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Brief Paper

SISO Controller Design to Minimize a Positive Combination of the l_1 and the $\mathcal{H}_2$ Norms*

MURTI V. SALAPAKA,† PETROS VOULGARIS‡ and MOHAMMED DAHLEH†

Key Words—Robust control; duality theory.

Abstract—We consider the problem of minimizing a given positive linear combination of the l_1 norm and the square of the $\mathcal{H}_2$ norm of the closed loop over all internally stabilizing controllers. The problem is analysed for the discrete-time, SISO, linear time-invariant case. It is shown that a unique optimal solution always exists, and can be obtained by solving a finite-dimensional convex optimization problem with an a priori determined dimension. It is also established that the solution is continuous with respect to changes in the coefficients of the linear combination. © 1997 Elsevier Science Ltd.

Notation

- $|x|_1$ 1-norm of the the vector $x \in \mathbb{R}^n$;
- $|x|_2$ 2-norm of the the vector $x \in \mathbb{R}^n$;
- $\hat{x}(\lambda)$ λ transform of a right-sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{\infty} x(k)\lambda^k$;
- l_1 Banach space of right-sided absolutely summable real sequences, with norm given by $\|x\|_1 := \sum_{k=0}^{\infty} |x(k)|$;
- l_∞ Banach space of right-sided bounded sequences, with norm given by $\|x\|_\infty := \sup_k |x(k)|$;
- c_0 subspace of l_∞ with elements x satisfying $\lim_{k\to\infty} x(k) = 0$;
- l_2 Banach space of right-sided square-summable sequences, with norm given by $\|x\|_2 := [\sum_{k=0}^{\infty} x(k)^2]^{1/2}$;
- $\mathcal{H}_2$ isometric isomorphic space of l_2 under the λ transform $\hat{x}(\lambda)$, with the norm given by $\|\hat{x}(\lambda)\|_2 = \|x\|_2$;
- X^* dual space of the Banach space X; $\langle x, x^*\rangle$ denotes the value of the bounded linear functional x^* at $x \in X$;
- $W(X^*, X)$ weak star topology on X^* induced by X;
- T^* adjoint operator of $T: X \to Y$, which maps Y^* to X^*.

Also, we have (see e.g. Luenberger, 1969) $(l_1)^* = l_\infty$, $(c_0)^* = l_1$ and $(l_2)^* = l_2$.

Introduction

Consider the standard feedback configuration of Fig. 1, and let ϕ_{zw} be the closed-loop transfer function that maps the exogenous input w to the regulated output z.

Many important control problems can be reduced to this setup, where the objective is to minimize a suitably defined measure of ϕ_{zw}. The standard l_1 problem addresses the design of an internally stabilizing controller, that minimizes the l_∞ norm of the regulated output z due to the worst-ease magnitude-bounded disturbance w. It is shown in Dahleh and Pearson (1987) that for the 1-block case the problem reduced to solving a finite-dimensional linear program. The analogous problem with the signal measures being the l_2 norm is the standard $\mathcal{H}_\infty$ problem. The standard $\mathcal{H}_2$ problem is concerned with the minimization of the energy contained in the pulse response of the closed loop, ϕ_{zw}. This can be viewed as minimizing the variance of the regulated output z due to a white noise input w. Both problems have been extensively analyzed in the past, and solutions have been provided (see e.g. Doyle *et al.*, 1989a).

All of the previous design problems refer to a single performance measure. It is well known (see e.g. Dahleh and Diaz-Bobillo, 1995) that optimization with respect to a particular norm may not necessarily yield good performance with respect to another. Thus if enhanced performance is required with respect to multiple measures then it is necessary to include all these measures directly in the design process. In recent years such considerations have led researchers to focus on the design of controllers to satisfy mixed performance criteria. One of the main problems in this class is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design. For this problem the interest is in the interplay between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures of the closed loop. Several state-space results associated with this problem and variants are available. The interested reader may consult Khargonekar and Rotea (1991), Doyle *et al.* (1989b), Mustafa *et al.* (1991) and Bernstein and Haddad (1989) to mention just a few references.

In Boyd and Barratt (1991) it is shown that a wide variety of control problems reduce to convex optimization problems, and it is argued that present technology makes it possible to deem the problem solved if it can be reduced to a convex problem. In this light, it is appropriate to exploit as much structure in the problem as possible, so that the standard software available becomes computationally more efficient. Within this context, several results on multiobjective functions involving the l_1 norm are becoming available. In Elia *et al.* (1993) the problem of minimizing the l_1 norm of the closed loop under linear inequality constraints is addressed. The problem of minimizing the l_1 norm of the closed loop while keeping the $\mathcal{H}_\infty$ norm under a prescribed level falls under the above category.

In Sznaier (1993) the problem of minimizing the l_1 norm of a single-input single-output transfer function while keeping the $\mathcal{H}_\infty$ norm of the closed-loop system under a specified value is reduced to solving a sequence of finite-dimensional convex optimization problems and an unconstrained $\mathcal{H}_\infty$ problem. In Sznaier and Blanchini (1993) a similar problem in continuous time of minimizing the maximum amplitude of the regulated variable due to a specified input while

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Corresponding author Professor Mohammed Dahleh. Tel. +1 805 893 2704; Fax +1 805 893 8651; E-mail dahleh@engineering.ucsb.edu.

† Mechanical and Environmental Engineering Department, University of California at Santa Barbara, CA 93106, U.S.A.

‡ University of Illinois at Urbana Champaign, Coordinated Science Laboratory, 1308 W. Main Street, Urbana, IL 61801, U.S.A.

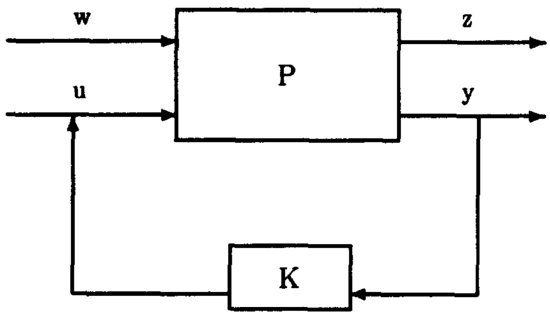

Fig. 1. Plant controller configuration.

keeping the $\mathcal{H}_\infty$ norm below a given level is reduced to solving a finite-dimensional convex constrained optimization problem and a standard unconstrained $\mathcal{H}_\infty$ problem.

In Voulgaris (1994) it is shown that the problem of minimizing the $\mathcal{H}_2$ norm of ϕ_{zw} while keeping its l_1 norm below a specified level reduced to a finite-dimensional quadratic programming problem. In Salapaka *et al.* (1995) it is shown that the problem of minimizing the l_1 norm of ϕ_{zw} while keeping its $\mathcal{H}_2$ norm below a prescribed level reduces to a finite-dimensional convex optimization problem with an a priori determined dimension. It is also shown that the optimal solution is unique whenever the $\mathcal{H}_2$-norm constraint is active.

In this paper we consider the problem of minimizing a given positive linear combination of the l_1 norm and the square of the $\mathcal{H}_2$ norm of ϕ_{zw} over all stabilizing controllers. This problem complements the set of problems considered in Voulgaris (1994) and Salapaka (1995), and can be used to provide the set of Pareto optimal solutions (see e.g. Boyd and Barratt, 1991) with respect to the l_1 and $\mathcal{H}_2$ norms. It is shown that the problem transforms to a tractable finite-dimensional convex problem of an a priori determined dimension. Although the underlying optimization principle that is employed is the same Lagrange duality theorem (Luenberger, 1969) used in Voulgaris (1994) and Salapaka (1995), the developments are substantially different.

The paper is organized as follows. In the next section the problem statement is made precise. In the following section it is shown that the problem has a unique solution, and then the problem is reduced to a finite-dimensional convex optimization problem. Conclusions are given in the last section.

Problem formulation

Consider the standard feedback problem represented in Fig. 1, where P and K are the plant and the controller respectively. Let w represent the exogenous input, z the output of interest, y the measured output and u the control input, where z and w are assumed scalar. Let ϕ be the closed-loop map $w \to z$. From Youla parametrization (Youla *et al.*, 1976), it is shown that all achievable closed-loop maps under stabilizing controllers are given by $\phi = h - u * q$ (where $*$ denotes convolution), where $h, u, q \in l_1$; h and u depend only on the plant P, and q is a free parameter in l_1. Throughout this paper, we make the following assumption:

Assumption 1. All the zeros of $\hat{u}$ (the λ transform of u) inside the unit disc are real and distinct. Also, $\hat{u}$ has no zeros on the unit circle.

The assumption that all zeros of $\hat{u}$ that are inside the open unit disc are real and distinct is not restrictive, and is made to streamline the presentation. Let the zeros of $\hat{u}$ that are inside the unit disc be $z_1, z_2, \ldots, z_n$. Let

$$\Phi := \{\phi : \text{there exists } q \in l_1 \text{ with } \phi = h - u * q\}.$$

Φ is the set of all achievable closed-loop maps under stabilizing controllers. Let $A: l_1 \to \mathbb{R}^n$ be given by

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \cdots \\ 1 & z_2 & z_2^2 & z_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \cdots \end{pmatrix},$$

and let $b \in \mathbb{R}^n$ be given by

$$b = \begin{pmatrix} \hat{h}(z_1) \\ \hat{h}(z_2) \\ \vdots \\ \hat{h}(z_n) \end{pmatrix}.$$

Theorem 1.

$$\begin{aligned}\Phi &= \{\phi \in l_1 : \hat{\phi}(z_i) = \hat{h}(z_i) \text{ for all } i = 1, \ldots, n\} \\ &= \{\phi \in l_1 : A\phi = b\}.\end{aligned}$$

Proof. This is given in Dahleh and Diaz-Bobillo (1995).

The problem of interest is as follows: given $c_1 > 0$ and $c_2 > 0$, obtain a solution to the following mixed objective problem:

$$\begin{aligned}\nu &:= \inf\{c_1 \|h - u * q\|_1 + c_2 \|h - u * q\|_2^2 : q \in l_1\} \\ &= \inf\{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 : \phi \in l_1, A\phi = b\}. \qquad (1)\end{aligned}$$

In the following sections we shall study the existence, structure and computation of the optimal solution. Before we initiate our study towards these goals, it is worthwhile to point out that the solution to the above problem generates Pareto-optimal solutions (see e.g. Boyd and Barratt, 1991) with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ of the closed loop ϕ. That is to say, if ϕ_0 is the optimal solution for the problem (1) then there does not exist a preferable alternative ϕ with $\phi = h - u * q$ for some $q \in l_1$ such that $\|\phi\|_1 \leq \|\phi_0\|_1$ and $\|\phi\|_2 < \|\phi_0\|_2$, or $\|\phi\|_1 < \|\phi_0\|_1$ and $\|\phi\|_2 \leq \|\phi_0\|_2$.

Existence, uniqueness and properties of the optimal solution

In the first part of this section we show that the problem (1) always has a solution. In the second part we show that any solution to (1) is a finite-impulse-response sequence. In the third we give an a priori bound on the length.

Existence of a solution

Here we show that a solution to (1) always exists. We use the following lemma to prove the main result of this subsection. This lemma is taken from page 128 of (Luenberger 1969).

Lemma 1. (*Banach–Alaoglu* (Luenberger, 1969).) Let X be a Banach space, with X^* its dual. Then the set $\{x^* : x^* \in X^*, \|x^*\| \leq M\}$ is $W(X^*, X)$ compact for any $M \in \mathbb{R}$.

Theorem 2. There exists a $\phi_0 \in \Phi$ such that

$$c_1 \|\phi_0\|_1 + c_2 \|\phi_0\|_2^2 = \inf_{\phi \in \Phi} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2\},$$

where $\Phi := \{\phi \in l_1 : A\phi = b\}$. Therefore the infimum in (1) is a minimum.

Proof. We denote the feasible set of our problem by $\Phi := \{\phi \in l_1 : A\phi = b\}$. Let $B := \{\phi \in l_1 : c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 \leq \nu + 1\}$. It is clear that

$$\nu = \inf_{\phi \in \Phi \cap B} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2\}.$$

Therefore, given $i > 0$, there exists $\phi_i \in \Phi \cap B$ such that $c_1 \|\phi_i\|_1 + c_2 \|\phi_i\|_2^2 \leq \nu + 1/i$. B is a bounded set in $l_1 = c_0^*$. It follows from the Banach–Alaoglu lemma that B is $W(c_0^*, c_0)$ compact. Using the fact that c_0 is separable, we know that there exists a subsequence $\{\phi_{i_k}\}$ of $\{\phi_i\}$ and $\phi_0 \in \Phi \cap B$ such that $\phi_{i_k} \to \phi_0$ in the $W(c_0^*, c_0)$ sense; that is, for all v in c_0,

$$\langle v, \phi_{i_k}\rangle \to \langle v, \phi_0\rangle \quad \text{as } k \to \infty. \tag{2}$$

Let the jth row of A be denoted by a_j and the jth element of b by b_j. Then, since $a_j \in c_0$, we have

$$\langle a_j, \phi_{i_k}\rangle \to \langle a_j, \phi_0\rangle \quad \text{as } k \to \infty \quad \text{for all } j = 1, 2, \ldots, n. \tag{3}$$

Since $A(\phi_{i_k}) = b$, we have $\langle a_j, \phi_{i_k}\rangle = b_j$ for all k and for all j, which implies that $\langle a_j, \phi_0\rangle = b_j$ for all j. Therefore we have $A(\phi_0) = b$, from which it follows that $\phi_0 \in \Phi$. This gives us $c_1 \|\phi_0\|_1 + c_2 \|\phi_0\|_2^2 \geq v$.

From (2), we have, for all N,

$$\sum_{j=0}^{N} \{c_1 |\phi_{i_k}(j)| + c_2[\phi_{i_k}(j)]^2\} \to \sum_{j=0}^{N} \{c_1 |\phi_0(j)| + c_2[\phi_0(j)]^2\} \quad \text{as } k \to \infty. \tag{4}$$

Since $c_1 \|\phi_{i_k}\|_1 + c_2 \|\phi_{i_k}\|_2^2 \leq v + 1/i_k$, we have

$$\sum_{j=0}^{N} \{c_1 |\phi_{i_k}(j)| + c_2[\phi_{i_k}(j)]^2\} \leq v + \frac{1}{i_k}. \tag{5}$$

Letting $k \to \infty$ in (5) and using (4), we have, for all N,

$$\sum_{j=0}^{N} \{c_1 |\phi_0(j)| + c_2[\phi_0(j)]^2\} \leq v.$$

Letting $N \to \infty$ in the above inequality, we conclude that $c_1 \|\phi_0\|_1 + c_2 \|\phi_0\|_2^2 \leq v$. Therefore it follows that $c_1 \|\phi_0\|_1 + c_2 \|\phi_0\|_2^2 = v$. This proves the theorem. □

Structure of optimal solutions

In this subsection we use a Lagrange duality result to show that every optimal solution is of finite length. First we give the following definitions, where we denote the interior of a set by int.

Definition 1. Let P be a convex cone in a vector space X. We write $x \geq y$ if $x - y \in P$. We write $x > 0$ if $x \in \operatorname{int}(P)$. Similarly, $x \leq y$ if $x - y \in -P := N$ and $x < 0$ if $x \in \operatorname{int}(N)$.

Definition 2. Let X be a vector space and Z a vector space with positive cone P. A mapping $G: X \to Z$ is *convex* if $G(tx + (1-t)y) \leq tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with $0 \leq t \leq 1$, and is *strictly convex* if $G(tx + (1-t)y) < tG(x) + (1-t)G(y)$ for all $x \neq y$ in X and t with $0 < t < 1$.

The following is a Lagrange duality theorem.

Theorem 3. (Luenberger (1969).) Let X be a Banach space, Ω a convex subset of X, Y a finite-dimensional space and Z a normed space with positive cone P. Let $f: \Omega \to \mathbb{R}$ be a real-valued convex functional, $g: X \to Z$ a convex mapping, $H: X \to Y$ an affine linear map and $0 \in \operatorname{int}[\operatorname{range}(H)]$. Define

$$\mu_0 := \inf\{f(x): g(x) \leq 0, H(x) = 0, x \in \Omega\}. \tag{6}$$

Suppose that there exists an $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$, and suppose that μ_0 is finite. Then

$$\mu_0 = \max\{\varphi(z^*, y): z^* \geq 0, z^* \in Z^*, y \in Y\}, \tag{7}$$

where $\varphi(z^*, y) := \inf\{f(x) + \langle g(x), z^*\rangle + \langle H(x), y\rangle : x \in \Omega\}$, and the maximum is achieved for some $z_0^* \geq 0$, $z_0^* \in Z^*$ and $y_0 \in Y$.

Furthermore, if the infimum in (6) is achieved by some $x_0 \in \Omega$ then

$$\langle g(x_0), z_0^*\rangle + \langle H(x_0), y_0\rangle = 0, \tag{8}$$

and

$$x_0 \text{ minimizes } f(x) + \langle g(x), z_0^*\rangle + \langle H(x), y_0\rangle \quad \text{over all } x \in \Omega. \tag{9}$$

We refer to (6) as the *Primal* problem and to (7) as the *Dual* problem.

Lemma 2.

$$v = \max_{y \in \mathbb{R}^n} \inf_{\phi \in l_1} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 + \langle b - A\phi, y\rangle\}. \tag{10}$$

Proof. We shall apply Theorem 3 to get the result. Let X, Ω, Y and Z in Theorem 3 correspond to $l_1, l_1, \mathbb{R}^n$ and $\mathbb{R}$ respectively. Let $\gamma := v + 1$, $g(\phi) := c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 - \gamma$ and $H(\phi) := b - A\phi$. With this notation, we have $Z^* = \mathbb{R}$.

A has full range, which implies that $0 \in \operatorname{int}[\operatorname{range}(H)]$. From Theorem 2, we know that there exists a ϕ_0 such that $g(\phi_0) = c_1 \|\phi_0\|_1 + c_2 \|\phi_0\|_2^2 - \gamma = -1 < 0$ and $H(\phi_0) = 0$. Therefore all the conditions of Theorem 3 are satisfied. From Theorem 3, we have

$$v = \max_{z \geq 0,\, y \in \mathbb{R}^n} \inf_{\phi \in l_1} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 + zg(\phi) + \langle b - A\phi, y\rangle\}.$$

Let $z_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$ be a maximizing solution of the right-hand side of the above equation. Since ϕ_0 is the solution of the primal, we have from (8) that $\langle g(\phi_0), z_0\rangle + \langle H(\phi_0), y_0\rangle = 0$, which implies that $\langle g(\phi_0), z_0\rangle = 0$. Since $g(\phi_0) \neq 0$, we conclude that $z_0 = 0$. This proves the lemma. □

Lemma 3.

$$v = \max_{y \in \mathbb{R}^n} \inf_{\phi \in l_1,\, \phi(i)v(i) \geq 0} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 - \langle \phi, v\rangle + \langle b, y\rangle\}, \tag{11}$$

where $v := A^* y$.

Proof. It follows easily from (10) that

$$v = \max_{y \in \mathbb{R}^n} \inf_{\phi \in l_1} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 - \langle \phi, v\rangle + \langle b, y\rangle\},$$

where $v := A^* y$. Suppose that $\phi \in l_1$ is such that $\phi(i)v(i) < 0$ for some i. Then choose $\phi_1 \in l_1$ such that $\phi_1(j) = \phi(j)$ for all $j \neq i$ and $\phi_1(i) = 0$. It follows that

$$c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 - \langle \phi, v\rangle > c_1 \|\phi_1\|_1 + c_2 \|\phi_1\|_2^2 - \langle \phi_1, v\rangle.$$

Therefore in the above infimization we can restrict ϕ to satisfy $\phi(i)v(i) \geq 0$ for all i. This proves the lemma. □

The following theorem shows that the solution of (1) is unique and that it is a finite-impulse-response sequence.

Theorem 4. Define $\mathcal{T} := \{\phi \in l_1 : \text{there exists } L^* \text{ with } \phi(i) = 0 \text{ if } i \geq L^*\}$. Then

$$v = \max_{y \in \mathbb{R}^n} \inf_{\phi \in \mathcal{T},\, \phi(i)v(i) \geq 0} \{c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 - \langle \phi, v\rangle + \langle b, y\rangle\}, \tag{12}$$

where $v(i) = (A^* y)(i)$. Also, the solution of the primal (1) is unique and belongs to $\mathcal{T}$.

Proof. Let $y_0 \in \mathbb{R}^n$ be the solution of the right-hand side of (12). Define $v_0 := (A^* y_0)(i)$ and let

$$L(\phi) := c_1 \|\phi\|_1 + c_2 \|\phi\|_2^2 - \langle \phi, v_0\rangle + \langle b, y_0\rangle.$$

It is clear from Lemma 3 that

$$v = \inf_{\phi \in l_1} L(\phi) = \inf_{\phi(i)v_0(i) \geq 0} L(\phi).$$

Since v_0 is in l_1, we know that there exists L^* such that $v_0(i)$ satisfies $|v_0(i)| < c_1$ if $i \geq L^*$. Suppose that ϕ is such that $\phi(i)v(i) \geq 0$. Then it follows that

$$L(\phi) = \sum_{i=0}^{\infty} \left(\phi(i)\{c_1 \operatorname{sgn}[v_0(i)] - v_0(i)\} + c_2[\phi(i)]^2\right) + \langle y_0, b\rangle.$$

If it is also true that $|v_0(i)| < c_1$ then it follows that

$$\phi(i)\{c_1 \operatorname{sgn}[v_0(i)] - v_0(i)\} + c_2[\phi(i)]^2 \geq 0,$$

with equality holding only when $\phi(i)=0$. Therefore in the infimization of

$$\inf_{\phi(i)v_0(i)\geq 0} L(\phi)$$

we can restrict ϕ to satisfy $\phi(i)=0$ whenever $|v_0(i)|<c_1$. It follows that we can restrict ϕ to $\mathcal{T}$ in the infimization, because if $i\geq L^*$ then $|v_0(i)|<c_1$.

We showed in Theorem 2 that there exists a solution ϕ_0 of the primal (1). We know from Theorem 3 that ϕ_0 is a solution of $\inf_{\phi\in l_1} L(\phi)$. Since $L(\phi)$ is strictly convex in ϕ, we conclude that the solution of the primal (1) is unique. It follows from the previous discussion that $\phi_0\in\mathcal{T}$ and that ϕ_0 is a solution of the problem

$$\inf_{\phi(i)v_0(i)\geq 0} \sum_{i=0}^{L^*} \{c_1|\phi(i)| + c_2[\phi(i)]^2 - \phi(i)v_0(i)\} + \langle y_0, b\rangle.$$

This proves the theorem. □

A priori bound on the length of the optimal solution.

In this section we give an a priori bound on the length of the solution of (1). First we establish the following three lemmas.

Lemma 4. Let ϕ_0 be a solution of the primal (1). Let y_0 represent the corresponding dual solution as obtained in (12). Let $v_0:=A^*y_0$. Then

$$c_2\phi_0(i)=\begin{cases}\frac{1}{2}[v_0(i)-c_1] & \text{if } v_0(i)>c_1,\\ \frac{1}{2}[v_0(i)+c_1] & \text{if } v_0(i)<-c_1,\\ 0 & \text{if } |v_0(i)|\leq c_1.\end{cases}$$

Also, $\|v_0\|_\infty\leq\alpha$, where $\alpha=2c_2[\|h\|_1+(c_2/c_1)\,\|h\|_2^2]+c_1$.

Proof. Let $L(\phi)$ be defined as in the proof of Theorem 4. We have shown that

$$v=\inf_{\phi\in l_1} L(\phi)=\inf_{\phi(i)v_0(i)\geq 0} L(\phi).$$

Suppose that $|v_0(i)|=c_1$. Since ϕ_0 minimizes $L(\phi)$, we have $\phi_0(i)=0$. In the proof of Theorem 4 we have shown that if $|v_0(i)|<c_1$ then $\phi_0(i)=0$. Therefore $c_2\phi_0(i)=0$ if $|v_0(i)|\leq c_1$.

Suppose that $v_0(i)>c_1$. Then it is easy to show that there exists a $\phi(i)$ such that $\phi(i)\geq 0$ and $\phi(i)\{c_1\,\mathrm{sgn}\,[v_0(i)]-v_0(i)\}+c_2[\phi(i)]^2<0$. Since any optimal minimizes $L(\phi)$, we know that $\phi_0(i)\{c_1\,\mathrm{sgn}\,[v_0(i)]-v_0(i)\}+c_2[\phi_0(i)]^2<0$, which implies that $\phi_0(i)>0$. Solving for the optimal by putting the derivative equal to zero, we have $c_1-v_0(i)+2c_2\phi_0(i)=0$ (differentiation is valid because $\phi_0(i)>0$). This implies that $c_2\phi_0(i)=\frac{1}{2}[v_0(i)-c_1]$. Similarly, it follows that $c_2\phi_0(i)=\frac{1}{2}[v_0(i)+c_1]$ when $v_0(i)<-c_1$. It also follows that

$$\begin{aligned}\|v_0\|_\infty &\leq 2c_2\,\|\phi_0\|_\infty + c_1 \leq 2c_2\,\|\phi_0\|_1 + c_1\\ &\leq 2c_2\left(\|h\|_1+\frac{c_2}{c_1}\|h\|_2^2\right)+c_1.\end{aligned}$$

The last inequality follows from the fact that h is an achievable closed-loop map. This implies that $\alpha:=2c_2[\|h\|_1+(c_2/c_1)\,\|h\|_2^2]+c_1$ is an a priori upper bound on $\|v_0\|_\infty$. This proves the lemma. □

Lemma 5. (Dahleh and Diaz-Bobillo (1995).) If $y\in\mathbb{R}^n$ is such that $\|A^*y\|_\infty\leq\alpha$ then there exists a positive integer L^* independent of y such that $|(A^*y)(i)|<c_1$ for all $i\geq L^*$.

Proof. Define

$$A_L^*=\begin{pmatrix}1 & 1 & 1 & \cdots & 1\\ z_1 & z_2 & z_3 & \cdots & z_n\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ z_1^L & z_2^L & z_3^L & \cdots & z_n^L\end{pmatrix},$$

$A_L^*:\mathbb{R}^n\to\mathbb{R}^{L+1}$. With this definition, we have $A_\infty^*=A^*$. Let $y\in\mathbb{R}^n$ be such that $\|A^*y\|_\infty\leq\alpha$. Choose any L such that $L\geq n-1$. Since z_i, $i=1,\ldots,n$, are distinct, A_L^* has full column rank. A_L^* can be regarded as a linear map taking $(\mathbb{R}^n,\|\cdot\|_1)\to(\mathbb{R}^{L+1},\|\cdot\|_\infty)$. Since A_L^* has full column rank, we can define the left inverse of A_L^*, $(A_L^*)^{-1}$, which takes $(R^{L+1},\|\cdot\|_\infty)\to(\mathbb{R}^n,\|\cdot\|_1)$. Let the induced norm of $(A_L^*)^{-l}$ be given by $\|(A_L^*)^{-l}\|_{\infty,1}$. $y\in\mathbb{R}^n$ is such that $\|A^*y\|_\infty\leq\alpha$ and therefore $\|A_L^*y\|_\infty\leq\alpha$. It follows that

$$\begin{aligned}\|y\|_1 &\leq \|(A_L^*)^{-l}\|_{\infty,1}\,\|A_L^*y\|_\infty\\ &\leq \|(A_L^*)^{-l}\|_{\infty,1}\,\alpha.\end{aligned}\tag{13}$$

Choose L^* such that

$$\max_{k=1,\ldots,n} |z_k|^{L^*}\,\|(A_L^*)^{-l}\|_{\infty,1}\,\alpha<c_1.\tag{14}$$

There always exists such an L^*, because $|z_k|<1$ for all $k=1,\ldots,n$. Note that L^* does not depend on y. For any $i\geq L^*$, we have

$$\begin{aligned}|(A^*y)(i)| &= \left|\sum_{k=1}^n z_k^i y(k)\right| \leq \max_{k=1,\ldots,n} |z_k|^i\,\|y\|_1\\ &\leq \max_{k=1,\ldots,n} |z_k|^i\,\|(A_L^*)^{-l}\|_{\infty,1}\,\alpha\\ &\leq \max_{k=1,\ldots,n} |z_k|^{L^*}\,\|(A_L^*)^{-l}\|_{\infty,1}\,\alpha.\end{aligned}$$

The second inequality follows from (13). From (14), we have $|(A^*y)(i)|<c_1$ if $i\geq L^*$. This proves the lemma. □

Theorem 5. Every solution ϕ_0 of the primal (1) is such that $\phi(i)=0$ if $i\geq L^*$, where L^* given in Lemma 5 can be determined a priori.

Proof. Let y_0 be the dual solution of (1) and let $v_0:=A^*y_0$. We know from Lemma 4 that $\|v_0\|_\infty\leq\alpha$, where $\alpha=2c_2[\|h\|_1+(c_2/c_1)\,\|h\|_2^2]+c_1$. Applying Lemma 5, we conclude that there exists an L^* (which can be determined a-priori) such that $|v_0(i)|<c_1$ if $i\geq L^*$. We have shown in the proof of Theorem 4 that $\phi_0(i)=0$ if $|v_0(i)|<c_1$. We conclude that $\phi_0=0$ if $i\geq L^*$. This proves the theorem. □

The above theorem shows that the problem (1) is a finite-dimensional convex minimization problem. In fact, it is a quadratic programming problem, which can be solved numerically with very efficient methods (see e.g. Luenberger, 1969).

Continuity of the optimal solution

In this section we show that the optimal is continuous with respect to changes in the parameters c_1 and c_2. First we state the following lemma, which is easy to prove.

Lemma 6. Let $\{f_k\}$ be a sequence of functions mapping $\mathbb{R}^m$ to $\mathbb{R}$. If f_k converges uniformly to a function f on a set $S\subset\mathbb{R}^m$ then

$$\lim_{k\to\infty}\min_{x\in S} f_k(x)=\min_{x\in S} f(x),$$

provided that the minima exist.

Theorem 6. Let $c_1^k\in[a_1,b_1]$ and $c_2^k\in[a_2,b_2]$, where $a_1>0$ and $a_2>0$. Let ϕ_k be the unique solution of the problem

$$v_k:=\min_{A\phi=b} c_1^k\,\|\phi\|_1+c_2^k\,\|\phi\|_2^2,\tag{15}$$

and let ϕ_0 be the solution of the problem

$$v:=\min_{A\phi=b} c_1\,\|\phi\|_1+c_2\,\|\phi\|_2^2,\tag{16}$$

where $c_1\in[a_1,b_1]$ and $c_2\in[a_2,b_2]$. If $c_1^k\to c_1$ and $c_2^k\to c_2$ then $\phi_k\to\phi_0$.

Proof. We prove this theorem in three parts: first we show that we can restrict the proof to a finite-dimensional space, second we show that $v_k\to v$, and finally we show that $\phi_k\to\phi_0$. Let y_k represent the dual solution of (15) and let

$v_k := A^* y_k$. Let α_k, the upper bound on $\|v_k\|_\infty$, be as given by Lemma 4. Therefore

$$\alpha_k = 2c_2^k\Big(\|h\|_1 + \frac{c_2^k}{c_1^k}\|h\|_2^2\Big) + c_1^k \le 2b_2\Big(\|h\|_1 + \frac{b_2}{a_1}\|h\|_2^2\Big) + b_1.$$

Let this bound be denoted by d. Choose L^* such that

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \|(A_L^{\dagger})^{-l}\|_{\infty,1}\, d < a_1.$$

where L is such that $L \ge n-1$. Therefore it follows that

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \|(A_L^{\dagger})^{-l}\|_{\infty,1}\, \alpha_k < c_1^k$$

for all k. It follows from arguments similar to those used in Lemma 5 and Theorem 5 that $\phi_k(i) = 0$ if $i \ge L^*$ for all k. Therefore we can assume that $\phi_k \in \mathbb{R}^{L^*}$.

We prove that $v_k \to v$. Let ϕ_1 be the solution of the problem

$$v_1 := \min_{A\phi=b} b_1\|\phi\|_1 + b_2\|\phi\|_2^2.$$

Since $c_1^k \le b_1$ and $c_2^k \le b_2$, we have $v_k \le v_1$ for all k. Therefore, for any k, we have $c_1^k\|\phi_k\|_1 + c_2^k\|\phi\|_2^2 \le v_1$, which implies that $\|\phi_k\|_1 \le v_1/c_1^k \le v_1/a_1$ and $\|\phi_k\|_2^2 \le v_1/c_2^k \le v_1/a_2$.

Let $f_k(\phi) := c_1^k\|\phi\|_1 + c_2^k\|\phi\|_2^2$ and $f(\phi) := c_1\|\phi\|_1 + c_2\|\phi\|_2^2$. Let

$$S := \Big\{\phi \in R^{L^*} : A\phi = b,\ \|\phi\|_1 \le \frac{v_1}{a_1}, \|\phi\|_2^2 \le \frac{v_1}{a_2}\Big\}.$$

Then it is clear that

$$v_k := \min_{\phi \in S} c_1^k\|\phi\|_1 + c_2^k\|\phi\|_2^2.$$

We now prove that f_k converges to f uniformly on S. Given $\epsilon > 0$, choose K such that if $k > K$ then $|c_1^k - c_1| < \epsilon a_1/2v_1$ and $|c_2^k - c_2| < \epsilon a_2/2v_1$. Then, for any $\phi \in S$, we have

$$|f_k(\phi) - f(\phi)| = |(c_1^k - c_1)\|\phi\|_1 + (c_2^k - c_2)\|\phi\|_2^2|$$
$$\le |c_1^k - c_1|\frac{v_1}{a_1} + |c_2^k - c_2|\frac{v_1}{a_2} < \epsilon.$$

Therefore, it follows that f_k converges uniformly to f on S. It follows from Lemma 6 that $v_k \to v$.

We now prove that $\phi_k \to \phi_0$. Let $B := \{\phi \in \mathbb{R}^{L^*} : \|\phi\|_1 \le v_1/a_1\}$. Then we know that $\phi_k \in B$, which is compact in $(\mathbb{R}^{L^*}, \|\cdot\|_1)$. Therefore there exists a subsequence ϕ_{k_i} of ϕ_k and $\bar\phi \in \mathbb{R}^{L^*}$ such that $\phi_{k_i} \to \bar\phi$.

Since $c_1^k \to c_1$, $c_2^k \to c_2$ and $\phi_{k_i} \to \bar\phi$, we have $f_{k_i}(\phi_{k_i}) \to f(\bar\phi)$. Since v_k converges to v, it follows that $f_{k_i}(\phi_{k_i}) \to f(\phi_0)$ (note that $v_{k_i} = f_{k_i}(\phi_{k_i})$ and $v = f(\phi_0)$), and therefore $f(\bar\phi) = f(\phi_0)$. Since $A\phi_{k_i} = b$ for all i, we have $A\bar\phi = b$. It follows from the uniqueness of the solution of (16) that $\bar\phi = \phi_0$. Therefore we have established that $\phi_{k_i} \to \phi_0$. It also follows from the uniqueness of the solution of (16) that $\phi_k \to \phi_0$. This proves the theorem. □

Conclusions

In this paper we have solved the problem of minimizing a linear positive combination of the l_1 norm and the square of the $\mathscr{H}_2$ norm of the closed-loop transfer function for discrete-time SISO feedback systems. The unique solution of the problem is readily obtained by a quadratic programming problem of a priori determined dimension. Also, continuity of the solution with respect to the linear combination coefficients of the problem has been established.

The main tool for the development in the paper is the Lagrange duality theory. Such a tool can also be used for MIMO problems, for which the above SISO results seem to have natural extensions. This remains the subject of current research.

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