\|_\infty + c_4\|\phi_i\|_\infty \leq v + 1/i.$$

Let

$$\bar{B} := \{\phi \in l_1 : c_1\|\phi\|_1 \leq v + 1\}.$$

$\bar{B}$ is a bounded set in $l_1 = c_0^*$. It follows from the Banach–Alaoglu lemma that $\bar{B}$ is $W(c_0^*, c_0)$ compact. Using the fact that c_0 is separable and that $\{\phi_i\}$ is a sequence in $\bar{B}$, we know that there exists a subsequence $\{\phi_{i_k}\}$ of $\{\phi_i\}$ and $\phi_0 \in \bar{B}$ such that $\phi_{i_k} \to \phi_0$ in the $W(c_0^*, c_0)$ sense; that is, for all v in c_0,

$$\langle v, \phi_{i_k}\rangle \to \langle v, \phi_0\rangle, \qquad \text{as } k \to \infty. \tag{3}$$

Let the jth row of A be denoted by a_j, and let the jth element of b be given by b_j. Then, as $a_j \in c_0$, we have

$$\langle a_j, \phi_{i_k}\rangle \to \langle a_j, \phi_0\rangle, \qquad \text{as } k \to \infty, \text{ for all } j = 1, 2, \ldots, n. \tag{4}$$

As $A(\phi_{i_k}) = b$, we have

$$\langle a_j, \phi_{i_k}\rangle = b_j, \qquad \text{for all } k \text{ and for all } j,$$

which implies that

$$\langle a_j, \phi_0 \rangle = b_j, \qquad \text{for all } j.$$

Therefore, we have $A(\phi_0) = b$, from which it follows that $\phi_0 \in \Phi$. This gives us

$$c_1\|\phi_0\|_1 + c_2\|\phi_0\|_2^2 + c_3\|\phi_0 * w_1\|_\infty + c_4\|\phi_0\|_\infty \geq v.$$

From (3), we can deduce that, for all t, $\phi_{i_k}(t) \to \phi_0(t)$. An easy consequence of this is that, for all N,

$$\sum_{t=0}^{N} |\phi_{i_k}(t)| \to \sum_{t=0}^{N} |\phi_0(t)|, \qquad \sum_{t=0}^{N} |\phi_{i_k}(t)|^2 \to \sum_{t=0}^{N} |\phi_0(t)|^2, \tag{5}$$

$$\max_{0 \leq t \leq N} \left| \sum_{j=0}^{t} \phi_{i_k}(j) \right| \to \max_{0 \leq t \leq N} \left| \sum_{j=0}^{t} \phi_0(j) \right|, \tag{6a}$$

$$\max_{0 \leq t \leq N} |\phi_{i_k}(t)| \to \max_{0 \leq t \leq N} |\phi_0(t)|. \tag{6b}$$

Now, by (5) and (6), we have that, for all N as $k \to \infty$,

$$\begin{aligned} &\sum_{t=0}^{N} \{c_1|\phi_{i_k}(t)| + c_2(\phi_{i_k}(t))^2\} + c_3 \max_{0 \leq t \leq N} |(\phi_{i_k} * w_1)(t)| + c_4 \max_{0 \leq t \leq N} |\phi_{i_k}(t)| \\ \to &\sum_{t=0}^{N} \{c_1|\phi_0(t)| + c_2(\phi_0(t))^2\} + c_3 \max_{t \leq N} |(\phi_0 * w_1)(t)| + c_4 \max_{0 \leq t \leq N} |\phi_0(t)|. \end{aligned} \tag{7}$$

As

$$c_1\|\phi_{i_k}\|_1 + c_2\|\phi_{i_k}\|_2^2 + c_3\|\phi_{i_k} * w_1\|_\infty + c_4\|\phi_{i_k}\|_\infty \leq v + 1/i_k,$$

we have that, for all N,

$$\begin{aligned} &\sum_{t=0}^{N} \{c_1|\phi_{i_k}(t)| + c_2(\phi_{i_k}(t))^2\} \\ &+ c_3 \max_{t < N} |(\phi_{i_k} * w_1)(t)| + c_4 \max_{0 \leq t \leq N} |\phi_{i_k}(t)| \leq v + 1/i_k. \end{aligned} \tag{8}$$

Letting $k \to \infty$ in (8) and using (7), we have that, for all N,

$$\begin{aligned} &\sum_{t=0}^{N} \{c_1|\phi_0(t)| + c_2(\phi_0(t))^2\} \\ &+ c_3 \max_{t \leq N} |(\phi_0 * w_1)(t)| + c_4 \max_{0 \leq t \leq N} |\phi_0(t)| \leq v. \end{aligned}$$

By letting $N \to \infty$ in the above inequality, we conclude that

$$c_1\|\phi_0\|_1 + c_2\|\phi_0\|_2^2 + c_3\|\phi_0 * w_1\|_\infty + c_4\|\phi_0\|_\infty \leq v.$$

This proves the theorem. $\square$

3.2. Structure of Optimal Solutions. In this section, we use a Lagrange duality result to show that every optimal solution is of finite length. First, we give the following definitions, where we denote the interior of a set by int.

Definition 3.1. Let P be a convex cone in a vector space X. We write

$$x \geq y, \qquad \text{if } x - y \in P.$$

We write

$$x > 0, \qquad \text{if } x \in \text{int}(P).$$

Similarly,

$$x \leq y, \qquad \text{if } x - y \in -P := N,$$

and

$$x < 0, \qquad \text{if } x \in \text{int}(N).$$

Definition 3.2. Let X be a vector space, and let Z be a vector space with positive cone P. A mapping $G: X \to Z$ is convex if

$$G(tx + (1-t)y) \leq tG(x) + (1-t)G(y),$$

for all $x \neq y$ in X and t with $0 \leq t \leq 1$.

It is strictly convex if

$$G(tx + (1-t)y) < tG(x) + (1-t)G(y),$$

for all $x \neq y$ in X and t with $0 < t < 1$.

The following is a Lagrange duality theorem.

Theorem 3.2. See Ref. 15. Let X be a Banach space, let Ω be a convex subset of X, let Y be a finite-dimensional space, let Z be a normed space with positive cone P. Let $f: \Omega \to R$ be a real-valued convex functional, let $g: X \to Z$ be a convex mapping, let $H: X \to Y$ be an affine linear map, and let $0 \in \text{int}[\text{range}(H)]$. Define

$$\mu_0 := \inf\{f(x): g(x) \leq 0,\ H(x) = 0,\ x \in \Omega\}. \tag{9}$$

Suppose that there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$, and suppose that μ_0 is finite. Then,

$$\mu_0 = \max\{\varphi(z^*, y): z^* \geq 0,\ z^* \in Z^*,\ y \in Y\}, \tag{10}$$

where

$$\varphi(z^*, y) := \inf\{f(x) + \langle g(x), z^*\rangle + \langle H(x), y\rangle : x \in \Omega\},$$

and the maximum is achieved for some $z_0^* \geq 0$, $z_0^* \in Z^*$, $y_0 \in Y$. Furthermore, if the infimum in (9) is achieved by some $x_0 \in \Omega$, then

$$\langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle = 0, \tag{11}$$

$$x_0 \text{ minimizes } f(x) + \langle g(x), z_0^*\rangle + \langle H(x), y_0\rangle, \text{ over all } x \in \Omega. \tag{12}$$

We refer to (9) as the primal problem and to (10) as the dual problem. Based on the above result, the following can be established in relation to our problem at hand.

Lemma 3.2. The following result holds:

$$v = \max_{y \in R^n} \inf_{\phi \in l_1} \{f(\phi) + \langle b - A\phi, y\rangle\}. \tag{13}$$

Proof. We will apply Theorem 3.2 to get the result. Let X, Ω, Y, Z in Theorem 3.2 correspond to l_1, l_1, R^n, R respectively. Let

$$\gamma := v + 1, \qquad g(\phi) := f(\phi) - \gamma, \qquad H(\phi) := b - A\phi.$$

With this notation, we have $Z^* = R$. A has full range, which implies $0 \in \text{int}[\text{range}(H)]$. From Theorem 3.1, we know that there exists ϕ_0 such that

$$g(\phi_0) = f(\phi_0) - \gamma = -1 < 0 \quad \text{and} \quad H(\phi_0) = 0.$$

Therefore, all the conditions of Theorem 3.2 are satisfied. From Theorem 3.2, we have

$$v = \max_{z \geq 0, y \in R^n} \inf_{\phi \in l_1} \{f(\phi) + \langle g(\phi), z\rangle + \langle b - A\phi, y\rangle\}.$$

Let $z_0 \in R$, $y_0 \in R^n$ be a maximizing solution to the right-hand side of the above equation. ϕ_0 being the solution of the primal, we have from (11) that

$$\langle g(\phi_0), z_0\rangle + \langle H(\phi_0), y_0\rangle = 0,$$

which implies that

$$\langle g(\phi_0), z_0\rangle = 0.$$

As $g(\phi_0) \neq 0$, we conclude that $z_0 = 0$. This proves the lemma. □

The following theorem shows that the solution to (2) is unique and that it is a finite impulse response sequence.

Theorem 3.3. Define

$$\mathscr{T} := \{\phi \in l_1 : \text{there exists } L^* \text{ with } \phi(i) = 0 \text{ if } i \geq L^*\}.$$

The following result is true:

$$v = \max_{y \in R^n} \inf_{\phi \in \mathscr{T}} \{f(\phi) - \langle \phi, v \rangle + \langle b, y \rangle\}, \tag{14}$$

where $v(i) := (A^* y)(i)$. Also, the solution to the primal (2) is unique, and the solution belongs to $\mathscr{T}$.

Proof. Let $y_0 \in R^n$ be the solution to the right-hand side of (14). Define

$$v_0 := A^* y_0,$$

and let

$$J(\phi) := f(\phi) - \langle \phi, v_0 \rangle + \langle b, y_0 \rangle.$$

It is immediate that

$$v = \inf_{\phi \in l_1} \sum_{i=0}^{\infty} \{c_1 |\phi(i)| + c_2 (\phi(i))^2 - \phi(i) v_0(i)\}$$
$$+ c_3 \sup_i |(\phi * w_1)(i)| + c_4 \sup_i |\phi(i)| + \langle b, y_0 \rangle.$$

As v_0 is in l_1, we know that there exists L^* such that $v_0(i)$ satisfies

$$|v_0(i)| < c_1, \qquad \text{if } i \geq L^*.$$

We now show that, for ϕ to be optimal, it is necessary that

$$\phi(i) = 0, \qquad \text{for all } i \geq L^*.$$

Indeed, if $\phi(i) \neq 0$, while $|v_0(i)| < c_1$ for some i, note that

$$c_1 |\phi(i)| + c_2 (\phi(i))^2 - \phi(i) v_0(i) > c_1 |\phi_1(i)| + c_2 (\phi_1(i))^2 - \phi_1(i) v_0(i),$$

for any $\phi_1 \in l_1$ such that $\phi_1(i) = 0$. Moreover, if ϕ_1 is such that

$$\phi_1(j) = \phi(j), \qquad \text{whenever } j < L^*,$$

$$\phi_1(j) = 0, \qquad \text{whenever } j \geq L^*,$$

it follows that

$$\sup_i \left| \sum_{j=0}^{i} \phi(j) \right| \geq \sup_i \left| \sum_{j=0}^{i} \phi_1(j) \right|, \qquad \sup_i |\phi(i)| \geq \sup_i |\phi_1(i)|,$$

or equivalently,

$$\|\phi * w_1\|_\infty \geq \|\phi_1 * w_1\|_\infty, \qquad \|\phi\|_\infty \geq \|\phi_1\|_\infty.$$

Hence, we have that $J(\phi) > J(\phi_1)$, which proves our claim. In Theorem 3.1, we showed that there exists a solution ϕ_0 to the primal (2). From Theorem 3.2, we know that ϕ_0 is a solution to $\inf_{\phi \in l_1} J(\phi)$. As $J(\phi)$ is strictly convex in ϕ, we conclude that the solution to the primal (2) is unique. From the previous discussion, it follows that the solution, ϕ_0 to the primal (2) is in $\mathscr{T}$. □

3.3. A Priori Bound on the Length of the Optimal Solution. In this section, we give an a priori bound on the length of the solution to (2). First, we establish the following two lemmas.

Lemma 3.3. Let ϕ_0 be a solution of the primal (2). Let y_0 represent a corresponding dual solution as obtained in (14). Let $v_0 := A^* y_0$. Then,

$$\|v_0\|_\infty \leq \alpha, \qquad \text{where } \alpha = c_1 + c_3 + c_4 + 2(c_2/c_1) f(h).$$

Proof. From the proof of Theorem 3.2, it is clear that ϕ_0 should be such that it minimizes

$$J(\phi) := \sum_{i=0}^{L^*} \{c_1|\phi(i)| + c_2(\phi(i))^2 - \phi(i) v_0(i)\}$$
$$+ c_3 \max_{i \leq L^*} \left|\sum_{j=0}^{i} \phi(j)\right| + c_4 \max_{i \leq L^*} |\phi(i)|,$$

where L^* is such that

$$|v_0(i)| < c_1, \qquad \text{if } i \geq L^*.$$

Let i be any integer such that $i \leq L^*$. Consider the perturbation ϕ of ϕ_0 given as

$$\phi(i) = \phi_0(i) + \epsilon, \text{ and } \phi(j) = \phi_0(j), \qquad \text{for } j \neq i.$$

Then, for all ϵ, it can be shown that

$$\max_{0 \leq t \leq L^*} \left|\sum_{j=0}^{t} \phi(j)\right| - \max_{0 \leq t \leq L^*} \left|\sum_{j=0}^{t} \phi_0(j)\right| \leq |\epsilon|, \tag{15}$$

$$\max_{0 \leq t \leq L^*} |\phi(t)| - \max_{0 \leq t \leq L^*} |\phi_0(t)| \leq |\epsilon|. \tag{16}$$

Indeed, assume that

$$\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi_0(j)\right|=\left|\sum_{j=0}^{N}\phi_0(j)\right|,\qquad \text{for some } N\le L^*.$$

For the given ϵ, let

$$\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi(j)\right|=\left|\sum_{j=0}^{M}\phi(j)\right|,\qquad \text{for some } M\le L^*. \tag{17}$$

If $M\le i$, then

$$\begin{aligned}&\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi(j)\right|-\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi_0(j)\right|\\&=|\phi_0(0)+\cdots+\phi_0(M)|-|\phi_0(0)+\cdots+\phi_0(N)|\\&\le 0\le|\epsilon|;\end{aligned}$$

and if $M>i$, then

$$\begin{aligned}&\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi(j)\right|-\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi_0(j)\right|\\&=|\phi_0(0)+\cdots+(\phi_0(i)+\epsilon)+\cdots+\phi_0(M)|-|\phi_0(0)+\cdots+\phi_0(N)|\\&\le|\phi_0(0)+\cdots+\phi_0(M)|+|\epsilon|-|\phi_0(0)+\cdots+\phi_0(N)|\\&\le|\epsilon|.\end{aligned}$$

Inequality (16) can be proved easily. It follows easily from (15) and (16) that

$$\begin{aligned}J(\phi)-J(\phi_0)&=c_1(|\phi_0(i)+\epsilon|-|\phi_0(i)|)+c_2(\epsilon^2+2\epsilon\phi_0(i))\\&\quad+c_3\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi(j)\right|-c_3\max_{0\le t\le L^*}\left|\sum_{j=0}^{t}\phi_0(j)\right|\\&\quad+c_4\left(\max_{0\le t\le L^*}|\phi(t)|-\max_{0\le t\le L^*}|\phi_0(t)|\right)-\epsilon v_0(i)\\&\le c_1|\epsilon|+c_2(\epsilon^2+2\epsilon\phi_0(i))+c_3|\epsilon|+c_4|\epsilon|-\epsilon v_0(i).\end{aligned}$$

As ϕ_0 is the unique minimum, we have that

$$J(\phi)-J(\phi_0)>0;$$

therefore, it follows that

$$c_1|\epsilon|+c_2(\epsilon^2+2\epsilon\phi_0(i))+c_3|\epsilon|+c_4|\epsilon|-\epsilon v_0(i)>0,\qquad \text{for all } \epsilon.$$

Dividing both sides of the above inequality by $|\epsilon|$ we get

$$c_1+c_3+c_4+c_2|\epsilon|+2c_2(\epsilon/|\epsilon|)\phi_0(i)-(\epsilon/|\epsilon|)v_0(i)>0, \qquad \text{for all } \epsilon.$$

Letting $\epsilon\to 0^+$ and $\epsilon\to 0^-$ in the above inequality, we have

$$\begin{aligned} v_0(i)&\le c_1+c_3+c_4+2c_2|\phi_0(i)|,\\ -v_0(i)&\le c_1+c_3+c_4-2c_2|\phi_0(i)|\\ &\le c_1+c_3+c_4+2c_2|\phi_0(i)|, \end{aligned}$$

respectively. This implies that

$$\begin{aligned} |v_0(i)|&\le c_1+c_3+c_4+2c_2|\phi_0(i)|\\ &\le c_1+c_3+c_4+2c_2\|\phi_0\|_1. \end{aligned}$$

As this holds for any $i\le L^*$, we have

$$\begin{aligned} \|v_0\|_\infty&\le c_1+c_3+c_4+2(c_2/c_1)f(\phi_0)\\ &\le c_1+c_3+c_4+2(c_2/c_1)f(h), \end{aligned}$$

where we have used

$$\begin{aligned} 2c_2\|\phi_0\|_1&\le 2(c_2/c_1)f(\phi_0)\\ &\le c_1+c_3+c_4+2(c_2/c_1)f(h),\\ f(\phi_0)&\le f(h), \end{aligned}$$

since h is feasible ($q=0$), but not necessarily optimal. This proves the lemma. □

Lemma 3.4. See Ref. 4. If $y\in R^n$ is such that $\|A^*y\|_\infty\le\alpha$, then there exists a positive integer L^* independent of y such that

$$|(A^*y)(i)|<c_1, \qquad \text{for all } i\ge L^*.$$

Proof. Define

$$A_L^*=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1\\ z_1 & z_2 & z_3 & \cdots & z_n\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ z_1^L & z_2^L & z_3^L & \cdots & z_n^L \end{bmatrix},$$

$A_L^*: R^n\to R^{L+1}$. With this definition, we have $A_\infty^*=A^*$. Let $y\in R^n$ be such that $\|A^*y\|_\infty\le\alpha$. Choose any L such that $L\ge n-1$. As z_i, $i=1,\ldots,n$, are distinct, A_L^* has full column rank. A_L^* can be regarded as a linear map taking $(R^n, \|\cdot\|_1)\to(R^{L+1}, \|\cdot\|_\infty)$. As A_L^* has full column rank, we can define $(A_L^*)^{-l}$, the left inverse of A_L^*, which takes $(R^{L+1}, \|\cdot\|_\infty)\to(R^n, \|\cdot\|_1)$. Let the induced norm of $(A_L^*)^{-l}$ be given by $\|(A_L^*)^{-l}\|_{\infty,1}$. $y\in R^n$ is such that $\|A^*y\|_\infty\leq\alpha$; therefore, $\|A_L^*y\|_\infty\leq\alpha$. It follows that

$$\|y\|_1\leq\|(A_L^*)^{-l}\|_{\infty,1}\|A_L^*y\|_\infty\leq\|(A_L^*)^{-l}\|_{\infty,1}\alpha. \tag{18}$$

Choose L^* such that

$$\max_{k=1,\ldots,n}|z_k|^{L^*}\|(A_L^*)^{-l}\|_{\infty,1}\alpha<c_1. \tag{19}$$

There always exists such an L^*, because

$$|z_k|<1, \qquad \text{for all } k=1,\ldots,n.$$

Note that L^* does not depend on y. For any $i\geq L^*$, we have

$$\begin{aligned}|(A^*y)(i)|&=\left|\sum_{k=1}^{k=n}z_k^iy(k)\right|\leq\max_{k=1,\ldots,n}|z_k|^i\|y\|_1\\&\leq\max_{k=1,\ldots,n}|z_k|^i\|(A_L^*)^{-l}\|_{\infty,1}\alpha\\&\leq\max_{k=1,\ldots,n}|z_k|^{L^*}\|(A_L^*)^{-l}\|_{\infty,1}\alpha.\end{aligned}$$

The second inequality follows from (18). From (19), we have

$$|(A^*y)(i)|<c_1, \qquad \text{if } i\geq L^*.$$

This proves the lemma. □

We now summarize the main result of the section

Theorem 3.4. The unique solution ϕ_0 of the primal (2) is such that $\phi(i)=0$, if $i\geq L^*$, where L^* given in Lemma 3.3 can be determined a priori.

Proof. Let y_0 be the dual solution to (2), and let

$$v_0:=A^*y_0.$$

From Lemma 3.3, we know that

$$\|v_0\|_\infty\leq\alpha, \qquad \text{where } \alpha=c_1+c_3+c_4+2(c_2/c_1)f(h).$$

Applying Lemma 3.3, we conclude that there exists L^*, which can be determined a priori, such that

$$|v_0(i)|<c_1, \qquad \text{if } i\geq L^*.$$

Therefore,

$$\phi_0(i) = 0, \qquad \text{if } i \geq L^*.$$

This proves the theorem. □

The above theorem shows that the Problem (2) is a finite-dimensional convex minimization problem. Such problems can be solved efficiently using standard numerical methods.

At this point, we would like to make a few remarks. It should be clear that the uniqueness property of the optimal solution is due to the nonzero coefficient c_2. This makes the problem strictly convex. The finite impulse response property of the optimal solution is due to the nonzero c_1. Also, it should be noted that, in the case where c_3 and/or c_4 are allowed to be zero, all of the previous results apply by setting respectively c_3 and/or c_4 to zero in the appropriate expressions for the upper bounds.

4. Example

In this section, we illustrate the theory developed in the previous sections with an example taken from Ref. 11. Consider the SISO plant

$$\hat{P}(\lambda) = \lambda - 1/2,$$

where we are interested in the sensitivity map

$$\phi := (I - PK)^{-1}.$$

Using Youla parametrization, we get that all achievable transfer functions are given by

$$\hat{\phi} = (I - \hat{P}\hat{K})^{-1} = 1 - (\lambda - 1/2)\hat{q},$$

where $\hat{q}$ is a stable map. Therefore,

$$h = 1 \quad \text{and} \quad u = \lambda - 1/2.$$

The matrix A and b are given by

$$A = (1, 1/2, 1/2^2, \ldots), \qquad b = 1.$$

We consider the case where

$$c_1 = 1, \qquad c_2 = 1, \qquad c_3 = 1, \qquad c_4 = 1.$$

Therefore,

$$\begin{aligned}\alpha &= c_1 + c_3 + c_4 + 2(c_2/c_1)(c_1\|h\|_1 + c_2\|h\|_2^2 + c_3\|h * w_1\|_\infty + c_4\|\phi\|_\infty)\\ &= 11.\end{aligned}$$

For this example,

$$n=1 \quad \text{and} \quad z_1=1/2.$$

L^*, the a priori bound on the length of the optimal, is chosen to satisfy

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \|(A_L^*)^{-l}\|_{\infty,1}\alpha < c_1, \tag{20}$$

where L is any positive integer such that $L \geq n-1$. We choose $L=0$; therefore,

$$A_L=1 \quad \text{and} \quad \|(A_L^*)^{-l}\|_{\infty,1}=1.$$

We choose $L^*=4$, which satisfies (20). Therefore, the optimal solution ϕ_0 satisfies

$$\phi_0(i)=0, \qquad \text{if } i \geq 4.$$

The problem reduces to the following finite-dimensional convex optimization problem:

$$v=\min_{A_{L^*}\phi=1} \left\{ \sum_{k=0}^{3} (|\phi(k)|+(\phi(k))^2) \right.$$

$$\left. + \max_{0\leq k\leq 3} |(\phi * w_1)(k)| + \max_{0\leq k\leq 3} |\phi(k)| : \phi \in R^4 \right\},$$

where

$$A_{L^*}=(1, 1/2, 1/4, 1/8).$$

We obtain (using the Matlab Optimization Toolbox) the optimal solution ϕ_0 to be

$$\hat{\phi}_0(\lambda)=0.9+0.2\lambda.$$

5. Continuity of the Optimal Solution

In this section, we show that the optimal is continuous with respect to changes in the parameters c_1, c_2, c_3, c_4. First, we prove the following lemma.

Lemma 5.1. Let $\{f_k\}$ be a sequence of functions which maps R^m to R. If f_k converges uniformly to a function f on a set $S \subset R^m$, then

$$\lim_{k\to\infty} \min_{x\in S} f_k(x) = \min_{x \in S} f(x),$$

provided the minima exist.

Proof. Let

$$\min_{x \in S} f(x) = f(x_0), \qquad \text{for some } x_0 \in S.$$

Given $\epsilon > 0$, we know from the convergence of the sequence $\{f_k\}$ to f that there exists an integer K such that, if $k > K$, then

$$|f_k(x_0) - f(x_0)| < \epsilon$$

$$\Rightarrow f_k(x_0) < \epsilon + f(x_0)$$

$$\Rightarrow \min_{x \in S} f_k(x) < \epsilon + f(x_0)$$

$$\Rightarrow \lim_{k \to \infty} \min_{x \in S} f_k(x) < \epsilon + f(x_0).$$

As ϵ is arbitrary, we have

$$\lim_{k \to \infty} \min_{x \in S} f_k(x) \leq f(x_0).$$

Now, we prove the other inequality. Given $\epsilon > 0$, we know that there exists an integer K such that, if $k > K$, then

$$|f_k(x) - f(x)| < \epsilon, \qquad \text{for any } x \in S,$$

$$\Rightarrow f_k(x) > f(x) - \epsilon \geq f(x_0) - \epsilon, \qquad \text{for any } x \in S,$$

$$\Rightarrow \min_{x \in S} f_k(x) > f(x_0) - \epsilon$$

$$\Rightarrow \lim_{k \to \infty} \min_{x \in S} f_k(x) > f(x_0) - \epsilon.$$

As ϵ is arbitrary, we have

$$\lim_{k \to \infty} \min_{x \in S} f_k(x) \geq f(x_0).$$

This proves the lemma. □

Theorem 5.1. Let

$$c_1^k \in [a_1, b_1], \qquad c_2^k \in [a_2, b_2], \qquad c_3^k \in [a_3, b_3], \qquad c_4^k \in [a_4, b_4],$$

where

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad a_4 > 0.$$

Let ϕ_k be the unique solution to the problem

$$v_k := \min_{A\phi = b} c_1^k \|\phi\|_1 + c_2^k \|\phi\|_2^2 + c_3^k \|\phi * w_1\|_\infty + c_4^k \|\phi\|_\infty, \tag{21}$$

and let ϕ_0 be the solution to the problem

$$v := \min_{A\phi=b} c_1\|\phi\|_1 + c_2\|\phi\|_2^2 + c_3\|\phi * w_1\|_\infty + c_4\|\phi\|_\infty, \tag{22}$$

If $c_1^k \to c_1$, $c_2^k \to c_2$, $c_3^k \to c_3$, $c_4^k \to c_4$, then $\phi_k \to \phi_0$.

Proof. We prove this theorem in three parts. First, we show that we can restrict the proof to a finite-dimensional space; second, we show that $v_k \to v$; finally, we show that $\phi_k \to \phi_0$. Let y_k represent the dual solution of (21), and let $\upsilon_k := A^* y_k$. Let

$$f_k(\phi) := c_1^k\|\phi\|_1 + c_2^k\|\phi\|_2^2 + c_3^k\|\phi_k * w_1\|_\infty + c_4^k\|\phi\|_\infty,$$

$$f(\phi) := c_1\|\phi\|_1 + c_2\|\phi\|_2^2 + c_3\|\phi * w_1\|_\infty + c_4\|\phi\|_\infty.$$

Let α_k, the upper bound on $\|\upsilon_k\|_\infty$, be as given by Lemma 3.3. Therefore,

$$\begin{aligned}\alpha_k &= c_1^k + c_3^k + c_4^k + 2(c_2^k/c_1^k) f_k(h)\\ &\le b_1 + b_3 + b_4\\ &\quad + 2b_2(\|h\|_1 + (b_2/a_1)\|h\|_2^2\\ &\qquad + (b_3/a_1)\|h * w_1\|_\infty + (b_4/a_1)\|h\|_\infty).\end{aligned}$$

Let this bound be denoted by d. Choose L^* such that

$$\max_{i=1,\dots,n} |z_i|^{L^*} \|(A_L^*)^{-l}\|_{\infty,1} d < a_1,$$

where L is such that $L \ge n-1$. Therefore, it follows that

$$\max_{i=1,\dots,n} |z_i|^{L^*} \|(A_L^*)^{-l}\|_{\infty,1} \alpha_k < c_1^k, \qquad \text{for all } k.$$

From arguments similar to those of Lemma 3.3 and Theorem 3.3, it follows that

$$\phi_k(i) = 0, \qquad \text{if } i \ge L^*, \text{ for all } k.$$

Therefore, we can assume that

$$\phi_k \in R^{L^*}.$$

Now, we prove that $v_k \to v$. Let ϕ_1 be the solution of the problem

$$v_1 := \min_{A\phi=b} b_1\|\phi\|_1 + b_2\|\phi\|_2^2 + b_3\|\phi * w_1\|_\infty + b_4\|\phi\|_\infty.$$

As

$$c_1^k \ge b_1, \qquad c_2^k \le b_2, \qquad c_3^k \le b_3, \qquad c_4^k \le b_4,$$

we have that

$$v_k \le v_1, \qquad \text{for all } k.$$

Therefore, for any k, we have

$$c_1^k \|\phi_k\|_1 + c_2^k \|\phi_k\|_2^2 + c_3^k \|\phi_k * w_1\|_\infty + c_4^k \|\phi_k\|_\infty \le v_1,$$

which implies that

$$\|\phi_k\|_1 \le v_1/c_1^k \le v_1/a_1,$$

$$\|\phi_k\|_2^2 \le v_1/c_2^k \le v_1/a_2,$$

$$\|\phi_k * w_1\|_\infty \le v_1/c_3^k \le v_1/a_3,$$

$$\|\phi_k\|_\infty \le v_1/a_4.$$

Let

$$S := \{\phi \in R^{L^*} : A\phi = b,\ \|\phi\|_1 \le v_1/a_1, \|\phi\|_2^2 \le v_1/a_2,$$
$$\|\phi * w_1\|_\infty \le v_1/a_3,\ \|\phi\|_\infty \le v_1/a_4\}.$$

Then, it is clear that

$$v_k := \min_{\phi \in S} c_1^k \|\phi\|_1 + c_2^k \|\phi\|_2^2 + c_3^k \|\phi * w_1\|_\infty + c_4^k \|\phi\|_\infty .$$

We now prove that f_k converges to f uniformly on S. Given $\epsilon > 0$, we choose K such that, if $k > K$, then

$$|c_1^k - c_1| < \epsilon a_1/4v_1,$$

$$|c_2^k - c_2| < \epsilon a_2/4v_c,$$

$$|c_3^k - c_3| < \epsilon a_3/4v_1,$$

$$|c_4^k - c_4| < \epsilon a_4/4v_1.$$

Then, for any $\phi \in S$, we have

$$|f_k(\phi) - f(\phi)| = |(c_1^k - c_1)\|\phi\|_1 + (c_2^k - c_2)\|\phi\|_2^2$$
$$+ (c_3^k - c_3)\|\phi * w_1\|_\infty + (c_4^k - c_4)\|\phi\|_\infty|,$$

and thus,

$$|f_k(\phi) - f(\phi)| \le |c_1^k - c_1|(v_1/a_1) + |c_2^k - c_2|(v_1/a_2)$$
$$+ |c_3^k - c_3|(v_1/a_3) + |c_4^k - c_4|(v_1/a_4) < \epsilon.$$

Therefore, it follows that f_k converges uniformly to f on S. From Lemma 5.1, it follows that $v_k \to v$.

We now prove that $\phi_k \to \phi_0$. Let

$$B := \{\phi \in R^{L^*}: \|\phi\|_1 \leq v_1/a_1\}.$$

Then, we know that $\phi_k \in B$ which is compact in $(R^{L^*}, \|\cdot\|_1)$. Therefore, there exists a subsequence ϕ_{k_i} of ϕ_k and $\bar{\phi} \in R^{L^*}$ such that $\phi_{k_i} \to \bar{\phi}$. As

$$c_1^k \to c_1, \qquad c_2^k \to c_2, \qquad c_3^k \to c_3, \qquad c_4^k \to c_4,$$

and $\phi_{k_i} \to \bar{\phi}$, we have that $f_{k_i}(\phi_{k_i}) \to f(\bar{\phi})$. As v_k converges to v, it follows that $f_{k_i}(\phi_{k_i}) \to f(\phi_0)$ [note that $v_{k_i} = f_{k_i}(\phi_{k_i})$ and $v = f(\phi_0)$]; therefore, $f(\bar{\phi}) = f(\phi_0)$. As $A\phi_{k_i} = b$, for all i, we have that $A\bar{\phi} = b$. From uniqueness of the solution of (22), it follows that $\bar{\phi} = \phi_0$. Therefore, we have established that $\phi_{k_i} \to \phi_0$. From uniqueness of the solution of (22), it also follows that $\phi_k \to \phi_0$. This proves the theorem. □

6. Conclusions

In this paper, we considered the mixed objective problem of minimizing a given linear combination of the l_1-norm, the square of the $\mathcal{H}_2$-norm, and the l_∞-norms of the step and pulse responses respectively of the closed loop. Employing a variant of the Kuhn–Tucker–Lagrange duality theorem, it was shown that this problem is equivalent to a finite-dimensional convex optimization problem with an a priori known dimension. The solution is unique and represents a Pareto-optimal point with respect to the individual measures involved. It was also shown that the optimal solution is continuous with respect to changes in the coefficients of the composite measure.

The duality theorem employed here can be used in other mixed objective problems. Also, the generalization of this theory to the multiple-input multiple-output case is possible. These topics are the subjects of future research.

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