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An algorithmic approach for lessening conservativeness of criteria determining absolute stability

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Abstract— **Lur'e systems are the feedback interconnection of a linear time-invariant system with a memoryless static operator. In this paper, we derive an algorithmic method to incorporate the information given by additional Lyapunov functions in order to enlarge the estimate of the domain of attraction of a Lur'e system. The methodology, limited for simplicity to quadratic Lyapunov functions, relies on the solution of a sequence of convex optimization programs. The ideas formulated in this article have general validity and can be extended to other scenarios (for example considering Lyapunov functions in the form of sum of squares.**

Keywords: Domains of Attraction, Lur'e systems, Local Quadratic Constraints.

I. Introduction

Lur'e models are defined as the feedback interconnections of a linear time-invariant system $\mathcal{L}$ with a memoryless static operator $\mathcal{N}$ [1]. Models with this structure have been widely used to describe numerous real systems. Recent examples include, but are not limited to, Atomic Force Microscopes [2], microelectromechanical systems [3], [4] and power devices [5]. Accurate knowledge about the domains of attraction of equilibria is relevant to all these applications, since it provides information about the intensity of impulsive perturbations ("shocks") that a system can tolerate and recover its current operating behavior.

Obtaining non-conservative estimates of the domain of attraction of a system is a challenging task. In particular when the dimension of the phase space is large, numerically computing the domain of attraction becomes difficult. Indeed any simulation based approach has to compute the solution of the nonlinear system for every initial condition of interest in the phase space. Here the number and choice of initial conditions selected has a direct influence on the quality of the estimate of the attraction domain.

The number of such simulations grows exponentially with the dimension of the state space [6]. An analytical, but more conservative method to estimate the domain of attraction employs the LaSalle theorem [7], [8] where Lyapunov functions need to satisfy only local conditions.

Many methods for establishing global asymptotic stability of Lur'e systems have utilized Linear Matrix Inequalities (LMI's) as a means to construct Lyapunov functions that satisfy global properties. The LMI based approach can be modified to determine Lyapunov functions that satisfy stability related properties only locally given the nonlinearity. Thus determined Lyapunov functions can then be used with LaSalle's theorem to provide less conservative estimates of domain of attraction [7]. The LMI based appoach also facilitates the use of multiple Lyapunov functions.

The combination of multiple Lyapunov functions to improve the estimate of the domain of attraction was proposed in [9]. A specialization of the techniques of [9] to Lur'e systems was advanced in [10] by exploiting the specific structure of this class of systems and using LMIs to test multiple candidate Lyapunov functions. However the results derived in [10], as in [9], are only sufficient conditions to determine if a certain ellipsoid is in the attraction domain of an equilibrium and no algorithmic or systematic method to obtain the estimates was suggested. More systematic methods were proposed in [11] where an optimization technique based on LMI's is proposed, based on the computation of the largest "contractively invariant ellipsoid" of a fixed shape containing the equilibrium. An ellipsoid is contractively invariant if the derivative of its related quadratic form is negative along the system trajectories. The authors combine multiple estimates of attraction domains by proving that the convex hull of contractively invariant ellipsoids is an invariant contractive set contained in the domain of attraction. However contractively invariant ellipsoids and their convex hull can still lead to a conservative estimate of the attraction domain. This is due to the strong requirement of contractive invariance: the ellipsoid must be an invariant set and also the derivative of the Lyapunov function along the trajectories must be strictly negative.

In this paper, we obtain results that are more specialized than [9]. Indeed, we fully exploit the structure of Lur'e systems and rely on numerical methods for the determination of Lyapunov functions. We also provide more appliable criteria than the ones provided in [10] by deriving an algorithmic and systematic methodology for the estimate of domains of attraction. At the same time, compared to other methods, our technique to combine Lyapunov functions has relatively mild requirements on the regions where the time derivative is required to be negative definite. Thus it has the potential of providing significantly less conservative estimates.

The conservativeness of estimates of the domain of attraction of equilibrium points is reduced by iteratively building the domain of attraction. The main concept exploits the fact that a particular set is part of the attraction domain if all the trajectories starting from the set reach the a prior estimate of the attraction domain in a finite time. This concept provides a method to iteratively improve the attraction domain estimate without limiting the analysis to contractively invariant sets.

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As a main contribution, we obtain an algorithmic method for the estimate of the attraction domain of an equilibrium.

The paper is organized in the following manner. In Section II we introduce some preliminary notions while in Section III we develop the main contribution of the paper by casting the problem of determining an estimate of an attraction domain in terms of a sequence of convex optimization programs.

II. Preliminary definitions and results

We start introducing certain required preliminary notions and results. We extend the concept of distance between two points to the distance of a point from a set.

Definition 1: Given a space X with a metric d, we define the distance between a point $x \in X$ and a set $A \subseteq X$ as

$$d(x, A) := \inf\{d(x, y) | y \in A\}. \tag{1}$$

We also define the neighborhood of A with radius ε as

$$\mathcal{I}(A, \varepsilon) := \{x \in X | d(x, A) < \varepsilon\}. \tag{2}$$

We introduce the concepts of flow, positively invariance, attracting set and attractor.

Definition 2: Consider a system $\mathcal{S}$ described by

$$\begin{array}{l} \dot{x} = f(x) \\ x(0) = x_0 \end{array} \tag{3}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ "regular enough" in order to guarantee the uniqueness of solutions. Let $x(t) := \phi(t, x_0)$ denote the solution (flow) to (3). We introduce the following definitions.

- $A \subseteq X$ is a *Positively Invariant* (PI) set for $\mathcal{S}$ if for any $x_0 \in A$
$$x(t) = \phi(t, x_0) \in A \qquad \text{for all } t > 0.$$
- A is an *attracting set* for $\mathcal{S}$ if, for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $T = T(\varepsilon) < \infty$ such that
$$x_0 \in \mathcal{I}(A, \delta) \text{ implies } x(t) = \phi(t, x_0) \in \mathcal{I}(A, \varepsilon) \text{ for all } t > T. \tag{4}$$
- A is an *attractor* if it is an attracting set that does not contain any other proper attracting subset.

The following definition introduces the concept of domain of attraction.

Definition 3: Given an attracting set A, we define its domain of attraction as

$$\begin{array}{l} \mathcal{D}(A) := \{x_0 \in \mathbb{R}^n | \\ \forall \varepsilon > 0 \ \exists T = T(\varepsilon, x_0), t > T \text{ implies } \phi(t, x_0) \in \mathcal{I}(A, \varepsilon)\} \end{array}$$

For the level sets of a Lyapunov function we use the following notation.

Definition 4: Given a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ we define

$$\Omega_V(H) := \{x \in \mathbb{R}^n : V(x) \le H\}.$$

In the case of a quadratic function $V(x) = x^T P x$, defined by the positive definite matrix P, we adopt the notation

$$\mathcal{E}_P(H) := \Omega_V(H). \tag{5}$$

The quadratic form defined by a matrix Σ will be denoted as the function $\sigma_\Sigma(\cdot, \cdot)$.

Definition 5: Given a matrix Σ, we define the quadratic form σ_Σ

$$\sigma_\Sigma(y, u) = \begin{pmatrix} y \\ u \end{pmatrix}^T \Sigma \begin{pmatrix} y \\ u \end{pmatrix}.$$

We also say that the matrix Σ is the multiplier of the quadratic form.

The following lemma establishes a bound on the output Cx when x is restricted to the ellipsoid $\mathcal{E}_p(H)$.

Lemma 2.1: Let us consider a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$, and a vector $C \in \mathbb{R}^{1\times n}$. It holds that

$$\max_{x \in \mathcal{E}_P(H)} \|Cx\|_2^2 = HCP^{-1}C^T \tag{6}$$

Lemma 2.2: Assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and that $\Omega_V(H)$ is a compact set for any H. Then, for every $\varepsilon > 0$ there exists $H_\varepsilon > H$ such that $x \in \Omega_V(H_\varepsilon)$ implies that $d(x, \Omega_V(H)) \le \varepsilon$.

Proof: By contradiction there exists $\varepsilon > 0$ such that for any $H_\varepsilon > H$ it is possible to find a $x \in \Omega_V(H_\varepsilon)$ with $d(x, \Omega_V(H_\varepsilon)) \ge \varepsilon$. Consider the sequence $H_n = H + 1/n$, for n positive and integer, and the associated sequence $x_n \in H_n$ such that $d(x_n, \Omega_V(H)) \ge 1/n$. Since $\Omega_V(H + 1)$ is a compact set, there exists a subsequence x_{k_n} converging to $\hat{x} \in \Omega_V(H + 1)$. By the continuity of V we have that

$$V(\hat{x}) = \lim_{n\to+\infty} V(x_{k_n}) \le \lim_{n\to+\infty} \left(H + \frac{1}{k_n}\right) = H \tag{7}$$

implying that $\hat{x} \in \Omega_V(H)$. On the other hand we have

$$0 = d(\hat{x}, \hat{x}) = \lim_{n\to+\infty} d(x_{k_n}, \hat{x}) \ge \liminf_{n\to+\infty} d(x_{k_n}, \Omega_V(H)) \ge \varepsilon > 0.$$

which is a contradiction. ■

The following theorem employs a scalar function V, defined on the state space, for the definition of positively invariant and attracting sets.

Theorem 2.3: Consider a dynamical system as in (3) and a scalar function $V : \mathbb{R}^n \to \mathbb{R}, V \in C^1(\mathbb{R})$ such that

$$x \in \Omega_V(\overline{H}) \backslash \Omega_V(\underline{H}) \text{ implies that } \frac{\partial V}{\partial x} \cdot f(x) < 0 \tag{8}$$

for some $\overline{H} > \underline{H}$ and assume that $\Omega_V(H)$ is compact for every $\underline{H} < H < \overline{H}$. Then

- $\Omega_V(H)$ is positively invariant
- $\Omega_V(\underline{H})$ is an attracting set
- $\Omega_V(H) \subseteq \mathcal{D}(\Omega_V(\underline{H}))$.

Proof: First, let us prove that $\Omega_V(H)$ is positively invariant. Fix $\varepsilon > 0$ and consider $x_0 \in \Omega_V(H)$. By contradiction, there exists t_3 such that $H < V(x(t_3)) := H^* < \overline{H}$. Consider

$$t_1 := \sup\{t | 0 < t < t_3, V(x(t)) \le H\} \tag{9}$$
$$t_2 := \inf\{t | t_1 < t < t_3, H^* \le V(x(t)) \le \overline{H}\} \neq t_1. \tag{10}$$

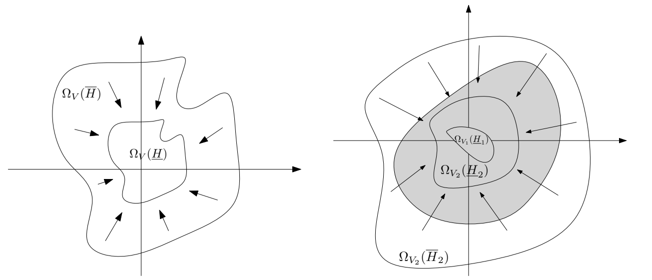
Since $V(x(t))$ is a continuous function the sets are not empty and the definitions are well-posed. Note also that $t_1 < t < t_2$ implies $H < V(x(t)) < H^*$. By the mean value theorem [12], we have that

$$0 < V(x(t_2)) - V(x(t_1)) = \int_{t_1}^{t_2} \dot{V}(x(t))dt = \dot{V}(x(\hat{t}))(t_2 - t_1)$$

for some $t_1 < \hat{t} < t_2$, but this contradicts $\dot{V}(x(\hat{t})) < 0$.

Now, we will show that any trajectory with $x_0 \in \Omega_V(H)$ is attracted to $\Omega_V(\underline{H})$. From Lemma 2.2 we know that, given $\varepsilon > 0$, there exists $H_\varepsilon > \underline{H}$ such that

$$x \in \Omega_V(H_\varepsilon) \Rightarrow d(x, \Omega_V(\underline{H})) < \varepsilon. \tag{11}$$



(a) (b)

Fig. 1. (a) An intuitive representation of a "contractive shell" in the phase space. (b) An intuitive phase space representation of Theorem 2.4. The gray area represents the region where $\dot{V}_1$ is negative.

Fig. 2.

We now assert that given $x_0 \in \Omega_V(H)$, there exists $t > t_0$ such that $x(t) \in \Omega_V(H_\varepsilon)$. Assume by contradiction that it does not exist. This means that $V(x(t)) > H_\varepsilon$ for all t. Then we have

$$dV/dt \leq -r := \max\{\dot{V}(x(t)) | H_\varepsilon \leq V(x(t)) \leq H\} < 0. \quad (12)$$

Integrating both sides, we find a contradiction, since $V(x(t))$ should diverge to $-\infty$. Therefore there exists $t > t_0$ such that $x(t) \in \Omega_V(H_\varepsilon)$. The fact that $\mathcal{E}_P(\underline{H})$ is an attracting set follows from the positively invariance of $\mathcal{E}_P(H_\varepsilon)$. ■

The key property in Theorem 2.3 is to have a negative time derivative of V on the set $\Omega_V(\overline{H}) \backslash \Omega_V(\underline{H})$ which can be interpreted as a "shell" in the phase space. Then, Theorem 2.3 states that such a shell is "contractive" in the sense that all trajectories in it approach its inner surface $\{x : V(x) = \underline{H}\}$. A graphical representation of this intuition is given in Figure 1. Note that Theorem 2.3 also provides a generalization of the classical Lyapunov theorem for asymptotic stability when $\underline{H} = 0$, $V(x) > 0$ for any $x \neq 0$.

The following theorem provides a method to include level sets described by a scalar function V_2 into the attraction domain of attracting sets described by a different scalar function V_1.

Theorem 2.4: Consider a dynamical system as in (3) and two scalar functions $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}, V_1, V_2 \in C^1(\mathbb{R}^n)$. Assume that

- $x \in \Omega_{V_2}(\overline{H}_2) \backslash \Omega_{V_2}(\underline{H}_2)$ implies $\frac{\partial V_2(x)}{\partial x} \cdot f(x) < 0$
- $x \in \Omega_{V_2}(\underline{H}_2) \backslash \Omega_{V_1}(H_1)$ implies $\frac{\partial V_1(x)}{\partial x} \cdot f(x) < 0$
- $\Omega_{V_1}(H_1)$ is an attracting set

for some $\overline{H}_2 > \underline{H}_2$ and H_1. Then, for any $\overline{H}_2 > H_2 > \underline{H}_2$,

$$\Omega_{V_2}(H_2) \subseteq \mathcal{D}(\Omega_{V_1}(H_1)). \quad (13)$$

Proof: Given H_2 and $\varepsilon_2 > 0$, from Lemma 2.3 we know that any solution $x(t)$ with initial condition in $\Omega_{V_2}(H_2)$ reaches the positively invariant set $\Omega_{V_2}(\underline{H}_2 + \varepsilon_2)$ at a time t_1. By the continuity of $\dot{V}_1(x(t))$ and Lemma 2.2 it is possible to choose ε_2 small enough, such that $\dot{V}_1(x(t)) < 0$ for any $x(t) \in \Omega_{V_2}(\underline{H}_2 + \varepsilon_2) \backslash \Omega_{V_1}(H_1)$. Now, we will prove that $x(t) \to \Omega_{V_1}(H_1)$ as $t \to +\infty$. By contradiction, assume that $x(t)$ is not attracted by $\Omega_{V_1}(H_1)$. Thus, there is a $\varepsilon_1 > 0$ such that $d(x(t), \Omega_{V_1}(H_1)) \geq \varepsilon_1$ for any t implying that $V(x(t)) > H_1$ for any t. Since $\Omega_{V_2}(\underline{H}_2 + \varepsilon_2)$ is a positively invariant set we have that $\dot{V}(x(t)) < 0$ for any $t > t_1$. Thus, for some $\eta > 0$

$$V(x(t)) - V(x(t_1)) =$$
$$= \int_{t_1}^{t} \dot{V}(x(t))dt = \dot{V}(x(\hat{t}))(t - t_1) < -\eta(t - t_1) \qquad \text{for } t > t_1.$$

This is a contradiction because it implies that $V(x(t))$ diverges to $-\infty$. ■

Theorem 2.4 provides a method to enlarge the estimate of the attraction domain of the attracting set $\Omega_{V_1}(H_1)$ using level sets of a different Lyapunov function V_2. An intuitive graphical representation is provided in Figure 1. It is important to stress that the set $\Omega_{V_2}(\overline{H}_2)$ is not required to be contractively invariant in order to be added to the estimate of the attraction domain.

III. Main Theoretical Results

Definition 6: Consider a class of SISO nonlinearities $\mathcal{N}$ and a symmetric matrix $\Sigma \in \mathbb{R}^{2\times 2}$, such that $\det(\Sigma) < 0$. We define the *Bias Function* $M_\Sigma : \mathbb{R}^+ \to \mathbb{R}^+$ of $\mathcal{N}$ with respect to Σ in the following manner

$$M_\Sigma(Y) := -\inf_y \left\{ \begin{pmatrix} n(y) \\ y \end{pmatrix}^T \Sigma \begin{pmatrix} n \\ y \end{pmatrix}, 0 \right\} \quad (14)$$

$$\text{subject to} \quad (15)$$

$$y^2 \leq Y^2 \quad (16)$$

$$n(\cdot) \in \mathcal{N} \quad (17)$$

Observation 1: The Bias Function $M_\Sigma(Y)$ is monotonically nondecreasing in Y.

The bias function defined by a multiplier Σ generalizes the concept of static Local Quadratic Constraints (LQC) as defined in [13]. Indeed, the class of nonlinear functions $\mathcal{N}$ satisfies the LQC with multiplier Σ if the associated Bias Function is identically zero. The introduction of the Bias Function M_Σ allows the functions in $\mathcal{N}$ to violate, at least locally, the LQC. In the Appendix we show how it is possible to practically compute a Bias Function for a class of nonlinearities in a sector delimited by linear spline functions.

We now define a feasibility problem.

Definition 7: Consider $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $\Sigma \in \mathbb{R}^{2\times 2}$ partitioned as

$$\Sigma = \begin{pmatrix} Q & S \\ S^T & R. \end{pmatrix} \quad (18)$$

and $M_\Sigma(\cdot)$, a monotonic non-negative function on the positive real line. We say that $(P, r, Y) \in FEAS(A, B, C, \Sigma, M_\Sigma, Y')$, if the following conditions are met

$$\begin{bmatrix} A^T P + PA + C^T QC + rP & PB + C^T S \\ B^T P + S^T C & R \end{bmatrix} < 0 \quad (19)$$

$$P > 0; \quad r > 0; \quad (Y')^2 \leq Y^2 \quad (20)$$

$$\frac{CP^{-1}C^T}{r} M_\Sigma(Y) < (Y')^2. \quad (21)$$

The following theorem specializes the results of Theorem 2.3 to the case of a Lur'e system.

Theorem 3.1: Let $\mathcal{S}$ be the Lur'e system described by the equation

$$\dot{x} = Ax + Bn(y) \quad (22)$$
$$y = Cx$$

where y is the system output, and $n(\cdot)$ is a nonlinear function with Bias Function $M_\Sigma(Y)$ defined by the matrix Σ. Assume that $(P, r, \hat{Y}) \in FEAS(A, B, C, \Sigma, M_\Sigma, \hat{Y}')$, for some $\hat{Y}'$ Define

$$\underline{H} := \frac{M_\Sigma(\hat{Y})}{r}; \qquad \overline{H} := \frac{\hat{Y}^2}{CP^{-1}C^T}. \quad (23)$$

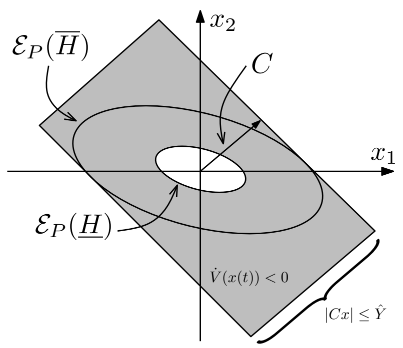


Fig. 3. Graphical interpretation of Theorem 3.1.

Then, for every $H \in (\underline{H}, \overline{H})$, it holds that

- $\mathcal{E}_P(H)$ is a positively invariant set
- $\mathcal{E}_P(\underline{H})$ is an attracting set
- $\mathcal{E}_P(H)$ is in the attraction domain of $\mathcal{E}_P(\underline{H})$
- $dV(x(t))/dt < 0$ in $L(\hat{Y}) \setminus \mathcal{E}_P(\underline{H})$ where $L(\hat{Y}) := \{x \in \mathbb{R}^n$ such that $|C^T x| \leq \hat{Y}\}$ and $V(x) = x^T P x$.

Proof: Partitioning Σ as in Equation 18, and defining $\xi := n(y)$, the derivative of $V(x(t))$ along any trajectory, for $x(t) \neq 0$, satisfies

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= x^T A^T P x + x^T P A x + \xi^T B^T P x + x^T P B \xi = \\ &= x^T A^T P x + x^T P A x + \xi^T B^T P x + x^T P B \xi + \\ &\quad + x^T C^T Q C x + 2x^T C^T S \xi + \xi^T R \xi - \sigma(y, \xi) < \\ &< -r x^T P x - \sigma_\Sigma(y, \xi)\end{aligned}$$

Observe that $dV(x(t))/dt < 0$ in $\mathcal{E}(\overline{H}) \backslash \mathcal{E}(\underline{H})$. As the hypotheses of Theorem 2.3 are satisfied, the first three assertions follow. The fourth assertion follows from the expression of the derivative of $V(t)$ by observing that it is negative if $x(t)$ is out of the ellipsoid $\mathcal{E}_P(\underline{H})$, but in the strip where $|C^T x| \leq \hat{Y}$. ■

A graphical representation of the results of Theorem 3.1 is given in Figure 3.

Theorem 3.1 suggests a way to enlarge the estimate of the attraction domain of a Lur'e system which is asymptotically stable in the origin. Assume that there exists a quadratic function $V_1 = x^T P_1 x$ that proves the local asymptotic stability of the origin and such that its derivative along the system trajectories is negative ($dV_1(x(t))/dt < 0$) in $L(\hat{Y}_1) \setminus \{0\}$, for some scalar $\hat{Y}_1 > 0$. Assume that, for a given multiplier Σ_2 with the bias function $M_{\Sigma_2}(Y)$ of the nonlinearity $n(\cdot)$, we have that $(P_2, r_2, \hat{Y}_2) \in FEAS(A, B, C, \Sigma, M_{\Sigma_2}, \hat{Y}_1)$. Then, by using Theorem 2.4, we can conclude that $\mathcal{E}_{P_2}(\hat{Y}_2^2 / C^T P_2^{(-1)} C)$ is in the attraction domain of the origin. Since Theorem 3.1 guarantees that $dV_2(x(t))/dt < 0$ in $L_2(\hat{Y}_2) \setminus \mathcal{E}_{P_2}(M(\hat{Y}_2/r_2))$, this procedure can be iterated again in order to improve the estimate of the domain of attraction with ellipsoids of a different shape. The following theorem formalizes this idea.

Theorem 3.2: Let $\mathcal{S}$ be the Lur'e system described by the equations (22) where y is the system output, and $n(\cdot)$ is a nonlinear function with Bias Functions $M_{\Sigma_k}(Y)$, $k = 1, ..., l$ for each matrix $\{\Sigma_1, ..., \Sigma_l\}$

$$\Sigma_k = \begin{pmatrix} Q_k & S_k \\ S_k^T & R_k \end{pmatrix} \qquad k = 1, ..., l. \tag{24}$$

Assume that there is a sequence $\{a_k\}_{k=1}^N$ with $a_k \in \{1, ..., l\}$ for $k = 1, ..., N$, such that

$$(P_k, r_k, \hat{Y}_k) \in FEAS(A, B, C, \Sigma_{a_k}, \hat{Y}_{k-1}) \tag{25}$$

for some scalars $0 = \hat{Y}_0 \leq \hat{Y}_1 \leq ...\hat{Y}_N$, $r_1, ..., r_N$ and some matrices $P_1, ..., P_N$. Then $\mathcal{E}_{P_k}(\hat{Y}_k^2 / C^T P_k^{(-1)} C)$ is in the attraction domain of the origin for every $k = 1, ..., N$.

Proof: Let us define, for $k = 1, ..., N$,

$$\underline{H}_k := \frac{M(\hat{Y}_k)}{r_k}; \qquad \overline{H}_k := \frac{\hat{Y}_k^2}{C P_k^{-1} C^T} \tag{26}$$

$$V_k(x) := x^T P_k x \tag{27}$$

$$L(Y) := \{x \text{ such that } |Cx| < Y\}. \tag{28}$$

The proof proceeds by induction. At the first step, for $k = 1$, the conditions of the theorem guarantee that the function $V_1(x)$ is globally positive and that $dV_1(x(t))/dt < 0$ in $L(\hat{Y}_1) \setminus \{0\}$. Thus, V_1 is a standard Lyapunov function. From Lemma 2.1, the ellipsoid $\mathcal{E}(\overline{H}_1)$ is contained in the strip $L(\hat{Y}_1)$. Using standard Lyapunov arguments we conclude that the $\mathcal{E}(\overline{H}_1)$ is in the attraction domain of the origin. At the generic step k we have that the matrix inequality condition and Lemma 2.1 guarantee that $dV_{k-1}(x(t))/dt < 0$ in $L(\hat{Y}_{k-1}) \setminus \mathcal{E}_{P_{k-1}}(\underline{H}_{k-1})$. Also, using Theorem 3.1, we have that $\mathcal{E}_{P_k}(\overline{H}_k)$ contracts in $\mathcal{E}_{P_k}(\underline{H}_k) \subseteq L(\hat{Y}_{k-1})$. Thus, by Theorem 2.4, $\mathcal{E}_{P_k}(\overline{H}_k)$ is in the attraction domain of $\mathcal{E}_{P_{k-1}}(\underline{H}_{k-1})$ which is in the attraction domain of the origin. ■

Theorem 3.2 amounts to a sufficient condition to be verified in order to add multiple ellipsoids associated with different quadratic Lyapunov functions to the domain of attraction of an equilibrium point in a Lur'e system. However, the theorem does not provide any algorithmic procedure to choose the sequence $\{a_k\}$ in order to enlarge the domain of attraction. In the following, we address this issue showing that, for a fixed set of multipliers $\Sigma_1, .., \Sigma_N$, with known associated bias functions, there is a way of choosing the sequence $\{a_k\}$ that is optimal in some sense. This optimal choice can be obtained by solving a sequence of convex optimization problems.

Let us focus on the feasibility problem $FEAS(\cdot, \cdot, \cdot, \cdot, \cdot)$. Such a feasibility problem can be decomposed in two parts. We define an optimization problem.

Definition 8: For a matrix Σ as in Equation 18, consider the optimization problem

$$w := \min_{P,r} \frac{C P^{-1} C^T}{r} \tag{29}$$

$$\text{subject to} \tag{30}$$

$$r > 0; \quad P > 0; \tag{31}$$

$$\begin{bmatrix} A^T P + PA + CQC^T + rP & PB + C^T S \\ B^T P + S^T C & R \end{bmatrix} < 0. \tag{32}$$

We use the notation $w = OPT1(A, B, C, \Sigma)$ to denote the minimum value of this program and $(P, r) = \arg OPT1(A, B, C, \Sigma)$ to denote the associated optimal solution.
We observe that program (29) can be cast as an equivalent program.

Lemma 3.3: Program (29) is equivalent to

$$w = \min_{q,P,r} \frac{q}{r} \tag{33}$$

subject to (34)

$$r > 0 \tag{35}$$

$$\begin{bmatrix} A^T P + PA + C^T QC + rP & PB + C^T S \\ B^T P + S^T C & R \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} P & C \\ C^T & q \end{bmatrix} > 0. \tag{37}$$

Proof: The equivalence is obtained by using the Schur complement. ∎

For every $r > 0$ the problem is convex. Since r is a scalar quantity, a gridding strategy in order to optimize with respect to it is not going to be computationally too expensive.

We define a second optimization problem

Definition 9: Let w be $OPT1(A, B, C, \Sigma)$ for a multiplier Σ and some matrices (A, B, C) defining a Lur'e system with nonlinearity $n(\cdot)$ and bias function $M_\Sigma(\cdot)$. We denote by $FEAS1(w, M_\Sigma(\cdot), \hat{Y})$ the set of $Y \geq 0$ such that

$$wM_\Sigma(Y) < \hat{Y}^2 \tag{38}$$

$$(\hat{Y})^2 \leq Y^2 \tag{39}$$

and we denote the maximum value of this set with

$$OPT2(w, M_\Sigma(\cdot), \hat{Y}) := \max FEAS1(w, M_\Sigma(\cdot), \hat{Y}).$$

Observation 2: Given the monotonicity of $M_\Sigma(Y)$ the problem can be efficiently solved by using a bisection strategy.

The following lemma proves that solutions to the feasibility problem $FEAS(A, B, C, \Sigma, M_\Sigma(\cdot), Y')$ can be equivalently obtained by solving $w = OPT1(A, B, C, \Sigma)$ and considering the feasibility problem $FEAS1(w, M_\Sigma(\cdot), Y')$.

Lemma 3.4: There exists $P > 0$ and $r > 0$ such that $(P, r, Y) \in FEAS(A, B, C, \Sigma, M_\Sigma(\cdot), Y')$ if and only if $Y \in FEAS1(OPT1(A, B, C, \Sigma), M_\Sigma(\cdot), Y')$.

Proof: The independent minimization of the quantity $CP^{-1}C^T/r$ in the problem OPT1 for the multiplier Σ guarantees the fulfillment of the condition $M_\Sigma(\hat{Y})CPC^T/r < \hat{Y}^2$, for the largest range of Y that is possible. At the same time, any pair P, r solving $OPT1$ provide P and r satisfying the constraints of $FEAS$. Thus, the two feasibility problems admit the same solution set. ∎

For reasons that will be clear in the following, among all the possible solution (P, r, Y), we are going to prefer the ones with larger Y, thus the ones provided by $OPT2(OPT1(A, B, C, \Sigma), M_\Sigma(\cdot), \hat{Y})$.

We provide the following algorithm.

Algorithm 1

1) Choose a set of multiplier $\Sigma_1, ..., \Sigma_\ell$ with known bias functions $M_{\Sigma_1}(\cdot)$,... ,$M_{\Sigma_\ell}(\cdot)$
2) For $i = 1, ..., \ell$ solve Program (29)
   - Let $w_i = OPT1(A, B, C, \Sigma_i)$, for $i = 1, ..., \ell$
3) Initialize $\hat{Y}_0$=0
4) Until $\hat{Y}_k > \hat{Y}_{k-1}$ (that is, as long the strip $L(\hat{Y}_k)$ can be made larger)
   - $\hat{Y}_k^{(i)}$= $OPT2(w_i, M_{\Sigma_i}(\cdot), \hat{Y}_{k-1})$
   - $\hat{Y}_k$=$\max_i$ $\hat{Y}_k^{(i)}$
   - $a_k = \arg\max_i \hat{Y}_k^{(i)}$
5) Record the last (and largest) $\hat{Y}_k$: $\hat{Y}_\infty \leftarrow \hat{Y}_k$
6) The output is $L(\hat{Y}_\infty)$

The following theorem summarizes the main contributions of the article linking them to the described algorithm for the determination of the estimates of domains of attraction.

Theorem 3.5: For every given set of multipliers, $\Sigma_1, ..., \Sigma_\ell$, Algorithm 1 provides a sequence $\hat{Y}_k$ converging to $\hat{Y}_\infty$. Every Positively Invariant set strictly in $L(\hat{Y}_\infty)$ is in the domain of attraction of the origin.

Proof: The convergence of $\hat{Y}_k$ comes from the fact that the sequence is monotonically increasing. The strip $L(\hat{Y}_\infty)$ is the union of an ellipsoid $\mathcal{E}$ that is in the attraction domain of the origin and a region where the derivative of a Lyapunov function proving the attractiveness of $\mathcal{E}$ is strictly negative. Thus, each x is a PI set strictly contained in $L(\hat{Y}_\infty)$ ∎

The following lemma provides a way to determine Positively Invariant sets in the strip $L(\hat{Y}_\infty)$.

Lemma 3.6: Consider the Lur'e systems $\dot{x} = Ax + Bn(Cx)$ where the nonlinearity $n(\cdot)$ has bias functions $M_{\Sigma_i}(\cdot)$ corresponding to the the multiplier Σ_i, for $i = 1, ..., \ell$. Consider the iterations

$$\Upsilon_0^{(i)} = \hat{Y}_\infty$$

$$\Upsilon_{j+1}^{(i)} = M_\Sigma^{-1}\left(\frac{\Upsilon_j^{(i)}}{w_i}\right)$$

for $w_i = OPT1(A, B, C, \Sigma_i)$, where $(P_i, r_i) = \arg OPT1(A, B, C, \Sigma_i)$. The sequence $\Upsilon_j^{(i)}$ is monotonically decreasing and converges to $\Upsilon^{(i)}$. Also, it holds that $\mathcal{E}(\frac{|\hat{\Upsilon}^{(i)}|^2}{CP_i^{-1}C^T})$ is a positively invariant set in $L(\hat{Y}_\infty)$.

Proof: Each function $M_{\Sigma_i}^{-1}\left(\frac{(\cdot)}{w_i}\right)$ is monotonically non-decreasing and defines a 1-st order discrete-time dynamic system, for $i = 1, ..., \ell$. Since Y_∞ corresponds to the strip $L(Y_\infty)$, we have that

$$Y_\infty \geq M_{\Sigma_i}^{-1}\left(\frac{Y_\infty}{w}\right),$$

for $i = 1, ..., \ell$. As a consequence, each sequence $\Upsilon_j^{(i)}$ converges monotonically to $\Upsilon^{(i)}$. Such a point is the largest fixed point of $M_{\Sigma_i}^{-1}\left(\frac{(\cdot)}{w}\right)$ that is less or equal to Y_∞

$$w_i M_{\Sigma_i}(\Upsilon^{(i)}) = |\Upsilon^{(i)}|^2.$$

By computing the derivative of $V_i(x) = x^T P_i x$ we find that the ellipsoid $\mathcal{E}(\frac{|\hat{\Upsilon}^{(i)}|^2}{CP_i^{-1}C^T})$ is a positively invariant set. ∎

Thus, once the strip $L(\hat{Y}_\infty)$ has been found an estimate of the domain of attraction of the Lur'e system can be computed as

$$\bigcup_{i=1}^{\ell} \mathcal{E}\left(\frac{\hat{\Upsilon}_i^2}{CP_i^{-1}C^T}\right).$$

IV. Conclusions

In this paper we have developed the idea of determining the domain of attraction of a Lur'e system by using a description of the feedback nonlinearity that is based on multiple Integral Quadratic Constraints that can be locally violated according to a Bias Function. The multipliers of the Integral Quadratic Constraints are used to compute Lyapunov functions via linear matrix inequalities, while the Bias function is used to obtain bounds on the time derivative of the Lyapunov functions along the system trajectories. The problem of estimating the domain of attraction is reduced to the solution of a sequence of convex programs. The proposed approach does not require stringent conditions such as contractively invariance for the found ellipsoids to be included in the attraction domain estimate. Thus, the conservativeness of the estimate can be significantly reduced.

Appendix: Specialization to the case of nonlinearities bounded in piecewise affine-sectors

An issue that hasn't been touched yet is how to determine appropriate bias functions to describe the feedback nonlinearities of the Lur'e system. We consider the following class of nonlinearities for which the determination of a bias function is easily achievable.

Definition 10: The class of scalar nonlinearities in the linear spline sector with boundaries $\underline{n}(\cdot)$ and $\overline{n}(\cdot)$ is defined as $\mathcal{N}(\underline{n}(\cdot),\overline{n}(\cdot)) := \{n(\cdot)| \quad y\underline{n}(y) \le yn(y) \le y\overline{n}(y)\}$, where $\underline{n}(y)$ and $\overline{n}(y)$ are two odd linear spline functions. An odd linear spline function $\hat{n}$ with node points $0 < \gamma_1 < \gamma_2 < ...$ is defined as $\hat{n}(y) = a_i + b_i y \qquad \text{for } |y| \in [\gamma_{i-1}, \gamma_i)$ for some constants a_i and b_i, with $i = 1, 2, ...$
A representation of this class of nonlinearities is given in Figure 4. Observe that there is no loss of generality if it is

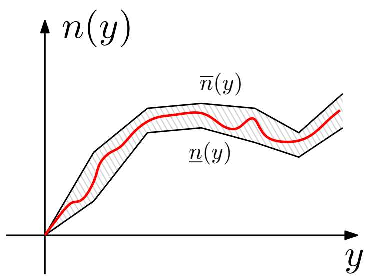


Fig. 4. Class of scalar nonlinearities in the linear spline sector with boundaries $\underline{n}(\cdot)$ and $\overline{n}(\cdot)$

assumed that the two boundary splines $\underline{n}(y)$ and $\overline{n}(y)$ have the same node points.

Lemma 4.1: Consider the class of nonlinearities $\mathcal{N}(\underline{n}(\cdot),\overline{n}(\cdot))$ with boundaries $\underline{n}(\cdot)$ and $\overline{n}(\cdot)$. Given a matrix $\Sigma \in \mathbb{R}^{2\times 2}$, such that $det(\Sigma) < 0$, the Bias Function $M(Y)$ has the form of a piecewise quadratic function.

Proof: Consider the optimization problem defining $M_\Sigma(Y)$

$$M_\Sigma(Y) := -\inf_{y,n}\left\{\left(\begin{array}{c} n \\ y \end{array}\right)^T \Sigma_s \left(\begin{array}{c} n \\ y \end{array}\right), 0\right\}$$
$$\text{subject to}$$
$$|y| \le Y; \qquad [n - \underline{n}(y)][\overline{n}(y) - n] \ge 0.$$

Since Σ_s has negative determinant, the quadratic form $\sigma_{\Sigma_s}(n, y)$ is concave in n. As a consequence, the inequality constraint $[n - \underline{n}(y)][\overline{n}(y) - n] \ge 0$ can be replaced with the equality constraint $[n - \underline{n}(y)][\overline{n}(y) - n] = 0$. In other words, for a fixed y, the maximum of the quadratic form defined by Σ is achieved for $n = \overline{n}(y)$ or $n = \underline{n}(y)$. With no loss of generality, assume that $\overline{n}(\cdot)$ and $\underline{n}(\cdot)$ have the same node points $0, \gamma_1, ...$ and Let I_i be the interval $[\gamma_{i-1}, \gamma_i]$, for $i = 1, ...$ Let us determine the functions

$$\overline{\phi}_i(y) := \max_{y_{i-1}\le y} \sigma(\overline{n}(y), y); \quad \underline{\phi}_i(y) := \max_{y_{i-1}\le y} \sigma(\underline{n}(y), y). \qquad (40)$$

for $y \in I_i$. We cast the optimization problem

$$\overline{\phi}_i(y) := -\inf_{y'}\left\{\left(\begin{array}{c} ay' + b \\ y' \end{array}\right)^T \Sigma_s \left(\begin{array}{c} ay' + b \\ y' \end{array}\right), 0\right\}$$
$$\text{subject to} \quad \gamma_{i-1} \le y' \le y$$

and notice that the optimum $\overline{\phi}_i(y)$ can be achieved only at the border ($y' = \gamma_{i-1}$, $y' = y$) or in the interior of the interval (γ_{i-1}, y). By differentiating, the extremal point is in the interior of (γ_{i-1}, y) if and only if

$$y_i^{(ex)} = -\frac{b\,S + b\,a\,Q}{2\,a\,S + R + a^2\,Q} \in (\gamma_{i-1}, y)$$

and in such a case the value assumed by the quadratic function is

$$\sigma(ay_i^{(ex)} + b, y_i^{(ex)}) = -\frac{b^2\,S^2 - b^2\,Q\,R}{2\,a\,S + R + a^2\,Q}.$$

Thus, we have that, for $y \in I_i$

$$\overline{\phi}_i(y) = \max\{0, \sigma(a\gamma_{i-1} + b, \gamma_{i-1}),$$
$$\sigma(ay + b, y), 1(y - y_i^{(ex)})\sigma(ay_i^{(ex)} + b, y_i^{(ex)})\}$$

where $1(\cdot)$ is the Heaviside step function. The function $\overline{\phi}_i(y)$ is evidently a nondecreasing quadratic spline. In an analogous manner it is possible to find that $\underline{\phi}_i(y)$ is a continuous quadratic spline, as well. The bias function $M(y)$ can now be determined iteratively by computing its restrictions $M^{(i)}(y)$ in the intervals I_i, for $i = 1, 2, ...$

$$M^{(i)} = \max\{M^{(i-1)}(\gamma_{i-1}), \overline{\phi}_i(y), \underline{\phi}_i(y)\}$$

initializing the computations with $M^{(0)} = 0$. Observe that is again a nondecreasing piecewise quadratic spline. ∎

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