SIAM J. Control Optim.
Vol. 37, No. 6, pp. 1865–1873

SOLUTION OF MIMO $\mathcal{H}_2/\ell_1$ PROBLEM WITHOUT ZERO INTERPOLATION*

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Abstract. In this paper we present a methodology to obtain converging lower and upper bounds to a multiple objective problem where an $\mathcal{H}_2$ performance objective is minimized subject to an ℓ_1 constraint. This methodology gives a computationally efficient synthesis procedure by avoiding many of the problems that are present in methods that employ zero interpolation techniques to characterize achievable closed loop maps.

Key words. robust control, ℓ_1 optimization, discrete time

AMS subject classifications. 49N05, 49N10, 49N15, 49N35, 93C55

PII. S0363012998332573

1. Notation.

$(X, \|\| \cdot \|\|)$	The set X endowed with the norm $\|\cdot\|$.
R	The real number system.
R^n	The n-dimensional Euclidean space.
$\hat{x}(\lambda)$	The λ transform of a right sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{\infty} x(k)\lambda^k$.
ℓ	The vector space of sequences.
$\ell^{m\times n}$	The vector space of matrix sequences of size $m \times n$.
ℓ_1	The Banach space of right sided absolutely summable real sequences with the norm given by $\|x\|_1 := \sum_{k=0}^{\infty} \|x(k)\|$.
$\ell_1^{m\times n}$	The Banach space of matrix valued right sided real sequences with the norm $\|x\|_1 := \max_{1\le i\le m} \sum_{j=1}^{n} \|x_{ij}\|_1$, where $x \in \ell_1^{m\times n}$ is the matrix (x_{ij}) and each x_{ij} is in ℓ_1.
c_0	The subspace of ℓ_∞ with elements x that satisfy $\lim_{k\to\infty} x(k) = 0$.
ℓ_2	The Banach space of right sided square summable sequences with the norm given by $\|x\|_2 := (\sum_{k=0}^{\infty} x(k)^2)^{1/2}$.
$\mathcal{H}_2$	The isometric isomorphic image of ℓ_2 under the λ transform $\hat{x}(\lambda)$ with the norm given by $\|\hat{x}(\lambda)\|_2 = \|x\|_2$.
P_n	The truncation operator on the space of sequences; $P_n(x(0)x(1)\ldots) = (x(0)\ x(1)\ldots x(n)0\ 0\ldots)$.
X^*	The dual space of the Banach space X. $\langle x, x^*\rangle$ denotes the value of the bounded linear functional x^* at $x \in X$.
$W(X^*, X)$	The weak star topology on X^* induced by X.
$\mathcal{D}$	The closed unit disc in the complex plane.
A'	The transpose of the matrix A.

*Received by the editors January 2, 1998; accepted for publication (in revised form) December 23, 1998; published electronically October 8, 1999. This research was supported by National Science Foundation grants ECS-9733802, ECS-9457485, ECS-9632820, and ECS-9457485; AFOSR grant F49620-97-1-0168; and a SPRIG grant by Iowa State University.

http://www.siam.org/journals/sicon/37-6/33257.html

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2. Introduction. Consider Figure 3.1, where a generalized linear time-invariant plant G is depicted with a controller K connected in a feedback arrangement. Also shown are exogenous inputs w_1 and w_2 and regulated variables z_1 and z_2. The controller determines the control effort based on the measured variable y. A number of control objectives can be cast into the setup shown in Figure 3.1. The controller's objective is to stabilize the system and enhance its performance with respect to external inputs. The nature of the exogenous inputs and the particular characteristics of the regulated variables determine the appropriate systems' "measures" that must be used to quantify performance. The $\mathcal{H}_2$, $\mathcal{H}_\infty$, and ℓ_1 measures are frequently used as objectives for control synthesis. The $\mathcal{H}_2$ measure of a system is the variance of the regulated output when the exogenous input is modeled as white noise, whereas the $\mathcal{H}_\infty$ measure is the maximum energy (the ℓ_2 norm) of the regulated variable when the exogenous signal is any signal with unit energy. Another measure of a system is the ℓ_1 measure, which is the maximum magnitude of the regulated variable when the exogenous input is allowed to be any signal with unity maximum magnitude. The design of controllers which minimize these measures have been extensively studied [1].

It is known that performance with respect to a measure (usually the ℓ_1, $\mathcal{H}_2$, or the $\mathcal{H}_\infty$ norm of the closed loop) is not a guarantee of good performance with respect to some other measure. For example, in Figure 3.1 a control design objective may be stated in terms of the ℓ_1 performance between w_1 and z_1 and the minimization of the variance of z_2 with w_2 as white noise. A standard ℓ_1 solution or a standard $\mathcal{H}_2$ solution might fail to address such multiobjective concerns. Motivated by such issues researchers have focused their attention on multiobjective problems that incorporate two or more different measures in their problem definition.

An important class of problems that falls under this category is the one that incorporates time domain objectives and the $\mathcal{H}_2$ objective. In [5] it was shown that the single-input single-output problem of minimizing the ℓ_1 norm of the closed loop subject to an $\mathcal{H}_2$ constraint can be solved via finite-dimensional convex programming. In a related result it was shown in [6] that problems that incorporate the ℓ_1 norm and the $\mathcal{H}_2$ norm of the various transfer functions in a closed loop map of a multiple-input–multiple-output (MIMO) system can be formulated and solved via finite-dimensional quadratic programming. In [2] a method based on positive cones was used to minimize an $\mathcal{H}_2$ measure of the closed loop map subject to an ℓ_1 constraint.

Most approaches that incorporate the ℓ_1 objective characterize the achievability of a closed loop map through a stabilizing controller by using zero interpolation conditions on the closed loop map [1]. Computation of the zeros and the zero directions can be done by finding the nullspaces of certain Toeplitz-like matrices. Once the optimal closed loop map is determined, the task of determining the controller still remains. The closed loop map needs to satisfy the zero interpolation conditions exactly to guarantee that the correct cancellations take place while solving for the controller. However, numerical errors are always present and there exists a need to determine which poles and zeros cancel. These difficulties exist even for the pure MIMO ℓ_1 problem when zero interpolation methods are employed. However, recently in [3] it was shown that converging upper and lower bounds can be determined to the ℓ_1 problem by solving an auxiliary problem that does not require zero interpolation and thus avoids the above mentioned problems.

In this paper we formulate an auxiliary problem to the one given in [6]. We show that converging upper and lower bounds can be computed without zero interpolation

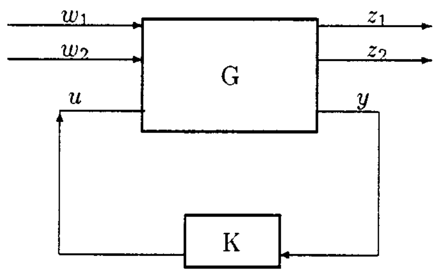

FIG. 3.1. *Closed loop system.*

for the most general MIMO case. This provides an attractive method for solving the problem.

The paper is organized as follows. In section 3 we present the preliminary material. In section 4 we formulate the multiobjective control problem and an auxiliary problem which regularizes it. In section 5 we solve the problem through lower and upper bound approximations. Section 6 contains conclusions and future directions.

3. Preliminaries.

In this section we present a brief summary of mathematical and system results that will be utilized later in the paper.

3.1. System preliminaries. Consider the system in Figure 3.1, where $w := (w_1\ w_2)'$ is the exogenous disturbance, $z := (z_1\ z_2)'$ is the regulated output, u is the control input, and y is the measured output. In feedback control design the objective is to design a controller K such that with $u = Ky$ the resulting closed loop map Φ_{zw} from w to z is stable (see Figure 3.1) and satisfies certain performance criteria. In [7] a parametrization of all closed loop maps that are achievable via stabilizing controllers was derived. A good treatment of the issues involved is presented in [1]. Following the notation used in [1] we denote by n_u, n_w, n_z, and n_y the number of control inputs, exogenous inputs, regulated outputs, and measured outputs, respectively, of the plant G. We represent by Θ the set of impulse responses of closed loop maps of the plant G that are achievable through stabilizing controllers. $H \in \ell_1^{n_z \times n_3}$, $U \in \ell_1^{n_z \times n_u}$, and $V \in \ell_1^{n_y \times n_w}$ characterize the Youla parametrization of the plant [7]. The following theorem follows from the Youla parametrization.

THEOREM 3.1. *$\Theta = \{\Phi \in \ell_1^{n_z \times n_w}$: there exists a $Q \in \ell_1^{n_u \times n_y}$ with $\hat{\Phi} = \hat{H} - \hat{U}\hat{Q}\hat{V}\}$, where $\hat{f}$ denotes the λ transform (see [1]) of f.*

If Φ is in Θ we say that Φ is an *achievable* closed loop map. We assume throughout the paper that $\hat{U}$ has normal rank n_u and $\hat{V}$ has normal rank n_y. There is no loss of generality in making this assumption [1].

3.2. Mathematical preliminaries. In this subsection we summarize the mathematical results that are relevant to the paper. An exhaustive treatment of the subject matter of this subsection is given in [4]. The reader may skip this part of the paper and refer to this subsection whenever required.

DEFINITION 3.2 (convex sets). *A subset Ω of a vector space X is said to be convex if for any two elements c_1 and c_2 in Ω and for a real number λ with $0 < \lambda < 1$ the element $\lambda c_1 + (1-\lambda)c_2 \in \Omega$.*

LEMMA 3.3. *Let Ω be a convex subset of a Banach space X and $f : \Omega \to R$ be strictly convex. If f achieves its minimum on Ω, then the minimizer is unique.*

THEOREM 3.4 (Banach–Alaoglu). *Let $(X, \|\cdot\|_x)$ be a normed vector space with X^* as its dual. The set*

$$B^* := \{x^* \in X^* : \|x^*\| \leq M\} \tag{3.1}$$

is compact in the weak-star topology for any $M \in R$.

LEMMA 3.5. *Suppose ϕ_k is a sequence in $\ell_2 \phi \in \ell_2$ and $\phi_k(t) \to \phi_0(t)$ for all t. Suppose also that $\|\phi_k\|_2 \nearrow \|\phi_0\|_2$. Then $\|\phi_k - \phi_0\|_2 \to 0$.*

Proof. Given $\epsilon > 0$ choose n such that

$$\|(I - P_n)\phi_0\|_2^2 \leq \min\left\{\frac{\epsilon}{8}, \left(\frac{\epsilon}{8(\|\phi_0\|_2 + 1)}\right)^2\right\}, \tag{3.2}$$

where P_n is the truncation operator. As $\phi_k(t) \to \phi_0(t)$ we can choose K_2 such that

$$k > K_2 \Rightarrow \|P_n(\phi_k - \phi_0)\|_2^2 \leq \frac{\epsilon}{4}. \tag{3.3}$$

We know that $\|P_n(\phi_k)\|_2 \to \|P_n(\phi_0)\|_2$ as $k \to \infty$. From the above and the fact that $\|\phi_k\|_2 \to \|\phi_0\|_2$ it follows that we can choose K_3 such that

$$k > K_3 \Rightarrow |\, \|(I - P_n)\phi_k\|_2^2 - \|(I - P_n)\phi_0\|_2^2 \,| \leq \frac{\epsilon}{4}. \tag{3.4}$$

Let $K \geq \max\{K_2, K_3\}$; then $k > K$ implies

$$\begin{aligned}
\|\phi_k - \phi_0\|_2^2 &= \|P_n(\phi_k - \phi_0)\|_2^2 + \|(I - P_n)(\phi_k - \phi_0)\|_2^2 \\
&\leq \frac{\epsilon}{4} + \|(I - P_n)(\phi_k)\|_2^2 + \|(I - P_n)(\phi_0)\|_2^2 + 2\sum_{t=n+1}^{\infty} |\phi_k(t)|\,|\phi_0(t)| \\
&\leq \frac{\epsilon}{4} + 2\|(I - P_n)(\phi_0)\|_2^2 + \frac{\epsilon}{4} + 2\sum_{t=n+1}^{\infty} |\phi_k(t)|\,|\phi_0(t)| \\
&\leq \frac{\epsilon}{4} + 2\frac{\epsilon}{8} + \frac{\epsilon}{4} + 2\|(I - P_n)\phi_k\|_2\,\|(I - P_n)\phi_0\|_2 \\
&\leq \frac{\epsilon}{4} + 2\frac{\epsilon}{8} + \frac{\epsilon}{4} + 2\|\phi_0\|_2 \frac{\epsilon}{8(\|\phi_0\|_2 + 1)} \\
&\leq \epsilon. \qquad \square
\end{aligned}$$

4. Problem statement. Let H, U, and V in the Youla parametrization be partitioned into submatrices according to the equation

$$H - U * Q * V = \begin{pmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{pmatrix} - \begin{pmatrix} U^1 \\ U^2 \end{pmatrix} * Q * (V^1 \quad V^2),$$

where $Q \in \ell_2^{n_u \times n_y}$. The problem statement is as follows: *Given a plant G and a positive real number γ solve the problem*

$$\inf_{Q \in \ell_1^{n_u \times n_y}} \|H^{22} - U^2 * Q * V^2\|_2^2$$

subject to

$$\|H^{11} - U^1 * Q * V^1\|_1 \leq \gamma.$$

We denote by μ the optimal value obtained from the above problem.

Now we define an auxiliary problem that is intimately related to the one defined above. The auxiliary problem statement is: *Given a plant G and positive real numbers α and γ solve the problem*

$$\begin{aligned} &\inf_{Q\in \ell_1^{n_u\times n_y}} \|H^{22}-U^2*Q*V^2\|_2^2 \\ \textit{subject to} \quad & \\ &\|H^{11}-U^1*Q*V^1\|_1 \le \gamma, \\ &\|Q\|_1 \le \alpha. \end{aligned} \tag{4.1}$$

The optimal value obtained from the above problem is denoted by ν.

Note that in the problem statement of μ the allowable Youla parameter Q, which is in $\ell_1^{n_u\times n_y}$, needs to satisfy $\|H^{11}-U^1*Q*V^1\|_1 \le \gamma$. Therefore, it follows that $\|U^1*Q*V^1\|_1 = \|H^{11}-U^1*Q*V^1-H^{11}\|_1 \le \|H^{11}-U^1*Q*V^1\|_1+\|H^{11}\|_1 \le \|H^{11}\|_1+\gamma$. Suppose $\hat{U}^1$ has more rows than columns and $\hat{V}^1$ has more columns than rows and both have full normal rank. Thus the left inverse of $\hat{U}^1$ exists (given by $(\hat{U}^1)^{-1}$) and the right inverse of $\hat{V}^1$ exists (given by $(\hat{V}^1)^{-r}$). Further, suppose that $\hat{U}^1$ and $\hat{V}^1$ have no zeros on the unit circle. Then it can be shown (see Lemma 4.2 and the discussion below) that there exists a β (which depends only on $(\hat{U}^1)^{-1}$ and $(\hat{V}^1)^{-r}$) such that $\|Q\|_1 \le \beta$. Thus if in the auxiliary problem we choose $\alpha \ge \beta$, then the constraint $\|Q\|_1 \le \alpha$ is redundant in the problem statement of ν and we get $\mu=\nu$. The extra constraint in the problem statement of ν is useful because it regularizes the problem (as will be seen). The following lemma is useful in estimating β.

LEMMA 4.1 (see [1]). *Let* $\mathrm{int}(\mathcal{D})$ *denote the interior of the unit disc in the complex plane. Given a function* $\hat{f}(\,.\,)$ *of the complex variable* λ *analytic in* $\mathrm{int}(\mathcal{D})$*, then* $\frac{d^k f}{d\lambda}|_{\lambda_0}=0$ *for* $k=0,1,\ldots,(\sigma-1)$ *and* $\lambda_0\in\mathrm{int}(\mathcal{D})$ *if and only if* $\hat{f}(\lambda)=(\lambda-\lambda_0)^\sigma \hat{g}(\lambda)$*, where* $g(\,.\,)$ *is analytic in* $\mathrm{int}(\mathcal{D})$.

LEMMA 4.2. *Let* ϕ *be an element of* ℓ_1 *such that* $\|\phi\|_1\le\gamma$ *for some* $\gamma>0$*. Let* $\hat{\phi}(\lambda)$ *be the* λ *transform of* ϕ*. Suppose,* $\hat{\phi}(\lambda)$ *has a zero at* $\lambda=a$ *with* $|a|<1$*. If* $\hat{\phi}(\lambda)=(\lambda-a)\hat{\psi}(\lambda)$*, then* $\|\hat{\psi}(\lambda)\|_1\le\frac{\gamma}{1-|a|}$.

Proof. As $\|\phi\|_1\le\gamma$ it follows that $\|(\lambda-a)\hat{\psi}(\lambda)\|_1\le\gamma$. This implies that $\sum_{t=-\infty}^{\infty}|\psi(t-1)-a\psi(t)|\le\gamma$. This is true only if $\sum_{t=-\infty}^{\infty}(|\psi(t-1)|-|a|\,|\psi(t)|)\le\gamma$, which implies that $\|\psi\|_1(1-|a|)\le\gamma$. Therefore, $\|\psi\|_1\le\frac{\gamma}{1-|a|}$. □

In the discussion above we have obtained an upper bound on the one norm of $R:=U^1*Q*V^1$ for any $Q\in\ell_1^{n_u\times n_y}$, which satisfies $\|H^{11}-U^1*Q*V^1\|_1\le\gamma$. As U^1 and V^1 are left and right invertible it follows that $\hat{Q}=(\hat{U}^1)^{-l}\hat{R}(\hat{V}^1)^{-r}$. As Q is in $\ell_1^{n_u\times n_y}$ it is true that $\hat{R}$ interpolates the unstable poles of $(\hat{U}^1)^{-l}$ and $(\hat{V}^1)^{-r}$ none of which are on the unit circle by assumption. Using Lemma 4.2 one can obtain an upper bound on the one norm of Q that depends only on the upper bound of the one norm of R, $(U^1)^{-l}$, and $(V^1)^{-r}$.

The following lemma is a result on the uniqueness of the solution to (4.1).

LEMMA 4.3. *Let* $Q^0\in\ell_1^{n_u\times n_y}$ *be a solution to* (4.1)*. Let* $\Phi^0=H-U*Q^0*V$ *with* $\Phi^{22,o}=H^{22}-U^2*Q^0*V^2$ *and* $\Phi^{11,o}=H^{11}-U^1*Q^0*V^1$*. Then* $\Phi^{22,o}$ *is unique. Furthermore, if* $\hat{U}^2$ *and* $\hat{V}^2$ *have full normal column and row ranks, respectively, then* Q^0 *is unique.*

Proof. Note that the problem statement of ν given by (4.1) can be recast as

$$\nu=\inf\{\|\Phi^{22}\|_2^2 : \Phi^{22}\in A_{al}\}, \tag{4.2}$$

where A_{al} is the following set:

$$\{\Phi^{22} : \text{there exists } Q \in \ell_1^{n_u \times n_y} \text{ with } \Phi^{22} = H^{22} - U^2 * Q * V^2,$$
$$\|H^{11} - U^1 * Q * V^1\|_1 \leq \gamma, \text{ and } \|Q\|_1 \leq \alpha\}.$$

Its clear that A_{al} is a convex set. It is also true that $\|\,.\,\|_2^2$ is a strictly convex function. It follows from Lemma 3.3 that the minimizer of (4.2) given by $\Phi^{22,o}$, if it exists, is unique. If $\hat{U}^2$ and $\hat{V}^2$ have full column and row ranks, then it follows that

$$\hat{Q}^0 = (\hat{U}^2)^{-l} \hat{\Phi}^{22,o} (\hat{V}^2)^{-r},$$

where $(\hat{U}^2)^{-l}$ and $(\hat{V}^2)^{-r}$ represent the left and the right inverses of $\hat{U}^2$ and $\hat{V}^2$, respectively. Thus $\hat{Q}^0$ is unique. This proves the lemma. ◻

5. Converging lower and upper bounds.

5.1. Converging lower bounds. Let ν_n be defined by

$$\inf_{Q \in \ell_1^{n_u \times n_y}} \|P_n(H^{22} - U^2 * Q * V^2)\|_2^2$$

(5.1) subject to

$$\|P_n(H^{11} - V^1 * Q * V^1)\|_1 \leq \gamma,$$
$$\|Q\|_1 \leq \alpha.$$

It is clear that only the parameters of $Q(0), \ldots, Q(n)$ enter into the optimization problem and therefore (5.1) is a finite-dimensional quadratic programming problem. Once optimal $Q(0), \ldots, Q(n)$ are found, $Q = \{Q(0), \ldots, Q(n), 0, \ldots\}$ will be a finite impulse response (FIR) optimal solution to (5.1).

THEOREM 5.1. *Suppose the constraint set in problem* (4.1) *is nonempty. Then problem* (4.1) *always has an optimal solution* $Q^0 \in \ell_1^{n_u \times n_y}$. *Furthermore,*

$$\nu_n \nearrow \nu.$$

Also, if $\Phi^{22,o} := H^{22} - U^2 * Q^0 * V^2$ *and* $\Phi^{22,n} := H^{22} - U^2 * Q^n * V^2$, *where* Q^n *is a solution to* (5.1), *then there exists a subsequence* $\{\Phi^{22,n_m}\}$ *of the sequence* $\{\Phi^{22,n}\}$ *such that*

$$\|\Phi^{22,n_m} - \Phi^{22,o}\|_2 \to 0 \text{ as } m \to \infty.$$

If $\hat{U}^2$ *and* $\hat{V}^2$ *have full normal column and row ranks, respectively, then* Q^0 *is unique and*

$$\|\Phi^{22,n} - \Phi^{22,o}\|_2 \to 0 \text{ as } n \to \infty.$$

Proof. We know that for any $Q \in \ell_1^{n_u \times n_y}$, $\|P_n(H^{11} - U^1 * Q * V^1)\|_1 \leq \|P_{n+1}(H^{11} - U^1 * Q * V^1)\|_1$, and $\|P_n(H^{22} - U^2 * Q * V^2)\|_2^2 \leq \|P_{n+1}(H^{22} - U^2 * Q * V^2)\|_2^2$. Therefore, $\nu_n \leq \nu_{n+1}$ for all $n = 1, 2, \ldots$. Thus $\{\nu_n\}$ forms an increasing sequence. Similarly it can be shown that for all n, $\nu_n \leq \nu$.

For $n = 1, 2, \ldots$, let $\{Q^n\} \in \ell_1^{n_u \times n_y}$ be FIR solutions of (5.1). As the sequence $\{Q^n\}$ is uniformly bounded by α in $\ell_1^{n_u \times n_y}$ it follows from the Banach–Alaoglu theorem that there exists a subsequence $\{Q^{n_m}\}$ of $\{Q^n\}$ and $Q^0 \in \ell_1^{n_u \times n_y}$ such that

$Q_{ij}^{n_m} \to Q_{ij}^0$ in the $W(c_0^*, c_0)$ topology. This implies that $Q^{n_m}(t) \to Q^0(t)$ for all $t = 0, 1, \ldots$. Therefore, for all n, $P_n(U * Q^{n_m} * V)$ converges to $P_n(U * Q^0 * V)$ as $m \to \infty$. Now for any $n > 0$ and for any $n_m > n$, $\|P_n(H^{11} - U^1 * Q^{n_m} * V^1)\|_1 \leq \gamma$. This implies that $\|P_n(H^{11} - U^1 * Q^0 * V^1)\|_1 \leq \gamma$. Since n is arbitrary, we have

$$\|H^{11} - U^1 * Q^0 * V^1\|_1 \leq \gamma.$$

Similarly for any $n > 0$ and for any $n_m > n$, $\|P_n(H^{22} - U^2 * Q^{n_m} * V^2)\|_2^2 \leq \nu$. Again, this implies that $\|P_n(H^{22} - U^2 * Q^0 * V^2)\|_2^2 \leq \nu$. Since n is arbitrary, it follows that

$$\|H^{22} - U^2 * Q^0 * V^2\|_2^2 \leq \nu.$$

It follows that Q^0 is an optimal solution for (4.1).

To prove that $\nu_n \nearrow \nu$, we note that

$$\|P_n(H^{22} - U^2 * Q^{n_m} * V^2)\|_2^2 \leq \|P_{n_m}(H^{22} - U^2 * Q^{n_m} * V^2)\|_2^2 = \nu_{n_m} \qquad \forall n > 0,\ \forall n_m > n.$$

Taking the limit as m goes to infinity we have

$$\|P_n(H^{22} - U^2 * Q^0 * V^2)\|_2^2 \leq \lim_{m\to\infty} \nu_{n_m}\ \forall n > 0.$$

It follows that

$$\|H^{22} - U^2 * Q^0 * V^2\|_2^2 \leq \lim_{m\to\infty} \nu_{n_m}.$$

Thus we have shown that $\lim_{m\to\infty} \nu_{n_m} = \nu$. Since ν_n is a monotonically increasing sequence, it follows that $\nu_n \nearrow \nu$.

It is clear from Lemma 4.3 that $\Phi^{22,o} := H^{22} - U^2 * Q^0 * V^2$ is unique. If $\Phi^{22,n} := P_n(H^{22} - U^2 * Q^n * V^2)$, then from the discussion above it follows that $\nu_{n_m} = \|\phi^{22,n_m}\|_2^2$ converges to $\nu = \|\Phi^{22,o}\|_2^2$. Also, $\Phi^{22,n_m}(t)$ converges to $\Phi^{22,o}(t)$. It follows from Lemma 3.5 that

$$\|\Phi^{22,n_m} - \Phi^{22,o}\|_2 \to 0 \text{ as } m \to \infty.$$

From Lemma 4.3 we also have that if $\hat{U}^2$ and $\hat{V}^2$ have full normal column and row ranks, respectively, then Q^0 is unique. From the uniqueness of Q^0 it follows that the original sequence $\{\Phi^{22,n}\}$ converges to $\Phi^{22,o}$ in the two norm. This proves the theorem. $\square$

5.2. Converging upper bounds. Let $\nu^n(\gamma)$ be defined by

$$\begin{aligned} &\inf_{Q \in \ell_1^{n_u \times n_y}} \|H^{22} - U^2 * Q * V^2\|_2^2 \\ &\text{subject to} \\ &\|H^{11} - U^1 * Q * V^1\|_1 \leq \gamma, \\ &\|Q\|_1 \leq \alpha, \\ &Q(k) = 0 \text{ if } k > n. \end{aligned} \tag{5.2}$$

We will assume that γ, which characterizes the ℓ_1 constraint level, is in the interior of the domain of the function ν. The following theorem shows that $\{\nu^n(\gamma)\}$ defines a sequence of upper bounds to $\nu(\gamma)$ which converge to $\nu(\gamma)$.

THEOREM 5.2. *For all n, $\nu^n(\gamma) \geq \nu^{n+1}(\gamma) \geq \nu(\gamma)$. Also*

$$\nu^n(\gamma) \searrow \nu(\gamma).$$

Proof. It is clear that $\nu^n(\gamma) \geq \nu^{n+1}(\gamma)$ because any $Q \in \ell_1^{n_u \times n_y}$ that satisfies the constraints in the problem definition of $\nu^n(\gamma)$ will satisfy the constraints in the problem definition of $\nu^{n+1}(\gamma)$. For the same reason we also have $\nu^n(\gamma) \geq \nu(\gamma)$ for all relevant n.

Thus $\{\nu^n(\gamma)\}$ is a decreasing sequence of real numbers bounded below by $\nu(\gamma)$. It can be shown that $\nu(\gamma)$ is a continuous function of γ (see Theorem 6.5 in [5]).

Given $\epsilon > 0$ choose $\delta > 0$ such that

$$\nu(\gamma - \delta) - \nu(\gamma) < \frac{\epsilon}{2}. \tag{5.3}$$

Such a δ exists from the continuity of $\nu(\gamma)$ in γ. Let $Q^{\gamma-\delta}$ be a solution to the problem $\nu(\gamma - \delta)$, which is guaranteed to exist from Theorem 5.1. Let M be large enough so that $m \geq M$ implies that

$$|\,\|H^{22} - U^2 * P_m(Q^{\gamma-\delta}) * V^2\|_2^2 - \|H^{22} - U^2 * Q^{\gamma-\delta} * V^2\|_2^2| < \frac{\epsilon}{2} \tag{5.4}$$

and

$$|\,\|H^{11} - U^1 * P_m(Q^{\gamma-\delta}) * V^1\|_1 - \|H^{11} - U^1 * Q^{\gamma-\delta} * V^1\|_1| < \frac{\delta}{2}. \tag{5.5}$$

As $Q^{\gamma-\delta}$ is a solution to the problem $\nu(\gamma - \delta)$ it is also true that

$$\|H^{22} - U^2 * Q^{\gamma-\delta} * V^2\|_2^2 = \nu(\gamma - \delta),$$
$$\|H^{11} - U^1 * Q^{\gamma-\delta} * V^1\|_1 \leq \gamma - \delta,$$

and

$$\|Q^{\gamma-\delta}\|_1 \leq \alpha.$$

From the above and (5.4), (5.5) it follows that for all $m \geq M$,

$$\|H^{22} - U^2 * P_m(Q^{\gamma-\delta}) * V^2\|_2^2 - \nu(\gamma - \delta) \leq \frac{\epsilon}{2}, \tag{5.6}$$

$$\|H^{11} - U^1 * P_m(Q^{\gamma-\delta}) * V^1\|_1 \leq \gamma, \tag{5.7}$$

and

$$\|P_m(Q^{\gamma-\delta})\|_1 \leq \alpha. \tag{5.8}$$

From (5.3) and the above it follows that for all $m \geq M$, $P_m(Q^{\gamma-\delta})$ satisfies all the constraints of problem $\nu^m(\gamma)$ and

$$\|H^{22} - U^2 * P_m(Q^{\gamma-\delta}) * V^2\|_2^2 - \frac{\epsilon}{2} - \nu(\gamma) \leq \frac{\epsilon}{2}.$$

Thus for all $m \geq M$ it follows that

$$\nu^m(\gamma) - \nu(\gamma) \leq \epsilon.$$

This proves the theorem. $\square$

6. Conclusions. In this paper, we have formulated a problem that incorporates the $\mathcal{H}_2$ performance measure and the ℓ_1 measure. It is shown that converging upper and lower bounds can be obtained via finite-dimensional convex programming problems. This methodology avoids many of the problems of the zero interpolation based methods previously employed.

Ongoing research has indicated that the method developed here can be generalized to solve multiple-objective problems that involve the $\mathcal{H}_2$ measure and various time domain measures (including the ℓ_1 norm). Future research involves implementation of the method developed.

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