These are notes for the directed study on Fourier Analysis from Stein and Shakarchi [1].

1 The Genesis of Fourier Analysis

Add any questions about Chapter 1 here.

Exercises: Complete 3, 4, 5, 11.

3. A sequence of complex numbers $\{w\}_{n=1}^\infty$ is said to converge if there exists a $w \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |w_n - w| = 0,$$

and we say that w is a limit of the sequence.
(a) Show that a converging sequence of complex numbers has a unique limit.

Proof. Suppose by way of contradiction that the sequence $\{w\}_{n=1}^\infty$ converges to complex numbers r and l. That is,

$$\lim_{n\to\infty} |w_n - r| = 0, \qquad \lim_{n\to\infty} |w_n - l| = 0$$

Consider the following,

$$|r - l| = |(r - w_n) + (w_n - l)| \leq |r - w_n| + |w_n - l| = |w_n - r| + |w_n - l|$$
$$\implies \lim_{n\to\infty} |r - l| \leq \lim_{n\to\infty} |w_n - r| + \lim_{n\to\infty} |w_n - l|$$
$$\implies |r - l| \leq 0 + 0 = 0.$$

Now since $|r - l| \geq 0$ and as we just demonstrated it is also $|r - l| \leq 0$ then $|r - l| = 0$ which implies $r = l$. Then by way of contradicition the limit of a converging sequence of complex numbers has unique limit. $\square$

The sequence $\{w_n\}_{n=1}^\infty$ is said to be a **Cauchy sequence** if for every $\epsilon > 0$ there exists a positive integer N such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n, m > N.$$

(b) Prove that a sequence of complex numbers converges if and only if it is a Cauchy sequence. [Hint: A similar theorem exists for the convergence of a sequence of real numbers. Why does it carry over to sequences of complex numbers?]

Proof. ($\implies$) Let $\epsilon > 0$ be given. Suppose the sequence of complex numbers $\{w_n\}_{n=1}^\infty$ converges to w, that is,

$$\forall \epsilon > 0, \exists N \text{ such that } \forall n(n > N) \implies |w_n - w| < \epsilon$$

So take $n, m \in \mathbb{N}$ with $n > m \geq N$ such that

$$|w_n - w| < \frac{\epsilon}{2} \qquad \text{and} \qquad |w_m - w| < \frac{\epsilon}{2}$$

And consider the following,

$$|w_n - w_m| = |(w_n - w) + (w - w_m)| \leq |w_n - w| + |w - w_m| = |w_n - w| + |w_m - w|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$< \epsilon$$
$$\implies |w_n - w_m| < \epsilon$$

($\impliedby$) Suppose the sequence $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\exists n, m, N \in \mathbb{N}$ such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n > m, \geq N$$

And suppose $w_n = x_n + iy_n$ and $w_m = x_m + iy_m$ for any arbitrary real sequences $x_n, x_m y_n, y_m$. Then

$$\implies |w_n - w_m|^2 = (x_n - x_m)^2 + (y_n - y_m)^2 < \epsilon^2$$
$$\implies (x_n - x_m)^2 < \epsilon^2 \qquad \text{and} \qquad (y_n - y_m)^2 < \epsilon^2$$
$$\implies |x_n - x_m| < \epsilon \qquad \text{and} \qquad |y_n - y_m| < \epsilon$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Therefore, there exists real numbers x and y such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Then we define $w = x + iy$. Then by definition of a Cauchy sequence there exists a $P \in \mathbb{N}$ such that for any $n > P$ we have $|x_n - x| < \frac{\epsilon}{\sqrt{2}}$ and there exits a $Q \in \mathbb{N}$ such that for any $n > Q$ we have $|y_n - y| < \frac{\epsilon}{\sqrt{2}}$. Now take the bigger number between P and Q and denoted it R. And consider $n \in \mathbb{N}$ such that $n > R$ then,

$$|w_n - w|^2 = (x_n - x)^2 + (y_n - y)^2 < \epsilon^2$$
$$\implies < \left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2$$
$$\implies < \epsilon^2$$
$$\implies |w_n - w| < \epsilon$$

Therefore the Cauchy sequence $\{w_n\}_{n=1}^{\infty}$ converges. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge if the sequence formed by the partial sums

$$S_N = \sum_{n=1}^{N} z_n$$

converges. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers such that the series $\sum_n a_n$ converges.

(c) Show that if $\{z_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying $|z_n| \leq a_n$ for all n, then the series $\sum_n z_n$ converges. [Hint: Use the Cauchy criterion].

Proof. Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_n a_n$ converges. Assume $\{z_n\}$ is a series of complex numbers such that $|z_n| \leq a_n$ for all n.

For any real sequence $\{a_n\}$, the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N}) \text{ such that } \forall(m, l \in \{m, l : m \geq l > N\}),$$

$$|s_m - s_l| = \left|\sum_{n=l+1}^{m} a_n\right| < \epsilon,$$

Where s_n is the nth partial sum of the series $\sum a_n$.

a_n is a convergent sequence of real numbers. Thus, by the Cauchy criterion, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, l \in \mathbb{N}$ such that $m \geq l > N$,

$$|s_m - s_l| = \left| \sum_{n=l+1}^{m} a_n \right| = \sum_{n=l+1}^{m} a_n < \epsilon,$$

where s_m and s_l are partial sums of the series $\sum a_n$.

Considering that

$$|z_n| \leq a_n$$

for each $n \in \{l+1, l+2, \cdots m\}$,

$$\implies \sum_{n=l+1}^{m} |z_n| \leq \sum_{n=l+1}^{m} a_n < \epsilon.$$

Next, we would like to show that

$$\left| \sum_{n=l+1}^{m} z_n \right| \leq \sum_{n=l+1}^{m} |z_n|.$$

Intuitively, (geometrically,) this should follow, because we can think of each z_i as a vector in the complex plane. The total distance of the journey taken by adding each vector z_i tip to tail is greater than or equal to the straightline distance from the origin to the tip of the last vector z_i

The proof is by induction on m, (starting from $m = l+1$, not $m = 1$.)

(Base case) For $m = l+1$,

$$\left| \sum_{n=l+1}^{l+1} z_n \right| = |z_{l+1}| = \sum_{n=l+1}^{l+1} |z_n|$$

(Induction step) Assume it to be true that for some $k \in \mathbb{N}$ s.t. $l+2 \leq k$,

$$\left| \sum_{n=l+1}^{k} z_n \right| \leq \sum_{n=l+1}^{k} |z_n|.$$

$$\implies |z_{k+1}| + \left| \sum_{n=l+1}^{k} z_n \right| \leq |z_{k+1}| + \sum_{n=l+1}^{k} |z_n|$$

On the right-hand side, we have

$$|z_{k+1}| + \sum_{n=l+1}^{k} |z_n| = \sum_{n=l+1}^{k+1} |z_n|$$

On the left-hand side, by the triangle inequality (Exercise 1d), we have

$$|z_{k+1}| + \left| \sum_{n=l+1}^{k} z_n \right| \geq \left| \sum_{n=l+1}^{k+1} z_n \right|$$

3

$$\implies \left| \sum_{n=l+1}^{k+1} z_n \right| \le \sum_{n=l+1}^{k+1} |z_n|.$$

Thus, the induction hypothesis holds for all $n \in N$.

And recall that, for all $m \ge l+1$,

$$\sum_{n=l+1}^{m} |z_n| \le \sum_{n=l+1}^{k+1} a_n \le \left| \sum_{n=l+1}^{k+1} a_n \right| < \epsilon$$

Thus we have shown that, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $l > N, m \ge l$,

$$\left| \sum_{n=l+1}^{m} z_n \right| < \epsilon$$

$$\Leftrightarrow \left| \sum_{n=1}^{m} z_n - \sum_{n=1}^{l} z_n \right| < \epsilon$$

Or in other words, the sequence (of partial sums of a series) $S_N = \sum_{n=1}^{N} z_n$ is Cauchy, which (by exercise 3b,) implies that the sequence S_N converges.

Therefore, (by the fact given at the start of Exercise 3c,) the series

$$\sum_{n=1}^{\infty} z_n$$

converges. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. For $z \in \mathbb{C}$, we define the complex exponential by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(a) Prove that the above definition makes sense, by showing that the series converges for every complex number z. Moreover, show that the convergence is uniform on every bounded subset of $\mathbb{C}$.

Proof. Define the following partial sums

$$S_p = \sum_{n=0}^{p} \frac{z^n}{n!} \qquad S_q = \sum_{n=0}^{q} \frac{z^n}{n!} \qquad \text{where } p, q \in \mathbb{N} \text{ and } p > q.$$

Now consider the following:

$$S_p - S_q = \sum_{n=0}^{p} \frac{z^n}{n!} - \sum_{n=0}^{q} \frac{z^n}{n!} = \sum_{n=q+1}^{p} \frac{z^n}{n!}$$

$$\implies |S_p - S_q| = \left| \sum_{n=q+1}^{p} \frac{z^n}{n!} \right| \le \sum_{n=q+1}^{p} \frac{|z|^n}{n!} \qquad \text{Let } b = |z| \text{ where } b \in \mathbb{R} \text{ and } b \ge 0.$$

$$\implies |S_p - S_q| \le \sum_{n=q+1}^{p} \frac{b^n}{n!} \le \sum_{n=q+1}^{\infty} \frac{b^n}{n!}$$

4

Where we are able to extend the infinite sum to positive infinity as all terms in sum are non-negative. Furthermore the infinite series converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{b^{n+1}}{(n+1)!} \cdot \frac{n!}{b^n} = \frac{b}{n+1} \to 0 \quad \text{which shows convergence by Ratio Test.}$$

Therefore we choose an $N \in \mathbb{N}$ such that $p > q \geq N$ and such that

$$\sum_{n=N+1}^{\infty} \frac{b^n}{n!} < \epsilon$$

ensuring the following,

$$|S_p - S_q| \leq \sum_{n=q+1}^{p} \frac{b^n}{n!} \leq \sum_{n=q+1}^{\infty} \frac{b^n}{n!} \leq \sum_{n=N+1}^{\infty} \frac{b^n}{n!} < \epsilon$$

$$\implies |S_p - S_q| < \epsilon$$

Therefore, by the Cauchy criterion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely. Now showing convergence on a bounded subset S of $\mathbb{C}$ is the same proof but just stating that $b = \sup\{|z| : z \in S\}$. $\qquad \square$

(b) If z_1, z_2 are two complex numbers, prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. [Hint: use the binomial theorem to expand $(z_1 + z_2)^n$, as well as the formula for binomial coefficients.]

(c) Show that if z is purely imaginary, that is, $z = iy$ with $y \in \mathbb{R}$, then

$$e^{iy} = \cos(y) + i\sin(y)$$

This is Euler's identity. [Hint: Use power series.]

Proof. Let $y \in \mathbb{R}$ then we can express e^{iy} as the following power series

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!}$$

Now we separate our infinite sum into two infinite sums of even and odd as follows:

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} y^{2k+1}}{(2k+1)!}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

Now we recognize both of these infinite sums as sine and cosine power series and arrive at the following:

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} = \cos(y) + i\sin(y)$$

$\qquad \square$

(e) prove that $e^z = 1$ if and only if $z = 2\pi k i$ for some integer k.

Proof. ($\implies$) Assume $z = 2\pi ki$. By Euler's identity,

$$e^z = e^{2\pi ki} = \cos(2\pi k) + i\sin(2\pi k).$$

By the cyclic nature of trigonometric functions,

$(\cos(x) = \cos(2\pi k + x)$ for $k \in \mathbb{Z}$,)

$$e^z = \cos(0) + i\sin(0)$$
$$= 1 + i(0)$$
$$= 1.$$

Conversely, assume $e^z = 1$.

Let $z = x + iy$ for some $x, y \in \mathbb{R}$. By Euler's identity,

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y) = 1.$$

Since the RHS has no imaginary part, this implies that

$$e^x i\sin y = 0, \quad e^x \cos y = 1.$$
$$\implies \sin y = 0, \quad \cos y = e^{-x}$$
$$\implies y \in \{2\pi k : k \in \mathbb{Z}\}$$
$$\implies \cos y = 1$$
$$\implies 1 = e^{-x}$$
$$\implies x = 0$$
$$\implies z = x + iy = iy,$$

for some $y \in \{2\pi k : k \in \mathbb{Z}\}$.

$$\implies z = 2\pi ki,$$

for some $k \in \mathbb{Z}$. $\qquad\qquad\qquad\square$

(f) show that every complex number $z = x + iy$ can be written in the form

$$z = re^{i\theta},$$

where r is unique and in the range $0 \le r < \infty$, and $\theta \in \mathbb{R}$ is unique up to an integer multiple of 2π. Check that

$$r = |z| \text{ and } \theta = \arctan(y/x)$$

whenever these formulas make sense.}

5. Verify that $f(x) = e^{inx}$ is periodic with period 2π, and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases}$$

Using Euler's identity,

$$f = e^{inx} = \cos nx + i\sin nx$$

6

Since $\cos z = \cos(z + 2\pi n)$ for all $z \in \mathbb{R}$, $n \in \mathbb{N}$, and the same is true for sin, we have that, for all $n \in \mathbb{N}$,

$$f(x + 2\pi) = \cos(n(x + 2\pi)) + i\sin(n(x + 2\pi))$$

$$= \cos nx + i\sin nx = f(x),$$

which is the definition of having period 2π.

For $n = 0$,

$$e^{inx} = 1$$

$$\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx$$

$$= \frac{1}{2\pi}(\pi - (-\pi)) = 1.$$

For $n \neq 0$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi in} \left[e^{inx} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi in} \left[\cos nx + i \sin nx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi in} \left[\cos(\pi n) - \cos(-\pi n) + i\sin(\pi n) - i\sin(-\pi n) \right]$$

Then, since $\cos(x) = \cos(-x)$, and $\sin(x) = -\sin(-x)$,

$$= \frac{1}{2\pi in} \left[(\cos \pi n - \cos \pi n) + (i \sin \pi n + i \sin \pi n) \right]$$

$$= \frac{1}{2\pi in} (2i \sin \pi n)$$

$$= \frac{1}{\pi n} \sin \pi n$$

For any integer multiple of π we have $\sin(\pi n) = 0$. Therefore,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = 0 \qquad \text{Whenever } n \neq 0$$

.

Use this fact to prove that, for $n.m \geq 1$, we have

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos mx \ dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Since $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$, then $\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$. Then,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{e^{inx} + e^{-inx}}{2} \right) \left(\frac{e^{imx} + e^{-imx}}{2} \right)$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(e^{ix(m+n)} + e^{ix(n-m)} + e^{ix(m-n)} + e^{-ix(n+m)} \right) dx$$

If $n = m$ then any integral with exponent not equal 0 will equal 0 by our first verification. Therefore,

$$\frac{1}{4\pi}\left[0 + \int_{-\pi}^{\pi} e^{ix(0)}dx + \int_{-\pi}^{\pi} e^{ix(0)}dx + 0\right]$$

and similarly

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \ dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

and

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \ dx = 0 \quad \text{for any } n, m.$$

[Hint: use $e^{inx}e^{-imx} + e^{inx}e^{imx}$ and $e^{inx}e^{-imx} - e^{inx}e^{imx}$.]

(11) Show that if $n \in \mathbb{Z}$ the only solutions of the differential equation

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0,$$

which are twice differentiable when $r > 0$, are given by linear combinations of r^n and r^{-n} when $n \neq 0$, and 1 and $\log r$ when $n = 0$.

[Hint: If F solves the equation, write $F(r) = g(r)r^n$, find the equation satisfied by g, and conclude that $rg'(r) + 2ng(r) = c$ where c is a constant.]

Given $F(r)$ is twice differentiable and solves

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0,$$

Define g such that

$$F(r) =: g(r)r^n$$
$$\implies F'(r) = ng(r)r^{n-1} + g'(r)r^n$$
$$\implies F''(r) = n(n-1)g(r)r^{n-2} + ng'(r)r^{n-1} + ng'(r)r^{n-1} + g''(r)r^n$$
$$= (n^2 - n)g(r)r^{n-2} + 2ng'(r)r^{n-1} + g''(r)r^n$$
$$\implies 0 = r^2\left((n^2 - n)g(r)r^{n-2} + 2ng'(r)r^{n-1} + g''(r)r^n\right) + r\left(ng(r)r^{n-1} + g'(r)r^n\right) - n^2\left(g(r)r^n\right)$$
$$= (n^2 - n)g(r)r^n + ng(r)r^n + 2ng'(r)r^{n+1} + g'(r)r^{n+1} + g''(r)r^{n+2} - n^2 g(r)r^n$$
$$= (n^2 - n + n - n^2)g(r)r^n + (2n + 1)g'(r)r^{n+1} + g''(r)r^{n+2}$$
$$\implies 0 = (2n + 1)g'(r) + g''(r)r$$

Solve the second order ODE by reducing to first order. Let

$$u(r) := g'(r)$$
$$\implies u'(r) = g''(r)$$
$$\implies ru' = -(2n+1)u$$
$$\int \frac{1}{u}\frac{du}{dr}dr = -(2n+1)\int\frac{1}{r}dr$$
$$\log|u| = (-2n+1)\log|r| + (2n+1)C$$
$$u = e^{-(2n+1)\log|r|}e^{(2n+1)C}$$
$$= e^{-(2n+1)}\left(e^{\log|r|}\right)^{-(2n+1)}$$
$$= Dr^{-(2n+1)}$$
$$\int u = g = D\int r^{-(2n+1)}dr$$
$$F(r) = r^n D\int r^{-(2n+1)}dr$$

For $n = 0$,

$$F(r) = D\int r^{-1}dr$$
$$= D\log|r| + DE$$
$$= A\log|r| + B.$$

For $n \neq 0$,

$$F(r) = Dr^n\int r^{-(2n+1)}dr$$
$$= Dr^n r^{-2n}\frac{1}{-2n} + Dr^n C$$
$$= Ar^{-n} + Br^n.$$

Problem 1 Consider the Dirichlet problem illustrated in Figure 11. More precisely, we look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x,y) : 0 \le x \le \pi, 0 \le y \le 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x,0) = f_0(x) \qquad \text{and} \qquad u(x,1) = f_1(x)$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle. Use separation of variables to show that if f_0 and f_1 have Fourier expressions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \qquad \text{and} \qquad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

9

then

$$u(x,y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Solution: Suppose $u(x,y) = P(x)Q(y)$ then using this in the Laplacian $\Delta u = 0$ we get:

$$P''(x)Q(y) = -P(x)Q''(y)$$
$$\frac{P''(x)}{P(x)} = -\frac{Q''(y)}{Q(y)}$$
$$\implies \frac{P''(x)}{P(x)} = -\lambda$$
$$\implies P''(x) + \lambda P(x) = 0$$
$$\implies -\frac{Q''(y)}{Q(y)} = -\lambda$$
$$\implies Q''(y) - \lambda Q(y) = 0$$

Now we must check what sign λ must have. We first check the case $\lambda = 0$ and input our initial conditions:

$$P''(x) = 0$$
$$\implies P'(x) = K$$
$$\implies P(x) = Kx + G$$
$$\implies u(0,t) = P(0) = G = 0 \qquad\qquad \implies G = 0$$
$$\implies P(x) = Kx$$
$$\implies u(\pi,t) = P(\pi) = K\pi = 0 \qquad\qquad \implies K = 0$$
$$\implies u(x,t) = 0$$

Therefore, $\lambda \neq 0$. Now we check the case where $\lambda < 0$ that is $\lambda = -r^2$:

$$P''(x) - r^2 P(x) = 0$$
$$\implies e^{kx} \left(k^2 - r^2 \right) = 0$$
$$\implies k = \pm r$$
$$\implies P(x) = Ae^{rx} + Be^{-rx}$$
$$\implies u(0,t) = P(0) = A + B = 0$$
$$\implies u(\pi,t) = P(\pi) = Ae^{r\pi} - Ae^{-r\pi}$$
$$\implies e^{r\pi} = e^{-r\pi} \implies 2r = 0$$
$$\implies r = 0$$
$$\implies P(x) = A - A = 0$$

Therefore, $\lambda > 0$. Let $\lambda = k^2$:

$$P''(x) + k^2 P(x) = 0$$
$$\implies e^{mx}\left(m^2 + k^2\right) = 0$$
$$\implies k = \pm ik$$
$$\implies P(x) = Ae^{ikx} + Be^{-ikx} = A\cos(kx) + iA\sin(kx) + B\cos(kx) - iB\sin(kx)$$
$$= (A+B)\cos(kx) + (iA - iB)\sin(kx)$$
$$\implies P(x) = C\cos(kx) + D\sin(kx)$$
$$\implies u(0,t) = P(0) = C = 0 \implies C = 0$$
$$\implies u(\pi, t) = P(\pi) = 0$$
$$\implies P(x) = D\sin(kx)$$

Since $\lambda = k^2$ does not lead to any $u(x,t) = 0$ solutions then we use thing in our ODE in terms of y:

$$Q''(y) - k^2 Q(y) = 0$$
$$\implies e^{my}\left(m^2 - k^2\right) = 0$$
$$\implies m = \pm k$$
$$\implies Q(y) = Ae^{-ky} + Be^{ky}$$

Now we introduce sinh and cosh in the following manner:

$$\sinh(ky) = \frac{1}{2}e^{ky} - \frac{1}{2}e^{-ky}$$
$$\cosh(ky) = \frac{1}{2}e^{ky} + \frac{1}{2}e^{-ky}$$
$$e^{ky} = \sinh(ky) + \cosh(ky)$$
$$e^{-ky} = \cosh(ky) - \sinh(ky)$$

Therefore, we can write $Q(y)$ as follows:

$$Q(y) = A\left(\cosh(ky) - \sinh(ky)\right) + B\left(\sinh(ky) + \cosh(ky)\right)$$
$$Q(y) = (A+B)\cosh(ky) + (B-A)\sinh(ky)$$
$$Q(y) = E\cosh(ky) + F\sinh(ky)$$

Therefore, we can write $u(x,y)$ as follows and extend to the its Fourier sum as well:

$$u(x,y)_k = D\sin(kx)\left(E\cosh(ky) + F\sinh(ky)\right)$$
$$u(x,y) = \sum_{k=1}^{\infty} D_k \sin(kx)\left(E_k \cosh(ky) + F_k \sinh(ky)\right)$$

Now we'll ensure our function $u(x, y)$ satisfies the initial conditions in terms of y:

$$u(x, 0) = \sum_{k=1}^{\infty} D_k E_k \sin(kx) = \sum_{k=1}^{\infty} A_k \sin(kx)$$

$$\implies A_k = D_k E_k$$

$$u(x, y) = \sum_{k=1}^{\infty} A_k \sin(kx) \cosh(ky) + D_k F_k \sin(kx) \sinh(ky)$$

$$u(x, 1) = \sum_{k=1}^{\infty} A_k \sin(kx) \cosh(k) + D_k F_k \sin(kx) \sinh(k) = \sum_{k=1}^{\infty} B_k \sin(kx)$$

$$\implies D_k F_k \sin(kx) \sinh(k) = B_k \sin(kx) - A_k \sin(kx) \cosh(k)$$

$$\implies D_k F_k = \frac{B_k}{\sinh(k)} - \frac{A_k \cosh(k)}{\sinh(k)}$$

$$u(x, y) = \sum_{k=1}^{\infty} A_k \sin(kx) \cosh(ky) + \left(\frac{B_k}{\sinh(k)} - \frac{A_k \cosh(k)}{\sinh(k)} \right) \cdot \sin(kx) \sinh(ky)$$

$$u(x, y) = \sum_{k=1}^{\infty} A_k \sin(kx) \cosh(ky) + B_k \frac{\sin(kx) \sinh(ky)}{\sinh(k)} - A_k \frac{\cosh(k) \sin(kx) \sinh(ky)}{\sinh(k)}$$

$$u(x, y) = \sum_{k=1}^{\infty} \left(\sin(kx) \cosh(ky) - \frac{\cosh(k) \sin(kx) \sinh(ky)}{\sinh(k)} \right) A_k + B_k \left(\frac{\sin(kx) \sinh(ky)}{\sinh(k)} \right)$$

$$u(x, y) = \sum_{k=1}^{\infty} \sin(kx) \left[\left(\frac{\cosh(ky) \sinh(k) - \cosh(k) \sinh(ky)}{\sinh(k)} \right) A_k + B_k \left(\frac{\sinh(ky)}{\sinh(k)} \right) \right]$$

Now we compute the coeffecients of A_k and B_k in the following manner:

$$\frac{\cosh(ky) \sinh(k) - \cosh(k) \sinh(ky)}{\sinh(k)} = \frac{(\frac{1}{2}e^{ky} + \frac{1}{2}e^{-ky})(\frac{1}{2}e^{k} - \frac{1}{2}e^{-k}) - (\frac{1}{2}e^{k} + \frac{1}{2}e^{-k})(\frac{1}{2}e^{ky} - \frac{1}{2}e^{-ky})}{\sinh(k)}$$

$$= \frac{(\frac{1}{4}e^{k(1+y)} - \frac{1}{4}e^{k(y-1)} + \frac{1}{4}e^{k(1-y)} - \frac{1}{4}e^{-k(1+y)}) - (\frac{1}{4}e^{k(1+y)} - \frac{1}{4}e^{k(1-y)} + \frac{1}{4}e^{k(y-1)} - \frac{1}{4}e^{-k(1+y)})}{\sinh(k)}$$

$$= \frac{\frac{1}{4}e^{k(1-y)} - \frac{1}{4}e^{-k(1-y)} + \frac{1}{4}e^{k(1-y)} - \frac{1}{4}e^{-k(1-y)}}{\sinh(k)}$$

$$= \frac{\frac{\sinh(k(1-y))}{2} + \frac{\sinh(k(1-y))}{2}}{\sinh(k)} = \frac{\sinh(k(1-y))}{\sinh(k)}$$

Therefore we arrive at the following simplified expression for $u(x, y)$:

$$\boxed{u(x, y) = \sum_{k=1}^{\infty} \left[\frac{\sinh(k(1-y))}{\sinh(k)} A_k + \frac{\sinh(ky)}{\sinh(k)} B_k \right] \sin(kx)}$$

2   Basic Properties of Fourier Series

Read sections 1 – 3.

Exercises: Complete 2, 4, 5, 6, 8

References

[1] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, vol. 1 of Princeton Lectures in Analysis, Princeton University Press, 2003.