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Revisiting Kalman and Aizerman Conjectures

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Abstract— **The article revisits Aizerman and Kalman conjectures for absolute stability through the lens of a novel graphical interpretation. Even though these conjectures are false in the general case, such a graphical interpretation suggests natural ways to introduce additional conditions in order to obtain valid absolute stability criteria. As an illustration, the article proves a new absolute stability criterion obtained by the iterative application of a variation of the circle criterion.**

I. Introduction

Robust stability analysis is the study of stability properties of interconnected systems in the presence of modeling uncertainties [1]. In most typical frameworks, model uncertainties are taken into account by assuming that each uncertain system is represented by an operator that belongs to a prespecified class [2]. In nonlinear control, the problem of absolute stability, formulated by Lur'e and Postnikov [3], can be interpreted as a robust stability problem. Indeed, in this case, global asymptotic stability is sought for a Single-Input/Single-Output (SISO) Linear Time Invariant (LTI) system in feedback with a function belonging to a class of memoryless static operators. If the class of operators is restricted to only linear static gains, Nyquist criterion provides necessary and sufficient conditions for the global asymptotic stability of the interconnection. The relative simplicity of the conditions given by the Nyquist criterion has led to several conjectures and attempts to extend the result to more general classes including nonlinear static operators. For example, Aizerman conjectured that if a SISO LTI system $\mathcal{L}$ in feedback interconnection with a linear static gain k is asymptotically stable for all $k \in (\alpha,\beta)$, then the system $\mathcal{L}$ in feedback interconnection with any static operator $n(\cdot)$ in the linear sector (α,β), namely satisfying the condition:

$$\alpha y^2 < yn(y) < \beta y^2 \qquad \text{for all } y \in \mathbb{R}, y \neq 0, \qquad (1)$$

is globally asymptotically stable. Counterexamples disprove Aizerman's conjecture [4], [5], [6]. Kalman [7] imposed stronger conditions on the function $n(\cdot)$, conjecturing that if a SISO LTI system $\mathcal{L}$ in feedback interconnection with a linear static gain k is asymptotically stable for all $k \in (\alpha,\beta)$, then the system $\mathcal{L}$ in feedback interconnection with any differentiable static operator $n(\cdot)$ satisfying the condition

$$\alpha < \frac{d}{dy}n(y) < \beta \qquad \text{for all } y \in \mathbb{R}, \qquad (2)$$

is globally asymptotically stable. However, counterexamples have disproved Kalman's conjecture, as well [4], [5].

Positive results guaranteeing the stability of a SISO LTI system $\mathcal{L}$ in feedback with a sector nonlinearity $n(\cdot)$ were first found by Popov and Zames. Popov's criterion [8] provides sufficient conditions for the stability of a SISO LTI system in feedback with a time-invariant nonlinearity in the linear sector $[0,k]$. Zames' circle criterion [9] states that a SISO LTI system in feedback with a nonlinearity in the sector (α,β) is globally asymptotically stable if (i) the LTI system in feedback with the linear gain k is asymptotically stable for all $k \in (\alpha,\beta)$; and (ii) the circle with diameter on the segment $(-\frac{1}{\alpha}, -\frac{1}{\beta})$ in the complex plane does not intersect the Nyquist plot of the LTI system. Zames' circle criterion extends Nyquist criterion in a natural way, but its conditions are quite conservative. Less conservative conditions are typically obtained by using multipliers methods [10], [11], [12], [13], [14], [15], [16]. A fundamental generalization of multiplier methods is given by the Integral Quadratic Constrain (IQC) approach pioneered by Yakubovich [17], [18], [19] and formalized by Megretski and Rantzer via the use of Linear Matrix Inequalities (LMI's) [20]. The IQC approach provides a unifying framework that encompasses most of the absolute stability criteria: Popov criterion, the circle criterion, criteria based on Zames-Falb multipliers. Moreover, many other criteria can be formulated in the form of IQC's and numerically implemented via LMI's. The use of IQC's as a tool to obtain robust stability results is prevalent [21], [22], [23], [24], [25] and the problem of absolute stability is receiving significant attention recently. The conservativeness of classical absolute stability theory can be reduced by replacing the linear sector in the classical absolute stability theory to a sector bounded by concave/convex functions [26], [27]. A systematic procedure to obtain estimates, via extended Lyapunov functions, of attracting sets of a class of nonlinear systems as well as an estimate of their stability regions is provided in [28]. In [29] absolute stability was extended to situations where the sector of the nonlinearity is not symmetric. In [30] the authors propose a new Lyapunov function based on LMIs and gave analysis to show that the suggested Lyapunov function would result in a less conservative estimate of the basin of attraction. The absolute stability problem for Lur'e singularly perturbed systems with multiple nonlinearities was analyzed in [31]. The authors proposed a parameter dependent criterion based on LMI's showed that it is less conservative than the other existing methods. In [32] nonlinear systems are analyzed using absolute stability theory and sum of squares programming.

In this article we provide a novel graphical interpretation of the problem of absolute stability as a robust stability problem. In this graphical representation, the class of nonlinearities considered by both Aizerman conjecture and Kalman conjecture admit a straightforward visual characterization. As previously mentioned, both conjectures are known to be false in the general case and they indicate the conclusion that the classes of nonlinearities these conjectures considered are in general too broad to provide any global asymptotic stability. Thus, in order to find sufficient conditions for global asymptotic stability, additional conditions need to be assumed

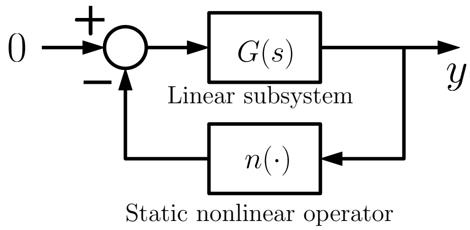

Fig. 1. Lur'e System: $G(s)$ is a linear time invariant system and $n(\cdot)$ is a nonlinear memoryless operator.

on the class of nonlinearities. The graphical representation advanced in the article is capable of suggesting possibles ways to introduce additional conditions in Aizerman and Kalman conjectures in order to find a class of nonlinear functions in the sector (α,β), where absolute stability can be guaranteed. In particular, we use our graphical representation to formulate the conjecture that for every LTI SISO system with transfer function $G(s)$ in feedback with the static operator $n(\cdot)$ such that $G(s)$ is asymptotically stable for $n(y) = ky$ for all $k \in (\alpha,\beta)$, there exists $\epsilon > 0$ such that all operators $n(\cdot)$ satisfying

$$\alpha y^2 < yn(y) < \beta y^2 \quad \text{and} \quad \left|\frac{\partial n}{\partial y} - \frac{n}{y}\right| < \epsilon \quad \forall y \in \mathbb{R}$$

the feedback interconnection of $G(s)$ with $n(\cdot)$ is globally asymptotically stable. In the rest of the article we show that the conjecture is correct combining recent absolute stability and Lyapunov methods developed in [33], [34], [35]. We also provide a quantitative lower bound for ϵ that can be obtained from the knowledge of $G(s)$ and the use of standard LMI tools. Analysis of conservativeness of this bound is out of scope of this paper, though.

II. Comparison of Nyquist stability criterion and absolute stability conjectures

A system $\mathcal{S}$ is a Lur'e model when it is given by the feedback interconnection of a SISO linear time-invariant system $G(s)$ and a nonlinear, possibly time-varying, static block $n(\cdot)$ as represented in Figure 1. The problem posed by Lur'e and Postnikov in [3] was about finding, for a fixed transfer function $G(s)$, a class of nonlinearities making $\mathcal{S}$ Globally Asymptotically Stable (GAS). In this section, we are mainly focusing on the nature of the function $n(\cdot)$ providing a graphical tool to describe different classes of nonlinearities in an intuitive and evocative way.

To this end, fix a transfer function $G(s)$ for the linear part of a Lur'e systems and consider the plane $\mathcal{P}$ where a function $n(\cdot)$ is described by plotting the values of $n(y)/y$ versus dn/dy. Observe that any differentiable function $n(\cdot)$ describes a continuous trajectory on $\mathcal{P}$. On such a plane it is possible to compare the Nyquist criterion with the statements of the Aizerman and Kalman conjectures.

A. Nyquist Criterion

As is well-known, Nyquist criterion considers a SISO LTI system with transfer function $G(s)$ and a linear negative feedback of the form $n(y) = ky$ stating that the interconnection is stable if and only if the winding number of the curve $G(i\omega)$ around $-1/k$ matches the number of unstable poles of $G(s)$ [36].

Definition 1: We say that (α,β) is a Hurwitz interval for the transfer function $G(s)$ iff $G(s)$ in negative feedback with $n(y) = ky$ is stable for any $k \in (\alpha,\beta)$.

Assume that (α,β) is a Hurwitz interval for $G(s)$ and let us interpret Nyquist criterion on the plane $\mathcal{P}$. Observe that the function $n(y) = ky$ is represented on $\mathcal{P}$ by the point (k, k). Therefore, Nyquist criterion states that each function $n(\cdot)$, represented by a single point on the diagonal of the region $(\alpha,\beta)\times(\alpha,\beta)$ in the plane $\mathcal{P}$, makes $\mathcal{S}$ asymptotically stable. See Figure 2.

B. Aizerman Conjecture

Now, let us consider Aizerman conjecture which states that the Lur'e system $\mathcal{S}$ is GAS for every $n(y)$ such that

$$\alpha y^2 < n(y)y < \beta y^2 \qquad \forall y \neq 0 \tag{3}$$

if every feedback interconnection of $G(s)$ with k for any $k \in (\alpha,\beta)$ is stable. The conjecture is known to be false in the general case. Nevertheless, the class of nonlinearity can be easily visualized on $\mathcal{P}$. Indeed, a function $n(\cdot)$ satisfying Relation (3) is depicted by a trajectory on $\mathcal{P}$ lying in the strip $(\alpha,\beta)\times\mathbb{R}$. See Figure 3.

C. Kalman Conjecture

In order to garantee stability, Kalman introduced an additional constraint to Aizerman condition [7]. His conjecture was that a Lur'e system is globally asymptotically stable for every $n(\cdot)$ such that

$$\begin{cases} \alpha y^2 < ny < \beta y^2 \qquad \forall y \neq 0 \\ \\ \alpha < \frac{dn(y)}{dy} < \beta. \end{cases} \tag{4}$$

According to our graphical interpretation, this is equivalent to saying that the trajectory described by $n(\cdot)$ on the plane $\mathcal{P}$ lies in the square $(\alpha,\beta)\times(\alpha,\beta)$. See Figure 4. Such a conjecture is false in the general case, as well. Thus, as in the case of Aizerman, the region on the plane $\mathcal{P}$ where the trajectory of $n(\cdot)$ is allowed to lie is in general too "large".

D. Formulating alternative conjectures

A problem that can now be posed is the determination of an appropriate region on the plane $\mathcal{P}$ such that every nonlinear function $n(\cdot)$ with a trajectory lying in such a region is guaranteed to make the system $\mathcal{S}$ globally asymptotically stable (see Figure 5). From this perspective, the graphical interpretation of the trajectory of a static operator $n(y)$ on the plane $\mathcal{P}$ helps choosing a suitable region. Indeed, while the diagonal set $D_0 := \{(k, k)|k \in (\alpha,\beta)\}$ in the plane $\mathcal{P}$ for which Nyquist criterion guarantees global asymptotic stability can be considered "too small", the sets described by Aizerman and Kalman conjectures are "too large". A possible approach is to somehow enlarge the diagonal set D_0. A natural choice is, for example, to consider the following set of nonlinearities.

Definition 2: The set D_ϵ, for $\epsilon > 0$, is the set of differentiable nonlinearities $n(y)$ defined as

$$D_\epsilon := \left\{ n(y) : \mathbb{R} \to \mathbb{R}, \left|\frac{\mathrm{dn(y)}}{\mathrm{dy}} - \frac{\mathrm{n(y)}}{\mathrm{y}}\right| \leq \epsilon,\ \forall \mathrm{y} \in \mathbb{R}, \mathrm{y} \neq 0 \right\}. \tag{5}$$

In the plane $\mathcal{P}$ the static operators in D_ϵ have trajectories that are close to the diagonal set D_0, as shown in Figure 6. We advance the following conjecture.

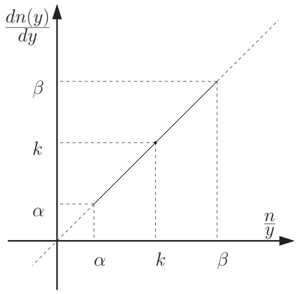

Fig. 2. Graphical interpretation of the Nyquist criterion where (α, β) is a Hurwitz sector. A linear static operator $n(y) = ky$ is represented by a point in the plane $\mathcal{P}$.

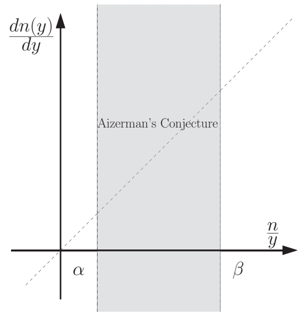

Fig. 3. Graphical interpretation of Aizerman conjecture. A differentiable static operator $n(\cdot)$ in the sector (α, β) describes a trajectory on the plane $\mathcal{P}$ that lies in the shaded infinite strip.

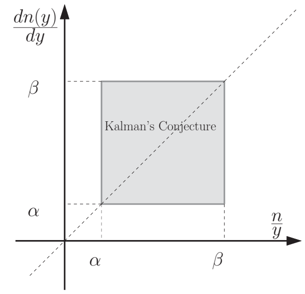

Fig. 4. Graphical interpretation of Kalman conjecture. A differentiable static operator $n(\cdot)$ with slope restricted to the interval (α, β) describes a trajectory on the plane $\mathcal{P}$ that lies in the shaded square.

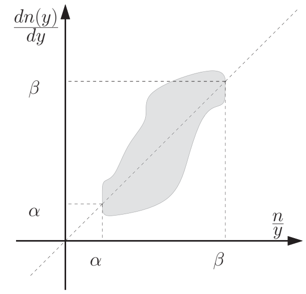

Fig. 5. Is it possible to determine a region in the plane $\mathcal{P}$ guaranteeing that all the static operators with a trajectory in such a region make a Lur'e system globally asymptotically stable?

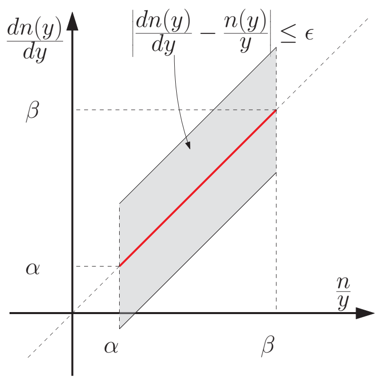

Fig. 6. Graphical interpretation of nonlinearities based on Definition 2.

Conjecture 2.1: **For every transfer function $G(s)$ with Hurwitz sector (α, β) there exists $\epsilon > 0$ such that $n(\cdot) \in D_\epsilon$ implies that the negative feedback interconnection of $G(s)$ with $n(\cdot)$ leads to a globally asymptotically stable system.**

III. Proof of Conjecture 2.1

In this section we prove that Conjecture 2.1 is true and, for a given $G(s)$ with Hurwitz sector (α, β), we also provide a lower bound for ϵ.

A. Preliminary definitions and results

We start introducing preliminary notions and results. For the level sets of a quadratic function we use the following notation.

Definition 3: Given a scalar quadratic function $V(x) = x^T P x : \mathbb{R}^n \to \mathbb{R}$, defined by the positive definite matrix P, and a positive real number H, we define

$$\mathcal{E}_P(H) := \left\{ x \in \mathbb{R}^n : V(x) = x^T P x \leq H \right\}. \tag{6}$$

Definition 4: Given a matrix Σ, we define the quadratic form σ_Σ via

$$\sigma_\Sigma(y, u) := \begin{pmatrix} y \\ u \end{pmatrix}^T \Sigma \begin{pmatrix} y \\ u \end{pmatrix}. \tag{7}$$

We also say that the matrix Σ is the multiplier of the quadratic form.

The following Lemma establishes a bound on the scalar quantity $y = Cx$ when x is restricted to the ellipsoid $\mathcal{E}_P(H)$.

Lemma 3.1: Let us consider a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$, and a vector $C \in \mathbb{R}^{1 \times n}$. It holds that

$$\max_{x \in \mathcal{E}_P(H)} \|Cx\|_2^2 = HCP^{-1}C^T. \tag{8}$$

Proof: The proof is omitted. ■

The set of states with output bounded by Y is denoted by $L(Y)$.

Definition 5: Let C be a vector in $\mathbb{R}^{1 \times n}$. We define the strip $L(Y)$ in the following way

$$L(Y) := \{ x \in \mathbb{R}^n,\ |Cx| \leq Y \}. \tag{9}$$

The notion of Bias Function for a static operator will play an important role in the development of this section.

Definition 6: Consider $\mathcal{N}$ a class of SISO static operators and $n(\cdot) \in \mathcal{N}$ such that $n(0) = 0$ and a symmetric matrix $\Sigma \in \mathbb{R}^{2 \times 2}$. We define the *Bias Function* $M_\Sigma : \mathbb{R}^+ \to \mathbb{R}^+$ of $\mathcal{N}$ with respect to Σ in the following manner:

$$M_\Sigma(Y) := -\inf_y \left\{ \begin{pmatrix} y \\ n(y) \end{pmatrix}^T \Sigma \begin{pmatrix} y \\ n(y) \end{pmatrix}, 0 \right\} \tag{10}$$

subject to

$$y^2 \leq Y^2 \text{ and } n(\cdot) \in \mathcal{N}. \tag{11}$$

Observation 3.2: The Bias Function $M_\Sigma(Y)$ is monotonically nondecreasing in Y and $M_\Sigma(0) = 0$.

The Bias Function defined by a multiplier Σ generalizes the concept of static Local Quadratic Constraints (LQC) as defined in [37]. Indeed, the class of nonlinear functions $\mathcal{N}$ satisfies the LQC with multiplier Σ if the associated Bias Function is identically zero. The introduction of the Bias Function, M_Σ, allows the functions in $\mathcal{N}$ to violate, at least locally, the LQC.

B. Main Proof

The following Lemma provides sufficient conditions to establish that an ellipsoid is a positively invariant set for a Lur'e system.

Lemma 3.3: Let $\mathcal{S}$ be the Lur'e system described by the equations

$$\begin{aligned} \dot{x} &= Ax - Bn(y) \\ y &= Cx \\ x(t_0) &= x_0 \end{aligned} \tag{12}$$

where y is the system's output, x_0 is the system's initial condition at time t_0 and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ such that (A,C) are observable and (A,B) are controllable. Suppose $n(\cdot)$ is a continuous function defined on $\mathbb{R}$, with Bias Function $M_\Sigma(Y)$ for the matrix $\Sigma \in \mathbb{R}^{2\times 2}$, partitioned as

$$\Sigma = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}. \tag{13}$$

Suppose that there exists $P = P^T$ positive definite and $r > 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T QC + rP & -PB + C^T S \\ -B^T P + S^T C & R \end{bmatrix} < 0 \tag{14}$$

$$\frac{CP^{-1}C^T}{r} M_\Sigma(\widehat{Y}) < \widehat{Y}^2 \tag{15}$$

for a scalar $\widehat{Y} > 0$. If $x_0 \in \mathcal{E}_P(\widehat{Y}^2/CP^{-1}C^T)$ at t_0, then $x(t) \in \mathcal{E}_P(\widehat{Y}^2/CP^{-1}C^T) \quad \forall t \geq t_0$.

Proof: The proof is omitted. ■

The following Lemma provides sufficient conditions to conclude that if the output of the system is bounded by $\overline{Y}$, $\forall t \geq \underline{t}$, then $\exists\, \bar{t}: \quad |y(t)| \leq \underline{Y} < \overline{Y}$ for $t \geq \bar{t}$.

Lemma 3.4: Let $\mathcal{S}$ be the Lur'e system described by Lemma 3.3 with the same conditions on A, B, C, x_0. Suppose $n(\cdot)$ is a nonlinear function defined on $\mathbb{R}$, continuous in its parameter with Bias Function $M_\Sigma(Y)$ for the matrix $\Sigma \in \mathbb{R}^{2\times 2}$, partitioned in equation (13). Suppose that there exist positive definite matrix $P = P^T \in \mathbb{R}^{n\times n}$, and positive scalars r, $\underline{Y}$, and $\overline{Y}$ such that inequality (14) holds and

$$\frac{CP^{-1}C^T}{r} M_\Sigma(\overline{Y}) < (\underline{Y})^2 \tag{16}$$

$$0 < \underline{Y} < \overline{Y}. \tag{17}$$

Suppose there exists $\underline{t}$ such that $x(t) \in L(\overline{Y})$, $\quad \forall t \geq \underline{t}$. Then there exists $\bar{t}$ such that $x(t) \in L(\underline{Y})$, $\quad \forall t \geq \bar{t}$.

Proof: The proof is omitted. ■

A graphical representation of Lemma 3.4 is given in Figure 7. The following Lemma computes an upper bound for the Bias Function of a function $n(\cdot) \in D_\epsilon$.

Lemma 3.5: Suppose $n(\cdot) \in D_\epsilon$ and assume that

$$\alpha_\ell y^2 < yn(y) < \beta_\ell y^2 \;, \;\; \forall |y| \in (\underline{Y}, \overline{Y}] \tag{18}$$

for some α_ℓ, β_ℓ and for real positive values $\underline{Y} < \overline{Y}$. Then $M_\Sigma(\overline{Y}) \leq \overline{M}_\Sigma$ in which

$$\overline{M}_\Sigma := \frac{\frac{1}{2}(\beta_\ell\text{-}\alpha_\ell)^2 \underline{Y}^2}{\sqrt{1+\left[\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\right]^2}\text{-}1}\, exp\left(\left[\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\right]\text{-}1\text{-}\sqrt{1+\left[\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\right]^2}\right) \tag{19}$$

where

$$\Sigma = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} := \begin{bmatrix} -\alpha_\ell\beta_\ell & (\alpha_\ell+\beta_\ell)/2 \\ (\alpha_\ell+\beta_\ell)/2 & -1 \end{bmatrix}. \tag{20}$$

Proof: The proof is omitted. ■

Observation 3.6: $\overline{M}_\Sigma$ in Equation (19) is a strictly increasing function of ϵ such that $\lim_{\epsilon\to 0} \overline{M}_\Sigma = 0$. In Figure 9, $\overline{M}_{\Sigma_k}/\underline{Y}^{\,2}$ for $\beta - \alpha = 2$ with respect to ϵ has been shown which can be used to find the least conservative ϵ.

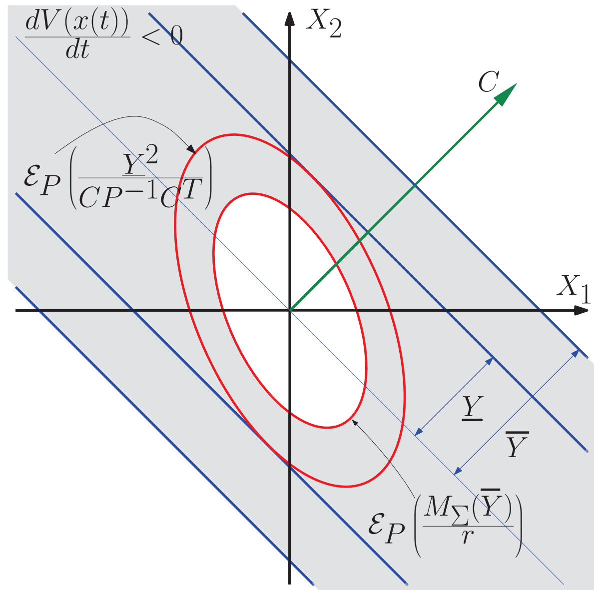

Fig. 7. Graphical interpretation of Lemma 3.4.

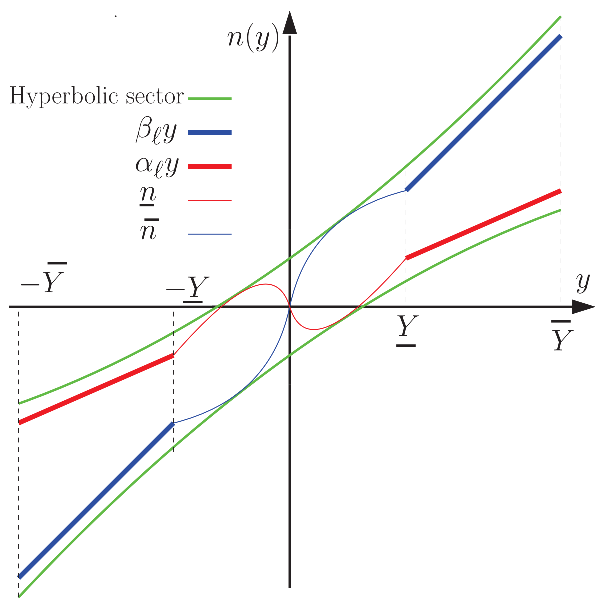

Fig. 8. Graphical description of Lemma 3.5.

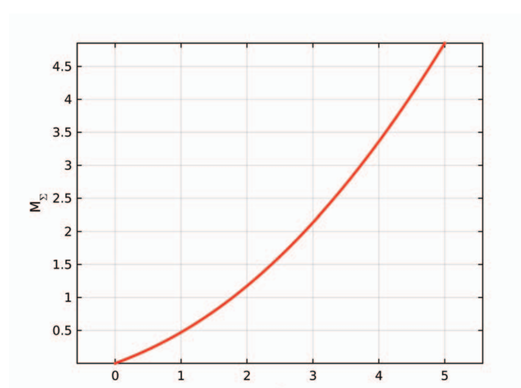

Fig. 9. $\overline{M}_\Sigma/\underline{Y}^2$ which is a strictly increasing function of ϵ, $\beta - \alpha = 2$.

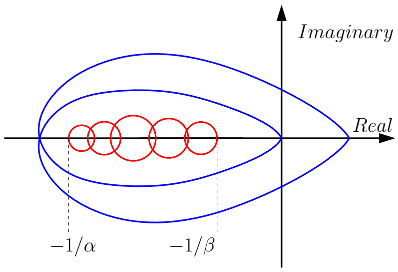

Fig. 10. Nyquist plot of a system along with sub-sectors (circles) which cover the Hurwitz sector (α,β).

The following Lemma provides regularity properties about a function $n(\cdot) \in D_\epsilon$.

Lemma 3.7: Suppose that there exists a sequence of intervals $\{[\alpha_\ell,\beta_\ell]\}$ and an interval $[\alpha,\beta]$ such that

$$\alpha = \alpha_1, \quad \beta = \beta_N, \tag{21}$$

$$\alpha_\ell < \alpha_{\ell+1} < \beta_\ell < \beta_{\ell+1} \quad \forall \ell = 1,2,3,...,N-1. \tag{22}$$

Suppose that $n(y) \in D_\epsilon$ and that

$$\alpha y^2 \le yn(y) \le \beta y^2, \quad \forall y \in \mathbb{R} \tag{23}$$

Then for every $\overline{Y} > 0$ there exists ℓ, $1 \le \ell \le N$ such that

$$\alpha_\ell y^2 \le yn(y) \le \beta_\ell y^2, \quad \forall |y| \in [\underline{Y}, \overline{Y}] \tag{24}$$

where $\underline{Y} = \overline{Y} exp(-\theta/\epsilon)$ and θ, the margin of sequence, is defined as

$$\theta := \frac{1}{2} \min_{1\le \ell < N} (\beta_\ell - \alpha_{\ell+1}) > 0. \tag{25}$$

Proof: The proof is omitted. ■

The following theorem gives sufficient conditions for GAS of a Lur'e system where $n(\cdot) \in D_\epsilon$.

Theorem 3.8: Consider the Lur'e system represented by equation (12) where y is the system's output, x_0 is the system's initial condition at time t_0 and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ such that (A, C) are observable and (A, B) are controllable. Suppose that there exists α_ℓ and β_ℓ for $\ell = 1,2,...,N$ such that

$$\alpha = \alpha_1, \quad \beta = \beta_N \tag{26}$$

$$\alpha_\ell < \alpha_{\ell+1} < \beta_\ell < \beta_{\ell+1}, \quad \forall \ell = 1,2,3,...,N-1. \tag{27}$$

Suppose that $n(y) \in D_\epsilon$ and

$$\alpha y^2 \le yn(y) \le \beta y^2, \quad \forall y \in \mathbb{R}. \tag{28}$$

Assume that $\forall \ell = 1,2,...,N$ there exist P_ℓ and r_ℓ such that

$$\begin{bmatrix} A^T P_\ell + P_\ell A + C^T Q_\ell C + r_\ell P_\ell & -P_\ell B + C^T S_\ell \\ -B^T P_\ell + S_\ell^T C & R_\ell \end{bmatrix} < 0 \tag{29}$$

$$P_\ell > 0; \quad r_\ell > 0; \tag{30}$$

where

$$\Sigma_\ell = \begin{bmatrix} Q_\ell & S_\ell \\ S_\ell^T & R_\ell \end{bmatrix} := \begin{bmatrix} -\alpha_\ell \beta_\ell & (\alpha_\ell + \beta_\ell)/2 \\ (\alpha_\ell + \beta_\ell)/2 & -1 \end{bmatrix}. \tag{31}$$

Suppose that $\epsilon > 0$ is such that

$$\frac{\frac{1}{2}(\beta_\ell\text{-}\alpha_\ell)^2}{\sqrt{1+\left(\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\right)^2}\text{-}1} exp\left[\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\text{-}1\text{-}\sqrt{1+\left(\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\right)^2}\right] < \frac{r_\ell}{CP_\ell^{-1}C^T} \tag{32}$$

for every $\ell = 1,2,...,N$. Then the origin of the Lur'e system defined by (12) is globally asymptotically stable.

Proof: Define

$$\overline{Y}_0 := \max_{\ell=1,...,N} \sqrt{x_0^T P_\ell x_0 C P_\ell^{-1} C^T}. \tag{33}$$

By the definition of $\overline{Y}_0$, $x_0 \in \mathcal{E}_{P_\ell}(\overline{Y}_0^2/CP^{-1}C^T)$ for all $\ell = 1,2,...N$. By applying Lemma 3.7 for $\overline{Y} = \overline{Y}_0$, we have that $\exists \ell \in \{1,2,...N\}$ such that

$$\alpha_\ell y^2 \le n(y)y \le \beta_\ell y^2, \quad \forall |y| \in [\underline{Y}_0, \overline{Y}_0] \tag{34}$$

with $\underline{Y}_0 := \overline{Y}_0 exp(-\theta/\epsilon)$. By using Lemma 3.5 we have that the Bias Function of $n(\cdot)$ with respect to the multiplier Σ_ℓ satisfies

$$M_{\Sigma_\ell}(\overline{Y}_0) \le \frac{\frac{1}{2}(\beta_\ell\text{-}\alpha_\ell)^2 \underline{Y}_0^2}{\sqrt{1+(\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon})^2}\text{-}1} exp\left[\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon}\text{-}1\text{-}\sqrt{1+(\frac{\beta_\ell\text{-}\alpha_\ell}{\epsilon})^2}\right]. \tag{35}$$

Because of Equation (32), we have

$$M_{\Sigma_\ell}(\overline{Y}_0) < \frac{r_\ell \underline{Y}_0^2}{CP_\ell^{-1}C^T}. \tag{36}$$

Since $x_0^T P_\ell x_0 C P_\ell^{-1} C^T \le \overline{Y}_0^2$ then $x_0 \in \mathcal{E}_{P_\ell}(\overline{Y}_0^2/CP_\ell^{-1}C^T)$. If we use Lemma 3.3 for $\Sigma = \Sigma_\ell$ and $\widehat{Y} = \overline{Y}_0$, we can guarantee that since the initial condition of the system is in $\mathcal{E}_{P_\ell}(\overline{Y}_0^2/CP_\ell^{-1}C^T)$, then $x(t) \in \mathcal{E}_{P_\ell}(\overline{Y}_0^2/CP_\ell^{-1}C^T)$ $\forall t \ge t_0$. By applying Lemma 3.1, $x(t) \in L(\overline{Y}_0)$ $\forall t \ge t_0$. Now by applying Lemma 3.4, we obtain that $\exists t_1 : x(t) \in L(\underline{Y}_0), \forall t \ge t_1$. By iteratively applying Lemma 3.7 and then Lemma 3.4 for $\overline{Y} = \overline{Y}_{k+1} = \underline{Y}_k$, $k = 0,1,2,...,\overline{N}-1$ we get

$$\underline{Y}_{\overline{N}} = \underline{Y}_0 exp\left(-\frac{\overline{N}\theta}{\epsilon}\right) \Longrightarrow \lim_{\overline{N}\to\infty} \underline{Y}_{\overline{N}} = 0 \tag{37}$$

Considering the fact that $|y(t)| \to 0$ and the full observability of the linear system, convergence of the state x(t) to the origin is guaranteed. ■

Theorem 3.8 Provides sufficient conditions for global asymptotic stability of a Lur'e system in feedback with a static function $n(\cdot) \in D_\epsilon$ such that $\alpha y^2 < yn(y) < \beta y^2$ for $y \ne 0$. The key assumption is the existence of sequence of intervals $[\alpha_\ell,\beta_\ell]$ for $\ell = 1,2,...,N$ satisfying (26) and (27) for which the LMIs (29) and (30) admits a solution. Conditions (26) and (27) amount to covering the internal $[\alpha,\beta]$ with intervals $[\alpha_\ell,\beta_\ell]$ for $\ell = 1,2,...,N$ such that two contiguous intervals are partially overlapping. The existence of solutions, P , r, for (29) and (30) is equivalent, from Kalman-Yakubovich-Popov Lemma, to having no intersections between the Nyquist plot of $G(s)$ and the circle with diameter $(-1/\alpha, -1/\beta)$ over the Real line [9]. In Figure 10 we provide a graphical presentation of a choice of intervals satisfying (26), (27), (29), and (30). These observations lead to the following Theorem.

Theorem 3.9: Let $\mathcal{S}$ be the Lur'e system described by the Equations (12) where y is the system's output, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ are minimal representation of the transfer function $G(s)$. Suppose $G(s)$ is Hurwitz in sector (α,β) and $\exists \delta > 0$ such that $inf_{\omega\in\mathbb{R}} |G(j\omega + 1/k)| \ge \delta$, $\forall k \in (\alpha,\beta)$. Then

$\exists\epsilon > 0$ such that $n(\cdot) \in D_\epsilon$ implies that the negative feedback interconnection of $G(s)$ with $n(\cdot)$ is a globally asymptotically stable system.

Proof: Since $inf_{\omega\in\mathbb{R}}|G(j\omega+1/k)| \geq \delta > 0, \quad \forall k \in (\alpha,\beta)$, then we are able to cover the Hurwitz sector (α,β) with circles having their diameters on the segment $(-1/\alpha_\ell, -1/\beta_\ell)$, where α_ℓ and β_ℓ satisfy (26) and (27) such that they do not intersect the Nyquist plot of $G(s)$. Kalman-Yakubovich-Popov Lemma guarantees the existence of solutions for (29) and (30). Now using Theorem 3.8, the assertion is proven. ■

IV. Conclusions

The article has revisited Aizerman and Kalman Conjectures through the lens of a novel graphical interpretation. This new interpretation is capable of highlighting some of the weaknesses of these conjectures, making their statements not valid in the general case. In particular it is found that the classes of nonlinearities considered by Aizerman and Kalman Conjectures are too large. Our new graphical interpretation enables the formulation of new Aizerman/Kalman-like Conjectures for absolute stability of the Lur'e system by considering "smaller" classes of nonlinearities. We have proven the correctness of one of these conjectures.

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