Controller Design to Optimize a Composite Performance Measure *

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Abstract

This paper studies a "mixed" objective problem of minimizing a composite measure of the ℓ_1, $\mathcal{H}_2$, and ℓ_∞ norms together with the ℓ_∞ norm of the step response of the closed loop . This performance index can be used to generate Pareto optimal solutions with respect to the individual measures. The problem is analysed for the discrete time, single-input single-output (SISO), linear time invariant systems. It is shown via the Lagrange duality theory that the problem can be reduced to a convex optimization problem with *a priori* known dimension. In addition, continuity of the unique optimal solution with respect to changes in the coefficients of the linear combination is established.

Keywords: duality theory, ℓ_1 optimization, multiobjective control.

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1 Notation

The following notation is employed in this paper:

$\|x\|_1$	The 1-norm of the the vector $x \in R^n$.
$\|x\|_2$	The 2-norm of the the vector $x \in R^n$.
$\hat{x}(\lambda)$	The λ transform of a right sided real sequence $x = (x(k))_{k=0}^{\infty}$ defined as $\hat{x}(\lambda) := \sum_{k=0}^{k=\infty} x(k)\lambda^k$.
ℓ_1	The Banach space of right sided absolutely summable real sequences with the norm given by $\parallel x \parallel_1 := \sum_{k=0}^{\infty} \|x(k)\|$.
ℓ_∞	The Banach space of right sided, bounded sequences with the norm given by $\parallel x \parallel_\infty := sup_k \|x(k)\|$.
c_0	The subspace of ℓ_∞ with elements x that satisfy $\lim_{k\to\infty} x(k) = 0$.
ℓ_2	The Banach space of right sided square summable sequences with the norm given by $\parallel x \parallel_2 := \left[\sum_{k=0}^{k=\infty} x(k)^2\right]^{\frac{1}{2}}$.
$\mathcal{H}_2$	The isometric isomorphic image of ℓ_2 under the λ transform $\hat{x}(\lambda)$ wih the norm given by $\parallel \hat{x}(\lambda) \parallel_2 = \parallel x \parallel_2$.
X^*	The dual space of the Banach space X. $< x, x^* >$ denotes the value of the bounded linear functional x^* at $x \in X$.
$W(X^*, X)$	The weak star topology on X^* induced by X.
T^*	The adjoint operator of $T : X \to Y$ which maps Y^* to X^*.

We have from functional analysis that $(\ell_1)^* = \ell_\infty$, $(c_0)^* = \ell_1$, $(\ell_2)^* = \ell_2$.

2 Introduction

Consider the standard feedback configuration of Figure 1 and let ϕ be the closed loop transfer function which maps the exogenous input w to the regulated output z. The aim in feedback control is to design a controller K such that the system is internally stable and the closed loop map between w and z meets certain specifications. In many cases the objective is to determine a controller K which minimizes a relevant norm of the closed loop over all stabilizing controllers. For example, in the standard $\mathcal{H}_2$ problem the norm being minimized is the $\mathcal{H}_2$ norm of the map ϕ. This is a measure of the variance in the regulated output z for a white noise input w. This problem is studied in detail in [7]. In the standard ℓ_1 problem the design of an internally stabilizing controller such that the ℓ_∞ norm of the regulated output z due to a worst case ℓ_∞ bounded disturbance w, is addressed. It is shown in [3] that for the 1-block case, the problem reduces to solving a finite dimensional linear program. Another relevant measure can be the ℓ_∞ norm of of the closed loop response to specific inputs w (e.g., steps, impulses, sinusoids, etc.), i.e., the maximum magnitude of the regulated output z to a specific w. Such type of problems have been considered in [15] where approximate solutions within any apriori established tolerance are obtained via linear programming.

In several situations however, it becomes necessary to consider different measures of the regulated output together since, it is well known that a controller that gives good performance with respect to any particular measure may not guarantee good performance

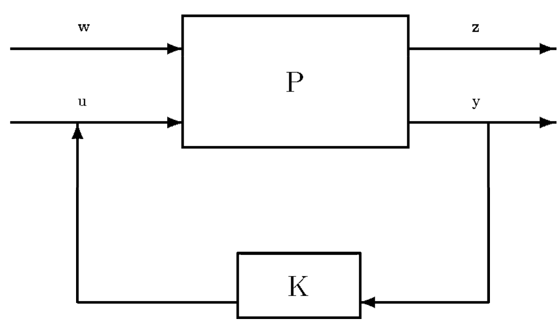

Figure 1: Plant Controller Configuration

with respect to some other measure (e.g., [2]). Thus, a "mixed" type of performance index is of importance. In recent years results were obtained for problems incorporating mixed performance indices. Many state space results are available on the interplay of the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norms of the closed loop (e.g., [6, 13]). In [4, 9] the problem of minimizing the ℓ_1 norm of the closed loop while keeping its $\mathcal{H}_\infty$ norm below a prespecified level is considered. It is shown that approximate solutions can be obtained via finite dimensional convex optimization. In [10] it is shown that approximate solutions to a continuous time problem of minimizing the maximum magnitude of z due to a specified input w while keeping the $\mathcal{H}_\infty$ norm of the closed loop below a given level can be obtained by solving a finite dimensional convex constrained optimization problem and a standard unconstrained $\mathcal{H}_\infty$ problem.

A wide variety of mixed objective optimization problems can be reduced to convex optimization problems [5]. However, these problems are in general infinite dimensional and only approximate solutions can be obtained by using methods given in [5]. In addition, it may be hard to check how close to optimal the approximation is. Hence, it is only appropriate to exploit as much structure in the problem as possible so that certain properties of the problem at hand are revieled and computation of solutions becomes more efficient. In particular, it is important to identify classes of problems that, although initially casted as infinite dimensional, they are in fact finite dimensional. In this case it is also helpful to give, if possible, *a priori* bounds on the dimension of the problem. Within this context, several results on multiobjective (mixed) optimization involving the ℓ_1 norm are becoming available. In [8] the problem of minimizing $\mathcal{H}_2$ norm of the closed loop while keeping its one norm below a prespecified level is reduced to a finite dimensional quadratic optimization problem. In [11] the converse problem of minimizing the one norm of the closed loop while constraining its two norm to lie below a prespecified level is considered. It is shown that the problem reduces to a finite dimensional convex optimization problem with an *a priori* known dimension.

In a similar vein, we consider in this paper the problem of minimizing a given linear combination of the ℓ_1 norm, the square of the $\mathcal{H}_2$ norm, and the ℓ_∞ norms of the step and pulse responses respectively of the closed loop over all stabilizing controllers. This performance index can be associated with the performance to inputs that are deterministic worst-case amplitude-bounded (hence the ℓ_1-norm), random (hence the $\mathcal{H}_2$-norm), and, fixed steps and pulses (hence the ℓ_∞ norms of the step and pulse responses). The problem is reduced to a finite dimensional convex optimization with *a priori* known dimension which can be readily solved via standard numerical algorithms. Its solution, represents a Pareto optimal point with respect to the individual measures involved. Continuity of the unique optimal solution with respect to changes in the coefficients of the linear combination is also established.

The paper is organized as follows. In section 3 the statement of the problem is made precise and its relation to Pareto optimality is established. In section 4 it is shown that the problem has a unique solution which has a finite impulse response. The problem is then reduced to a finite dimensional convex optimization problem. In section 5 an example is given to illustrate the theory developed. In section 6 continuity properties of the optimal solution are studied. Finally, we conclude in section 7.

3 Problem Formulation

Consider Figure 1 where P and K are the plant and the controller respectively. Let w represent the exogenous input, z represent the output of interest, y is the measured output and u is the control input where z and w are assumed scalar. Let ϕ be the closed loop map which maps $w \to z$. From Youla parametrization [14] it is known that all achievable closed loop maps under stabilizing controllers are given by $\phi = h - u * q$ ($*$ denotes convolution), where $h, u, q \in \ell_1$; h, u depend only on the plant P and q is a free parameter in ℓ_1. Throughout the paper we make the following assumption:

Assumption 1 *All the zeros of $\hat{u}$ (the λ transform of u) inside the unit disc are real and distinct. Also, $\hat{u}$ has no zeros on the unit circle.*

The assumption that all zeros of $\hat{u}$ which are inside the open unit disc are real and distinct is not restrictive and is made to streamline the presentation of the paper. Let the zeros of

$\hat{u}$ which are inside the unit disc be given by $z_1, z_2, \ldots, z_n$. Let

$$\Phi := \{\phi : \text{ there exists } q \in \ell_1 \text{ with } \phi = h - u * q\}.$$

Φ is the set of all achievable closed loop maps under stabilizing controllers. Let $A : \ell_1 \to R^n$ be given by

$$A = \begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \ldots \\ 1 & z_2 & z_2^2 & z_2^3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_n & z_n^2 & z_n^3 & \ldots \end{pmatrix},$$

and $b \in R^n$ be given by

$$b = \begin{pmatrix} \hat{h}(z_1) \\ \hat{h}(z_2) \\ \vdots \\ \hat{h}(z_n) \end{pmatrix}.$$

Theorem 1 *The following is true:*

$$\begin{aligned} \Phi &= \{\phi \in \ell_1 : \hat{\phi}(z_i) = \hat{h}(z_i) \text{ for all } i = 1, \ldots, n\} \\ &= \{\phi \in \ell_1 : A\phi = b\}. \end{aligned}$$

Proof: Given in [2]. ■

Let w_1 be the unit step input i.e., $w_1 = (1, 1, \ldots)$. Let the corresponding output be denoted as z_1 i.e., $z_1(i) = (\phi * w_1)(i) = \sum_{k=0}^{i} \phi(i-k) w_1(k) = \sum_{k=0}^{i} \phi(k)$. Then the problem of interest can be stated as:

Given $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$ obtain a solution to the following mixed objective problem:

$$\begin{aligned} \nu &:= \inf_{\phi \text{ Achievable}} \{c_1 \|\, \phi \,\|_1 + c_2 \|\, \phi \,\|_2{}^2 + c_3 \|\, \phi * w_1 \,\|_\infty + c_4 \|\, \phi \,\|_\infty \} \\ &= \inf_{\phi \in \ell_1,\ A\phi = b} \{c_1 \|\, \phi \,\|_1 + c_2 \|\, \phi \,\|_2{}^2 + c_3 \|\, \phi * w_1 \,\|_\infty + c_4 \|\, \phi \,\|_\infty \}. \end{aligned} \tag{1}$$

We define $f : \ell_1 \to R$ by,

$$f(\phi) := c_1 \|\, \phi \,\|_1 + c_2 \|\, \phi \,\|_2{}^2 + c_3 \|\, \phi * w_1 \,\|_\infty + c_4 \|\, \phi \,\|_\infty,$$

which is the objective functional in the optimization given by (1).

In the following sections we will study the existence, structure and computation of the optimal solution. Before we initiate our study towards these goals it is worthwhile to point out certain connections between the cost under consideration and the notion of Pareto optimality.

3.1 Relation to Pareto Optimality

The notion of Pareto optimality can be stated as follows (see for example, [5]). Given a set of m nonnegative functionals $\overline{f}_i$, $i = 1, \ldots, m$ on a normed linear space X, a point $x_0 \in X$ is Pareto optimal with respect to the vector valued criterion $\overline{f} := (\overline{f}_1, \ldots, \overline{f}_m)$ if there does not exist any $x \in X$ such that

$$\overline{f}_i(x) \leq \overline{f}_i(x_0) \ \forall i \in \{1, \ldots, m\} \ \text{ and } \ \overline{f}_i(x) < \overline{f}_i(x_0) \text{ for some } i \in \{1, \ldots, m\}.$$

Under certain conditions the set of all Pareto optimal solutions can be generated by solving a minimization of weighted sum of the functionals as the following theorem indicates.

Theorem 2 *[12] Let X be a normed linear space and each nonnegative functional $\overline{f}_i$ be convex. Also let*

$$S_m := \{c \in R^m : c_i \geq 0, \ \sum_{i=1}^{m} c_i = 1\},$$

and for each $c \in R^m$ consider the following scalar valued optimization:

$$\inf_{x \in X} \sum_{i=1}^{m} c_i \overline{f}_i(x).$$

If $x_0 \in X$ is Pareto optimal with respect to the vector valued criterion $\overline{f}(x)$, then there exists some $c \in S_m$ such that x_0 solves the above minimization. Conversely, given $c \in S_m$, if the above minimization has at most one solution x_0 then x_0 is Pareto optimal with respect to $\overline{f}(x)$. ■

In the next section we show that there is a unique solution ϕ_0 to Problem (1). Furthermore, since u is assumed to be a scalar, there is a unique optimal $q \in \ell_1$. Hence, in view of the aforementioned theorem we have that if we restrict our attention to parameters c_1, c_2, c_3 and

c_4 such that $(c_1, c_2, c_3, c_4) \in \Sigma_4 := \{(c_1, c_2, c_3, c_4) : c_1 + c_2 + c_3 + c_4 = 1,\ c_1, c_2, c_3, c_4 > 0\}$, we will produce a set of Pareto optimal solutions with respect to the vector valued function

$$\begin{aligned} \overline{f}(q) &:= (\| h - u * q \|_1, \| h - u * q \|_2{}^2, ||(h - u * q) * w_1||_\infty, \| h - u * q \|_\infty) \\ &=: (\overline{f}_1(q), \overline{f}_2(q), \overline{f}_3(q), \overline{f}_4(q)). \end{aligned}$$

where $q \in \ell_1$. Thus, if ϕ_0 is the optimal solution for Problem (1) with a corresponding q_0 for some given $(c_1, c_2, c_3, c_4) \in \Sigma_4$, then there does not exist a preferable alternative ϕ with $\phi = h - u * q$ for some $q \in \ell_1$ such that

$$\overline{f}_i(q) \leq \overline{f}_i(q_0)\ \forall i \in \{1, \ldots, 4\} \text{ and } \overline{f}_i(q) < \overline{f}_i(q_0) \text{ for some } i \in \{1, \ldots, 4\}.$$

As a final note we mention that if (c_1, c_2, c_3, c_4) do not satisfy $c_1+c_2+c_3+c_4 = 1$ then we can define a new set of parameters $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ and $\bar{c}_4$ by $\bar{c}_1 = \frac{c_1}{c_1+c_2+c_3+c_4}$, $\bar{c}_2 = \frac{c_2}{c_1+c_2+c_3+c_4}$, $\bar{c}_3 = \frac{c_3}{c_1+c_2+c_3+c_4}$ and $\bar{c}_4 = \frac{c_4}{c_1+c_2+c_3+c_4}$ with $\bar{c}_1 + \bar{c}_2 + \bar{c}_3 + \bar{c}_4 = 1$. These new parameters would yield the same optimal solution as with (c_1, c_2, c_3, c_4).

4 Existence, Uniqueness and Properties of the Optimal Solution

In the first part of this section we show that Problem (1) always has a solution. In the second part we show that any solution to Problem (1) is a finite impulse response sequence and in the third we give an *a priori* bound on the length.

4.1 Existence of a Solution

Here we show that a solution to (1) always exists. We use the following well-known lemma (see for example [1]) to prove the main result of this subsection.

Lemma 1 *(***Banach Alaoglu***) Let X be a separable Banach space with X^* as its dual then the set $\{x^* : x^* \in X^*, \| x^* \| \leq M\}$ is $W(X^*, X)$ sequentially compact for any $M \in R$.*

Theorem 3 *There exists $\phi_0 \in \Phi$ such that*

$$f(\phi_0) = \inf_{\phi \in \Phi} \{c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 + c_3 \| \phi * w_1 \|_\infty + c_4 \| \phi \|_\infty\},$$

where Φ*:=*$\{\phi \in \ell_1 : A\phi = b\}$*. Therefore the infimum in (1) is a minimum.*

Proof: We denote the feasible set of our problem by $\Phi := \{\phi \in \ell_1 : A\phi = b\}$. Let

$$B := \{\phi \in \ell_1 : c_1\| \phi \|_1 + c_2\| \phi \|_2{}^2 + c_3\| \phi * w_1 \|_\infty + c_4\| \phi \|_\infty \leq \nu + 1\}.$$

It is clear that

$$\nu = \inf_{\phi \in \Phi \cap B} \{c_1\| \phi \|_1 + c_2\| \phi \|_2{}^2 + c_3\| \phi * w_1 \|_\infty + c_4\| \phi \|_\infty\}.$$

Therefore given $i > 0$ there exists $\phi_i \in \Phi \cap B$ such that

$$c_1\| \phi_i \|_1 + c_2\| \phi_i \|_2{}^2 + c_3\| \phi_i * w_1 \|_\infty + c_4\| \phi_i \|_\infty \leq \nu + \frac{1}{i}.$$

Let

$$\overline{B} := \{\phi \in \ell_1 : c_1\| \phi \|_1 \leq \nu + 1\}.$$

$\overline{B}$ is a bounded set in $\ell_1 = c_0^*$. It follows from the Banach Alaoglu lemma that $\overline{B}$ is $W(c_0^*, c_0)$ compact. Using the fact that c_0 is separable and that $\{\phi_i\}$ is a sequence in $\overline{B}$ we know that there exists a subsequence $\{\phi_{i_k}\}$ of $\{\phi_i\}$ and $\phi_0 \in \overline{B}$ such that $\phi_{i_k} \to \phi_0$ in the $W(c_0^*, c_0)$ sense, that is for all v in c_0

$$< v, \phi_{i_k} > \to < v, \phi_0 > \text{ as } k \to \infty. \tag{2}$$

Let the j^{th} row of A be denoted by a_j and the j^{th} element of b be given by b_j . Then as $a_j \in c_0$ we have,

$$< a_j, \phi_{i_k} > \to < a_j, \phi_0 > \text{ as } k \to \infty \text{ for all } j = 1, 2, \ldots, n. \tag{3}$$

As $A(\phi_{i_k}) = b$ we have $< a_j, \phi_{i_k} >= b_j$ for all k and for all j which implies $< a_j, \phi_0 >= b_j$ for all j. Therefore we have $A(\phi_0) = b$ from which it follows that $\phi_0 \in \Phi$. This gives us $c_1\| \phi_0 \|_1 + c_2\| \phi_0 \|_2{}^2 + c_3\| \phi_0 * w_1 \|_\infty + c_4\| \phi_0 \|_\infty \geq \nu$.

From (2) we can deduce that for all t, $\phi_{i_k}(t) \to \phi_0(t)$. An easy consequence of this is that for all N,

$$\sum_{t=0}^{N} |\phi_{i_k}(t)| \to \sum_{t=0}^{N} |\phi_0(t)|, \; \sum_{t=0}^{N} |\phi_{i_k}(t)|^2 \to \sum_{t=0}^{N} |\phi_0(t)|^2, \tag{4}$$

$$\max_{0\le t\le N} |\sum_{j=0}^{t} \phi_{i_k}(j)| \to \max_{0\le t\le N} |\sum_{j=0}^{t} \phi_0(j)| \text{ and } \max_{0\le t\le N} |\phi_{i_k}(t)| \to \max_{0\le t\le N} |\phi_0(t)|. \tag{5}$$

Now, by (4) and (5) we have for all N as $k \to \infty$,

$$\sum_{t=0}^{N}\{c_1|\phi_{i_k}(t)| + c_2(\phi_{i_k}(t))^2\} + c_3 \max_{0\le t\le N} |(\phi_{i_k} * w_1)(t)| + c_4 \max_{0\le t\le N} |\phi_{i_k}(t)| \to$$
$$\sum_{t=0}^{N}\{c_1|\phi_0(t)| + c_2(\phi_0(t))^2\} + c_3 \max_{t\le N} |(\phi_0 * w_1)(t)| + c_4 \max_{0\le t\le N} |\phi_0(t)|. \tag{6}$$

As

$$c_1\|\ \phi_{i_k}\ \|_1 + c_2\|\ \phi_{i_k}\ \|_2{}^2 + c_3\|\ \phi_{i_k} * w_1\ \|_\infty + c_4\|\ \phi_{i_k}\ \|_\infty) \le \nu + \frac{1}{i_k}$$

we have that for all N

$$\sum_{t=0}^{N}\{c_1|\phi_{i_k}(t)| + c_2(\phi_{i_k}(t))^2\} + c_3 \max_{t\le N} |(\phi_{i_k} * w_1)(t)| + c_4 \max_{0\le t\le N} |\phi_{i_k}(t)| \le \nu + \frac{1}{i_k}. \tag{7}$$

Letting $k \to \infty$ in (7) and using (6) we have that for all N

$$\sum_{t=0}^{N}\{c_1|\phi_0(t)| + c_2(\phi_0(t))^2\} + c_3 \max_{t\le N} |(\phi_0 * w_1)(t)| + c_4 \max_{0\le t\le N} |\phi_0(t)| \le \nu.$$

By letting $N \to \infty$ in the above inequality we conclude that $c_1\|\ \phi_0\ \|_1 + c_2\|\ \phi_0\ \|_2{}^2 + c_3\|\ \phi_0 * w_1\ \|_\infty + c_4\|\ \phi_0\ \|_\infty \le \nu$. This proves the theorem. ■

4.2 Structure of Optimal Solutions

In this subsection we use a Lagrange duality result to show that every optimal solution is of finite length. First we give the following definitions, where we denote the interior of a set by *int*.

Definition 1 *Let P be a convex cone in a vector space X. We write $x \ge y$ if $x - y \in P$. We write $x > 0$ if $x \in int(P)$. Similarly $x \le y$ if $x - y \in -P := N$ and $x < 0$ if $x \in int(N)$.*

Definition 2 *Let X be a vector space and Z be a vector space with positive cone P. A mapping $G : X \to Z$ is convex if $G(tx + (1-t)y) \le tG(x) + (1-t)G(y)$ for all $x \ne y$ in X and t with $0 \le t \le 1$. It is strictly convex if $G(tx + (1-t)y) < tG(x) + (1-t)G(y)$ for all $x \ne y$ in X and t with $0 < t < 1$.*

The following is a Lagrange duality theorem.

Theorem 4 *[1] Let X be a Banach space, Ω be a convex subset of X, Y be a finite dimensional space, Z be a normed space with positive cone P. Let $f : \Omega \to R$ be a real valued convex functional, $g : X \to Z$ be a convex mapping, $H : X \to Y$ be an affine linear map and $0 \in int[range(H)]$. Define*

$$\mu_0 := inf\{f(x) : \; g(x) \leq 0, \; H(x) = 0, \; x \in \Omega\}. \tag{8}$$

Suppose there exists $x_1 \in \Omega$ such that $g(x_1) < 0$ and $H(x_1) = 0$ and suppose μ_0 is finite. Then,

$$\mu_0 = max\{\varphi(z^*, y) : z^* \geq 0, \; z^* \in Z^*, \; y \in Y\}, \tag{9}$$

where $\varphi(z^*, \; y) := inf \{ \; f(x) + < g(x), z^* > + < H(x), y >: x \in \Omega \; \}$ and the maximum is achieved for some $z_0^* \geq 0$, $z_0^* \in Z^*$, $y_0 \in Y$.

Furthermore if infimum in (8) is achieved by some $x_0 \in \Omega$ then

$$< g(x_0), z_0^* > + < H(x_0), y_0 >= 0, \tag{10}$$

and

$$x_0 \; minimizes \; f(x) + < g(x), z_0^* > + < H(x), y_0 >, \quad over \; all \; x \in \Omega. \tag{11}$$

We refer to (8) as the **Primal** problem and to (9) as the **Dual** problem. Based on the above result the following can be established in relation to our problem at hand.

Lemma 2

$$\nu = \max_{y \in R^n} \inf_{\phi \in \ell_1} \{f(\phi) + < b - A\phi, y >\}. \tag{12}$$

Proof: We will apply Theorem 4 to get the result. Let X, Ω, Y, Z in Theorem 4 correspond to ℓ_1, ℓ_1, R^n, R respectively. Let $\gamma := \nu + 1$, $g(\phi) := f(\phi) - \gamma$ and $H(\phi) := b - A\phi$. With this notation we have $Z^* = R$. A has full range which implies $0 \in int[range(H)]$. ¿From Theorem 3 we know that there exists ϕ_0 such that $g(\phi_0) = f(\phi_0) - \gamma = -1 < 0$ and

$H(\phi_0) = 0$. Therefore all the conditions of Theorem 4 are satisfied. From Theorem 4 we have

$$\nu = \max_{z \geq 0, y \in R^n} \inf_{\phi \in \ell_1} \{f(\phi) + < g(\phi), z > + < b - A\phi, y >\}.$$

Let $z_0 \in R$, $y_0 \in R^n$ be a maximizing solution to the right hand side of the above equation. ϕ_0 being the solution of the primal we have from (10) that $< g(\phi_0), z_0 > + < H(\phi_0), y_0 >=$ 0 which implies that $< g(\phi_0), z_0 >= 0$. As $g(\phi_0) \neq 0$ we conclude that $z_0 = 0$. This proves the theorem. ■

The following theorem shows that the solution to (1) is unique and that it is a finite impulse response sequence.

Theorem 5 *Define $\mathcal{T}$:=$\{\phi \in \ell_1 :$ there exists L^* with $\phi(i) = 0$ if $i \geq L^*\}$. The following is true:*

$$\nu = \max_{y \in R^n} \inf_{\phi \in \mathcal{T}} \{f(\phi) - < \phi, v > + < b, y >\}, \tag{13}$$

where $v(i) := (A^* y)(i)$. Also, the solution to the primal (1) is unique and the solution belongs to $\mathcal{T}$.

Proof: Let $y_0 \in R^n$ be the solution to the right hand side of (13). Define $v_0 := (A^* y_0)(i)$ and let

$$J(\phi) := f(\phi) - < \phi, v_0 > + < b, y_0 > .$$

It is immediate that

$$\nu = \inf_{\phi \in \ell_1} \sum_{i=0}^{\infty} \{c_1 |\phi(i)| + c_2 (\phi(i))^2 - \phi(i) v_0(i)\} + c_3 \sup_i |(\phi * w_1)(i)| + c_4 \sup_i |\phi(i)| + < b, y_0 > .$$

As v_0 is in ℓ_1 we know that there exists L^* such that $v_0(i)$ satisfies $|v_0(i)| < c_1$ if $i \geq L^*$. We now show that for ϕ to be optimal it is necessary that $\phi(i) = 0$ for all $i \geq L^*$. Indeed, if $\phi(i) \neq 0$ while $|v_0(i)| < c_1$ for some i, note that,

$$c_1 |\phi(i)| + c_2 (\phi(i))^2 - \phi(i) v_0(i) > c_1 |\phi_1(i)| + c_2 (\phi_1(i))^2 - \phi_1(i) v_0(i),$$

for any $\phi_1 \in \ell_1$ such that $\phi_1(i) = 0$. Moreover, if ϕ_1 is such that $\phi_1(j) = \phi(j)$ whenever $j < L^*$ and $\phi_1(j) = 0$ whenever $j \geq L^*$ it follows that

$$\sup_i |\sum_{j=0}^{i} \phi(j)| \geq \sup_i |\sum_{j=0}^{i} \phi_1(j)| \text{ and } \sup_i |\phi(i)| \geq \sup_i |\phi_1(i)|$$

or equivalently,

$$\| \phi * w_1 \|_\infty \geq \| \phi_1 * w_1 \|_\infty \text{ and } \| \phi \|_\infty \geq \| \phi_1 \|_\infty .$$

Hence, we have that $J(\phi) > J(\phi_1)$ which proves our claim. In Theorem 3 we showed that there exists a solution ϕ_0 to the primal (1). From Theorem 4 we know that ϕ_0 is a solution to $\inf_{\phi \in \ell_1} J(\phi)$. As $J(\phi)$ is strictly convex in ϕ we conclude that the solution to the primal (1) is unique. From the previous discussion it follows that the solution to the primal (1), ϕ_0 is in $\mathcal{T}$. This proves the theorem. ∎

4.3 *A priori* Bound on the Length of the Optimal Solution

In this section we give an *a priori* bound on the length of the solution to (1). First we establish the following two lemmas.

Lemma 3 *Let ϕ_0 be a solution of the primal (1). Let y_0 represent a corresponding dual solution as obtained in (13). Let $v_0 := A^* y_0$ then, $\| v_0 \|_\infty \leq \alpha$ where $\alpha = c_1 + c_3 + c_4 + 2\frac{c_2}{c_1} f(h)$.*

Proof: From the proof of Theorem 5 it is clear that ϕ_0 should be such that it minimizes

$$J(\phi) := \sum_{i=0}^{L^*} \{c_1 |\phi(i)| + c_2 (\phi(i))^2 - \phi(i) v_0(i)\} + c_3 \max_{i \leq L^*} |\sum_{j=0}^{i} \phi(j)| + c_4 \max_{i \leq L^*} |\phi(i)|,$$

where L^* is such that $|v_0(i)| < c_1$ if $i \geq L^*$.

Let i be any integer such that $i \leq L^*$. Consider perturbation ϕ of ϕ_0 given as $\phi(i) = \phi_0(i) + \epsilon$ and $\phi(j) = \phi_0(j)$ for $j \neq i$. Then, for all ϵ, it can be shown that

$$\max_{0 \leq t \leq L^*} |\sum_{j=0}^{t} \phi(j)| - \max_{0 \leq t \leq L^*} |\sum_{j=0}^{t} \phi_0(j)| \leq |\epsilon| \tag{14}$$

and

$$\max_{0\le t\le L^*}|\phi(t)| - \max_{0\le t\le L^*}|\phi_0(t)| \le |\epsilon|. \qquad (15)$$

Indeed, assume that

$$\max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi_0(j)| = |\sum_{j=0}^{N}\phi_0(j)| \text{ for some } N \le L^*.$$

For the given ϵ let

$$\max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi(j)| = |\sum_{j=0}^{M}\phi(j)| \text{ for some } M \le L^*. \qquad (16)$$

If $M \le i$ then

$$\begin{aligned}\max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi(j)| - \max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi_0(j)| &= |\phi_0(0)+\ldots+\phi_0(M)| - |\phi_0(0)+\ldots+\phi_0(N)| \\ &\le 0 \le |\epsilon|\end{aligned}$$

and if $M > i$ then

$$\begin{aligned}\max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi(j)| - \max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi_0(j)| &= |\phi_0(0)+\ldots+(\phi_0(i)+\epsilon)+\ldots+\phi_0(M)| \\ &\quad -|\phi_0(0)+\ldots+\phi_0(N)| \\ &\le |\phi_0(0)+\ldots+\phi_0(M)| + |\epsilon| - |\phi_0(0)+\ldots+\phi_0(N)| \\ &\le |\epsilon|.\end{aligned}$$

(15) can be proved easily. It follows easily from (14) and (15) that

$$\begin{aligned}J(\phi)-J(\phi_0) &= c_1(|\phi_0(i)+\epsilon| - |\phi_0(i)|) + c_2(\epsilon^2+2\epsilon\phi_0(i)) \\ &\quad + c_3\max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi(j)| - c_3\max_{0\le t\le L^*}|\sum_{j=0}^{t}\phi_0(j)|) \\ &\quad + c_4(\max_{0\le t\le L^*}|\phi(t)| - \max_{0\le t\le L^*}|\phi_0(t)|) - \epsilon v_0(i) \\ &\le c_1|\epsilon| + c_2(\epsilon^2+2\epsilon\phi_0(i)) + c_3|\epsilon| + c_4|\epsilon| - \epsilon v_0(i).\end{aligned}$$

As ϕ_0 is the unique minimum we have that $J(\phi)-J(\phi_0) > 0$ and therefore it follows that

$$c_1|\epsilon| + c_2(\epsilon^2+2\epsilon\phi_0(i)) + c_3|\epsilon| + c_4|\epsilon| - \epsilon v_0(i) > 0 \text{ for all } \epsilon.$$

Dividing both sides of the above inequality by $|\epsilon|$ we get

$$c_1+c_3+c_4+c_2|\epsilon| + 2c_2\frac{\epsilon}{|\epsilon|}\phi_0(i) - \frac{\epsilon}{|\epsilon|}v_0(i) > 0 \text{ for all } \epsilon.$$

Letting $\epsilon \to 0^+$ and $\epsilon \to 0^-$ in the above inequality we have

$$v_0(i) \leq c_1 + c_3 + c_4 + 2c_2|\phi_0(i)|$$

and

$$-v_0(i) \leq c_1 + c_3 + c_4 - 2c_2|\phi_0(i)| \leq c_1 + c_3 + c_4 + 2c_2|\phi_0(i)|,$$

respectively. This implies that

$$|v_0(i)| \leq c_1 + c_3 + c_4 + 2c_2|\phi_0(i)| \leq c_1 + c_3 + c_4 + 2c_2\| \phi_0 \|_1$$

As this holds for any $i \leq L^*$ we have

$$\| v_0 \|_\infty \leq c_1 + c_3 + c_4 + 2\frac{c_2}{c_1} f(\phi_0) \leq c_1 + c_3 + c_4 + 2\frac{c_2}{c_1} f(h),$$

where we have used that $2c_2\| \phi_0 \|_1 \leq 2\frac{c_2}{c_1} f(\phi_0) \leq c_1 + c_3 + c_4 + 2\frac{c_2}{c_1} f(h)$, and, $f(\phi_0) \leq f(h)$ since h is feasible $(q = 0)$ but not necessarily optimal. This proves the lemma. ∎

Lemma 4 *[2] If $y \in R^n$ is such that $\| A^* y \|_\infty \leq \alpha$ then there exists a positive integer L^* independent of y such that $|(A^* y)(i)| < c_1$ for all $i \geq L^*$.*

Proof: Define

$$A_L^* = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ z_1 & z_2 & z_3 & \dots & z_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^L & z_2^L & z_3^L & \dots & z_n^L \end{pmatrix},$$

$A_L^* : R^n \to R^{L+1}$. With this definition we have $A_\infty^* = A^*$. Let $y \in R^n$ be such that $\| A^* y \|_\infty \leq \alpha$. Choose any L such that $L \geq (n-1)$. As $z_i, i = 1, \dots, n$ are distinct A_L^* has full column rank. A_L^* can be regarded as a linear map taking $(R^n, \| . \|_1) \to (R^{L+1}, \| . \|_\infty)$. As A_L^* has full column rank we can define the left inverse of A_L^*, $(A_L^*)^{-l}$ which takes $(R^{L+1}, \| . \|_\infty) \to (R^n, \| . \|_1)$. Let the induced norm of $(A_L^*)^{-l}$ be given by $\| (A_L^*)^{-l} \|_{\infty,1}$. $y \in R^n$ is such that $\| A^* y \|_\infty \leq \alpha$ and therefore $\| A_L^* y \|_\infty \leq \alpha$. It follows that,

$$\| y \|_1 \leq \| (A_L^*)^{-l} \|_{\infty,1} \| A_L^* y \|_\infty \leq \| (A_L^*)^{-l} \|_{\infty,1} \alpha. \tag{17}$$

Choose L^* such that

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \ \| (A_L^*)^{-l} \|_{\infty,1} \alpha < c_1. \tag{18}$$

There always exists such an L^* because $|z_k| \ < 1$ for all $k = 1, \ldots, n$. Note that L^* does not depend on y. For any $i \geq L^*$ we have

$$\begin{aligned} |(A^* y)(i)| &= |\sum_{k=1}^{k=n} z_k^i y(k)| \leq \max_{k=1,\ldots,n} |z_k|^i \| y \|_1 \\ &\leq \max_{k=1,\ldots,n} |z_k|^i \ \| (A_L^*)^{-l} \|_{\infty,1} \alpha \\ &\leq \max_{k=1,\ldots,n} |z_k|^{L^*} \ \| (A_L^*)^{-l} \|_{\infty,1} \alpha. \end{aligned}$$

The second inequality follows from 17. From 18 we have $|(A^* y)(i)| < c_1$ if $i \geq L^*$. This proves the lemma. ■

We now summarize the main result of the section

Theorem 6 *The unique solution ϕ_0 of the primal (1) is such that $\phi(i) = 0$ if $i \geq L^*$ where L^* given in Lemma 4 can be determined a priori.*

Proof: Let y_0 be the dual solution to (1) and let $v_0 := A^* y_0$. From Lemma 3 we know that $\| v_0 \|_\infty \leq \alpha$ where $\alpha = c_1 + c_3 + c_4 + 2\frac{c_2}{c_1} f(h)$. Applying Lemma 4 we conclude that there exists L^* (which can be determined apriori) such that $|v_0(i)| < c_1$ if $i \geq L^*$. Therefore, $\phi_0(i) = 0$ if $i \geq L^*$. This proves the theorem. ■

The above theorem shows that the Problem (1) is a finite dimensional convex minimization problem. Such problems can be solved efficiently using standard numerical methods.

At this point we would like to make a few remarks. It should be clear that the uniqueness property of the optimal solution is due the the non-zero coefficient c_2. This makes the problem strictly convex. The finite impulse response property of the optimal solution is due to the nonzero c_1. Also, it should be noted that in the case where c_3 and/or c_4 are allowed to be zero, all of the previous results apply by setting respectively c_3 and/or c_4 to zero in the appropriate expressions for the upper bounds.

5 An Example

In this section we illustrate the theory developed in the previous sections with an example taken from [8]. Consider the SISO plant,

$$\hat{P}(\lambda) = \lambda - \frac{1}{2}, \tag{19}$$

where we are interested in the sensitivity map $\phi := (I - PK)^{-1}$. Using Youla parametrization we get that all achievable transfer functions are given by $\hat{\phi} = (I - \hat{P}\hat{K})^{-1} = 1 - (\lambda - \frac{1}{2})\hat{q}$ where $\hat{q}$ is a stable map. Therefore, $h = 1$ and $u = \lambda - \frac{1}{2}$. The matrix A and b are given by

$$A = (1, \frac{1}{2}, \frac{1}{2^2}, \ldots), \quad b = 1.$$

We consider the case where $c_1 = 1, c_2 = 1, c_3 = 1$ and $c_4 = 1$. Therefore,

$$\alpha = c_1 + c_3 + c_4 + 2\frac{c_2}{c_1}(c_1\| \ h \ \|_1 + c_2\| \ h \ \|_2{}^2 + c_3||h * w_1||_\infty + c_4\| \ \phi \ \|_\infty) = 11.$$

For this example $n = 1$ and $z_1 = \frac{1}{2}$. L^* the *a priori* bound on the length of the optimal is chosen to satisfy

$$\max_{k=1,\ldots,n} |z_k|^{L^*} \ \ \| \ (A_L^*)^{-l} \ \|_{\infty,1} \ \alpha < c_1, \tag{20}$$

where L is any positive integer such that $L \geq (n-1)$. We choose $L = 0$ and therefore $A_L = 1$ and $\| \ (A_L^*)^{-l} \ \|_{\infty,1} = 1$. We choose $L^* = 4$ which satisfies (20). Therefore, the optimal solution ϕ_0 satisfies $\phi_0(i) = 0$ if $i \geq 4$. The problem reduces to the following finite dimensional convex optimization problem:

$$\nu = \min_{A_{L^*}\phi = 1} \{\sum_{k=0}^{3}(|\phi(k)| + (\phi(k))^2) + \max_{0 \leq k \leq 3} |(\phi * w_1)(k)| + \max_{0 \leq k \leq 3} |\phi(k)| : \phi \in R^4\},$$

where $A_{L^*} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8})$. We obtain (using Matlab Optimization Toolbox) the optimal solution ϕ_0 to be:

$$\hat{\phi}_0(\lambda) = 0.9 + 0.2\lambda.$$

6 Continuity of the Optimal Solution

In this section we show that the optimal is continuous with respect to changes in the parameters c_1, c_2, c_3 and c_4. First, we prove the following lemma:

Lemma 5 *Let $\{f_k\}$ be a sequence of functions which map R^m to R. If f_k converges uniformly to a function f on a set $S \subset R^m$ then*

$$\lim_{k \to \infty} \min_{x \in S} f_k(x) = \min_{x \in S} f(x),$$

provided that the minima exist.

Proof : Let $\min_{x \in S} f(x) = f(x_0)$ for some $x_0 \in S$. Given $\epsilon > 0$ we know from convergence of the sequence $\{f_k\}$ to f that there exists an integer K such that if $k > K$ then

$$|f_k(x_0) - f(x_0)| < \epsilon,$$

$$\Rightarrow f_k(x_0) < \epsilon + f(x_0),$$

$$\Rightarrow \min_{x \in S} f_k(x) < \epsilon + f(x_0),$$

$$\Rightarrow \lim_{k \to \infty} \min_{x \in S} f_k(x) < \epsilon + f(x_0).$$

As ϵ is arbitrary we have $\lim_{k \to \infty} \min_{x \in S} f_k(x) \leq f(x_0)$. Now we prove the other inequality. Given $\epsilon > 0$ we know that there exists an integer K such that if $k > K$ then

$$|f_k(x) - f(x)| < \epsilon \text{ for any } x \in S$$

$$\Rightarrow f_k(x) > f(x) - \epsilon \geq f(x_0) - \epsilon \text{ for any } x \in S$$

$$\Rightarrow \min_{x \in S} f_k(x) > f(x_0) - \epsilon$$

$$\Rightarrow \lim_{k \to \infty} \min_{x \in S} f_k(x) > f(x_0) - \epsilon.$$

As ϵ is arbitrary we have $\lim_{k \to \infty} \min_{x \in S} f_k(x) \geq f(x_0)$. This proves the lemma ■

Theorem 7 *Let $c_1^k \in [a_1, b_1]$, $c_2^k \in [a_2, b_2]$, $c_3^k \in [a_3, b_3]$ and $c_4^k \in [a_4, b_4]$ where $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$. Let ϕ_k be the unique solution to the problem*

$$\nu_k := \min_{A\phi=b} c_1^k \| \phi \|_1 + c_2^k \| \phi \|_2{}^2 + c_3^k \| \phi * w_1 \|_\infty + c_4^k \| \phi \|_\infty, \tag{21}$$

and let ϕ_0 be the solution to the problem

$$\nu := \min_{A\phi=b} c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 + c_3 \| \phi * w_1 \|_\infty + c_4 \| \phi \|_\infty, \tag{22}$$

If $c_1^k \to c_1$, $c_2^k \to c_2$, $c_3^k \to c_3$ and $c_4^k \to c_4$ then $\phi_k \to \phi_0$.

Proof : We prove this theorem in three parts; first we show that we can restrict the proof to a finite dimensional space, second we show that $\nu_k \to \nu$ and finally we show that $\phi_k \to \phi_0$. Let y_k represent the dual solution of (21) and let $v_k := A^* y_k$. Let $f_k(\phi) := c_1^k \| \phi \|_1 + c_2^k \| \phi \|_2{}^2 + c_3^k \| \phi_k * w_1 \|_\infty + c_4^k \| \phi \|_\infty$ and $f(\phi) := c_1 \| \phi \|_1 + c_2 \| \phi \|_2{}^2 + c_3 \| \phi * w_1 \|_\infty + c_4 \| \phi \|_\infty$. Let α_k the upper bound on $\| v_k \|_\infty$ be as given by Lemma 3. Therefore,

$$\begin{aligned} \alpha_k &= c_1^k + c_3^k + c_4^k + 2\tfrac{c_2^k}{c_1^k} f_k(h) \\ &\leq b_1 + b_3 + b_4 + 2b_2(\| h \|_1 + \tfrac{b_2}{a_1} \| h \|_2{}^2 + \tfrac{b_3}{a_1} \| h * w_1 \|_\infty + \tfrac{b_4}{a_1} \| h \|_\infty). \end{aligned}$$

Let this bound be denoted by d. Choose L^* such that

$$\max_{i=1,\ldots,n} |z_i|^{L^*} \; \| (A_L^*)^{-l} \|_{\infty,1} \, d < a_1.$$

where L is such that $L \geq (n-1)$. Therfore, it follows that

$$\max_{i=1,\ldots,n} |z_i|^{L^*} \; \| (A_L^*)^{-l} \|_{\infty,1} \, \alpha_k < c_1^k.$$

for all k. From arguments similar to that of Lemma 4 and Theorem 6 it follows that $\phi_k(i) = 0$ if $i \geq L^*$ for all k. Therefore we can assume that $\phi_k \in R^{L^*}$.

Now, we prove that $\nu_k \to \nu$. Let ϕ_1 be the solution of the problem

$$\nu_1 := \min_{A\phi=b} b_1 \| \phi \|_1 + b_2 \| \phi \|_2{}^2 + b_3 \| \phi * w_1 \|_\infty + b_4 \| \phi \|_\infty.$$

As $c_1^k \leq b_1$, $c_2^k \leq b_2$, $c_3^k \leq b_3$ and $c_4^k \leq b_4$ we have that $\nu_k \leq \nu_1$ for all k. Therefore, for any k we have $c_1^k \| \phi_k \|_1 + c_2^k \| \phi_k \|_2{}^2 + c_3^k \| \phi_k * w_1 \|_\infty + c_4^k \| \phi_k \|_\infty \leq \nu_1$ which implies $\| \phi_k \|_1 \leq \frac{\nu_1}{c_1^k} \leq \frac{\nu_1}{a_1}$, $\| \phi_k \|_2{}^2 \leq \frac{\nu_1}{c_2^k} \leq \frac{\nu_1}{a_2}$, $\| \phi_k * w_1 \|_\infty \leq \frac{\nu_1}{c_3^k} \leq \frac{\nu_1}{a_3}$ and $\| \phi_k \|_\infty \leq \frac{\nu_1}{a_4}$. Let

$$S := \{\phi \in R^{L*} : A\phi = b, \| \phi \|_1 \leq \frac{\nu_1}{a_1}, \| \phi \|_2{}^2 \leq \frac{\nu_1}{a_2}, \| \phi * w_1 \|_\infty \leq \frac{\nu_1}{a_3}, \| \phi \|_\infty \leq \frac{\nu_1}{a_4}\}.$$

Then it is clear that

$$\nu_k := \min_{\phi \in S} c_1^k \| \phi \|_1 + c_2^k \| \phi \|_2{}^2 + c_3^k \| \phi * w_1 \|_\infty + c_4^k \| \phi \|_\infty.$$

We now prove that f_k converges to f uniformly on S. Given $\epsilon > 0$ choose K such that if $k > K$ then $|c_1^k - c_1| < \frac{\epsilon a_1}{4\nu_1}$, $|c_2^k - c_2| < \frac{\epsilon a_2}{4\nu_1}$, $|c_3^k - c_3| < \frac{\epsilon a_3}{4\nu_1}$ and $|c_4^k - c_4| < \frac{\epsilon a_4}{4\nu_1}$. Then for any $\phi \in S$ we have

$$|f_k(\phi) - f(\phi)| = |(c_1^k - c_1)\| \phi \|_1 + (c_2^k - c_2)\| \phi \|_2{}^2 + (c_3^k - c_3)\| \phi * w_1 \|_\infty + (c_4^k - c_4)\| \phi \|_\infty|$$

and thus

$$|f_k(\phi) - f(\phi)| \leq |c_1^k - c_1|\frac{\nu_1}{a_1} + |c_2^k - c_2|\frac{\nu_1}{a_2} + |c_3^k - c_3||\frac{\nu_1}{a_3} + |c_4^k - c_4|\frac{\nu_1}{a_4} < \epsilon.$$

Therefore, it follows that f_k converges uniformly to f on S. From Lemma 5 it follows that $\nu_k \to \nu$.

We now prove that $\phi_k \to \phi_0$. Let $B := \{\phi \in R^{L*} : \| \phi \|_1 \leq \frac{\nu_1}{a_1}\}$ then we know that $\phi_k \in B$ which is compact in $(R^{L*}, \| \,.\, \|_1)$. Therefore there exists a subsequence ϕ_{k_i} of ϕ_k and $\overline{\phi} \in R^{L^*}$ such that $\phi_{k_i} \to \overline{\phi}$.

As $c_1^k \to c_1$, $c_2^k \to c_2$, $c_3^k \to c_3$, $c_4^k \to c_4$, and $\phi_{k_i} \to \overline{\phi}$ we have that $f_{k_i}(\phi_{k_i}) \to f(\overline{\phi})$. As ν_k converges to ν it follows that $f_{k_i}(\phi_{k_i}) \to f(\phi_0)$ (note that $\nu_{k_i} = f_{k_i}(\phi_{k_i})$ and $\nu = f(\phi_0)$) and therefore $f(\overline{\phi}) = f(\phi_0)$. As $A\phi_{k_i} = b$ for all i we have that $A\overline{\phi} = b$. From uniqueness of the solution of (22) it follows that $\overline{\phi} = \phi_0$. Therefore we have established that $\phi_{k_i} \to \phi_0$. From uniqueness of the solution of (22) it also follows that $\phi_k \to \phi_0$. This proves the theorem. ■

7 Conclusions

In this paper we considered a mixed objective problem of minimizing a given linear combination of the ℓ_1 norm, the square of the $\mathcal{H}_2$ norm, and the ℓ_∞ norms of the step and pulse responses respectively of the closed loop. Employing a variant of the Khun-Tucker Lagrange duality theorem it was shown that this problem is equivalent to a finite dimensional convex optimization problem with an *a priori* known dimension. The solution is unique and represents a Pareto optimal point with respect to the individual measures involved. It was also shown that the optimal solution is continuous with respect to changes in the coefficients of the composite measure.

The duality theorem employed here can be used in other mixed objective problems. Also, the generalization of this theory to the multiple-input multiple-output case is possible. These topics are the subjects of future research.

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