Architectures for Distributed Controller with Sub-controller Communication Uncertainty

Vikas Yadav, Murti V. Salapaka, and Petros G. Voulgaris

Abstract— **Recent technology has led to systems with architectures that distribute the controller effort over sub-controllers to respect information flow and/or resource constraints. The associated communication uncertainty between sub-controllers partly governs the performance of the controller. The related controller synthesis methodology has to address internal stability concerns and has to incorporate the effect of communication uncertainty into the performance metric. In this article, it is demonstrated that different canonical distributed architectures derived from a centralized design can result in appreciably different performance. Architectures of information flow that result in convex optimization problems are developed and synthesis methods to incorporate robustness with respect to model uncertainty of communication channels are obtained. The efficacy of the approach is demonstrated on application problems.**

Keywords— **Structured system, distributed system, nested system, banded structure, coprime factorization, observer based controller, Youla parameter, communication noise, uncertainty.**

I. Introduction

RECENT technological demands of high performance in the presence of information flow constraints and large computational loads have posed new challenges for controller synthesis. An example where a large computational load is apparent is the recently proposed, massively parallel probe based data storage device, where thousands of cantilevers operating in parallel have to be controlled [1]. Large computational load may also result as a consequence of stringent performance specifications on the system that leads to a controller that is complex. In these situations, the controller cannot be realized at one location and the large load has to be shared by multiple stations. Distributed controller design is also motivated by recent advances in computational hardware for real-time applications which are being realized with a mix and match of various components like FPGA's and DSP's. In such cases it is often possible to abstract the hardware into distinct regions with communication between the regions that is possibly uncertain. The need to address sub-controller communication uncertainty arises naturally in distributed systems, like sensor networks and the power grid, where a sub-controller associated with a sub-system needs to be locally realized at a station with structured information exchange with other substations.

In all examples outlined above, the task of determining how the controller task can be divided into various sub-controllers such that plant-controller interconnection is stable and the effect of sub-controller communication uncertainty on the performance is minimized is of interest. Furthermore, information flow constraints of distributed systems lead to structural constraints on controllers. The design of controllers that achieve optimal performance when structural constraints are present, even without considering uncertain communication, is a difficult problem. Recently, identification of specific classes of problems where structural constraints can be addressed via convex optimization methods was reported in [2], [3], [4]. In [3] a necessary and sufficient condition on the controller structure was derived that ensures that the optimization problem remains convex in the Youla parameter Q [5]. In these works, even though controller transfer matrices satisfying structural constraints are obtained, the realization that implements the transfer matrix into various sub-controllers remains unaddressed. As is shown in this article, seemingly natural ways of distributing controller effort may even destabilize the interconnection. In [6], [7] distributed spatially invariant systems are studied using a state space approach and a convex method based on solving constraints in the form of LMIs is presented to obtain structured controllers. In [8], a heuristic method for structured H_2 controllers is presented based on low dimensional LMIs. These efforts do not consider the effect of uncertainty affecting the communication between different sub-controllers. Other related work can be found in [9], [10].

Motivated by the concerns outlined above, in this article, a framework for designing architectures for distributed implementation of controllers, that incorporates sub-controller communication uncertainty is developed. A central issue studied in this article is the identification of signals that need to be transmitted between sub-controllers. Earlier work by the authors can be found in [11], [12], [13], [14].

This paper is organized as follows: in Section II the framework is provided . In Section III systems without any structure are considered. These results are further specialized for banded structures and nested structures in Section IV. In Section V analysis and synthesis of distributed robust controllers in presence of uncertainty in the channel are presented. This is followed in Section VI by two examples illustrating the results. Section VII is the conclusion which also discusses future directions.

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II. Framework

In this section the framework, preliminary results and the notation employed are presented. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ represents a state-space realization of a finite-dimensional discrete time, linear time-invariant operator with a transfer matrix P if $P = C(zI - A)^{-1}B + D = \lambda C(I - \lambda A)^{-1}B + D$ where z and λ represent the forward shift and the unit delay operators. The standard interconnection of the generalized plant G with the controller K and the interconnection of the plant G_{22} and the controller K are shown in Fig. 1 and Fig. 2 respectively. The transpose of a vector x is denoted by x'. When needed a convenient notation to represent matrices is used where, for example, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is denoted by $[a, b; c, d]$. Φ_{oi} denotes the closed-loop transfer matrix with input i and output o.

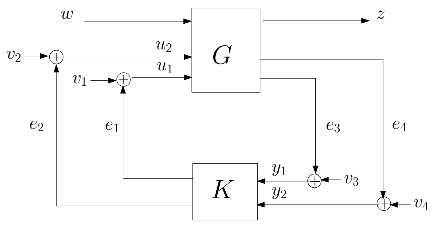

Fig. 1. The (G, K) interconnection

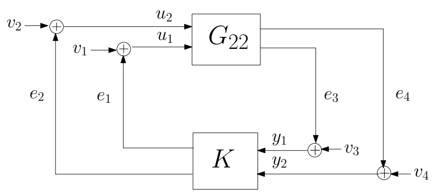

Fig. 2. The (G_{22}, K) interconnection

Definition II.1: Consider transfer function matrices G and K in an interconnection shown in Fig. 1 referred to as the (G, K) interconnection. The (G, K) interconnection is internally stable if any stabilizable and detectable realizations of G and K with states x_G and x_K, respectively, are such that $(x'_G\ x'_K)'(t) \to 0$ as $t \to \infty$ for any initial condition $(x'_G\ x'_K)'(0)$ with external signals w, $v = (v'_1\ v'_2 v'_3\ v'_4)'$ set to zero. We will say that K internally stabilizes the (G, K) interconnection if (G, K) is internally stable. The internal stability of (G_{22}, K) interconnection shown in Fig. 2 is defined similarly.

The definition of internal stability above is provided in terms of state space realizations. However, it can be shown (see [15]) that the internal stability of (G, K) in Fig. 1, is equivalent to the stability of the closed-loop map from the input $(w'\ v')'$ to the output $(z'\ u'\ y')'$ with $u = (u'_1\ u'_2)'$ and $y = (y'_1\ y'_2)'$. Similarly the (G_{22}, K) pair is internally stable if the closed-loop map from v to $(u'\ y')'$ shown in Fig. 2 is stable.

A state space realization $\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$ of the generalized plant transfer matrix $G : (w'\ u')' \to (z'\ y')'$, is assumed throughout the article. Note that from the above realization, an induced realization of the transfer matrix G_{22} is given by $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$. The following theorem holds.

Theorem II.1: [15] Assume that the induced realization $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ of G_{22} is stabilizable and detectable. Then the (G, K) interconnection is internally stable if and only if (G_{22}, K) interconnection is internally stable. If $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ is not stabilizable and/or detectable then (G, K) interconnection is not internally stable for any K.

Motivated by Theorem II.1 the following is assumed throughout the article

Assumption II.1: All interconnections are well-posed and the inherited realization of G_{22}, $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ is stabilizable and detectable.

The general framework for a distributed interconnection, considered in this article, is illustrated in Fig. 3 where G represents the generalized plant, sub-controllers K_1 and K_2 represent distributed controller implementation where $n := (n'_1\ n'_2)'$ represents the sub-controller communication noise, with w, z, $(e'_1\ e'_2)'$ and y representing, exogenous input, regulated output, control effort and the measured output respectively. G, K_1 and K_2 are assumed to be finite, discrete time, linear and time-invariant systems and are assumed to represent the transfer matrices in the z or the λ domain. The plant and controller interconnection with distributed implementation using sub-controllers K_1 and K_2 is shown in Fig. 4 where G_{22} is part of G that maps u to $(e'_3\ e'_4)'$. Even though this article presents the controller broken only into two sub-controllers, results can be generalized to more than two sub-controllers.

The internal stability with the distributed implementation of the controller is defined as follows.

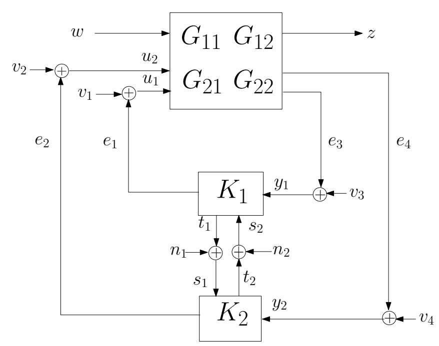

Fig. 3. The (G, K_1, K_2) interconnection

Definition II.2: Consider transfer function matrices G and K_1 and K_2 in an interconnection shown in Fig. 3 referred to as the (G, K_1, K_2) interconnection. The

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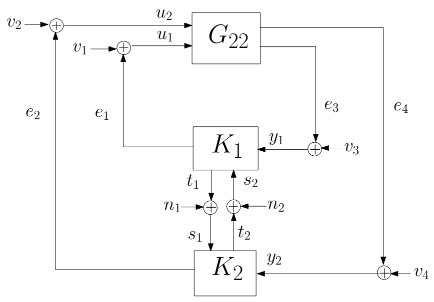

Fig. 4. (G_{22}, K_1, K_2) interconnection

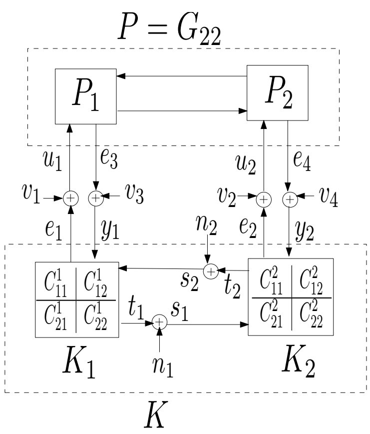

Fig. 5. The controller with communication noise n_1 and n_2

(G, K_1, K_2) interconnection is internally stable if any stabilizable and detectable realizations of G, K_1 and K_2 with states x_G, x_{K_1} and x_{K_2}, respectively, are such that $(x'_G\ x'_{K_1}\ x'_{K_2})'(t) \to 0$ with $t \to \infty$ for any initial condition $(x'_G\ x'_{K_1}\ x'_{K_2})'(0)$ and with inputs w, v and n set to zero. We will say that (K_1, K_2) internally stabilizes the (G, K_1, K_2) interconnection if (G, K_1, K_2) is internally stable. The internal stability of (G_{22}, K_1, K_2) interconnection shown in Fig. 4 is defined similarly.

For an input-output characterization of internal stability, it can be shown that (G, K_1, K_2) is internally stable if and only if the closed-loop map in Fig. 3 from $(w'\ v'\ n')'$ to $(z'\ u'\ y'\ t')'$ is internally stable. Similarly, (G_{22}, K_1, K_2) is internally stable if and only if the closed-loop map in Fig. 4 from $(v'\ n')'$ to $(u'\ y'\ t')'$ is internally stable.

Note that with the signal $n = (n'_1\ n'_2)'$ in Fig. 3 and Fig. 4 set to zero, the sub-controllers K_1 and K_2 provide a map K from $(y'_1\ y'_2)'$ to $(e'_1\ e'_2)'$, and thus reduce to the interconnection shown in Fig. 1 and Fig. 2 respectively. K is termed the induced realization from K_1 and K_2 if it is the map from $(y'_1\ y'_2)'$ to $(e'_1\ e'_2)'$ obtained in Fig. 3 or Fig. 4 by setting $n = 0$ and by eliminating the sub-controller communication signal t.

Definition II.3: If the (G, K_1, K_2) interconnection shown in Fig. 3 is such that the map from $(y'_1\ y'_2)'$ to $(e'_1\ e'_2)'$ is equal to K, then (G, K_1, K_2) is called a distributed implementation of the (G, K) interconnection as shown in Fig.1. Similarly, the (G_{22}, K_1, K_2) interconnection shown in Fig. 4 is defined as a distributed implementation of the (G_{22}, K) interconnection as shown in Fig. 2.

The main issue addressed by this article is the effect of communication uncertainty between sub-controllers K_1 and K_2 and means of incorporating these effects into controller synthesis. The sub-controllers might have to satisfy information flow constraints. Two specific structures considered in this article are the nested structure that is characterized by a block triangular structure where K_1 does not receive any information from K_2 and banded structure characterized by delays between sub-systems. Nested and banded structure appear in many applications (see [4]). Fig. 6(a) shows a nested structure controller where t is the signal transmitted from K_1 to K_2. The resulting controller K that maps $y = (y_1\ y_2)'$ to $(u_1\ u_2)'$ has lower triangular structure such that:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = K \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Another structure considered in this paper is the banded structure which is characterized by one step delay in interactions between nearest subsystems as depicted in Fig. 6(b). Controllers with this information constraint are called banded controllers. The resulting transfer matrix K that maps $(y'_1\ \dots, y'_n)'$ to $(u'_1 \dots u'_n)'$ will have the banded structure as given below, where λ denotes delay:

$$\begin{bmatrix} K_{11}(\lambda) & \lambda K_{12}(\lambda) & \cdots & \lambda^{n-1}K_{1n}(\lambda) \\ \lambda K_{21}(\lambda) & K_{22}(\lambda) & \cdots & \lambda^{n-2}K_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{n-1}K_{n1}(\lambda) & \lambda^{n-2}K_{n2}(\lambda) & \cdots & K_{nn}(\lambda) \end{bmatrix}. \quad (1)$$

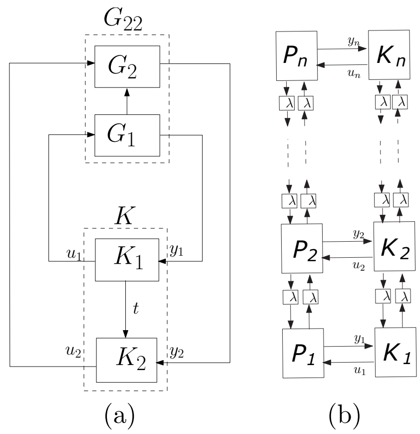

Fig. 6. (a) 2-nest system with signal t from inner nest to outer nest (b) System with banded structure

III. Architectures for distributed controllers leading to convex performance problems.

A. Internal stability of distributed architectures

In the recent past, results were reported where the the optimal controller K that internally stabilizes the (G, K) interconnection and optimizes some metric on the closed-loop map from $w \to z$ is sought with the additional constraint that the transfer matrix K has to satisfy structure. For example, for a nested controller $K : (y'_1\ y'_2)' \to$

$(e_1'\ e_2')'$ the structure is imposed on K by having $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ where $K_{12} = 0$.

However, the structural constraints on the controller K usually arise because of information flow constraints, that are not entirely met or specified by the lower triangular structure of K. For example the lower triangular structure of K might be required because of a hierarchical structure where the controller K has to be realized by different sub-controllers K_1 and K_2 where K_1 has only y_1 available whereas K_2 has y_2 and an additional input from K_1 available. Given a K, there is a non-unique way of determining K_1 and K_2 as this depends on what signals are being communicated between K_1 and K_2. For example, one choice is, $K_1 = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix}$ and $K_2 = \begin{bmatrix} I & K_{22} \end{bmatrix}$ while a second choice is $K_1 = \begin{bmatrix} K_{11} \\ I \end{bmatrix}$ and $K_2 = \begin{bmatrix} K_{21} & K_{22} \end{bmatrix}$. Both these choices realize, in the absence of sub-controller noise, $e_1 = K_{11}y_1$ and $e_2 = K_{21}y_1 + K_{22}y_2$. However a vital difference is that in the first implementation $K_{21}y_1$ is transmitted by the subcontroller K_1 to K_2 whereas in the second implementation y_1 is transmitted by K_1 as an input to K_2.

Existing work in the input-output setting provide means of obtaining the optimal K (with structural constraints, see [3] and [4]). However, they do not address, for example, the task of determining whether to implement the first or the second choice of K_1 and K_2 in the case delineated above. Indeed, it can be shown that a K that internally stabilizes the (G, K) interconnection (see Fig. 1) when implemented distributively in a (G, K_1, K_2) (see Fig. 3) interconnection can be unstable with K_1 and K_2 chosen according to the second choice (see the Appendix, section VIII-A for a complete description). This aspect is remarkable because it implies that certain ways of splitting the transfer matrix K into sub-controller transfer matrices K_1 and K_2 should be prohibited.

The above discussion motivates the study of conditions under which a internally stable (G, K) interconnection can be realized using a (G, K_1, K_2) that is internally stable. Let the sub-controllers $K_1 : (y_1'\ s_2')' \rightarrow (e_1'\ t_1')'$ and $K_2 : (y_2'\ s_1')' \rightarrow (e_2'\ t_2')'$ admit stabilizable and detectable realizations given by

$$\left[\begin{array}{c|c} A_{C_1} & B_{C_1} \\ \hline C_{C_1} & D_{C_1} \end{array}\right] = \left[\begin{array}{c|cc} A_{C_1} & B_{C_1,1} & B_{C_1,2} \\ \hline C_{C_1,1} & D_{C_1,11} & D_{C_1,12} \\ C_{C_1,2} & D_{C_1,21} & D_{C_1,22} \end{array}\right]$$

and

$$\left[\begin{array}{c|c} A_{C_2} & B_{C_2} \\ \hline C_{C_2} & D_{C_2} \end{array}\right] = \left[\begin{array}{c|cc} A_{C_2} & B_{C_2,1} & B_{C_2,2} \\ \hline C_{C_2,1} & D_{C_2,11} & D_{C_2,12} \\ C_{C_2,2} & D_{C_2,21} & D_{C_2,22} \end{array}\right],$$

respectively.

By eliminating the sub-controller to sub-controller communication signals an induced realization $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ of $K : (y_1'\ y_2')' \rightarrow (e_1'\ e_2')'$ from K_1 and K_2 can be obtained with A_K given by
$[A_{C_1} + B_{C_1,2}(\Delta_1)^{-1}D_{C_2,22}C_{C_1,2},\ B_{C_1,2}(\Delta_1)^{-1}C_{C_2,2};$
$B_{C_2,2}(\Delta_2)^{-1}C_{C_1,2},\ A_{C_2} + B_{C_2,2}(\Delta_2)^{-1}D_{C_1,22}C_{C_2,2}]$,
B_K given by
$[B_{C_1,1} + B_{C_1,2}(\Delta_1)^{-1}D_{C_2,22}D_{C_1,21},\ B_{C_1,2}(\Delta_1)^{-1}D_{C_2,21};$
$B_{C_2,2}(\Delta_2)^{-1}D_{C_1,21},\ B_{C_2,1} + B_{C_2,2}(\Delta_2)^{-1}D_{C_1,22}D_{C_2,21}]$,
C_K given by
$[C_{C_1,1} + D_{C_1,12}(\Delta_1)^{-1}D_{C_2,22}C_{C_1,2},\ D_{C_1,12}(\Delta_1)^{-1}C_{C_2,2};$
$D_{C_2,12}(\Delta_2)^{-1}C_{C_1,2},\ C_{C_2,1} + D_{C_2,12}(\Delta_2)^{-1}D_{C_1,22}C_{C_2,2}]$
and D_K given by
$[D_{C_1,11}+D_{C_1,12}(\Delta_1)^{-1}D_{C_2,22}D_{C_1,21},\ D_{C_1,12}(\Delta_1)^{-1}D_{C_2,21};$
$D_{C_2,12}(\Delta_2)^{-1}D_{C_1,21},\ D_{C_2,11}+D_{C_2,12}(\Delta_2)^{-1}D_{C_1,22}D_{C_2,21}]$.
where $\Delta_1 = I - D_{C_2,22}D_{C_1,22}$ and $\Delta_2 = I - D_{C_1,22}D_{C_2,22}$. Based on the expressions above the following theorem follows.

Theorem III.1: The (G, K_1, K_2) interconnection shown in Fig. 3, where K_1 and K_2 admit stabilizable and detectable realizations $\left[\begin{array}{c|c} A_{C_1} & B_{C_1} \\ \hline C_{C_1} & D_{C_1} \end{array}\right]$ and $\left[\begin{array}{c|c} A_{C_2} & B_{C_2} \\ \hline C_{C_2} & D_{C_2} \end{array}\right]$ respectively is internally stable if and only if the interconnection (G_{22}, K) shown in Fig. 2 is stable and the induced realizations $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ and $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ are stabilizable and detectable. Also, if $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ and $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ are stabilizable and detectable then (G, K_1, K_2) is internally stable if and only if (G_{22}, K_1, K_2) is internally stable.

Proof: It will be shown that if the interconnection (G_{22}, K) shown in Fig. 2 is stable and the induced realizations $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ and $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ are stabilizable and detectable then the (G_{22}, K_1, K_2) will be internally stable. The rest of the proof is left to the reader.

Consider the (G_{22}, K) interconnection of Fig. 2. It can be shown that with stabilizable and detectable realizations of G_{22} and K given by $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ and $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ respectively, a stabilizable and detectable realization of the (G_{22}, K) closed-loop map with input v and output $(u'\ y')'$ is given by $\left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$ where $A_{cl} = \begin{pmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{pmatrix}$ and thus the (G_{22}, K) interconnection in Fig. 2 is stable if and only if A_{cl} is a stable matrix. From the hypothesis (G_{22}, K) interconnection is stable and therefore A_{cl} is stable. The (G_{22}, K_1, K_2) interconnection in Fig. 4 is internally stable if and only if the (G_{22}, K_1, K_2) closed-loop map from $(v'\ n')'$ to $(u'\ y'\ s')'$ with $s = (s_1'\ s_2')'$ is stable. It can be shown that (G_{22}, K_1, K_2) closed-loop map admits a realization $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ where $\bar{A} = A_{cl}$. As A_{cl} is stable it follows that (G_{22}, K_1, K_2) closed-loop map is stable and therefore the (G_{22}, K_1, K_2) interconnection is stable. ■

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Remark: It can be shown that it is possible to have an induced realization $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ of K from minimal realizations of K_1 and K_2 that is not stabilizable and that in such a case, (G_{22}, K_1, K_2) is unstable even though (G_{22}, K) is stable; see the Appendix for an example.

The additional maps in the (G, K_1, K_2) interconnection for internal stability apart from the maps Φ_{uv} and Φ_{yv} of the (G, K) interconnection, that have to be stable are the closed-loop maps Φ_{tv}, Φ_{un}, Φ_{yn} and Φ_{tn} in Fig. 3. These maps are now derived in terms of the sub-controller maps.

Fig. 4 is redrawn as Fig. 5 with sub-controller K_1 described by :

$$\begin{pmatrix} e_1 \\ t_1 \end{pmatrix} = \begin{bmatrix} C^1_{11} & C^1_{12} \\ C^1_{21} & C^1_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ s_2 \end{pmatrix} \tag{2}$$

and K_2 described by

$$\begin{pmatrix} e_2 \\ t_2 \end{pmatrix} = \begin{bmatrix} C^2_{11} & C^2_{12} \\ C^2_{21} & C^2_{22} \end{bmatrix} \begin{pmatrix} y_2 \\ s_1 \end{pmatrix}, \tag{3}$$

where s_1 and s_2 are received signals at K_1 and K_2, t_1 and t_2 are transmitted signals which get corrupted by additive communication noise n_1 and n_2, respectively i.e. $s_1 = t_1 + n_1$ and $s_2 = t_2 + n_2$. The closed-loop maps Φ_{tv}, Φ_{un}, Φ_{yn} and Φ_{tn} in Fig. 3 are given by

$$\Phi_{un} = (I - KG_{22})^{-1}K_n, \tag{4}$$
$$\Phi_{yn} = G_{22}(I - KG_{22})^{-1}K_n, \tag{5}$$
$$\Phi_{tn} = K_t G_{22}(I - KG_{22})^{-1}K_n + K_{tn}, \tag{6}$$

and

$$\Phi_{tv} \;=\; K_t \left[\; (I - G_{22}K)^{-1}G_{22} \quad (I - G_{22}K)^{-1} \;\right] \tag{7}$$

where K_n is a map from n to $(e'_1, e'_2)'$, K_t is a map from y to t and K_{tn} is a map from n to t. K_n, K_t and K_{tn} are given by:

$$\begin{bmatrix} C^1_{12}(1 - C^2_{22}C^1_{22})^{-1}C^2_{22} & C^1_{12}(1 - C^2_{22}C^1_{22})^{-1} \\ C^2_{12}(1 - C^1_{22}C^2_{22})^{-1} & C^2_{12}(1 - C^1_{22}C^2_{22})^{-1}C^1_{22} \end{bmatrix}, \tag{8}$$

$$\begin{bmatrix} (1 - C^1_{22}C^2_{22})^{-1}C^1_{21} & (1 - C^1_{22}C^2_{22})^{-1}C^1_{22}C^2_{21} \\ (1 - C^2_{22}C^1_{22})^{-1}C^2_{22}C^1_{21} & (1 - C^2_{22}C^1_{22})^{-1}C^2_{21} \end{bmatrix}, \tag{9}$$

$$\begin{bmatrix} (1 - C^1_{22}C^2_{22})^{-1}C^1_{22}C^2_{22} & (1 - C^1_{22}C^2_{22})^{-1}C^1_{22} \\ (1 - C^2_{22}C^1_{22})^{-1}C^2_{22} & (1 - C^2_{22}C^1_{22})^{-1}C^2_{22}C^1_{22} \end{bmatrix}. \tag{10}$$

Thus, for internal stability of the (G_{22}, K_1, K_2), (G_{22}, K) has to be internally stable and the maps ϕ_{un}, ϕ_{yn}, ϕ_{tn}, ϕ_{tv} have to be stable.

B. Convexity of relevant closed-loop maps

In order to account for the communication uncertainty in the (G, K_1, K_2) interconnection, apart from the standard regulated variable z, the strength of the transmitted signal t and the effect of the sub-controller to sub-controller communication noise n on the regulated variable z are of interest. Thus, the closed-loop map that encapsulates the performance concerns of the distributed implementation shown in Fig. 3 is given by the closed-loop map

$$T(K_1, K_2) : \begin{pmatrix} w \\ n \end{pmatrix} \mapsto \begin{pmatrix} z \\ t \end{pmatrix} = \begin{bmatrix} \Phi_{zw} & \Phi_{zn} \\ \Phi_{tw} & \Phi_{tn} \end{bmatrix}. \tag{11}$$

It can be shown that $\Phi_{zn} = G_{12}\Phi_{un}$ and $\Phi_{tw} = K_t(I - G_{22}K)^{-1}G_{21}$.

The following is the well known Youla Parametrization result.

Lemma III.1: [5] Suppose the plant $G_{22} : (u'_1\ u'_2)' \to (e'_3\ e'_4)'$ shown in Fig. 2 has a double-coprime factorization

$$\begin{pmatrix} X_l & -Y_l \\ -N_l & M_l \end{pmatrix} \begin{pmatrix} M_r & Y_r \\ N_r & X_r \end{pmatrix} = I \tag{12}$$

where all terms in (12) are stable. Then (G_{22}, K) in Fig. 2 is internally stable if and only if there exists a stable Q such that $K = (Y_r - M_rQ)(X_r - N_rQ)^{-1} = (X_\ell - QN_\ell)^{-1}(Y_\ell - QM_\ell)$. Under the above parametrization of stabilizing controllers, it can be shown that a closed-loop map Φ_{zw} is achievable via stabilizing controller if and only if $\Phi_{zw} \in \{H - UQV|\ Q \text{ stable}\}$ where H, U, and V are stable transfer matrices determinable from G.

With $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}}_{K} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ the induced map of K in terms of the maps C^k_{ij} that describe K_1 and K_2 is given by

$$K_{11} = C^1_{11} + C^1_{12}(1 - C^2_{22}C^1_{22})^{-1}C^2_{22}C^1_{21} \tag{13}$$
$$K_{12} = C^1_{12}(1 - C^2_{22}C^1_{22})^{-1}C^2_{21} \tag{14}$$
$$K_{21} = C^2_{12}(1 - C^1_{22}C^2_{22})^{-1}C^1_{21} \tag{15}$$
$$K_{22} = C^2_{11} + C^2_{12}(1 - C^1_{22}C^2_{22})^{-1}C^1_{22}C^2_{21}. \tag{16}$$

From Lemma III.1 it follows that for the above controller to internally stabilize the (G_{22}, K) interconnection a stable Q should exist such that

$$\begin{aligned} \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] &= (Y_r - M_rQ)(X_r - N_rQ)^{-1}, \\ &= (X_\ell - QN_\ell)^{-1}(Y_\ell - QM_\ell) \end{aligned} \tag{17}$$

with K_{ij} satisfying (13), (14), (15) and (16). The condition above also ensures that the map Φ_{zw} is an affine and stable map in the variable Q and that the maps Φ_{uv} and Φ_{yv} in Fig. 4 are stable.

Let, $\overline{Y}_r = Y_r - M_rQ$, $\overline{Y}_l = Y_l - QM_l$, $\overline{X}_r = X_r - N_rQ$ and $\overline{X}_l = X_l - QN_l$. Note that $\overline{Y}_r$, $\overline{Y}_l$, $\overline{X}_r$ and $\overline{X}_l$ are stable and affine in Q. The theorem below follows.

Theorem III.2: Consider plant G_{22} shown in Fig. 5 that has double-coprime factorization given by Lemma III.1 with K stabilizing the (G_{22}, K) interconnection and parameterized in terms of Q. Assume a distributed implementation of K with K_1 and K_2 described by (2) and (3) and K_n, K_t and K_{tn} given by (8)-(10). Let $T_a = \overline{X}_l K_n, T_b = K_t \overline{X}_r$, and $\Phi_{tn} = K_t G_{22}(I - KG_{22})^{-1}K_n + K_{tn}$. Then the closed-loop map $T(K_1, K_2)$ given by (11) for the distributive implementation as shown in Fig. 5 is affine in Q with (G_{22}, K_1, K_2) internally stable if T_a, T_b and Φ_{tn} are stable and affine in Q.

Proof: From [16], Φ_{zw} is stable and affine in Q. By using the fact that $(I - G_{22}K)^{-1} = \overline{X}_r M_l, (I - G_{22}K)^{-1}G_{22} = \overline{X}_r N_l, (I - KG_{22})^{-1} = M_r\overline{X}_l$ and $G_{22}(I - KG_{22})^{-1} = N_r\overline{X}_l$, the closed-loop maps given by (6) and (7) are as follows: $\Phi_{un} = M_r\overline{X}_l K_n$, $\Phi_{yn} = N_r\overline{X}_l K_n$, $\Phi_{tn} = K_t N_r \overline{X}_l K_n + K_{tn}$, $\Phi_{tv} = K_t\left[\begin{array}{cc} \overline{X}_r N_l & \overline{X}_r M_l \end{array}\right]$. Since, $T_a = \overline{X}_l K_n$ and $T_b = K_t\overline{X}_r$. It follows that $\Phi_{un} = M_r T_a$, $\Phi_{yn} = N_r T_a$, $\Phi_{tn} = T_b\overline{X}_r^{-1}N_r T_a + K_{tn}$, $\Phi_{tv} = \left[\begin{array}{cc} T_b N_l & T_b M_l \end{array}\right]$. Since, M_r, N_r, M_l and N_l are stable and constant matrices, the proof is complete. ∎

Remark: Note that conditions above for affineness of relevant closed-loop maps are sufficient conditions. However, note that for affineness of T in Q it is necessary, for example, that $\Phi_{zn} = G_{12}\Phi_{un} = G_{12}M_r T_a(Q)$, where, G_{12} and M_r are not dependent on Q, be affine in Q. This in turn, apart from pathological cases, will imply that $T_a(Q)$ has to be an affine function of Q. A similar argument can be made for the necessity of $T_b(Q)$ to be affine in Q. Φ_{tn} is a part of the map $T(Q)$ and has to be affine in Q if T is affine in Q. Thus the requirements of affineness in Q of T_a, T_b and Φ_{tn} are not conservative.

Given affine in Q conditions of Theorem III.2 an optimization problem of interest is to obtain sub-controllers C_{ij}^k which minimize a convex measure of the performance map T given by the following problem

$$\mu_1 := \inf_{Q\in\mathcal{S}} \|T(Q)\| \tag{18}$$

where

$$T(Q) = \left[\begin{array}{cc} \Phi_{zw}(Q) & \Phi_{zn}(Q) \\ \Phi_{tw}(Q) & \Phi_{tn}(Q) \end{array}\right] = \left[\begin{array}{cc} H - UQV & G_{12}M_r T_a(Q) \\ T_b(Q)M_l G_{21} & T_b(Q)\overline{X(Q)}_r^{-1} N_r T_a(Q) + K_{tn}(Q) \end{array}\right]$$

with $T_a(Q) = \overline{X}_l(Q)K_n(Q)$ and $T_b(Q) = K_t(Q)\overline{X}_r(Q)$. Also, $Q \in \mathcal{S}$ if $C_{ij}^k(Q)$ satisfy (13), (14), (15), (16) and (17) and $T_a(Q), T_b(Q), \Phi_{tn}(Q)$ are stable and affine in Q, with Q stable.

C. Two stabilizing distributed implementations with convex performance problem

The main difficulty in solving the performance problem given in (18) is that it is not clear how to solve (17) to obtain all possible solutions for $C_{ij}^k(Q)$ for $k = 1,2; i = 1,2; j = 1,2$ such that $T_a(Q), T_b(Q)$ and $\Phi_{tn}(Q)$ are stable and affine in Q. Two sets of solutions for $C_{ij}^k(Q)$ are obtained in this section such that the performance problem remains convex. $\overline{X}_l, \overline{Y}_l, \overline{X}_r$ and $\overline{Y}_r$, which are parameterized in terms of Q, are partitioned in conformation with partitioning of K in Fig. 5 as $\overline{X}_l = \left(\begin{array}{cc} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{array}\right)$; $\overline{Y}_l = \left(\begin{array}{cc} \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{array}\right)$; $\overline{X}_r = \left(\begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array}\right)$; $\overline{Y}_r = \left(\begin{array}{cc} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{array}\right)$; $N_l = \left(\begin{array}{cc} \widetilde{N}_{11} & \widetilde{N}_{12} \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{array}\right)$. Similarly, M_l, N_l, M_r and N_r can be partitioned as $M_l = \left(\begin{array}{cc} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{array}\right)$; $N_r = \left(\begin{array}{cc} N_{11} & N_{12} \\ N_{21} & N_{22} \end{array}\right)$ and $M_r = \left(\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array}\right)$. A notation of $\widetilde{\ }$ is used to distinguish between blocks of left-coprime factors and right-coprime factors. Since $\overline{X}_l$, $\overline{Y}_l$, X_r, and Y_r are stable and affine in Q, $\widetilde{X}_{ij}, \widetilde{Y}_{ij}, X_{ij}$ and Y_{ij} for $i = 1,2$ and $j = 1,2$ are also stable and affine in Q. Following two corollaries present the main results of this paper.

Corollary III.1: (Left-coprime architecture)
The sub-controller $K_1 = \left[\begin{array}{c|c} C_{11}^1 & C_{12}^1 \\ \hline C_{21}^1 & C_{22}^1 \end{array}\right]$ given by

$$\left[\begin{array}{c|cc} \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & \widetilde{X}_{11}^{-1}\widetilde{Y}_{12} & -\widetilde{X}_{11}^{-1}\widetilde{X}_{12} \\ \hline I & 0 & 0 \\ \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & \widetilde{X}_{11}^{-1}\widetilde{Y}_{12} & -\widetilde{X}_{11}^{-1}\widetilde{X}_{12} \end{array}\right]; \tag{19}$$

and $K_2 = \left[\begin{array}{c|c} C_{11}^2 & C_{12}^2 \\ \hline C_{21}^2 & C_{22}^2 \end{array}\right]$ given by

$$\left[\begin{array}{c|cc} \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} & \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \\ \hline I & 0 & 0 \\ \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} & \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \end{array}\right] \tag{20}$$

satisfying the parametrization given by (17) and are such that T_a, T_b and Φ_{tn} are stable and affine in Q. The architecture given by (20) will be called the left-coprime architecture and is shown in Fig. 7. The two transmitted signals in this architecture are $(t_1', t_2')' = (t_{11}', t_{12}', t_{21}', t_{22}')' = (y_1', u_1', y_2', u_2')' = t$.

Proof: See Appendix VIII-B for proof. ∎

The left-coprime architecture obtained in above corollary can be used to reduce (18) to the following suboptimal problem:

$$\gamma_L = \underbrace{\inf}_{\substack{\text{Left-coprime architecture} \\ Q\text{ - stable}}} \|T(Q)\| \tag{21}$$

$$= \underbrace{\inf}_{Q\text{ - stable}} \left\| \left[\begin{array}{cc} H - UQV & H_1^L - U_1^L Q V_1^L \\ H_2^L - U_2^L Q V_2^L & H_3^L - U_3^L Q V_3^L \end{array}\right] \right\| \tag{22}$$

where $H_1^L, U_1^L, V_1^L, H_2^L, U_2^L, V_2^L, H_3^L, U_3^L$ and V_3^L are determined based on left-coprime architecture. Another set of sub-controllers can be obtained using right-coprime factors as discussed in following corollary.

Corollary III.2: (Right-coprime architecture)
The sub-controllers $K_1 = \left[\begin{array}{c|c} C_{11}^1 & C_{12}^1 \\ \hline C_{21}^1 & C_{22}^1 \end{array}\right]$ given by

$$\left[\begin{array}{c|cc} Y_{11}X_{11}^{-1} & I & Y_{11}X_{11}^{-1} \\ \hline Y_{21}X_{11}^{-1} & 0 & Y_{21}X_{11}^{-1} \\ -X_{21}X_{11}^{-1} & 0 & -X_{21}X_{11}^{-1} \end{array}\right]; \tag{23}$$

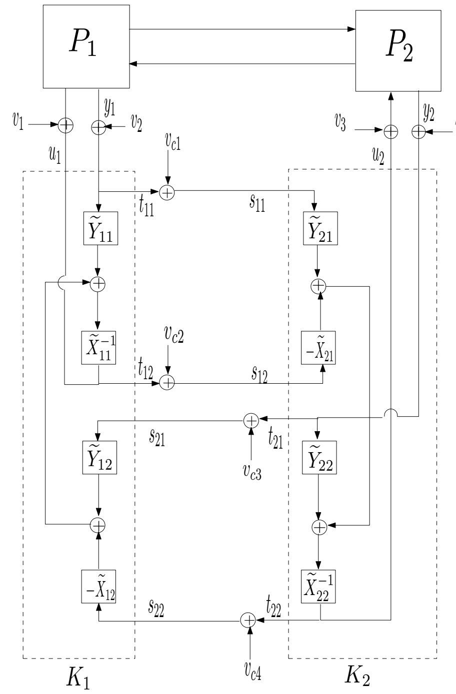

Fig. 7. Left-coprime architecture

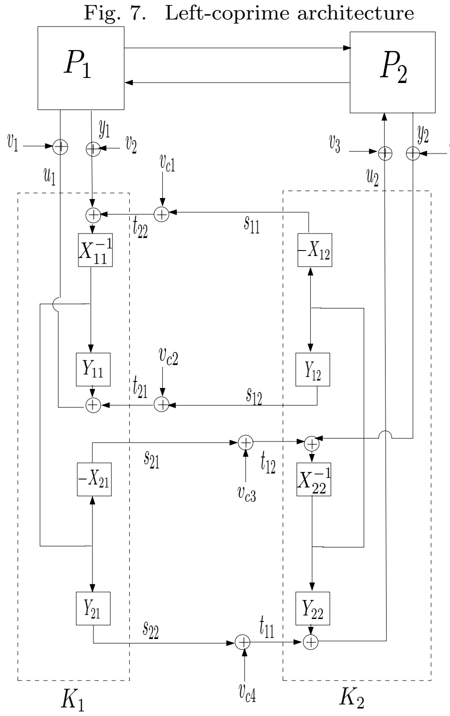

Fig. 8. Right-coprime architecture

and $K_2 = \left[\begin{array}{c|c} C_{11}^2 & C_{12}^2 \\ \hline C_{21}^2 & C_{22}^2 \end{array}\right]$ given by

$$\left[\begin{array}{c|cc} Y_{22}X_{22}^{-1} & I & Y_{22}X_{22}^{-1} \\ \hline Y_{12}X_{22}^{-1} & 0 & Y_{12}X_{22}^{-1} \\ -X_{12}X_{22}^{-1} & 0 & -X_{12}X_{22}^{-1} \end{array}\right] \tag{24}$$

satisfying the parametrization given by (17) and is such that T_a, T_b and Φ_{tn} are stable and affine in Q. This architecture given by (24) is called the right-coprime architecture and is shown in Fig. 8 where the transmitted signal is $t = (t_1', t_2')' = (t_{11}', t_{12}', t_{21}', t_{22}')' = (Y_{21}X_{11}^{-1}y_1', -X_{21}X_{11}^{-1}y_1', Y_{12}X_{22}^{-1}y_2', -X_{21}X_{22}^{-1}y_2')'$.

Proof: See Appendix VIII-C for proof. ∎

The right-coprime architecture obtained in above corollary can be used to reduce (18) to the following suboptimal problem:

$$\gamma_R = \underbrace{\inf}_{\substack{\text{Right-coprime architecture} \\ Q \text{ - stable}}} \|T(Q)\| \tag{25}$$

$$= \underbrace{\inf}_{Q \text{ - stable}} \left\| \begin{bmatrix} H - UQV & H_1^R - U_1^R Q V_1^R \\ H_2^R - U_2^R Q V_2^R & H_3^R - U_3^R Q V_3^R \end{bmatrix} \right\| \tag{26}$$

where $H_1^R, U_1^R, V_1^R, H_2^R, U_2^R, V_2^R, H_3^R, U_3^R$ and V_3^R are determined based on right-coprime architecture. It is within this (restricted) class of architectures that we consider optimality in Q.

IV. Distributed Design For Controllers With Special Structure

The two architectures obtained in the previous section for controllers without any structure can be specialized to controllers with banded structure and nested structure.

A. Banded Structure

In [4] it is shown that all stabilizing banded structure controllers K given by (1) can be parameterized in terms of Youla parameter Q which is also banded. This results in the parametrization of K in terms of Q having the same banded structure. Let Q be partitioned in conformation with partitioning of G_{22} and K as $Q = \begin{pmatrix} Q_{11} & \lambda Q_{12} \\ \lambda Q_{21} & Q_{22} \end{pmatrix}$. Let, $\overline{Y}_r = Y_r - M_r Q$, $\overline{Y}_l = Y_l - QM_l$, $\overline{X}_r = X_r - N_r Q$ and $\overline{X}_l = X_l - QN_l$. From [4] it follows that $\overline{Y}_r$, $\overline{Y}_l$, $\overline{X}_r$ and $\overline{X}_l$ are affine in Q, stable and have banded structure. Let, $\overline{X}_l = \begin{pmatrix} \widetilde{X}_{11} & \lambda\widetilde{X}_{12} \\ \lambda\widetilde{X}_{21} & \widetilde{X}_{22} \end{pmatrix}$; $\overline{Y}_l = \begin{pmatrix} \widetilde{Y}_{11} & \lambda\widetilde{Y}_{12} \\ \lambda\widetilde{Y}_{21} & \widetilde{Y}_{22} \end{pmatrix}$; $N_l = \begin{pmatrix} \widetilde{N}_{11} & \lambda\widetilde{N}_{12} \\ \lambda\widetilde{N}_{21} & \widetilde{N}_{22} \end{pmatrix}$; $M_l = \begin{pmatrix} \widetilde{M}_{11} & \lambda\widetilde{M}_{12} \\ \lambda\widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix}$; $\overline{X}_r = \begin{pmatrix} X_{11} & \lambda X_{12} \\ \lambda X_{21} & X_{22} \end{pmatrix}$; $\overline{Y}_r = \begin{pmatrix} Y_{11} & \lambda Y_{12} \\ \lambda Y_{21} & Y_{22} \end{pmatrix}$, $N_r = \begin{pmatrix} N_{11} & \lambda M_{12} \\ \lambda N_{21} & N_{22} \end{pmatrix}$ and $M_r = \begin{pmatrix} M_{11} & \lambda M_{12} \\ \lambda M_{21} & M_{22} \end{pmatrix}$. The coprime factors can be substituted in the architectures obtained in Section III-C to obtain two ways of implementing banded structured controller such that the performance problem is a convex problem in presence of sub-controller noise. The two sub-controllers $\left[\begin{array}{c|c} C_{11}^1 & C_{12}^1 \\ \hline C_{21}^1 & C_{22}^1 \end{array}\right]$ and $\left[\begin{array}{c|c} C_{11}^2 & C_{12}^2 \\ \hline C_{21}^2 & C_{22}^2 \end{array}\right]$ for the left-coprime architecture are given by

$$\left[\begin{array}{c|cc} \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & \lambda\widetilde{X}_{11}^{-1}\widetilde{Y}_{12} & -\lambda\widetilde{X}_{11}^{-1}\widetilde{X}_{12} \\ \hline I & 0 & 0 \\ \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & \lambda\widetilde{X}_{11}^{-1}\widetilde{Y}_{12} & -\lambda\widetilde{X}_{11}^{-1}\widetilde{X}_{12} \end{array}\right]; \quad (27)$$

$$\left[\begin{array}{c|cc} \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} & \lambda\widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\lambda\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \\ \hline I & 0 & 0 \\ \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} & \lambda\widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\lambda\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \end{array}\right] \quad (28)$$

and is shown in Fig. 9.

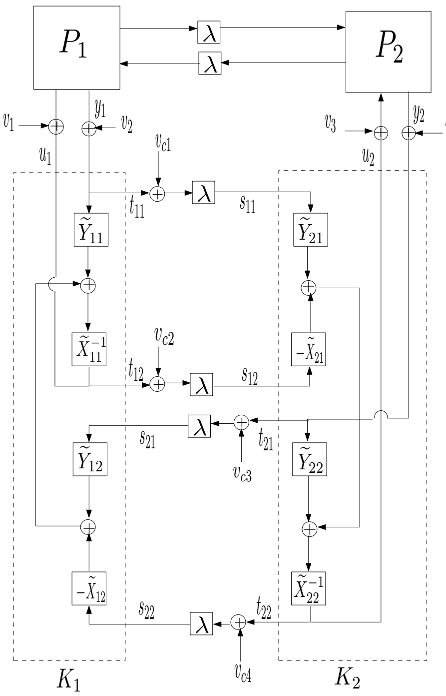

Fig. 9. Left-coprime architecture for banded structure

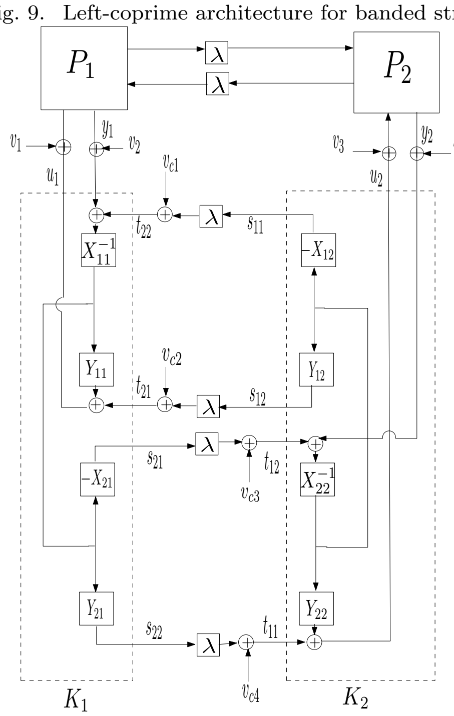

Fig. 10. Right-coprime architecture for banded structure

The two sub-controllers of right-coprime architecture for banded structure controller are given by

$$K_1 = \left[\begin{array}{c|cc} Y_{11}X_{11}^{-1} & I & Y_{11}X_{11}^{-1} \\ \hline \lambda Y_{21}X_{11}^{-1} & 0 & \lambda Y_{21}X_{11}^{-1} \\ -\lambda X_{21}X_{11}^{-1} & 0 & -\lambda X_{21}X_{11}^{-1} \end{array}\right]; \quad (29)$$

$$K_2 = \left[\begin{array}{c|cc} Y_{22}X_{22}^{-1} & I & Y_{22}X_{22}^{-1} \\ \hline \lambda Y_{12}X_{22}^{-1} & 0 & \lambda Y_{12}X_{22}^{-1} \\ -\lambda X_{12}X_{22}^{-1} & 0 & -\lambda X_{12}X_{22}^{-1} \end{array}\right] \quad (30)$$

and is shown in Fig. 10.

B. Nested Structure

The following result on internal stability holds in the case of nested (G_{22}, K_1, K_2) system shown in Fig. 6.

Corollary IV.1: Suppose K_1 and K_2 in Fig. 6 are transfer matrices such that corresponding transfer matrix from $(y_1' \ y_2')'$ to $(e_1' \ e_2')'$ is the transfer matrix K. Further suppose that K internally stabilizes the (G_{22}, K) interconnection. Let transfer matrices K_1 and K_2 have state space realizations given by $K_1 = \left[\begin{array}{c|c} A_{C_1} & B_{C_1} \\ \hline C_{C_1,1} & \multirow{2}{*}{0} \\ C_{C_1,2} & \end{array}\right]$ and $K_2 = \left[\begin{array}{c|cc} A_{C_2} & B_{C_2,1} & B_{C_2,2} \\ \hline C_{C_2} & \multicolumn{2}{c}{0} \end{array}\right]$ such that $(A_{C_1}, B_{C_1}, C_{C_1,1})$ and $(A_{C_2}, B_{C_2,1}, C_{C_2})$ are stabilizable and detectable. Then (G, K_1, K_2) is internally stable.

Remark: Note that the in the corollary above, it is not needed to derive the realization of the controller transfer matrix K from the realizations of K_1 and K_2 to infer internal stability of the distributed interconnection.

The following result from [4] translates the triangular structure restriction on the controller to the same structure on the Youla parameter Q.

Lemma IV.1: Consider 2-nest $G_{22} - K$ system shown in Fig. 6, where $G_{22} = \left[\begin{array}{c|c} G_{22a} & 0 \\ \hline G_{22c} & G_{22d} \end{array}\right] := P$ maps control inputs $u = (u_1', u_2')'$ to the measured output $y = (y_1', y_2')'$. Assume that $P_1 = \left[\begin{array}{cc} G_{22a} & 0 \end{array}\right]$ and $P_2 = \left[\begin{array}{cc} G_{22c} & G_{22d} \end{array}\right]$ have state space realizations $\left[\begin{array}{c|cc} A_1 & B_{11} & 0 \\ \hline C_1 & D_{11} & 0 \end{array}\right]$ and $\left[\begin{array}{c|cc} A_2 & B_{21} & B_{22} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right]$, respectively, and the inherited realizations of G_{22a} and G_{22d} are stabilizable and detectable. Let a double-coprime factorization of G_{22} be given by (12). Then K is lower triangular and it internally stabilizes the $G_{22} - K$ inter-connection if and only if there exists a stable Q that is lower triangular such that $K = (Y_r - M_r Q)(X_r - N_r Q)^{-1} = (X_\ell - QN_\ell)^{-1}(Y_\ell - QM_\ell)$
This results in the parametrization of K in terms of Q having the same structure. Let Q be partitioned according to the structure of G_{22} and K as $Q = \begin{pmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{pmatrix}$. Let, $\overline{Y}_r = Y_r - M_r Q$, $\overline{Y}_l = Y_l - QM_l$, $\overline{X}_r = X_r - N_r Q$ and $\overline{X}_l = X_l - QN_l$. Note that $\overline{Y}_r$, $\overline{Y}_l$, $\overline{X}_r$ and $\overline{X}_l$ are affine in Q, stable and have lower triangular structure. Let, $\overline{X}_l = \begin{pmatrix} \widetilde{X}_{11} & 0 \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{pmatrix}$; $\overline{Y}_l = \begin{pmatrix} \widetilde{Y}_{11} & 0 \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{pmatrix}$;

$N_l = \begin{pmatrix} \widetilde{N}_{11} & 0 \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{pmatrix}$; $M_l = \begin{pmatrix} \widetilde{M}_{11} & 0 \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix}$; $\overline{X}_r = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$; $\overline{Y}_r = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}$; $N_r = \begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}$ and $M_r = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$.

The sub-controllers $K_1 = \left[\begin{array}{c|c} C^1_{11} & C^1_{12} \\ \hline C^1_{21} & C^1_{22} \end{array}\right]$ and $K_2 = \left[\begin{array}{c|c} C^2_{11} & C^2_{12} \\ \hline C^2_{21} & C^2_{22} \end{array}\right]$ with left-coprime architecture are given by

$$\left[\begin{array}{c|cc} \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & 0 & 0 \\ \hline I & 0 & 0 \\ \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & 0 & 0 \end{array}\right] \text{ and } \tag{31}$$

$$\left[\begin{array}{c|cc} \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} & \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{32}$$

respectively. The architecture is shown in Fig. 11. The closed-loop transfer functions are all affine in Q and given by: $\Phi_{u_2n_1} = -M_{22}\widetilde{X}_{21}$, $\Phi_{u_2n_2} = M_{22}\widetilde{Y}_{21}$, $\Phi_{y_2n_1} = -N_{22}\widetilde{X}_{21}$, $\Phi_{y_2n_2} = N_{22}\widetilde{Y}_{21}$, $\Phi_{t_1v_1} = Y_{11}\widetilde{N}_{11}$, $\Phi_{t_2v_1} = X_{11}\widetilde{N}_{11}$, $\Phi_{t_1v_3} = Y_{11}\widetilde{M}_{11}$, $\Phi_{t_2v_3} = X_{11}\widetilde{M}_{11}$

Similarly sub-controllers K_1 and K_2 for the right coprime architecture are given by

$$K_1 = \left[\begin{array}{c|cc} Y_{11}X_{11}^{-1} & 0 & 0 \\ \hline Y_{21}X_{11}^{-1} & 0 & 0 \\ -X_{21}X_{11}^{-1} & 0 & 0 \end{array}\right]; \tag{33}$$

$$K_2 = \left[\begin{array}{c|cc} Y_{22}X_{22}^{-1} & I & Y_{22}X_{22}^{-1} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{34}$$

respectively. The architecture is shown in Fig. 12. The closed-loop transfer functions are all affine in Q and given by: $\Phi_{u_2n_1} = M_{22}\widetilde{X}_{22}$, $\Phi_{u_2n_2} = M_{22}\widetilde{Y}_{22}$, $\Phi_{y_2n_1} = -N_{22}\widetilde{X}_{22}$, $\Phi_{y_2n_2} = N_{22}\widetilde{Y}_{22}$, $\Phi_{t_1v_1} = Y_{21}\widetilde{N}_{11}$, $\Phi_{t_2v_1} = -X_{21}\widetilde{N}_{11}$, $\Phi_{t_1v_3} = Y_{21}\widetilde{M}_{11}$, $\Phi_{t_2v_3} = -X_{21}\widetilde{M}_{11}$.

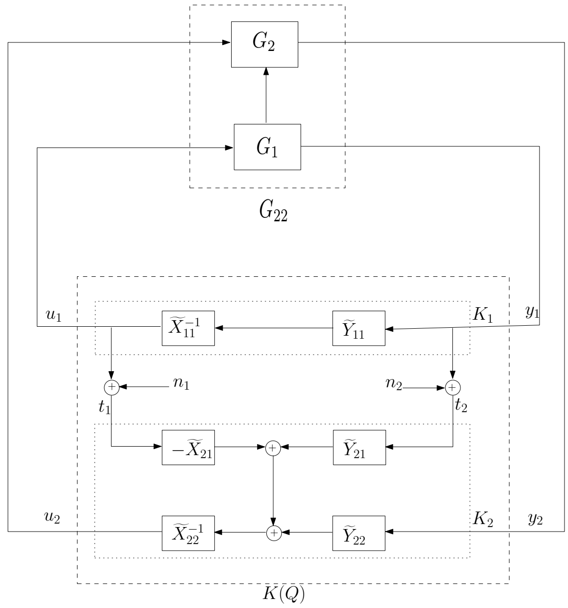

Fig. 11. Left-coprime architecture for nested structure

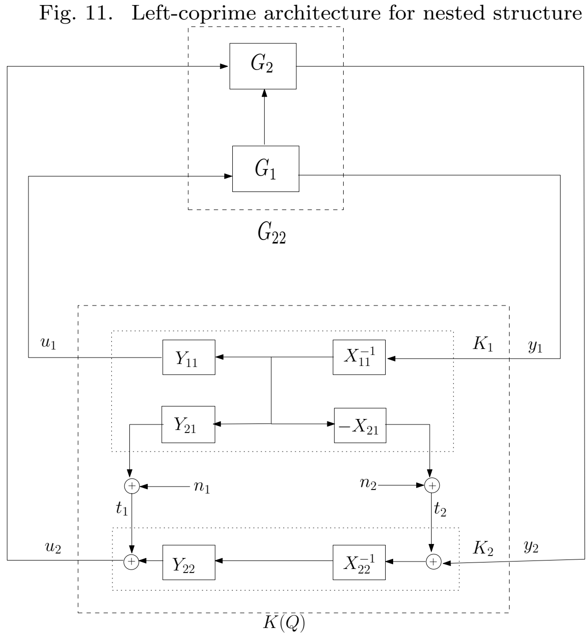

Fig. 12. Right-coprime architecture for nested structure

V. Robust Stability Framework

In this section controller synthesis in the presence of communication and plant uncertainty is considered.

For the illustration purpose the discussion is restricted to the nested structure systems with uncertainty only in sub-controller communication but this formulation can be generalized to banded structure as well as plants with no information structure with uncertainty in all possible external links.

Consider the descriptions of nested systems given in Fig. 13 that shows uncertainty affecting the link from the sub-controller K_1 to K_2. Fig. 13 implements K_1 and K_2 with two-channel transmission from K_1 to K_2 based on one of the architectures obtained in Section IV. This uncertainty characterizations can be cast into the standard (M, Δ) framework as shown in Fig. 14, where $n = \binom{n_1}{n_2}$, $s = \binom{s_1}{s_2}$, $\Delta_n = \begin{pmatrix} \Delta_{n_1} & 0 \\ 0 & \Delta_{n_2} \end{pmatrix}$ and $M(Q) : \begin{pmatrix} w \\ n \end{pmatrix} \mapsto \begin{pmatrix} z \\ s \end{pmatrix} = \left[\begin{array}{c|cc} H - UQV & H'_{a1} - U'_{a1}QV'_{a1} & H'_{a2} - U'_{a2}QV'_{a2} \\ \hline H'_{b1} - U'_{b1}QV'_{b1} & 0 & \\ H'_{b2} - U'_{b2}QV'_{b2} & & \end{array}\right]$ where Q is a stable lower triangular transfer function. Following class for uncertainty description is considered: $\Delta_{LTV} = \{\Delta \in \mathcal{S}$ is linear time varying and $\|\cdot\| < \infty\}$ where $\mathcal{S}$ characterizes the structure and the norm is either ℓ_∞ or ℓ_2 induced norm. Note that when $\mathcal{S}$ is given by the block diagonal structure $diag(\Delta_z, \Delta_n)$ with Δ_z, and Δ_n being unstructured then the (M, Δ) interconnection is robustly stable with respect to all $\Delta \in B\Delta_{LTV} := \{\Delta \in \Delta_{LTV} | \|\Delta\| \leq 1\}$ if and only if $\inf_{D\in\mathcal{D}} \|DM(Q)D^{-1}\|_{\ell_1} < 1$ where $\mathcal{D} := \{D = diag(1, d_1, d_2)$ with $d_i > 0\}$ [17]. Thus

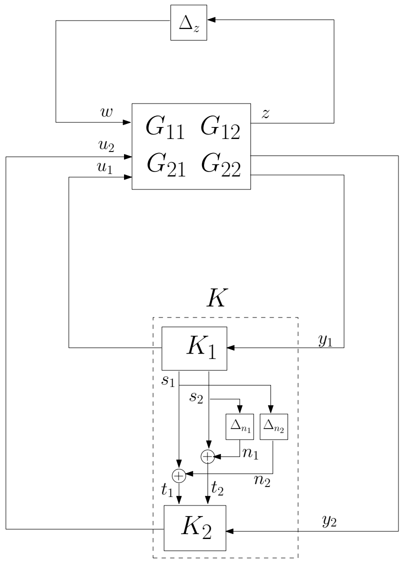

Fig. 13. (G, K) nested system with noise modelled as multiplicative uncertainty

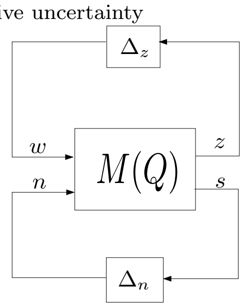

Fig. 14. The (M, Δ) configuration for two channel case shown in Fig. 13

the problem for robust synthesis in this case reduces to the problem

$$\inf_{Q\in\ell_1}\inf_{D\in\mathcal{D}}\|DM_2(Q)D^{-1}\|_{\ell_1}$$

where $M(Q)$ is affine in Q. Thus for the architectures identified, the above from is convex in Q for fixed D or fixed Q and the synthesis problem has the same complexity as the standard robust performance problem. However, this problem is nonconvex jointly in the variables D and the Youla parameter Q. Recently in [18] a global solution to the above synthesis problem was achieved. This provides an effective procedure to address the problem of synthesizing controllers for ℓ_1 robust synthesis when there is uncertainty in the sub-controller to sub-controller communication.

VI. Examples

A. *Optimal Distributed Controller Design for* 2*-node ABR Network: Robust Synthesis Framework*

Consider the nested system shown in Fig. 15 of 2-node Available Bit Rate (ABR) communication network, with the problem of congestion of data packets at two nodes. This example illustrates design of an optimal controller with nested structure in presence of sub-controller noise using robust control synthesis technique as discussed in Section VI.

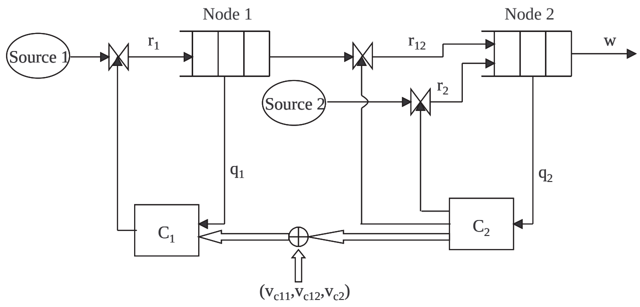

Fig. 15. 2-nodal ABR network with congestion control

In Fig. 15, r_1 and r_2 are rates with which source 1 and 2 transmit data packets to node 1 and 2, respectively. The rate of flow from node 1 to node 2 is r_{12}. The total available capacity (bit-rate) for the two sources is represented by w. The queue lengths at node 1 and 2 are denoted by q_1 and q_2, respectively. The network traffic is controlled by regulating rates r_1, r_{12} and r_2. The two sub-controllers C_1 and C_2, control rates r_1 and $(r'_{12}, r'_2)'$, respectively. Note that, the flow of information is only from C_2 to C_1, similar to the 2-nest system of Fig. 6, with $K_2 = C_1$ and $K_1 = C_2$. The objectives are to avoid the two queues from overflowing by avoiding congestion, to maximize the utilization factor of the network, i.e. to make $r_1 + r_2$ match w as close as possible, to minimize the affect of sub-controller to sub-controller noise on the queue lengths and rates of transmission, and to regulate the signal power transmitted between two sub-controllers.

The controller K is implemented using the left-coprime architecture derived in Section IV. The information transmitted from C_2 to C_1 are $(r'_{12}\ r'_2)'$ and q_2, and they get corrupted by noise viz. $(v'_{c11}\ v'_{c12})'$ and v_{c2}, respectively. The exogenous signals are identified as the available capacity w and the noise $(v'_{c11}, v'_{c12}, v'_{c2})'$. The regulated variables are two queue lengths q_1 and q_2, and the difference between the data rate of the source and the fraction of w allocated to that source, i.e. $r_1 - a_1 w$ and $r_2 - a_2 w$. Thus, $z = [\ (r_2 - a_1 w)'\ \ (r_1 - a_2 w)'\ \ q'_2\ \ q'_1\ ]'$. The controlled input is $u = [\ r'_2\ \ r'_{12}\ \ r'_1\ ]'$, with $u_1 = (r'_2, r'_{12})'$ and $u_2 = r_1$. The measured outputs is $y = [\ y'_1\ \ y'_2\ ]' = [\ q'_2\ \ q'_1\ ]'$. We assume tha $a_1 = a_2 = 0.5$. The dynamics of the network is described by:

- Node 1: $q_1(k+1) = q_1(k) + r_1(k) - r_{12}(k)$
- Sub-controller C_1: $r_1 = f_1(q_1, r_2 + v_{c11}, r_{12} + v_{c12}, q_2 + v_{c2})$
- Node 2: $q_2(k+1) = q_2(k) + r_2(k) + r_{12}(k) - w(k)$
- Sub-controller C_2: $r_2 = f_2(q_2); r_{12} = f_{12}(q_2)$

where f_1, f_2 and f_{12} are causal and linear operators. Clearly, plant G_{22} (the part of generalized plant G : $[t; u] \mapsto [z; y]$ which maps u to y) and controller K in this case are lower triangular operators i.e.: $G_{22} := \left[\begin{array}{cc|c} * & * & 0 \\ \hline * & * & * \end{array}\right]$ and $K := \left[\begin{array}{c|c} * & 0 \\ * & 0 \\ \hline * & * \end{array}\right]$. The state-space description of G_{22} is given by: $G_{22} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] =$

$\left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & B_{21} & B_2 \\ \hline C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \end{array}\right]$ where $A_1 = A_2 = 1$, $B_1 = [\ 1 \quad 1\ ], B_{21} = [\ 0 \quad -1\ ], B_2 = 1$, and $C_1 = C_2 = 1$. The state-feedback matrix F, and observer gain matrix L for G_{22} are chosen such that $A+LC$ and $A+BF$ are Hurwitz: $F = \left[\begin{array}{cc} F_1 & 0 \\ 0 & F_2 \end{array}\right], L = \left[\begin{array}{cc} L_1 & 0 \\ 0 & L_2 \end{array}\right]$, where $F_1 = [\ -0.9 \quad 0\ ]^T, F_2 = -0.9, L_1 = L_2 = -0.9$. The doubly coprime factorization of G_{22} is described by $X_r = \left[\begin{array}{c|c} \frac{1+0.8\lambda}{1-0.1\lambda} & 0 \\ \hline 0 & \frac{1+0.8\lambda}{1-0.1\lambda} \end{array}\right]$; $Y_r = \left[\begin{array}{c|c} \frac{-0.81\lambda}{1-0.1\lambda} & 0 \\ 0 & 0 \\ \hline 0 & \frac{-0.81\lambda}{1-0.1\lambda} \end{array}\right]$; $X_l = \left[\begin{array}{cc|c} \frac{-1-0.8\lambda}{1-0.1\lambda} & \frac{-0.9\lambda}{1-0.1\lambda} & 0 \\ 0 & -1 & 0 \\ \hline 0 & \frac{0.9\lambda}{1-0.1\lambda} & \frac{-1-0.8\lambda}{1-0.1\lambda} \end{array}\right]$; $Y_l = \left[\begin{array}{c|c} \frac{0.81\lambda}{1-0.1\lambda} & 0 \\ 0 & 0 \\ \hline 0 & \frac{0.81\lambda}{1-0.1\lambda} \end{array}\right]$;

$M_r = \left[\begin{array}{cc|c} \frac{1-\lambda}{1-0.1\lambda} & \frac{0.9\lambda}{1-0.1\lambda} & 0 \\ 0 & -1 & 0 \\ \hline 0 & \frac{-0.9\lambda}{1-0.1\lambda} & \frac{1-\lambda}{1-0.1\lambda} \end{array}\right]$;

$N_r = \left[\begin{array}{cc|c} \frac{-\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} & 0 \\ \hline 0 & \frac{\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} \end{array}\right]$; $M_l = \left[\begin{array}{c|c} \frac{1-\lambda}{1-0.1\lambda} & 0 \\ \hline 0 & \frac{1-\lambda}{1-0.1\lambda} \end{array}\right]$

$N_l = \left[\begin{array}{cc|c} \frac{\lambda}{1-0.1\lambda} & \frac{\lambda}{1-0.1\lambda} & 0 \\ \hline 0 & \frac{-\lambda}{1-0.1\lambda} & \frac{\lambda}{1-0.1\lambda} \end{array}\right]$. Note that all right coprime factors are lower triangular.

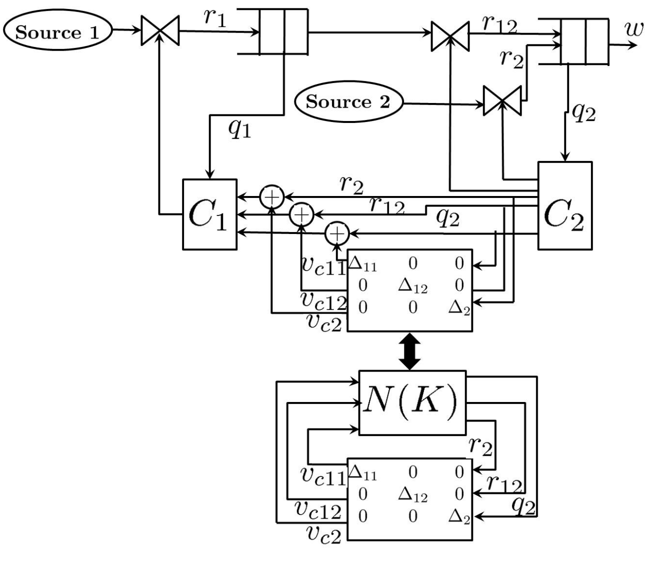

Fig. 16. 2-nodal ABR network with uncertain communication channel

The communication channel uncertainty is assumed to enter in a multiplicative manner as shown in Fig. 16. As shown in Fig. 17 the problem can be written in standard (M,Δ) form, where $M(K) = \left[\begin{array}{c|c} \Phi_{zw} & \Phi_{zv} \\ \hline \Phi_{rw} & \Phi_{rv} \end{array}\right]$. Φ_{zw} is the 4×1 closed-loop transfer matrix from w to z, Φ_{zv} is the 4×3 closed-loop transfer matrix from $v = (v'_{c11}\ v'_{c12}\ v'_{c2})'$ to z, Φ_{rw} is the 3×1 closed-loop transfer function from w to $r = (u'_1\ y'_1)'$ and Φ_{rv} is the 3×3 closed-loop transfer

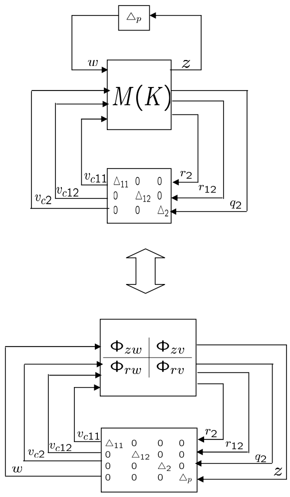

Fig. 17. $M - \triangle$ form of 2-nodal ABR network

function from v to r. $M(K)$ is affine in the parameter Q. Indeed Φ_{zw} and Φ_{rw} can be written as affine functions of Q using coprime factors of G_{22} and Φ_{zv} is affine in Q as established in Section V. In this example, Φ_{rv} is equal to 0. These four transfer functions can be written in terms of Youla parameter Q (in $H - UQV$ form) such that the design problem can be written as following optimization problem:

$$\nu = \inf_{Q\in \ell_1,\ lower\ triangular} \inf_{D\in\mathcal{D}} \|DM(Q)D^{-1}\|_{\ell_1} \tag{35}$$

which can be solved using the technique given in [18] to obtain a global solution.

The closed loop map Φ_{zw} can be obtained by rewriting dynamics of network in terms of λ transform as follows:

$$\begin{pmatrix} z \\ y \end{pmatrix} = \left(\begin{array}{c|ccc} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 0 & 1 \\ \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & 0 \\ 0 & 0 & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \\ \hline \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & 0 \\ 0 & 0 & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \end{array}\right) \begin{pmatrix} w \\ u \end{pmatrix} \equiv$$

$\left(\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right)\begin{pmatrix} w \\ u \end{pmatrix} \equiv G\begin{pmatrix} w \\ u \end{pmatrix}$, $u = Ky$. This implies that $\Phi_{zw} = H_1 - U_1 Q V_1$ where, $H_1 = G_{11} + G_{12}Y_r M_l G_{21}$, $U_1 = G_{12}M_r$ and $V_1 = M_l G_{21}$ which is given $H_1 = [-0.5 + \frac{0.81\lambda^2}{(1-0.1\lambda)^2},\ -0.5,\ \frac{\lambda(1-0.8\lambda)}{(1-0.1\lambda)^2},\ 0]'$,

$$U_1 = \left[\begin{array}{ccc} \frac{-(1-\lambda)}{1-0.1\lambda} & \frac{0.9\lambda}{1-0.1\lambda} & 0 \\ 0 & \frac{-0.9\lambda}{1-0.1\lambda} & \frac{-(1-\lambda)}{1-0.1\lambda} \\ \frac{-\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} & 0 \\ 0 & \frac{\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} \end{array}\right],$$

$V_1 = [\frac{-\lambda}{1-0.1\lambda},\ 0]'$. Since, the controller is implemented using left-coprime architecture of Section IV, and coprime factors X_l and Y_l are affine in Q, Φ_{zv} can be written as

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$\Phi_{zv} = H_2 - U_2QV_2$ where $\Phi_{zv} = \begin{bmatrix} 0 & 0 \\ -M_{22}\widetilde{X}_{21} & M_{22}\widetilde{Y}_{21} \\ 0 & 0 \\ -N_{22}\widetilde{X}_{21} & N_{22}\widetilde{Y}_{21} \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 & 0 \\ -M_{22}\widetilde{X}^0_{21} & M_{22}\widetilde{Y}^0_{21} \\ 0 & 0 \\ -N_{22}\widetilde{X}^0_{21} & N_{22}\widetilde{Y}^0_{21} \end{bmatrix}$, $U_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & M_{22} \\ 0 & 0 & 0 \\ 0 & 0 & N_{22} \end{bmatrix}$ and $V_2 = \begin{bmatrix} -\widetilde{N}_{11} & \widetilde{M}_{11} \\ -\widetilde{N}_{21} & \widetilde{M}_{21} \end{bmatrix}$ where $\widetilde{X}^0_{21}$ and $\widetilde{Y}^0_{21}$ are lower off-diagonal parts of X_l and Y_l. By substituting for values of coprime factors, H_2, U_2 and V_2 can be written as: $H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{0.9\lambda(1-\lambda)}{(1-0.1\lambda)^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{0.9\lambda^2}{(1-0.1\lambda)^2} & \end{bmatrix}$, $U_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-(1-\lambda)}{1-0.1\lambda} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-\lambda}{1-0.1\lambda} \end{bmatrix}$, $V_2 = \begin{bmatrix} -\frac{\lambda}{1-0.1\lambda} & -\frac{\lambda}{1-0.1\lambda} & \frac{1-\lambda}{1-0.1\lambda} \\ 0 & \frac{\lambda}{1-0.1\lambda} & 0 \end{bmatrix}$.

In order to obtain Q parametrization of Φ_{rw}, the dynamic equations of the network with $r = (r'_2\ r'_{12}\ q'_2)'$ as regulated variable is written in terms of λ as following: $\begin{pmatrix} r \\ y \end{pmatrix} = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & 0 \\ \hline \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & 0 \\ 0 & 0 & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \end{array}\right)$ $\begin{pmatrix} w \\ u \end{pmatrix} \equiv \left(\begin{array}{c|c} \bar{G}_{11} & \bar{G}_{12} \\ \hline \bar{G}_{21} & \bar{G}_{22} \end{array}\right) \begin{pmatrix} w \\ u \end{pmatrix} \equiv \bar{G} \begin{pmatrix} w \\ u \end{pmatrix}, u = Ky$. This implies that $\Phi_{zw} = H_3 - U_3QV_3$ where, $H_3 = \bar{G}_{11} + \bar{G}_{12}Y_rM_l\bar{G}_{21}$, $U_3 = \bar{G}_{12}M_r$ and $V_3 = M_l\bar{G}_{21}$ which can be computed to be the following: $H_3 = [\frac{0.81\lambda^2}{(1-0.1\lambda)^2}, 0, \frac{-\lambda(1+0.8\lambda)}{(1-0.1\lambda)^2}]'$, $U_3 = \begin{bmatrix} \frac{-(1-\lambda)}{1-0.1\lambda} & \frac{0.9\lambda}{1-0.1\lambda} & 0 \\ 0 & -1 & 0 \\ \frac{-\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} & 0 \end{bmatrix}$, $V_3 = [\frac{-\lambda}{1-0.1\lambda}, 0]'$. Thus, M transfer matrix can be written as an affine function of Q as follows: $M = \left[\begin{array}{c|c} \Phi_{zw} & \Phi_{zv} \\ \hline \Phi_{rw} & \Phi_{rv} \end{array}\right] = \left[\begin{array}{c|c} H_1 - U_1QV_1 & H_2 - U_2QV_2 \\ \hline H_3 - U_3QV_3 & 0 \end{array}\right]$.

The optimal controller obtained by solving the robust synthesis problem in (35) is $Q_{opt}(\lambda) = \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right]$ with $Q_{11}(\lambda) = [\frac{-0.6-0.283\lambda-0.0689\lambda^2+0.0217\lambda^3+0.016\lambda^4-0.002\lambda^5}{1+0.786\lambda+0.362\lambda^2+0.085\lambda^3+0.005\lambda^4}, \frac{0.198\lambda-0.156\lambda^2-0.074\lambda^3-0.018\lambda^4-0.002\lambda^5}{1+0.786\lambda+0.362\lambda^2+0.085\lambda^3+0.005\lambda^4}]'$, $Q_{12} = [0,0]'$, $Q_{21} = \frac{-0.001\lambda^3-0.001\lambda^4-0.001\lambda^5}{1+0.786\lambda+0.362\lambda^2+0.085\lambda^3+0.005\lambda^4}$ and $Q_{22} = \frac{0.9+0.706\lambda+0.325\lambda^2+0.077\lambda^3+0.004\lambda^4}{1+0.786\lambda+0.362\lambda^2+0.085\lambda^3+0.005\lambda^4}$.

B. *Optimal Distributed Controller Design for 2-node ABR Network: Search over two different architectures*

In this example, consider the 2-node ABR network as shown in Fig. 15. The objectives are to design a distributed controller for given ABR network which not only avoids the congestion in the network while keeping the channel utilization ratio as large as possible, but also minimizes the affect of the sub-controller to sub-controller noise on the queue lengths (q_1 and q_2) and the regulated rates (r_1, r_{12} and r_2) of transmission of packets by regulating rates r_1 and r_2. It should also minimize the signal power of transmitted signal between sub-controllers. Thus, it is required to find a stabilizing controller K which is lower triangular and minimizes $\|T(K)\|$, where $T(K) : \begin{pmatrix} w \\ n \end{pmatrix} \mapsto \begin{pmatrix} z \\ t \end{pmatrix} = \begin{bmatrix} \Phi_{zw} & \Phi_{zn} \\ \Phi_{tw} & \Phi_{tn} \end{bmatrix}$, where Φ_{zw} captures the performance requirement of system, Φ_{zn} captures the affect of sub-controller communication noise on performance and Φ_{tw} denotes the power of transmitted signal with respect to power of external input signals where $n = v$ and $t = n + v$. From Section IV, $T(K) = T(Q)$, where Q is lower triangular and stable. This parametrization makes the closed-loop map Φ_{zw} affine in Q, i.e. $\Phi_{zw} = H - UQV$, where $H = G_{11} + G_{12}Y_rM_lG_{21}, U = G_{12}M$ and $V = M_lG_{21}$. $\Phi_{tn} = I$ depend on the architecture. Thus, the two convex performance optimization problems to be solved are: $\mu_l = \inf_{\substack{K-\text{stabilizing, lower triangular,}\\ \text{based on the left-coprime architecture}}} \|T(K)\|_1$, $\mu_r = \inf_{\substack{K-\text{stabilizing, lower triangular,}\\ \text{based on the right-coprime architecture}}} \|T(K)\|_1$

Solving above two optimization problems for these two architectures, two different optimal controllers $Q_{opt,l}$ and $Q_{opt,r}$ are obtained as shown in Fig. 18 - 19 with $\mu_l = 1.5$ and $\mu_r = 3.7$, respectively. Clearly, in this example it is better to use left-coprime architecture with $\mu_l = 1.5$. For the centralized optimal controller the ℓ_1 norm from exogenous input w to regulated variable z is 1, however, if this centralized solution is implemented in a distributive manner using the left-coprime architecture the ℓ_1 norm from noise to regulated variable becomes 1.87, which is 25% higher than μ_l in this example. Similarly, a distributive implementation of the centralized solution using the right-coprime architecture leads to an ℓ_1 norm from noise to regulated variable of 6.45, which is 74% higher than μ_r in this example. In Fig. 18 - 19 $In(1)$ and $In(2)$ are inputs q_2 and q_1, respectively; and $Out(1), Out(2)$, and $Out(3)$ are outputs r_2, r_{12}, and r_1, respectively.

VII. Conclusion

In this paper, the problem to design optimal distributed controllers is formulated in terms of Youla parameter Q. A sufficient condition is obtained which assures internal stability of distributed implementation and guarantees that all relevant closed loop maps are stable and affine in Q. Two

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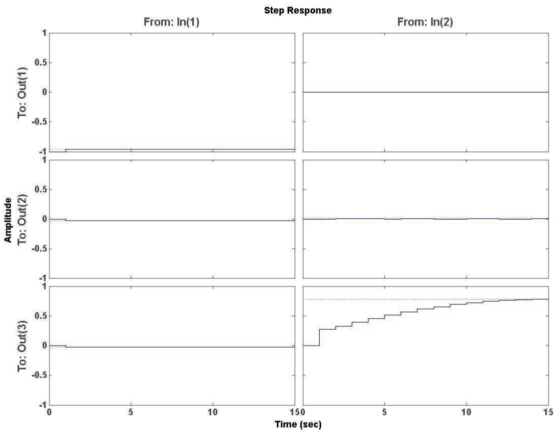

Fig. 18. $Q_{opt,r}$ optimal controller for left-coprime architecture with $\mu_l = 1.5$

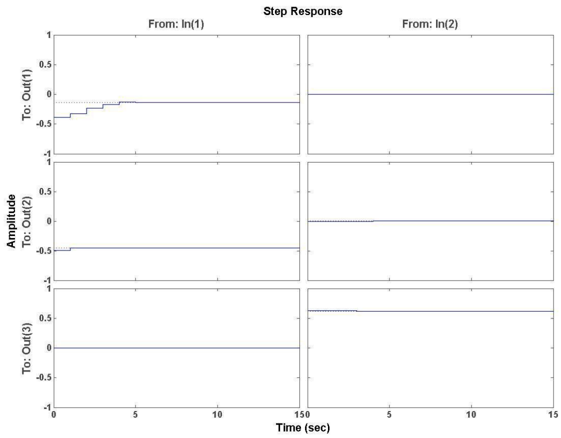

Fig. 19. $Q_{opt,l}$ optimal controller for right-coprime architecture with $\mu_r = 3.74$

architectures are obtained for distributed implementation of controller such that the closed loop is internally stable and the optimal performance problem is convex. These architectures are further refined for controllers with specific information structures viz. nested and banded structures. These two architectures provide a subclass of controllers which can be implemented distributively such that the performance including the affect of sub-controller noise and the power of sub-controller transmission signal results in a convex optimization problem.

VIII. Appendix

A. Example: Unstable closed-loop system with sub-controller communication

Consider a generalized plant G with exogenous input w, control input u, measured output y and regulated output $z = y$ given by.

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \left(\begin{array}{c|ccc} \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & 0 \\ 0 & 0 & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \\ \hline \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} & 0 \\ 0 & 0 & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \end{array}\right),$$

$$y = G_{22}u + G_{21}w, u = Ky.$$

The plant G_{22} has triangular structure. Consider controller K with $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ that is also triangular. K is implemented distributively as shown in Fig. 4 where t_1 is the sub-controller transmission signal from K_1 to K_2 whereas no signal is transmitted from K_2 to K_1 and therefore $t_2 = 0$. K is distributively implemented such that $\begin{pmatrix} u_1 \\ t_1 \end{pmatrix} = K_1 \begin{pmatrix} y_1 \\ t_2 \end{pmatrix} = \begin{bmatrix} K_{11} & 0 \\ I & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ t_2 \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ t_2 \end{pmatrix} = K_2 \begin{pmatrix} y_2 \\ t_1 \end{pmatrix} = \begin{bmatrix} K_{22} & K_{21} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_2 \\ t_1 \end{pmatrix}$. In this implementation, $t_1 = y_1$ and $\Phi_{tw} = K_{21}\Phi_{y_1 w}$. A minimal state space representation of G_{22} is given by $G_{22} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right)$. With

$$F = \begin{pmatrix} -0.8 & 0 \\ -0.1 & 0 \\ 0 & -0.9 \end{pmatrix} \text{ and } L = \begin{pmatrix} -1.9 & 0 \\ 0 & -1.9 \end{pmatrix},$$

$A + BF$ and $A + LC$ are stable. A doubly coprime factorization of G_{22} is given by $X_r = \left(\begin{array}{c|c} \frac{1+1.8\lambda}{1-0.1\lambda} & 0 \\ \hline \frac{0.19\lambda^2}{(1-0.1\lambda)^2} & \frac{1+1.8\lambda}{1-0.1\lambda} \end{array}\right)$, $Y_r = \left(\begin{array}{c|c} \frac{-1.52\lambda}{1-0.1\lambda} & 0 \\ \frac{-0.19\lambda}{1-0.1\lambda} & 0 \\ \hline \frac{-0.17\lambda^2}{(1-0.1\lambda)^2} & \frac{-1.71\lambda}{1-0.1\lambda} \end{array}\right)$,

$$X_l = \left(\begin{array}{cc|c} \frac{-1-1.7\lambda}{1+0.9\lambda} & \frac{-0.8\lambda}{1+0.9\lambda} & 0 \\ \frac{-0.1\lambda}{1+0.9\lambda} & \frac{-1-\lambda}{1+0.9\lambda} & 0 \\ \hline 0 & \frac{0.9\lambda}{1+0.9\lambda} & \frac{-1-1.8\lambda}{1+0.9\lambda} \end{array}\right), Y_l = \left(\begin{array}{c|c} \frac{1.52\lambda}{1+0.9\lambda} & 0 \\ \frac{1.9\lambda}{1+0.9\lambda} & 0 \\ \hline 0 & \frac{1.71\lambda}{1+0.9\lambda} \end{array}\right)$$

$$M_l = \left(\begin{array}{c|c} \frac{1-\lambda}{1+0.9\lambda} & 0 \\ \hline 0 & \frac{1-\lambda}{1+0.9\lambda} \end{array}\right), N_l = \left(\begin{array}{cc|c} \frac{\lambda}{1+0.9\lambda} & \frac{\lambda}{1+0.9\lambda} & 0 \\ \hline 0 & \frac{-\lambda}{1+0.9\lambda} & \frac{\lambda}{1+0.9\lambda} \end{array}\right)$$

$$M_r = \left(\begin{array}{cc|c} \frac{1-0.9\lambda}{1-0.1\lambda} & \frac{-0.8\lambda}{1-0.1\lambda} & 0 \\ \frac{-0.1\lambda}{1-0.1\lambda} & \frac{1-0.2\lambda}{1-0.1\lambda} & 0 \\ \hline \frac{-0.09\lambda^2}{(1-0.1\lambda)^2} & \frac{0.9\lambda(1-0.2\lambda)}{(1-0.1\lambda)^2} & \frac{1-\lambda}{1-0.1\lambda} \end{array}\right), \quad N_r = \left(\begin{array}{cc|c} \frac{\lambda}{1-0.1\lambda} & \frac{\lambda}{1-0.1\lambda} & 0 \\ \hline \frac{0.1\lambda^2}{(1-0.1\lambda)^2} & \frac{-\lambda(1-0.2\lambda)}{(1-0.1\lambda)^2} & \frac{\lambda}{1-0.1\lambda} \end{array}\right)$$

Controller $K = Y_r X_r^{-1}$ is stabilizing which is

$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] = \left[\begin{array}{c|c} -\frac{1.52\lambda}{1+1.8\lambda} & 0 \\ \frac{-0.19\lambda}{1+1.8\lambda} & 0 \\ \hline -\frac{0.171\lambda^2}{(1+1.8\lambda)^2} & \frac{-1.71\lambda}{1+1.8\lambda} \end{array}\right],$$

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and the closed-loop map $\Phi_{zw} = H - UQV = H = G_{11} + G_{12}Y_rM_lG_{21}$ can be written as

$$\Phi_{zw} = \begin{pmatrix} \Phi_{z_1w} \\ \Phi_{z_2w} \end{pmatrix} = \begin{pmatrix} \Phi_{y_1w} \\ \Phi_{y_2w} \end{pmatrix} = \begin{pmatrix} \frac{-\lambda(1+1.8\lambda)}{(1-0.1\lambda)(1+0.9\lambda)} \\ \frac{-0.19\lambda^3}{(1-0.1\lambda)^2(1+0.9\lambda)} \end{pmatrix}.$$

Note that $z = y$ i.e. $z = (z_1', z_2')' = (y_1', y_2')'$ and Φ_{y_1w} has only one unstable zero at $z = -1.8$. Thus, the closed loop map from w to z is stable. Note that the map Φ_{tw} from w to sub-controller transmission signal t is not stable. Indeed, $\Phi_{tw} = K_{21}\Phi_{y_1w} = \widetilde{X}_{22}^{-1}[\widetilde{X}_{22}Y_{21} - \widetilde{Y}_{22}X_{21}]X_{11}^{-1}\Phi_{y_1w}$ $= \frac{-\lambda^2(0.17-0.2\lambda)}{(1-0.1\lambda)(1+1.8\lambda)^2}\frac{-\lambda(1+1.8\lambda)}{(1-0.1\lambda)(1+0.9\lambda)} = \frac{\lambda^3(0.17-0.2\lambda)}{(1-0.1\lambda)^2(1+1.8\lambda)(1+0.9\lambda)}$ which is unstable, because K_{21} has two unstable poles at -1.8 while Φ_{y_1w} has only one unstable zero at $z = -1.8$.

Note that the sub-controllers $K_1 : \begin{pmatrix} y_1 \\ t_2 \end{pmatrix} \to \begin{pmatrix} u_1 \\ t_1 \end{pmatrix}$ and $K_2 : \begin{pmatrix} y_2 \\ t_1 \end{pmatrix} \to \begin{pmatrix} u_2 \\ t_2 \end{pmatrix}$ admit minimal realizations given by

$$K_1 = \left[\begin{array}{c|c} A_{C_1} & B_{C_1} \\ \hline C_{C_1} & D_{C_1} \end{array}\right] = \left[\begin{array}{c|cc} -1.8 & 1 & 0 \\ \hline -1.51 & 0 & 0 \\ -0.19 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

and $K_2 = \left[\begin{array}{c|c} A_{C_2} & B_{C_2} \\ \hline C_{C_2} & D_{C_2} \end{array}\right]$ with $A_{C_1} = -1.8$, $B_{C_1,1} = 1$, $B_{C_1,2} = 0$, $C_{C_1,1} =: \begin{pmatrix} -1.52 \\ -0.19 \end{pmatrix}$, $C_{C_1,2} = 0$, $D_{C_1,11} = D_{C_1,12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $D_{C_1,21} = 1$, $D_{C_1,22} = 0$ and $A_{C_2} = \begin{pmatrix} -1.8 & 0 \\ -1.71 & 0 \end{pmatrix}$, $B_{C_2,1} = \begin{pmatrix} 0 \\ -1.71 \end{pmatrix}$, $B_{C_2,2} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$, $C_{C_2,1} = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $C_{C_2,2} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $D_{C_2,11} = D_{C_2,12} = D_{C_2,21} = D_{C_2,22} = 0$.

It can be shown that the induced realization of the controller $K : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ as given in Theorem III.1 by $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ is not stabilizable. Thus the conditions of Theorem III.1 are not met.

B. Proof of Corollary III.1: Left-coprime architecture

Let $K_0 = \begin{bmatrix} C_{11}^1 & 0 \\ 0 & C_{11}^2 \end{bmatrix}$ and by using definitions for K_n and K_t given by (8)-(9), the controller K can be written as:

$$K = K_0 + K_n \begin{bmatrix} C_{21}^1 & 0 \\ 0 & C_{21}^2 \end{bmatrix}. \quad (36)$$

K_n can be partitioned according to the dimensions of n_1 and n_2 as $K_n = \begin{bmatrix} K_{n1} & K_{n2} \end{bmatrix}$ that leads to

$$T_a = \overline{X}_lK_n = \begin{bmatrix} \overline{X}_lK_{n1} & \overline{X}_lK_{n2} \end{bmatrix} =: \begin{bmatrix} T_{a1} & T_{a2} \end{bmatrix}. \quad (37)$$

Therefore, by replacing K by its Q-parameterized form $\overline{X}_l^{-1}\overline{Y}_l$, multiplying (36) from left by $\overline{X}_l$ and then replacing $\overline{X}_lK_n$ by T_a following can be obtained:

$$(\overline{Y}_l - \overline{X}_lK_0) = \begin{bmatrix} T_{a1}C_{21}^1 & T_{a2}C_{21}^2 \end{bmatrix}. \quad (38)$$

By suitably choosing K_0, i.e., suitably choosing C_{11}^1 and C_{11}^2, a solution for (38) can be obtained leading to a stabilizing distributed implementations with convex performance problem as discussed next. T_a can be further partitioned in conformation with $\overline{X}_l$ as $T_a = \begin{pmatrix} T_{a1}^u & T_{a2}^u \\ T_{a1}^d & T_{a2}^d \end{pmatrix}$. Consider the case when two communication channels for transmission in each direction are used between sub-controllers i.e. $t_1 = \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix}$, $t_2 = \begin{pmatrix} t_{21} \\ t_{22} \end{pmatrix}$, $n_1 = \begin{pmatrix} n_{11} \\ n_{12} \end{pmatrix}$ and $n_2 = \begin{pmatrix} n_{21} \\ n_{22} \end{pmatrix}$. Set $C_{11}^1 = \widetilde{Y}_{11}\widetilde{X}_{11}^{-1}$ and $C_{11}^2 = \widetilde{Y}_{22}\widetilde{X}_{22}^{-1}$. (38) is further simplified to:

$$\begin{bmatrix} 0 & (\widetilde{Y}_{12} - \widetilde{X}_{12}\widetilde{X}_{22}^{-1}\widetilde{Y}_{22}) \\ (\widetilde{Y}_{21} - \widetilde{X}_{21}\widetilde{X}_{11}^{-1}\widetilde{Y}_{11}) & 0 \end{bmatrix} = \begin{bmatrix} T_{a1}C_{21}^1 & T_{a2}C_{21}^2 \end{bmatrix}. \quad (39)$$

Remark: In order to solve (39) such that T_a is stable and affine in Q, it is required to factorize $(\widetilde{Y}_{12} - \widetilde{X}_{12}\widetilde{X}_{22}^{-1}\widetilde{Y}_{22})$ and $(\widetilde{Y}_{21} - \widetilde{X}_{21}\widetilde{X}_{11}^{-1}\widetilde{Y}_{11})$ into two factors such that at least the left factor is stable and affine in Q. Thus, if there is a way to factorize (39) to obtain T_a (which is stable and affine in Q), C_{21}^1 and C_{21}^2, then rest of sub-controller parts viz. $C_{12}^1, C_{12}^2, C_{22}^1$ and C_{22}^2 can be obtained by using definition of K_n given by (8). Furthermore, if the closed loop map Φ_{tn} is stable and affine in Q, then this implementation gives an architecture with $C_{11}^1 = \widetilde{Y}_{11}\widetilde{X}_{11}^{-1}$ and $C_{11}^2 = \widetilde{Y}_{22}\widetilde{X}_{22}^{-1}$ for which the performance problem in the presence of sub-controller communication uncertainty is a convex problem.

Writing C_{21}^1 and C_{21}^2 in block form compatible with dimension of T_{a1}, i.e. $C_{21}^1 = \begin{bmatrix} C_{21}^{1u} \\ C_{21}^{1d} \end{bmatrix}$ and $C_{21}^2 = \begin{bmatrix} C_{21}^{2u} \\ C_{21}^{2d} \end{bmatrix}$, (39) becomes:

$$\begin{aligned} T_{a11}^uC_{21}^{1u} + T_{a12}^uC_{21}^{1d} &= 0 \\ T_{a11}^dC_{21}^{1u} + T_{a12}^dC_{21}^{1d} &= (\widetilde{Y}_{21} - \widetilde{X}_{21}\widetilde{X}_{11}^{-1}\widetilde{Y}_{11}) \\ T_{a21}^uC_{21}^{2u} + T_{a22}^uC_{21}^{2d} &= (\widetilde{Y}_{12} - \widetilde{X}_{12}\widetilde{X}_{22}^{-1}\widetilde{Y}_{22}) \\ T_{a21}^dC_{21}^{2u} + T_{a22}^dC_{21}^{2d} &= 0 \end{aligned}$$

One set of solution for above equations is $T_{a11}^d = \widetilde{Y}_{21}, C_{21}^{1u} = I, T_{a12}^d = -\widetilde{X}_{21}, C_{21}^{1d} = \widetilde{X}_{11}^{-1}\widetilde{Y}_{11}, T_{a11}^u = 0$ and $T_{a12}^u = 0$ which can be readily obtained by visual inspection. Similarly, $T_{a21}^u = \widetilde{Y}_{12}, C_{21}^{2u} = I, T_{a22}^u = -\widetilde{X}_{12}, C_{21}^{2d} = \widetilde{X}_{22}^{-1}\widetilde{Y}_{22}, T_{a21}^d = 0$ and $T_{a22}^d = 0$ are also obtained. This implies $T_a = \begin{bmatrix} 0 & 0 & \widetilde{Y}_{12} & -\widetilde{X}_{12} \\ \widetilde{Y}_{21} & -\widetilde{X}_{21} & 0 & 0 \end{bmatrix}$ with parts of sub-controllers $C_{21}^1 = \begin{bmatrix} I \\ \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} \end{bmatrix}$ and $C_{21}^2 = \begin{bmatrix} I \\ \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} \end{bmatrix}$. Other parts of controllers are obtained by using definition

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of K_n given by (8) as follows

$$K_n = \begin{bmatrix} 0 & C_{12}^1 \\ C_{12}^2 & 0 \end{bmatrix} \begin{bmatrix} I & -C_{22}^1 \\ -C_{22}^2 & I \end{bmatrix}^{-1},$$

This implies that $K_n \begin{bmatrix} I & -C_{22}^1 \\ -C_{22}^2 & I \end{bmatrix} = \begin{bmatrix} 0 & C_{12}^1 \\ C_{12}^2 & 0 \end{bmatrix}$, and therefore $T_a \begin{bmatrix} I & -C_{22}^1 \\ -C_{22}^2 & I \end{bmatrix} = \overline{X}_l K_n \begin{bmatrix} I & -C_{22}^1 \\ -C_{22}^2 & I \end{bmatrix} = \overline{X}_l \begin{bmatrix} 0 & C_{12}^1 \\ C_{12}^2 & 0 \end{bmatrix}$.

Substituting for T_a in above equation gives
$\begin{bmatrix} 0 & \begin{bmatrix} \widetilde{Y}_{12} & -\widetilde{X}_{12} \end{bmatrix} \\ \begin{bmatrix} \widetilde{Y}_{21} & -\widetilde{X}_{21} \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} I & -C_{22}^1 \\ -C_{22}^2 & I \end{bmatrix} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{bmatrix} \begin{bmatrix} 0 & C_{12}^1 \\ C_{12}^2 & 0 \end{bmatrix}$, which implies that

$$\begin{bmatrix} -\begin{bmatrix} \widetilde{Y}_{12} & -\widetilde{X}_{12} \end{bmatrix} C_{22}^2 & \begin{bmatrix} \widetilde{Y}_{12} & -\widetilde{X}_{12} \end{bmatrix} \\ \begin{bmatrix} \widetilde{Y}_{21} & -\widetilde{X}_{21} \end{bmatrix} & -\begin{bmatrix} \widetilde{Y}_{21} & -\widetilde{X}_{21} \end{bmatrix} C_{22}^1 \end{bmatrix} = \begin{bmatrix} \widetilde{X}_{12} C_{12}^2 & \widetilde{X}_{11} C_{12}^1 \\ \widetilde{X}_{22} C_{12}^2 & \widetilde{X}_{21} C_{12}^1 \end{bmatrix}. \quad (40)$$

Corresponding elements in matrix equation given by (40) are compared to obtain: $\widetilde{X}_{11} C_{12}^1 = \begin{bmatrix} \widetilde{Y}_{12} & -\widetilde{X}_{12} \end{bmatrix}$ which implies that $C_{12}^1 = \begin{bmatrix} \widetilde{X}_{11}^{-1}\widetilde{Y}_{12} & -\widetilde{X}_{11}^{-1}\widetilde{X}_{12} \end{bmatrix}$

$\widetilde{X}_{21} C_{12}^1 = -\begin{bmatrix} \widetilde{Y}_{21} & -\widetilde{X}_{21} \end{bmatrix} C_{22}^1$ which implies that $C_{22}^1 = \begin{bmatrix} 0 & 0 \\ \widetilde{X}_{11}^{-1}\widetilde{Y}_{12} & -\widetilde{X}_{11}^{-1}\widetilde{X}_{12} \end{bmatrix}$

$\widetilde{X}_{22} C_{12}^2 = \begin{bmatrix} \widetilde{Y}_{21} & -\widetilde{X}_{21} \end{bmatrix}$ implies that $C_{12}^2 = \begin{bmatrix} \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \end{bmatrix}$.

$\widetilde{X}_{12} C_{12}^2 = -\begin{bmatrix} \widetilde{Y}_{12} & -\widetilde{X}_{12} \end{bmatrix} C_{22}^2$ implies that $C_{22}^2 = \begin{bmatrix} 0 & 0 \\ \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & -\widetilde{X}_{22}^{-1}\widetilde{X}_{21} \end{bmatrix}$. Thus, two sub-controllers with this architecture are given by (20) and is called left-coprime architecture as shown in Fig 7. It can be verified that sub-controllers C_{11}^1, C_{12}^1, C_{21}^1, C_{22}^1, C_{11}^2, C_{12}^2, C_{21}^2, and C_{22}^2 given by (20) satisfy (17). By substituting C_{11}^1, C_{12}^1, C_{21}^1, C_{22}^1, C_{11}^2, C_{12}^2, C_{21}^2, and C_{22}^2 in the definitions of K_n, K_t and K_{tn} given by (8)-(10), following is obtained: $K_n = \overline{X}_l^{-1} \begin{bmatrix} 0 & 0 & \widetilde{Y}_{12} & -\widetilde{X}_{12} \\ \widetilde{Y}_{21} & -\widetilde{X}_{21} & 0 & 0 \end{bmatrix}$, $K_t = \Pi \begin{bmatrix} K \\ I \end{bmatrix}$ and $K_{tn} = \Pi \begin{bmatrix} K_n \\ 0 \end{bmatrix}$. Π is a permutation matrix given by $\Pi = \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{bmatrix}$. Furthermore for left-coprime architecture, the closed loop map $\Phi_{tn} = \Pi \left(\begin{bmatrix} K \\ I \end{bmatrix} G(I-KG)^{-1} + \begin{bmatrix} I \\ 0 \end{bmatrix} \right) K_n$.

By noticing that $T_a = \overline{X}_l K_n$, $T_b = K_t \overline{X}_r$, and $K = \overline{X}_l^{-1} \overline{Y}_l$, following is obtained:

$$T_a = \begin{bmatrix} 0 & -\widetilde{X}_{12} & 0 & \widetilde{Y}_{12} \\ -\widetilde{X}_{21} & 0 & \widetilde{Y}_{21} & 0 \end{bmatrix} \Pi^T,$$

$T_b = \Pi \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \\ X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ and $\Phi_{tn} = \Pi \begin{bmatrix} M_r \\ N_r \end{bmatrix} T_a$ which are stable and affine in Q. The transmitted signals in this architecture are given by $t = (t_1', t_2')' = (t_{11}', t_{12}', t_{21}', t_{22}')' = (y_1', u_1', y_2', u_2')' = \Pi(u_1', u_2', y_1', y_2')'$.

C. Proof of Corollary III.2: Right-coprime architecture

Rewrite controller K as

$$K = K_0 + \begin{bmatrix} 0 & C_{12}^1 \\ C_{12}^2 & 0 \end{bmatrix} K_t. \quad (41)$$

K_t can be partitioned according to the dimensions of t_1 and t_2 as $K_t = \begin{bmatrix} K_{t1} \\ K_{t2} \end{bmatrix}$, that leads to

$$T_b = K_t \overline{X}_r = \begin{bmatrix} K_{t1}\overline{X}_r \\ K_{t2}\overline{X}_r \end{bmatrix} =: \begin{bmatrix} T_{b1} \\ T_{b2} \end{bmatrix}. \quad (42)$$

Therefore, by replacing K by its Q-parameterized form $\overline{Y}_r \overline{X}_r^{-1}$, multiplying (41) from right by $\overline{X}_r$ and then replacing $K_t \overline{X}_r$ by T_b following can be obtained:

$$(\overline{Y}_r - K_0 \overline{X}_r) = \begin{bmatrix} C_{12}^1 T_{b2} \\ C_{12}^2 T_{b1} \end{bmatrix}. \quad (43)$$

T_b can be further partitioned in conformation with $\overline{X}_r$ as $T_b = \begin{pmatrix} T_{b1}^l & T_{b1}^r \\ T_{b2}^l & T_{b2}^r \end{pmatrix}$. With two communication channels for transmission in each direction being used between sub-controllers set $C_{11}^1 = Y_{11}X_{11}^{-1}$ and $C_{11}^2 = Y_{22}X_{22}^{-1}$, and simplify (43) to $\begin{bmatrix} 0 & (Y_{12} - Y_{11}X_{11}^{-1}X_{12}) \\ (Y_{21} - Y_{22}X_{22}^{-1}X_{21}) & 0 \end{bmatrix} = \begin{bmatrix} C_{12}^1 T_{b2} \\ C_{12}^2 T_{b1} \end{bmatrix}$. Similarly, the right-coprime architecture, (24) can be obtained. Right-coprime architecture is shown in Fig 8. It can be verified that sub-controllers C_{11}^1, C_{12}^1, C_{21}^1, C_{22}^1, C_{11}^2, C_{12}^2, C_{21}^2, and C_{22}^2 given by (24) satisfy (17). T_a, T_b and Φ_{tn} for this architecture are given by: $T_a = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} & \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ \widetilde{X}_{21} & \widetilde{X}_{22} & \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{bmatrix} \Pi$, $T_b = \Pi^T \begin{bmatrix} 0 & Y_{12} \\ Y_{21} & 0 \\ 0 & -X_{12} \\ -X_{21} & 0 \end{bmatrix}$ and $\Phi_{tn} = T_b \begin{bmatrix} N_l & M_l \end{bmatrix} \Pi$ which are stable and affine in Q. The transmitted signals in this architecture are given by $t = (t_1', t_2')' = (t_{11}', t_{12}', t_{21}', t_{22}')' = (y_1', u_1', y_2', u_2')' = \Pi(u_1', u_2', y_1', y_2')'$.

D. Nested Structure Controller: Condition to obtain stabilizing architectures

It can be shown that the result presented in [13] to obtain above two architectures for nested system can be derived from (38)-(43). This is done by by setting $C_{11}^1 = \widetilde{X}_{11}^{-1}\widetilde{Y}_{11}$ and $C_{11}^2 = \widetilde{X}_{22}^{-1}\widetilde{Y}_{22}$ in (38) and by setting $C_{11}^1 = Y_{11}X_{11}^{-1}$ and $C_{11}^2 = Y_{22}X_{22}^{-1}$, (43). As noted in the above two architectures for nested structure controllers given by (33) and (34), $C_{12}^1 = 0, C_{22}^1 = 0, C_{21}^2 =$

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0 and $C_{22}^2 = 0$. This is in conformation with the definition of $C_{12}^1, C_{22}^1, C_{21}^2$ and C_{22}^2 for nested structure as there is no signal being transmitted from K_2 to K_1. Using this and right multiplying (39) by $\overline{X}_r$ and left multiplying (43) by $\overline{X}_l$, following two equations are obtained:

$$T_a T_b = \begin{bmatrix} 0 \\ \overline{T}_a \end{bmatrix} \begin{bmatrix} \overline{T}_b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (\widetilde{Y}_{21} X_{11} - \widetilde{X}_{21} Y_{11}) & 0 \end{bmatrix}$$

$$T_a T_b = \begin{bmatrix} 0 \\ \overline{T}_a \end{bmatrix} \begin{bmatrix} \overline{T}_b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (\widetilde{X}_{22} Y_{21} - \widetilde{Y}_{22} X_{21}) & 0 \end{bmatrix}.$$

This implies that (a) $\overline{T}_a \overline{T}_b = \widetilde{Y}_{21} X_{11} - \widetilde{X}_{21} Y_{11}$, and (b) $\overline{T}_a \overline{T}_b = \widetilde{X}_{22} Y_{21} - \widetilde{Y}_{22} X_{21}$, where $\overline{T}_a = \widetilde{X}_{22} C_{12}^2$ and $\overline{T}_b = C_{21}^1 X_{11}$. Equation (a) is same as the set of equations used in [13] to derive the architectures shown in Fig. 11 and 12. Note that if (a) (and (b)) can be factorized to obtain affine in Q factors $\overline{T}_a$ and $\overline{T}_b$, then it will result in all possible architectures with $C_{11}^1 = \widetilde{Y}_{11} \widetilde{X}_{11}^{-1}$ and $C_{11}^2 = \widetilde{Y}_{22} \widetilde{X}_{22}^{-1}$ (and architecture with $C_{11}^1 = Y_{11} X_{11}^{-1}$ and $C_{11}^2 = Y_{22} X_{22}^{-1}$) for which the performance problem in presence of sub-controller communication is a convex problem.

The following corollary follows

Corollary VIII.1: Consider 2-nest $G_{22} - K$ system shown in Fig. 6 having double-coprime factorization given by Lemma IV.1 with nested-structure K being a stabilizing controller parameterized in terms of nested-structure Q. K is implemented distributively such that $C_{11}^1 = Y_{11} X_{11}^{-1}$ and $C_{11}^2 = Y_{22} X_{22}^{-1}$. Then there exists an architecture for distributive implementation such that the closed-loop map $T(K)$ is a stable and affine in Q if and only if there exists a factorization of $(\widetilde{Y}_{21} X_{11} - \widetilde{X}_{21} Y_{11})$ such that the two factors are stable and affine in Q.

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