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Solution of MIMO $\mathcal{H}_2/\ \ell_1$ Problem without Zero Interpolation [1]

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Abstract

In this paper we present a methodology to obtain converging lower and upper bounds to multiple objective problems which incorporate the $\mathcal{H}_2$ performance measure and time domain criteria. This methodology avoids many of the problems which are present in methods which employ zero interpolation techniques to characterize achievable closed loop maps.

1 Introduction

It is well known that performance with respect to a measure (usually the ℓ_1, $\mathcal{H}_2$, or the $\mathcal{H}_\infty$ norm of the closed loop) is not a guarantee of good performance with respect to some other measure. Motivated by this concern researchers have focussed their attention on multi-objective problems which incorporate two or more different measures in their problem definition.

An important class of problems which falls under this category are the ones which incorporate time domain objectives and the $\mathcal{H}_2$ objective. In [5] it was shown that the single input single output problem of minimizing the ℓ_1 norm of the closed loop subject to an $\mathcal{H}_2$ constraint can be solved via finite dimensional convex programming. In a related result it was shown in [4] that problems which incorporate the ℓ_1 norm and the $\mathcal{H}_2$ norm of the various transfer functions in a closed loop map of a multiple input multiple output system can be formulated and solved via finite dimensional quadratic programming.

Most of approaches which incorporate the ℓ_1 objective characterize the achievability of a closed loop map from a stabilizing controller by using zero interpolation conditions on the closed loop map [1]. Computation of the zeros and the zero directions can be done by finding the nullspaces of certain Toeplitz like matrices. Once the optimal closed loop map is determined the task of determining the controller still remains. The closed loop map needs to satisfy the zero interpolation conditions exactly to guarantee that the correct cancellations take place while solving for the controller. However, numerical errors are always present and there exists a need to determine which poles and zeros cancel. These difficulties exist even for the pure MIMO ℓ_1 problem, when zero interpolation methods are employed for a solution. However, recently in [2] it was shown that converging lower bounds can be determined to the ℓ_1 problem by solving an auxiliary problem which does not require zero interpolation and thus avoids the above mentioned problems.

In this paper we formulate an auxiliary problem to the one given in [4]. We show that converging lower bounds can be computed without zero interpolation for the most general MIMO

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case. This provides an attractive method for solving multi-objective problems which incorporate time domain and $\mathcal{H}_2$ objectives.

The paper is organized as follows. In Section 2 we present the preliminary material relevant to the discussion here. In Section 3 we formulate the problem and define an auxiliary problem which regularizes the original one. In Section 4 we present lower and upper bounds for the problem. Finally, we conclude in Section 5.

2 Preliminaries

In this section we present a brief summary of mathematical and systems results to be utilized later in the paper.

2.1 System preliminaries

Consider the system of Figure 1 where $w := (w_1\ w_2)'$ is the exogenous disturbance, $z := (z_1\ z_2)'$ is the regulated output, u is the control input and y is the measured output. In feedback control design the objective is to design a controller K such that with $u = Ky$ the resulting closed loop map Φ_{zw} from w to z is stable (see Figure 1) and satisfies certain performance criteria. In [6] a nice parametrization of all closed loop maps which are achievable via stabilizing controllers was first derived. A good treatment of the issues involved is presented in [1]. Following the notation used in [1] we denote by n_u, n_w, n_z and n_y the number of control inputs, exogenous inputs, regulated outputs and measured outputs respectively of the plant G. We represent by Θ, the set of closed loop maps of the plant G which

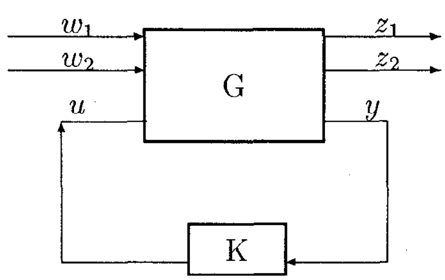

Figure 1: Closed Loop System.

are achievable through stabilizing controllers. $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$ and $V \in \ell_1^{n_y \times n_w}$ characterize the Youla parametrization of the plant [6]. The following theorem follows from Youla parametrization.

Theorem 1 $\Theta = \{\Phi \in \ell_1^{n_z \times n_w}$: there exists a $Q \in \ell_1^{n_u \times n_y}$ with $\hat{\Phi} = \hat{H} - \hat{U}\hat{Q}\hat{V}\}$, *where $\hat{f}$ denotes the λ transform (see [1]) of f.*

If Φ is in Θ we say that Φ is an *achievable* closed loop map. We assume throughout the paper that $\hat{U}$ has normal rank n_u and $\hat{V}$ has normal rank n_y. There is no loss of generality in making this assumption [1].

2.2 Mathematical preliminaries

In this subsection we summarize the mathematical results which are relevant to the paper. An exhaustive treatment of the subject matter of this subsection is given in [3]. The reader may skip this part of the paper and refer to this subsection whenever required.

Definition 1 (Convex Sets) *A subset Ω of a vector space X is said to be convex if for any two elements c_1 and c_2 in Ω and for a real number λ with $0 < \lambda < 1$ the element $\lambda c_1 + (1-\lambda)c_2 \in \Omega$.*

Lemma 1 *Let Ω be a convex subset of a Banach space X and $f : \Omega \to R$ be strictly convex. If f achieves its minimum on Ω then the minimizer is unique.*

Theorem 2 (Banach Alaoglu) *Let $(X, ||\cdot||_X)$ be a normed vector space with X^* as its dual. The set*

$$B^* := \{x^* \in X^* : ||x^*|| \leq M\}, \qquad (1)$$

is compact in the weak-star topology for any $M \in R$.

Lemma 2 *Suppose ϕ_k is a sequence in ℓ_2 and $\phi_k(t) \rightarrow \phi_0(t)$ for all t. Suppose also that $||\phi_k||_2 \nearrow ||\phi_0||_2$. Then $||\phi_k - \phi_0||_2 \rightarrow 0$.*

Proof: Given $\epsilon > 0$ choose n such that

$$||(I - P_n)\phi_0||_2^2 \leq \min\{\frac{\epsilon}{8}, (\frac{\epsilon}{8(||\phi_0||_2 + 1)})^2\} \quad (2)$$

As $\phi_k(t) \rightarrow \phi_0(t)$ we can choose K_2 such that

$$k > K_2 \Rightarrow ||P_n(\phi_k - \phi_0)||_2^2 \leq \frac{\epsilon}{4}. \quad (3)$$

We know that $||P_n(\phi_k)||_2 \rightarrow ||P_n(\phi_0)||_2$ as $k \rightarrow \infty$. From above and the fact that $||\phi_k||_2 \rightarrow ||\phi_0||_2$ it follows that we can choose K_3 such that

$$k > K_3 \Rightarrow |\, ||(I-P_n)\phi_k||_2^2 - ||(I-P_n)\phi_0||_2^2 \,| \leq \frac{\epsilon}{4}. \quad (4)$$

Let $K \geq \max\{K_2, K_3\}$ the $k > K$ implies

$$\begin{aligned}
||\phi_k - \phi_0||_2^2 &= ||P_n(\phi_k - \phi_0)||_2^2 \\
&\quad + ||(I - P_n)(\phi_k - \phi_0)||_2^2 \\
&\leq \frac{\epsilon}{4} + ||(I - P_n)(\phi_k)||_2^2 \\
&\quad + ||(I - P_n)(\phi_0)||_2^2 + \\
&\quad 2 \sum_{t=n+1}^{\infty} |\phi_k(t)|\, |\phi_0(t)| \\
&\leq \frac{\epsilon}{4} + 2||(I - P_n)(\phi_0)||_2^2 + \frac{\epsilon}{4} \\
&\quad + 2 \sum_{t=n+1}^{\infty} |\phi_k(t)|\, |\phi_0(t)| \\
&\leq \frac{\epsilon}{4} + 2\frac{\epsilon}{8} + \frac{\epsilon}{4} \\
&\quad + 2||(I - P_n)\phi_k||_2 \\
&\quad ||(I - P_n)\phi_0||_2 \\
&\leq \frac{\epsilon}{4} + 2\frac{\epsilon}{8} + \frac{\epsilon}{4} \\
&\quad + 2||\phi_0||_2 \frac{\epsilon}{8(||\phi_0||_2 + 1)} \\
&\leq \epsilon.
\end{aligned}$$

■

3 Problem statement

Let H, U and V be partitioned into submatrices as given below $H = \begin{pmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{pmatrix}$, $U = \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}$ and $V = \begin{pmatrix} V^1 & V^2 \end{pmatrix}$. Then the Youla parametrization is given by $H - U * Q * V$ which is equal to

$$\begin{pmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{pmatrix} - \begin{pmatrix} U^1 \\ U^2 \end{pmatrix} * Q * \begin{pmatrix} V^1 & V^2 \end{pmatrix},$$

for some $Q \in \ell_1^{n_u \times n_y}$. The problem statement is: *Given a plant G, positive real number γ solve the following problem,*

$$\inf_{Q \in \ell_1^{n_u \times n_y}} ||H^{22} - U^2 * Q * V^2||_2^2$$

subject to

$$||H^{11} - U^1 * Q * V^1||_1 \leq \gamma.$$

We denote by μ the optimal value obtained from the above problem.

Now we define an auxiliary problem which is intimately related to the one defined above. The auxiliary problem statement is: *Given a plant G, positive real numbers α and γ solve the following problem.*

$$\begin{aligned}
&\inf_{Q \in \ell_1^{n_u \times n_y}} ||H^{22} - U^2 * Q * V^2||_2^2 \\
&\text{subject to} \\
&\qquad ||H^{11} - U^1 * Q * V^1||_1 \leq \gamma \\
&\qquad ||Q||_1 \leq \alpha.
\end{aligned} \quad (5)$$

The optimal value obtained from the above problem is denoted by ν.

Note that in the problem statement of μ the allowable Youla parameter Q which is in $\ell_1^{n_u \times n_y}$ needs to satisfy $||H^{11} - U^1 * Q * V^1||_1 \leq \gamma$. Therefore it follows that $||U^1 * Q * V^1||_1 = ||H^{11} - U^1 * Q * V^1 - H^{11}||_1 \leq ||H^{11} - U^1 * Q * V^1||_1 + ||H^{11}||_1 \leq ||H^{11}||_1 + \gamma$. Suppose, U^1 has more rows than columns and V^1 has more columns than rows and both have full normal rank. Thus the left inverse of U^1 exists (given by $(U^1)^{-l}$) and the right inverse of V^1 exists (given by $(V^1)^{-r}$). Further suppose that U^1 and V^1 have no zeros on the unit circle. Then it can be shown (see Lemma 4 and the discussion below) that there exists a β (which depends only on $(U^1)^{-l}$ and $(V^1)^{-r}$) such that

$||Q||_1 \leq \beta$. Thus if in the auxiliary problem we choose $\alpha = \beta$ then the constraint $||Q||_1 \leq \alpha$ is redundant in the problem statement of ν and we get $\mu = \nu$. The extra constraint in the problem statement of ν is useful because it regularizes the problem (as will be seen). The following lemma is useful in estimating β.

Lemma 3 *[1] Let $\mathcal{D}$ denote the open unit disc in the complex plane. Given a function $\hat{f}(.)$ of the complex variable λ analytic in $\mathcal{D}$, then $\left.\frac{d^k \hat{f}}{d\lambda}\right|_{\lambda_0} = 0$ for $k = 0, 1, \ldots, (\sigma - 1)$ and $\lambda_0 \in \mathcal{D}$ if and only if $\hat{f}(\lambda) = (\lambda - \lambda_0)^\sigma \hat{g}(\lambda)$ where $g(.)$ is analytic in $\mathcal{D}$.*

Lemma 4 *Let ϕ be an element of ℓ_1 such that $||\phi||_1 \leq \gamma$ for some $\gamma > 0$. Let $\hat{\phi}(\lambda)$ be the λ transform of ϕ. Suppose, $\hat{\phi}(\lambda)$ has a zero at $\lambda = a$ with $|a| < 1$. If $\hat{\phi}(\lambda) = (\lambda - a)\hat{\psi}(\lambda)$ then $||\hat{\psi}(\lambda)||_1 \leq \frac{\gamma}{1 - |a|}$.*

Proof: From assumption $||(\lambda - a)\hat{\psi}(\lambda)||_1 \leq \gamma$. This implies that $\sum_{t=-\infty}^{\infty} |\psi(t-1) - a\psi(t)| \leq \gamma$. This is true only if $\sum_{t=-\infty}^{\infty}(|\psi(t-1)| - |a|\ |\psi(t)|) \leq \gamma$, which implies that $||\psi||_1(1 - |a|) \leq \gamma$. Therefore, $||\psi||_1 \leq \frac{\gamma}{1-|a|}$. ∎

In the discussion above we have obtained an upper bound on the one norm of $R := U^1 * Q * V^1$ for any $Q \in \ell_1^{n_u \times n_y}$ which satisfies $||H^{11} - U^1 * Q * V^1||_1 \leq \gamma$. As U^1 and V^1 are left and right invertible it follows that $\hat{Q} = (\hat{U}^1)^{-l}\hat{R}(\hat{V}^1)^{-r}$. As Q is in $\ell_1^{n_u \times n_y}$ it is true that $\hat{R}$ interpolates the unstable poles of $(\hat{U}^1)^{-l}$ and $(\hat{V}^1)^{-r}$ none of which are on the unit circle by assumption. Using Lemma 4 one can obtain an upper bound on the one norm of Q which depends only on the upper bound on the one norm of R, $(U^1)^{-l}$ and $(V^1)^{-r}$.

The following lemma is a result on the uniqueness of the solution to (5).

Lemma 5 *Let $Q^0 \in \ell_1^{n_u \times n_y}$ be a solution to (5). Let $\Phi^0 = H - U * Q^0 * V$ with $\Phi^{22,o} = H^{22} - U^2 * Q^0 * V^2$ and $\Phi^{11,o} = H^{11} - U^1 * Q^0 * V^1$. Then $\Phi^{22,o}$ is unique. Furthermore, if $\hat{U}^2$ and $\hat{V}^2$ have full normal column and row ranks respectively then Q^0 is unique.*

Proof: Note that the problem statement of ν given by (5) can be recast as,

$$\nu = \inf\{||\Phi^{22}||_2^2 : \Phi^{22} \in A_{al}\}, \qquad (6)$$

where A_{al} is the following set

$\{\Phi^{22}$: there exists $Q \in \ell_1^{n_u \times n_y}$ with $\Phi^{22} = H^{22} - U^2 * Q * V^2, ||H^{11} - U^1 * Q * V^1||_1 \leq \gamma$, and $||Q||_1 \leq \alpha\}$.

Its clear that A_{al} is a convex set. It is also true that $||.||_2^2$ is a strictly convex function. It follows from Lemma 1 that the minimizer of (6) given by $\Phi^{22,o}$, if it exists is unique. If $\hat{U}^2$ and $\hat{V}^2$ have full column and row ranks then it follows that

$$\hat{Q}^0 = (\hat{U}^2)^{-l}\hat{\Phi}^{22,o}(\hat{V}^2)^{-r},$$

where $(\hat{U}^2)^{-l}$ and $(\hat{V}^2)^{-r}$ represent the left and the right inverses of $\hat{U}^2$ and $\hat{V}^2$ respectively. Thus $\hat{Q}^0$ is unique. This proves the lemma. ∎

4 Converging lower and upper bounds

4.1 Converging lower bounds

Let ν_n be defined by

$$\begin{aligned} &\inf_{Q \in \ell_1^{n_u \times n_y}} ||P_n(H^{22} - U^2 * Q * V^2)||_2^2 \\ \text{s.t.}\ & \\ &||P_n(H^{11} - U^1 * Q * V^1)||_1 \leq \gamma \\ &||Q||_1 \leq \alpha. \end{aligned} \qquad (7)$$

It is clear that only the parameters of $Q(0), \ldots, Q(n)$ enter into the optimization problem and therefore (7) is a finite dimensional quadratic programming problem. Once optimal $Q(0), \ldots, Q(n)$ are found, $Q = \{Q(0), \ldots, Q(n), 0, \ldots\}$ will be an FIR optimal solution to (7).

Theorem 3 *Suppose the constraint set in problem (5) is nonempty. Then problem (5) always has an optimal solution $Q^0 \in \ell_1^{n_u \times n_y}$. Furthermore,*

$$\nu_n \nearrow \nu.$$

Also, if $\Phi^{22,o} := H^{22} - U^2 * Q^0 * V^2$ and $\Phi^{22,n} := H^{22} - U^2 * Q^n * V^2$ where Q^n is a solution to (7) then there exists a subsequence $\{\Phi^{22,n_m}\}$ of the sequence $\{\Phi^{22,n}\}$ such that

$$||\Phi^{22,n_m} - \Phi^{22,o}||_2 \to 0 \text{ as } m \to \infty.$$

If $\hat{U}^2$ and $\hat{V}^2$ have full normal column and row ranks respectively then Q^0 is unique and

$$||\Phi^{22,n} - \Phi^{22,o}||_2 \to 0 \text{ as } m \to \infty.$$

Proof: We know that for any $Q \in \ell_1^{n_u \times n_y}$, $||P_n(H^{11} - U^1 * Q * V^1)||_1 \leq ||P_{n+1}(H^{11} - U^1 * Q * V^1)||_1$ and $||P_n(H^{22} - U^2 * Q * V^2)||_2^2 \leq ||P_{n+1}(H^{22} - U^2 * Q * V^2)||_2^2$. Therefore $\nu_n \leq \nu_{n+1}$ for all $n = 1, 2, \ldots$. Thus $\{\nu_n\}$ forms an increasing sequence. Similarly it can be shown that for all n, $\nu_n \leq \nu$.

For $n = 1, 2, \ldots$, let $\{Q^n\} \in \ell_1^{n_u \times n_y}$ be FIR solutions of (7). As the sequence $\{Q^n\}$ is uniformly bounded by α in $\ell_1^{n_u \times n_y}$ it follows from Banach-Alaoglu theorem that there exists a subsequence $\{Q^{n_m}\}$ of $\{Q^n\}$ and $Q^0 \in \ell_1^{n_u \times n_y}$ such that $Q_{ij}^{n_m} \to Q_{ij}^0$ in the $W(c_0^*, c_0)$ topology. This implies that $Q^{n_m}(t) \to Q^0(t)$ for all $t = 0, 1, \ldots$. Therefore $\forall n$, $P_n(U * Q^{n_m} * V)$ converges to $P_n(U * Q^0 * V)$ as $m \to \infty$. Now for any $n > 0$ and for any $n_m > n$, $||P_n(H^{11} - U^1 * Q^{n_m} * V^1)||_1 \leq \gamma$. This implies that $||P_n(H^{11} - U^1 * Q^0 * V^1)||_1 \leq \gamma$. Since n is arbitrary, we have

$$||H^{11} - U^1 * Q^0 * V^1||_1 \leq \gamma.$$

Similarly for any $n > 0$ and for any $n_m > n$, $||P_n(H^{22} - U^2 * Q^{n_m} * V^2)||_2^2 \leq \nu$. Again, this implies that $||P_n(H^{22} - U^2 * Q^0 * V^2)||_2^2 \leq \nu$.. Since n is arbitrary, it follows that

$$||H^{22} - U^2 * Q^0 * V^2||_2^2 \leq \nu.$$

It follows that Q^0 is an optimal solution for (5).

To prove that $\nu_n \nearrow \nu$, we note that

$||P_n(H^{22} - U^2 * Q^{n_m} * V^2)||_2^2 \leq ||P_{n_m}(H^{22} - U^2 * Q^{n_m} * V^2)||_2^2 = \nu_{n_m} \qquad \forall n > 0, \ \ \forall n_m > n.$

Taking the limit as m goes to infinity we have

$$||P_n(H^{22} - U^2 * Q^0 * V^2)||_2^2 \leq \lim_{m \to \infty} \nu_{n_m} \qquad \forall n > 0.$$

It follows that

$$||H^{22} - U^2 * Q^0 * V^2||_2^2 \leq \lim_{m \to \infty} \nu_{n_m}.$$

Thus we have shown that $\lim_{m \to \infty} \nu_{n_m} = \nu$. Since ν_n is a monotonically increasing sequence, it follows that $\nu_n \nearrow \nu$.

It is clear from Lemma 5 that $\Phi^{22,o} := H^{22} - U^2 * Q^0 * V^2$ is unique. If $\Phi^{22,n} := P_n(H^{22} - U^2 * Q^n * V^2)$ then from the discussion above it follows that $\nu_{n_m} = ||\Phi^{22,n_m}||_2^2$ converges to $\nu = ||\Phi^{22,o}||_2^2$. Also, $\Phi^{22,n_m}(t)$ converges to $\Phi^{22,o}(t)$. It follows from Lemma 2 that

$$||\Phi^{22,n_m} - \Phi^{22,o}||_2 \to 0 \text{ as } m \to \infty.$$

From Lemma 5 we also have that if $\hat{U}^2$ and $\hat{V}^2$ have full normal column and row ranks respectively then Q^0 is unique. From the uniqueness of Q^0 it follows that the original sequence, $\{\Phi^{22,n}\}$ converges to $\Phi^{22,o}$ in the two norm. This proves the theorem. ∎

4.2 Converging upper bounds

Let ν^n be defined by

$$\inf_{Q \in \ell_1^{n_u \times n_y}} ||H^{22} - U^2 * Q * V^2||_2^2$$

subject to

$$\begin{aligned} &||H^{11} - U^1 * Q * V^1||_1 \leq \gamma \\ &||Q||_1 \leq \alpha \\ &Q(k) = 0 \text{ if } k > n. \end{aligned} \tag{8}$$

It is clear that $\nu^n \geq \nu^{n+1}$ because any $Q \in \ell_1^{n_u \times n_y}$ which satisfies the constraints in the problem definition of ν^n will satisfy the constraints in the problem definition of ν^{n+1}. For the same reason we also have $\nu^n \geq \nu$ for all relevant n.

Similar results as developed for the lower bounds can be proven for these upper bounds. The details are left to the reader.

5 Conclusions

In this paper we have formulated a problem which incorporates the $\mathcal{H}_2$ performance measure and the ℓ_1 measure. It is shown that converging upper and lower bounds can be obtained via finite dimensional convex programming problems. This methodology avoids many of the problems of the zero interpolation based methods previously employed.

Ongoing research has indicated that the method developed here can be generalized to solve multiple-objective problems which involve the $\mathcal{H}_2$ measure and various time domain measures (including the ℓ_1 norm). Future research involves implementation of the method developed.

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