Fourier Analysis

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1 The Genesis of Fourier Analysis

Add any questions about Chapter 1 here.

Exercises: Complete 3, 4, 5, 11.

3. A sequence of complex numbers $\{w\}_{n=1}^{\infty}$ is said to converge if there exists a $w \in \mathbb{C}$ such that

$$\lim_{n\to\infty} |w_n - w| = 0,$$

and we say that w is a limit of the sequence.
(a) Show that a converging sequence of complex numbers has a unique limit.

Proof. Suppose by way of contradiction that the sequence $\{w\}_{n=1}^{\infty}$ converges to complex numbers r and l. That is,

$$\lim_{n\to\infty} |w_n - r| = 0, \qquad \lim_{n\to\infty} |w_n - l| = 0$$

Consider the following,

$$\begin{aligned}
&|r-l| = |(r-w_n) + (w_n - l)| \le |r - w_n| + |w_n - l| = |w_n - r| + |w_n - l| \\
\implies &\lim_{n\to\infty} |r-l| \le \lim_{n\to\infty} |w_n - r| + \lim_{n\to\infty} |w_n - l| \\
\implies &|r-l| \le 0 + 0 = 0.
\end{aligned}$$

Now since $|r-l| \ge 0$ and as we just demonstrated it is also $|r-l| \le 0$ then $|r-l| = 0$ which implies $r = l$. Then by way of contradicition the limit of a converging sequence of complex numbers has unique limit. □

The sequence $\{w_n\}_{n=1}^{\infty}$ is said to be a **Cauchy sequence** if for every $\epsilon > 0$ there exists a positive integer N such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n, m > N.$$

(b) Prove that a sequence of complex numbers converges if and only if it is a Cauchy sequence. [Hint: A similar theorem exists for the convergence of a sequence of real numbers. Why does it carry over to sequences of complex numbers?]

Proof. ($\implies$) Let $\epsilon > 0$ be given. Suppose the sequence of complex numbers $\{w_n\}_{n=1}^{\infty}$ converges to w, that is,

$$\forall \epsilon > 0, \exists N \text{ such that } \forall n(n > N) \implies |w_n - w| < \epsilon$$

So take $n, m \in \mathbb{N}$ with $n > m \ge N$ such that

$$|w_n - w| < \frac{\epsilon}{2} \qquad \text{and} \qquad |w_m - w| < \frac{\epsilon}{2}$$

And consider the following,

$$\begin{aligned}
|w_n - w_m| = |(w_n - w) + (w - w_m)| \le |w_n - w| + |w - w_m| &= |w_n - w| + |w_m - w| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon \\
&\implies |w_n - w_m| < \epsilon
\end{aligned}$$

($\Longleftarrow$) Suppose the sequence $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\exists n, m, N \in \mathbb{N}$ such that

$$|w_n - w_m| < \epsilon \qquad \text{whenever } n > m, \geq N$$

And suppose $w_n = x_n + iy_n$ and $w_m = x_m + iy_m$ for any arbitrary real sequences $x_n, x_m y_n, y_m$. Then

$$\begin{aligned}
&\Longrightarrow |w_n - w_m|^2 = (x_n - x_m)^2 + (y_n - y_m)^2 < \epsilon^2 \\
&\Longrightarrow (x_n - x_m)^2 < \epsilon^2 \qquad \text{and} \qquad (y_n - y_m)^2 < \epsilon^2 \\
&\Longrightarrow |x_n - x_m| < \epsilon \qquad \text{and} \qquad |y_n - y_m| < \epsilon
\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Therefore, there exists real numbers x and y such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Then we define $w = x + iy$. Then by definition of a Cauchy sequence there exists a $P \in \mathbb{N}$ such that for any $n > P$ we have $|x_n - x| < \frac{\epsilon}{\sqrt{2}}$ and there exits a $Q \in \mathbb{N}$ such that for any $n > Q$ we have $|y_n - y| < \frac{\epsilon}{\sqrt{2}}$. Now take the bigger number between P and Q and denoted it R. And consider $n \in \mathbb{N}$ such that $n > R$ then,

$$\begin{aligned}
|w_n - w|^2 &= (x_n - x)^2 + (y_n - y)^2 < \epsilon^2 \\
&\Longrightarrow < \left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2 \\
&\Longrightarrow < \epsilon^2 \\
\Longrightarrow |w_n - w| < \epsilon
\end{aligned}$$

Therefore the Cauchy sequence $\{w_n\}_{n=1}^{\infty}$ converges. $\square$

A series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to converge if the sequence formed by the partial sums

$$S_N = \sum_{n=1}^{N} z_n$$

converges. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers such that the series $\sum_n a_n$ converges.

(c) Show that if $\{z_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying $|z_n| \leq a_n$ for all n, then the series $\sum_n z_n$ converges. [Hint: Use the Cauchy criterion].

Proof. Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_n a_n$ converges. Assume $\{z_n\}$ is a series of complex numbers such that $|z_n| \leq a_n$ for all n.

For any real sequence $\{a_n\}$, the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N}) \text{ such that } \forall(m, l \in \{m, l : m \geq l > N\}),$$

$$|s_m - s_l| = \left| \sum_{n=l+1}^{m} a_n \right| < \epsilon,$$

Where s_n is the nth partial sum of the series $\sum a_n$.

a_n is a convergent sequence of real numbers. Thus, by the Cauchy criterion, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, l \in \mathbb{N}$ such that $m \geq l > N$,

$$|s_m - s_l| = \left| \sum_{n=l+1}^{m} a_n \right| = \sum_{n=l+1}^{m} a_n < \epsilon,$$

where s_m and s_l are partial sums of the series $\sum a_n$.

Considering that

$$|z_n| \leq a_n$$

for each $n \in \{l+1, l+2, \cdots m\}$,

$$\implies \sum_{n=l+1}^{m} |z_n| \leq \sum_{n=l+1}^{m} a_n < \epsilon.$$

Next, we would like to show that

$$\left| \sum_{n=l+1}^{m} z_n \right| \leq \sum_{n=l+1}^{m} |z_n|.$$

Intuitively, (geometrically,) this should follow, because we can think of each z_i as a vector in the complex plane. The total distance of the journey taken by adding each vector z_i tip to tail is greater than or equal to the straightline distance from the origin to the tip of the last vector z_i

The proof is by induction on m, (starting from $m = l+1$, not $m = 1$.)

(Base case) For $m = l+1$,

$$\left| \sum_{n=l+1}^{l+1} z_n \right| = |z_{l+1}| = \sum_{n=l+1}^{l+1} |z_n|$$

(Induction step) Assume it to be true that for some $k \in \mathbb{N}$ s.t. $l+2 \leq k$,

$$\left| \sum_{n=l+1}^{k} z_n \right| \leq \sum_{n=l+1}^{k} |z_n|.$$

$$\implies |z_{k+1}| + \left| \sum_{n=l+1}^{k} z_n \right| \leq |z_{k+1}| + \sum_{n=l+1}^{k} |z_n|$$

On the right-hand side, we have

$$|z_{k+1}| + \sum_{n=l+1}^{k} |z_n| = \sum_{n=l+1}^{k+1} |z_n|$$

On the left-hand side, by the triangle inequality (Exercise 1d), we have

$$|z_{k+1}| + \left| \sum_{n=l+1}^{k} z_n \right| \geq \left| \sum_{n=l+1}^{k+1} z_n \right|$$

$$\implies \left|\sum_{n=l+1}^{k+1} z_n\right| \leq \sum_{n=l+1}^{k+1} |z_n|.$$

Thus, the induction hypothesis holds for all $n \in N$.

And recall that, for all $m \geq l+1$,

$$\sum_{n=l+1}^{m} |z_n| \leq \sum_{n=l+1}^{k+1} a_n \leq \left|\sum_{n=l+1}^{k+1} a_n\right| < \epsilon$$

Thus we have shown that, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $l > N, m \geq l$,

$$\left|\sum_{n=l+1}^{m} z_n\right| < \epsilon$$

$$\Leftrightarrow \left|\sum_{n=1}^{m} z_n - \sum_{n=1}^{l} z_n\right| < \epsilon$$

Or in other words, the sequence (of partial sums of a series) $S_N = \sum_{n=1}^{N} z_n$ is Cauchy, which (by exercise 3b,) implies that the sequence S_N converges.

Therefore, (by the fact given at the start of Exercise 3c,) the series

$$\sum_{n=1}^{\infty} z_n$$

converges. □

4. For $z \in \mathbb{C}$, we define the complex exponential by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(a) Prove that the above definition makes sense, by showing that the series converges for every complex number z. Moreover, show that the convergence is uniform on every bounded subset of $\mathbb{C}$.

Proof. Define the following partial sums

$$S_p = \sum_{n=0}^{p} \frac{z^n}{n!} \qquad S_q = \sum_{n=0}^{q} \frac{z^n}{n!} \qquad \text{where } p, q \in \mathbb{N} \text{ and } p > q.$$

Now consider the following:

$$S_p - S_q = \sum_{n=0}^{p} \frac{z^n}{n!} - \sum_{n=0}^{q} \frac{z^n}{n!} = \sum_{n=q+1}^{p} \frac{z^n}{n!}$$

$$\implies |S_p - S_q| = \left|\sum_{n=q+1}^{p} \frac{z^n}{n!}\right| \leq \sum_{n=q+1}^{p} \frac{|z|^n}{n!} \quad \text{Let } b = |z| \text{ where } b \in \mathbb{R} \text{ and } b \geq 0.$$

$$\implies |S_p - S_q| \leq \sum_{n=q+1}^{p} \frac{b^n}{n!} \leq \sum_{n=q+1}^{\infty} \frac{b^n}{n!}$$

Where we are able to extend the infinite sum to positive infinity as all terms in sum are non-negative. Furthermore the infinite series converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{b^{n+1}}{(n+1)!} \cdot \frac{n!}{b^n} = \frac{b}{n+1} \to 0 \quad \text{which shows convergence by Ratio Test.}$$

Therefore we choose an $N \in \mathbb{N}$ such that $p > q \geq N$ and such that

$$\sum_{n=N+1}^{\infty} \frac{b^n}{n!} < \epsilon$$

ensuring the following,

$$|S_p - S_q| \leq \sum_{n=q+1}^{p} \frac{b^n}{n!} \leq \sum_{n=q+1}^{\infty} \frac{b^n}{n!} \leq \sum_{n=N+1}^{\infty} \frac{b^n}{n!} < \epsilon$$

$$\implies |S_p - S_q| < \epsilon$$

Therefore, by the Cauchy criterion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely. Now showing convergence on a bounded subset S of $\mathbb{C}$ is the same proof but just stating that $b = \sup\{|z| : z \in S\}$. □

(b) If z_1, z_2 are two complex numbers, prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. [Hint: use the binomial theorem to expand $(z_1 + z_2)^n$, as well as the formula for binomial coefficients.]

(c) Show that if z is purely imaginary, that is, $z = iy$ with $y \in \mathbb{R}$, then

$$e^{iy} = \cos(y) + i\sin(y)$$

This is Euler's identity. [Hint: Use power series.]

Proof. Let $y \in \mathbb{R}$ then we can express e^{iy} as the following power series

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!}$$

Now we separate our infinite sum into two infinite sums of even and odd as follows:

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} y^{2k+1}}{(2k+1)!}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

Now we recognize both of these infinite sums as sine and cosine power series and arrive at the following:

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} = \cos(y) + i\sin(y)$$

□

(e) prove that $e^z = 1$ if and only if $z = 2\pi k i$ for some integer k.

Proof. ($\implies$) Assume $z = 2\pi ki$. By Euler's identity,

$$e^z = e^{2\pi ki} = \cos(2\pi k) + i\sin(2\pi k).$$

By the cyclic nature of trigonometric functions,
($cos(x) = cos(2\pi k + x)$ for $k \in \mathbb{Z}$,)

$$\begin{aligned} e^z &= \cos(0) + i\sin(0) \\ &= 1 + i(0) \\ &= 1. \end{aligned}$$

Conversely, assume $e^z = 1$.
Let $z = x + iy$ for some $x, y \in \mathbb{R}$. By Euler's identity,

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y) = 1.$$

Since the RHS has no imaginary part, this implies that

$$e^x i\sin y = 0, \quad e^x\cos y = 1.$$

$$\begin{aligned} &\implies \sin y = 0, \quad \cos y = e^{-x} \\ &\implies y \in \{2\pi k : k \in \mathbb{Z}\} \\ &\implies \cos y = 1 \\ &\implies 1 = e^{-x} \\ &\implies x = 0 \\ &\implies z = x + iy = iy, \end{aligned}$$

for some $y \in \{2\pi k : k \in \mathbb{Z}\}$.

$$\implies z = 2\pi ki,$$

for some $k \in \mathbb{Z}$. □

(f) show that every complex number $z = x + iy$ can be written in the form

$$z = re^{i\theta},$$

where r is unique and in the range $0 \leq r < \infty$, and $\theta \in \mathbb{R}$ is unique up to an integer multiple of 2π. Check that

$$r = |z| \text{ and } \theta = \arctan(y/x)$$

whenever these formulas make sense.}

5. Verify that $f(x) = e^{inx}$ is periodic with period 2π, and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Using Euler's identity,

$$f = e^{inx} = \cos nx + i\sin nx$$

Since $\cos z = \cos(z + 2\pi n)$ for all $z \in \mathbb{R}$, $n \in \mathbb{N}$, and the same is true for sin, we have that, for all $n \in \mathbb{N}$,

$$f(x+2\pi) = \cos(n(x+2\pi)) + i\sin(n(x+2\pi))$$
$$= \cos nx + i \sin nx = f(x),$$

which is the definition of having period 2π.

For $n = 0$,

$$e^{inx} = 1$$
$$\implies \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1dx$$
$$= \frac{1}{2\pi}(\pi - (-\pi)) = 1.$$

For $n \neq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi in}\left[e^{inx}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi in}\left[\cos nx + i\sin nx\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi in}\left[\cos(\pi n) - \cos(-\pi n) + i\sin(\pi n) - i\sin(-\pi n)\right]$$

Then, since $\cos(x) = \cos(-x)$, and $\sin(x) = -\sin(-x)$,

$$= \frac{1}{2\pi in}[(\cos \pi n - \cos \pi n) + (i\sin \pi n + i \sin \pi n)]$$
$$= \frac{1}{2\pi in}(2i\sin \pi n)$$
$$= \frac{1}{\pi n}\sin \pi n$$

For any integer multiple of π we have $\sin(\pi n) = 0$. Therefore,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = 0 \qquad \text{Whenever } n \neq 0$$

.

Use this fact to prove that, for $n.m \geq 1$, we have

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \; dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Since $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, then $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$. Then,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{e^{inx}+e^{-inx}}{2}\right)\left(\frac{e^{imx}+e^{-imx}}{2}\right)$$
$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{ix(m+n)} + e^{ix(n-m)} + e^{ix(m-n)} + e^{-ix(n+m)}\right)dx$$

If $n = m$ then any integral with exponent not equal 0 will equal 0 by our first verification. Therefore,

$$\frac{1}{4\pi}\left[0+\int_{-\pi}^{\pi} e^{ix(0)}dx+\int_{-\pi}^{\pi} e^{ix(0)}dx+0\right]$$

and similarly

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \ dx = \begin{cases} 1, & n=m \\ 0, & n\neq m \end{cases}$$

and

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \ dx = 0 \quad \text{for any } n, m.$$

[Hint: use $e^{inx}e^{-imx} + e^{inx}e^{imx}$ and $e^{inx}e^{-imx} - e^{inx}e^{imx}$.]

11. Show that if $n \in \mathbb{Z}$ the only solutions of the differential equation

$$r^2F''(r) + rF'(r) - n^2F(r) = 0,$$

which are twice differentiable when $r > 0$, are given by linear combinations of r^n and r^{-n} when $n \neq 0$, and 1 and $\log r$ when $n = 0$.

[Hint: If F solves the equation, write $F(r) = g(r)r^n$, find the equation satisfied by g, and conclude that $rg'(r) + 2ng(r) = c$ where c is a constant.]

Given $F(r)$ is twice differentiable and solves

$$r^2F''(r) + rF'(r) - n^2F(r) = 0,$$

Define g such that

$$\begin{aligned}
F(r) &=: g(r)r^n \\
\implies F'(r) &= ng(r)r^{n-1} + g'(r)r^n \\
\implies F''(r) &= n(n-1)g(r)r^{n-2} + ng'(r)r^{n-1} + ng'(r)r^{n-1} + g''(r)r^n \\
&= (n^2 - n)g(r)r^{n-2} + 2ng'(r)r^{n-1} + g''(r)r^n \\
\implies 0 &= r^2\Big((n^2 - n)g(r)r^{n-2} + 2ng'(r)r^{n-1} + g''(r)r^n\Big) + r\Big(ng(r)r^{n-1} + g'(r)r^n\Big) - n^2\Big(g(r)r^n\Big) \\
&= (n^2 - n)g(r)r^n + ng(r)r^n + 2ng'(r)r^{n+1} + g'(r)r^{n+1} + g''(r)r^{n+2} - n^2g(r)r^n \\
&= (n^2 - n + n - n^2)g(r)r^n + (2n+1)g'(r)r^{n+1} + g''(r)r^{n+2} \\
\implies 0 &= (2n+1)g'(r) + g''(r)r
\end{aligned}$$

Solve the second order ODE by reducing to first order. Let

$$\begin{aligned}
u(r) &:= g'(r) \\
\implies u'(r) &= g''(r) \\
\implies ru' &= -(2n+1)u \\
\int \frac{1}{u}\frac{du}{dr}dr &= -(2n+1)\int \frac{1}{r}dr \\
\log|u| &= (-2n+1)\log|r| + (2n+1)C \\
u &= e^{-(2n+1)\log|r|}e^{(2n+1)C} \\
&= e^{-(2n+1)}\left(e^{\log|r|}\right)^{-(2n+1)} \\
&= Dr^{-(2n+1)} \\
\int u = g &= D\int r^{-(2n+1)}dr \\
F(r) &= r^nD\int r^{-(2n+1)}dr
\end{aligned}$$

For $n = 0$,

$$\begin{aligned} F(r) &= D \int r^{-1} dr \\ &= D \log |r| + DE \\ &= A \log |r| + B. \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} F(r) &= Dr^n \int r^{-(2n+1)} dr \\ &= Dr^n r^{-2n} \frac{1}{-2n} + Dr^n C \\ &= Ar^{-n} + Br^n. \end{aligned}$$

Problem 1 Consider the Dirichlet problem illustrated in Figure 11. More precisely, we look for a solution of the steady-state heat equation $\Delta u = 0$ in the rectangle $R = \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \qquad \text{and} \qquad u(x, 1) = f_1(x)$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle. Use separation of variables to show that if f_0 and f_1 have Fourier expressions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \qquad \text{and} \qquad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Solution: Suppose $u(x, y) = P(x)Q(y)$ then using this in the Laplacian $\Delta u = 0$ we get:

$$\begin{aligned} &P''(x)Q(y) = -P(x)Q''(y) \\ &\frac{P''(x)}{P(x)} = -\frac{Q''(y)}{Q(y)} \\ &\implies \frac{P''(x)}{P(x)} = -\lambda \\ &\implies P''(x) + \lambda P(x) = 0 \\ &\implies -\frac{Q''(y)}{Q(y)} = -\lambda \\ &\implies Q''(y) - \lambda Q(y) = 0 \end{aligned}$$

Now we must check what sign λ must have. We first check the case $\lambda = 0$ and input our initial conditions:

$$\begin{aligned}
&P''(x) = 0 \\
&\implies P'(x) = K \\
&\implies P(x) = Kx + G \\
&\implies u(0,t) = P(0) = G = 0 && \implies G = 0 \\
&\implies P(x) = Kx \\
&\implies u(\pi,t) = P(\pi) = K\pi = 0 && \implies K = 0 \\
&\implies u(x,t) = 0
\end{aligned}$$

Therefore, $\lambda \neq 0$. Now we check the case where $\lambda < 0$ that is $\lambda = -r^2$:

$$\begin{aligned}
&P''(x) - r^2 P(x) = 0 \\
&\implies e^{kx}\left(k^2 - r^2\right) = 0 \\
&\implies k = \pm r \\
&\implies P(x) = Ae^{rx} + Be^{-rx} \\
&\implies u(0,t) = P(0) = A + B = 0 \\
&\implies u(\pi,t) = P(\pi) = Ae^{r\pi} - Ae^{-r\pi} \\
&\implies e^{r\pi} = e^{-r\pi} \implies 2r = 0 \\
&\implies r = 0 \\
&\implies P(x) = A - A = 0
\end{aligned}$$

Therefore, $\lambda > 0$. Let $\lambda = k^2$:

$$\begin{aligned}
&P''(x) + k^2 P(x) = 0 \\
&\implies e^{mx}\left(m^2 + k^2\right) = 0 \\
&\implies k = \pm ik \\
&\implies P(x) = Ae^{ikx} + Be^{-ikx} = A\cos(kx) + iA\sin(kx) + B\cos(kx) - iB\sin(kx) \\
&= (A + B)\cos(kx) + (iA - iB)\sin(kx) \\
&\implies P(x) = C\cos(kx) + D\sin(kx) \\
&\implies u(0,t) = P(0) = C = 0 \implies C = 0 \\
&\implies u(\pi,t) = P(\pi) = 0 \\
&\implies P(x) = D\sin(kx)
\end{aligned}$$

Since $\lambda = k^2$ does not lead to any $u(x,t) = 0$ solutions then we use thing in our ODE in terms of y:

$$\begin{aligned}
&Q''(y) - k^2 Q(y) = 0 \\
&\implies e^{my}\left(m^2 - k^2\right) = 0 \\
&\implies m = \pm k \\
&\implies Q(y) = Ae^{-ky} + Be^{ky}
\end{aligned}$$

Now we introduce sinh and cosh in the following manner:

$$\sinh(ky) = \frac{1}{2}e^{ky} - \frac{1}{2}e^{-ky}$$
$$\cosh(ky) = \frac{1}{2}e^{ky} + \frac{1}{2}e^{-ky}$$
$$e^{ky} = \sinh(ky) + \cosh(ky)$$
$$e^{-ky} = \cosh(ky) - \sinh(ky)$$

Therefore, we can write $Q(y)$ as follows:

$$Q(y) = A\left(\cosh(ky) - \sinh(ky)\right) + B\left(\sinh(ky) + \cosh(ky)\right)$$
$$Q(y) = (A+B)\cosh(ky) + (B-A)\sinh(ky)$$
$$Q(y) = E\cosh(ky) + F\sinh(ky)$$

Therefore, we can write $u(x,y)$ as follows and extend to the its Fourier sum as well:

$$u(x,y)_k = D\sin(kx)\left(E\cosh(ky) + F\sinh(ky)\right)$$
$$u(x,y) = \sum_{k=1}^{\infty} D_k\sin(kx)\left(E_k\cosh(ky) + F_k\sinh(ky)\right)$$

Now we'll ensure our function $u(x,y)$ satisfies the initial conditions in terms of y:

$$u(x,0) = \sum_{k=1}^{\infty} D_k E_k \sin(kx) = \sum_{k=1}^{\infty} A_k \sin(kx)$$
$$\implies A_k = D_k E_k$$
$$u(x,y) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(ky) + D_k F_k \sin(kx)\sinh(ky)$$
$$u(x,1) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(k) + D_k F_k \sin(kx)\sinh(k) = \sum_{k=1}^{\infty} B_k\sin(kx)$$
$$\implies D_k F_k \sin(kx)\sinh(k) = B_k\sin(kx) - A_k\sin(kx)\cosh(k)$$
$$\implies D_k F_k = \frac{B_k}{\sinh(k)} - \frac{A_k\cosh(k)}{\sinh(k)}$$
$$u(x,y) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(ky) + \left(\frac{B_k}{\sinh(k)} - \frac{A_k\cosh(k)}{\sinh(k)}\right)\cdot\sin(kx)\sinh(ky)$$
$$u(x,y) = \sum_{k=1}^{\infty} A_k\sin(kx)\cosh(ky) + B_k\frac{\sin(kx)\sinh(ky)}{\sinh(k)} - A_k\frac{\cosh(k)\sin(kx)\sinh(ky)}{\sinh(k)}$$
$$u(x,y) = \sum_{k=1}^{\infty}\left(\sin(kx)\cosh(ky) - \frac{\cosh(k)\sin(kx)\sinh(ky)}{\sinh(k)}\right)A_k + B_k\left(\frac{\sin(kx)\sinh(ky)}{\sinh(k)}\right)$$
$$u(x,y) = \sum_{k=1}^{\infty}\sin(kx)\left[\left(\frac{\cosh(ky)\sinh(k) - \cosh(k)\sinh(ky)}{\sinh(k)}\right)A_k + B_k\left(\frac{\sinh(ky)}{\sinh(k)}\right)\right]$$

Now we compute the coeffecients of A_k and B_k in the following manner:

$$\frac{\cosh(ky)\sinh(k)-\cosh(k)\sinh(ky)}{\sinh(k)}=\frac{(\frac{1}{2}e^{ky}+\frac{1}{2}e^{-ky})(\frac{1}{2}e^{k}-\frac{1}{2}e^{-k})-(\frac{1}{2}e^{k}+\frac{1}{2}e^{-k})(\frac{1}{2}e^{ky}-\frac{1}{2}e^{-ky})}{\sinh(k)}$$
$$=\frac{(\frac{1}{4}e^{k(1+y)}-\frac{1}{4}e^{k(y-1)}+\frac{1}{4}e^{k(1-y)}-\frac{1}{4}e^{-k(1+y)})-(\frac{1}{4}e^{k(1+y)}-\frac{1}{4}e^{k(1-y)}+\frac{1}{4}e^{k(y-1)}-\frac{1}{4}e^{-k(1+y)})}{\sinh(k)}$$
$$=\frac{\frac{1}{4}e^{k(1-y)}-\frac{1}{4}e^{-k(1-y)}+\frac{1}{4}e^{k(1-y)}-\frac{1}{4}e^{-k(1-y)}}{\sinh(k)}$$
$$=\frac{\frac{\sinh(k(1-y))}{2}+\frac{\sinh(k(1-y))}{2}}{\sinh(k)}=\frac{\sinh(k(1-y))}{\sinh(k)}$$

Therefore we arrive at the following simplified expression for $u(x,y)$:

$$\boxed{u(x,y)=\sum_{k=1}^{\infty}\left[\frac{\sinh(k(1-y))}{\sinh(k)}A_k+\frac{\sinh(ky)}{\sinh(k)}B_k\right]\sin(kx)}$$

References

[1] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, vol. 1 of Princeton Lectures in Analysis, Princeton University Press, 2003.

Homework 2

January 22, 2026

These are notes for the directed study on Fourier Analysis from Stein and Shakarchi [1].

Exercises: Complete 2, 4, 5, 6, 8

2.)

4.)

5.)

6.)

8.)

Verify that $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}$ *is the Fourier series of the* 2π*-periodic sawtooth function, defined by* $f(0)=0$*, and*

$$f(x)=\begin{cases}-\frac{\pi}{2}-\frac{x}{2} & \textit{if } -\pi<x<0,\\ \frac{\pi}{2}-\frac{x}{2} & \textit{if } 0<x<\pi.\end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x *(by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of* $f(x)$ *as* x *approaches the origin from the left and the right.*

References

[1] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, vol. 1 of Princeton Lectures in Analysis, Princeton University Press, 2003.

Homework 2b

January 23, 2026

Exercises: Complete

Bibliography

[1] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, vol. 1 of Princeton Lectures in Analysis, Princeton University Press, 2003.