Proceedings of the American Control Conference
Arlington, VA June 25-27, 2001

# Optimal Controller Synthesis with Multiple Objectives *

**Xin Qi** **Mustafa H. Khammash** **Murti V. Salapaka** †

Department of Electrical and Computer Engineering
Iowa State University
Ames, IA 50011

**Abstract** We present a methodology to address the general multiobjective (GMO) control problem involving the $\ell_1$ norm, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm and time-domain constraint (TDC). We show that the problem resulted from imposing a regularizing condition always admits an optimal solution, and suboptimal solutions with performance arbitrarily close to the optimal cost can be obtained by constructing two sequences of finite dimensional convex optimization problems whose objective values converge to the optimum from below and above. A numerical example is presented to illustrate the effectiveness of the proposed methodology.

**Keywords** $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$, multiobjective control, robust optimal control, time-domain constraint, linear matrix inequality (LMI), semidefinite programming (SDP)

## 1 Introduction

In the last twenty years, designing engineering systems that are insensitive to uncertainties has attracted considerable attention. Various robust control theories (e.g, $\mathcal{H}_\infty$ theory, $\ell_1$ theory) have been proposed to deal with the effects of uncertainties. The common practice in these methodologies is to optimize the closed-loop system for a given measure of the system with respect to all the stabilizing controllers. In practice, however, diversity of the uncertainties exerted on engineering systems renders it impossible to evaluate controllers' performance by using a single measure. Thus, in a typical controller synthesis procedure, multiple quantifiers are employed to judge the quality of a controller. An example where the multiobjective concerns exist naturally is the suspension control for transport vehicles ([6]). In these systems, suspensions are designed to achieve several conflicting goals that can be translated into three norm-based objectives: $\mathcal{H}_2$ minimization to optimize the driver and cargo comfort for stochastic road disturbances; $\ell_1/L_1$ optimization to prevent certain variables like control action exceeding specified limits; bounding the $\mathcal{H}_\infty$ norm to deal with the variability in the system parameters and model structure errors. The suspension controller design may be reduced to a search for a suitable tradeoff among the above three norm-based objectives. Such multiobjective concerns on controller design are quite common and thus, there is a need for developing a multiobjective control design framework in which such concerns can be naturally addressed.

As indicated in the previous example, a problem with multiple objectives can be cast as an optimization problem with mixed frequency- and time-domain specifications imposed on the $\mathcal{H}_2$ performance, $\mathcal{H}_\infty$ performance, peak-to-peak closed-loop gain, and transient time response due to exogenous inputs (such as a step). Although it is desirable to have all four types of specifications present in the multiobjective formalism, most current approaches address the problems combining a subset of the objectives listed above. For example, various approaches have been proposed to compute and improve the upper bounds to the $\mathcal{H}_2/\mathcal{H}_\infty$ combination problem. In [7], an LMI-based approach was presented to compute a sequence of bounds that converge to the optimum from below.

In the $\ell_1/\mathcal{H}_\infty$ problem, the objective is to minimize the worst case peak output due to persistent disturbances while at the same time satisfying a bound on the $\mathcal{H}_\infty$ norm of a certain given closed-loop transfer matrix. In [8], linear programming and duality theory were used to solve this problem by approximating the $\mathcal{H}_\infty$ constraint with a finite set of linear constraints obtained by sampling the unit circle. In [9], the solution to the four-block $\ell_1/\mathcal{H}_\infty$ problem was obtained by solving a finite-dimensional convex optimization problem together with an unconstrained $\mathcal{H}_\infty$ problem. In [10], the existence of an optimal solution to the multiblock $\ell_1/\mathcal{H}_\infty$ problem was established. Moreover, [10] showed that the optimal solution can be approximated arbitrarily closely by real-rational transfer matrices.

For the mixed-norm optimization problems involving $\mathcal{H}_2$ and $\ell_1$ objectives, two main lines of approaches have been developed to obtain the solution. In [11] and [12], solutions to the $\mathcal{H}_2/\ell_1$ problem were developed by using quadratic programming techniques combined with duality theory. Nevertheless inasmuch as the achievable

*This research was supported by the National Science Foundation under Grants No. ECS-9457485 and ECS-9733802.

†Emails: xinqi@iastate.edu, khammash@iastate.edu, murti@iastate.edu.

closed-loop maps are characterized by using zero interpolation, this line of approaches will lead to a heavy computational burden. Also since the inversion of certain rational matrices is required in the recovery of optimal controller from the resulting optimal closed-loop map, these approaches commonly suffer from numerical difficulties given the finite precision accompanying any numerical method. Recently, a new approach was proposed in [13] to deal with the problems involving $\ell_1$ optimization. This method, which is referred as the Scaled-$Q$ method, avoids the utilization of zero interpolation to characterize the admissible closed-loop maps. Also it yields the impulse response sequence of the minimizing Youla parameter as the optimal solution. This makes straightforward the task of controller recovery. More noticeably, this approach suggests that introducing an $\ell_1$ norm bound on the Youla parameter $Q$ in the optimization may lead to a well regularized optimization problem. Motivated by this idea and by appealing to the Banach-Alaoglu theorem, solution to the $\mathcal{H}_2/\ell_1$ problem has been developed in [1] and [2].

Often performance requirements on the transient time response of the closed-loop system to a given test signal (such as a step) are imposed. It is well recognized that standard single-norm optimal control ($\ell_1$, $\mathcal{H}_2$, or $\mathcal{H}_\infty$) strategies cannot handle specifications or constraints on the time response of a closed-loop system exactly. Thus, there exists a need to consider the time response specifications explicitly in the multiobjective problem setup. Solutions were obtained in [14] to the problem of $\mathcal{H}_\infty$ optimization with infinite-horizon template constraints. In [15], the problem of $\mathcal{H}_2$ minimization with constraints on the time-domain response of a closed-loop transfer function was studied. In [11], the problem of $\ell_1$ optimization with TDC was addressed by a method which needed zero interpolation.

In this paper, we consider the general multiobjective (GMO) control problem involving the $\ell_1$ norm, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm and time-domain template constraints. The approach we pursue here evolves from the solution to the $\mathcal{H}_2/\ell_1$ problem presented in [1] and [2], where the idea of introducing Youla parameter $Q$ as the optimizing variables was used. This has similarity to the $Q$-Parameter design mentioned in [3]. To accommodate the inclusion of $\mathcal{H}_\infty$ norm objectives in the GMO problem formalism, we make use of the LMI relations proposed in [4] and [5]. For special cases of the GMO problem, we present simplified solutions whose computation does not call for the LMI tools and therefore the computational burden is significantly reduced.

The outline of the paper is as follows. In Section 2, we formulate the setup of the multiobjective control problem studied in the paper. In Section 3, we show that the problem of interest always admits a minimizing solution in $\ell_1$. We also show that it can be solved by constructing two sequences of suboptimal problems whose objective values converge to the optimal cost from below and above respectively. In Section 4 and 5, we discuss the numerical realization of the proposed approach and present an illustrative example. Finally, in Section 6, we give a brief summary on the results of this paper.

# 2 Problem Formulation

Consider the system shown in Figure 1, where $G$ is the generalized discrete-time linear time-invariant plant and $K$ is the controller. $w$, $z$, $u$, and $v$ are the exogenous input, regulated output, control input, and measured output, respectively. $r$ is a given scalar reference input (such as a step) and $y$ is the time response output.

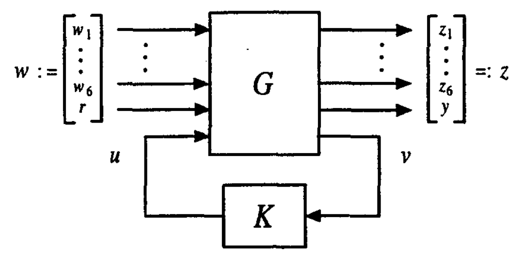


Figure 1: Closed-loop system

By Youla parametrization ([8]), all the achievable closed-loop maps can be characterized as follows:

$$\{R \in \ell_1^{n_z \times n_w} | R = H - U * Q * V \text{ with } Q \in \ell_1^{n_u \times n_v}\}$$

where $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$, $V \in \ell_1^{n_v \times n_w}$, $Q$ is a free parameter in $\ell_1^{n_u \times n_v}$ and '$*$' denotes the convolution operation. In what follows, without any loss of generality, we shall always assume that $\hat{U}$ and $\hat{V}$, the $\lambda$ transforms ([8]) of $U$ and $V$, have full column and row ranks respectively. Also it can be assumed that $\hat{H}$, $\hat{U}$, and $\hat{V}$ are polynomial matrices in $\lambda$, i.e. impulse response sequences of $H$, $U$, and $V$ are finitely supported. To simplify the notations, we use $\hat{R}^i$ ($i = 1, \ldots, 6$) to denote the closed-loop transfer matrix from $w_i$ to $z_i$ and $\hat{R}^7$ the transfer function from $r$ to $y$. For simplicity, and without loss of generality, we shall consider the case when the reference input $r$ is a step sequence. Define $A_{temp}$ as:

$$A_{temp} := \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Then the time response of the closed-loop system due to the reference input $r$ is given by $y = R^7 * r = A_{temp} R^7$. The GMO problem studied in this paper can then be stated as follows: *Given a plant $P$, constants $c_i > 0$, $i = 1, \ldots, 6$, and two sequences $\{a_{temp}(k)\}_{k=0}^{\infty}$ and $\{b_{temp}(k)\}_{k=0}^{\infty}$, solve the following problem,*

$$
\begin{aligned}
&\inf_{Q \in \ell_1^{n_u \times n_v}} f(Q) \\
&\text{subject to} \\
&\qquad \|R^4(Q)\|_1 \le c_4 \\
&\qquad \|R^5(Q)\|_2^2 \le c_5 \\
&\qquad \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \le c_6 \\
&\qquad a_{temp}(k) \le [A_{temp} R^7(Q)](k) \le b_{temp}(k), \ \forall k.
\end{aligned} \tag{1}
$$

where $f(Q) := c_1\|R^1(Q)\|_1 + c_2\|R^2(Q)\|_2^2 + c_3\|\hat{R}^3(Q)\|_{\mathcal{H}_\infty}$, $R^i(Q) = H^{ii} - U^i * Q * V^i, i = 1, \ldots, 7$. Let $\mu$ denote the optimal value of the above problem. From now on, we will always assume that problem (1) has a nonempty feasible set, which includes the requirement that the optimal cost $\mu$ be finite.

The GMO problem defined above represents a large class of multiobjective control problems. Many extensively studied multiobjective problems are special cases of the GMO setup, e.g., $\mathcal{H}_2/\ell_1$, $\ell_1/\mathcal{H}_\infty$, $\mathcal{H}_2/\mathcal{H}_\infty$, $\ell_1$/TDC, $\mathcal{H}_2$/TDC, $\mathcal{H}_\infty$/TDC. Furthermore, for the first time, the $\mathcal{H}_\infty/\ell_1$ problem and $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ problem are addressed. The problem formulation in (1) provides a uniform framework for the performance tradeoff study involving the $\ell_1$, $\mathcal{H}_2$, $\mathcal{H}_\infty$, and TDC. By solving the GMO problem for various combinations of the parameters $c_i (i = 1, \ldots, 6)$ and $\{a_{temp}(k)\}$ and $\{b_{temp}(k)\}$, important information on the limits of system performance can be obtained both qualitatively and quantitatively.

### 2.1 An auxiliary problem

In the general case, (1) is difficult to solve. To facilitate the solution of this problem, we define an auxiliary problem closely related to it: *Given constants $\gamma > 0$, $c_i > 0$, $i = 1, \ldots, 6$, and two sequences $\{a_{temp}(k)\}_{k=0}^{\infty}$ and $\{b_{temp}(k)\}_{k=0}^{\infty}$, solve the following problem,*

$$
\begin{aligned}
&\inf_{Q \in \ell_1^{n_u \times n_v}} f(Q) \\
&\text{subject to} \\
&\qquad \|Q\|_1 \le \gamma \\
&\qquad \|R^4(Q)\|_1 \le c_4 \\
&\qquad \|R^5(Q)\|_2^2 \le c_5 \\
&\qquad \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \le c_6 \\
&\qquad a_{temp}(k) \le [A_{temp} R^7(Q)](k) \le b_{temp}(k), \ \forall k.
\end{aligned} \tag{2}
$$

Let $\nu$ denote the optimal value obtained from this problem. Note there is an extra one norm bound on the Youla parameter $Q$ in the auxiliary problem compared with the original GMO problem (1). This extra constraint plays an essential role in obtaining solution to (1). Also, introducing $Q$ as an optimization variable greatly facilitates the computation of the optimal controller. This avoids the numerical difficulties involved with zero interpolation methods.

### 2.2 Relationship between the GMO problem and the auxiliary problem

In the problem formulation of (1), $Q$ needs to satisfy the constraint $\|R^4(Q)\|_1 = \|H^{44} - U^4 * Q * V^4\|_1 \le c_4$. Suppose $\hat{U}^4$ and $\hat{V}^4$ have full normal column and row rank and have no zeros on the unit circle. Then $U^4$ and $V^4$ are left- and right-invertible in $\ell_1$ and it follows that $\|Q\|_1 \le \|(U^4)^{-l}\|_1(\|H^{44}\|_1 + c_4)\|(V^4)^{-r}\|_1 := \beta$, where $(U^4)^{-l}$ and $(V^4)^{-r}$ denote the left and right inverse of $U^4$ and $V^4$, respectively. Consequently if we choose $\gamma \ge \beta$ in the auxiliary problem, the constraint $\|Q\|_1 \le \gamma$ is redundant in GMO problem and we get $\nu = \mu$. In the case where $\hat{U}^4$ or $\hat{V}^4$ has zeros on the unit circle, there is a possibility that the original GMO problem does not admit an optimal solution and the one norm of the optimization variable $Q$ can not be restricted to any bounded set. Thus, from a computational point of view, it would be desirable to impose a reasonable bound on $\|Q\|_1$ in the optimization for this case as well.

## 3 Problem Solution

In what follows, we shall focus our attention on the auxiliary problem. In proving the main results of the paper, we make the following assumption on the TDC.

**Assumption** *For all $k$, $a_{temp}(k) < b_{temp}(k)$. Furthermore, there exists $N_1$, $N_2$ so that $a_{temp}(k) = a_{temp}(N_1)$ for all $k \ge N_1$ and $b_{temp}(k) = b_{temp}(N_2)$ for all $k \ge N_2$.*

### 3.1 Existence of an optimal solution and converging lower bounds

In this subsection, we develop a sequence of finite dimensional convex optimization problems whose objective values converge to $\nu$ monotonically from below and prove the existence of an optimal solution to (2).

Define a candidate lower bound of $\nu$ as

$$
\begin{aligned}
&\nu_n := \inf_{Q \in \ell_1^{n_u \times n_v}} f_n(Q) \\
&\text{subject to} \\
&\qquad \|Q\|_1 \le \gamma \\
&\qquad \|P_n(R^4(Q))\|_1 \le c_4 \\
&\qquad \|P_n(R^5(Q))\|_2^2 \le c_5 \\
&\qquad \|T_{6,n}(Q)\| \le c_6 \\
&\qquad a_{temp}(k) \le [A_{temp} R^7(Q)](k) \le b_{temp}(k) \\
&\qquad\qquad k = 0, 1, \ldots, n.
\end{aligned} \tag{3}
$$

where $f_n(Q) := c_1\|P_n(R^1(Q))\|_1 + c_2\|P_n(R^2(Q))\|_2^2 + c_3\|T_{3,n}(Q)\|$, $P_n$ denotes the truncation operator on the space of sequences defined as $P_n(x(0)\ x(1)\ \ldots) = (x(0)\ x(1)\ \ldots\ x(n)\ 0\ 0\ \ldots)$, and

$$T_{i,k}(Q) := \begin{bmatrix} R^i(0) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ R^i(k) & \cdots & \cdots & R^i(0) \end{bmatrix}, \ i = 3, 6.$$

**Theorem 1** *There is an optimal solution $Q^0$ in $\ell_1^{n_u \times n_v}$ to problem (2). Moreover, $\nu_n \nearrow \nu$, as $n \to \infty$.*

## 3.2 Converging upper bounds

In the last subsection, we have shown that $\nu_n$ provides a lower bound for $\nu$ and that the sequence $\{\nu_n\}$ converges monotonically to $\nu$. However, it is clear that $\nu_n$ itself does not provide any information on its distance to the optimal cost $\nu$. This motivates the computation of an upper bound of $\nu$. Let $\nu^n$ be defined by

$$\inf_{Q \in \ell_1^{n_u \times n_v}} f(Q)$$

subject to

$$\begin{aligned} &\|Q\|_1 \leq \gamma \\ &\|R^4(Q)\|_1 \leq c_4 \\ &\|R^5(Q)\|_2^2 \leq c_5 \\ &\|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \leq c_6 \\ &a_{temp}(k) \leq [A_{temp} R^7(Q)](k) \leq b_{temp}(k), \ \forall k \\ &Q(k) = 0 \text{ if } k > n. \end{aligned} \tag{4}$$

**Theorem 2** *$\{\nu^n\}$ forms a monotonically decreasing sequence of upper bounds of $\nu$. Furthermore,*

$$\nu^n \searrow \nu, \ \text{as } n \to \infty.$$

## 3.3 Uniqueness and convergence of the optimal solution

**Theorem 3** *Suppose $\hat{U}^2$ and $\hat{V}^2$ have full column and row rank on the unit circle respectively. Let $Q^n$ denote an optimal solution to $\nu^n$. Let $Q^0$ denote an optimal solution to $\nu$. Let $R^n := H - U * Q^n * V$, $n = 0, 1, \ldots$. Then the following is true:*

(1) *$R^n$ $(n = 0, 1, \ldots)$ is unique.*
(2) *$Q^n$ $(n = 0, 1, \ldots)$ is unique.*

If no $\mathcal{H}_\infty$ term is present in the objective $f(Q)$ of (1), the conclusion of Theorem 3 can be made stronger.

**Theorem 4** *Suppose $\hat{U}^2$ and $\hat{V}^2$ have full column and row rank on the unit circle respectively. Let $Q^n$ denote an optimal solution to $\nu^n(c_3 = 0)$. Let $Q^0$ denote an optimal solution to $\nu$. Define $R^n := H - U * Q^n * V$ and $R^{i,n} := H^{ii} - U^i * Q^n * V^i$, $i = 1, \ldots, 7$, $n = 0, 1, \ldots$. Then the following is true:*

(1) *$R^n$ $(n = 0, 1, \ldots)$ is unique.*
(2) *$Q^n$ $(n = 0, 1, \ldots)$ is unique.*
(3) *$\|R^{2,n} - R^{2,0}\|_2 \to 0$, as $n \to \infty$.*

# 4 Numerical Realization

It is clear from the definitions (3) and (4) that only the parameters of $Q(0), \ldots, Q(n)$ enter into the optimization of $\nu_n$ and $\nu^n$. Thus (3) and (4) are actually two finite dimensional convex programming problems and by appealing to the LMI formulas proposed in [4] and [5], they can be readily transformed into solvable SDP forms and be effectively solved by using some well-developed semidefinite programming techniques. In the case where there is no $\mathcal{H}_\infty$ norm involved in the problem setup (i.e. $c_3 = 0$, $c_6 = \infty$) and no $\mathcal{H}_2$ norm constraint imposed on the closed-loop system($c_5 = 0$), the corresponding GMO problem can be solved in a less computationally expensive manner by using quadratic programming techniques. For the case where neither $\mathcal{H}_2$ norm nor $\mathcal{H}_\infty$ norm is involved in the problem setup, the GMO problem can be efficiently solved by using linear programming (LP) techniques.

It should be noted that the GMO control design framework we have developed here is flexible; given any finite numbers of $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ norm objectives and TDCs, they can be directly stacked into the GMO problem formalism and the theoretical and numerical schemes established in this and the previous sections can be extended in a straightforward manner to obtain the solution.

# 5 An Illustrative Example

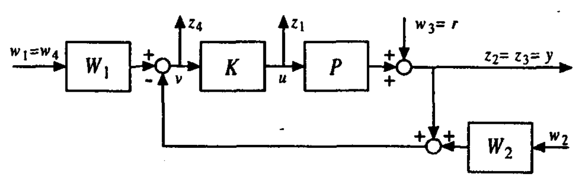


Figure 2: System configuration

As an example, we consider a multi-block problem shown in Figure 2, where

$$W_1(\lambda) = \frac{0.2452 + 0.2452\lambda}{1 - 0.5095\lambda}, \ W_2(\lambda) = \frac{0.6 - 0.54\lambda}{1 - 0.1\lambda},$$
$$P(\lambda) = \frac{0.5\lambda(1 + 0.2\lambda)}{(1 + 1.5\lambda)(1 - 0.1\lambda)}.$$

Weighted with a second order low-pass filter $W_1$, $w_1$ denotes a slowly varying but otherwise arbitrary reference input signal with magnitude less than or equal to one. The actuator which yields the control action $u$ to the plant $P$ saturates at a certain constant value $u_{max}$ of 4 units. Thus it is desired to design a controller capable of preventing the magnitude of $u$ exceeding 4 units. This engineering objective can be captured by imposing an $\ell_1$ norm constraint on the transfer function from $w_1$ to $z_1(=u)$. Together with the high-pass weighting function $W_2$, the signal $w_2$ denotes the high frequency measurement noise introduced by the sensor. To attenuate the effect of $w_2$ on the regulated output $z_2$, an $\mathcal{H}_2$ norm constraint is imposed on the transfer function from $w_2$ to $z_2$. To reject step disturbance signal $w_3(=r)$ injected at the plant output, we choose the TDC templates $\{a_{temp}(k)\}$ and $\{b_{temp}(k)\}$ in such a manner that time response $y(k)$ of the resulting optimal system due to $r$ is forced to converge to zero as the system proceeds into the steady state. To accommodate this requirement, in this example, the template parameters are chosen such that $a_{temp}(k) < b_{temp}(k)$ for $0 \leq k \leq 11$ and $a_{temp}(k) = b_{temp}(k) = 0$ for $k \geq 12$. To address the performance requirement on tracking the reference input signal $w_1(=w_4)$, an $\ell_1$ norm constraint $\alpha = 0.4$ is placed on the weighted sensitivity function $R^4(Q)$.

In order to synthesize controllers that incorporate the above objectives, it is conducive to perform a limits of performance study that will indicate the achievable and unachievable closed-loop performance specifications. Such an analysis can be carried out by solving the following GMO problem:

$$\begin{aligned} \nu := \inf_{Q \in \ell_1^{n_u \times n_v}} \; & c_1\|R^1(Q)\|_1 + c_2\|R^2(Q)\|_2^2 \\ \text{subject to} \quad & \\ & a_{temp}(k) \leq y(k) \leq b_{temp}(k),\ k = 0,1,2,\ldots \\ & \|R^4(Q)\|_1 \leq \alpha. \end{aligned} \tag{5}$$

Given the problem setup (5) and the discussion in Section 4, it is clear that the sequences of upper and lower bounds for $\nu$ can be computed efficiently by appealing to quadratic programming techniques. For our purpose, a large enough bound $\gamma = 20$ was chosen such that problem (5) and its auxiliary problem admit the same objective values. For a prescribed increasing sequence of nonnegative ratios of $c_1/c_2$ (11 points), the auxiliary problems of (5) were solved by computing the converging upper and lower bounds respectively and the optimal solutions, i.e, Youla parameter $Q$'s, and the values of $\|R^1(Q)\|_1$ and $\|R^2(Q)\|_2^2$ were obtained. For all pairs of $c_1$ and $c_2$, the $\ell_1$ norms of the optimal $Q$'s are far less than $\gamma$ (usually $\|Q\|_1 \leq 6$). This shows that the extra $\ell_1$ norm constraint on $Q$ is inactive and problem (5) and its auxiliary problem admit the same optimal cost. The plots of $\|R^1(Q)\|_1$ versus $\|R^2(Q)\|_2^2$ are shown in Figure 3, where the dashed curve ABCD denotes the cases with no template constraint while the solid curve EFGH denotes the cases where there exists time-domain template constraint on the step response $y$. From these two curves, important information on the tradeoff among system performance specifications are obtained both qualitatively and quantitatively.

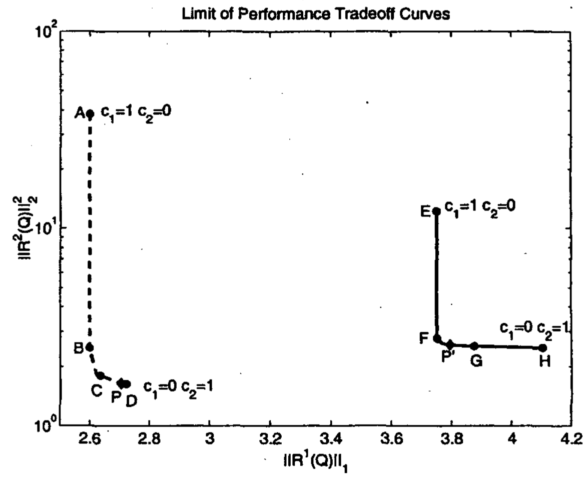


Figure 3: Tradeoff curves between $\mathcal{H}_2$ and $\ell_1$ norms

First, it is clear that each of these two curves denote exactly the boundary between achievable and unachievable performance specifications of the closed-loop system. The region above the curve denotes the performance requirement that can be achieved by some stabilizing controller while the region below the curve represents the specifications that cannot be obtained by any stabilizing controller. Furthermore, by observing the tradeoff plots, the task of choosing system operating points can be readily accomplished. Here by system operating points, we refer to those points that have reasonable values of $\|R^1(Q)\|_1/\|R^2(Q)\|_2^2$ and whose slopes are insensitive to the variation of $c_1/c_2$. To illustrate the idea, consider points from the plot EFGH. The points on segment FG are better candidates over points on segments EF and GH since the latter points are sensitive to the deviation of ratios $c_1/c_2$. By comparing the relative position of ABCD and EFGH in Figure 3, it is clear that the systems' $\ell_1$ and $\mathcal{H}_2$ performance are compromised since one extra performance specification (referring to the template constraints put on $y$) is introduced in the problem setup of (5).

In Figure 4, the time response plot of $y$ with respect to the step disturbance $r$ (case: $c_1 = c_2 = 1$) is shown. It is clear from the plots that the time response satisfies the requirement of zero steady state value, i.e, $y(k) = 0$ for $k \geq 12$. In contrast to this, in the case where there is no template constraints imposed on the time response, the system step response yields a nonzero steady-state value of $y(\infty) = 0.4177$, which is often not desirable.

By model reduction, the following low-order suboptimal controllers (case: $c_1 = c_2 = 1$) are obtained

$$K_P = \frac{-0.8728(1 - 2.865\lambda)(1 - 0.6618\lambda)}{(1 + 0.2046\lambda)(1 - 1.545\lambda + 0.63\lambda^2)}$$

Table 1: Comparison of the optimal solutions

| Point | TDC | CONTROLLER ORDER | $\|Q\|_1$ | $\|R^1(Q)\|_1$ | $\|R^2(Q)\|_2$ | $\|R^4(Q)\|_1$ |
|---|---|---|---|---|---|---|
| P | Not Present | 21 | 5.40 | 2.7040 | 1.6325 | 0.4000 |
| P' | Present | 22 | 5.73 | 3.7937 | 2.5793 | 0.4000 |

Table 2: Comparison of the suboptimal solutions

| Point | TDC | CONTROLLER ORDER | $\|Q\|_1$ | $\|R^1(Q)\|_1$ | $\|R^2(Q)\|_2$ | $\|R^4(Q)\|_1$ |
|---|---|---|---|---|---|---|
| P | Not Present | 5 | 5.40 | 2.6212 | 1.6660 | 0.4221 |
| P' | Present | 8 | 5.73 | 3.8000 | 2.5891 | 0.4117 |

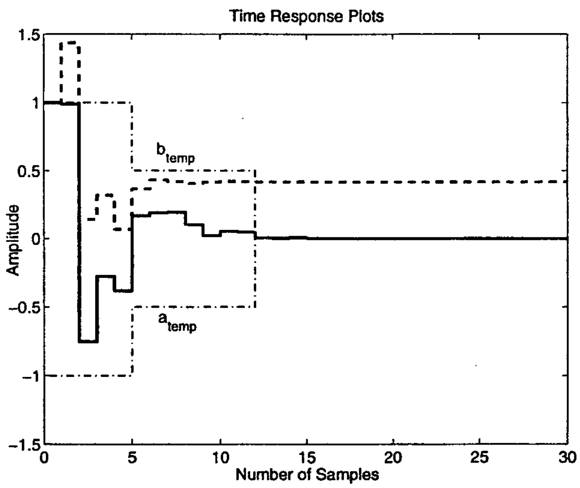


Figure 4: Time responses of the system with TDC (solid) and without TDC (dashed)

$$K_{P'} = \frac{(1+0.2899\lambda+0.4832\lambda^2)(1-0.0996\lambda)}{(1+0.2075\lambda+0.4382\lambda^2)} \frac{0.023012(1+151.7\lambda)(1-0.09944\lambda)}{(1-\lambda)(1+0.1998\lambda)} \frac{(1-1.536\lambda+0.6284\lambda^2)(1+0.8303\lambda+0.5779\lambda^2)}{(1-1.721\lambda+0.8152\lambda^2)(1+0.7838\lambda+0.5497\lambda^2)} \frac{(1+0.01055\lambda+0.596\lambda^2)}{(1+0.004397\lambda+0.5593\lambda^2)}.$$

It is clear from Table 1 and Table 2 that, at the expense of slight loss of system performance, we can obtain suboptimal controllers of manageable orders. It is interesting to note that there is an integrator in the controller $K_{P'}$ and this verifies the result shown in Figure 4 from a different point of view.

# 6 Conclusion

In this paper, we present an attractive methodology to solve the general multiobjective (GMO) control problem concerning the $\ell_1$ norm, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm and time-domain constraint. We show that the regularized GMO problem admits a minimizing solution in $\ell_1$ and, more importantly, from an engineering viewpoint, the optimal performance can be approximated arbitrarily closely by FIR suboptimal controllers. Furthermore, the GMO problem presented here furnishes us with a uniform framework under which many multiobjective problems can be addressed and solved by using the techniques proposed in this paper.

# References


[1] M. V. Salapaka, M. Khammash, and M. Dahleh, "Solution of MIMO $\mathcal{H}_2/\ell_1$ problem without zero interpolation," *CDC97, pp. 1546-1551*, 1997.

[2] M. V. Salapaka, M. Khammash, and M. Dahleh, "Solution of MIMO $\mathcal{H}_2/\ell_1$ Problem without Zero Interpolation," *SIAM J. Control Optim., vol.37, No.6, pp.1865-1873*, Nov., 1999.

[3] S. P. Boyd and C. H. Barratt, *Linear Controller Design: Limits of Performance.* Prentice Hall, Englewood Cliffs, New Jersey 07632, 1991.

[4] L. Vandenberghe and S. Boyd, "Semidefinite Programming," *SIAM Review*, Mar., 1996.

[5] H. A. Hindi, B. Hassibi, and S. P. Boyd, "Multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$-Optimal Control via Finite Dimensional $Q$-Parametrization and Linear Matrix Inequalities," *ACC98, pp.3244-3248*, June, 1998.

[6] B. D. Jager, "Multiobjective suspension control problem," *CDC95, pp.3652-3657*, Dec., 1995.

[7] C. W. Scherer, "Lower Bounds in Multi-Objective $\mathcal{H}_2/\mathcal{H}_\infty$ Problems," *CDC99, pp.3605-3610*, Dec., 1999.

[8] M. A. Dahleh and I. J. Diaz-Bobillo, *Control of Uncertain Systems: A Linear Programming Approach.* Prentice Hall, Englewood Cliffs, New Jersey 07632, 1995.

[9] M. Sznaier and J. Bu, "Mixed $\ell_1/\mathcal{H}_\infty$ Control of MIMO Systems via Convex Optimization," *IEEE TAC, vol.43, No.9, pp.1229-1241*, Sept., 1998.

[10] Hector Rotstein and Bension Shklyar, "Optimal Solution and Approximation to the $\ell_1/\mathcal{H}_\infty$ Control Problem," *IEEE TAC, vol.44, No.6, pp.1240-1243*, 1999.

[11] N. Elia and M. A. Dahleh, "Controller Design with Multiple Objectives," *IEEE TAC, vol.42, No.5, pp.596-613*, May, 1997.

[12] M. V. Salapaka, M. Dahleh, and P. G. Voulgaris, "MIMO Optimal Control Design: The Interplay Between the $\mathcal{H}_2$ and the $\ell_1$ Norms," *IEEE TAC, vol.43, No.10, pp.1374-1388*, Oct., 1998.

[13] M. Khammash, "A New Approach to the Solution of the $\ell_1$ Control Problem: The Scaled-$Q$ Method," *IEEE TAC, vol.45, No.2, pp.180-187*, Feb., 2000.

[14] A. Sideris and H. Rotstein, "Constrained $\mathcal{H}_\infty$ Optimal Control over an Infinite Horizon," *SIAM J. Control Optim., vol.35, No.4, pp.1244-1262*, 1997.

[15] M. Sznaier and T. Amishima, "$\mathcal{H}_2$ Control with Time Domain Constraints," *ACC98, pp.3229-3233*, 1998.