# A Less Conservative Circle Criterion

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*Abstract*— **A weak form of the Circle Criterion for Lur'e systems is stated. The result allows prove global boundedness of all system solutions. Moreover such a result can be employed to enlarge the set of nonlinearities for which the standard Circle Criterion can guarantee absolute stability.**

## I. INTRODUCTION

Although linear models can be successfully employed to locally describe the behaviour of a physical system, they often fail to provide a satisfactory global characterization [1]. An important class of nonlinear models is given by the feedback interconnection of a SISO linear time-invariant system $G(s)$ and a nonlinear, possibly time-varying, static block $N$ as represented in Figure 1.
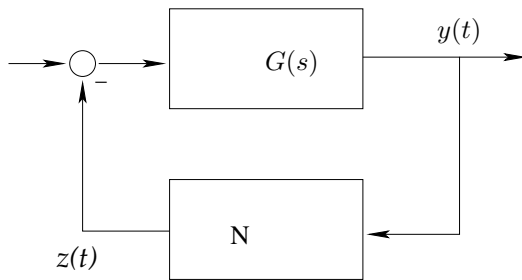


Fig. 1. Block diagram of a Lur'e system

Models of this kind are known in literature as Lur'e Systems [2] and their properties have been extensively studied in the last few decades. They reveal to be a very "expressive" class of models, since they can exhibit a wide variety of behaviours that linear systems cannot show (for example the presence of isolate equilibria, limit cycles, quasi-periodic and chaotic attractors [3] .

From this perspective (and considering also their relative simplicity), Lur'e systems can be considered a very attractive class to try to give a more accurate and complete description of a physical system. [4]

Such a consideration shows the importance of having some analytical tools allowing us to investigate qualitatively and quantitatively their properties.

One of the significant results for Lur'e systems is the Circle Criterion [2], [5] which may be employed to estabilish global asymptotical stability of the origin when the nonlinear feedback block satisfies a sector condition.

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In this paper, we first provide a relaxed form of the Circle Criterion which guarantees only a quantitative bound for all solutions after a finite time, but under conditions easier to fulfill. This result can be considered an useful analysis tool by itself. It can be employed to find quantitative bounds for any attractor in the system.

We also exploit this tool to develop a criterion to estabilish Absolute Stability for a set of nonlinearity wider than the simple sector class taken into account by the Standard Circle Criterion. The paper is organized as follows. Section II contains all the main theoretical results, Section III gives a numerical example of the developed techniques and finally in Section IV main results are briefly summarized.

## II. MAIN RESULTS

Let us consider the feedback interconnection of a SISO linear block with transfer function $G(s)$ and a nonlinear static block $N$ defined by a possibly time-varying function $n(t, \cdot)$. We assume that $n(t, \cdot)$ is regular enough to guarantee the uniqueness and the existence of the solutions. Let us call $u(t)$ and $y(t)$ respectively the input and the output signals of the linear subsystem. Since we are considering an unforced system, the relation $u(t) = -n(t, y(t))$ holds.

For sake of completeness we present some definitions.

**Definition 1 (Positive Real Rational Function).** *A proper rational function $G(s)$ is called Positive Real if it satisfies the following conditions*

- *all poles of $G(s)$ are in $\{s \mid Re[s] \leq 0\}$*
- *if $i\omega$ ($\omega \in \mathrm{R}$) is not a pole of $G(s)$ then $Re[G(i\omega)] \geq 0$*
- *if $i\omega$ ($\omega \in \mathrm{R}$) is a pole of $G(s)$ then it is a simple pole with positive residual.*

**Definition 2.** *A proper rational function $G(s)$ is called $\epsilon$-Positive Real if $\epsilon > 0$ and $G(s - \epsilon)$ is Positive Real.*

We first introduce a simple generalization of a well-known result in passivity theory.

**Lemma II.1.** *Let $S$ be a Lur'e system made of a SISO linear block with transfer function $G(s)$ and nonlinear static block $n(t, y)$. Let us assume that*

- *$G(s)$ is $\epsilon$-Positive Real*
- *$\exists \, M > 0 : \quad -M < n(t, y)y(t)$.*

*Let $(A, B, C, D)$ be any minimal realization of $G(s)$ with a state $x$. Let $(P, L, J)$ be the matrices satisfing the Kalman-Yakubovich-Popov (KYP) lemma conditions for the realization $(A, B, C, D)$.*

*Then, for all $\eta > 0$ the state $x$ of the system $S$ reaches the region $\Omega_{2M/\epsilon + \eta} := \{x \mid x^T P x \leq 2M/\epsilon + \eta\}$ in a finite time. Also this set is positively invariant.*
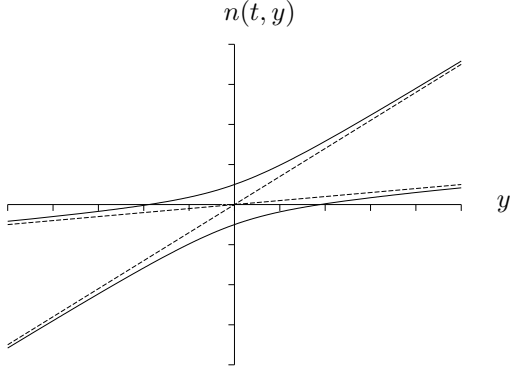
*Proof:* As $(P, L, J)$ is a solution to the KYP problem, the following the relations are satisfied

$$PA + A^T P = -L^T L - \frac{\epsilon}{2} P$$
$$PB = C^T - L^T J$$
$$J = \sqrt{2D},$$

with $P = P^T > 0$. Consider the Lyapunov function $V(x) = \frac{1}{2} x^T P x$. We find that

$$\dot{V}(x) = -\frac{\epsilon}{2} V(x) - \frac{\epsilon}{2}(Lx + Ju)^T (Lx + Ju) - n(t, y)y \leq$$
$$\leq -\frac{\epsilon}{2} V(x) - n(t, y)y < M - \frac{\epsilon}{2} V(x). \quad (1)$$

If $V(x) > \frac{2M}{\epsilon}$ then $\dot{V}$ is strictly negative. We use standard arguments to show that, if the initial condition $x(0)$ is outside $\Omega_{2M/\epsilon+\eta}$, then the state $x$ enters $\Omega_{2M/\epsilon+\eta}$ in a finite time. Suppose $x(t)$ remains outside $\Omega_{2M/\epsilon+\eta}$ for all $t$. Then, $V(x(t)) \geq \frac{2M}{\epsilon} + \eta$ for all $t$. $V(x(t))$ is monotonically decreasing and bounded from below by $\frac{2M}{\epsilon} + \eta$. Thus, $V(x(t))$ converges to $\underline{V} \geq \frac{2M}{\epsilon} + \eta > 0$. Note that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) \mathrm{d}\tau \leq$$
$$\leq V(x(0)) + Mt - \frac{\epsilon}{2} \underline{V} t \leq V(x(0)) - \frac{\epsilon \eta}{2} t \to -\infty.$$

Thus, we have a contradiction. We have concluded that for all $\eta > 0$ there exists a time $t_\eta$ such that $x(t_\eta) \in \Omega_{2M/\epsilon+\eta}$. It can be shown, exploiting the continuity of $V(x(t))$, that the solution $x(t)$ cannot leave the region $\Omega_{2M/\epsilon+\eta}$ once it has entered it. ∎

As a direct consequence of the previous lemma we have the following theorem which can be interpreted as a weak form of the Circle Criterion.

**Definition 3.** *We introduce the notation $[G(s) \circlearrowleft \alpha]$ to indicate the transfer function of the system defined by $G(s)$ with a linear static negative feedback gain $\alpha$*

$$[G(s) \circlearrowleft \alpha] := \frac{G(s)}{1 + \alpha G(s)}. \quad (2)$$

**Theorem II.2 (Weak Circle Criterion).** *Let $S$ be a Lur'e system made of a SISO linear block with transfer function $G(s)$ and nonlinear static block $n(t, y)$. Let $\alpha, \beta$ be two real scalars. Suppose*

$$[G(s) \circlearrowleft \alpha] + \frac{1}{\beta - \alpha} \text{ is } \epsilon - PositiveReal \quad (3)$$

*and there exists $M > 0$ such that*

$$-M < \frac{1}{\beta - \alpha}(n(t, y) - \alpha y)(\beta y - n(t, y)) \quad (4)$$

*Let $(A, B, C, D)$ be any minimal realization of $G(s)$ with a state $x$. Let $(P, L, J)$ be a solution of the associated KYP problem associated to the matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A - \alpha BC, B, C/(1 + \alpha D), D/(1 + \alpha D) + \frac{1}{\beta - \alpha})$.*
*Then, for all $\eta > 0$ the state $x$ of the system $S$ reaches the region $\Omega_{2M/\epsilon+\eta} := \{x \mid x^T P x < 2M/\epsilon + \eta\}$ in a finite time without leaving it anymore.*



Fig. 2. Zero and Pole Shifting

*Proof:* Let us apply sequentially to the system $S$ an $\alpha$-poleshift and a $\frac{1}{\beta - \alpha}$-zeroshift as depicted in the Figure 2. We obtain a new Lur'e system $\hat{S}$ made of a linear system $\hat{G} := [[G(s) \circlearrowleft \alpha]] + \frac{1}{\beta - \alpha}$ with a nonlinear feedback $\hat{n}(t, y) := [n(t, y) - \alpha y]$. It is straightfoward to note that $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a realization of $\hat{G}(s)$ and also that the new output $\hat{y}$ is equal to $\frac{1}{\beta - \alpha}(\beta y - n(t, y))$. Now we can apply the Lemma II.1 to conclude that there exists $t_\eta$ such that the state $x(t)$ is in the region $\Omega_{2M/\epsilon+\eta}$ for all $t > t_\eta$. ∎

The condition (3) is a Circle Criterion Condition. It has the following well-known characterization:

**Lemma II.3.** *There exists $\epsilon > 0$ such that $[G(s) \circlearrowleft \alpha] + \frac{1}{\beta - \alpha}$ is $\epsilon$-Positive Real if and only if, given on the complex plane the circle $\Gamma$ whose diameter is the segment $\left[\min\left\{-\frac{1}{\alpha}, -\frac{1}{\beta}\right\}, \max\left\{-\frac{1}{\alpha}, -\frac{1}{\beta}\right\}\right]$, we have that*
- *for $\alpha < \beta$ the inner part of $\Gamma$ is contained in the stability region of the Nyquist plot of $G(i\omega)$.*
- *for $\beta < \alpha$ the outer part of $\Gamma$ is contained in the stability region of the Nyquist plot of $G(i\omega)$.*

The inequality (4) is a condition weaker than the usual sector condition. However, it is important to point out that a graphical interpretation is still possible. For a fixed $M$, condition (4) is satisfied by all functions lying in a "hyperbolic sector" that contains the linear sector $[\alpha, \beta]$ (see Figure 3). It is straightforward to note that in (4) if $M = 0$ then the condition reduces to the usual sector condition and the theorem becomes equivalent to the Circle Criterion.

It is of interest to find a bound for the signal $|y(t)|$ when $x(t)$ is assured to remain in the region $\Omega_{2M/\epsilon+\eta}$. This can be achieved by the following result.

$$n(t, y)$$

Fig. 3. A hyperbolic sector

**Lemma II.4.** *Let us consider a positive matrix $P$ that, without loss of generality, can be supposed symmetrical. The following optimization problem*

$$\begin{aligned} \max \quad & |y| \\ \text{subject to} \quad & \begin{cases} y = Cx \\ \frac{1}{2}x^T P x \le \gamma \end{cases} \end{aligned}$$

*has a solution*

$$Y := \sqrt{2\gamma C P^{-1} C^T} \qquad (5)$$

We have the following corollary

**Corollary II.5.** *Under the hypotheses of Theorem II.2 and if $G(s)$ is strictly proper we have that the system output is bounded by $Y := 2\sqrt{\frac{M}{\epsilon}C P^{-1} C^T}$. Thus is*

$$\forall \, \eta > 0 \; \exists \, t_\eta : t > t_\eta \; \Rightarrow \; |y(t)| < Y + \eta. \qquad (6)$$

Summarizing, the Theorem II.2 gives an analytical sufficient condition to prove the global boundedness of all orbits for a Lur'e system after a sufficiently long time. A bounding region can be easily evaluated solving a KYP problem and leads to a bound on the output if the function $G(s)$ is strictly proper. In addition, the well known Circle Criterion is a special case of the derived result.

For a given fixed nonlinearity, the obtained bound $Y$ on $|y(t)|$ depends on the choice of the parameters $\alpha, \beta$ and $M$. Since such a choice is not unique, in order to obtain the best bound offered by the Theorem II.2, we can consider the following optimization problem

$$\begin{aligned} \min \quad & Y = \sqrt{\frac{2M}{\epsilon}C P^{-1} C^T} \\ \text{subject to} \quad & \begin{cases} \frac{1}{\beta-\alpha}(n(t,y)-\alpha y)(\beta y - n(t,y)) > -M \\ P\hat{A}(\alpha,\beta) + \hat{A}(\alpha,\beta)^T P = -L^T L - \frac{\epsilon}{2}P \\ P\hat{B}(\alpha,\beta) = \hat{C}(\alpha,\beta)^T - L^T J \\ J = \sqrt{2D}. \end{cases} \end{aligned} \qquad (7)$$

Unfortunately, the problem (7) is difficult to tackle because it involves the nonlinear function $n(t,\cdot)$. A plausible approach is to find an upper bound to the above problem that would be pursued in future works. The knowledge of a bound on

$y$ can be used to reduce the conservativeness of the Circle Criterion. In every case, the possibility of obtaining a bound for $\max |y(t)|$ (even if suboptimal or non-optimal) suggests an idea to enlarge the set of nonlinearities providing absolute stability which can be normally found applying the canonical Circle Criterion.

**Theorem II.6.** *Let $S$ be a Lur'e system made of a SISO linear block with a strictly proper transfer function $G(s)$ and nonlinear static block $n(t,y)$. Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be four scalars Assume that*

(a) $[G(s) \circlearrowleft \alpha_i] + \frac{1}{\beta_i - \alpha_i}$ *is $\epsilon_i$-Positive Real for i=1,2*
(b) *there exists $M_1 > 0$ such that*

$$M_1 < \frac{1}{\beta_1 - \alpha_1}(n(t,y) - \alpha_1 y)(\beta_1 y - n(t,y))$$

(c) *there exists $\eta_1 > 0$ such that*

$$|y| < 2\sqrt{\frac{M_1}{\epsilon_1}C P_1^{-1} C^T} + \eta_1 \Rightarrow \qquad (8)$$
$$0 \le \frac{1}{\beta_2 - \alpha_2}(n(t,y) - \alpha_2 y)(\beta_2 y - n(t,y))$$

*Then we can conclude Global Asymptotical Stability for the system.*

*Proof:* Using (a) and (b) from the Corollary II.2 follows that there exists a time $t_1$ after which the system output $y$ is such that $|y|$ will be bounded by $Y_1 + \eta_1$ with $Y_1 := 2\sqrt{\frac{M_1}{\epsilon_1}C P_1^{-1} C^T}$. For all $t > t_1$ we will have that the system "explores" the nonlinearity $n(t, \cdot)$ only in the linear sector $[\alpha_2, \beta_2]$. Using (c) it follows from the standard Circle Criterion that the system is absolutely stable. ∎

The previous theorem can be immediately extended to cover a more general case.

**Theorem II.7.** *Let $S$ be a Lur'e system made of a SISO linear block with a strictly proper transfer function $G(s)$ and nonlinear static block $n(t,y)$. Let $\alpha_1...\alpha_K, \beta_1, ...\beta_K$ be $2K$ scalars. Let us assume that*

- $[G(s) \circlearrowleft \alpha_i] + \frac{1}{\beta_i - \alpha_i}$ *is $\epsilon_i$-Positive Real for $i = 1, K$*
- *there exists $M_1 > 0 :$ $-M_1 < \frac{1}{\beta_1 - \alpha_1}(n(t,y) - \alpha_1 y)(\beta_1 y - n(t,y))$*
- *there exists $\eta_i > 0 :$ $|y| < 2\sqrt{\frac{M_i}{\epsilon_i}C P_i^{-1} C^T} + \eta_i$ $\Rightarrow -M_{i+1} \le \frac{1}{\beta_{i+1} - \alpha_{i+1}}(n(t,y) - \alpha_{i+1} y)(\beta_{i+1} y - n(t,y))$ for all $i = 1...K-2$*
- *there exists $\eta_{K-1} > 0$ such that*

$$|y| < 2\sqrt{\frac{M_{K-1}}{\epsilon_{K-1}}C P_{K-1}^{-1} C^T} + \eta_{K-1}$$

*implies*

$$0 \le \frac{1}{\beta_K - \alpha_K}(n(t,y) - \alpha_K y)(\beta_K y - n(t,y))$$

*Then we can conclude Global Asymptotical Stability for the system.*

*Proof:* It is a trivial extension of the previous theorem. ∎

## III. A NUMERICAL EXAMPLE

Let us consider the stable and minimum phase transfer function

$$G(s) := 0.25 \frac{(\frac{s}{5}+1)(\frac{s}{5}+1)(\frac{s}{5}+1)}{(\frac{s}{20}+1)(\frac{s}{21}+1)(\frac{s}{22}+1)(\frac{s}{23}+1)} \quad (9)$$

modeling a physical system of interest. By the Nyquist Criterion, we obtain that $G(s) \circlearrowleft K$ is stable for every $K > 0$. Suppose the task is to to design a simple proportional gain to improve the system performances. However, the actuator realizing the control law is subject to saturation. A noise $\xi(t)$ acting on the input of the plant is also present. This scenario is depicted in Figure 4. We define the saturation
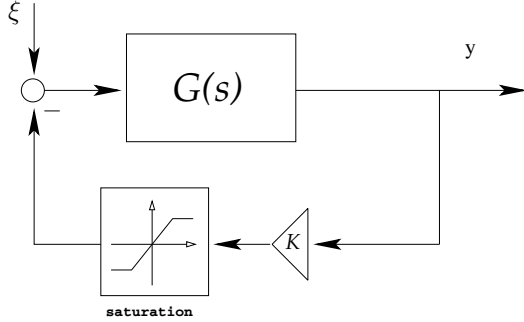
Fig. 4. Nyquist Plot of $G(s)$

function in the following way

$$\text{sat}(y) = \begin{cases} -1 & \text{if } y \le -1 \\ y & \text{if } |y| < 1 \\ 1 & \text{if } y \ge 1. \end{cases} \quad (10)$$

A more sophisticated model for the controlled system is a Lur'e model $G(s)$ and a static nonlinear feedback

$$n_K(t, y) := S \, \text{sat}\left(\frac{Ky}{S}\right) \quad (11)$$

where $S$ is the actuator saturation level. In this situation, the Nyquist Criterion can be applied only to estabilish local stability of the origin for any $K > 0$. In order to obtain global results, it is possible to use the Circle Criterion. Since the nonlinearity $n_K(t, \cdot) \in \text{sect}[0, K]$ and $G(s)$ is a stable transfer function, the application of the Circle Criterion to our problem is immediate. It can assure global asymptotical stability for all nonlinearities in $\text{sect}[0, K_{cc}]$ where $K_{cc} := -1/\min\{\text{Re}G(i\omega)\}$. In our particular case we find that $K_{cc} \simeq 18.75$. However, for $K > K_{cc}$ the Circle Criterion does not provide any conclusion.
Conversely, consider $n_K(t, y) = S\text{sat}(Ky/S)$ for $K > K_{cc}$, choose $\alpha = 0$ and $\beta = 3.2$. The transfer function $G(s) + 1/\beta$ is $\epsilon$-Positive Real with $\epsilon > 0.8$. Also note that

$$-M < \frac{1}{\beta} n_K(\beta y - n_K) < \frac{S^2}{\beta} \Rightarrow M > \frac{S^2}{\beta}. \quad (12)$$

Thus, we can apply Theorem II.2. Solving the related $KYP$-problem, we find an upper bound for $y(t)$ given by

$$y(t) \le \ Y := 2S\sqrt{\frac{CP^{-1}C^T}{\beta\epsilon}} \simeq 32.5 \ S \quad (13)$$

which depends on the value of the parameter $S$. From (13) we conclude that the system eventually enters a bounded region. The nonlinearity "explored" in this region lies in the sector $[\frac{S}{Y}, +\infty] \simeq [0.03, +\infty]$. Since this sector satisfies the Circle Condition we can conclude global asymptotical stability of the system for every value $K > 0$. Thus we can assert that a simple relay can be used as a global controller for the system. The developed results also allow us to take into account bounded noise effects. Suppose the disturbance enters the interconnection in the following manner

$$n_{+\infty}(t, y) := \text{sgn}(y) + \xi(t). \quad (14)$$

Let us assume that $|\xi(t)| < aS$. In this case (4) is equivalent to

$$-M < \frac{1}{\beta} n_{+\infty}(\beta y - n_{+\infty}) < \frac{S^2(1+a)^2}{\beta} \quad (15)$$

$$\text{thus } M > \frac{S^2(1+a)^2}{\beta}. \quad (16)$$

Thus, we obtain

$$y(t) \le \ Y := S\sqrt{\frac{CP^{-1}C^T}{\beta\epsilon}} \simeq 32.5(1+a)S. \quad (17)$$

The nonlinearity, in the noisy case, gets explored only in the sector $[\frac{S(1-a)}{Y}, +\infty] \simeq [0.03\frac{1-a}{1+a}, +\infty]$. We numerically find that for $a < 0.2$ the circle condition is satisfied and the system is globally asymptotically stable. Summarizing, we have proved that the system can be successfully controlled by any linear static controller even in presence of saturation on the actuators. Note that such an assert cannot be made using the Circle Criterion. In addition, it was shown that in presence of a low bounded noise the relay controller performs a complete rejection of the disturbance.

## IV. CONCLUSIONS

We have presented a relaxed form of the Circle Criterion in order to prove a global boundedness of the trajectories of a Lur'e System. The result generalizes the classical Circle Criterion considering hyperbolic sectors instead of linear ones. This tool can be employed to extend the class of nonlinearities which the Circle Criterion can conclude Absolute Stability.

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