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# Integrated Parameter and Control Design *

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**Abstract** In this paper, the integrated parameter and control (IPC) problem is considered where the system parameters are assumed to enter into the state-space realization in a polynomial manner. Converging finite-dimensional sub-optimal problems are constructed and solved via a linear relaxation technique, whereby a global optimal solution to the IPC problem can be computed to any prescribed tolerance.

**Keywords** $\ell_1$, $\mathcal{H}_2$, robust optimal control, linear/quadratic program, Integrated Structure and Control (ISC) Design

## 1 Introduction

It is a well-known fact that system structure design and feedback control synthesis are not isolated processes ([1]). The two are naturally iterative in a sense that good modelling should take into consideration the knowledge of the control input generated by the controller, and a good control design should (ideally) yield directions on how to modify the model to achieve the best possible performance. Moreover, on designing today's engineering systems, more and more demanding performance requirements, often diverse or even conflicting in nature, have rendered it impossible to meet these criteria without redesigning the plant itself. Thus it is well-motivated to develop a systematic approach to conduct system and control design simultaneously.

Research efforts towards this direction have yielded algorithms to synthesize a stabilizing controller and to select corresponding system structure parameters that affinely enter into the system dynamics. Specifically, the performance objectives include output RMS([2]), $\mathcal{H}_\infty$([3]), and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance([4]). The design approaches proposed are usually carried out in an iterative way. The control design and plant design are repeated one after another until a certain prescribed tolerance is achieved. Due to the non-convex nature of the problem formulations tackled in these approaches, the above-mentioned iterative algorithms usually yield a sequence of non-increasing upper bounds and does not guarantee the convergence to the optimal solution.

In this paper, the simultaneous system and control design problem is considered for the case where plant parameters enter into the system in a general polynomial manner. Via Youla parameterization, finite-dimensional sub-problems are constructed to provide converging upper and lower bounds. After certain algebraic operations, these non-convex sub-problems are equivalently transformed into a much more manageable nonlinear programming formulations. Whence a linear relaxation scheme is combined with an effective branch and bound algorithm to obtain the solution. The outline of the paper is as follows. In Section 2, we introduce the notations to be used. In Section 3, we formulate the problem setup and the converging sub-problems. In Section 4, we transform the nonlinear sub-problems into a more manageable expression. In Section 5, we show that the solutions to these sub-problems can be effectively computed by solving a relaxed linear programming problem combined with a branch and bound algorithm. In Section 6, we summarize the paper.

## 2 Nomenclature

| | |
|---|---|
| $c_0$ | The subspace of sequences of real numbers whose element $x$ satisfies $\lim_{k\to\infty} x(k) = 0$. |
| $c_0^{m\times n}$ | The space of matrix-valued right-sided real sequences such that each element $x \in c_0^{m\times n}$ is the matrix $(x_{ij})$ and each $x_{ij}$ is in $c_0$. |
| $\ell_1$ | The Banach space of right sided absolutely summable real sequences with the norm given by $\|x\|_1 := \sum_{k=0}^{\infty} |x(k)|$. |
| $\ell_1^{m\times n}$ | The Banach space of matrix-valued right-sided real sequences with the norm given by $\|x\|_1 := \max_{1\le i\le m} \sum_{j=1}^{n} \|x_{ij}\|_1$, where $x \in \ell_1^{m\times n}$ is the matrix $(x_{ij})$ and each $x_{ij}$ is in $\ell_1$. |
| $\ell_2$ | The Banach space of right-sided square summable real sequences with the norm given by $\|x\|_2 := \left(\sum_{k=0}^{\infty} x^2(k)\right)^{1/2}$. |

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| | |
|---|---|
| $\hat{x}(\lambda)$ | The $\lambda$ transform of a right-sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{\infty} x(k)\lambda^k$. |
| $\mathcal{H}_2$ | The isometrically isomorphic image of $\ell_2$ under the $\lambda$ transform with the norm given by $\|\hat{x}(\lambda)\|_{\mathcal{H}_2} = \|x\|_2$. |
| $\mathcal{R}$ | The real number system. |
| $\mathcal{R}^n$ | The $n$ dimensional Euclidean space. |
| $P_n$ | The truncation operator on the space of sequences defined as $P_n(x(0)\ x(1)\ \ldots) = (x(0)\ x(1)\ \ldots\ x(n)\ 0\ 0\ \ldots)$. |
| $X^*$ | The dual space of a normed linear vector space $X$. For $x^* \in X^*$, $< x, x^* >$ denotes the value of the bounded linear functional $x^*$ at $x \in X$([6]). For example, it can be verified that $(c_0^{m\times n})^* = \ell_1^{m\times n}$ and for $x \in c_0^{m\times n}$ and $x^* \in \ell_1^{m\times n}$, $< x, x^* > = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{\infty} x_{ij}(k)x_{ij}{}^*(k)$. |

# 3 Problem Formulation

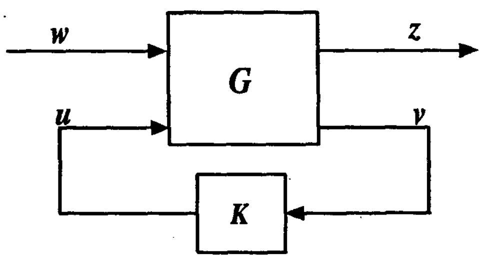


Figure 1: Closed-loop system

Consider the setup in Figure 1, where $G : [w; u] \to [z; v]$ is the generalized discrete-time linear time-invariant plant, $K$ is the controller. $w$, $z$, $u$, and $v$ are the exogenous input, regulated output, control input, and measured output of dimensions $n_w$, $n_z$, $n_u$, and $n_v$, respectively.

Suppose $G$ has the following realization:

$$G(\rho) := \left[\begin{array}{c|cc} A(\rho) & B_1(\rho) & B_2(\rho) \\ \hline C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{array}\right].$$

where $\rho = [\rho_1 \cdots \rho_m]^T \in R^m$, and each entry $g(\rho)$ of $G(\rho)$ is a $p$-degree polynomial of the form:

$$g(\rho) = \sum f_\theta \rho^\theta, \ \rho^\theta = \prod_{j=1}^{m} \rho_j^{\theta_j}$$

$$0 \le \theta_j \le p, \ \textstyle\sum_{j=1}^{m} \theta_j = \phi \in \{0, 1, 2, \ldots, p\}.$$

where $f_\theta$ is the coefficient of the $\phi$-degree monomial $\rho^\theta$.

Given two $m$-dimensional real vectors $\underline{\rho} = [\underline{\rho}_1 \cdots \underline{\rho}_m]^T$ and $\overline{\rho} = [\overline{\rho}_1 \cdots \overline{\rho}_m]^T$. In the sequel, we use the notation $\underline{\rho} \le \rho \le \overline{\rho}$ or $\rho \in [\underline{\rho}, \overline{\rho}]$ to denote the set of inequality relations $\{\underline{\rho}_i \le \rho_i \le \overline{\rho}_i, i = 1, \ldots, m\}$. Assume that $(A(\rho), B_2(\rho))$ is stabilizable and $(A(\rho), C_2(\rho))$ is detectable for any $\rho \in [\underline{\rho}, \overline{\rho}]$. The problem to be solved is formulated as:

$$\mu := \inf\{\|\hat{R}(\hat{K}, \rho)\|_1 : \hat{K} \text{ stabilizing}, \rho \in [\underline{\rho}, \overline{\rho}]\} \quad (1)$$

where $\hat{R}$ denotes the closed-loop transfer matrix from $w$ to $z$. From now on, we assume that the feasible set of problem (1) is non-empty, which includes the requirement that the optimal cost $\mu$ be finite.

Via Youla parametrization([7]), problem (1) is equivalently transformed into the following form:

$$\begin{aligned} \mu \ := \ & \inf_{Q,\rho} \ \|R(Q,\rho)\|_1 \\ & \text{subject to} \ R(Q,\rho) = H(\rho) - U(\rho) * Q * V(\rho) \\ & \qquad \underline{\rho} \le \rho \le \overline{\rho} \end{aligned}$$

where $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$, $V \in \ell_1^{n_v \times n_w}$, $Q$ is a free parameter in $\ell_1^{n_u \times n_v}$, and '$*$' denotes the convolution operation. In the sequel, without loss of generality, we shall assume that $H$, $U$, and $V$ are finitely supported.

Introducing an extra $\ell_1$ norm bound on $Q$([8]), we obtain the following auxiliary problem of $\mu$:

$$\begin{aligned} \nu \ := \ & \inf_{Q,\rho} \ \|R(Q,\rho)\|_1 \\ & \text{subject to} \ \|Q\|_1 \le \alpha \\ & \qquad R(Q,\rho) = H(\rho) - U(\rho) * Q * V(\rho) \\ & \qquad \underline{\rho} \le \rho \le \overline{\rho}. \end{aligned}$$

It is clear that $\mu$ and $\nu$ are closely related. If problem $\mu$ has an optimal solution, say, $Q_o$, then $\mu = \nu$ for any $\alpha \ge \|Q_o\|_1$. If $\mu$ doesn't bear an optimal solution, then the constraint $\|Q\|_1 \le \alpha$ plays the role of a regularizing condition such that $\nu$ always has an optimal solution with a reasonable bounded gain. Thus in what follows, we shall solely focus on problem $\nu$. Two sequences of lower and upper bounds of $\nu$ are then given by:

$$\begin{aligned}\nu_n \;:=\; & \inf_{Q,\rho} && \|P_n R(Q,\rho)\|_1 \\ & \text{subject to} && \|Q\|_1 \le \alpha \\ &&& R(Q,\rho) = H(\rho) - U(\rho) * Q * V(\rho) \\ &&& \underline{\rho} \le \rho \le \overline{\rho}. \\ \nu^n \;:=\; & \inf_{Q,\rho} && \|R(Q,\rho)\|_1 \\ & \text{subject to} && \|Q\|_1 \le \alpha \\ &&& R(Q,\rho) = H(\rho) - U(\rho) * Q * V(\rho) \\ &&& \underline{\rho} \le \rho \le \overline{\rho}. \\ &&& Q(k) = 0 \text{ if } k > n.\end{aligned}$$

Following the same argument as in [10], it can be be shown that $\nu_n$ and $\nu^n$ monotonically converge to $\nu$ from below and above as $n$ goes to infinity. In what follows, we shall demonstrate how to solve these finite-dimensional non-convex problems. The development will be based solely on $\nu_n$, but the same technique also applies to the solution of $\nu^n$.

## 4 Reformulation

In this section, we shall demonstrate that, by introducing two sets of auxiliary variables, the non-convex problem to be solved can be reformulated as an optimizing problem with linear and non-linear constraints, where the non-linear constraints are of the type $x = yz$ for variables $x$, $y$, and $z$.

Let the following be the corresponding state-space representation([11]) of $H$, $U$, and $V$:

$$H_{ss}(\rho) = \left[\begin{array}{c|c} A_H(\rho) & B_H(\rho) \\ \hline C_H(\rho) & D_H(\rho)\end{array}\right]$$

$$= \left[\begin{array}{cc|c} A(\rho)+B_2(\rho)F & -B_2(\rho)F & B_1(\rho) \\ 0 & A(\rho)+LC_2(\rho) & B_1(\rho)+LD_{21}(\rho) \\ \hline C_1(\rho)+D_{12}(\rho)F & -D_{12}(\rho)F & D_{11}(\rho)\end{array}\right]$$

$$U_{ss}(\rho) = \left[\begin{array}{c|c} A_U(\rho) & B_U(\rho) \\ \hline C_U(\rho) & D_U(\rho)\end{array}\right]$$

$$= \left[\begin{array}{c|c} A(\rho)+B_2(\rho)F & -B_2(\rho) \\ \hline C_1(\rho)+D_{12}(\rho)F & -D_{12}(\rho)\end{array}\right]$$

$$V_{ss}(\rho) = \left[\begin{array}{c|c} A_V(\rho) & B_V(\rho) \\ \hline C_V(\rho) & D_V(\rho)\end{array}\right]$$

$$= \left[\begin{array}{c|c} A(\rho)+LC_2(\rho) & B_1(\rho)+LD_{21}(\rho) \\ \hline C_2(\rho) & D_{21}(\rho)\end{array}\right]$$

where we assume the existence of a pair of feedback gain $F$ and observer gain $L$ that stabilize the system for any $\rho \in [\underline{\rho}, \overline{\rho}]$. Note that if $A(\rho)$ is assumed to be stable for any feasible parameter vector $\rho$, then the zero controller ($F = 0$, $L = 0$) are to be chosen in the above realizations. By the definition of the impulse response for discrete-time systems, we infer from the above state-space representations that any entry $H_{ij}(k)$ of $H$ is a polynomial of $\rho$, and so are $U_{ij}(k)$ and $V_{ij}(k)$. In what follows, for ease of notation, we shall use $S_k^{ij}$ to denote $S_{ij}(k)$ for any variable $S$ in $\ell_1^{m\times n}$ or $c_0^{m\times n}$.

It is easy to see from the definition of $v_n$ that only the parameters of $R_0^{bc}, \ldots, R_k^{bc}, \ldots, R_n^{bc}$ involves in the optimization of $v_n$ and so, in what follows, we shall develop a new formulation for these variables. By Lemma 1 of [8], the $b^{th}$-row $c^{th}$-column entry $R^{bc}$ of the closed-loop map $R$ can be characterized as follow:

$$R_k^{bc}(Q,\rho) = H_k^{bc}(\rho) - \langle W^{bck}(\rho), Q\rangle$$

where

$$\begin{aligned} W^{bck}(\rho) &:= \{Z_k^{bc}(\rho), \ldots, Z_0^{bc}(\rho), 0, \ldots\} \in c_0^{n_u\times n_v} \\ Z^{bc}(\rho) &:= U_{b,\cdot}^T(\rho) * V_{\cdot,c}^T(\rho) \\ &= \{Z_0^{bc}(\rho), \ldots, Z_k^{bc}(\rho), Z_{k+1}^{bc}(\rho), \ldots\} \in \ell_1^{n_u\times n_v},\end{aligned}$$

$U_{b,\cdot}$ is the $b^{th}$-row of of $U$, and $V_{\cdot,c}$ is the $c^{th}$-column of $V$. Then it is clear from above that $R_k^{bc}$ ($k = 0, 1, \ldots, n$) is a polynomial of $\rho_1, \ldots, \rho_m$ (up to the degree of a constant, say, $o_n$) and $Q_0^{st}, \ldots, Q_k^{st}$.

Let

$$1,\ \rho_1,\ \ldots,\ \rho_m,\ \rho_1^2,\ \rho_1\rho_2,\ \ldots,\ \rho_1^{o_n},\ \ldots,\ \rho_m^{o_n}$$

be a basis for the $o_n$-degree polynomials and let $d$ be its dimension. Define

$$\begin{aligned}\Gamma &= [1\ \rho_1\ \cdots\ \rho_m\ \rho_1^2\ \rho_1\rho_2\ \cdots\ \rho_1^{o_n}\ \cdots\ \rho_m^{o_n}]^T \\ &= [\tau_1\ \tau_2\ \ldots\ \tau_d]^T.\end{aligned}$$

Then each element $\tau_i$ of $\Gamma$ is a $d_i$-degree monomial of the form

$$\tau_i = \prod_{j=1}^m \rho_j^{\theta_{ij}},\ 0 \le \theta_{ij} \le d_i,\ \sum_{j=1}^m \theta_{ij} = d_i \le o_n. \quad (4)$$

Moreover, there exist indices $i_l \in \{1,2,\ldots,d\}$ and $j_l \in \{1,2,\ldots,m\}$ $(l=0,\ldots,d_i)$ such that (4) is equivalently characterized by the following set of equations:

$$\begin{aligned}
\tau_i = \tau_{i_0} &= \tau_{i_1}\rho_{j_1}, \\
&\vdots \\
\tau_{i_l} &= \tau_{i_{l+1}}\rho_{j_{l+1}} \\
&\vdots \\
\tau_{i_{d_i-1}} &= \tau_{i_{d_i}}\rho_{j_{d_i}} \\
\tau_{d_i} &= 1.
\end{aligned} \tag{5}$$

It follows that there exist constant coefficients $f_i$ and $g_{istl}$ such that $R_k^{bc}$ can be expressed as

$$\begin{aligned}
&R_k^{bc}(Q,\rho) \\
&= \sum_{i=1}^{d} f_i\tau_i + \sum_{i=1}^{d}\sum_{s=1}^{n_u}\sum_{t=1}^{n_v}\sum_{l=0}^{k} g_{istl}\tau_i Q_l^{st} \\
&= \sum_{i} f_i\{\prod_{j=1}^{m}\rho_j^{\theta_{ij}}\} + \sum_{i,s,t,l} g_{istl}\{\prod_{j=1}^{m}\rho_j^{\theta_{ij}}\}Q_l^{st}
\end{aligned} \tag{6}$$

Note that in (6), $f_i$ and $g_{istl}$ are functions of the indices $b$, $c$, and $k$ as well. But for the sake of notational simplicity, these three indices are omitted in the symbolic expressions of $f_i$ and $g_{istl}$.

So the problem of interest becomes

$$\begin{aligned}
\nu_n = \ &\inf \gamma \\
&\text{subject to} \\
&\sum_{t=1}^{n_v}\sum_{l=0}^{n}[Q_l^{st,+} + Q_l^{st,-}] \le \alpha \\
&\textstyle\sum_{c=1}^{n_w}\sum_{k=0}^{n}[R_k^{bc,+} + R_k^{bc,-}] \le \gamma \\
&Q_l^{st} = Q_l^{st,+} - Q_l^{st,-} \\
&R_k^{bc} = R_k^{bc,+} - R_k^{bc,-} \\
&(6) \\
&R_k^{bc,+} \ge 0,\ R_k^{bc,-} \ge 0 \\
&Q_l^{st,+} \ge 0,\ Q_l^{st,-} \ge 0,\ \underline{\rho} \le \rho \le \overline{\rho}.
\end{aligned} \tag{7}$$

where we have used a standard change of variables from linear programming (see for instance [7]) to reformulate the variables and constraints of $\nu_n$. Specifically, the variable $x$ is replaced by nonnegative variables $x^+$ and $x^-$ such that $x = x^+ - x^-$. Then the $\ell_1$ norm constraint $\|Q\|_1 \le \alpha$ is replaced by the constraint $\sum_{t=1}^{n_v}\sum_{k=0}^{n}[Q_l^{st,+} + Q_l^{st,-}] \le \alpha$, and $\|P_n R(Q,\rho)\|_1$, the objective function to be minimized, is replaced by introducing an auxiliary variable $\gamma$ such that $\sum_{c=1}^{n_w}\sum_{k=0}^{n}[R_k^{bc,+} + R_k^{bc,-}] \le \gamma$. It is also useful to mention that the optimal solution of problem (7) always satisfies that either $R_k^{bc,+}$ or $R_k^{bc,-}$ is zero.

To set the stage for the branch and bounding algorithm, we suppose that the rectangle-type set $[\underline{\rho},\overline{\rho}]$ is partitioned into $M$ subsets $[\underline{\rho}^r,\overline{\rho}^r](r=1,\ldots,M)$ such that $[\underline{\rho},\overline{\rho}] = \bigcup_{r=1}^{M}[\underline{\rho}^r,\overline{\rho}^r]$, where $\underline{\rho}^r = [\underline{\rho}_1^r \cdots \underline{\rho}_m^r]^T \in R^m$ and $\overline{\rho}^r = [\overline{\rho}_1^r \cdots \overline{\rho}_m^r]^T \in R^m$. Then a finer grid version of problem (7) is defined as:

$$\begin{aligned}
\nu_{n,r} := \ &\inf \gamma \\
&\text{subject to} \\
&\sum_{t=1}^{n_v}\sum_{l=0}^{n}[Q_l^{st,+} + Q_l^{st,-}] \le \alpha \\
&\textstyle\sum_{c=1}^{n_w}\sum_{k=0}^{n}[R_k^{bc,+} + R_k^{bc,-}] \le \gamma \\
&Q_l^{st} = Q_l^{st,+} - Q_l^{st,-} \\
&R_k^{bc} = R_k^{bc,+} - R_k^{bc,-} \\
&(6) \\
&R_k^{bc,+} \ge 0,\ R_k^{bc,-} \ge 0 \\
&Q_l^{st,+} \ge 0,\ Q_l^{st,-} \ge 0,\ \underline{\rho}^r \le \rho \le \overline{\rho}^r.
\end{aligned} \tag{8}$$

For notational convenience, we shall use the symbol $\Psi$ to denote the set of $(\gamma,\rho,R_k^{bc},R_k^{bc,+},R_k^{bc,-},Q_l^{st},Q_l^{st,+},Q_l^{st,-}) \in R^N$ $(N = 1+m+3n_z n_w(n+1) + 3n_u n_v(n+1))$ such that all the other constraints except the non-linear constraint (6) in problem (8) are satisfied. Thus problem $\nu_{n,r}$ is equivalently expressed as:

$$\begin{aligned}
\nu_{n,r} = \ &\inf \gamma \\
&\text{subject to} \\
&(6),\ (\gamma,\rho,R_k^{bc},R_k^{bc,+},R_k^{bc,-},Q_l^{st},Q_l^{st,+},Q_l^{st,-}) \in \Psi.
\end{aligned} \tag{9}$$

To prepare for the linear relaxation scheme introduced in the next section, let us further introduce the following set of variables:

$$\lambda_{istl} \ := \ \tau_i Q_l^{st} \tag{10}$$

and it follows from (6) that

$$R_k^{bc}(Q,\rho) = \sum_i f_i\tau_i + \sum_{i,s,t,l} g_{istl}\lambda_{istl}. \tag{11}$$

So problem (9) becomes

$$\begin{aligned}\nu_n \ := \ & \inf \gamma \\ & \text{subject to} \\ & (5),\ (10),\ (11) \\ & (\gamma,\rho,R_k^{bc},R_k^{bc,+},R_k^{bc,-},Q_l^{st},Q_l^{st,+},Q_l^{st,-}) \in \Psi.\end{aligned} \tag{12}$$

Clearly that problem (12) is a non-linear optimization problem and hard to solve in general.

# 5 Problem Solution

From the formulation of problem (12), we can infer that the crux of solving this problem is how to deal with those non-convex product terms present in (5) and (10). For this purpose, we introduce the following result from [10]:

**Lemma 1** *If the variables $x_j \in R$ satisfy the conditions $l_j \le x_j \le u_j$ and $t_{ij} := x_i x_j$, then*

$$\begin{aligned} t_{ij} &\ge u_j x_i + u_i x_j - u_i u_j \\ t_{ij} &\le l_j x_i + u_i x_j - u_i l_j \\ t_{ij} &\le u_j x_i + l_i x_j - l_i u_j \\ t_{ij} &\ge l_j x_i + l_i x_j - l_i l_j. \end{aligned} \tag{13}$$

*Furthermore, if variables $t_{ij} \in R$ satisfy (13) and $x_k$ satisfy $l_k \le x_k \le u_k$, then*

$$|t_{ij} - x_i x_j| \le \tfrac{1}{4}(u_i - l_i)(u_j - l_j).$$

Following (5) and (10), define

$$\begin{aligned}\Omega_{ijl} \ := \ & \{(\tau_{i_l},\tau_{i_{l+1}},\rho_{j_{l+1}}) \in R^3 |\ (13) \text{ are satisfied} \\ & \text{with } (t_{ij},x_i,x_j,u_i,l_i,u_j,l_j) \text{ replaced by} \\ & (\tau_{i_l},\tau_{i_{l+1}},\rho_{j_{l+1}},\overline{\tau}_{i_{l+1}},\underline{\tau}_{i_{l+1}},\overline{\rho}^{\tau}_{j_{l+1}},\underline{\rho}^{\tau}_{j_{l+1}})\}. \\ \Lambda_{istl} \ := \ & \{(\lambda_{istl},\tau_i,Q_l^{st}) \in R^3 |\ (13) \text{ are satisfied} \\ & \text{with } (t_{ij},x_i,x_j,u_i,l_i,u_j,l_j) \text{ replaced by} \\ & (\lambda_{istl},\tau_i,Q_l^{st},\overline{\tau}_i,\underline{\tau}_i,\alpha,-\alpha)\}.\end{aligned}$$

where $\overline{\tau}_i$ and $\underline{\tau}_i$ are upper and lower bounds for $\tau_i$ and they can be a priori computed. Hence from (12) and Lemma 1, we have

$$\begin{aligned}\nu_{n,r} \ = \ & \inf \gamma \\ & \text{subject to} \\ & (\tau_{i_l},\tau_{i_{l+1}},\rho_{j_{l+1}}) \in \Omega_{ijl}, (\lambda_{istl},\tau_i,Q_l^{st}) \in \Lambda_{istl} \\ & (5),\ (10),\ (11) \\ & (\gamma,\rho,R_k^{bc},R_k^{bc,+},R_k^{bc,-},Q_l^{st},Q_l^{st,+},Q_l^{st,-}) \in \Psi.\end{aligned}$$

Removing the the nonlinear constraints (5) and (10), we have the following relaxed linear programming problem:

$$\begin{aligned}\nu_{n,r}^R \ = \ & \inf \gamma \\ & \text{subject to} \\ & (\tau_{i_l},\tau_{i_{l+1}},\rho_{j_{l+1}}) \in \Omega_{ijl}, (\lambda_{istl},\tau_i,Q_l^{st}) \in \Lambda_{istl} \\ & (11) \\ & (\gamma,\rho,R_k^{bc},R_k^{bc,+},R_k^{bc,-},Q_l^{st},Q_l^{st,+},Q_l^{st,-}) \in \Psi.\end{aligned}$$

It is clear that if the relaxed problem $\nu_{n,r}^R$ is infeasible, then so is the problem $\nu_{n,r}$. If $\nu_{n,r}^R$ is a finite real number, then $\nu_{n,r}^R \le \nu_{n,r}$. Now we are ready to prove the main result of the paper.

**Theorem 1** *Suppose an optimal solution of the relaxed problem $\nu_{n,r}^R$ is given by:*

$$(\widehat{\gamma},\widehat{\rho},\widehat{\tau}_i,\widehat{\lambda_{istl}},\widehat{R_k^{bc}},\widehat{R_k^{bc,+}},\widehat{R_k^{bc,-}},\widehat{Q_l^{st}},\widehat{Q_l^{st,+}},\widehat{Q_l^{st,-}}).$$

*Then there exists a feasible solution*

$$(\gamma_{feas}, \widehat{\rho}, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, \widehat{Q_l^{st}}, \widehat{Q_l^{st,+}}, \widehat{Q_l^{st,-}}).$$

*for problem $\nu_{n,r}$ (as defined in (9)) such that*

$$\nu_{n,r}^R = \widehat{\gamma} \leq \nu_{n,r} \leq \gamma_{feas} \quad (14)$$

$$\gamma_{feas} - \widehat{\gamma} \leq C|\overline{\rho}^r - \underline{\rho}^r|_\infty \quad (15)$$

*where $C$ is a finite positive constant and* $|\overline{\rho}^r - \underline{\rho}^r|_\infty = \max\{|\overline{\rho}_i^r - \underline{\rho}_i^r| : i = 1, \ldots, m\}$.

Having established the relationship between $\widehat{\gamma}$ and $\gamma_{feas}$, we can now make use of certain well-established branch and bound algorithms (see, for example, [9]) to compute the optimal solution of $\mu$ within any prescribed performance tolerance $\epsilon > 0$. More specifically, for a fixed tapping length $n$ of $Q$, $\nu_n$ and $\nu^n$ are computed by combining the linear programs with the branching and bounding algorithm. If $\nu^n - \nu_n \leq \epsilon$ then the globally optimal controller is recovered from the corresponding optimizing variables and the optimizations are terminated. Otherwise $n$ is increased until the desired performance is achieved.

If the performance measure used in problem (1) is $\mathcal{H}_2$ norm instead of $\ell_1$ norm, then the exactly same procedure as above would enable us the arrive at the same conclusion of Theorem 1 by additionally observing the fact that

$$\begin{aligned} & |\ [R_k^{bc,+} + R_k^{bc,-}]^2 - [\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}]^2\ | \\ \leq\ & C_{\mathcal{H}_2}|\ [R_k^{bc,+} + R_k^{bc,-}] - [\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}]\ | \end{aligned}$$

where $C_{\mathcal{H}_2}$ is a finite constant that can be computed a priori from the known parameters.

# 6 Summary

We present a global optimal solution to the IPC problem in the paper. The solutions are obtained by solving linear/quadratic programming problems for which powerful numerical softwares exist.

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