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Synthesis of Globally Optimal Controllers in ℓ_1 using Linear Relaxation [1]

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Abstract

This paper solves the problem of synthesis of controllers achieving *globally optimal* robust performance against structured time-varying and/or nonlinear uncertainty. The performance measure considered is the infinity to infinity induced norm of a system's transfer function. The solution utilizes linear relaxation for finding the global optimal solution.

1 Introduction

The issue of system robustness has been the subject of extensive research. Various results concerning the stability as well as performance robustness analysis for several types of uncertainty models have been obtained. For the time-invariant 2-norm bounded structured perturbations, the Structured Singular Value (SSV) theory provides a nonconservative measure for stability. For the time-varying induced ∞-norm bounded structured perturbations, [5, 6] provide computable necessary and sufficient conditions for robust stability and performance. The robustness synthesis problem, however, remains largely unsolved for any norm. Controllers can be designed using the so called $D-K$ iterations for the SSV (see e.g. [4]). When perturbations with induce ∞-norm bound are present, a $D-K$ iteration type procedure appears in [6]. However, neither method guarantees that a global minimum is achieved, and in general a local minimum is reached. The problem appears to be an inherently nonconvex one in either norm. In the Full Information or State-Feedback cases, however, the synthesis problem for H_∞ has been solved for any number of uncertainty blocks. See [9]. Recently, Yamada and Hara [11] provided an algorithm for approximately finding a global solution to the constantly scaled H_∞ control synthesis problem.

In this paper, we address the output feedback synthesis problem when the signal norm is the infinity norm and the perturbations are structured time-varying or nonlinear systems with an induced ∞-norm bound. A *globally optimal* solution to the robustness synthesis problem is obtained. It is shown that the solution involves only solving certain linear programming problems.

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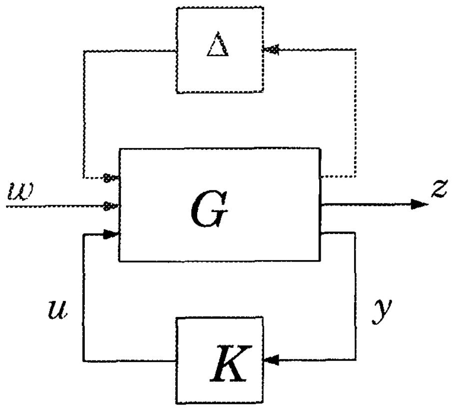

Figure 1: Robust Performance Problem

2 Problem Statement

Before presenting the problem statement we provide the notation employed. ℓ_∞ is the space of bounded sequences of real numbers, ℓ_1 is the space of absolutely summable sequences of real numbers. $\ell_1^{p\times q}$ the $p \times q$ matrices of elements of ℓ_1. If $x \in \ell_1^{p\times q}$, $\|x\|_1 := \max_i \sum_j \|x_{ij}\|_1$. P_N is the truncation operator so that for any sequence x, $P_N x = y$ where $y(k) = x(k)$ whenever $k \leq N$ and $y(k) = 0$ for $k > N$.

Consider the system, which appears in Figure 1. As before, G is a generalized LTI plant, and K is an LTI controller, and both are in discrete-time. In this system modeling uncertainty is also considered. The perturbation block describes the uncertainty Δ belonging to the class of admissible perturbations:

$\underline{\Delta} := \{\Delta = diag(\Delta_1, \ldots, \Delta_n) \ : \ \Delta_i : \ell_\infty \Rightarrow \ell_\infty$ is causal, and $\|\Delta_i\| := \sup_{u\neq 0} \frac{\|\Delta_i u\|_\infty}{\|u\|_\infty} \leq 1\}$,

where the norm used is the ℓ_∞ norm. The perturbation may therefore be nonlinear or time-varying. The system in the figure is said to be robustly stable if it is ℓ_∞-stable for all admissible perturbations, i.e. for all $\Delta \in \underline{\Delta}$. The problem we shall address in this paper is as follows:

Problem Statement: Find a linear finite dimensional controller K such that:

1. The system achieves robust stability, and
2. The system achieves robust performance, i.e.
$$\|\mathcal{T}_{zw}\| < 1 \qquad \forall\, \Delta \in \underline{\Delta}$$
where $\mathcal{T}_{zw}$ is the map from w to z, and the norm used above is the induced ℓ_∞ operator norm.

The robustness *synthesis* problem can be stated as follows:

$$\inf_{D\in\mathcal{D}} \ \inf_{Q\in\ell_1} \left\|D^{-1}\Phi(Q)D\right\|_1 =: \gamma_* \qquad (1)$$

where $\Phi(Q) = H - U * Q * V$ is the standard parameterization of the closed-loop system using the Youla parameter Q. For each *fixed* $D = diag(d_1, \ldots, d_n)$, problem (1) is a standard ℓ_1 norm-minimization problem. We say Φ is an achievable closed loop map if there exists a stable Q such that $\Phi = H - U * Q * V$. It can be shown that Φ is achievable if and only if $\mathcal{A}\Phi = b$ where $\mathcal{A} : \ell_1^{n\times n} \to \ell_1$ is a linear operator and $b \in \ell_1$. Both $\mathcal{A}$ and b can be determined based on H, U and V. Let $\mathcal{D}$ denote the set of all diagonal $n \times n$ real matrices with positive elements. Before recasting the main problem given by (1) we define,

$$\gamma(D) := \inf \|D^{-1}\Phi D\|_1 \text{ subject to } \mathcal{A}\Phi = b, \qquad (2)$$

$$\overline{\gamma}^N(D) := \inf \|D^{-1}\Phi D\|_1 \text{ subject to } \mathcal{A}\Phi = b,\ \Phi(k) = 0 \text{ for all } k \geq N, \qquad (3)$$

$$\underline{\gamma}^N(D) := \inf \|D^{-1}\Phi D\|_1 \text{ subject to } \mathcal{A}_N\Phi = b \qquad (4)$$

where $\mathcal{A}_N = P_N\mathcal{A}$ and $b_N = P_N b$.

It is to be noted that problems (3) and (4) can be shown to be finite dimensional linear programming problems. Using the above definitions we further define

$$\overline{\gamma}_*^N := \inf_{D\in\mathcal{D}} \overline{\gamma}^N(D), \qquad (5)$$

$$\underline{\gamma}_*^N := \inf_{D \in \mathcal{D}} \underline{\gamma}^N(D) \quad (6)$$

$$\gamma_* := \inf_{D \in \mathcal{D}} \gamma(D), \quad (7)$$

The following results can be established.

Lemma 1 $\overline{\gamma}_*^N \searrow \gamma_*$ *and* $\underline{\gamma}_*^N \nearrow \gamma_*$. *as* $N \to \infty$

The previous lemma suggests that if an effective solution procedure exists to solve for $\overline{\gamma}_*^N$ and $\underline{\gamma}_*^N$ then we can obtain converging upper and lower bounds to γ_*. The rest of the development in this section will be focussed in deriving a method to obtain $\overline{\gamma}_*^N$ for any given N. The methodology for finding $\underline{\gamma}_*^N$ can be identically developed. Note that in the problem statement of (3) only a finite number of variables are present. However, because of the special structure of the matrix $\mathcal{A}$ it can be shown that if only finite number of variables $\Phi(k)$ are involved then only a finite number of constraints posed by the equation $\mathcal{A}\Phi = b$ are relevant to the optimization. Thus it can be shown that

$$\begin{array}{rcl} \overline{\gamma}^N(D) & = & \inf \|D^{-1}\Phi D\|_1 \\ & & \text{subject to} \\ & & \mathcal{A}^{N'}\Phi = b \\ & & \Phi(k) = 0 \text{ for all } k \geq N, \end{array} \quad (8)$$

where N' depends on N. The optimization associated with (8) can be cast into the finite dimensional linear program:

$$\begin{array}{rcl} \overline{\gamma}^N(D) & = & \inf \alpha \\ & & \text{subject to} \\ & & \sum_j \sum_{k=1}^N d_j(\Phi_{ij}^+(k) + \Phi_{ij}^-(k)) \leq d_i\alpha, \\ & & \mathcal{A}^{N'}(\Phi^+ - \Phi^-) = b, \\ & & \Phi_{ij}^+ \in R^N, \Phi_{ij}^- \in R^N, \\ & & \Phi_{ij}^+(k) \geq 0,\ \Phi_{ij}^-(k) \geq 0, \alpha \geq 0. \end{array}$$

Note that $\overline{\gamma}_*^N = \inf_{D \in \mathcal{D}} \overline{\gamma}^N(D)$. Thus the resulting finite dimensional optimization problem that needs to be solved has the following structure;

$$\begin{array}{rcl} \overline{\gamma}_*^N & = & \inf \alpha \\ & & \text{subject to} \\ & & \sum_j d_j p_{ij} \leq d_i\alpha, \\ & & p_{ij} = \sum_{k=1}^N \Phi_{ij}^+(k) + \Phi_{ij}^-(k), \\ & & \mathcal{A}^N(\Phi^+ - \Phi^-) = b, \\ & & \Phi_{ij}^+ \in R^N, \Phi_{ij}^- \in R^N, \\ & & \Phi_{ij}^+(k) \geq 0,\ p_{ij} \geq 0, \\ & & \Phi_{ij}^-(k) \geq 0, \alpha \geq 0, d_j > 0, \end{array} \quad (9)$$

For what follows, we will asssume that *a priori* upper and lower bounds are available on the variables d_j, α and p_{ij} involved in the optimization. Accordingly we assume that $L_d \leq d_j \leq U_d$ for all $j = 1, \ldots, n$. An upper bound on α is immediate because any feasible solution to the problem will yield an upper bound on α. We will assume that similar bounds exist on p_{ij}. Thus the optimization problem of interest is

$$\begin{array}{rl} \mu := & \inf \alpha \\ & \text{subject to} \\ & \sum_j d_j p_{ij} \leq d_i\alpha, \\ & p_{ij} = \sum_{k=1}^N \Phi_{ij}^+(k) + \Phi_{ij}^-(k)), \\ & \mathcal{A}^N(\Phi^+ - \Phi^-) = b, \\ & \Phi_{ij}^+ \in R^N, \Phi_{ij}^- \in R^N, \\ & L_{ij} \leq p_{ij} \leq U_{ij} \\ & L_d \leq d_j \leq U_d, L_\alpha \leq \alpha \leq U_\alpha, \\ & \Phi_{ij}^+(k) \geq 0,\ p_{ij} \geq 0, \\ & \Phi_{ij}^-(k) \geq 0, \alpha \geq 0, d_j > 0. \end{array} \quad (10)$$

3 Problem solution

In this section we provide the problem solution. We make the assumption that $L_d > 0$ in problem (10). Consider problem (10) posed on a smaller grid;

$$\begin{aligned}
\mu(\ell_d,u_d) := \quad & \inf \alpha \\
& \text{subject to} \\
& \textstyle\sum_j d_j p_{ij} \le d_i \alpha, \\
& p_{ij} = \textstyle\sum_{k=1}^{N} \Phi_{ij}^{+}(k) + \Phi_{ij}^{-}(k), \\
& \mathcal{A}^{N'}(\Phi^{+} - \Phi^{-}) = b, \\
& \Phi_{ij}^{+} \in R^N, \Phi_{ij}^{-} \in R^N, \\
& L_{ij} \le p_{ij} \le U_{ij}, \\
& \ell_{d_j} \le d_j \le u_{d_j},\ L_\alpha \le \alpha \le U_\alpha, \\
& \Phi_{ij}^{+}(k) \ge 0,\ p_{ij} \ge 0, \\
& \Phi_{ij}^{-}(k) \ge 0, \alpha \ge 0, d_j > 0.
\end{aligned} \tag{11}$$

where $\ell_d = (\ell_{d_1}, \ldots, \ell_{d_n})$, $u_d = (u_{d_1}, \ldots, u_{d_n})$. For our purposes the interval $[\ell_{d_j}, u_{d_j}]$ is a subset of the interval $[L_d, U_d]$. Note that since the variables in the statement of $\mu(\ell_d, u_d)$ are confined to smaller region than in μ, the above problem is being solved for a sub-problem on a grid.

In solving the subproblem, a relaxation scheme will be employed. For this purpose, the following lemma is needed:

Lemma 2 *If the variables $x_j \in R$ satisfy the conditions $\ell_j \le x_j \le u_j$ and $t_{ij} := x_i x_j$ then*

$$t_{ij} \ge u_j x_i + u_i x_j - u_i u_j, \tag{12}$$
$$t_{ij} \le \ell_j x_i + u_i x_j - u_i \ell_j, \tag{13}$$
$$t_{ij} \le u_j x_i + \ell_i x_j - \ell_i u_j, \tag{14}$$
$$t_{ij} \ge \ell_j x_i + \ell_i x_j - \ell_i \ell_j. \tag{15}$$

Furthermore, if variables $t_{ij} \in R$ satisfy (12), (13), (14), (15) and x_k satisfy $\ell_k \le x_k \le u_k$ then

$$|t_{ij} - x_i x_j| \le \frac{1}{4}(u_i - \ell_i)(u_j - \ell_j). \tag{16}$$

Define $W_{ij} := \{(p_{ij}, d_j, w_{ij}) \in R^3 |$ (12), (13), (14), (15) are satisfied with t_{ij}, x_i, x_j, u_i, ℓ_i, u_j, ℓ_j replaced by w_{ij}, p_{ij}, d_j, U_{ij}, L_{ij}, u_{d_j}, $\ell_{d_j}\}$,

$W_i := \{(\alpha, d_i, w_i) \in R^3 |$ (12), (13), (14), (15) are satisfied with $t_{ij}, x_i, x_j, u_i, \ell_i, u_j, \ell_j$ replaced by $w_i, \alpha, d_i, U_\alpha, L_\alpha, u_{d_j}, \ell_{d_j}\}$.

Thus it follows that

$$\begin{aligned}
\mu(\ell_d,u_d) = \quad & \inf \alpha \\
& \text{subject to} \\
& \textstyle\sum_j w_{ij} \le w_i, \\
& (p_{ij}, d_j, w_{ij}) \in W_{ij}, \\
& (\alpha, d_i, w_i) \in W_i, \\
& w_{ij} = d_j p_{ij},\ w_i = d_i \alpha \\
& p_{ij} = \textstyle\sum_{k=1}^{N} \Phi_{ij}^{+}(k) + \Phi_{ij}^{-}(k)), \\
& \mathcal{A}^{N'}(\Phi^{+} - \Phi^{-}) = b, \\
& \Phi_{ij}^{+} \in R^N, \Phi_{ij}^{-} \in R^N, \\
& L_{ij} \le p_{ij} \le U_{ij}, \\
& \ell_{d_j} \le d_j \le u_{d_j},\ L_\alpha \le \alpha \le U_\alpha, \\
& \Phi_{ij}^{+}(k) \ge 0,\ p_{ij} \ge 0, \\
& \Phi_{ij}^{-}(k) \ge 0, \alpha \ge 0, d_j > 0.
\end{aligned} \tag{17}$$

Note that in the above equation all the constraints are linear constraints except for the constraints $w_{ij} = p_{ij} d_j$ and $w_i = \alpha d_i$. Let $\mu_R(\ell_d, u_d)$ be the *relaxed* problem obtained by removing the nonlinear constraints from (17). Then

Lemma 3 *If the problem $\mu_R(\ell_d, u_d)$ is infeasible then so is the problem $\mu(\ell_d, u_d)$. If $\mu_R(\ell_d, u_d)$ is feasible then $\mu_R(\ell_d, u_d) \le \mu(\ell_d, u_d)$.*

Now we will obtain an upper bound on μ from the solution to μ_R. Suppose $(p_{ij}^*, \alpha^*, d_j^*, w_{ij}^*, w_i^*, \Phi^{+,*}, \Phi^{-,*})$ are feasible variables for the relaxed problem with $\alpha^* = \mu_R$. We will construct feasible variables for (17) from these variables. Let $w_{ij}^f := p_{ij}^* d_j^*$, and let $c := \max_i \sum_j w_{ij}^f$. Further suppose that $c = \sum_j w_{i_0 j}^f$. Let $w_{i_0} = c$. Note that $d_{i_0}^* > L_d > 0$. Choose $\alpha^f = \frac{c}{d_{i_0}^*}$ and let $w_i^f = d_i^* \alpha^f$. Then it follows that $(p_{ij}^*, \alpha^f, d_j^*, w_{ij}^f, w_i^f, \Phi^{+,*}, \Phi^{-,*})$ are feasible variables for (11). Thus $\alpha^f \ge \mu(\ell_d, u_d)$. One can establish the following result,

Lemma 4 $|\alpha^f - \alpha^*| \le$

$$\frac{\frac{1}{4}\sum_j (u_{i_0 j} - \ell_{i_0 j})(u_d - \ell_d) + \frac{1}{4}(u_\alpha - \ell_\alpha)(u_d - \ell_d)}{d_{i_0}}.$$

Note that α^* is a lower bound to $\mu(\ell_d, u_d)$ and α^f is an upper bound to $\mu(\ell_d, u_d)$. Thus

$\alpha^* \leq \mu(\ell_d, u_d) \leq \alpha^f$. The above lemma gives an estimate on how good an approximation is given by α^f and α^*. Note that for any optimization on a smaller grid the optimal value $\mu(\ell_d, u_d)$ is an upper bound on μ (because the optimization for μ is on the whole variable space whereas it is only on a portion of the variable space for $\mu(\ell_d, u_d)$). Thus the upper bound obtained on $\mu(\ell_d, u_d)$ is indeed an upper bound on μ.

Suppose, the gridding is performed to obtain a mesh such that for each grid $\|u_d - \ell_d\|_\infty = \epsilon$. Then the total number of problems to be solved equals $(\frac{U_d - L_d}{\epsilon})^n$ where n is the number of d variables. Let μ^* represent the minimum of the optimal values $\mu(\ell_d, u_d)$ on each grid. Then it is clear that $\mu = \mu^*$. Let μ_R^* denote the relaxation associated with μ^*. Let α^f denote the upper bound as obtained in Lemma 4. Then it follows that $\mu^* \leq \mu \leq \alpha^f$. Also, it follows that $\alpha^f - \mu \leq$

$$\frac{\frac{1}{4}\sum_j (U_{i_0 j} - L_{i_0 j})(u_d - \ell_d) + \frac{1}{4}(U_\alpha - L_\alpha)(u_d - \ell_d)}{d_{i_0}} \leq \epsilon\left(\frac{\frac{1}{4}\sum_j (U_{i_0 j} - L_{i_0 j}) + \frac{1}{4}(U_\alpha - L_\alpha)}{L_d}\right).$$

Thus to obtain a value with an ϵ tolerance the number of problems to be solved is in the order of $\frac{1}{\epsilon^n}$. Also for any grid associated with a given ℓ_d, u_d, if $\mu_R(\ell_d, u_d)$ is greater than any upper bound on any other grid then that grid can be discarded. This can be used as the basis for a branch and bound algorithm whereby branches corresponding to regions in the d-parameter space can be fathomed as soon as the lower bound obtained from solving the relaxed problem for these regions becomes larger than the best available global upper bound for the problem. If this does not happen, further gridding on that region is performed, and hence new branches are formed. The process continues until all the branches which have not been fathomed have lower bounds equal to (or up to a given tolerance of) the best available upper bound, which also will be the global optimal value.

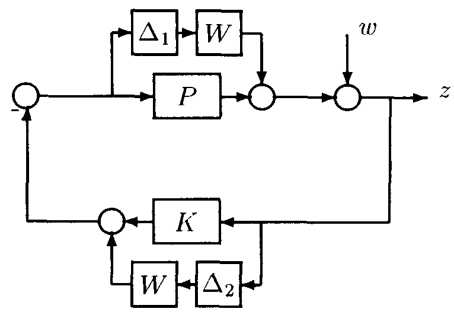

Figure 2: Example system

4 Robust Control Example

Consider the system in the figure above where

$$P = \frac{z+2}{z^2(z-0.5)(z+0.5)}$$

and

$$W = \frac{0.02z^2 - 0.04z + 0.02}{z^2 + 1.56z + 0.64}.$$

The objective is to design (if possible) a controller which makes the worst case norm of the mapping from w to z less than one. This can posed as a robust stability problem by connecting z to w through a fictitious perturbation whose norm is allowed to be less than or equal to one, and then requiring the resulting system with 3 perturbation blocks to be robustly stable. The synthesis problem for that system is:

$$\inf_{D \in \mathcal{D}} \inf_{Q \in \ell_1} \left\| D^{-1}\Phi(Q)D \right\|_1 =: \gamma_* \qquad (18)$$

where Φ is the impulse response matrix of the system "seen" by the three uncertainty blocks. When applying the RLT algorithm to this problem, the globally optimal value ($\pm$ 0.01) of the objective function was found to be 2.93. This was obtained after 71 iterations. After only 35 iterations the best available upper bound ($\pm$ 0.01) coincides with the global solution. However, the best available lower bound was 2.8, and additional iterations were needed only to verify that the best upper bound value cannot be improved upon. The figure below shows the

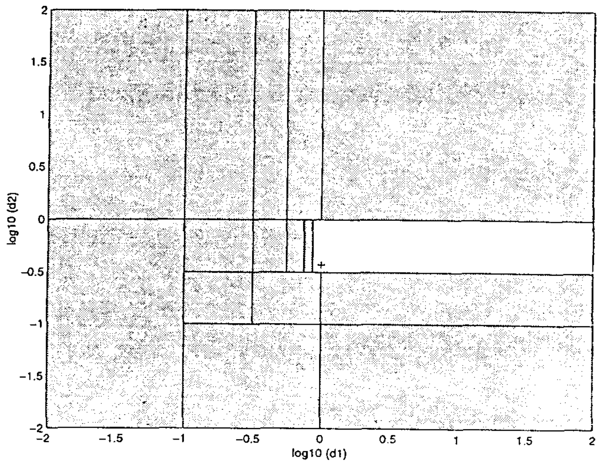

Figure 3: Fathomed branches in the d parameter space (after 35 iterations)

branches that have been fathomed after 35 iterations (gray color). The lower bounds obtained for these branches guarantee that the global optimal cannot be achieved in that region. Further branching on the white region allows us to zoom in on the region where the global optimal is achieved.

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