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MIMO OPTIMAL CONTROL DESIGN: THE INTERPLAY OF THE $\mathcal{H}_2$ AND THE ℓ_1 NORMS

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ABSTRACT

In this paper we consider the problem of minimizing the $\mathcal{H}_2$ performance of the closed loop subject to a ℓ_1 constraint. The development is devoted to multi-input multi-output (MIMO) systems. It is shown that approximating solutions to the optimal within any *a priori* given tolerance can be obtained via a finite dimensional quadratic optimization problem whose dimension is known *a priori* .

1. Notation

In this section we present the notation employed in this paper. $\hat{x}(\lambda)$ is the λ transform of a right sided real sequence (possibly a matrix sequence) $(x(k))_{k=0}^{\infty}$. $\ell_1, \ell_2, \ell_\infty$ are the well known banach spaces of sequences. X^* is the dual space of the banach space X. $< x, x^* >$ denotes the value of the bounded linear functional x^* at $x \in X$.

2. Introduction

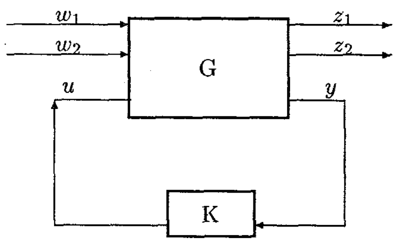

Figure 1: Closed Loop System.

Consider the system of Figure 1 where $w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is the exogenous disturbance, $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is the regulated output, u is the control input and y is the measured output. In feedback control design the objective is to design a controller, K such that with $u = Ky$ the resulting closed loop map Φ_{zw} from w to z is stable (see Figure) and satisfies certain performance criteria. Such criteria may be posed in terms of a measure on Φ_{zw} which depends on the signal norms of w and z that may be of interest in a particular situation.

The two norm of the closed loop, $\| \Phi_{zw} \|_2$ measures the energy in the regulated output z for a unit pulse input, w. The standard $\mathcal{H}_2$ problem finds a stabilizing controller K which results in a closed loop map which has the minimum $\mathcal{H}_2$ norm when compared to all other closed loop maps achievable through stabilizing controllers. State space solutions for this problem can be found in [4].

The ℓ_1 norm of the closed loop, $\| \Phi_{zw} \|_1$ is the infinity norm of the regulated output z, for the worst magnitude bounded disturbance, w i.e.

$$\| \Phi_{zw} \|_1 = \sup_{\| w \|_\infty \leq 1} \| \Phi_{zw} w \|_\infty .$$

The standard ℓ_1 problem finds a controller which minimizes this norm over all closed loop maps that are achievable through stabilizing controllers. Thus the standard ℓ_1 problem is to determine a controller K which solves the following problem:

$$\min_{\text{K stabilizing}} \| w \to z \|_1 .$$

It is shown in [3] that this problem reduces to a finite dimensional linear program for the 1-block case.

Both of the previous criteria refer to a single performance measure of the closed loop. It is well known (see for example [2]) that minimization with respect to one norm may not necessarily yield good performance with respect to another. This has led to research in problems where multiple measures of the closed loop are incorporated directly into the design. Recently, results were obtained for mixed objective problems involving the ℓ_1 and the $\mathcal{H}_2$ norms of the closed loop for single-input single-output (SISO) systems. In [5] the problem of minimizing the two norm of the closed loop while keeping its one norm below a prespecified level was reduced to a finite dimensional

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quadratic optimization problem. The converse problem of minimizing the ℓ_1 norm of the closed loop over all internally stabilizing controllers while keeping its two norm below a prespecified level was reduced to a finite dimensional convex optimization problem in [6]. It was also shown that the dimension of the equivalent convex optimization problem can be determined *a priori*.

This paper explores the interplay of the ℓ_1 and the $\mathcal{H}_2$ norms of the closed loop for the multi-input multi-output (MIMO) case which is much richer in its complexity and its applicability than the SISO case. Consider for example Figure 1 where the part of the regulated output given by z_2 is used to reflect the performance with respect to a unit pulse input and it is also required that the the maximum magnitude of z_1 due to a worst magnitude bounded input stays below a prespecified level. This objective can be captured by the following problem:

$$\min_{K \text{ stabilizing}} \{\| w \to z_2 \|_2 : \| w \to z_1 \|_1 \leq \gamma\}, \quad (1)$$

where γ is the level over which the infinity norm of z_2 is not allowed to cross for the worst bounded disturbance. Or, it may be that the disturbance w is such that a part of it, w_1 is a white noise while another part, w_2 is magnitude bounded. A relevant objective is the minimization of the effect of these disturbances on the regulated output. The problem,

$$\min_{K \text{ stabilizing}} \{\| w_1 \to z \|_2 : \| w_2 \to z \|_1 \leq \gamma\}, \quad (2)$$

where γ is the level over which the infinity norm of z is not allowed to cross for the worst bounded disturbance is then a problem of interest.

Both problems (1) and (2) mentioned previously fall under a general framework of a problem which we call the *mixed problem*. The MIMO problem in the mixed $\mathcal{H}_2$ and ℓ_1 setting poses many questions which are not addressed in the SISO setting of [5]. It is shown that MIMO problems need to be handled differently in a significant way. The optimal solutions for the 1-block are not in general finite impulse responses nor are they unique (as will be shown) unlike the SISO case. However, it is established that the approximating solutions to the optimal within any given tolerance can be obtained via finite dimensional quadratic optimization. We show that it is possible to obtain an *a priori* bound on the dimension of the suboptimal problems (no *a priori* bound is given in [5]where the SISO case is considered).

The paper is organized as follows: In section 2 we give system and mathematical preliminaries. In section 3 we study the mixed problem. In section 4 conclusions are given.

3. Preliminaries

In the first part of this section we give system preliminaries and in the second part we give mathematical preliminaries. It is to be noted that all the assumptions made in this section are valid throughout the paper.

3.1 System Preliminaries

In this subsection we state theorems, assumptions and give notation which will be relevant to the rest of the paper. A good reference for this subsection is [2]. We denote by n_u, n_w, n_z and n_y the number of control inputs, exogenous inputs, regulated outputs and measured outputs respectively of the plant G. We represent by Θ, the set of closed loop maps of the plant G which are achievable through stabilizing controllers. $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$ and $V \in \ell_1^{n_y \times n_w}$ characterize the Youla parametrization of the plant [7]. The following theorem follows from Youla Parametrization.

Theorem 1 $\Theta = \{\Phi \in \ell_1^{n_z \times n_w}$: there exists a $Q \in \ell_1^{n_u \times n_y}$ with $\hat{\Phi} = \hat{H} - \hat{U}\hat{Q}\hat{V}\}$.

If Φ is in Θ we say that Φ is an *achievable* closed loop map. We now state all the assumptions made (which are valid throughout this paper).

Assumption 1 *$\hat{U}$ has normal rank n_u and $\hat{V}$ has normal rank n_y.*

Let Λ_{UV} denote the set of zeros of $\hat{U}$ and $\hat{V}$ in $\mathcal{D}$ where $\mathcal{D}$ is the open unit disc.

Assumption 2 *$\hat{U}$ and $\hat{V}$ have no zeros which lie on the unit circle, that is $\Lambda_{UV} \subset int(\mathcal{D})$. Also. all the zeros of $\hat{U}$ and $\hat{V}$ which lie inside the unit disc are real.*

See [2] for the definitions of $\sigma_{U_i}(\lambda_0)$ and $\sigma_{V_j}(\lambda_0)$. We define

$$N_{ij\lambda_0} := \sigma_{U_i}(\lambda_0) + \sigma_{V_j}(\lambda_0) - 1.$$

Theorem 2 *[2] $\Phi \in \ell_1^{n_z \times n_w}$ is in Θ if and only if the following conditions hold:*

$$< \Phi, F^{ijk\lambda_0} >=< H, F^{ijk\lambda_0} > \ \forall \begin{cases} \lambda_0 \in \Lambda_{UV} \\ i = 1, \ldots, n_u \\ j = 1, \ldots, n_y \\ k = 0, \ldots, N_{ij\lambda_0} \end{cases}$$

and

$$\begin{array}{l} < \Phi, G_{\alpha_i qt} >=< H, G_{\alpha_i qt} > \\ < \Phi, G_{\beta_j pt} >=< H, G_{\beta_j pt} > \end{array} \ \forall \begin{cases} i = n_u + 1, \ldots, n_z \\ j = n_y + 1, \ldots, n_w \\ q = 1, \ldots, n_w \\ p = 1, \ldots, n_z \\ t = 0, 1, 2, \ldots \end{cases}$$

$F^{ijk\lambda_0}$, $G_{\alpha_i qt}$ and $G_{\beta_j pt}$ are matrix sequences in $\ell_1^{n_z \times n_w}$ (See [2] for a detailed description of $F^{ijk\lambda_0}$, $G_{\alpha_i qt}$ and $G_{\beta_j pt}$.)

The first set of conditions constitute the *zero interpolation* conditions whereas the second set consists of the *rank interpolation conditions.* The plant G is called **square**, or equivalently, we have a 1-block problem, if the rank interpolation conditions are absent (i.e., when $n_u = n_z$ and $n_y = n_w$). Otherwise, the plant is **non-square**, or equivalently, we have a 4-block problem.

We define $b^{ijk\lambda_0} :=< H, F^{ijk\lambda_0} >$ and $c_z := \sum_{\lambda_0 \in \Lambda_{UV}} \sum_{i=1}^{n_u} \sum_{j=1}^{n_y} N_{ij\lambda_0}$. c_z is the total number of zero interpolation conditions. The following problem

$$\nu_{0,1} = \inf_{\Phi \text{ Achievable}} \{||\Phi||_1\}, \tag{3}$$

is the standard multiple input multiple output $\ell_1^{n_z \times n_w}$ problem. In [3] it is shown that this problem for a square plant has a solution, possibly nonunique but the solution is a finite impulse response matrix sequence. Let

$$\mu_{0,2} := \inf_{\Phi \text{ Achievable}} \{||\Phi||_2^2\}, \tag{4}$$

which is the standard $\mathcal{H}_2$ problem. The solution to this problem is unique and is an infinite impulse response sequence.

3.2 Mathematical Preliminaries

In this section we collect all the relevant theorems from convex optimization theory. [1] is an excellent reference for this subsection. The following is a Lagrange duality theorem.

Theorem 3 *Let X be a Banach space, Ω be a convex subset of X, Y be a finite dimensional space, Z be a normed space with positive cone P. Let $f : \Omega \to R$ be a real valued convex functional, $g : X \to Z$ be a convex mapping, $H : X \to Y$ be an affine linear map and $0 \in int[range(H)]$. Define*

$$\mu_0 := inf\{f(x) : \ g(x) \leq 0, \ H(x) = 0, \ x \in \Omega\}. \tag{5}$$

Suppose there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$ and suppose μ_0 is finite. Then,

$$\mu_0 = max\{\varphi(z^*, y) : z^* \geq 0, \ z^* \in Z^*, \ y \in Y\}, \tag{6}$$

where $\varphi(z^*, \ y) := inf \ \{ \ f(x) + < g(x), z^* > + < H(x), y >: \ x \in \Omega \ \}$ and the maximum is achieved for some $z_0^* \geq 0$, $z_0^* \in Z^*$, $y_0 \in Y$.

Furthermore if infimum in (5) is achieved by some $x_0 \in \Omega$ then

$$< g(x_0), z_0^* > + < H(x_0), y_0 >= 0 \tag{7}$$

and

$$x_0 \text{ minimizes } f(x) + < g(x), z_0^* > + < H(x), y_0 > . \tag{8}$$

Proof: See [1]

We refer to (5) as the **Primal** problem and (6) as the **Dual** problem.

Corollary 1 *Let X, Y, Z, f, H, g, Ω be as in Theorem 3. Let x_0 be the solution to the problem*

$$\begin{gathered} \textit{minimize } f(x) \\ \textit{subject to } x \in \Omega, \ H(x) = 0, \ g(x) \leq z_0 \end{gathered}$$

with (z_0^*, y_0) as the dual solution. Let x_1 be the solution to the problem

$$\begin{gathered} \textit{minimize } f(x) \\ \textit{subject to } x \in \Omega, \ H(x) = 0, \ g(x) \leq z_1 \end{gathered}$$

with (z_1^*, y_1) as the dual solution. Then,

$$< z_1 - z_0, z_1^* > \leq f(x_0) - f(x_1) \ \leq < z_1 - z_0, z_0^* > . \tag{9}$$

Proof : See [1]

4. The Mixed Problem

In this section we make the statement for the mixed problem precise. We solve the mixed problem via a related problem called the approximate problem. For both the mixed and the approximate problems the following notation is relevant: Let $N_w := \{1, \ldots, n_w\}$ and let $N_z := \{1, \ldots, n_z\}$. Let $\mathcal{S}$ be a given subset of N_z. $\mathcal{S}$ corresponds to those rows of the closed loop which have some part constrained in the one norm. We denote the cardinality of $\mathcal{S}$ by c_n. Let N_p for $p \in \mathcal{S}$ be a subset of N_w. N_p characterizes the part of the p^{th} row of the closed loop that is constrained in the one norm. γ_p in R for $p \in \mathcal{S}$ are such that $\gamma_p > \nu_{0,1}$. γ_p is the constraint level on the p^{th} row. Let $\gamma \in R^{c_n}$ contain γ_p for $p \in \mathcal{S}$ as its elements.

We define a set $\Gamma_\gamma \subset \ell_1^{n_z \times n_w}$ of feasible solutions as follows: $\Phi \in \ell_1^{n_z \times n_w}$ is in Γ_γ if and only if it satisfies the following conditions:

a) $\displaystyle\sum_{q \in N_p} \| \Phi_{pq} \|_1 \leq \gamma_p$ for all $p \in \mathcal{S}$,

b) $\Phi \in \Theta$ (i.e Φ is an achievable closed loop map).

Φ is said to be *feasible* if $\Phi \in \Gamma_\gamma$. Let $\overline{M}$ be a given subset of $N_z \times N_w$.

The problem statements for the mixed and the approximate problems are now presented.

Given a plant G the **mixed problem** *is the following optimization:*

$$\mu_\gamma := \inf_{\Phi \in \Gamma_\gamma} \{ \sum_{(p,q) \in \overline{M}} ||\Phi_{pq}||_2^2\}. \tag{10}$$

Given a plant G the **approximate problem** *of order δ is the following optimization:*

$$\mu_\gamma^\delta := \inf_{\Phi \in \Gamma_\gamma} \{ \sum_{(p,q)\in \overline{M}} ||\Phi_{pq}||_2^2 + \delta \sum_{p\in S} \sum_{q\in N_p} \| \Phi_{pq} \|_1 \}. \tag{11}$$

We will further assume that for all $(p,q) \in N_z \times N_w$ the component Φ_{pq} appears in the ℓ_1 constraint or in the objective function or in both. Note that $\overline{M}$ is the set of transfer function pairs whose two norms have to be minimized in the problem. The problem is set up so that one can include the constraint of a complete row in the closed loop map Φ or part of a row. This way we can easily incorporate constraints of the form $\| \Phi \|_1 \leq 1$ which is equivalent to each row having one norm less than 1. Also, the $\mathcal{H}_2$ norm of Φ can be included in the cost as a special case.

We also define the following sets which help in isolating various cases in the dual formulation:

$$\overline{N} := \cup_{i\in S}(i, N_i),$$

which is set of indices (i,j) such that Φ_{ij} occur in the ℓ_1 constraint,

$$MN := \overline{M} \cap \overline{N},$$

which is the set of indices (i,j) such that Φ_{ij} occurs in the ℓ_1 constraint and its two norm appears in the objective,

$$M := \overline{M} \backslash MN,$$

which is the set of indices (i,j) such that two norm of Φ_{ij} occurs in the objective but it does not appear in the ℓ_1 constraint and

$$N := \overline{N} \backslash MN,$$

which is the set of indices (i,j) such that Φ_{ij} occurs in the ℓ_1 constraint but its two norm does not appear in the objective. With this notation we have, $\overline{M} = (MN) \cup M$ and $\overline{N} = (MN) \cup N$. We assume that $MN \cup M \cup N$ equals $N_z \times N_w$. This implies that for all $(p,q) \in N_z \times N_w$ Φ_{pq} appears in the ℓ_1 constraint or in the objective function or in both.

We define $f_m : \ell_1^{n_z \times n_w} \to R$ and $f_a^\delta : \ell_1^{n_z \times n_w} \to R$ by

$$f_m(\Phi) := \sum_{(p,q)\in \overline{M}} ||\Phi_{pq}||_2^2 = \sum_{(p,q)\in (MN)\cup M} ||\Phi_{pq}||_2^2,$$

$$f_a^\delta(\Phi) := \sum_{(p,q)\in \overline{M}} ||\Phi_{pq}||_2^2 + \delta \sum_{p\in S} \sum_{q\in N_p} \| \Phi_{pq} \|_1 = \sum_{(p,q)\in MN\cup M} ||\Phi_{pq}||_2^2 + \delta \sum_{(p,q)\in MN\cup N} \| \Phi_{pq} \|_1,$$

which are the objective functions of the mixed and the approximate problems respectively. We make the following assumption.

Assumption 3 *The plant is square i.e., $n_z = n_u$ and $n_y = n_w$.*

We comment at the end of this section on the non-square case. We now solve the approximate problem and later we give the relation of the mixed problem to the approximate problem.

4.1 The Approximate Problem

In this subsection we present results on the approximate problem of order δ. The importance of this problem comes from its connection to the mixed problem. In the rest of the paper Z is given by the following where $y_{ijk\lambda_0}$ is defined by the next theorem.

$$Z_{pq}(t) := \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} F_{pq}^{ijk\lambda_0}(t).$$

Theorem 4 *There exists $\Phi^0 \in \Gamma_\gamma$ such that*

$$\mu_\gamma^\delta = \sum_{(p,q)\in MN\cup M} \| \Phi_{pq}^0 \|_2{}^2 + \sum_{(p,q)\in MN\cup N} \delta \| \Phi_{pq}^0 \|_1.$$

Therefore, the infimum in (11) is a minimum. Moreover, the following is true:

$$\mu_\gamma^\delta = \max\{ \sum_{(p,q)\in M} \sum_{t=0}^{\infty} -\Phi_{pq}(t)^2 + \sum_{(p,q)\in MN} \sum_{t=0}^{\infty} -\Phi_{pq}(t)^2 + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} b^{ijk\lambda_0} - \sum_{p\in S} \overline{y}_p \gamma_p \},$$

subject to

$\overline{y} \in R^{c_n}$, $\overline{y} \geq 0$, $y \in R^{c_z}, \Phi_{pq} \in \ell_1$ *for all* (p,q) *in*$MN \cup M$,

$$\begin{aligned}
-(\delta + \overline{y}_p) &\leq Z_{pq}(t) \leq (\delta + \overline{y}_p) && \textit{if } (p,q) \in N \\
2\Phi_{pq}(t) &= Z_{pq}(t) - (\delta + \overline{y}_p) && \textit{if } (p,q) \in MN \\
& && \textit{and } Z_{pq}(t) > (\delta + \overline{y}_p), \\
&= Z_{pq}(t) + (\delta + \overline{y}_p) && \textit{if } (p,q) \in MN \\
& && \textit{and } Z_{pq}(t) < -(\delta + \overline{y}_p), \\
&= 0 && \textit{if } (p,q) \in MN \\
& && \textit{and } |Z_{pq}(t)| \leq (\delta + \overline{y}_p), \\
&= Z_{pq}(t) && \textit{if } (p,q) \in M,
\end{aligned}$$

for all $t = 0, 1, 2, \ldots$.

In addition, the optimal Φ_{pq}^0 is unique for all $(p,q) \in (MN) \cup M$.

Proof : The proof follows using the Banach-Alaoglu Lemma on weak-star compactness and Theorem 3. ■

The following Lemma establishes an upper bound on the dual variable $\overline{y}$.

Lemma 1 *Let $\Phi^{0,1}$ denote a solution of the standard ℓ_1 problem (3). $f_a^\delta(\Phi^{0,1})$ is the objective of the approximate problem evaluated at a solution of the* *standard ℓ_1 problem. If $(\overline{y}^\gamma,\ y^\gamma)$ is the solution to the approximate problem as given in Theorem 4 then*

$$\overline{y}_p^\gamma \leq \frac{f_a^\delta(\Phi^{0,1})}{\gamma_p - \nu_{0,1}} \text{ for all } p \in \mathcal{S}.$$

Proof : Follows from Corollary 1. ■

The following lemma relates the primal solution to the dual solution.

Lemma 2 *Let Φ^0 be a solution to the primal problem (11) and let $\overline{y}^0,\ y^0, Z_{pq}^0$ be solutions to the dual. Then the following is true:*

$$\begin{aligned}
-(\delta + \overline{y}_p) &\leq Z_{pq}^0(t) \leq (\delta + \overline{y}_p) && \textit{if } (p,q) \in N, \\
\Phi_{pq}^0(t) &= 0 && \textit{if } (p,q) \in N \\
& && \textit{and } |Z_{pq}^0(t)| < (\delta + \overline{y}_p), \\
2\Phi_{pq}^0(t) &= Z_{pq}^0(t) - (\delta + \overline{y}_p) && \textit{if } (p,q) \in MN \\
& && \textit{and } Z_{pq}^0(t) > (\delta + \overline{y}_p), \\
&= Z_{pq}^0(t) + (\delta + \overline{y}_p) && \textit{if } (p,q) \in MN \\
& && \textit{and } Z_{pq}^0(t) < -(\delta + \overline{y}_p), \\
&= 0 && \textit{if } (p,q) \in MN \\
& && \textit{and } |Z_{pq}^0(t)| \leq (\delta + \overline{y}_p), \\
&= Z_{pq}^0(t) && \textit{if } (p,q) \in M.
\end{aligned}$$

Φ_{pq}^0 is unique for all $(p,q) \in (MN) \cup M$. Also, there exists an a priori bound α_a such that $||Z_{pq}^0||_\infty \leq \alpha_a$ where

$$\alpha_a := \frac{f_a^\delta(\Phi^{0,1})}{\gamma_p - \nu_{0,1}} + \delta + \frac{2}{\delta} f_a^\delta(\Phi^{0,1}) + 2\sqrt{f_a^\delta(\Phi^{0,1})}.$$

for all $(p,q) \in N_z \times N_w$. $\Phi^{0,1}$ is the solution to (3).

Proof : The first half of the above lemma follows by exploiting the structure of the dual. The upper bound on Z_{pq} can be deduced using the upper bound on $\overline{y}$. ■

By using the fact that the zero-interpolation conditions form an independent set which can be characterized by elements in ℓ_1 and using the upper bound on Z we can determine L_a^*, *a priori* such that

$$|Z_{pq}(t)| < \delta \text{ for all } t \geq L_a^*.$$

The following theorem follows easily.

Theorem 5 *The following is true:*

$$\mu_\gamma^\delta =$$

$$\max\{ \sum_{(p,q)\in M} \sum_{t=0}^{\infty} -\frac{1}{4} Z_{pq}(t)^2 + \sum_{(p,q)\in MN} \sum_{t=0}^{L_a^*} -\Phi_{pq}(t)^2 + \sum_{i,j,k,\lambda_0} y_{ijk\lambda_0} b^{ijk\lambda_0} - \sum_{p\in\mathcal{S}} \overline{y}_p \gamma_p \}$$

subject to

$\overline{y} \in R^{c_n},\ \overline{y} \geq 0,\ y \in R^{c_z}, \Phi_{pq}(t) \in R^{L_a^*}\ \forall\ (p,q) \in MN.$

$$-(\delta + \overline{y}_p) \leq Z_{pq}(t) \leq (\delta + \overline{y}_p) \quad \textit{if } (p,q) \in N$$

$$-(\delta + \overline{y}_p) \leq 2\Phi_{pq}(t) - Z_{pq}(t) \leq (\delta + \overline{y}_p) \quad \textit{if } (p,q) \in MN$$

for all $t = 0, 1, 2, \ldots, L_a^*$.

Furthermore, the optimal Φ_{pq}^0 of the primal (11) is unique for all $(p,q) \in (MN) \cup M$.

Thus, we have reduced the approximate problem to a finite dimensional quadratic optimization problem with *a priori* known dimension. Note that for the optimal solution Φ^0 the Φ_{pq}^0's with (p,q) in $MN \cup N$, are always FIR while the Φ_{pq}^0's with (p,q) in M, are IIR.

4.2 Relation between the Approximate and the Mixed Problem

In this section we show how to solve the mixed problem using the results of the approximate problem. Note that the approximate problem reduces to a finite dimensional quadratic optimization problem with *a priori* known dimension.

For the mixed problem (1-block) a similar Lagrange duality approach can be used to show that the problem can be converted to a finite dimensional convex problem with some of the optimal Φ_{pq}^0 being possibly FIR and some IIR as in the approximate problem (see Theorem 4). Nonetheless, even in the single input-single-output-case, an *a priori* bound on the dimension of the equivalent quadratic problem has proved elusive [5]. In addition, the MIMO problem is substantially more complex, for one cannot determine *a priori* which of the optimal dual variables $\overline{y}_p^0$ corresponding to the ℓ_1 constraint is active (i.e., $\overline{y}_p^0 > 0$.) Hence, the *a priori* determination of which (if any) of the otpimal Φ_{pq}^0 is FIR is not possible. This can make the solution procedure extremely complicated and virtually intractable by trying to examine all possibilities.

This difficulty can be circumvented by considering the approximate problem. The results in this section show that a suboptimal solution to the mixed problem can be obtained by solving an approximate problem. The following theorem shows that we can design a controller K for the mixed problem which achieves an objective value within any given tolerance of the optimal value by solving a corresponding approximate problem. The existence of a solution for the mixed problem and the optimal Φ_{pq}^0 being unique for $(p,q) \in (MN) \cup M$ can be proved in a similar manner as was done for the approximate problem.

Theorem 6

$$\mu_\gamma \leq \mu_\gamma^\delta \leq \mu_\gamma + \delta|\gamma|_1.$$

Proof : It is easy to show that $\mu_\gamma \leq \mu_\gamma^\delta$. Note that if $\Phi \in \Gamma_\gamma$ then

$$\sum_{q \in N_p} \| \Phi_{pq} \|_1 \leq \gamma_p \text{ for all } p \in S.$$

This implies that

$$f_a^\delta(\Phi) \leq \sum_{(p,q)\in(MN)\cup M} \| \Phi_{pq} \|_2{}^2 + \delta|\gamma|_1.$$

Taking infimum over Γ_γ on both sides in the above inequality the theorem follows. ■

The next theorem is a result on the convergence of the optimal solutions of the approximate problems to the solution of the mixed problem.

Theorem 7 *Let Φ^n be a solution of the approximate problem of order $\frac{1}{n}$. Then, there exists a subsequence $\{\Phi^{n_k}\}$ of Φ^n and $\Phi^0 \in \ell_1^{n_z \times n_w}$ such that Φ^0 is a solution of the mixed problem and*

$$\Phi^{n_k} \rightarrow \Phi^0 \textit{ in the } W((\ell_2^{n_z \times n_w})^*, \ell_2^{n_z \times n_w}) \textit{ topology.}$$

Furthermore,
$\Phi_{pq}^n \rightarrow \Phi^0$ in the $W((\ell_2)^*, \ell_2)$ topology $\forall\ (p,q) \in (MN) \cup M$.

At this point we would like to comment briefly on the nonsquare problem. Note that the result of Theorem 6 is valid for the square as well as the nonsquare case. Therefore, we would like to obtain a suboptimal solution to the nonsquare mixed problem by solving the corresponding nonsquare approximate problem (as defined in this section) which is more tractable. The delay augmentation method as described in [2] can be employed to obtain converging lower bounds. What is needed is a converging upper bound to μ_γ^δ. A simple way to do this is to solve directly the approximate problem by truncating Φ. This is what is called the finite-many-variables (FMV) approach [2]. Convergence of the optimal cost of the truncated problem to μ_γ^δ as the number of (untruncated) variables increases is easy to establish [2].

5. Conclusions

In this paper the $\mathcal{H}_2$ performance of the closed loop is minimized subject to a ℓ_1 constraint. It was shown that suboptimal solutions within any given tolerance of the optimal value can be obtained via a finite dimensional quadratic optimization problem. The dimension of the equivalent quadratic optimization problem can be determined *a priori*. Numerical implementation, and tighter bounds on the dimensions of the equivalent problems is the subject of future research.

6. References

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