MIMO Optimal Control Design: The Interplay Between the $\mathcal{H}_2$ and the ℓ_1 Norms

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***Abstract*—In this paper we consider controller design methods which can address directly the interplay between the $\mathcal{H}_2$ and ℓ_1 performance of the closed loop. The development is devoted to multi-input/multi-output (MIMO) systems. Two relevant multiobjective performance problems are considered each being of interest in its own right. In the first, termed as the combination problem, a weighted sum of the ℓ_1 norms and the square of the $\mathcal{H}_2$ norms of a given set of input–output transfer functions constituting the closed loop is minimized. It is shown that, in the one-block case, the solution can be obtained via a finite-dimensional quadratic optimization problem which has an *a priori* known dimension. In the four-block case, a method of computing approximate solutions within any *a priori* given tolerance is provided. In the second, termed as the mixed problem, the $\mathcal{H}_2$ performance of the closed loop is minimized subject to an ℓ_1 constraint. It is shown that approximating solutions within any *a priori* given tolerance can be obtained via the solution to a related combination problem.**

***Index Terms*—Duality, ℓ_1 optimization, mixed objectives, robust control.**

Notation

The following notation will be employed in this paper.

$\|x\|_1$	1-norm of the vector x in R^n.
$\|x\|_2$	2-norm of the vector x in R^n.
$\hat{x}(\lambda)$	λ transform of a right-sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \Sigma_{k=0}^{\infty} x(k)\lambda^k$.
ℓ_1	Banach space of right sided absolutely summable real sequences with the norm given by $\|\|x\|\|_1 := \Sigma_{k=0}^{\infty}\|x(k)\|$.
$\ell_1^{m\times n}$	Banach space of matrix-valued right-sided real sequences with the norm $\|\|x\|\|_1 := \max_{1\le i\le m} \Sigma_{j=1}^{n}\|\|x_{ij}\|\|_1$ where x in $\ell_1^{m\times n}$ is the matrix (x_{ij}) and each x_{ij} is in ℓ_1.
ℓ_∞	Banach space of right sided, bounded sequences with the norm given by $\|\|x\|\|_\infty := \sup_k \|x(k)\|$.
$\ell_\infty^{m\times n}$	Banach space of matrix valued right sided real sequences with the norm $\|\|x\|\|_\infty := \Sigma_{i=1}^{m} \max_{1\le j\le n}\|\|x_{ij}\|\|_\infty$ where x in $\ell_\infty^{m\times n}$ is the matrix (x_{ij}) and each x_{ij} is in ℓ_∞.
c_0	Subspace of ℓ_∞ with elements x that satisfy $\lim_{k\to\infty} x(k) = 0$.
$c_0^{m\times n}$	Banach subspace of $\ell_\infty^{m\times n}$ with elements x which satisfy $\lim_{k\to\infty} x(k) = 0$.
ℓ_2	Banach space of right sided square summable sequences with the norm given by $\|\|x\|\|_2 := [\Sigma_{k=0}^{\infty}x(k)^2]^{1/2}$.
$\ell_2^{m\times n}$	Banach space of matrix valued right sided real sequences with the norm $\|\|x\|\|_2 := [\Sigma_{i=1}^{m} \Sigma_{j=1}^{n}\|\|x_{ij}\|\|_2^2]^{1/2}$ where x in $\ell_2^{m\times n}$ is the matrix (x_{ij}) and each x_{ij} is in ℓ_2.
$\mathcal{H}_2$	Isometric isomorphic image of ℓ_2 under the λ transform $\hat{x}(\lambda)$ with the norm given by $\|\|\hat{x}(\lambda)\|\|_2 = \|\|x\|\|_2$.
X^*	Dual space of the Banach space X. $\langle x, x^*\rangle$ denotes the value of the bounded linear functional x^* at x in X.
$W(X^*, X)$	Weak star topology on X^* induced by X.
T^*	Adjoint operator of T: $X \to Y$ which maps Y^* to X^*.
$\mathrm{int}(A)$	Interior of the set A.
$\mathcal{D}$	Closed unit disc in the complex plane.
A'	Transpose of the matrix A.

We have from functional analysis that $(\ell_1)^* = \ell_\infty, (c_0)^* = \ell_1, (\ell_2)^* = \ell_2, (\ell_1^{m\times n})^* = \ell_\infty^{m\times n}, (c_0^{m\times n})^* = \ell_1^{m\times n}, (\ell_2^{m\times n})^* = \ell_2^{m\times n}$. If X is a Banach space of scalar sequences (example ℓ_1) and X^* represents its dual then $\langle x, y\rangle := \Sigma_{k=0}^{\infty}x(k)y(k)$ where x is in X and y is in X^*. If X is a Banach space of matrix sequences of dimension $m \times n$ (example $\ell_1^{m\times n}$) and X^* is its dual then $\langle x, y\rangle := \Sigma_{i=1}^{m} \Sigma_{j=1}^{n}\langle x_{ij}, y_{ij}\rangle$.

I. Introduction

Consider the system of Fig. 1 where $w = (w_1' \; w_2')'$ is the exogenous disturbance, $z = (z_1' \; z_2')'$ is the regulated output, u is the control input, and y is the measured output. In feedback control design the objective is to design a controller K such that with $u = Ky$ the resulting closed-loop map Φ_{zw} from w to z is stable (see Fig. 1) and satisfies certain performance criteria. Such criteria may be posed in terms of a measure on Φ_{zw} which depends on the signal norms of w and z, which are of interest in a particular situation. For example, the $\mathcal{H}_\infty$ norm of the closed loop measures the energy of the regulated output z for the worst disturbance w which has unit energy. The standard $\mathcal{H}_\infty$ problem minimizes

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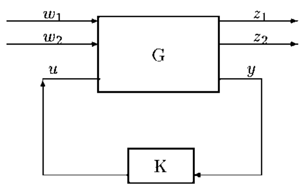

Fig. 1. Framework.

this norm over all achievable closed-loop maps. The two norm of the closed loop measures the energy in the regulated output z for a unit pulse input w. The standard $\mathcal{H}_2$ problem finds a stabilizing controller, which results in a closed-loop map which has the minimum two norm when compared to all other closed-loop maps achievable through stabilizing controllers. State-space solutions for both the above-mentioned problems are provided in [7]. The ℓ_1 norm of the closed loop is the infinity norm of the regulated output z for the worst disturbance w whose maximum magnitude is less than or equal to one. The standard ℓ_1 problem finds a controller which minimizes the ℓ_1 norm over all closed-loop maps that are achievable through stabilizing controllers. It is shown in [3] that this problem reduces to a finite-dimensional linear program for the one-block case.

All of the previous criteria refer to a single performance measure of the closed loop. It is well known (see for example [2]) that minimization with respect to one norm may not necessarily yield good performance with respect to another. This has led researchers to consider problems where multiple measures of the closed loop are incorporated directly into the design. One of the important classes of designs in this category is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design where the focus is on problems which include the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norms of the closed loop in their definitions. Several state-space results are available in this class (e.g., [6] and [15]). Another important class of problems considers the interplay between the ℓ_1 and the $\mathcal{H}_\infty$ norm of the closed loop. Problems from this class are addressed in [4] and [10].

Recently, results were obtained for mixed objective problems involving the ℓ_1 and the $\mathcal{H}_2$ norms of the closed loop for single-input/single-output (SISO) systems. In [8] the problem of minimizing the two norm of the closed loop while keeping its one norm below a prespecified level was reduced to a finite-dimensional quadratic optimization problem. The converse problem of minimizing the ℓ_1 norm of the closed loop over all internally stabilizing controllers while keeping its two norm below a prespecified level was reduced to a finite-dimensional convex optimization problem in [14]. It was also shown that the dimension of the equivalent convex optimization problem can be determined *a priori*. Related SISO problems are addressed in [12] and [13].

This paper explores the interplay of the ℓ_1 and the $\mathcal{H}_2$ norms of the closed loop for the multi-input/multi-output (MIMO) case which is much richer in its complexity and its applicability than the SISO case. Consider for example Fig. 1 where the part of the regulated output given by z_2 is used to reflect the performance with respect to a unit pulse input. It is also required that the maximum magnitude of z_1, due to any disturbance with maximum magnitude less than or equal to one, stays below a prespecified level. This objective can be captured by the following problem:

$$\min_{K \text{ stabilizing}} \{\|w \to z_2\|_2 \colon \|w \to z_1\|_1 \leq \gamma\} \tag{1}$$

where γ is the prescribed level. Or, it may be that the disturbance w is such that a part of it, w_1, is a white noise while another part, w_2, is magnitude-bounded. A relevant objective is the minimization of the effect of these disturbances on the regulated output. The problem

$$\min_{K \text{ stabilizing}} \{\|w_1 \to z\|_2 \colon \|w_2 \to z\|_1 \leq \gamma\} \tag{2}$$

where γ is the level over which the infinity norm of z is not allowed to cross for any disturbance whose maximum magnitude is less than or equal to one, is then a problem of interest.

Both problems mentioned previously fall under a general framework of a problem which we call the *mixed problem*. This problem is addressed and solved in the paper along with a related problem which minimizes a combination of the various input–output maps of the closed loop which we call the *combination problem*. The latter is of interest by itself and in relation to the mixed problem.

Unlike the SISO case, the optimal solutions for the one-block are not, in general, finite impulse responses, nor are they unique (as will be shown). However, it is established that the solution can be obtained via finite-dimensional quadratic optimization and linear programming. We show that it is possible to obtain an *a priori* bound on the dimension of the suboptimal problems even for the MIMO version of the problem addressed in [8] (no *a priori* bound is given in [8]).

The paper is organized as follows: In Section II we give system and mathematical preliminaries. In Section III we show that the combination problem can be solved exactly via a finite-dimensional quadratic optimization and linear programming for the square case. The nonsquare case is handled by the delay augmentation method [2]. In Section IV we study the mixed problem and its relation to the combination problem. In Section V we give an example to illustrate the theory developed. In Section VI conclusions are given. Section VII is the Appendix which contains the proofs of some of the facts stated in earlier sections.

II. Preliminaries

In the first part of this section we give system preliminaries, and in the second part we give mathematical preliminaries. It is to be noted that all the assumptions made in this section are valid throughout the paper.

A. System Preliminaries

A good reference for this section is [2]. We denote by n_u, n_w, n_z, and n_y the number of control inputs, exogenous inputs, regulated outputs, and measured outputs, respectively, of the plant G. We represent by Θ the set of closed-loop maps of the plant G which are achievable through stabilizing controllers. H in $\ell_1^{n_z\times n_u}$, U in $\ell_1^{n_x\times n_u}$, and V in $\ell_1^{n_y\times n_w}$ characterize the Youla parameterization of the plant [16]. The following theorem follows from Youla parameterization.

Theorem 1: $\Theta = \{\Phi$ in $\ell_1^{n_z\times n_w}$: there exists a Q in $\ell_1^{n_u\times n_y}$ with $\hat{\Phi} = \hat{H} - \hat{U}\hat{Q}\hat{V}\}$.

If Φ is in Θ we say that Φ is an *achievable* closed-loop map. We assume throughout the paper that $\hat{U}$ has normal rank n_u and $\hat{V}$ has normal rank n_y. There is no loss of generality in making this assumption [2]. Let the Smith–McMillan decomposition of $\hat{U}$ and $\hat{V}$ be given by $\hat{U} = \hat{L}_U\hat{M}_U\hat{R}_U$, and $\hat{V} = \hat{L}_V\hat{M}_V\hat{R}_V$, respectively, where L_U in $\ell_1^{n_z\times n_z}$, R_U in $\ell_1^{n_u\times n_u}$, L_V in $\ell_1^{n_y\times n_y}$ and R_V in $\ell_1^{n_w\times n_w}$ are unimodular matrices. M_U in $\ell_1^{n_z\times n_u}$ and M_V in $\ell_1^{n_y\times n_w}$ can be written as

$$\hat{M}_U = \begin{pmatrix} \frac{\hat{\epsilon}_1}{\hat{\psi}_1} & & \\ & \ddots & \\ & & \frac{\hat{\epsilon}_{n_u}}{\hat{\psi}_{n_u}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$\hat{M}_V = \begin{pmatrix} \frac{\hat{\epsilon}'_1}{\hat{\psi}'_1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \frac{\hat{\epsilon}'_{n_y}}{\hat{\psi}'_{n_y}} & 0 & \cdots & 0 \end{pmatrix} \tag{3}$$

where $\{\hat{\epsilon}_i, \hat{\psi}_i\}$ and $\{\hat{\epsilon}'_i, \hat{\psi}'_i\}$ are coprime monic polynomial pairs. Let Λ_{UV} denote the set of zeros $\hat{U}$ and $\hat{V}$ in $\mathcal{D}$ (i.e., the zeros of $\Pi_{i=1}^{n_u}\,\hat{\epsilon}_i\,\Pi_{j=1}^{n_y}\,\hat{\epsilon}'_j$ which lie inside the closed unit disc). For λ_0 in Λ_{UV} define

$$\begin{aligned} \sigma_{U_i}(\lambda_0) :=& \text{ multiplicity of } \lambda_0 \\ &\cdot \text{ as a root of } \hat{\epsilon}_i(\lambda) \text{ for } i = 1,\cdots,n_u, \\ \sigma_{V_j}(\lambda_0) :=& \text{ multiplicity of } \lambda_0 \\ &\cdot \text{ as a root of } \hat{\epsilon}'_j(\lambda) \text{ for } j = 1,\cdots,n_y. \end{aligned}$$

As $\hat{L}_U, \hat{R}_V$ are unimodular we can define the following polynomial row and column vectors:

$$\begin{aligned} \hat{\alpha}_i(\lambda) &= (\hat{L}_U^{-1})_i(\lambda), &&\text{for } i = 1,\cdots,n_z \\ \hat{\beta}_j(\lambda) &= (\hat{R}_V^{-1})^j(\lambda), &&\text{for } j = 1,\cdots,n_w \end{aligned}$$

where $(M)_i$ denotes the ith row of the matrix M and $(M)^j$ denotes the jth column of a matrix M. Note that α_i is in $\ell_1^{1\times n_z}$ and β_j is in $\ell_1^{n_w\times 1}$. We assume that $\hat{U}$ *and* $\hat{V}$ *have no zeros which lie on the unit circle*. This is a standard assumption in the optimal model-matching approach we employ and is crucial for the one-block development. For a detailed discussion about the case where this assumption fails see [2]. We now present the main interpolation theorem for a closed-loop map to be achievable. We denote the kth derivative of $(\cdot)$ with respect to λ by $(\cdot)^{(k)}$ whereas the kth power of $(\cdot)$ is denoted by $(\cdot)^k$.

Theorem 2 [2]: Φ in $\ell_1^{n_z\times n_w}$ is achievable if and only if the following conditions hold for all λ_0 in Λ_{UV} :

1) $(\hat{\alpha}_i\hat{\Phi}\hat{\beta}_j)^{(k)}(\lambda_0) = (\hat{\alpha}_i\hat{H}\hat{\beta}_j)^{(k)}(\lambda_0)$

$$\text{for} \begin{cases} i = 1,\cdots,n_u \\ j = 1,\cdots,n_y \\ k = 0,\cdots,\sigma_{U_i}(\lambda_0) + \sigma_{v_j}(\lambda_0) - 1. \end{cases}$$

2) $\begin{cases} (\hat{\alpha}_i\hat{\Phi})(\lambda) = (\hat{\alpha}_i\hat{H})(\lambda), & \text{for } i = n_u+1,\cdots,n_z \\ (\hat{\Phi}\hat{\beta}_j)(\lambda) = (\hat{H}\hat{\beta}_j)(\lambda), & \text{for } j = n_y+1,\cdots,n_w. \end{cases}$

The first set of conditions constitutes the *zero interpolation* conditions, whereas the second set consists of the *rank interpolation conditions*. The plant G is called *square*, or equivalently, we have a one-block problem, if the rank interpolation conditions are absent (i.e., when $n_u = n_z$ and $n_y = n_w$). Otherwise, the plant is *nonsquare*, or equivalently, we have a four-block problem. Define $F^{ijk\lambda_0}$ in $\ell_\infty^{n_z\times n_w}$ by

$$F_{pq}^{ijk\lambda_0}(s) := \sum_{l=0}^{\infty}\sum_{t=0}^{\infty} \alpha_{ip}(t-l)\beta_{jq}(l-s)\,(\lambda^t)^{(k)}\Big|_{\lambda=\lambda_0}. \tag{4}$$

It can be verified that for any Φ in $\ell_1^{n_z\times n_w}$, $(\hat{\alpha}_i, \hat{\Phi}\hat{\beta}_j)^{(k)}(\lambda_0) = \langle \Phi, F^{ijk\lambda_0}\rangle$. Thus $F^{ijk\lambda_0}$'s characterize the zero interpolation conditions. We state the following lemma, which will be of use in the next section (the proof is omitted).

Lemma 1: For all λ_0 in Λ_{UV}

$F^{ijk\lambda_0}$ is in $\ell_1^{n_z\times n_w}$(and hence is in $\ell_2^{n_z\times n_w}$) for

$$\begin{cases} i = 1,\cdots,n_u \\ j = 1,\cdots,n_y \\ k = 0,\cdots,\sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) - 1. \end{cases}$$

This means that the zero interpolation conditions can be characterized via elements in $\ell_1^{n_z\times n_w}$. Similarly, the rank interpolation conditions can be characterized via elements $G_{\alpha_i qt}$ and $G_{\beta_j pt}$ in $\ell_1^{n_z\times n_w}$ (see Appendix) as the following theorem states. In the following theorem $G_{\alpha_i qt}$ and $G_{\beta_j pt}$ (elements in $\ell_1^{n_z\times n_w}$) are defined in the Appendix.

Theorem 3 [2]: Define $RF^{ijk\lambda_0} := \text{Real}(F^{ijk\lambda_0})$ and $IF^{ijk\lambda_0} = \text{Imaginary}(F^{ijk\lambda_0})$. Suppose $\Lambda_{UV} \subset \text{int}(\mathcal{D})$, then Φ in $\ell_1^{n_z\times n_w}$ is in Θ if and only if the following conditions hold:

$$\begin{cases} \langle \Phi, RF^{ijk\lambda_0}\rangle = \langle H, RF^{ijk\lambda_0}\rangle \\ \langle \Phi, IF^{ijk\lambda_0}\rangle = \langle H, IF^{ijk\lambda_0}\rangle \end{cases}$$

$$\text{for} \begin{cases} \lambda_0 \text{ in } \Lambda_{UV} \\ i = 1,\cdots,n_u \\ j = 1,\cdots,n_y \\ k = 0,\cdots,\sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) - 1 \end{cases}$$

and

$$\begin{cases} \langle \Phi, G_{\alpha_i qt}\rangle = \langle H, G_{\alpha_i qt}\rangle \\ \langle \Phi, G_{\beta_j pt}\rangle = \langle H, G_{\beta_j pt}\rangle \end{cases}$$

$$\text{for} \begin{cases} i = n_u+1,\cdots,n_z \\ j = n_y+1,\cdots,n_w \\ q = 1,\cdots,n_w \\ p = 1,\cdots,n_z \\ t = 0,1,2,\cdots. \end{cases}$$

Furthermore, $F^{ijk\lambda_0}$, $G_{\alpha_i qt}$ and $G_{\beta_j pt}$ are matrix sequences in $\ell_1^{n_z\times n_w}$.

Proof: The proof is omitted. ∎

We assume without loss of generality that $F^{ijk\lambda_0}$ is a real sequence. Further, we define

$$b^{ijk\lambda_0} := \langle H, F^{ijk\lambda_0}\rangle$$

and

$$c_z := \sum_{\lambda_0 \in \Lambda_{UV}} \sum_{i=1}^{n_u} \sum_{j=1}^{n_y} \sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0).$$

c_z is the total number of zero interpolation conditions. The following problem:

$$\nu_{0,1} = \inf_{\Phi \text{Achievable}} \{\|\Phi\|_1\} \tag{5}$$

is the standard MIMO $\ell_1^{n_z \times n_w}$ problem. In [3] it is shown that this problem for a square plant has a solution, possibly nonunique, but the solution is a finite impulse response matrix sequence. Let

$$\mu_{0,2} := \inf_{\Phi \text{Achievable}} \{\|\Phi\|_2^2\} \tag{6}$$

which is the standard $\mathcal{H}_2$ problem. The solution to this problem is unique and is an infinite impulse response (IIR) sequence. We now collect all the assumptions made (which are valid throughout this paper) for easy reference.

Assumption 1: $\hat{U}$ has normal rank n_u and $\hat{V}$ has normal rank n_y.

Assumption 2: $\hat{U}$ and $\hat{V}$ have no zeros which lie on the unit circle, that is $\Lambda_{UV} \subset \text{int}(\mathcal{D})$.

Assumption 3: $F^{ijk\lambda_0}$ is a real sequence.

B. Mathematical Preliminaries

In this section we collect all the relevant theorems and definitions from convex optimization theory. Reference [1] is an excellent reference for this section. We present a Lagrange duality theorem that applies to the minimization of a convex functional subject to both equality and inequality constraints. A sensitivity result which follows directly from the Lagrange duality theorem is presented. This section can be skipped and can be referred to whenever required by the reader. We employ the terminology used in [1]. First, we need the following definitions.

Definition 1: Let P be a convex cone in a vector space X. We write $x \geq y$ if $x - y$ is in P. We write $x > 0$ if x is in $\text{int}(P)$. Similarly $x \leq y$ if $x - y$ is in $-P =: N$ and $x < 0$ if x is in $\text{int}(N)$.

Definition 2: Let X be a vector space and Z be a vector space with positive cone P. A mapping G: $X \to Z$ is convex if $G(tx + (1-t)y) \leq tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with $0 \leq t \leq 1$. It is strictly convex if $G(tx + (1-t)y) < tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with $0 < t < 1$.

The following is a Lagrange duality theorem.

Theorem 4: Let X be a Banach space, Ω a convex subset of X, Y a finite-dimensional space, Z a normed space with positive cone P. Let f: $\Omega \to R$ be a real valued convex functional, g: $X \to Z$ be a convex mapping, H: $X \to Y$ be an affine linear map and $0 \in \text{int}[\text{range}(H)]$. Define

$$\mu_0 := \inf\{f(x) \colon g(x) \leq 0, H(x) = 0, x \text{ in } \Omega\}. \tag{7}$$

Suppose there exists x_1 in Ω such that $g(x_1) < 0$ and $H(x_1) = 0$ and suppose μ_0 is finite. Then

$$\mu_0 = \max\{\varphi(z^*, y) \colon z^* \geq 0, z^* \text{ in } Z^*, y \text{ in } Y\} \tag{8}$$

where $\varphi(z^*, y) := \inf\{f(x) + \langle g(x), z^*\rangle + \langle H(x), y\rangle \colon x$ in $\Omega\}$ and the maximum is achieved for some $z_0^* \geq 0$, z_0^* in Z^*, y_0 in Y.

Furthermore, if the infimum in (7) is achieved by some x_0 in Ω, then

$$\langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle = 0 \tag{9}$$

and

$$x_0 \text{ minimizes } f(x) + \langle g(x), z_0^*\rangle + \langle H(x), y_0\rangle, \text{ over all } x \text{ in } \Omega. \tag{10}$$

Proof: See [1]. ■

We refer to (7) as the *Primal* problem and (8) as the *Dual* problem.

Corollary 1: Let X, Y, Z, f, H, g, Ω be as in Theorem 4. Let x_0 be the solution to the problem

$$\text{minimize } f(x)$$
$$\text{subject to } x \text{ in } \Omega, H(x) = 0, g(x) \leq z_0$$

with (z_0^*, y_0) as the dual solution. Let x_1 be the solution to the problem

$$\text{minimize } f(x)$$
$$\text{subject to } x \text{ in } \Omega, H(x) = 0, g(x) \leq z_1$$

with (z_1^*, y_1) as the dual solution. Then

$$\langle z_1 - z_0, z_1^*\rangle \leq f(x_0) - f(x_1) \leq \langle z_1 - z_0, z_0^*\rangle. \tag{11}$$

Proof: See [1]. ■

III. The Combination Problem

In this section we state and solve the combination problem. We first make the problem statement precise. Next we show the existence of an optimal solution. We then solve the problem for the square case. Finally, we study the nonsquare case.

Let N_w and N_z denote the sets $\{1, \cdots, n_w\}$ and $\{1, \cdots, n_z\}$, respectively. Let M, N, and MN be subsets of $N_z \times N_w$ such that the intersection between any two of these sets is empty and their union is $N_z \times N_w$. Let $\bar{c}_{pq}$ and c_{pq} be given positive constants for (p,q) in $MN \cup M$ and for (p,q) in $MN \cup N$, respectively. The problem of interest is the following: *Given a plant G solve the following optimization problem*:

$$\nu = \inf_{\Phi \text{Achievable}} \sum_{(p,q) \in MN \cup M} \bar{c}_{pq} \|\Phi_{pq}\|_2^2 + \sum_{(p,q) \in MN \cup N} c_{pq} \|\Phi_{pq}\|_1. \tag{12}$$

Note that for all (p,q) in M only the $\mathcal{H}_2$ norm of Φ_{pq} appears in the objective, for all (p,q) in N only the ℓ_1 norm of Φ_{pq} appears in the objective, and for all (p,q) in MN a combination of the $\mathcal{H}_2$ and the ℓ_1 norm of Φ_{pq}

appears in the objective. For notational convenience we define f: $\ell_2^{n_z \times n_w} \to R$ by

$$f(\Phi) := \sum_{(p,q) \in MN \cup M} \overline{c}_{pq} \|\Phi_{pq}\|_2^2 + \sum_{(p,q) \in MN \cup N} c_{pq} \|\Phi_{pq}\|_1$$

which is the objective functional being minimized. As it can be seen, the objective functional of the combination problem constitutes a weighted sum of the square of the $\mathcal{H}_2$ norm and the ℓ_1 norm of individual elements Φ_{pq} of the closed-loop map Φ. Note that with this type of functional the overall $\mathcal{H}_2$ norm of the closed loop as well as ℓ_1 norms of individual rows can be incorporated as special cases.

For technical reasons explained in the sequel we define the space

$$\mathcal{A} := \{\Phi \text{ in } \ell_2^{n_z \times n_w}\text{: } \Phi_{pq} \text{ in } \ell_1 \text{ for all } (p,q) \text{ in } MN \cup N\}.$$

The following set is an extension of Θ:

$$\begin{aligned}\overline{\Theta} := \{&\Phi \text{ in } \mathcal{A}\text{: } \Phi \text{ satisfies the zero and} \\ &\cdot \text{ the rank interpolation conditions}\}.\end{aligned}$$

Note that Θ is given by

$$\begin{aligned}\Theta = \{&\Phi \text{ in } \ell_1^{n_z \times n_w}\text{: } \Phi \text{ satisfies the zero and} \\ &\cdot \text{ the rank interpolation conditions}\}.\end{aligned}$$

Also, note that when M is empty then $\Theta = \overline{\Theta}$. Finally, we define the following optimization problem:

$$\nu_e := \inf_{\Phi \in \overline{\Theta}} f(\Phi). \tag{13}$$

Now, we show that a solution to (13) always exists.

Lemma 2: There exists Φ^0 in $\overline{\Theta}$ such that

$$\nu_e = \sum_{(p,q) \in MN \cup M} \overline{c}_{pq} \|\Phi_{pq}^0\|_2^2 + \sum_{(p,q) \in MN \cup N} c_{pq} \|\Phi_{pq}^0\|_1.$$

Therefore, the infimum in (13) is a minimum.

Proof: See the Appendix. ■

A. Square Case

In this part of the paper we solve the combination problem for the square case. Throughout this section the following assumption holds.

Assumption 4: $n_u = n_z$ and $n_y = n_w$.

In the sequel y in R^{c_z} is indexed by $ijk\lambda_0$ where i, j, k, λ_0 vary as in the zero interpolation conditions. The following lemma gives the dual problem for the square case.

Lemma 3: $\nu_e = |!\max\{\psi(y), y \text{ in } R^{c_z}\}$, where $\psi(y) = \inf_{\Phi \in \mathcal{A}} L(\Phi)$ with

$$\begin{aligned}L(\Phi) := &\sum_{(p,q) \in (MN) \cup M} \overline{c}_{pq} \|\Phi_{pq}\|_2^2 \\ &+ \sum_{(p,q) \in (MN) \cup N} c_{pq} \|\Phi_{pq}\|_1 \\ &+ \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} (b^{ijk\lambda_0} - \langle F^{ijk\lambda_0}, \Phi \rangle).\end{aligned}$$

Proof: We will apply Theorem 4 to get the result. Let X, Ω, Y, Z in Theorem 4 correspond to $\mathcal{A}, \mathcal{A}, R^{c_z}, R$, respectively. Let $\gamma = \nu_e + 1$ and let g: $\mathcal{A} \to R$ be given by $g(\Phi) := f(\Phi) - \gamma$. Let $\overline{H}$: $\mathcal{A} \to R^{c_z}$ be given by $\overline{H}_{ijk\lambda_0}(\Phi) := b^{ijk\lambda_0} - \langle F^{ijk\lambda_0}, \Phi \rangle$.

We index the equality constraints of $\overline{H}$ by $ijk\lambda_0$ where i, j, k, λ_0 vary as in the zero interpolation conditions. In [2] it is shown that the map $\overline{H}$ is onto R^{c_z} (it is shown that the zero interpolation conditions are independent). This means that zero is in $\operatorname{int}(\operatorname{Range}(\overline{H}))$. From Lemma 2 we know that there exists a Φ^1 in $\mathcal{A}$ such that $\overline{H}(\Phi^1) = 0$ (that is Φ^1 satisfies the zero interpolation conditions) and $f(\Phi^1) = \nu_e$ which implies that $g(\Phi^1) = -1 < 0$. Thus all the conditions of Theorem 4 are satisfied. The lemma follows by applying Theorem 4 to (13). ■

For notational convenience we define the functionals Z_{pq} in ℓ_1 by

$$Z_{pq}(t) := \Sigma_{i,j,k,\lambda_0} y_{ijk\lambda_0} F_{pq}^{ijk\lambda_0}(t).$$

In what follows we show that the dual problem is in fact a finite-dimensional convex programming problem. A bound on its dimension is also furnished.

Theorem 5: The following is true:

$$\begin{aligned}\nu = &\nu_e \\ = &\max \Bigg\{ \sum_{(p,q) \in M} \sum_{t=0}^{\infty} -\overline{c}_{pq} \Phi_{pq}(t)^2 \\ &+ \sum_{(p,q) \in MN} \sum_{t=0}^{\infty} -\overline{c}_{pq} \Phi_{pq}(t)^2 + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} b^{ijk\lambda_0} \Bigg\}\end{aligned}$$

subject to

$$y \text{ in } R^{c_z}, \Phi_{pq} \text{ in } \ell_1 \text{ for all } (p,q) \text{ in } MN \cup M$$

and (I), shown at the bottom of the page.

Furthermore, it holds that the infimum in (12) is a minimum, and Φ^0 is a solution of (13) if and only if it is a solution of (12). In addition, Φ_{pq}^0 is unique for all (p,q) in $(MN) \cup M$.

$$\left.\begin{aligned} -c_{pq} \le\ & Z_{pq}(t) \le c_{pq}, && \text{if } (p,q) \text{ in } N, \\ 2\overline{c}_{pq}\Phi_{pq}(t) =\ & Z_{pq}(t) - c_{pq}, && \text{if } (p,q) \text{ in } MN \text{ and } Z_{pq}(t) > c_{pq}, \\ =\ & Z_{pq}(t) + c_{pq}, && \text{if } (p,q) \text{ in } MN \text{ and } Z_{pq}(t) < -c_{pq}, \\ =\ & 0, && \text{if } (p,q) \text{ in } MN \text{ and } |Z_{pq}(t)| \le c_{pq}, \\ =\ & Z_{pq}(t), && \text{if } (p,q) \text{ in } M, \\ & \text{for all } t = 0, 1, 2, \cdots \end{aligned}\right\} \quad \text{(I)}$$

Proof: In Lemma 2 we showed that an optimal solution Φ^0 always exists for problem (13). From Theorem 4 we know that if y in R^{c_z} is optimal for the dual problem, then Φ^0 minimizes

$$\begin{aligned}L(\Phi) = &\sum_{(p,q)\in M}\sum_{t=0}^{\infty}(\overline{c}_{pq}\Phi_{pq}(t)^2 - Z_{pq}(t)\Phi_{pq}(t))\\ &+\sum_{(p,q)\in MN}\sum_{t=0}^{\infty}(\overline{c}_{pq}\Phi_{pq}(t)^2\\ &+c_{pq}|\Phi_{pq}(t)| - Z_{pq}(t)\Phi_{pq}(t))\\ &+\sum_{(p,q)\in N}\sum_{t=0}^{\infty}(c_{pq}|\Phi_{pq}(t)|\\ &-Z_{pq}(t)\Phi_{pq}(t)) + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0}b^{ijk\lambda_0}.\end{aligned}$$

Therefore, if (p,q) is in M then $\Phi^0_{pq}(t)$ minimizes $\overline{c}_{pq}\Phi_{pq}(t)^2 - Z_{pq}(t)\Phi_{pq}(t)$ which is strictly convex in $\Phi_{pq}(t)$ and therefore $\Phi^0_{pq}(t)$ is unique. Differentiating $\overline{c}_{pq}\Phi_{pq}(t)^2 - Z_{pq}(t)\Phi_{pq}(t)$ with respect to $\Phi_{pq}(t)$ and equating the result to zero, we conclude that if (p,q) is in M then $2\overline{c}_{pq}\Phi^0_{pq}(t) = Z_{pq}(t)$. As Z_{pq} is in ℓ_1 we have that for all (p,q) in M, Φ^0_{pq} is in ℓ_1.

If (p,q) is in MN, then $\Phi^0_{pq}(t)$ minimizes $\overline{c}_{pq}\Phi_{pq}(t)^2 + c_{pq}|\Phi_{pq}(t)| - Z_{pq}(t)\Phi_{pq}(t)$, which is strictly convex and therefore the minimizer is unique. This also implies that if (p,q) is in MN then $\Phi^0_{pq}(t)$ satisfies conditions stipulated in (I).

Suppose (p,q) is in N, then $\Phi^0_{pq}(t)$ minimizes $\overline{f}(\Phi_{pq}(t)) := \{c_{pq}|\Phi_{pq}(t)| - Z_{pq}(t)\Phi_{pq}(t)\}$. Note that if $Z_{pq}(t)\Phi_{pq}(t) < 0$ then $\overline{f}(\Phi_{pq}(t)) \geq 0$. We know that $\overline{f}(0) = 0$. Therefore, the optimal minimizes $\Phi_{pq}(t)(c_{pq}\ \operatorname{sgn}(Z_{pq}(t)) - Z_{pq}(t))$, over all $\Phi_{pq}(t)$ in R such that $Z_{pq}(t)\Phi_{pq}(t) \geq 0$. Now, if $|Z_{pq}(t)| > c_{pq}$, then given any $K > 0$ we can choose $\Phi_{pq}(t)$ in R that satisfies $Z_{pq}(t)\Phi_{pq}(t) > 0$ and $\overline{f}(\Phi_{pq}(t)) < -K$ and thus the infimum value would be $-\infty$. Therefore, we can restrict $Z_{pq}(t)$ in the maximization of the dual to satisfy $|Z_{pq}(t)| \leq c_{pq}$. If, $|Z_{pq}(t)| < c_{pq}$ then $\overline{f}(\Phi_{pq}(t)) \geq 0$ for any $\Phi_{pq}(t)$ in R that satisfies $Z_{pq}(t)\Phi_{pq}(t) \geq 0$ and is equal to zero only for $\Phi_{pq}(t) = 0$. Therefore, we conclude that if we have (p,q) in N, then we can restrict $Z_{pq}(t)$ in the maximization of the dual to satisfy $|Z_{pq}(t)| \leq c_{pq}$ and that the optimal $\Phi^0_{pq}(t)$ minimizes $\overline{f}(\Phi_{pq}(t))$ with a minimum value of zero. It also follows that if $|Z_{pq}(t)| < c_{pq}$ then $\Phi_{pq}(t)$ in R that minimizes $\overline{f}(\Phi_{pq}(t))$ is equal to zero.

The expression for ν_e follows by substituting the value of $\Phi^0_{pq}(t)$ obtained in the above discussion for various indexes in the functional $L(\Phi)$. Note that $\Theta = \overline{\Theta} \cap \ell_1^{n_z\times n_w}$. But in the previous steps we have shown by construction that the optimal solution to (13), Φ^0, is such that Φ^0_{pq} is in ℓ_1 for all (p,q) in M. This means that Φ^0 is in Θ. From the above discussion the theorem follows easily. ∎

Note that the above theorem demonstrates that the problem at hand is finite-dimensional. Indeed, at an optimal point y^0, Φ^0 the constraint $|Z^0_{pq}(t)| \leq c_{pq}$ will be satisfied for sufficiently large t since Z^0_{pq} is in ℓ_1. Thus, $\Phi^0_{pq}(t) = 0$ for (p,q) in $MN \cup N$ and large t, i.e., Φ^0_{pq} is FIR for (p,q) in $MN \cup N$. The following lemmas provide a way to compute *a priori* bounds on the dimension of the problem.

Lemma 4: Let Φ^0 be a solution to the primal problem (12) and let y^0, Z^0_{pq} be solutions to the dual. Then the following is true:

$$\begin{aligned}-c_{pq} \leq Z^0_{pq}(t) \leq c_{pq}, &\quad \text{if } (p,q) \text{ belongs to } N,\\ \Phi^0_{pq}(t) = 0, &\quad \text{if } (p,q) \text{ belongs to } N \text{ and}\\ &\quad \cdot |Z^0_{pq}(t)| < c_{pq},\\ 2\overline{c}_{pq}\Phi^0_{pq}(t) = Z^0_{pq}(t) - c_{pq}, &\quad \text{if } (p,q) \text{ belongs to } MN \text{ and}\\ &\quad \cdot Z^0_{pq}(t) > c_{pq},\\ = Z^0_{pq}(t) + c_{pq}, &\quad \text{if } (p,q) \text{ belongs to } MN\\ &\quad \cdot \text{and } Z^0_{pq}(t) < -c_{pq},\\ = 0, &\quad \text{if } (p,q) \text{ belongs to } MN \text{ and}\\ &\quad \cdot |Z^0_{pq}(t)| \leq c_{pq},\\ = Z^0_{pq}(t), &\quad \text{if } (p,q) \text{ belongs to } M.\end{aligned}$$

Φ^0_{pq} is unique for all (p,q) in $(MN) \cup M$. Also, there exists an *a priori* bound α such that $||Z^0_{pq}||_\infty \leq \alpha$ for all (p,q) in $N_z \times N_w$.

Proof: The first part of the lemma follows from the arguments used in Theorem 5. We now determine the *a priori* bound. For all (p,q) in MN the following is true:

$$\begin{aligned}|Z^0_{pq}(t)| &\leq c_{pq} + 2\overline{c}_{pq}|\Phi^0_{pq}(t)| \leq c_{pq} + 2\overline{c}_{pq}||\Phi^0_{pq}||_1\\ &\leq c_{pq} + \frac{2\overline{c}_{pq}}{c_{pq}}f(H)\end{aligned}$$

where the last inequality follows since H is a feasible solution and hence $c_{pq}||\Phi^0_{pq}||_1 \leq f(\Phi^0) \leq f(H)$. For all (p,q) in N the following is true: $|Z^0_{pq}(t)| \leq c_{pq}$, and for all (p,q) in M

$$\begin{aligned}|Z^0_{pq}(t)| &\leq 2\overline{c}_{pq}|\Phi^0_{pq}(t)| \leq 2\overline{c}_{pq}||\Phi^0_{pq}||_2\\ &\leq 2\overline{c}_{pq}\sqrt{\frac{f(\Phi^0)}{\overline{c}_{pq}}} \leq 2\overline{c}_{pq}\sqrt{\frac{f(H)}{\overline{c}_{pq}}}.\end{aligned}$$

Denote by d_{pq} the upper bounds determined above, all of which can be determined *a priori*. Let $\alpha := \max_{(p,q)\in N_z\times N_w}\ d_{pq}$. Thus α is an *a priori* upperbound on $||Z^0_{pq}||_\infty$ for all (p,q) in $N_z \times N_w$. This proves the lemma. ∎

Lemma 5: Let $c_{\min} := \min_{(p,q)\in MN\cup N}\ c_{pq}$. If y in R^{c_z} is such that for all (p,q) in $N_z \times N_w$ $||Z_{pq}||_\infty \leq \alpha$, where $Z_{pq}(t) := \Sigma_{ijk\lambda_0}\ y_{ijk\lambda_0}F^{ijk\lambda_0}_{pq}(t)$, then there exists an *a priori* computable positive integer L^* such that

$$|Z_{pq}(t)| < c_{\min}, \qquad \text{for all } t \geq L^*.$$

Proof: For notational convenience we index $F^{ijk\lambda_0}_{pq}$ and $b^{ijk\lambda_0}$ where $ijk\lambda_0$ vary as in the zero interpolation conditions by F^n_{pq} and b^n, respectively, where $n = 1, \cdots, c_z$. The vector in R^{c_z} whose nth element is given by b^n is denoted by b. We interpret F_{pq}^n as an $\infty \times 1$ column vector equal to $(F_{pq}^n(0), F_{pq}^n(1), \cdots)'$. With this notation $Z_{pq} = (F_{pq}^1, F_{pq}^2, \cdots, F_{pq}^{c_z})y$, where Z_{pq} is viewed as a infinite column vector with the tth element equal to $Z_{pq}(t)$. Therefore, the condition $\|Z_{pq}\|_\infty \leq \alpha$ for all (p,q) in $N_z \times N_w$, is equivalent to the condition $\|A'y\|_\infty \leq \alpha$, where

$$A' = \begin{pmatrix} F_{11}^1 & F_{11}^2 & \cdots & F_{11}^{c_z} \\ F_{12}^1 & F_{12}^2 & \cdots & F_{12}^{c_z} \\ \vdots & \vdots & \vdots & \vdots \\ F_{1n_w}^1 & F_{1n_w}^2 & \cdots & F_{1n_w}^{c_z} \\ F_{21}^1 & F_{21}^2 & \cdots & F_{21}^{c_z} \\ \vdots & \vdots & \vdots & \vdots \\ F_{n_z n_w}^1 & F_{n_z n_w}^2 & \cdots & F_{n_z n_w}^{c_z} \end{pmatrix}.$$

The matrix $A := (A')'$ is the matrix which has c_z rows each for one zero interpolation condition. If Φ in $\ell_1^{n_z \times n_w}$ is stringed into a vector as follows:

$$\Phi = (\Phi_{11}\Phi_{12}\cdots\Phi_{1n_w}\Phi_{21}\cdots\Phi_{n_z n_w})'$$

then $A\Phi = b$ gives the zero interpolation conditions. It is known that the zero interpolation conditions are independent and therefore A has full row rank. Equivalently, A' has full column rank. Choose $L \geq c_z$ such that D in $R^{c_z \times c_z}$ with rows from the first L rows of A' is invertible. Consider D as a map from $(R^{c_z}, |\cdot|_1)$ to $(R^{c_z}, |\cdot|_\infty)$. Now, as $y = D^{-1}Dy$ we have

$$|y|_1 = |D^{-1}Dy|_1 \leq |D^{-1}|_{\infty,1}|Dy|_\infty \leq |D^{-1}|_{\infty,1}\alpha \quad (14)$$

where $|D^{-1}|_{\infty,1}$ is the induced norm of D^{-1}. The last inequality follows because $\|A'y\|_\infty \leq \alpha$ which implies $|Dy|_\infty \leq \alpha$.

From the note after Lemma 1 we know that for any n and for any (p,q) in $N_z \times N_w$ there exists an integer T_{pqn} such that $t > T_{pqn}$ implies that $|F_{pq}^n(t)| < (c_{\min}/|D^{-1}|_{\infty,1}\alpha)$. Let $L^* = \max_{pqn} T_{pqn}$. Note that L^* is determined *a priori*. Let $t > L^*$ then

$$\begin{aligned} |Z_{pq}(t)| &= \left|\sum_{n=1}^{c_z} F_{pq}^n(t)y_n\right| \leq \sum_{n=1}^{c_z} |y_n||F_{pq}^n(t)| \\ &\leq \frac{c_{\min}}{|D^{-1}|_{\infty,1}\alpha}\sum_{n=1}^{c_z}|y_n| \leq c_{\min}. \end{aligned}$$

The last inequality follows from (14). This proves the lemma. ■

We now state the main result of this section.

Theorem 6: The following is true:

$$\begin{aligned} \nu = \max \Bigg\{ & \sum_{(p,q)\in M}\sum_{t=0}^{\infty} -\frac{1}{4\bar{c}_{pq}} Z_{pq}(t)^2 \\ & + \sum_{(p,q)\in MN}\sum_{t=0}^{L^*} -\bar{c}_{pq}\Phi_{pq}(t)^2 + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} b^{ijk\lambda_0} \Bigg\} \end{aligned}$$

subject to

$$\left.\begin{array}{ll} y \text{ in } R^{c_z}, \Phi_{pq} \text{ in } R^{L^*}, & \text{for all } (p,q) \text{ in } MN, \\ -c_{pq} \leq Z_{pq}(t) \leq c_{pq}, & \text{for all } (p,q) \text{ in } N \\ -c_{pq} \leq 2\bar{c}_{pq}\Phi_{pq}(t) & \\ \quad -Z_{pq}(t) \leq c_{pq}, & \text{for all } (p,q) \text{ in } MN \end{array}\right\} \quad \text{(II)}$$

for all $t = 0, 1, 2, \cdots, L^*$ where L^* is determined *a priori* as given in Lemma 5.

Furthermore, the optimal of the primal (12) Φ_{pq}^0 is unique for all (p,q) in $(MN) \cup M$.

Proof: A conclusion of Lemmas 4 and 5 is that

$$\begin{aligned} \nu = \max \Bigg\{ & \sum_{(p,q)\in M}\sum_{t=0}^{\infty} -\frac{1}{4\bar{c}_{pq}} Z_{pq}(t)^2 \\ & + \sum_{(p,q)\in MN}\sum_{t=0}^{L^*} -\bar{c}_{pq}\Phi_{pq}(t)^2 + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} b^{ijk\lambda_0} \Bigg\} \end{aligned}$$

subject to (III), as shown at the bottom of the page, for all $t = 0, 1, 2, \cdots, L^*$ where L^* is determined *a priori* as given in Lemma 5.

Indeed, from Lemma 4 we know that we can restrict the maximization in Theorem 5 over the set y in R^{c_z} such that $\|Z_{pq}\|_\infty \leq \alpha$. Now we conclude from Lemma 5 that there exists an integer L^* such that $t > L^*$ implies that $|Z_{pq}(t)| < c_{\min} \leq c_{pq}$. One of the conditions stipulated in (I) of Theorem 5 is that $\Phi_{pq}(t) = 0$ if (p,q) is in MN and $|Z_{pq}(t)| < c_{pq}$. Note that the condition $2\Phi_{pq}(t) = Z_{pq}(t)$ for any (p,q) in M has been incorporated into the objective functional of this theorem. The fact that Φ_{pq}^0 is unique for all (p,q) in $(MN) \cup M$ is due to Theorem 5.

To bring the problem into the form stated in the theorem, denote the right-hand side of the equation in the statement of the theorem by $\bar{\nu}$. Suppose y in $R^{c_z}, Z_{pq}(t)$ (determined by y), and $\Phi_{pq}(t)$ satisfy condition (III) above for all the appropriate indexes. If (p,q) is in MN and $Z_{pq}(t) > c_{pq}$ then $-c_{pq} = 2\bar{c}_{pq}\Phi_{pq}(t) - Z_{pq}(t) \leq c_{pq}$, because $c_{pq} \geq 0$. Similarly it can be checked that all the other conditions of

$$\left.\begin{array}{rl} y \text{ in } R^{c_z}, \Phi_{pq} \text{ in } R^{L^*} & \text{for all } (p,q) \text{ in } MN, \\ -c_{pq} \leq Z_{pq}(t) \leq c_{pq} & \text{for all } (p,q) \text{ in } N \\ 2\bar{c}_{pq}\Phi_{pq}(t) = Z_{pq}(t) - c_{pq} & \text{for all } (p,q) \text{ in } MN \text{ and } Z_{pq}(t) > c_{pq} \\ = Z_{pq}(t) + c_{pq} & \text{for all } (p,q) \text{ in } MN \text{ and } Z_{pq}(t) < -c_{pq} \\ = 0 & \text{for all } (p,q) \text{ in } MN \text{ and } |Z_{pq}(t)| \leq c_{pq} \end{array}\right\} \quad \text{(III)}$$

(II) are satisfied. This implies that $\overline{\nu} \geq \nu$. Suppose y in $R^{c_z}, Z_{pq}(t)$ (determined by y) and $\Phi_{pq}(t)$ satisfy condition (II) of Theorem 6. Let $\overline{\Phi}_{pq}(t)$ be defined as follows:

$$\begin{aligned}
2\overline{c}_{pq}\overline{\Phi}_{pq}(t) &= Z_{pq}(t) - c_{pq} \quad \text{for all } (p,q) \text{ in } MN \text{ and} \\
&\cdot Z_{pq}(t) > c_{pq}, \\
2\overline{c}_{pq}\overline{\Phi}_{pq}(t) &= Z_{pq}(t) + c_{pq} \quad \text{for all } (p,q) \text{ in } MN \text{ and} \\
&\cdot Z_{pq}(t) < -c_{pq}, \\
\overline{\Phi}_{pq}(t) &= 0 \quad \text{for all } (p,q) \text{ in } MN \text{ and } |Z_{pq}(t)| \leq c_{pq}, \\
&\text{for all } 0 \leq t \leq L^* \text{ (i.e., } \overline{\Phi}_{pq}(t) \\
&\text{satisfies constraints (III)).}
\end{aligned}$$

Suppose (p,q) is in MN and $Z_{pq}(t) > c_{pq}$ then $0 \leq 2\overline{c}_{pq}\overline{\Phi}_{pq}(t) = Z_{pq}(t) - c_{pq} \leq 2\overline{c}_{pq}\Phi_{pq}(t)$. Therefore, $-\overline{\Phi}_{pq}^2(t) \geq -\Phi_{pq}^2(t)$. Similarly, the above condition follows for other indexes. Thus, given variables satisfying (II), we have constructed variables satisfying (III) which achieve a greater objective value. This proves that $\overline{\nu} \leq \nu$. This proves the theorem. ■

Thus, we have shown that the problem (12) for a square plant is equivalent to the finite-dimensional quadratic programming problem of Theorem 6 with the dimension known *a priori*. Such optimization problems are well studied in the literature and efficient numerical methods are available (e.g., [5]). We should point out that the sum $\Sigma_{t=0}^{\infty} -(1/4\overline{c}_{pq})Z_{pq}(t)^2$ appearing in the quadratic program above is a quadratic function of $y_{ijk\lambda_0}$ with coefficients of the form $\langle F_{pq}^{ijk\lambda_0}, F_{pq}^{\overline{\lambda}_0}\rangle$, which can be computed.

The solution procedure consists of solving the quadratic program of Theorem 6 to obtain the optimal variables $y_{ijk\lambda_0}^0$ and $\Phi_{pq}^0(t)$ with $t = 0, \cdots, L^*$ for all (p,q) in MN. The latter set completely determines Φ_{pq}^0 for all (p,q) in MN. From $y_{ijk\lambda_0}^0$ the optimal Φ_{pq}^0 for (p,q) in M can be computed as $\Phi_{pq}^0 = (1/2\overline{c}_{pq})Z_{pq}^0$ (see Lemma 4). The quadratic program of Theorem 6 does not yield immediately any information on Φ_{pq}^0 for any (p,q) in N. Nonetheless, Φ_{pq}^0 for (p,q) in N can be easily obtained once Φ_{pq}^0 for (p,q) in $MN \cup M$ are found, through the following (finite-dimensional) optimization:

$$\begin{aligned}
&\text{minimize} \sum_{(p,q)\in N} c_{pq}\|\Phi_{pq}\|_1 \\
&\text{subject to} \\
&\quad \Phi_{pq} \text{ in } R^{L^*} \\
&\quad \cdot \sum_{(p,q)\in N} \langle F^{ijk\lambda_0}, \Phi_{pq}\rangle = b^{ijk\lambda_0} \\
&\quad - \sum_{(p,q)\in (MN)\cup M} \langle F^{ijk\lambda_0}, \Phi_{pq}^0\rangle.
\end{aligned}$$

This problem can be readily solved via linear programming [2].

From the developments above it follows that the structure of an optimal solution Φ^0 to the primal problem (12) has in general an IIR. The parts of Φ^0, however, that are contained in the cost via their ℓ_1 norm, i.e., the Φ_{pq}^0's with (p,q) in $MN \cup N$, are always FIR.

Finally, it should be noted that the optimal solution has certain properties related to the notion of Pareto optimality (e.g., [5]). In particular, from the uniqueness properties of Φ^0 it is clear that there is no other feasible Φ such that $\|\Phi_{pq}\|_2 < \|\Phi_{pq}^0\|_2$ for some (p,q) in $MN \cup M$ while $\|\Phi_{pq}\|_1 \leq \|\Phi_{pq}^0\|_1$; or, conversely, there is no Φ such that $\|\Phi_{pq}\|_1 < \|\Phi_{pq}^0\|_1$ for some (p,q) in $MN \cup M$ while $\|\Phi_{pq}\|_2 \leq \|\Phi_{pq}^0\|_2$.

B. The Nonsquare Case

In the previous section we analyzed problem (12) when the plant is square. In this section we study the problem when the plant is allowed to be nonsquare, that is, we allow for $n_z > n_u$ and $n_w > n_y$. We employ the Delay Augmentation method which converts the nonsquare plant to a square plant. We borrow heavily from [2, Sec. 12.2.2]. First we introduce some notation. We denote block submatrices by $(\cdot)^{ij}$, so Φ^{ij} will denote the ijth block matrix in contrast to Φ_{ij} which will represent the ijth element of Φ. The dimension of the block matrices will be clear from the context. S denotes a unit shift, that is, $S(x(0), x(1), x(2), \cdots) = (0, x(0), x(1), \cdots)$, and S^T denotes a Tth order shift. When the plant is nonsquare the rank interpolation conditions are no longer absent. The delay augmentation method converts the nonsquare plant to a square plant by using delays. The augmented problem being square has no rank interpolation conditions and can be handled by the theory developed in Section III-A.

Suppose that the Youla parameterization of the plant yields H in $\ell_1^{n_z \times n_w}$, U in $\ell_1^{n_z \times n_u}$, and V in $\ell_1^{n_y \times n_w}$. Partition $\hat{U}$ into

$$\hat{U} = \begin{pmatrix} \hat{U}^1 \\ \hat{U}^2 \end{pmatrix}$$

where U^1 in $\ell_1^{n_u \times n_u}$. Similarly, partition $\hat{V}$ into $(\hat{V}^1, \hat{V}^2)$ where V^1 is in $\ell_1^{n_y \times n_y}$. Let Φ and H be partitioned according to the following equation:

$$\begin{pmatrix} \hat{\Phi}^{11} & \hat{\Phi}^{12} \\ \hat{\Phi}^{21} & \hat{\Phi}^{22} \end{pmatrix} = \begin{pmatrix} \hat{H}^{11} & \hat{H}^{12} \\ \hat{H}^{21} & \hat{H}^{22} \end{pmatrix} - \begin{pmatrix} \hat{U}^1 \\ \hat{U}^2 \end{pmatrix} \hat{Q}^{11} (\hat{V}^1 \; \hat{V}^2). \tag{15}$$

We augment $\hat{U}$ and $\hat{V}$ by Tth order delays and augment $\hat{Q}^{11}$ as given by the following:

$$\begin{aligned}
\begin{pmatrix} \hat{\Phi}^{11,T} & \hat{\Phi}^{12.T} \\ \hat{\Phi}^{21,T} & \hat{\Phi}^{22,T} \end{pmatrix} &= \begin{pmatrix} \hat{H}^{11} & \hat{H}^{12} \\ \hat{H}^{21} & \hat{H}^{22} \end{pmatrix} - \begin{pmatrix} \hat{U}^1 & 0 \\ \hat{U}^2 & \hat{S}^T \end{pmatrix} \\
&\cdot \begin{pmatrix} \hat{Q}^{11} & \hat{Q}^{12} \\ \hat{Q}^{21} & \hat{Q}^{22} \end{pmatrix} \begin{pmatrix} \hat{V}^1 & \hat{V}^2 \\ 0 & \hat{S}^T \end{pmatrix}
\end{aligned} \tag{16}$$

or equivalently, $\hat{\Phi}^T := \hat{H} - \hat{U}^T\hat{Q}\hat{V}^T =: \hat{H} - \hat{R}^T$. Alternatively, another expansion of $\hat{\Phi}^T$ is given by $\hat{\Phi}^T = \hat{H} - \hat{U}\hat{Q}^{11}\hat{V} - \hat{S}^T\tilde{R}^T$, where

$$\tilde{R}^T := \begin{pmatrix} 0 & \hat{U}^1\hat{Q}^{12} \\ \hat{Q}^{21} & \hat{Q}^{21}\hat{V}^2 + \hat{U}^2\hat{Q}^{12} + \hat{S}^T\hat{Q}^{22} \end{pmatrix}.$$

We define the Delay Augmentation problem of order T by

$$\begin{aligned}
\underline{\nu}^T := \inf\{f(\Phi^T)\colon \hat{\Phi}^T = \hat{H} - \hat{U}^T\hat{Q}\hat{V}^T \\
\text{for some } Q \text{ in } \ell_1^{n_z \times n_w}\}
\end{aligned} \tag{17}$$

where $f(\Phi) = \Sigma_{(p,q)\in(MN)\cup M}\ \bar{c}_{pq}\|\Phi_{pq}\|_2^2 + \Sigma_{(p,q)\in(MN)\cup N}\ c_{pq}\|\Phi_{pq}\|_1$. Problem (17) is a square problem because U^T, V^T belong to $\ell_1^{n_z\times n_z}$ and $\ell_1^{n_w\times n_w}$, respectively. From Section III-A we know that problem (17) has a solution. Suppose Φ^T is optimal for (17) where $\hat{\Phi}^T = \hat{H} - \hat{U}^T\hat{Q}^T\hat{V}^T = \hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V} - \hat{S}^T\tilde{R}^T$. We define

$$\overline{\nu}^T := f(\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}). \tag{18}$$

With these definitions the following lemma is true.

Lemma 6:

$$\underline{\nu}^T \le \nu \le \overline{\nu}^T \tag{19}$$

for all integers T.

Proof: The proof is omitted. ■

It is also true that $\underline{\nu}^T$ *is a nondecreasing sequence in* T, because the only difference between the problem definition of $\underline{\nu}^T$ and the problem definition of $\underline{\nu}^{T+1}$ is an extra zero in (16). This results in a more restricted set of interpolation conditions for the problem $\underline{\nu}^{T+1}$ (see [2, p. 284] for a more detailed discussion). Our intention is to show that $\underline{\nu}^T \to \nu$ and $\overline{\nu}^T \to \nu$ as $T \to \infty$. We make the following assumptions.

Assumption 5: $\hat{U}^1(\lambda)$ and $\hat{V}^1(\lambda)$ have no zeros on the unit circle.

See [2] for the implications of this assumption.

Assumption 6: M is empty, i.e., for all (p,q) in $N_z \times N_w, \|\Phi_{pq}\|_1$ appears in the cost $f(\Phi)$.

This assumption is made to guarantee the existence of solutions in $\ell_1^{n_z\times n_w}$ (see Lemma 2 and the discussion before it). Note that this assumption was not necessary for the one-block problem since the optimal solution was shown to be in $\ell_1^{n_z\times n_w}$.

Theorem 7: Let Φ^T be an optimal solution to the Delay Augmentation problem (17). Let

$$\hat{\Phi}^T = \hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V} - \hat{S}^T\tilde{R}^T$$

be the appropriate expansion. Then, there exists Φ^0 in $\ell_1^{n_z\times n_w}$ and a subsequence $\{\Phi^{T_s}\}$ of $\{\Phi^T\}$ such that $\Phi^{T_s} \to \Phi^0$ in the $W((c_0^{n_z\times n_w})^*, c_0^{n_z\times n_w})$ topology. Furthermore, the unaugmented problem's [see (12)] optimal solution is Φ^0 with $\|\hat{H} - \hat{U}\hat{Q}^{11,T_s}\hat{V} - \hat{\Phi}^0\|_1 \to 0$. Also

$$\underline{\nu}^T \to \nu \quad \text{and} \quad \overline{\nu}^{T_s} \to \nu.$$

Moreover, if N is empty then the above results hold for the original sequences.

Proof: The proof is provided in the Appendix. ■

Thus, the optimal solution can be approximated arbitrarily closely by using the controller that corresponds to the Youla parameter Q^{11,T_s}. Moreover, knowledge of the distance from an optimal can be obtained from the difference $\overline{\nu}^{T_s} - \underline{\nu}^{T_s}$.

IV. The Mixed Problem

In this section we make the statement for the mixed problem precise. We solve the mixed problem via a related problem called the approximate problem. For both the mixed and the approximate problems the following notation is relevant: let $N_w := \{1, \cdots, n_w\}$ and let $N_z := \{1, \cdots, n_z\}$. Let $\mathcal{S}$ be a given subset of N_z. $\mathcal{S}$ corresponds to those rows of the closed loop which have some part constrained in the one norm. We denote the cardinality of $\mathcal{S}$ by c_n. Let N_p for p in $\mathcal{S}$ be a subset of N_w. N_p characterizes the part of the pth row of the closed loop that is constrained in the one norm. The (positive) scalars γ_p for p in $\mathcal{S}$ represent the ℓ_1 constraint level on the pth row. It is assumed that $\gamma_p > \nu_{0,1}$. Finally, γ in R^{c_n} is a vector which has γ_p for p in $\mathcal{S}$ as its elements.

We define a set $\Gamma_\gamma \subset \ell_1^{n_x\times n_w}$ of feasible solutions as follows: Φ in $\ell_1^{n_s\times n_w}$ is in Γ_γ if and only if it satisfies the following conditions:

1) $\displaystyle\sum_{q\in N_p} \|\Phi_{pq}\|_1 \le \gamma_p$ for all p in $\mathcal{S}$.
2) Φ is in Θ (i.e Φ is an achievable closed loop map).

Φ is said to be *feasible* if Φ is in Γ_γ. Let $\overline{M}$ be a given subset of $N_z \times N_w$.

The problem statements for the mixed and the approximate problems are now presented. *Given a plant* G *the* mixed problem *is the following optimization:*

$$\mu_\gamma := \inf_{\Phi\in\Gamma_\gamma} \left\{ \sum_{(p,q)\in\overline{M}} \|\Phi_{pq}\|_2^2 \right\}. \tag{20}$$

Given a plant G *the* approximate problem *of order* δ *is the following optimization:*

$$\mu_\gamma^\delta := \inf_{\Phi\in\Gamma_\gamma} \left\{ \sum_{(p,q)\in\overline{M}} \|\Phi_{pq}\|_2^2 + \delta \sum_{p\in\mathcal{S}} \sum_{q\in N_p} \|\Phi_{pq}\|_1 \right\}. \tag{21}$$

We will further assume that for all (p,q) in $N_z \times N_w$ the component Φ_{pq} appears in the ℓ_1 constraint or in the objective function or in both. Note that $\overline{M}$ is the set of transfer function pairs whose two norms have to be minimized in the problem. The problem is set up so that one can include the constraint of a complete row in the closed loop map Φ or part of a row. This way we can incorporate constraints of the form $\|\Phi\|_1 \le 1$ which is equivalent to each row having one norm less than one. Also, the $\mathcal{H}_2$ norm of Φ can be included in the cost as a special case.

We also define the following sets which help in isolating various cases in the dual formulation: $\overline{N} := \cup_{i\in\mathcal{S}}(i, N_i)$, which is set of indexes (i,j) such that Φ_{ij} occur in the ℓ_1 constraint, $MN := \overline{M} \cap \overline{N}$, which is the set of indexes (i,j) such that Φ_{ij} occurs in the ℓ_1 constraint and its two norm appears in the objective, $M := \overline{M}\backslash MN$, which is the set of indexes (i,j) such that two norm of Φ_{ij} occurs in the objective but it does not appear in the ℓ_1 constraint and $N := \overline{N}\backslash MN$, which is the set of indexes (i,j) such that Φ_{ij} occurs in the ℓ_1 constraint but its two norm does not appear in the objective. With this notation we have $\overline{M} = (MN) \cup M$ and $\overline{N} = (MN) \cup N$. We assume that $MN \cup M \cup N$ equals $N_z \times N_w$. This implies that for all (p,q) in $N_z \times N_w, \Phi_{pq}$ appears in the ℓ_1 constraint or in the objective function or in both.

We define $f_m\colon \ell_1^{n_s\times n_w}\to R$ and $f_a^\delta\colon \ell_1^{n_s\times n_w}\to R$ by

$$f_m(\Phi):=\sum_{(p,q)\in\overline{M}}\|\Phi_{pq}\|_2^2=\sum_{(p,q)\in(MN)\cup M}\|\Phi_{pq}\|_2^2$$

$$\begin{aligned}f_a^\delta(\Phi)&:=\sum_{(p,q)\in\overline{M}}\|\Phi_{pq}\|_2^2+\delta\sum_{p\in\mathcal{S}}\sum_{q\in N_p}\|\Phi_{pq}\|_1\\&=\sum_{(p,q)\in MN\cup M}\|\Phi_{pq}\|_2^2+\delta\sum_{(p,q)\in MN\cup N}\|\Phi_{pq}\|_1\end{aligned}$$

which are the objective functions of the mixed and the approximate problems, respectively. We make the following assumption.

Assumption 7: The plant is square, i.e., $n_z = n_u$ and $n_y = n_w$.

We comment at the end of this section on the nonsquare case. We now solve the approximate problem, and later we give the relation of the mixed problem to the approximate problem.

A. The Approximate Problem

In this section we study the approximate problem of order δ. This problem is very similar to the combination problem. The techniques used in solving the combination problem are often identical to the ones used in solving the approximate problem. We state many facts without proof. These facts can be easily deduced in ways similar to the ones used in the solution of the combination problem. The importance of this problem comes from its connection to the mixed problem. As in the combination problem, we define for notational convenience $Z_{pq}(t):=\Sigma_{i,j,k,\lambda_0}\, y_{ijk\lambda_0}F_{pq}^{ijk\lambda_0}(t)$.

Theorem 8: There exists Φ^0 in Γ_γ such that

$$\mu_\gamma^\delta=\sum_{(p,q)\in MN\cup M}\|\Phi_{pq}^0\|_2^2+\sum_{(p,q)\in MN\cup N}\delta\|\Phi_{pq}^0\|_1.$$

Therefore, the infimum in (21) is a minimum. Moreover, the following is true:

$$\mu_\gamma^\delta=\max\left\{\sum_{(p,q)\in M}\sum_{t=0}^{\infty}-\Phi_{pq}(t)^2+\sum_{(p,q)\in MN}\sum_{t=0}^{\infty}-\Phi_{pq}(t)^2+\sum_{i,j,k,\lambda_0}y_{ijk\lambda_0}b^{ijk\lambda_0}-\sum_{p\in\mathcal{S}}\overline{y}_p\gamma_p\right\}$$

subject to

$$\begin{aligned}&\overline{y}\text{ in }R^{c_n},\overline{y}\ge 0,\\&y\text{ in }R^{c_z},\Phi_{pq}\text{ in }\ell_1\text{ for all }(p,q)\text{ in }MN\cup M,\end{aligned}$$

and (IV), as shown at the bottom of the page. In addition, the optimal Φ_{pq}^0 is unique for all (p,q) in $(MN)\cup M$.

Proof: The proof follows by utilizing results analogous to Lemmas 2 and 3 and similar arguments to Theorem 5. ■

To get an analogous result to Lemma 4 it is clear that we have to get an *a priori* bound on the dual variable $\overline{y}$.

Lemma 7: Let $\Phi^{0,1}$ denote a solution of the standard ℓ_1 problem (5). $f_a^\delta(\Phi^{0,1})$ is the objective of the approximate problem evaluated at a solution of the standard ℓ_1 problem. If $(\overline{y}^\gamma, y^\gamma)$ is the solution to the approximate problem as given in Theorem 8 then $\overline{y}_p^\gamma\le f_a^\delta(\Phi^{0,1})/(\gamma_p-\nu_{0,1})$ for all p in $\mathcal{S}$.

Proof: Take any c in R such that $\nu_{0,1}<c<\min_{p\in\mathcal{S}}\gamma_p$. Let γ^0 in R^{c_n} be given by $\gamma_p^0 = c$. Let $\mu_{\gamma^0}^\delta := \inf_{\Phi\in\Gamma_{\gamma^0}} f_a^\delta(\Phi)$. Then from Corollary 1 we have $\langle\gamma-\gamma^0,\overline{y}^\gamma\rangle\le\mu_{\gamma^0}^\delta-\mu_\gamma^\delta\le\mu_{\gamma^0}^\delta\le f_a^\delta(\Phi^{0,1})$. Therefore, $\Sigma_{p\in\mathcal{S}}(\gamma_p-c)\overline{y}_p^\gamma\le f_a^\delta(\Phi^{0,1})$. Thus we have $\overline{y}_p^\gamma\le f_a^\delta(\Phi^{0,1})/(\gamma_p-c)$ for all p in S. This holds for all $c>\nu_{0,1}$ and therefore the lemma follows. ■

The following lemma is analogous to Lemma 4.

Lemma 8: Let Φ^0 be a solution to the primal problem (21) and let $\overline{y}^0, y^0, Z_{pq}^0$ be solutions to the dual. Then the following is true: Φ_{pq}^0 is unique for all (p,q) in $(MN)\cup M$. Also, there exists an *a priori* bound α_a such that $\|Z_{pq}^0\|_\infty\le\alpha_a$ where

$$\alpha_a:=\frac{f_a^\delta(\Phi^{0,1})}{\gamma_p-\nu_{0,1}}+\delta+\frac{2}{\delta}f_a^\delta(\Phi^{0,1})+2\sqrt{f_a^\delta(\Phi^{0,1})}$$

and the equation shown at the bottom of the next page, for all (p,q) in $N_z\times N_w$. $\Phi^{0,1}$ is a solution to (5).

Proof: The proof is similar to the proof of Lemma 4. Note here that a feasible solution to the approximate problem is always the ℓ_1 optimal $\Phi^{0,1}$ as opposed to H (used in Lemma 4) which may not be in this case since it may not satisfy the ℓ_1 constraint. This is why $\Phi^{0,1}$ appears in the expression for α_a ■

Using arguments similar to the ones used in Lemma 5 we can determine L_a^* such that $|Z_{pq}(t)|<\delta$ for all $t\ge L_a^*$. The following theorem follows using arguments identical to that used in proving Theorem 6.

$$\left.\begin{aligned}-(\delta+\overline{y}_p)\le\ |Z_{pq}(t)|&\le(\delta+\overline{y}_p) && \text{if }(p,q)\text{ is in }N\\ 2\Phi_{pq}(t)&=Z_{pq}(t)-(\delta+\overline{y}_p) && \text{if }(p,q)\text{ is in }MN\text{ and }Z_{pq}(t)>(\delta+\overline{y}_p),\\ &=Z_{pq}(t)+(\delta+\overline{y}_p) && \text{if }(p,q)\text{ is in }MN\text{ and }Z_{pq}(t)<-(\delta+\overline{y}_p),\\ &=0 && \text{if }(p,q)\text{ is in }MN\text{ and }|Z_{pq}(t)|\le(\delta+\overline{y}_p),\\ &=Z_{pq}(t) && \text{if }(p,q)\text{ is in }M,\\ &\text{for all }t=0,1,2,\cdots.\end{aligned}\right\}\quad\text{(IV)}$$

Theorem 9: The following is true:

$$\mu_\gamma^\delta = \max \Bigg\{ \sum_{(p,q)\in M} \sum_{t=0}^{\infty} -\tfrac{1}{4} Z_{pq}(t)^2 + \sum_{(p,q)\in MN} \sum_{t=0}^{L_a^*} -\Phi_{pq}(t)^2 + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} b^{ijk\lambda_0} - \sum_{p\in\mathcal{S}} \overline{y}_p \gamma_p \Bigg\}$$

subject to

$$\begin{aligned} &\overline{y} \text{ in } R^{c_n}, \overline{y} \geq 0, y \text{ in } R^{c_z}, \Phi_{pq}(t) \text{ in } R^{L_a^*} \text{ for all } (p,q) \text{ in } MN, \\ &-(\delta + \overline{y}_p) \leq Z_{pq}(t) \leq (\delta + \overline{y}_p), \quad \text{if } (p,q) \text{ is in } N \\ &-(\delta + \overline{y}_p) \leq 2\Phi_{pq}(t) - Z_{pq}(t) \\ &\leq (\delta + \overline{y}_p), \quad \text{if } (p,q) \text{ is in } MN \\ &\text{for all } t = 0, 1, 2, \cdots, L_a^*. \end{aligned}$$

Furthermore, the optimal Φ_{pq}^0 of the primal (21) is unique for all (p,q) in $(MN) \cup M$.

Thus, we have reduced the approximate problem to a finite-dimensional quadratic optimization problem with *a priori* known dimension. The same remarks relative to the solution procedure hold as in the combination problem. Note that again for the optimal solution Φ^0 the Φ_{pq}^0's with (p,q) in $MN \cup N$ are always FIR, while the Φ_{pq}^0's with (p,q) in M are IIR.

B. Relation Between the Approximate and the Mixed Problems

In this section we show how to solve the mixed problem using the results of the approximate problem. Note that the approximate problem reduces to a finite-dimensional quadratic optimization problem with *a priori* known dimension. For the mixed problem (one-block), a similar Lagrange duality approach can be used to show that the problem can be converted to a finite-dimensional convex problem with some of the optimal Φ_{pq}^0 being possibly FIR and some IIR as in the approximate problem (see Theorem 8). Nonetheless, even in the SISO case, an *a priori* bound on the dimension of the equivalent quadratic problem has proved elusive [8]. In addition, the MIMO problem is substantially more complex, for one cannot determine *a priori* which of the optimal dual variables $\overline{y}_p^0$ corresponding to the ℓ_1 constraint is active (i.e., $\overline{y}_p^0 > 0$). Hence, the *a priori* determination of which (if any) of the optimal Φ_{pq}^0 is FIR is not possible. This can make the solution procedure extremely complicated and virtually intractable by trying to examine all possibilities.

This difficulty can be circumvented by considering the approximate problem. The results in this section show that a suboptimal solution to the mixed problem can be obtained by solving an approximate problem. The following theorem shows that we can design a controller K for the mixed problem which achieves an objective value within any given tolerance of the optimal value by solving a corresponding approximate problem. The existence of a solution for the mixed problem and the optimal Φ_{pq}^0 being unique for (p,q) in $(MN) \cup M$ can be proved in a similar manner as was done for the approximate problem.

Theorem 10: $\mu_\gamma \leq \mu_\gamma^\delta \leq \mu_\gamma + \delta|\gamma|_1$.

Proof: It can be shown that $\mu_\gamma \leq \mu_\gamma^\delta$. Note that if Φ is in Γ_γ then

$$\sum_{q\in N_p} \|\Phi_{pq}\|_1 \leq \gamma_p, \qquad \text{for all } p \text{ in } S.$$

This implies that

$$f_a^\delta(\Phi) \leq \sum_{(p,q)\in(MN)\cup M} \|\Phi_{pq}\|_2^2 + \delta|\gamma|_1.$$

Taking infimum over Γ_γ on both sides in the above inequality, the theorem follows. ■

The next theorem is a result on the convergence of the optimal solutions of the approximate problems to the solution of the mixed problem.

Theorem 11: Let Φ^n be a solution of the approximate problem of order $1/n$. Then, there exists a subsequence $\{\Phi^{n_k}\}$ of Φ^n and Φ^0 in $\ell_1^{n_z \times n_w}$ such that Φ^0 is a solution of the mixed problem and $\Phi^{n_k} \to \Phi^0$ in the $W((\ell_2^{n_z\times n_w})^*, \ell_2^{n_z\times n_w})$ topology. Furthermore, $\Phi_{pq}^n \to \Phi^0$ in the $W((\ell_2)^*, \ell_2)$ topology for all (p,q) in $(MN) \cup M$.

Proof: The proof utilizes the Banach–Alaouglu result on weak star compactness to construct the subsequence of $\{\Phi^n\}$. Most of the arguments used in proving this theorem are similar to the ones employed in proving Lemma 2. The details of the proof are omitted. ■

At this point we would like to comment briefly on the nonsquare problem. Note that the result of Theorem 10 is valid for the square as well as the nonsquare case. Therefore, we would like to obtain a suboptimal solution to the nonsquare mixed problem by solving the corresponding nonsquare approximate problem (as defined in this section) which is more tractable. Analogously to the method employed for the combination problem, we would like to employ the delay augmentation method to solve the nonsquare approximate problem. Indeed, by considering the approximate problem, a similar method (see Section III-B, Theorem 7) can be used

$$\begin{aligned} -(\delta + \overline{y}_p) \leq Z_{pq}^0(t) &\leq (\delta + \overline{y}_p) && \text{if } (p,q) \text{ is in } N, \\ \Phi_{pq}^0(t) &= 0 && \text{if } (p,q) \text{ is in } N \text{ and } |Z_{pq}^0(t)| < (\delta +_p), \\ 2\Phi_{pq}^0(t) &= Z_{pq}^0(t) - (\delta + \overline{y}_p) && \text{if } (p,q) \text{ is in } MN \text{ and } Z_{pq}^0(t) > (\delta + \overline{y}_p), \\ &= Z_{pq}^0(t) + (\delta + \overline{y}_p) && \text{if } (p,q) \text{ is in } MN \text{ and } Z_{pq}^0(t) < -(\delta + \overline{y}_p), \\ &= 0 && \text{if } (p,q) \text{ is in } MN \text{ and } |Z_{pq}^0(t)| \leq (\delta + \overline{y}_p), \\ &= Z_{pq}^0(t) && \text{if } (p,q) \text{ is in } M \end{aligned}$$

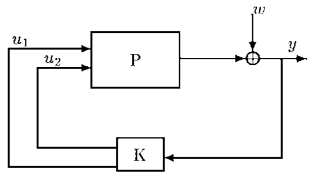

Fig. 2. A two-input/single-output example.

to generate a lower bound $f_a^\delta(\Phi^T)$ (analogous to $\underline{\nu}^T$ of the combination problem) which converges to the optimal with $\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V} \to \hat{\bar{\Phi}}^0$ in the $W((c_0^{n_z\times n_w})^*, c_0^{n_z\times n_w})$ topology. The main problem is that $\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}$ is guaranteed to satisfy the ℓ_1 constraint only in the limit as $T \to \infty$ which implies that $f_a^\delta(\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V})$ may not converge from *above* to μ_γ^δ (although it will converge). Hence, from $f_a^\delta(\Phi^T)$ and $f_a^\delta(\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V})$ alone it may not be clear how far away μ_γ^δ (and hence μ_γ) is. What is needed is a converging upper bound to μ_γ^δ. A simple way to do this is to solve directly the approximate problem by truncating Φ. This is what is called the finitely many variables (FMV) approach [2]. Convergence of the optimal cost of the truncated problem to μ_γ^δ as the number of (untruncated) variables increases is easy to establish [2]. For brevity however, we choose not to give a detailed analysis here.

V. An Illustrative Example

Here we illustrate the theory developed with an example. Consider the two-input/single-output plant P as depicted in Fig. 2 where $u := (u_1\ u_2)'$ is the input, w is the exogenous disturbance, and y is the measured output. The plant P is given by $\hat{P} = (\lambda - 0.5\ 1)$. The regulated output is given by $z := (y\ u_2)'$. Therefore

$$\begin{pmatrix}\hat{z}\\ \hat{y}\end{pmatrix} := \begin{pmatrix}\hat{y}\\ \hat{u}_2\\ \hat{y}\end{pmatrix} = \begin{pmatrix}1 & \lambda-0.5 & 1\\ 0 & 0 & 1\\ 1 & \lambda-0.5 & 1\end{pmatrix}\begin{pmatrix}\hat{w}\\ \hat{u}_1\\ \hat{u}_2\end{pmatrix} := \begin{pmatrix}\hat{G}_{11} & \hat{G}_{12}\\ \hat{G}_{21} & \hat{G}_{22}\end{pmatrix}\begin{pmatrix}\hat{w}\\ \hat{u}\end{pmatrix}.$$

As the plant P is stable, a valid Youla parameterization is given by

$$\hat{H}(\lambda) = \hat{G}_{11} = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \hat{U}(\lambda) = \hat{G}_{12} = \begin{pmatrix}\lambda-0.5 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \hat{V}(\lambda) = \hat{G}_{21} = 1.$$

For this problem $n_z = 2, n_w = 1, n_y = 1$, and $n_u = 2$. Let Φ be an achievable closed-loop map then $\hat{\Phi}(\lambda) = \hat{H}(\lambda) - \hat{U}(\lambda)\hat{Q}(\lambda)\hat{V}(\lambda)$, for some stable Q. This implies that

$$\begin{pmatrix}\hat{q}_1\\ \hat{q}_2\end{pmatrix} = \begin{pmatrix}\frac{1}{\lambda-0.5} & \frac{-1}{\lambda-0.5}\\ 0 & 1\end{pmatrix}\begin{pmatrix}(1-\hat{\Phi}_1)\\ -\hat{\Phi}_2\end{pmatrix}.$$

Therefore, q_1 and q_2 are in ℓ_1 only if $(1/(\lambda-0.5))(1 - \hat{\Phi}_1(\lambda) + \hat{\Phi}_2(\lambda))$ in ℓ_1. Thus Φ is an achievable closed-loop map if and only if Φ in $\ell_1^{2\times 1}$ and $1 - \hat{\Phi}_1(0.5) + \hat{\Phi}_2(0.5) = 0$. The above equation is the only interpolation condition. Following the notation developed in the earlier sections we define $F_1 := (1, \frac{1}{2}, (\frac{1}{2})^2, \cdots)'$ and $F_2 := (-1, -\frac{1}{2}, -(\frac{1}{2})^2)'$. It can be checked that the interpolation condition is equivalent to $\langle F, \Phi\rangle = \langle F_1, \Phi_1\rangle + \langle F_2, \Phi_2\rangle = 1$. As $n_u = n_z$ and $n_w = n_y$ the system is square and rank interpolation conditions are absent. First we solve the standard MIMO ℓ_1 problem for the given system G.

A. Standard ℓ_1 Solution

In this section we are interested in solving the following optimization:

$$\nu_{0,1} = \inf_{\Phi \text{Achievable}} \|\Phi\|_1 = \inf_{\langle F,\Phi\rangle = 1} \|\Phi\|_1.$$

We refer the reader to [2, Sec. 12.1.2] for the theory used to solve this problem. It can be easily verified that the above problem reduces to the following finite-dimensional linear program:

$$\min_{\nu, \psi, \Phi_i^+, \Phi_i^-} \nu$$

subject to

$$\psi(i) + \sum_{t=0}^{1} \Phi_i^+(t) + \Phi_i^-(t) = \nu \quad \text{for } i = 1, 2$$

$$\begin{aligned}&(\Phi_1^+(0) - \Phi_1^-(0)) + \tfrac{1}{2}(\Phi_1^+(1) - \Phi_1^-(1))\\ &\quad - (\Phi_2^+(0) - \Phi_2^-(0)) - \tfrac{1}{2}(\Phi_2^+(1) - \Phi_2^-(1)) = 1,\\ &\quad \cdot \psi, \Phi_i^+, \Phi_i^- \geq 0.\end{aligned}$$

Using the linear programming software of MATLAB™ we obtain that an optimal is given by $\Phi^{0,1} = (0.5\ \ -0.5)'$. This implies that $\nu_{0,1} = 0.5$.

B. Mixed Problem's Solution

In this section we are interested in solving the following optimization for the given system in Fig. 2:

$$\mu_\gamma := \inf_{\Phi \text{Achievable}} \{\|\Phi\|_2^2 \colon \|\Phi\|_1 \leq 1, \Phi \text{ in } \ell_1^{2\times 1}\}.$$

We give the corresponding Approximate problem of order 0.1 by the following optimization:

$$\mu_\gamma^{0.1} := \inf_{\Phi \text{Achievable}} \{\|\Phi\|_2^2 + 0.1\|\Phi_1\|_1 + 0.1\|\Phi_2\|_1 \colon \|\Phi\|_1 \leq 1, \Phi \text{ in } \ell_1^{2\times 1}\}.$$

The dual of the above problem using Theorem 4 is given by

$$\begin{aligned}\mu_\gamma^{0.1} := \max \inf_{\Phi \in \ell_1^{2\times 1}} \{&\|\Phi\|_2^2 + (0.1 + \overline{y}_1)\|\Phi_1\|_1\\ &+ (0.1 + \overline{y}_2)\|\Phi_2\|_1 - y\langle F, \Phi\rangle + y - \overline{y}_1 - \overline{y}_2\}\\ &\text{subject to} \quad \overline{y}_1 \geq 0, \overline{y}_2 \geq 0, y \text{ in } R.\end{aligned}$$

In keeping with the notation defined in earlier section we denote $Z := yF$, that is $Z_1 = yF_1$ and $Z_2 = yF_2$, and

$$f_a^{0.1}(\Phi) := \|\Phi_1\|_2^2 + \|\Phi_2\|_2^2 + 0.1\|\Phi_1\|_1 + 0.1\|\Phi_2\|_1.$$

Therefore, we have $f_a^{0.1}(H) = 1^2 + 0 + 0.1 + 0 = 1.1$ and $f_a^{0.1}(\Phi^{0,1}) = (0.5)^2 + (0.5)^2 + 0.1\,(0.5 + 0.5) = 0.6$. Let $\overline{y}^\gamma$

and y^γ be the solution to the dual problem stated above and let Φ^γ be the solution to the primal. We define L: $\ell_1^{2\times 1} \rightarrow R$ by

$$\begin{aligned} L(\Phi) := &\|\Phi\|_2^2 + (0.1 + \overline{y}_1^\gamma)\|\Phi_1\|_1 \\ &+ (0.1 + \overline{y}_2^\gamma)\|\Phi_2\|_1 - \langle Z_1^\gamma, \Phi_1\rangle - \langle Z_2^\gamma, \Phi_2\rangle. \end{aligned}$$

From Theorem 4 it follows that Φ^γ minimizes $L(\Phi)$ over all Φ in $\ell_1^{2\times 1}$. This implies that $\Phi^\gamma(t)$ minimizes

$$\Phi_1(t)^2 + (0.1 + \overline{y}_1^\gamma)|\Phi_1(t)| - Z_1^\gamma(t)\Phi_1(t) \tag{22}$$

over all $\Phi_1(t)$ in R. We can discard the $\Phi_1(t)$ in R which have the opposite sign to that of $Z_1^\gamma(t)$ because then $-\langle Z_1^\gamma, \Phi_1\rangle \geq 0$. Therefore $\Phi^\gamma(t)$ minimizes $\Phi_1(t)^2 + \Phi_1(t)((0.1 + \overline{y}_1^\gamma)\ \ \mathrm{sgn}(Z_1^\gamma(t)) - Z_1^\gamma(t))$ over all $\Phi_1(t)$ in R which satisfy $\Phi_1(t)Z_1^\gamma(t) \geq 0$. Without loss of generality assume that $Z_1^\gamma(t) \geq 0$. Then $\Phi^\gamma(t)$ minimizes, $\Phi_1(t)^2 + \Phi_1(t)((0.1 + \overline{y}_1^\gamma) - Z_1^\gamma(t))$, over all $\Phi_1(t)$ in R which are positive. Now if $(0.1 + \overline{y}_1^\gamma) \geq Z_1^\gamma(t)$ then objective is always positive as $\Phi_1(t)$ is restrained to be positive and therefore the minimizer $\Phi^\gamma(t)$ is forced to be equal to zero. If $0 < (0.1 + \overline{y}_1^\gamma) < Z_1^\gamma(t)$ then the coefficient of $\Phi(t)$ in the objective is negative, and therefore we can do better than achieving a zero objective value and this forces $\Phi^\gamma(t) > 0$. With this knowledge we can now differentiate the unconstrained objective function in (22) to get $2\Phi^\gamma(t) = Z_1^\gamma(t) - (0.1 + \overline{y}_1^\gamma)$. Similarly, if $Z_1^\gamma(t) <, -(0.1 + \overline{y}_1^\gamma) < 0$ then $2\Phi^\gamma(t) = Z_1^\gamma(t) + (0.1 + \overline{y}_1^\gamma)$. In any case the following holds:

$$\begin{aligned} |Z_1^\gamma(t)| &\leq 0.1 + \overline{y}_1^\gamma + 2|\Phi^\gamma(t)| \\ &\leq 0.1 + \frac{f_a^{0.1}(\Phi^{0,1})}{\gamma_p - \nu_{0,1}} + 2\|\Phi\|_1. \end{aligned}$$

The second inequality follows from Lemma 7. From the fact that $\|\Phi_1^\gamma\|_1 \leq 1$ we have $|Z_1^\gamma(t)| \leq 3.3$. Note that $Z_1^\gamma(t) = y^\gamma F_1(t)$. As $|Z_1^\gamma(0)| \leq 3.3$, it follows that $|y^\gamma F_1(0)| = |y^\gamma| \leq 3.3$. Now, $|Z_1^\gamma(t)| = |y^\gamma F_1(t)| \leq 3.3\ F_1(t) = 3.3(1/2^t)$. This implies that we can determine *a priori* L_a^* such that if $t \geq L_a^*$ then $|Z_1^\gamma(t)| \leq 0.1 \leq 0.1 + \overline{y}_1^\gamma$, which will imply that $\Phi_1^\gamma(t) = 0$ if $t \geq L_a^*$. $L_a^* = 5$ does satisfy this requirement. A similar development holds for Φ_2^γ. From Theorem 9 we have that the dual can be written as

$$\mu_\gamma^{0.1} = \max\left\{\sum_{i=1}^{2}\sum_{t=0}^{5} -\Phi_i(t)^2 + y - \sum_{i=1}^{2}\overline{y}_i\right\}$$

subject to

$$\begin{aligned} &\overline{y} \text{ in } R^2, \overline{y} \geq 0,\ y \text{ in } R, \Phi_i \text{ in } R^6 i = 1,2 \\ &-(0.1 + \overline{y}_i) \leq 2\Phi_i(t) - Z_i(t) \leq (0.1 + \overline{y}_i) \\ &\quad \text{for } i = 1,2, \\ &\quad \text{for all } t = 0,1,2,\cdots,6. \end{aligned}$$

Using MATLAB™ software we conclude that the optimal Φ^γ which is unique for this example is given by

$$\begin{aligned} \hat{\Phi}_1^\gamma(\lambda) &= 0.3972 + 0.1732\lambda + 0.0617(\lambda)^2 + 0.0058(\lambda)^3 \\ \hat{\Phi}_2^\gamma(\lambda) &= -\Phi_1^\gamma(\lambda). \end{aligned}$$

Therefore, $f_a^{0.1}(\gamma) = 0.5109, \|\Phi^\gamma\|_1 = 0.6379$ and $\|\Phi^\gamma\|_2 = 0.6191$. This implies from Theorem 10 that if Φ^0 represents the solution of the mixed problem then $|f_m(\Phi^0) - f_m(\Phi^\gamma)| \leq 0.2$. This completes the example.

VI. Conclusions

In this paper we considered two related problems of MIMO controller design which incorporate the $\mathcal{H}_2$ and the ℓ_1 norms of input–output maps constituting the closed loop directly in their definitions.

In the first problem, termed the combination problem, a positive linear combination of the square of the $\mathcal{H}_2$ norms and the ℓ_1 norms of the input–output maps was minimized over all stabilizing controllers. It was shown that for the one-block case, the optimal is possibly IIR and the solution can be nonunique. However, it was shown that the problem can be solved exactly via a finite-dimensional quadratic optimization problem and a linear programming problem of *a priori* known dimensions. For the four-block case a delay augmentation approach was employed to obtain suboptimal solutions.

In the second problem, termed the mixed problem, the $\mathcal{H}_2$ performance of the closed loop is minimized subject to a ℓ_1 constraint. It was shown that suboptimal solutions within any given tolerance of the optimal value can be obtained via the solution to a related combination problem.

Numerical implementation, and possibly tighter bounds on the dimensions of the equivalent problems, is the subject of future research. Moreover, a state-space approach based on recent results [17], [18] for the standard ℓ_1 problem is worthy of being investigated in order to construct suboptimal solutions for the mixed $\mathcal{H}_2/\ell_1$ problem.

Appendix

A. Interpolation Conditions

The elements $G_{\alpha_i q t}(l)$ and $G_{\beta_j p t}(l)$ corresponding to the rank conditions are defined as

$$\begin{array}{c} \qquad\qquad q\text{th column} \\ \begin{pmatrix} \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \alpha_i'(t-l) & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix} \end{array}, \quad \text{and}$$

$$\begin{array}{cc} \begin{pmatrix} \cdots & 0 & \cdots \\ & \vdots & \\ \cdots & 0 & \cdots \\ \cdots & \beta_j'(t-l) & \cdots \\ \cdots & 0 & \cdots \\ & \vdots & \\ \cdots & 0 & \cdots \end{pmatrix} & \begin{array}{c} \\ \\ \\ p\text{th row} \\ \\ \\ \\ \end{array} \end{array}$$

respectively (see [2]).

B. Existence of a Solution for the Combination Problem

Here we show that a solution to (12) always exists. The following lemma (e.g., [1]) will be used to prove this result.

Lemma 9 (Banach–Alaoglu): Let X be a Banach space with X^* as its dual, then the set $\{x^*: x^* \text{ in } X^*, \|x^*\| \leq M\}$ is $W(X^*, X)$ compact for any M in R.

Proof [Lemma 2]: As

$$f(\Phi) := \sum_{(p,q)\in MN\cup M} \overline{c}_{pq}\|\Phi_{pq}\|_2^2 + \sum_{(p,q)\in MN\cup N} c_{pq}\|\Phi_{pq}\|_1$$

we have $\nu_e = \inf_{\Phi \in \overline{\Theta}}\{f(\Phi)\colon f(\Phi) \le \nu_e + 1\}$. Thus there exist constants $\overline{C}$ and C such that with

$$B := \{\Phi \text{ in } \ell_2^{n_z \times n_w} \colon \|\Phi_{pq}\|_2^2 \le \overline{C} \text{ for all } (p,q) \text{ in } M \text{ and } \cdot \|\Phi_{pq}\|_1 \le C \text{ for all } (p,q) \text{ in } MN \cup N\}$$

$\nu_e = \inf_{\Phi \in \overline{\Theta} \cap B} f(\Phi)$. From above we conclude that there exists a sequence $\{\Phi^n\}$ in $\overline{\Theta} \cap B$ such that

$$f(\Phi^n) \le \nu_e + \frac{1}{n}. \tag{23}$$

As $\{\Phi^n\}$ in $\overline{\Theta} \cap B$ we have the following:

$$\langle F^{ijk\lambda_0}, \Phi^n \rangle = b^{ijk\lambda_0}, \tag{24}$$

$$\|\Phi_{pq}^n\|_1 \le C, \qquad \text{for all } (p,q) \text{ in } MN \cup N \tag{25}$$

$$\|\Phi_{pq}^n\|_2^2 \le \overline{C}, \qquad \text{for all } (p,q) \text{ in } M. \tag{26}$$

From (25) we conclude that for all (p,q) in $MN \cup N, \Phi_{pq}^n$ belongs to a bounded set in $(c_0)^*$. From the Banach–Alaoglu theorem and separability of c_0 we conclude that for all (p,q) in $MN \cup N$ there exists a subsequence $\{\Phi_{pq}^{n_k}\}$ of $\{\Phi_{pq}^n\}$ and Φ_{pq}^0 such that $\{\Phi_{pq}^{n_k}\} \to \Phi_{pq}^0$ in the $W((c_0)^*, c_0)$ topology. This implies that

$$\langle v, \Phi_{pq}^{n_k} \rangle \to \langle v, \Phi_{pq}^0 \rangle, \qquad \text{for all } v \text{ in } c_0. \tag{27}$$

Similarly, we conclude that for all (p,q) in M there exists a subsequence $\{\Phi_{pq}^{n_{k_s}}\}$ of $\{\Phi_{pq}^{n_k}\}$ and Φ_{pq}^0 such that $\{\Phi_{pq}^{n_{k_s}}\} \to \Phi_{pq}^0$ in the $W((\ell_2)^*, \ell_2)$ topology. This implies that

$$\langle v, \Phi_{pq}^{n_{k_s}} \rangle \to \langle v, \Phi_{pq}^0 \rangle, \qquad \text{for all } v \text{ in } \ell_2. \tag{28}$$

Thus, we have defined Φ^0 in $\mathcal{A}$ by the limits (27) and (28). Note that

$$\langle F^{ijk\lambda_0}, \Phi \rangle = \sum_{(p,q) \in N_z \times N_w} \langle F_{pq}^{ijk\lambda_0}, \Phi_{pq} \rangle.$$

Therefore, it follows from (24), (28), (27), and Lemma 1 that $\langle F^{ijk\lambda_0}, \Phi^0 \rangle = b^{ijk\lambda_0}$. Similarly, the rank interpolation conditions are also satisfied by Φ^0. From the above discussion it follows that Φ^0 is in $\overline{\Theta}$ and therefore

$$f(\Phi^0) = \sum_{(p,q) \in (MN) \cup M} \overline{c}_{pq} \|\Phi_{pq}^0\|_2^2 + \sum_{(p,q) \in N} c_{pq} \|\Phi_{pq}^0\|_1 \ge \nu_e.$$

From (27) and (28) it follows that $\Phi_{pq}^{n_{k_s}}(t) \to \Phi_{pq}^0(t)$, for all t in R and for all (p,q) in $N_z \times N_w$. This implies that for all T as $s \to \infty$, $f(P_T \Phi^{n_{k_s}}) \to f(P_T \Phi^0)$, where P_T is the truncation operator. We have from (23) that for all s and T $f(P_T \Phi^{n_{k_s}}) \le \nu_e + (1/n_{k_s})$. Letting $s \to \infty$ and then letting $T \to \infty$ we have that $f(\Phi^0) \le \nu_e$. This proves the lemma. ■

C. Results on the Nonsquare Case of the Combination Problem

Here, we prove Theorem 7 using techniques very similar to the ones employed in proving [2, Th. 12.2.4]. Toward this goal we state the following lemmas (proofs are omitted).

Lemma 10: Suppose $\{x^n\}$ is a sequence in ℓ_1. If there exists a constant C such that $\|x^n\|_1 \le C$ for all n then $S^n x^n \to 0$, in the $W((c_0)^*, c_0)$ topology.

Consider the expression for $f\colon \ell_2^{n_z \times n_w} \to R$ which was defined earlier

$$f(\Phi) = \sum_{(p,q) \in (MN) \cup N} \overline{c}_{pq} \|\Phi_{pq}\|_2^2 + \sum_{(p,q) \in (MN) \cup M} c_{pq} \|\Phi_{pq}\|_1.$$

Then we have the following.

Lemma 11: Suppose $\{\Phi^k\}$ is a sequence in $\ell_2^{n_z \times n_w}$ which converges pointwise to Φ^0 in $\ell_2^{n_z \times n_w}$ and $f(\Phi^k) \le f(\Phi^0) < \infty$ for all k. Then $f(\Phi^k) \to f(\Phi^0)$ and $f(\Phi^k - \Phi^0) \to 0$ as $k \to \infty$.

Lemma 12: Let Φ^T be an optimal solution to the Delay Augmentation problem (17). Let $\hat{\Phi}^T = \hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V} - \hat{S}^T \tilde{R}^T$ be the appropriate expansion. Then it follows that there exist constants C_1 and C_2 such that for all $T, \|\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}\|_1 \le C_1$ and $\|\tilde{R}^T\|_1 \le C_2$.

Proof: Suppose $\|\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}\|_1$ is unbounded in T. Then, $\|\hat{U}\hat{Q}^{11,T}\hat{V}\|_1$ is unbounded in T, which implies that $\|Q^{11,T}\|_1$ is unbounded in T. As $\hat{U}^1(\lambda)$ and $\hat{V}^1(\lambda)$ have no zeros on the unit circle it can be argued that $\|\hat{H}^{11} - \hat{U}^1\hat{Q}^{11,T}\hat{V}^1\|_1$ is unbounded in T. Note that

$$\begin{pmatrix} \hat{\Phi}^{11,T} & \hat{\Phi}^{12,T} \\ \hat{\Phi}^{21,T} & \hat{\Phi}^{22,T} \end{pmatrix} = \begin{pmatrix} \hat{H}^{11} - \hat{U}^1\hat{Q}^{11,T}\hat{V}^1 & * \\ * & * \end{pmatrix}$$

where $*'$s are irrelevant to the argument. Now from Assumption 6 it follows that $\|\hat{H}^{11} - \hat{U}^1\hat{Q}^{11,T}\hat{V}^1\|_1$ is uniformly bounded in T. Thus we have reached a contradiction. This proves that there exists a constant C_1 such that for all $T, \|\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}\|_1 \le C_1$. Notice that

$$\|\tilde{R}^T\|_1 = \|S^T \tilde{R}^T\|_1 \le \|\hat{\Phi}^T\|_1 + \|\hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}\|_1.$$

From Assumption 6 we know that $\|\hat{\Phi}^T\|_1$ is uniformly bounded in T. This implies that $\|\tilde{R}^T\|_1$ is uniformly bounded in T. This proves the lemma. ■

Proof [Theorem 7]: Note that with Assumption 6, Lemma 2 is the statement of the existence of the solution to the nonsquare problem. We have from Assumption 6 that there exists a constant C such that $\|\Phi^T\|_1 \le C$ for all T. From the Banach–Alaoglu lemma we conclude that there exists a subsequence $\{\Phi^{T_s}\}$ of $\{\Phi^T\}$ and Φ^0 in $c_0^{n_z \times n_w}$ such that

$$\Phi^{T_s} \to \Phi^0 \text{ in the } W((c_0^{n_z \times n_w})^*, c_0^{n_z \times n_w}) \text{ topology.} \tag{29}$$

This implies that $\hat{\Phi}^0 = \hat{H} - (\hat{U}\hat{Q}^{11,T_s}\hat{V})^{w^*} - (\hat{S}^{T_s}\tilde{R}^{T_s})^{w^*}$, where $(\cdot)^{w^*}$ represents the limit in the $W((c_0^{n_z \times n_w})^*, c_0^{n_z \times n_w})$ topology. From Lemmas 12 and 10 it follows that $(S^{T_s}\tilde{R}^{T_s})^{w^*} = 0$ and therefore $\hat{\Phi}^0 = \hat{H} - (\hat{U}\hat{Q}^{11,T_s}\hat{V})^{w^*} = \hat{H} - \hat{U}(\hat{Q}^{11,T_s})^{w^*}\hat{V}$. From (29) and because all interpolation conditions can be characterized by elements in $\ell_1 \subset c_0$, we have that Φ^0 satisfies the zero interpolation conditions. This implies that Φ^0 in Θ and therefore $f(\Phi^0) \ge \nu$. From arguments similar to the ones used in the proof of Lemma 2 it can be shown that $f(\Phi^0)$ is in fact equal to ν. As Φ^0 in Θ we know from Theorem 1 that there exists $Q^{11,0}$ in $\ell_1^{n_u \times n_y}$ such that $\hat{\Phi}^0 = \hat{H} - \hat{U}\hat{Q}^{11,0}\hat{V}$ which implies that $\hat{\Phi}_{11}^0 = \hat{H}_{11} - \hat{U}_1\hat{Q}^{11,0}\hat{V}_1$. As $\hat{U}_1$ and $\hat{V}_1$ have no zeros on the unit circle, it follows that they are invertible and therefore $Q^{11,0}$ is unique. This implies that $Q^{11,0} = (Q^{11,T_s})^{w^*}$.

Thus we have a sequence $\{\Phi^{T_s}\}$ which converges to Φ^0 pointwise and $f(\Phi^{T_s}) \leq f(\Phi^0)$. From Lemma 11 we have that $\underline{\nu}^{T_s} = f(\Phi^{T_s}) \to f(\Phi^0) = \nu$ and $f(\Phi^{T_s} - \Phi^0) \to 0$. As $\underline{\nu}^T$ is a nondecreasing sequence we conclude that $\underline{\nu}^T \to \nu$. As M is empty and $f(\Phi^{T_s} - \Phi^0) \to 0$, we know that $\|\Phi^{T_s} - \Phi^0\|_1 \to 0$ as $s \to \infty$. Therefore, $\|\Phi^{11,T_s} - \Phi^{11,0}\|_1 \to 0$ as $s \to \infty$. As $\hat{Q}^{11,T_s} - \hat{Q}^{11,0} = \hat{U}_1{}^{-1}(\hat{\Phi}^{11,T_s} - \hat{\Phi}^{11,0})\hat{V}_1{}^{-1}$ and because the map from Φ^{11} to Q^{11} is a continuous map, we conclude that

$$\|Q^{11,T_s} - Q^{11,0}\|_1 \to 0. \tag{30}$$

Define $\hat{\overline{\Phi}}^T := \hat{H} - \hat{U}\hat{Q}^{11,T}\hat{V}$. From (30) and the fact that $\hat{\overline{\Phi}}^{T_s} - \hat{\Phi}^0 = -\hat{U}(\hat{Q}^{11,T_s} - \hat{Q}^{11,0})\hat{V}$ we conclude that $\|\overline{\Phi}^{T_s} - \Phi^0\|_2 \to 0$ as $s \to \infty$. As $\|\overline{\Phi}^{T_s} - \Phi^0\|_2 \leq K\|\overline{\Phi}^{T_s} - \Phi^0\|_1$, where K is a constant, we conclude that $\|\overline{\Phi}^{T_s} - \Phi^0\|_2 \to 0$ as $s \to \infty$. Therefore, it follows that $\overline{\nu}^{T_s} = f(\overline{\Phi}^{T_s}) \to f(\Phi^0) = \nu$ as $s \to \infty$.

If N is also empty, then the convergence of the original sequence follows from the uniqueness of the Φ^0. This proves the theorem. ∎

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