These are notes for the directed study on Fourier Analysis from Stein and Shakarchi [1].

# 1 The Genesis of Fourier Analysis

Add any questions about Chapter 1 here.

**Exercises:** Complete 3, 4, 5, 11.

**3.** A sequence of complex numbers $\{w\}_{n=1}^{\infty}$ is said to converge if there exists a $w \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |w_n - w| = 0,$$

and we say that $w$ is a limit of the sequence.
**(a)** Show that a converging sequence of complex numbers has a unique limit.

*Proof.* Suppose by way of contradiction that the sequence $\{w\}_{n=1}^{\infty}$ converges to complex numbers $r$ and $l$. That is,

$$\lim_{n\to\infty} |w_n - r| = 0, \qquad \lim_{n\to\infty} |w_n - l| = 0$$

Consider the following,

$$\begin{aligned} &|r-l| = |(r-w_n)+(w_n-l)| \le |r-w_n| + |w_n-l| = |w_n-r| + |w_n-l| \\ \implies &\lim_{n\to\infty} |r-l| \le \lim_{n\to\infty} |w_n-r| + \lim_{n\to\infty} |w_n-l| \\ \implies &|r-l| \le 0+0 = 0. \end{aligned}$$

Now since $|r-l| \ge 0$ and as we just demonstrated it is also $|r-l| \le 0$ then $|r-l| = 0$ which implies $r = l$. Then by way of contradicition the limit of a converging sequence of complex numbers has unique limit. $\square$

The sequence $\{w_n\}_{n=1}^{\infty}$ is said to be a **Cauchy sequence** if for every $\epsilon > 0$ there exists a positive integer $N$ such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n, m > N.$$

**(b)** Prove that a sequence of complex numbers converges if and only if it is a Cauchy sequence. [Hint: A similar theorem exists for the convergence of a sequence of real numbers. Why does it carry over to sequences of complex numbers?]

*Proof.* ( $\implies$ ) Let $\epsilon > 0$ be given. Suppose the sequence of complex numbers $\{w_n\}_{n=1}^{\infty}$ converges to $w$, that is,

$$\forall \epsilon > 0, \exists N \text{ such that } \forall n(n > N) \implies |w_n - w| < \epsilon$$

So take $n, m \in \mathbb{N}$ with $n > m \ge N$ such that

$$|w_n - w| < \frac{\epsilon}{2} \qquad \text{and} \qquad |w_m - w| < \frac{\epsilon}{2}$$

And consider the following,

$$\begin{aligned} |w_n - w_m| = |(w_n - w) + (w - w_m)| \le |w_n - w| + |w - w_m| &= |w_n - w| + |w_m - w| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \\ &\implies |w_n - w_m| < \epsilon \end{aligned}$$

( $\Longleftarrow$ ) Suppose the sequence $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\exists n, m, N \in \mathbb{N}$ such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n > m, \geq N$$

And suppose $w_n = x_n + iy_n$ and $w_m = x_m + iy_m$ for any arbitrary real sequences $x_n, x_m y_n, y_m$. Then

$$\begin{aligned}
&\Longrightarrow |w_n - w_m|^2 = (x_n - x_m)^2 + (y_n - y_m)^2 < \epsilon^2 \\
&\Longrightarrow (x_n - x_m)^2 < \epsilon^2 \qquad \text{and} \qquad (y_n - y_m)^2 < \epsilon^2 \\
&\Longrightarrow |x_n - x_m| < \epsilon \qquad \text{and} \qquad |y_n - y_m| < \epsilon
\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Therefore, there exists real numbers $x$ and $y$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Then we define $w = x + iy$. Then by definition of a Cauchy sequence there exists a $P \in \mathbb{N}$ such that for any $n > P$ we have $|x_n - x| < \frac{\epsilon}{\sqrt{2}}$ and there exits a $Q \in \mathbb{N}$ such that for any $n > Q$ we have $|y_n - y| < \frac{\epsilon}{\sqrt{2}}$. Now take the bigger number between $P$ and $Q$ and denoted it $R$. And consider $n \in \mathbb{N}$ such that $n > R$ then,

$$\begin{aligned}
|w_n - w|^2 &= (x_n - x)^2 + (y_n - y)^2 < \epsilon^2 \\
&\Longrightarrow < \left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2 \\
&\Longrightarrow < \epsilon^2 \\
\Longrightarrow |w_n - w| < \epsilon
\end{aligned}$$

Therefore the Cauchy sequence $\{w_n\}_{n=1}^{\infty}$ converges. $\square$

A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge if the sequence formed by the partial sums

$$S_N = \sum_{n=1}^{N} z_n$$

converges. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers such that the series $\sum_n a_n$ converges.

**(c)** Show that if $\{z_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying $|z_n| \leq a_n$ for all $n$, then the series $\sum_n z_n$ converges. [Hint: Use the Cauchy criterion].

*Proof.* Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_n a_n$ converges. Assume $\{z_n\}$ is a series of complex numbers such that $|z_n| \leq a_n$ for all $n$.

For any real sequence $\{a_n\}$, the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N}) \text{ such that } \forall(m, l \in \{m, l : m \geq l > N\}),$$

$$|s_m - s_l| = \left|\sum_{n=l+1}^{m} a_n\right| < \epsilon,$$

Where $s_n$ is the $n$th partial sum of the series $\sum a_n$.

$a_n$ is a convergent sequence of real numbers. Thus, by the Cauchy criterion, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, l \in \mathbb{N}$ such that $m \geq l > N$,

$$|s_m - s_l| = \left| \sum_{n=l+1}^{m} a_n \right| = \sum_{n=l+1}^{m} a_n < \epsilon,$$

where $s_m$ and $s_l$ are partial sums of the series $\sum a_n$.

Considering that

$$|z_n| \leq a_n$$

for each $n \in \{l+1, l+2, \cdots m\}$,

$$\implies \sum_{n=l+1}^{m} |z_n| \leq \sum_{n=l+1}^{m} a_n < \epsilon.$$

Next, we would like to show that

$$\left| \sum_{n=l+1}^{m} z_n \right| \leq \sum_{n=l+1}^{m} |z_n|.$$

Intuitively, (geometrically,) this should follow, because we can think of each $z_i$ as a vector in the complex plane. The total distance of the journey taken by adding each vector $z_i$ tip to tail is greater than or equal to the straightline distance from the origin to the tip of the last vector $z_i$

The proof is by induction on $m$, (starting from $m = l+1$, not $m = 1$.)

(Base case) For $m = l+1$,

$$\left| \sum_{n=l+1}^{l+1} z_n \right| = |z_{l+1}| = \sum_{n=l+1}^{l+1} |z_n|$$

(Induction step) Assume it to be true that for some $k \in \mathbb{N}$ s.t. $l+2 \leq k$,

$$\left| \sum_{n=l+1}^{k} z_n \right| \leq \sum_{n=l+1}^{k} |z_n|.$$

$$\implies |z_{k+1}| + \left| \sum_{n=l+1}^{k} z_n \right| \leq |z_{k+1}| + \sum_{n=l+1}^{k} |z_n|$$

On the right-hand side, we have

$$|z_{k+1}| + \sum_{n=l+1}^{k} |z_n| = \sum_{n=l+1}^{k+1} |z_n|$$

On the left-hand side, by the triangle inequality (Exercise 1d), we have

$$|z_{k+1}| + \left| \sum_{n=l+1}^{k} z_n \right| \geq \left| \sum_{n=l+1}^{k+1} z_n \right|$$

$$\implies \left|\sum_{n=l+1}^{k+1} z_n\right| \leq \sum_{n=l+1}^{k+1} |z_n|.$$

Thus, the induction hypothesis holds for all $n \in N$.

And recall that, for all $m \geq l+1$,

$$\sum_{n=l+1}^{m} |z_n| \leq \sum_{n=l+1}^{k+1} a_n \leq \left|\sum_{n=l+1}^{k+1} a_n\right| < \epsilon$$

Thus we have shown that, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $l > N, m \geq l$,

$$\left|\sum_{n=l+1}^{m} z_n\right| < \epsilon$$

$$\Leftrightarrow \left|\sum_{n=1}^{m} z_n - \sum_{n=1}^{l} z_n\right| < \epsilon$$

Or in other words, the sequence (of partial sums of a series) $S_N = \sum_{n=1}^{N} z_n$ is Cauchy, which (by exercise 3b,) implies that the sequence $S_N$ converges.

Therefore, (by the fact given at the start of Exercise 3c,) the series

$$\sum_{n=1}^{\infty} z_n$$

converges. $\square$

**4.** For $z \in \mathbb{C}$, we define the complex exponential by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

**(a)** Prove that the above definition makes sense, by showing that the series converges for every complex number $z$. Moreover, show that the convergence is uniform on every bounded subset of $\mathbb{C}$.

*Proof.* Define the following partial sums

$$S_p = \sum_{n=0}^{p} \frac{z^n}{n!} \qquad S_q = \sum_{n=0}^{q} \frac{z^n}{n!} \qquad \text{where } p, q \in \mathbb{N} \text{ and } p > q.$$

Now consider the following:

$$S_p - S_q = \sum_{n=0}^{p} \frac{z^n}{n!} - \sum_{n=0}^{q} \frac{z^n}{n!} = \sum_{n=q+1}^{p} \frac{z^n}{n!}$$

$$\implies |S_p - S_q| = \left|\sum_{n=q+1}^{p} \frac{z^n}{n!}\right| \leq \sum_{n=q+1}^{p} \frac{|z|^n}{n!} \quad \text{Let } b = |z| \text{ where } b \in \mathbb{R} \text{ and } b \geq 0.$$

$$\implies |S_p - S_q| \leq \sum_{n=q+1}^{p} \frac{b^n}{n!} \leq \sum_{n=q+1}^{\infty} \frac{b^n}{n!}$$

Where we are able to extend the infinite sum to positive infinity as all terms in sum are non-negative. Furthermore the infinite series converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{b^{n+1}}{(n+1)!} \cdot \frac{n!}{b^n} = \frac{b}{n+1} \to 0 \quad \text{which shows convergence by Ratio Test.}$$

Therefore we choose an $N \in \mathbb{N}$ such that $p > q \geq N$ and such that

$$\sum_{n=N+1}^{\infty} \frac{b^n}{n!} < \epsilon$$

ensuring the following,

$$|S_p - S_q| \leq \sum_{n=q+1}^{p} \frac{b^n}{n!} \leq \sum_{n=q+1}^{\infty} \frac{b^n}{n!} \leq \sum_{n=N+1}^{\infty} \frac{b^n}{n!} < \epsilon$$
$$\implies |S_p - S_q| < \epsilon$$

Therefore, by the Cauchy criterion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely. Now showing convergence on a bounded subset $S$ of $\mathbb{C}$ is the same proof but just stating that $b = \sup\{|z| : z \in S\}$. □

**(b)** If $z_1, z_2$ are two complex numbers, prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. [Hint: use the binomial theorem to expand $(z_1 + z_2)^n$, as well as the formula for binomial coefficients.]

**(c)** Show that if $z$ is purely imaginary, that is, $z = iy$ with $y \in \mathbb{R}$, then

$$e^{iy} = \cos(y) + i\sin(y)$$

This is Euler's identity. [Hint: Use power series.]

*Proof.* Let $y \in \mathbb{R}$ then we can express $e^{iy}$ as the following power series

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!}$$

Now we separate our infinite sum into two infinite sums of even and odd as follows:

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} y^{2k+1}}{(2k+1)!}$$
$$e^{iy} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

Now we recognize both of these infinite sums as sine and cosine power series and arrive at the following:

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} = \cos(y) + i\sin(y)$$

□

**(e)** prove that $e^z = 1$ if and only if $z = 2\pi k i$ for some integer $k$.

*Proof.* ($\implies$) Assume $z = 2\pi k i$. By Euler's identity,

$$e^z = e^{2\pi k i} = \cos(2\pi k) + i\sin(2\pi k).$$

By the cyclic nature of trigonometric functions,
$(cos(x) = cos(2\pi k + x)$ for $k \in \mathbb{Z}$,)

$$\begin{aligned} e^z &= \cos(0) + i\sin(0) \\ &= 1 + i(0) \\ &= 1. \end{aligned}$$

Conversely, assume $e^z = 1$.
Let $z = x + iy$ for some $x, y \in \mathbb{R}$. By Euler's identity,

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y) = 1.$$

Since the RHS has no imaginary part, this implies that

$$e^x i\sin y = 0, \quad e^x\cos y = 1.$$

$$\begin{aligned} &\implies \sin y = 0, \quad \cos y = e^{-x} \\ &\implies y \in \{2\pi k : k \in \mathbb{Z}\} \\ &\implies \cos y = 1 \\ &\implies 1 = e^{-x} \\ &\implies x = 0 \\ &\implies z = x + iy = iy, \end{aligned}$$

for some $y \in \{2\pi k : k \in \mathbb{Z}\}$.

$$\implies z = 2\pi k i,$$

for some $k \in \mathbb{Z}$. □

**(f)** show that every complex number $z = x + iy$ can be written in the form

$$z = re^{i\theta},$$

where $r$ is unique and in the range $0 \leq r < \infty$, and $\theta \in \mathbb{R}$ is unique up to an integer multiple of $2\pi$. Check that

$$r = |z| \text{ and } \theta = \arctan(y/x)$$

whenever these formulas make sense.}

**5.** Verify that $f(x) = e^{inx}$ is periodic with period $2\pi$, and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Using Euler's identity,

$$f = e^{inx} = \cos nx + i\sin nx$$

Since $\cos z = \cos(z + 2\pi n)$ for all $z \in \mathbb{R}$, $n \in \mathbb{N}$, and the same is true for sin, we have that, for all $n \in \mathbb{N}$,

$$f(x + 2\pi) = \cos(n(x + 2\pi)) + i \sin(n(x + 2\pi))$$
$$= \cos nx + i \sin nx = f(x),$$

which is the definition of having period $2\pi$.

For $n = 0$,

$$e^{inx} = 1$$
$$\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx$$
$$= \frac{1}{2\pi}(\pi - (-\pi)) = 1.$$

For $n \neq 0$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi in} \left[e^{inx}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi in} \left[\cos nx + i \sin nx\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi in} \left[\cos(\pi n) - \cos(-\pi n) + i \sin(\pi n) - i \sin(-\pi n)\right]$$

Then, since $\cos(x) = \cos(-x)$, and $\sin(x) = -\sin(-x)$,

$$= \frac{1}{2\pi in}[(\cos \pi n - \cos \pi n) + (i \sin \pi n + i \sin \pi n)]$$
$$= \frac{1}{2\pi in}(2i \sin \pi n)$$
$$= \frac{1}{\pi n} \sin \pi n$$

For any integer multiple of $\pi$ we have $\sin(\pi n) = 0$. Therefore,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = 0 \qquad \text{Whenever } n \neq 0$$

.

Use this fact to prove that, for $n.m \geq 1$, we have

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Since $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, then $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$. Then,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{e^{inx} + e^{-inx}}{2}\right) \left(\frac{e^{imx} + e^{-imx}}{2}\right)$$
$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(e^{ix(m+n)} + e^{ix(n-m)} + e^{ix(m-n)} + e^{-ix(n+m)}\right) dx$$

If $n = m$ then any integral with exponent not equal 0 will equal 0 by our first verification. Therefore,

$$\frac{1}{4\pi}\left[0 + \int_{-\pi}^{\pi} e^{ix(0)} dx + \int_{-\pi}^{\pi} e^{ix(0)} dx + 0\right]$$

and similarly

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \; dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

and

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = 0 \quad \text{for any } n, m.$$

[Hint: use $e^{inx}e^{-imx} + e^{inx}e^{imx}$ and $e^{inx}e^{-imx} - e^{inx}e^{imx}$.]

**11.** Show that if $n \in \mathbb{Z}$ the only solutions of the differential equation

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0,$$

which are twice differentiable when $r > 0$, are given by linear combinations of $r^n$ and $r^{-n}$ when $n \neq 0$, and 1 and $\log r$ when $n = 0$.

[Hint: If $F$ solves the equation, write $F(r) = g(r)r^n$, find the equation satisfied by $g$, and conclude that $rg'(r) + 2ng(r) = c$ where $c$ is a constant.]

Given $F(r)$ is twice differentiable and solves

$$r^2 F''(r) + rF'(r) - n^2 F(r) = 0,$$

Define $g$ such that

$$\begin{aligned}
F(r) &=: g(r)r^n \\
\implies F'(r) &= ng(r)r^{n-1} + g'(r)r^n \\
\implies F''(r) &= n(n-1)g(r)r^{n-2} + ng'(r)r^{n-1} + ng'(r)r^{n-1} + g''(r)r^n \\
&= (n^2 - n)g(r)r^{n-2} + 2ng'(r)r^{n-1} + g''(r)r^n \\
\implies 0 &= r^2\Big((n^2 - n)g(r)r^{n-2} + 2ng'(r)r^{n-1} + g''(r)r^n\Big) + r\Big(ng(r)r^{n-1} + g'(r)r^n\Big) - n^2\Big(g(r)r^n\Big) \\
&= (n^2 - n)g(r)r^n + ng(r)r^n + 2ng'(r)r^{n+1} + g'(r)r^{n+1} + g''(r)r^{n+2} - n^2 g(r)r^n \\
&= (n^2 - n + n - n^2)g(r)r^n + (2n+1)g'(r)r^{n+1} + g''(r)r^{n+2} \\
\implies 0 &= (2n+1)g'(r) + g''(r)r
\end{aligned}$$

Solve the second order ODE by reducing to first order. Let

$$\begin{aligned}
u(r) &:= g'(r) \\
\implies u'(r) &= g''(r) \\
\implies ru' &= -(2n+1)u \\
\int \frac{1}{u}\frac{du}{dr}dr &= -(2n+1)\int \frac{1}{r}dr \\
\log|u| &= (-2n+1)\log|r| + (2n+1)C \\
u &= e^{-(2n+1)\log|r|}e^{(2n+1)C} \\
&= e^{-(2n+1)}\big(e^{\log|r|}\big)^{-(2n+1)} \\
&= Dr^{-(2n+1)} \\
\int u = g &= D\int r^{-(2n+1)}dr \\
F(r) &= r^n D\int r^{-(2n+1)}dr
\end{aligned}$$

For $n = 0$,

$$\begin{aligned} F(r) &= D \int r^{-1} dr \\ &= D \log|r| + DE \\ &= A \log|r| + B. \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} F(r) &= Dr^n \int r^{-(2n+1)} dr \\ &= Dr^n r^{-2n} \frac{1}{-2n} + Dr^n C \\ &= Ar^{-n} + Br^n. \end{aligned}$$

**Problem 1** Consider the Dirichlet problem illustrated in Figure 11. More precisely, we look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq 1\}$ that vanishes on the vertical sides of $R$, and so that

$$u(x, 0) = f_0(x) \qquad \text{and} \qquad u(x, 1) = f_1(x)$$

where $f_0$ and $f_1$ are initial data which fix the temperature distribution on the horizontal sides of the rectangle. Use separation of variables to show that if $f_0$ and $f_1$ have Fourier expressions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \qquad \text{and} \qquad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left( \frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

**Solution:** Suppose $u(x, y) = P(x)Q(y)$ then using this in the Laplacian $\Delta u = 0$ we get:

$$\begin{aligned} & P''(x)Q(y) = -P(x)Q''(y) \\ & \frac{P''(x)}{P(x)} = -\frac{Q''(y)}{Q(y)} \\ & \implies \frac{P''(x)}{P(x)} = -\lambda \\ & \implies P''(x) + \lambda P(x) = 0 \\ & \implies -\frac{Q''(y)}{Q(y)} = -\lambda \\ & \implies Q''(y) - \lambda Q(y) = 0 \end{aligned}$$

Now we must check what sign $\lambda$ must have. We first check the case $\lambda = 0$ and input our initial conditions:

$$\begin{aligned}
&P''(x) = 0 \\
&\implies P'(x) = K \\
&\implies P(x) = Kx + G \\
&\implies u(0,t) = P(0) = G = 0 && \implies G = 0 \\
&\implies P(x) = Kx \\
&\implies u(\pi,t) = P(\pi) = K\pi = 0 && \implies K = 0 \\
&\implies u(x,t) = 0
\end{aligned}$$

Therefore, $\lambda \neq 0$. Now we check the case where $\lambda < 0$ that is $\lambda = -r^2$:

$$\begin{aligned}
&P''(x) - r^2 P(x) = 0 \\
&\implies e^{kx}\left(k^2 - r^2\right) = 0 \\
&\implies k = \pm r \\
&\implies P(x) = Ae^{rx} + Be^{-rx} \\
&\implies u(0,t) = P(0) = A + B = 0 \\
&\implies u(\pi,t) = P(\pi) = Ae^{r\pi} - Ae^{-r\pi} \\
&\implies e^{r\pi} = e^{-r\pi} \implies 2r = 0 \\
&\implies r = 0 \\
&\implies P(x) = A - A = 0
\end{aligned}$$

Therefore, $\lambda > 0$. Let $\lambda = k^2$:

$$\begin{aligned}
&P''(x) + k^2 P(x) = 0 \\
&\implies e^{mx}\left(m^2 + k^2\right) = 0 \\
&\implies k = \pm ik \\
&\implies P(x) = Ae^{ikx} + Be^{-ikx} = A\cos(kx) + iA\sin(kx) + B\cos(kx) - iB\sin(kx) \\
&= (A + B)\cos(kx) + (iA - iB)\sin(kx) \\
&\implies P(x) = C\cos(kx) + D\sin(kx) \\
&\implies u(0,t) = P(0) = C = 0 \implies C = 0 \\
&\implies u(\pi,t) = P(\pi) = 0 \\
&\implies P(x) = D\sin(kx)
\end{aligned}$$

Since $\lambda = k^2$ does not lead to any $u(x,t) = 0$ solutions then we use thing in our ODE in terms of $y$:

$$\begin{aligned}
&Q''(y) - k^2 Q(y) = 0 \\
&\implies e^{my}\left(m^2 - k^2\right) = 0 \\
&\implies m = \pm k \\
&\implies Q(y) = Ae^{-ky} + Be^{ky}
\end{aligned}$$

Now we introduce sinh and cosh in the following manner:

$$\begin{aligned}
\sinh(ky) &= \frac{1}{2}e^{ky} - \frac{1}{2}e^{-ky} \\
\cosh(ky) &= \frac{1}{2}e^{ky} + \frac{1}{2}e^{-ky} \\
e^{ky} &= \sinh(ky) + \cosh(ky) \\
e^{-ky} &= \cosh(ky) - \sinh(ky)
\end{aligned}$$

Therefore, we can write $Q(y)$ as follows:

$$\begin{aligned}
Q(y) &= A\left(\cosh(ky) - \sinh(ky)\right) + B\left(\sinh(ky) + \cosh(ky)\right) \\
Q(y) &= (A + B)\cosh(ky) + (B - A)\sinh(ky) \\
Q(y) &= E\cosh(ky) + F\sinh(ky)
\end{aligned}$$

Therefore, we can write $u(x, y)$ as follows and extend to the its Fourier sum as well:

$$\begin{aligned}
u(x, y)_k &= D\sin(kx)\left(E\cosh(ky) + F\sinh(ky)\right) \\
u(x, y) &= \sum_{k=1}^{\infty} D_k\sin(kx)\left(E_k\cosh(ky) + F_k\sinh(ky)\right)
\end{aligned}$$

Now we'll ensure our function $u(x, y)$ satisfies the initial conditions in terms of $y$:

$$\begin{aligned}
&u(x, 0) = \sum_{k=1}^{\infty} D_kE_k\sin(kx) = \sum_{k=1}^{\infty} A_k\sin(kx) \\
&\implies A_k = D_kE_k \\
&u(x, y) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(ky) + D_kF_k\sin(kx)\sinh(ky) \\
&u(x, 1) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(k) + D_kF_k\sin(kx)\sinh(k) = \sum_{k=1}^{\infty} B_k\sin(kx) \\
&\implies D_kF_k\sin(kx)\sinh(k) = B_k\sin(kx) - A_k\sin(kx)\cosh(k) \\
&\implies D_kF_k = \frac{B_k}{\sinh(k)} - \frac{A_k\cosh(k)}{\sinh(k)} \\
&u(x, y) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(ky) + \left(\frac{B_k}{\sinh(k)} - \frac{A_k\cosh(k)}{\sinh(k)}\right)\cdot\sin(kx)\sinh(ky) \\
&u(x, y) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(ky) + B_k\frac{\sin(kx)\sinh(ky)}{\sinh(k)} - A_k\frac{\cosh(k)\sin(kx)\sinh(ky)}{\sinh(k)} \\
&u(x, y) = \sum_{k=1}^{\infty} \left(\sin(kx)\cosh(ky) - \frac{\cosh(k)\sin(kx)\sinh(ky)}{\sinh(k)}\right)A_k + B_k\left(\frac{\sin(kx)\sinh(ky)}{\sinh(k)}\right) \\
&u(x, y) = \sum_{k=1}^{\infty} \sin(kx)\left[\left(\frac{\cosh(ky)\sinh(k) - \cosh(k)\sinh(ky)}{\sinh(k)}\right)A_k + B_k\left(\frac{\sinh(ky)}{\sinh(k)}\right)\right]
\end{aligned}$$

Now we compute the coeffecients of $A_k$ and $B_k$ in the following manner:

$$\frac{\cosh(ky)\sinh(k) - \cosh(k)\sinh(ky)}{\sinh(k)} = \frac{(\frac{1}{2}e^{ky} + \frac{1}{2}e^{-ky})(\frac{1}{2}e^{k} - \frac{1}{2}e^{-k}) - (\frac{1}{2}e^{k} + \frac{1}{2}e^{-k})(\frac{1}{2}e^{ky} - \frac{1}{2}e^{-ky})}{\sinh(k)}$$

$$= \frac{(\frac{1}{4}e^{k(1+y)} - \frac{1}{4}e^{k(y-1)} + \frac{1}{4}e^{k(1-y)} - \frac{1}{4}e^{-k(1+y)}) - (\frac{1}{4}e^{k(1+y)} - \frac{1}{4}e^{k(1-y)} + \frac{1}{4}e^{k(y-1)} - \frac{1}{4}e^{-k(1+y)})}{\sinh(k)}$$

$$= \frac{\frac{1}{4}e^{k(1-y)} - \frac{1}{4}e^{-k(1-y)} + \frac{1}{4}e^{k(1-y)} - \frac{1}{4}e^{-k(1-y)}}{\sinh(k)}$$

$$= \frac{\frac{\sinh(k(1-y))}{2} + \frac{\sinh(k(1-y))}{2}}{\sinh(k)} = \frac{\sinh(k(1-y))}{\sinh(k)}$$

Therefore we arrive at the following simplified expression for $u(x, y)$:

$$\boxed{u(x,y) = \sum_{k=1}^{\infty} \left[ \frac{\sinh(k(1-y))}{\sinh(k)} A_k + \frac{\sinh(ky)}{\sinh(k)} B_k \right] \sin(kx)}$$

# References


[1] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, vol. 1 of Princeton Lectures in Analysis, Princeton University Press, 2003.