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Nominal $\mathcal{H}_2$ performance and ℓ_1 robust performance [1]

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Abstract

In this paper we formulate a problem which incorporates nominal $\mathcal{H}_2$ performance and robust ℓ_1 performance in the presence of a scalar perturbation. It is shown that this problem can be solved via a sequence of finite dimensional quadratic programming problems.

1 Introduction

Consider Figure 1 where, G is the nominal generalized plant, K is the controller, w is the exogenous disturbance, u is the control input, z is the regulated output and y is the measured output. Δ is the perturbation to the nominal generalized plant G which accounts for modelling errors. A number of control problems can be cast into the setup given by Figure 1, [1].

In the paradigm of Figure 1, three classes of problems can be defined. First, is the nominal performance problem where the system model is assumed to be accurate and the best possible controller K is sought to minimize the effect of w on z. Second, is the robust stability problem where the issue is the stability of the system for all perturbations which lie in a specified class. Third, is the robust performance problem which combines the objectives of the first and the second problem.

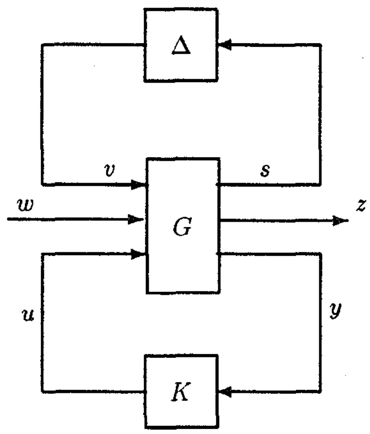

Figure 1: The Performance Problem

The nominal performance problem ignores plant modelling errors and therefore these problems are particularly relevant when accurate models for the physical system are available. The nominal performance problem of interest depends on the nature of the external disturbance w and the measure of interest on the regulated variable z.

It is well known that optimization with respect to one particular measure may compromise performance with respect to some other measure. This has motivated

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the study of problems which incorporate multiple objectives. A survey of a number of approaches to multi-objective design techniques is given in [2].

Recently, results were obtained for nominal performance problems which include time-domain specifications via the ℓ_1 norm and the $\mathcal{H}_2$ objective [6, 9, 8, 7, 10]. These results are of particular importance to the subject of this paper.

The robust performance problem in contrast to the nominal performance problem addresses the issue of synthesis of a controller K which minimizes the effect of w on z over all controllers which stabilize the system in Figure 1 for the worst case Δ belonging to a specified class. The ℓ_1 robust performance problem captures the objectives of ℓ_1 robust stability and ℓ_1 performance. In [3] it is shown that such a problem can be solved by employing sensitivity methods to linear programming, when there is only one perturbation block. However, ℓ_1 performance is no guarantee of acceptable $\mathcal{H}_2$ performance. Motivated by these concerns we formulate a problem which reflects the objectives of the ℓ_1 robust design and nominal $\mathcal{H}_2$ performance. We show that this problem can be solved via finite dimensional quadratic programming when there is only one perturbation block.

This paper is organized as follows. In section 2 we give results on robust performance when the perturabation block is bounded in the ℓ_∞ induced norm. In section 3 we formulate the problem of interest and give fomulations of problems which give upper and lower bounds to the main problem. We also show how these problems can be cast into quadratic programming problems which have certain structure. In section 4 we describe the solution method to the problem formulated in this paper. Finally, in section 5 we give the conclusions.

2 Robust Performance

Define,

$\mathbf{\Delta}_{LTV} := \{\Delta = diag\{\Delta_1, \ldots, \Delta_n\},\ \Delta_i \text{ is causal } LTV\}$,

$\mathbf{\Delta}_{NL} := \{\Delta = diag\{\Delta_1, \ldots, \Delta_n\},\ \Delta_i \text{ is causal } NLTI\}$,

where LTV and $NLTI$ mean linear time varying and nonlinear time invariant systems. Also, define

$$\mathbf{B}_{\mathbf{\Delta}_{LTV}} := \{\Delta \in \mathbf{\Delta}_{LTV} \mid \ ||\Delta||_{\infty-ind} \leq 1\},$$

and

$$\mathbf{B}_{\mathbf{\Delta}_{NL}} := \{\Delta \in \mathbf{\Delta}_{NL} \mid \ ||\Delta||_{\infty-ind} \leq 1\}.$$

$\mathbf{B}_{\mathbf{\Delta}_{LTV}}$ and $\mathbf{B}_{\mathbf{\Delta}_{NL}}$ incorporate the diagonal structure of the perturbations. Consider Figure 2(a). The ℓ_1 robust performance problem with respect to linear time-varying structured perturbations is the problem of synthesizing a controller K such that the closed-loop map is stable for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{LTV}}$ and the ℓ_1 norm of the map from w to z is less than one for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{LTV}}$. In [4] it is shown that this problem is the same as the problem of synthesizing a controller K such that the system in Figure 2(b) is stable for all causal (Δ, Δ_p) which lie in $\mathbf{B}_{\mathbf{\Delta}_p}$ which is defined as

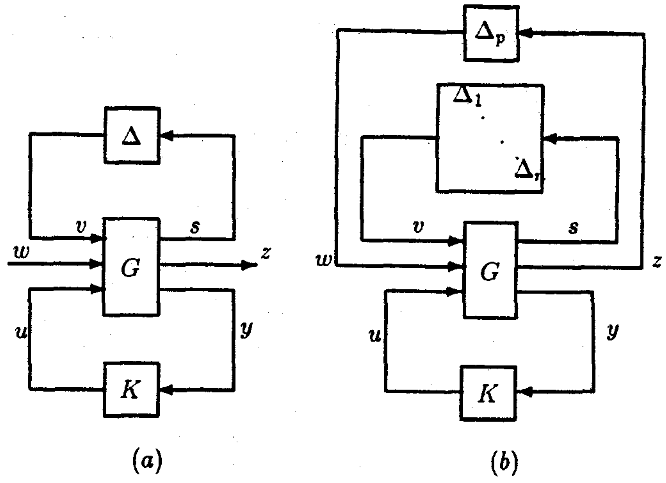

Figure 2: Robust Performance for (a) is equivalent to robust stability of (b)

$$\{(\Delta, \Delta_p) | \Delta \in \mathbf{B}_{\mathbf{\Delta}_{LTV}}, ||\Delta_p||_{\infty-ind} \leq 1, \Delta_p \text{ is } LTV\}.$$

This results in the following theorem [5],

Theorem 1 *The system in Figure 2 achieves robust performance with respect to* $\mathbf{B}_{\mathbf{\Delta}_{LTV}}$ *if and only if either of the following conditions is satisfied*

$$\begin{array}{ll} (a) & \rho(|\Phi|) < 1, \\ (b) & \inf_{D \in \mathcal{D}} \| D^{-1}\Phi D \|_1 < 1, \end{array}$$

where Φ *is the closed-loop map from* $\begin{pmatrix} v \\ w \end{pmatrix}$ *to* $\begin{pmatrix} s \\ z \end{pmatrix}$ *and* $\mathcal{D}$ *is the set of diagonal matrices with strictly positive elements which have compatible dimensions with the dimensions of* Φ.

The same theorem holds for the robust performance problem with respect to nonlinear time invariant structured perturbations. The extra block Δ_p is now restricted to be nonlinear time invariant or linear time varying.

3 Problem Formulation

In this section we formulate an optimization problem which captures the objectives of ℓ_1 robust performance and $\mathcal{H}_2$ nominal performance. After defining the problem of interest we formulate problems which give upper and lower bounds to the problem of interest. Youla parametrization [11] tells us that all closed loop maps achievable through stabilizing controllers are given by

$$\Phi = H - U * Q * V,$$

where H, U and V are fixed elements dependent only on the system G and Q is a stable free parameter. In this paper we assume the closed loop map is a two input two output map. We make the following definitions.

$$\mathcal{D} := \{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ d_i > 0\}, \tag{1}$$

$$\Theta(D) := \{\Phi = H - U*Q*V,\ Q \text{ stable}, \| D^{-1}\Phi D \|_1 \leq \gamma\}, \tag{2}$$

$$\mu(D) := \inf\{\| \Phi_{22} \|_2{}^2 \mid \Phi \in \Theta(D)\}, \tag{3}$$

$$\mu := \inf_{D \in \mathcal{D}} \mu(D). \tag{4}$$

Note that Φ_{22} is the map between w and z. Thus the problem optimizes nominal $\mathcal{H}_2$ performance between w and z while guaranteeing a specified level (given by γ) of ℓ_1 robust performance. In the following sections we obtain converging upper and lower bounds to μ.

3.1 Delay Augmentation Approach

Let n_u, n_w, n_v, n_y, n_z and n_s denote the dimension of u, w, v, y, z and s respectively. The constraint that Q is stable in the Youla parametrization can be translated into linear constraints on the closed loop map Φ. Therefore there exists an operator $\mathcal{A}$: $\ell_1^{n_w+n_v} \to \ell_1$ such that $\Phi = H - U * Q * V$ for some Q stable if and only if $\mathcal{A}(\Phi) = b$, where b is a fixed element in ℓ_1. The range space of the operator $\mathcal{A}$ is finite dimensional if $n_z + n_s = n_u$ and $n_w + n_v = n_y$. In this case the problem is called a **square problem**. We define $\mu^\delta(D)$ to be

$$\inf_{\Phi \in \Theta(D)} \| \Phi_{22} \|_2{}^2 + \delta \| (D^{-1}\Phi D)_1 \|_1 + \delta \| (D^{-1}\Phi D)_2 \|_1, \tag{5}$$

and

$$\mu^\delta := \inf_{D \in \mathcal{D}} \mu^\delta(D), \tag{6}$$

where $(D^{-1}\Phi D)_i$ is the i^{th} row of Φ. It is clear that we can approximate μ by μ^δ to the desired accuracy by choosing an appropriate δ. $\mu(D)$ is a quadratic programming problem. However, it is a difficult problem because it is not possible to show that it can be solved via *finite* dimensional quadratic programming even in the square case. In contrast it has been established for the square case that $\mu^\delta(D)$ can be solved via finite dimensional quadratic programming [7].

If $n_z + n_s > n_u$ or $n_w + n_v > n_y$ then the range space of $\mathcal{A}$ is not finite dimensional. In this case $\mu^\delta(D)$ is solved by converting it to a square problem. This is done by the *Delay Augmentation Method.* We give a brief description of this method here (for a detailed discussion see [1, 7]). Let S denote a unit shift, that is,

$$S(x(0), x(1), x(2), \ldots) = (0, x(0), x(1), \ldots),$$

and S^T denotes a T^{th} order shift. Suppose, that the Youla parametrization of the plant yields $H \in \ell_1^{n_z' \times n_w'}$, $U \in \ell_1^{n_z' \times n_u}$, and $V \in \ell_1^{n_y \times n_w'}$, where $n_z' = n_z + n_s$ and $n_w' = n_w + n_v$. Partition, $\hat{U}$ into

$$\hat{U} = \begin{pmatrix} \hat{U}^1 \\ \hat{U}^2 \end{pmatrix},$$

where $U^1 \in \ell_1^{n_u \times n_u}$. Similarly, partition $\hat{V}$ into $(\hat{V}^1, \hat{V}^2)$ where $V^1 \in \ell_1^{n_y \times n_y}$. Let

$$\hat{U}^N := \begin{pmatrix} \hat{U}^1 & 0 \\ \hat{U}^2 & \hat{S}^N \end{pmatrix} \text{ and } V^N := \begin{pmatrix} \hat{V}^1 & \hat{V}^2 \\ 0 & \hat{S}^N \end{pmatrix}.$$

where we have augmented $\hat{U}$ and $\hat{V}$ by N^{th} order delays. Let

$$\hat{\Phi}^N := \hat{H} - \hat{U}^N \hat{Q} \hat{V}^N,$$

where $\hat{Q}$ has compatible dimensions. We denote the feasible set for the delay augmented problem by $\Theta(D, N)$ which is the following set

$$\{\hat{\Phi}^N = \hat{H} - \hat{U}^N \hat{Q} \hat{V}^N,\ \hat{Q} \text{ stable } \| D^{-1}\Phi^N D \|_1 \leq \gamma\}.$$

We define the Delay Augmentation problem of order N by

$$\mu_N^\delta(D) := \inf\{l(\Phi^N) : \Phi^N \in \Theta(D, N)\}, \tag{7}$$

where

$$l(\Phi) := \| \Phi_{22} \|_2{}^2 + \delta(\| (D^{-1}\Phi D)_1 \|_1 + \| (D^{-1}\Phi D)_2 \|_1).$$

This is a square problem and can be solved via finite dimensional quadratic programming. Let the delay augmented problem corresponding to (6) be given by

$$\mu_N^\delta := \inf_{D \in \mathcal{D}} \mu_N^\delta(D) \tag{8}$$

Lemma 1

$$\lim_{N \to \infty} \mu_N^\delta = \mu^\delta,$$

where the limit on the left hand side of the equation above exists. Furthermore, μ_N^δ is an increasing sequence in N.

Notice that for a given $D \in \mathcal{D}$ we have the following:

$$D^{-1}\Phi D = \begin{pmatrix} \Phi_{11} & (d_2/d_1)\Phi_{12} \\ (d_1/d_2)\Phi_{21} & \Phi_{22} \end{pmatrix},$$

where $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. We denote d_2/d_1 by s and therefore we have

$$D^{-1}\Phi D = \begin{pmatrix} \Phi_{11} & s\Phi_{12} \\ (1/s)\Phi_{21} & \Phi_{22} \end{pmatrix}.$$

D appears in the problem only as $D^{-1}\Phi D$. Thus all problems that have been parametrized by D can be parametrized in terms of a single real variable $s := \frac{d_2}{d_1}$. Defining the vectors x and p appropriately it can be shown that $\mu_N^\delta(D)$ can be cast into the following form:

$$\begin{aligned} &\min \frac{1}{2}x'Cx - p'(s)x \\ &\text{subject to} \\ &\begin{pmatrix} A_{11}(s) & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x \le b \\ &Hx = e \\ &x \ge 0 \end{aligned} \qquad (QP1)$$

where $A_{11}(s) = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$ and $s > 0$. The achievability conditions are absorbed into H. C is positive semidefinite. $\mu_N^\delta(D)$ is a function of the variable s. Therefore, we denote $\mu_N^\delta(D)$ by $\gamma(s)$. Note that

$$\mu_N^\delta = \inf_{s \in R+} \gamma(s) =: \gamma_{opt}.$$

3.2 Finitely Many Variables Approach

Converging upper bounds can be obtained by Finitely Many Variables approach. In this approach the allowable closed loop maps achievable via stabilizing controllers are restricted to have a finite impluse response structure.

We define

$$\mu^N(D) = \inf\{\| \Phi_{22} \|_2{}^2 \mid \Phi \in \Theta^N(D)\}, \qquad (9)$$

where $\Theta^N(D)$ is defined as the set of achievable maps Φ, such that

$$\Phi(N+k) = 0 \text{ for } k > 0, \text{ and } \| D^{-1}\Phi D \|_1 \le \gamma.$$

It can be shown that $\mu^N(D)$ is a finite dimensional optimization problem. It is also true that $\mu^N := \inf_{D \in \mathcal{D}} \mu^N(D) \to \mu$ from above as $N \to \infty$. Defining x and C appropriately we can cast $\mu^T(D)$ into the following form

$$\begin{aligned} &\min \frac{1}{2}x'Cx \\ &\text{subject to} \\ &\begin{pmatrix} A_{11}(s) & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x \le b \\ &Hx = e \\ &x \ge 0 \end{aligned} \qquad (QP2)$$

Note that if we denote $\mu^N(D)$ by $\gamma(s)$ then

$$\mu^N = \inf_{s \in R+} \gamma(s) =: \gamma_{opt}.$$

4 Problem Solution

We saw in section 2 that converging upper and lower bounds to μ as defined in (4) can be obtained by solving problems which can be cast into the following form:

$$\begin{aligned} &\min \frac{1}{2}x'Cx - p'(s)x \\ &\text{subject to} \\ &\begin{pmatrix} A_{11}(s) & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x \le b \\ &Hx = e \\ &x \ge 0 \end{aligned} \qquad (QP(s))$$

$$\gamma_{opt} := \inf_{s \in R+} \gamma(s),$$

where γ_{opt} is the problem of interest. Note that $A_{11}(s)$ has the structure given by

$$A_{11}(s) = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$$

and $s > 0$. It can be shown using duality theory that the above problem has a solution if and only if there exists x, u, v, y, λ which satisfy the following constraints:

$$\begin{pmatrix} C & A'(s) & -I & 0 & H' \\ A(s) & 0 & 0 & I & 0 \\ H & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ u \\ v \\ y \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ b \\ e \end{pmatrix}, \qquad (10)$$

$$v'x + u'y = 0, \qquad (11)$$

$$\begin{pmatrix} x & u & v & y \end{pmatrix} \ge 0. \qquad (12)$$

λ is the dual variable associated with the equality constraint $Hx = e$, whereas v is the dual variable associated with the inequality $x \ge 0$. Using the structure of $A(s)$ the constraints given by equation (10) can be rearranged as given below:

$$\begin{pmatrix} s & 0 & 0 & 0 & * & * & * \\ 0 & \frac{1}{s} & 0 & 0 & * & * & * \\ * & * & -s & 0 & * & * & * \\ * & * & 0 & -\frac{1}{s} & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ \underline{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ p_1(s) \\ p_2(s) \\ \underline{b} \\ \underline{e} \end{pmatrix}$$

where the entries denoted by $*$ do not depend on s. We denote the matrix on the left hand side by $\mathbf{A}(s)$, the vector on the rightmost side of the equation by $\mathbf{b}(s)$ and $(x_1, x_2, u_1, u_2, \underline{x}, \lambda)'$ by $\mathbf{x}$ (note that λ is the last element in $\mathbf{x}$). If $\mathbf{x}$ satisfies Equations (10),(11) and (12) then it is a *feasible solution*. If $\mathbf{x}$ can be written as a combination of m independent columns of $\mathbf{A}$ which satisfies Equation (10) then we say that $\mathbf{x}$ is a *basic solution*. A basic solution which is feasible is a *basic feasible solution*. Note that a feasible solution is also an optimal solution of the quadratic programming problem.

We assume that $\mathbf{A}$ is a $m \times n$ matrix with $m \leq n$. Given an indexing set of positive integers no larger than n, say $\mathcal{J} = \{j_1, \ldots, j_m\}$, the notation $B_{\mathcal{J}}$ denotes the matrix formed by those columns of $\mathbf{A}$ indexed by the elements of $\mathcal{J}$. An indexing set is said to be *basis-index* if the $m \times m$ matrix $B_{\mathcal{J}}$ is invertible and is an *optimal-basis-index* if $(B_{\mathcal{J}})^{-1}b(s)$ corresponds to a feasible solution.

We can show that if $(QP(s))$ has a finite value for some fixed s then there exists a basic feasible solution z of $\mathbf{A}(s)\mathbf{x} = \mathbf{b}(s)$. Also note that $f(.)$ the objective of the quadratic programming problem can be wriiten as

$$f(\mathbf{x}) = c'(s)\mathbf{x}$$

if $\mathbf{x}$ is a feasible solution, for a suitably defined vector $c'(s)$.

Suppose, for some fixed value $s_0 > 0$ we have obtained a basic feasible solution of $(QP(s_0))$, given by z_{s_0}. This implies that z_{s_0} is a certain linear combination of m independent columns of $\mathbf{A}$. We call the m independent columns as the *optimal basis* associated with z_{s_0}. Our intention is to characterize the set of real numbers $0 < s$ such that $(QP(s))$ has a basic feasible solution which has the same optimal basis as the optimal basis of z_{s_0}. We introduce some notation now. c_B is the $1 \times m$ vector consisting of entries of c corresponding to the basic variables whereas c_D is the $1 \times (n-m)$ vector corresponding to the nonbasic variables. $\beta := B_{\mathcal{J}}^{-1} = [\beta_{ij}] = [\beta^1 \; \beta^2 \; \ldots \; \beta^m]$.

Definition 1 *Let $s_0 > 0$. Let $\mathcal{J}_0$ be an optimal basis for the problem $(QP(s_0))$. Define $\mathbf{x}_{\mathcal{J}_0}(.) : R \to R^m$, the* **solution function** *w.r.t $\mathcal{J}$ as follows*

$$\mathbf{x}_{\mathcal{J}_0}(s) := B_{\mathcal{J}_0}^{-1}(s)\mathbf{b}(s)$$

if $B_{\mathcal{J}_0}^{-1}(s)$ exists. Otherwise this function is given a value 0.

We **assume** throughout this paper that x_1 and x_2 are basic variables.

Theorem 2 *Let $s_0 > 0$. Let $\mathcal{J}_0$ be an optimal basis-index with x_B as the basic fesaible solution for the problem $(QP(s_0))$. Suppose u_1 and u_2 are basic variables in the optimal solution. Define $B := B_{\mathcal{J}_0}(s_0)$ and let $\beta := B^{-1}$. Then $B_{\mathcal{J}_0}(s)$ is invertible if and only*

$$\alpha(s) := det(I_4 + SYB^{-1}X) \neq 0$$

where S, Y, X are defined in the proof of the theorem. If $\alpha(s) \neq 0$ then $\mathbf{x}_{\mathcal{J}_0}(s)$ is equal to

$$x_B(s) - \begin{bmatrix} \beta^1 & \beta^2 & \beta^3 & \beta^4 \end{bmatrix} R(s) \begin{bmatrix} (x_B(s))_1 \\ (x_B(s))_2 \\ (x_B(s))_3 \\ (x_B(s))_4 \end{bmatrix}$$

where

$$R(s) := (I_4 + SYB^{-1}X)^{-1}S \text{ and } x_B(s) := B^{-1}\mathbf{b}(s).$$

If $\alpha(s) = 0$ then $\mathbf{x}_{\mathcal{J}_0}(s) = 0$.

Proof : Let $B := B_{\mathcal{J}_0}(s_0)$. As x_1, x_2, u_1 and u_2 are basic variables in the optimal we have,

$$B_{\mathcal{J}_0}(s) = B + XSY,$$

where XSY is given by

$$\begin{pmatrix} I_4 \\ 0 \end{pmatrix} \begin{pmatrix} s - s_0 & 0 & 0 & 0 \\ 0 & \frac{s_0 - s}{s_0 s} & 0 & 0 \\ 0 & 0 & s_0 - s & 0 \\ 0 & 0 & 0 & \frac{s - s_0}{s_0 s} \end{pmatrix} \begin{pmatrix} I_4 & 0 \end{pmatrix}.$$

Therefore, $det(B_{\mathcal{J}_0}(s)) = det[B(I + B^{-1}XSY)] = det(B)det(I + B^{-1}XSY) = det(B)det(I_4 + SYB^{-1}X)$. Note that $\alpha(s) = det(I_4 + SYB^{-1}X)$ and therefore, it is clear that the inverse of $B_{\mathcal{J}_0}(s)$ exists if and only if $\alpha(s) \neq 0$. Assuming $\alpha(s) \neq 0$ we find an expression for $B_{\mathcal{J}_0}^{-1}(s)$ as follows:

$$\begin{aligned} B_{\mathcal{J}_0}^{-1}(s) &= B^{-1}(I + XSYB^{-1})^{-1} \\ &= B^{-1}[I - (I + XSYB^{-1})^{-1}XSYB^{-1}] \\ &= B^{-1} - B^{-1}X(I_4 + SYB^{-1}X)^{-1}SYB^{-1} \end{aligned}$$

Now, $\mathbf{x}_{\mathcal{J}_0}(s)$

$$\begin{aligned} &= B_{\mathcal{J}_0}^{-1}(s)\mathbf{b}(s) \\ &= [B^{-1} - B^{-1}X(I_4 + SYB^{-1}X)^{-1}SYB^{-1}]\mathbf{b}(s) \\ &= B^{-1}\mathbf{b}(s) - B^{-1}X(I_4 + SYB^{-1}X)^{-1}SYB^{-1}\mathbf{b}(s) \\ &= x_B(s) - \begin{bmatrix} \beta^1 & \beta^2 & \beta^3 & \beta^4 \end{bmatrix} R(s) \begin{bmatrix} (x_B(s))_1 \\ (x_B(s))_2 \\ (x_B(s))_3 \\ (x_B(s))_4 \end{bmatrix} \end{aligned}$$

where we have defined $R(s) := (I_4 + SYB^{-1}X)^{-1}S$ and $x_B(s) := B^{-1}\mathbf{b}(s)$. An expression for the 4×4 matrix $R(s)$ can be found easily. ■

Definition 2 *Given $s_0 > 0$. Let $\mathcal{J}_0$ be an optimal basis-index for the problem $(QP(s_0))$. Define $Reg(\mathcal{J}_0)$ as the set of all poistive real numbers $s > 0$ such that*

$$\alpha(s) \neq 0, (\mathbf{x}_{\mathcal{J}_0}(s))_i \geq 0,$$

for all indices i excepting for indices which correspond to the dual variable associated with the equality constraint $Hx = e$.

Note that $\mathbf{x}_{\mathcal{J}_0}(s)$ is a rational function of s and therefore $Reg(\mathcal{J}_0)$ is a union of closed intervals except for the roots of $\alpha(s) = 0$. Determining $Reg(\mathcal{J}_0)$ is therefore an easy task.

Theorem 3 *Let $s_0 > 0$ and let $\mathcal{J}_0$ be an optimal basis-index with x_B as the basic feasible solution for the problem $(QP(s_0))$. Suppose u_1 and u_2 are basic variables in the optimal solution. Then $B_{\mathcal{J}_0}(s)$ is an optimal basis for $(QP(s))$ if and only if $s \in Reg(\mathcal{J}_0)$. Suppose, $s \in Reg(\mathcal{J}_0)$ then the objective value of $(QP(s))$ is given by $\gamma_{\mathcal{J}_0}(s)$ which is given by*

$$c_B^T(s)x_B(s) - c_B^T(s)\ \overline{\beta}R(s) \begin{bmatrix} (x_B(s))_1 \\ (x_B(s))_2 \\ (x_B(s))_3 \\ (x_B(s))_4 \end{bmatrix},$$

where

$$\overline{\beta} := \begin{bmatrix} \beta^1 & \beta^2 & \beta^3 & \beta^4 \end{bmatrix}.$$

We now present the following theorem which gives a way to compute γ_{opt}.

Theorem 4 *There exists a finite set of basis indices $\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_l$ such that $R^+ = \cup_{k=1}^{l} Reg(\mathcal{J}_k)$. Furthermore if*

$$f_k := \min_{s \in Reg(\mathcal{J}_k)} \gamma_{\mathcal{J}_k}(s)$$

then

$$\gamma_{opt} = \min_{k=0,\ldots,l} f_k.$$

We have assumed that for $(QP(s_0))$ the optimal is such that u_1, u_2 are there in the basis (we assume that x_1 and x_2 are always in the optimal basis) . This might not be so. In that case the expressions can be easily modified and they will be simpler than the ones derived.

5 Conclusions

A problem which incorporates $\mathcal{H}_2$ nominal performance and ℓ_1 robust performance was formulated. It was shown that this problem can be solved via quadratic programming using sensitivity techniques.

Future research involves the study of a similar problem where the $\mathcal{H}_2$ nominal performance and ℓ_1 robust performance are needed on different input-output maps. Effort is underway to obtain more efficient algorithms for solving these problems.

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