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Constructive Control of Quantum-Mechanical Systems

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Abstract

In this paper we develop a method for exact constructive controllability of quantum mechanical systems. The method has three steps, first a path from the initial state to the final state is determined and intermediate points chosen such that any two consecutive points are close, next small sinusoidal control signals are used to drive the system between the points, and finally quantum measurement technique is used to exactly achieve the desired state. The methodology is demonstrated for the control of spin-half particles in a Stern-Gerlach setting.

1 Introduction

Control of quantum mechanical systems has taken particular importance due to the recent theoretical and technological advances. It was shown in [9] that quantum mechanical principles can be exploited to solve problems that are intractable if classical means are employed. This has given significant boost to the field of quantum computing where quantum control is needed. At the same time laser technology has sufficiently advanced to prompt the study of control of molecular systems at the quantum regime.

Most of the studies on control of quantum-mechanical systems utilize the description of the dynamics given by Schrodingers equation:

$$i\hbar\frac{d\psi}{dt} = H(t)\psi; \quad \psi(0) = \psi_0, \tag{1}$$

where ψ is the state and H represents the Hamiltonian. In many studies it is assumed that the quantum mechanical system can be described in terms of the eigenstates of an observable with the observable having a discrete spectrum. In such a case, the underlying state space has a finite dimensional basis. In this paper we will restrict the study where such an approximation is valid. It can then be shown that the study of control of (1) is equivalent to the study of the following:

$$i\hbar\frac{dU}{dt} = H(t)U; \quad U(0) = I, \tag{2}$$

where $H = X_0 + \sum_{i=1}^{m} X_i u_i(t)$ with X_i being skew-Hermitian matrices. In this form control of quantum-mechanical systems can borrow results from the control of bilinear systems evolving on Lie groups. Even though tests based on the classical result given in [1] to assess the controllability of a bilinear system can be applied to quantum mechanical systems (see [7]) no result is known on how to synthesize a control law that drives an initial state to a desired final state. Thus the question of constructive controllability remains largely unsolved.

Recent results have been obtained on constructive controllability for specific systems. For example in [2] it was shown that in the case of a single spin $\frac{1}{2}$ particle sinusoidal control authority can be utilized to obtain any desired state in an optimal manner when the system is drift-less. In [12], radio frequency pulse sequences are designed to obtain a time optimal control of a unitary propagator in multiple spin systems.

In [4] averaging techniques were utilized to achieve approximate constructive controllability of drift-less left-invariant bilinear systems evolving on Lie groups. In this method small amplitude sinusoidal signals drive the system state to any desired final condition approximately. Once the state is near the desired state linear feedback strategies are employed to reach the desired state.

For quantum mechanical systems traditional feedback strategies cannot be directly applied (see [11]). This is due to the quantum mechanical principle that a measurement results in the state collapsing into one of the eigenstates of the observable corresponding to the measurement, thus destroying the state.

In this paper we derive the result in [4] by a simple and direct application of a classical result based on two time scale separation. We apply this result to quantum-mechanical systems. Finally we provide control techniques for spin states where a new method is introduced to obtain *exact* constructive controllability of the quantum-mechanical system. We demonstrate the effectiveness of the proposed strategy for the control of spin $\frac{1}{2}$ particles in a Stern-Gerlach experiment.

The rest of the paper is organized as follows: section

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2 presents mathematical and quantum mechanical preliminaries, section 3 presents the application of two-time scale method to quantum mechanical systems, section 4 describes the Stern-Gerlach experiment to which the developed methodology is applied, and finally we conclude in section 5.

2 Preliminaries

In this section we first present relevant quantum mechanical preliminaries outlining the basic axioms of the theory. We then present the needed mathematical preliminaries.

2.1 Quantum-mechanical preliminaries

In quantum mechanics it is assumed that the state of system can be described by a Hilbert space $\mathcal{H}$ with the complex numbers as the field. Each element of the Hilbert space is called a *ket* and the Hilbert space is called the *ket* space. Associated with the ket space is the *bra* space which is the dual space of the ket space.

We will assume that a complete orthonormal basis for the ket space is available. Note that the basis corresponds to the states that result when an observation is done. In other words they correspond to the eigenstates of an observable. Let this basis be given by $\{e_i\}_{i\in\alpha}$. One of the axioms of quantum-mechanics is that the probability that a given ket x, is in the eigenstate e_i after a measurement is given by $|<x,e_i>|^2$. Also the total probability that the ket x is in any of the eigenstates e_i after an measurement equals one, that is $\sum_{i\in\alpha}|<x,e_i>|^2 = <x,x> = 1$.

In this paper we restrict our study to cases where the ket space is a finite dimensional Hilbert space. Suppose $\psi(t)$ in the ket space describes the quantum mechanical state of a system. Then an axiom of quantum-mechanics is that the evolution of the ket is governed by Schrodingers equation

$$i\hbar\dot{\psi}(t) = H(t)\psi, \qquad (3)$$

where H is the Hamiltonian operator which maps $\mathcal{H}$ to $\mathcal{H}$. The Hamiltonian can be written as $H_0 + \sum_{k=0}^{m} u_i(t)H_i(t)$, where the functions $u_i : R \to R$ are the control signals. If we assume that the ket space $\mathcal{H}$ is a n dimensional Hilbert space then it follows that H admits a matrix representation also (with some abuse of notation) denoted by H, where H is Hermitian, and ψ can be represented by a n dimensional vector in R^n also denoted by ψ. The matrix H can be decomposed into $H_0+\sum_{k=0}^{m} u_i(t)H_i(t)$ where H_0 and H_i are all Hermitian. Associated with Equation (3) is the equation for the transition matrix

$$\begin{aligned}\dot{U} &= X_0U + \sum_{k=1}^{m} u_kX_kU \qquad (4)\\ U(t_0) &= I,\end{aligned}$$

where $X_0 := \frac{1}{i\hbar}H_0$ and $X_k := \frac{1}{i\hbar}H_k$. Note that X_0 and X_k, $k = 1,\ldots,m$ are skew-Hermitian matrices. If $U(t,t_0,u_i,\ldots,u_m,I)$ denotes the solution of (4) then $U(t,t_0,u_i,\ldots,u_m,I)\psi(t_0)$ is the solution of (3) with initial condition at time t_0 given by $\psi(t_0)$. It is to be noted that often the Hamiltonian is such that X_0 and X_k can be restricted to traceless skew-Hermitian matrices.

2.2 Mathematical preliminaries

Lemma 1 (Wei-Norman) *(see [10]) Let $\{X_i\}_{i=1}^{\ell}$ be a basis for Lie algebra $\mathcal{L}$ with $[X_i,X_j] = \sum_{k=1}^{\ell}\gamma_{ij}^kX_k$, then*

$$\left(\Pi_{j=1}^{r}e^{x_jX_j}\right)X_i\left(\Pi_{j=\ell}^{r}e^{-x_jX_j}\right) = \sum_{k=1}^{\ell}\xi_{ki}X_k, \qquad (5)$$

where $\xi_{ki}(x_1,x_2,\ldots,x_r)$ is an analytic function of x_j for $j = 1,\ldots,r$.

By expanding the left hand side of (5) and comparing coefficients of X_k we can see that

$$\begin{aligned}\xi_{ki}(x) = \delta_{ik} + \sum_{m=1}^{r}x_m\gamma_{mi}^k + \sum_{m=1}^{r}\sum_{n=1}^{\ell}\frac{x_m^2}{2}\gamma_{mi}^n\gamma_{mn}^k\\ + \sum_{m=1}^{r-1}\sum_{n=m+1}^{r}\sum_{p=1}^{\ell}x_mx_n\gamma_{ni}^p\gamma_{mp}^k + O(x_ix_jx_k).(6)\end{aligned}$$

Theorem 1 *(see [10]) Let $A(t) = \sum_{i=1}^{m}u_i(t)X_i$ and $\mathcal{L}$ be the Lie algebra generated by $\{X_i\}_{i=1}^{m}$ with the basis formed by $\{X_i\}_{i=1}^{\ell}$. Then there exits $\epsilon > 0$ such that for all t with $|t| < \epsilon$ the solution to*

$$\dot{U} = A(t)U, \quad U(0) = I;$$

is given by

$$U(t) = e^{x_1(t)X_1}e^{x_2(t)X_2}\ldots e^{x_\ell(t)X_\ell},$$

where $x(t) := (x_1,x_2,\ldots,x_\ell)$, satisfies the differential equation

$$\dot{x} = \xi^{-1}u, \qquad (7)$$

with $u = (u_1\ u_2\ \ldots\ u_m\ 0\ 0\ \ldots 0)' \in R^\ell$ and ξ is a matrix with elements ξ_{ij} defined by (5) with $r = i-1$.

We will say $\delta_1(\epsilon) = O(\delta_2(\epsilon))$ for $\epsilon \to 0$ if there exists a constant $k > 0$ and $\epsilon_0 > 0$ such that $|\delta_1(\epsilon)| \le k|\delta_2(\epsilon)|$ for all $0 < \epsilon < \epsilon_0$. Also, $e(t,\epsilon) = O(\epsilon)$ on a time scale $\frac{1}{\epsilon}$ as $\epsilon \to 0$ if there exists constants L, k and ϵ_0 such that $\sup_{0\le t\le\frac{L}{\epsilon}}|e(t,\epsilon)| \le k\epsilon$, for all $0 < \epsilon < \epsilon_0$.

The following result is based on two-variable expansion procedure [5].

Theorem 2 *(see [8]) Consider the initial value problem*

$$\dot{x} = \epsilon f(t,x),\ x(0) = x_0;$$

with f, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ defined, continuous and bounded in $D \times [0,\infty)$, $D \subset R^n$. Let $\tau := \epsilon t$ and $y(\tau)$ be the solution of

$$\frac{dy}{d\tau} = f_{av}(y);\ y(0) = x_0 + O(\epsilon^2).$$

Let $x_1 = u^1(t, y(\tau)) + z(\tau)$, with

$$u^1(t, y(\tau)) = \int_0^t [-\frac{dy}{d\tau} + f(s, y(\tau))]ds,$$

and $z(\tau)$ is the solution to

$$\frac{dz}{d\tau} = \frac{\partial f_{av}}{\partial y} z + \frac{1}{T}\int_0^T [\frac{-\partial u^1}{\partial \tau} + \frac{\partial f}{\partial y} u^1]dt,\ z(0) = 0$$

If $y(\tau)$ belongs to the interior of D then

$$x(t) = y(\tau) + \epsilon x_1(t, y(\tau)) + O(\epsilon^2)$$

on the time scale $\frac{1}{\epsilon}$.

Corollary 1 *Let $v(t)$ be periodic with period T and $v_{av} := \frac{1}{T}\int_0^T v(t)dt = 0$. Define $\tilde{v}(t) = \int_0^t v(\tau)d\tau$. Then the solution to the following dynamic system*

$$\dot{x} = \epsilon M(x) v(t);\ x(0) = x_0. \tag{8}$$

is given by,

$$x(t) = x_0 + \epsilon M(x_0)\tilde{v}(t) + \epsilon^2 t \sum_{i=1}^{\ell} G^i(x_0)\alpha^i + O(\epsilon^2)$$

on the time scale $\frac{1}{\epsilon}$, where $G^i(x) = \frac{\partial M_i(x)}{\partial x} M(x)$ with $M_i(x)$ denoting the i^{th} column of $M(x)$ and $\alpha^i = \frac{1}{T}\int_0^T v_i(t)\tilde{v}(t)dt$. Furthermore if $x_0 = O(\epsilon^2)$ then

$$x(t) = \epsilon\tilde{v}(t) + \epsilon^2 t \sum_{i=1}^{\ell} G^i(0)\alpha^i + O(\epsilon^2) \tag{9}$$

on the time scale $\frac{1}{\epsilon}$.

A drift-less quantum-mechanical system is described by equation (4) with $X_0 = 0$. If we assume that $\{X_k\}_{k=1}^m$ are independent such that $\{X_k\}_{k=1}^m$ can be extended to $\{X_k\}_{k=1}^{\ell}$ which forms the basis for the Lie algebra containing $X_1, X_2, \ldots, X_m$, we can write equation (4) as

$$\dot{U}(t) = \sum_{k=1}^{\ell} u_k(t) X_k U(t),\ U(0) = I. \tag{10}$$

with the understanding that $u_k = 0$ if $k > m$.

It is clear from Theorem 1 that for $|t| \leq \epsilon$ where ϵ is some positive number, the solution to (10) is given by

$$U(t) = \Pi_{k=1}^{\ell} e^{x_k(t)X_k}, \tag{11}$$

where $x := (x_1, x_2, \ldots, x_\ell)$ satisfies the differential equation (7). If we assume that the controls are small and periodic with period T that is $u(t) = \epsilon v(t)$ with $v(t)$ being a periodic signal with period T, from Corollary 1 we can write

$$x_n(t) = \epsilon\tilde{v}_n(t) + \epsilon^2 t \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} \gamma_{ji}^n \alpha_j^i + O(\epsilon^2) \tag{12}$$

on the time scale $\frac{1}{\epsilon}$, with $x = (x_1, x_2, \ldots, x_\ell)'$, $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_\ell)'$ and $\alpha^i = (\alpha_1^i, \alpha_2^i, \ldots, \alpha_\ell^i)$. Note that $M(x) := \xi^{-1}(x)$ is defined by (5) with $r = i - 1$ and from equations (6) and (7) $\frac{\partial M_{ni}(x)}{\partial x_j} = -\frac{\partial M_{ni}(x)}{\partial x_j} = \gamma_{ji}^n$ and $G^i(0) = \left.\frac{\partial M_i(x)}{\partial x}\right|_{x=0} M(0) = \left.\frac{\partial M_i(x)}{\partial x}\right|_{x=0}$.

Note that the above result agrees with the result obtained in [4] where the averaging technique was modified to obtain the same result. The higher order approximations derived in [4] can be directly derived using the study in [6].

3 Small amplitude sinusoidal control

In this section we present the methodology to obtain exact controllability for quantum mechanical system. That is we develop a technique to drive an initial quantum-mechanical state ψ^0 to a desired final state ψ^d. The basic steps involved are given below.

Step 1 Identify a path that connects ψ^0 to ψ^d in the state-space.

Step 2 On the path, choose intermediate points, ψ^1, ψ^2, ψ^3, ..., ψ^N such that the distance between the consecutive points is small.

Step 3 Construct control strategies to drive system state from ψ^{i-1} to ψ^i. When this control is applied to the quantum-mechanical system, it will result in a final state ψ^f which will be close to ψ^i.

Step 4 Perform a measurement which collapses the state ψ^f to ψ^i.

Step 5 Repeat the steps 3 and 4, N times, until state ψ^d is reached.

The procedure in Step 3 is done first by finding the unitary matrix U^i such that $\psi^i = U^i\psi^{i-1}$ and U^i is within $O(\epsilon)$ of I. Then devise the control law to drive (4) from $U(0) = I$ to U^f which is $O(\epsilon^2)$ close to U^i over a time scale of $\frac{1}{\epsilon}$. When this control is applied to the quantum-mechanical system, it will result in a final state ψ^f which will be close to ψ^i.

For the control of quantum-mechanical systems in the above methodology two questions remain to be answered:

- Given an arbitrary vector x^d which is in a $O(\epsilon)$ neighborhood of the origin, what is the control law $u(t) = \epsilon v(t)$, with $v(t)$ periodic with zero average, a time t_f such that $x(t_f) = x^d + O(\epsilon^2)$.
- Is there a mechanism by which quantum mechanical state can be made to reach a state x^d exactly. That is, is there a way to eliminate the error which is $O(\epsilon^2)$.

The first question is analyzed in [4]. For the second question we propose that if x^d happens to be an eigenstate of an observable then performing a measurement corresponding to the observable after $x(t_f)$ is reached (which is close to x^d) will collapse $x(t_f)$ onto x^d with a very high probability. The subsequent question is, whether every state x^d, an eigen state of some observable. We will answer these queries for the Stern-Gerlach setup for the control of a spin-$\frac{1}{2}$ particle.

4 Control of the spin-$\frac{1}{2}$ particle and the Stern-Gerlach experiment

The Stern-Gerlach apparatus consists of a beam of neutral paramagnetic particles (for example silver atoms) in a highly inhomogeneous magnetic field (see Figure 1) in a particular direction associated with the apparatus. Under the influence of the magnetic field generated by a magnet with a pointed pole tip, it is seen that the beam splits into two parts depending upon the spin of the particle. The particles deflected in the direction of the magnetic field are said to be in the plus state (spin up) state with respect to Stern-Gerlach apparatus S and the particles deflected down are in minus state (spin down) with respect to S. The plus state and the minus states of S are denoted by $|+S>$ and $|-S>$ respectively. These states would correspond to

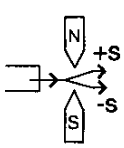

Figure 1: +S particles are deflected up and -S particles are deflected down while passing through the magnetic field

quantized values of the magnetic spin for the particles. In a modified Stern-Gerlach apparatus, either of the base states ($|+S>$ or $|-S>$) can be blocked and the other base state can be extracted. Consider any other Stern-Gerlach apparatus T which has a magnetic field in a direction different from that of the S device. Then the probability amplitude of $|iS>$ state being in the $|jT>$ state (where i, j can be + or −) is denoted by $< jT|iS >$. The probability that a $|iS>$ state results in a $|jT>$ state is given by the square of the magnitude of the probability amplitude $< jT|iS >$. It can be

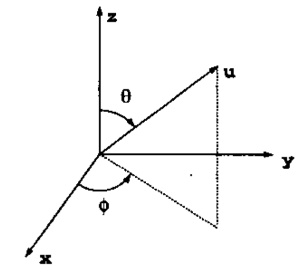

Figure 2: The S device has the magnetic field in the direction z whereas the magnetic field of the device T is in the direction u.

shown that if $< \chi|iS >$ is the probability amplitude of the state $|iS>$ being in the state $|\chi>$ after a particular measurement then the corresponding probability of a pure state of a device T whose magnetic field is oriented with respect to the magnetic field of S device as shown in Figure 2 is given by

$$\begin{bmatrix} \chi|+T \\ \chi|-T \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} \chi|+S \\ \chi|-S \end{bmatrix}$$

This can be deduced based on the symmetry of the space (see [3]). This implies

$$|+T> = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}|+S> + \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}|-S>,$$

with χ as any arbitrary state. The direction of the magnetic field of a device whose plus state corresponds to $a|+S> + b|-S>$, where a and b are complex numbers with $|a|^2 + |b|^2 = 1$, is given by,

$$\phi = Arg(b) - Arg(a),\ \theta = arctan(|b|/|a|) \qquad (13)$$

With this chosen ϕ and θ, we get

$$|+T> = e^{i\beta}(a|+S> + b|-S>),$$

where β is a real number. As $a|+S> + b|-S>$ differs from $|+T>$ by a constant phase factor $e^{i\beta}$ they are the same state physically and thus we can always construct an arbitrary superposition of the pure states of a particular device to correspond to the plus state of some other device. Thus Step 4 in the outline of the method presented in the previous section can be accomplished by performing a measurement which has the magnetic field in the appropriate direction given by equation (13).

Consider any particle subjected to an arbitrary magnetic field $\vec{B}$. The vector $\psi(t)$ of probability amplitudes corresponding to the particle being in the $|+S>$ and $|-S>$ satisfies the Schrodinger's equation (3) with $H(t) = -\mu \sum_{k=1}^{3} \sigma_k B_k$, where the direction of the magnetic field in the S device is e_3, $\vec{B} = B_1\vec{e}_1 + B_2\vec{e}_2 + B_3\vec{e}_3$, and σ_i are the Pauli matrices (see [3]). The associated differential equation for the transition matrix is given by following differential equation:

$$\dot{U} = -\frac{\gamma}{2}\sum_{k=1}^{3} u_k(t) X_k U, \tag{14}$$

where $X_1 = [0, -i; -i, 0]$, $X_2 = [0, -1; 1, 0]$ and $X_3 = [-i, 0; 0, i]$. Note that we have identified subscripts $1, 2$ and 3 with x, y and z directions where the magnetic field in the Stern-Gerlach experiment is in the z direction. Also, the Pauli matrices are given by $\frac{1}{i}X_i$. $[X_i, X_j] = 2\epsilon_{ijk}X_k$, where $\epsilon_{ijk} = 1$ if ijk is equal to 123, 231 or 312. It is -1 otherwise. X_1, X_2 and X_3 form a basis for the Lie Algebra of traceless skew-Hermitian matrices, $su(2)$. We are interested in solving the following problem;

$$\begin{aligned} \dot{U} &= -\frac{\gamma}{2}\sum_{i=1}^{3} u_i(t) X_i U \\ U(0) &= I,\ U(t_f) = U_d. \end{aligned} \tag{15}$$

We will now determine the matrix ξ given in (5) for the spin-$\frac{1}{2}$ system. Using the Baker-Hausdorf formula we can get that

$$e^{x_i(t)X_i} X_j e^{-x_i(t)X_i} = \cos 2x_i(t) X_j + \epsilon_{ijk} \sin 2x_i(t) X_k$$

Then using Wei-Norman equation (5) we can get that

$$\xi = \begin{bmatrix} 1 & 0 & \sin 2x_2 \\ 0 & \cos 2x_1 & -\cos 2x_2 \sin 2x_1 \\ 0 & \sin 2x_1 & \cos 2x_2 \cos 2x_1 \end{bmatrix},$$

which implies that

$$M = \begin{bmatrix} 1 & \tan 2x_2 \sin 2x_1 & -\tan 2x_2 \cos 2x_1 \\ 0 & \cos 2x_1 & \sin 2x_1 \\ 0 & -\frac{\sin 2x_1}{\cos 2x_2} & \frac{\cos 2x_1}{\cos 2x_2} \end{bmatrix} \tag{16}$$

Lets assume that we know $x^f = O(\epsilon)$ and that we want to drive $x = 0$ to $x = x^f$. We can get from Corollary 1 that $x(t_f) = \epsilon\tilde{v}(t) + \epsilon^2 t_f \sum_{i=1}^{3} G_i(0)\alpha_i + O(\epsilon^2)$ on the time scale $\frac{1}{\epsilon}$, where $G_1(0) = [0\ 0\ 0;\ 0\ 0\ 2;\ 0\ -2\ 0]$, $G_2(0) = [0\ 0\ -2;\ 0\ 0\ 0;\ 0\ 0\ 0]$ and $G_3(0) = 0$. If we assume that $v_1(t) = 0$, $v_2(t) = s_2\cos(\omega t)$ and $v_3(t) = s_3\cos(\omega t + \phi)$, then we can derive that $\tilde{v}(t) = \frac{1}{\omega}[0,\ s_2\sin(\omega t),\ s_3(\sin(\omega t + \phi) - \sin(\phi))]'$ and $\alpha^1 = [0,\ 0,\ 0]'$, $\alpha^2 = -\frac{1}{2\omega}[0,\ 0,\ 2s_2s_3\sin(\phi)]'$ and $\alpha^3 = -\frac{1}{2\omega}[0,\ -2s_2s_3\sin(\phi),\ 0]'$. Thus we have from equation (8)

$$x(t_f) = \begin{bmatrix} \frac{\epsilon^2 t_f}{2\omega} 2\sin(\phi) \\ \frac{\epsilon}{\omega}\sin(\omega t_f) \\ \frac{\epsilon}{\omega}(\sin(\omega t_f + \phi) - \sin(\phi)) \end{bmatrix} + O(\epsilon^2) \tag{17}$$

4.1 Illustrative example

We now provide an example of the proposed methodology. We will assume that the state of the spin-$\frac{1}{2}$ particle is described in terms of the basis $|+S>$ and $|-S>$. In this basis we assume that the initial condition is given by $\psi^o = [0.6,\ 0.8]'$. The desired state vector is $\psi^d = [0.5235 - 0.2640i,\ 0.8075 + 0.0652i]'$. It can be verified that $\psi^d = U^d\psi_0$ where,

$$U^d = \begin{bmatrix} 0.9601 - 0.2105i & -0.0657 - 0.1721i \\ 0.0657 - 0.1721i & 0.9601 + 0.2105i \end{bmatrix}.$$

Evaluating the corresponding Wei-Norman parameters we see that $U^d = e^{x_1^d X_1} e^{x_2^d X_2} e^{x_3^d X_3}$, where $x^d = [0.0785,\ 0.0500,\ 0.1000]'$.

We will assume that x^d is in an $O(\epsilon)$ neighborhood of 0. Thus we need to find small sinusoidal control signals $v_i(t)$ to drive $\dot{x} = \epsilon M(x)v$ from the origin to x^d where $M(x)$ is given by (16). We will further impose the condition that $v_1(t) = 0$. From (17) we can evaluate the state with ($O(\epsilon^2)$ error) to which the system evolves when small sinusoidal control are used. Thus we can use (17) to reach x^d with $O(\epsilon^2)$ error. An algorithm is given in [4]) to achieve this task. We will now devise the control using the algorithm given in [4]. Let $q = \frac{x_1^d}{\pi x_3^d x_2^d \tan(\psi)}$, where $0 < \psi < \pi/2$ is selected such that q is a positive integer. Define $T = \frac{t_f}{q+1}$, $t_1 = \frac{T}{4}$, $t_2 = t_1 + qT$, $t_3 = t_2 + \frac{T}{4}$ and $t_4 = t_f = t_3 + \frac{T}{2}$. We will then construct the controls as,

$$\begin{aligned} \epsilon v_1(t) &= 0 \\ \epsilon v_2(t) &= \begin{cases} x_2^d\omega\sin(\omega t) & 0 \le t \le t_1 \\ \frac{x_2^d\omega}{\cos(\psi)}\cos(\omega(t - t_1) - \psi) & t_1 < t \le t_2 \\ x_2^d\omega\cos(\omega(t - t_2)) & t_2 < t \le t_3 \\ \frac{1}{2}x_2^d\omega\sin(\omega(t - t_3)) & t_3 < t \le t_4 \end{cases} \\ \epsilon v_3(t) &= \begin{cases} x_3^d\omega\sin(\omega t) & 0 \le t \le t_3 \\ -\frac{1}{2}x_3^d\omega\sin(\omega(t - t_3)) & t_3 < t \le t_4 \end{cases} \end{aligned}$$

We have chosen $\epsilon = 0.1$ for small amplitude control and $T = 4$ for small frequency control and then $\omega = \pi/2$, $t_f = 24$ and $\phi = \pi/4$.The time plots of the control inputs $v_2(t)$ and $v_3(t)$ are given in the Figure 3. The time plots of the Wei-Norman Parameters (as obtained by solving the differential equation $\dot{x} = \epsilon M(x)v(t)$ numerically) $x_1(t)$, $x_2(t)$, and $x_3(t)$) are given in the Figure 4. It is seen that the numerical solution $x(t)$ evaluated at t_f is such that $x(t_f) = [0.0756,\ 0.0500,\ 0.1039]'$. The matrix $U(t_f)$ where $U(t)$ is a solution to (15) is given by

$$U(t_f) = \begin{bmatrix} 0.9594 - 0.2176i & -0.0657 - 0.1670i \\ 0.0657 - 0.1670i & 0.9594 + 0.2176i \end{bmatrix}.$$

Thus the numerical solution of (8) yields $\psi(t_f) = U(t_f)\psi(t_0) = [0.5231 - 0.2641i,\ 0.8069 + 0.0739i]'$.

Using the relationship (13) we can construct a Stern-Gerlach setup with its magneitic field aligned in such a manner that the plus state of the device corresponds to

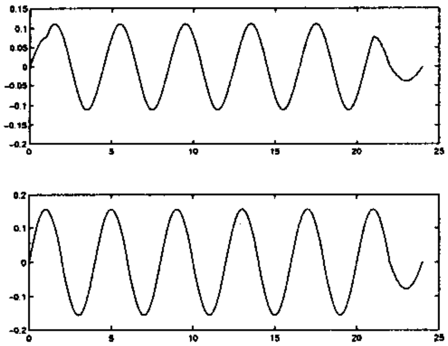

Figure 3: Control Input Signals v_2(top) and v_3(bottom)

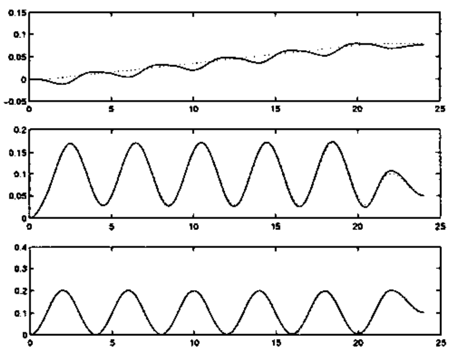

Figure 4: Actual (solid lines) and Average (dashed lines) plots of the Wei-Norman Parameters x_1(top), x_2(middle) and x_3(bottom)

the vector ψ^d. When a measurement is performed on the state $\psi(t_f)$ the probability that we will obtain the plus state is given by $| < \psi^d, \psi(t_f) > |^2 = 0.98$. Thus with the proposed method we can reach ψ^d state with a probability of 0.98.

5 Conclusions

It is evident that the method developed offers a number of advantages the primary one being that we obtain exact controllability. The result on the construction of small sinusoidal control has no restriction on the dimension of the quantum-mechanical state space. The notion of using measurements to obtain the state desired with very high probability has been demonstrated for two state systems. However, it holds considerable promise for generalization to higher dimensional state space. This is the topic of future research.

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