

# Solutions to Bain and Engelhardt's Introduction to Probability and Mathematical Statistics

## 06.01

**Given: the pdf of x**  $f_x(x) = \begin{cases} 4x^3 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of a)  $Y = X^4$**

**Setup:** Use the CDF technique to get the CDF of Y in terms of a CDF of X  
 $F_Y(y) = P[Y \leq y] = P[X^4 \leq Y] = P[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}] = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}})$

**Steps: i)** Differentiate with respect to y to find an equation given in terms of the pdf of x:  
 $f_y(y) = \frac{d}{dy}F_X(y^{\frac{1}{4}}) - \frac{d}{dy}F_X(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{d}{dy}y^{\frac{1}{4}} - f_x(-y^{\frac{1}{4}})\frac{d}{dy}(-y^{\frac{1}{4}}) = f_x(y^{\frac{1}{4}})\frac{y^{-\frac{3}{4}}}{4} - f_x(-y^{\frac{1}{4}})\frac{-y^{-\frac{3}{4}}}{4}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:**  $f_y(y) = \begin{cases} 4y^{\frac{3}{4}}\frac{1}{4y^{\frac{3}{4}}} & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases} = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of b)  $W = e^X$**

**Setup:** Use the CDF technique to get the CDF of W in terms of a CDF of X  
 $F_W(w) = P[W \leq w] = P[e^X \leq W] = P[X \leq \ln W] = F_X(\ln W)$

**Steps: i)** Differentiate with respect to w to find an equation given in terms of the pdf of x:  
 $f_w(w) = \frac{d}{dw}F_X(\ln W)\frac{d}{dw}(\ln w) = f_x(\ln w)\frac{1}{w}$

**ii)** Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:**  $f_w(w) = \begin{cases} \frac{4(\ln w)^3}{w} & , \quad 1 < w < e \\ 0 & , \quad o/w \end{cases}$

**Find: PDF of c)  $Z = \ln x$**

**Setup:** Use the CDF technique to get the CDF of Z in terms of a CDF of X  
 $F_Z(z) = P[Z \leq z] = P[\ln x \leq z] = P[X \leq e^z] = F_X(e^z)$

**Steps: i)** Differentiate with respect to z to find an equation given in terms of the pdf of x:  
 $f_z(z) = \frac{d}{dz}F_X(e^z) = f_x(e^z)\frac{de^z}{dz}$

ii) Plug in the original limits and function for the pdf of x, and compute the cdf for y

**Result:** 
$$f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$$

**Find: PDF of d)  $U = (X - 0.5)^2$**

**Setup:** Use the CDF technique to get the CDF of U in terms of a CDF of X

$$F_U(u) = P[U \leq u] = P[(X - 0.5)^2 \leq u] = P[|X - 0.5| \leq u^{0.5}] = F_X(u^{1/2} + 1/2) - F_X(-u^{1/2} + 1/2)$$

**Steps: i)** Differentiate with respect to u to find an equation given in terms of the pdf of x:

$$f_U(u) = \frac{d}{du} F_X(u^{1/2} + 1/2) = f_x(u^{1/2} + 1/2) \frac{d}{du} (u^{1/2} + 1/2) - f_x(-u^{1/2} + 1/2) \frac{d}{du} (-u^{1/2} + 1/2)$$

$$f_x(u^{1/2} + 1/2) 1/2 u^{-1/2} - f_x(-u^{1/2} + 1/2) 1/2 u^{-1/2}$$

ii) INCOMPLETE

**Result:** 
$$f_Z(z) = \begin{cases} 4e^{4z} & , \quad -\infty \leq z < 0 \\ 0 & , \quad o/w \end{cases}$$

06.02

**Given:**  $X \sim Unif(0, 1)$

**Find: a) PDF of  $Y = X^{1/4}$**

**Setup:**  $F_Y(y) = P[Y \leq y] = P[X^{1/4} \leq y] = P[X \leq y^4] = F_X(y^4)$

**Steps: i)** find the pdf of x. Because X is a Uniform distribution with parameters 1 and 0, the pdf, which for Unif(a,b) is  $1/(b-a)$  where  $a < x < b$ . Here, Unif(0,1) gives  $1/1-0 = 1$

ii) Differentiate with respect to y to find an equation given in terms of the pdf of x:

$$f_Y(y) = \frac{d}{dy} F_X(y^4) = 4y^3$$

**Result:** 
$$f_Y(y) = \begin{cases} 4y^3 & , \quad 0 < y < 1 \\ 0 & , \quad o/w \end{cases}$$

**Find: b) PDF of  $W = e^{-X}$**

**Setup:**  $F_W(z) = P[W \leq w] = P[e^{-X} \leq w] = P[-X \leq \ln w] = P[X \geq -\ln w] = 1 - F_x(-\ln w)$

**Steps: i)** find the pdf of x. See part a) for an explanation of why it is 1 when  $a < x < b$

ii) Differentiate with respect to  $w$  to find an equation given in terms of the pdf of  $x$ :  
 $f_W(w) = -\frac{d}{dw}F_X\frac{d}{dw}(-\ln w) = -f_X(-\ln w)\frac{-1}{w} \quad \text{for } e^{-1} < w < 1 = -\frac{1}{w}$

**Result:**  $f_W(w) = \begin{cases} \frac{1}{w} & , \quad e^{-1} < w < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: c) PDF of  $Z = 1 - e^{-X}$**

**Setup:**  $F_Z(z) = P[Z \leq z] = P[1 - e^{-X} \leq z] = P[-e^{-X} \leq z - 1] = P[e^{-X} \geq 1 - z] = P[-X \geq \ln(1 - z)] = P[X \leq -\ln(1 - z)] = F_X(-\ln(1 - z))$

**Steps: i)** find the pdf of  $x$ . See part a) for an explanation of why it is 1 when  $a < x < b$

ii) Differentiate with respect to  $w$  to find an equation given in terms of the pdf of  $x$ :  
 $f_Z(z) = -\ln(1 - z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad \text{for } 0 < z < 1 - e^{-1}$

**Result:**  $f_Z(z) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < 1 - e^{-1} \\ 0 & , \quad o/w \end{cases}$

**Find: d) PDF of  $U = X(1 - X)$**

**Setup:**  $F_U(u) = P[U \leq u] = P[X(1 - x) \leq u] = P[-X^2 + X \leq u] = P[-(X - 1/2)^2 \leq u - 1/4] = P[(X - 1/2)^2 \geq 1/4 - u] = P[|(X - 1/2)| \geq (1/4 - u)^{1/2}] =$

**Steps: i)** find the pdf of  $x$ . See part a) for an explanation of why it is 1 when  $a < x < b$

ii) INCOMPLETE:

$f_Z(z) = -\ln(1 - z) = -\frac{-1}{1-z} = \frac{1}{1-z} \quad \text{for } 0 < z < e^{-1}$

**Result:**  $f_W(w) = \begin{cases} \frac{1}{1-z} & , \quad 0 < z < e^{-1} \\ 0 & , \quad o/w \end{cases}$

06.03

**Given: PDF**  $f_R(r) = \begin{cases} 6r(1 - r) & , \quad 0 < r < 1 \\ 0 & , \quad o/w \end{cases}$

**Find: Distribution of the circumference**

**Setup:** The circumference is  $c = 2\pi r$ . We have the pdf in terms of  $x$ , so this is the transformation:

$F_C(c) = P[C \leq c] = P[2\pi r \leq c] = P[r \leq c/2\pi] = F_x(c/2\pi)$

**Steps: i)** Differentiate with respect to  $c$  to find an equation given in terms of the pdf of  $x$ .  
 $f_C(c) = \frac{d}{dc} F_R(c/2\pi) = f_R(c/2\pi) \frac{d}{dc} (c/2\pi) = f_R(c/2\pi)(1/2\pi)$

**ii)** Plug the original pdf back into this new form:

$$f_C(c) = \frac{6c}{2\pi} (1 - (c/2\pi))(1/2\pi) = \frac{6c(2\pi-c)}{(2\pi)^3} \quad \text{if } 0 < c < 2\pi$$

**Result:** 
$$f_C(c) = \begin{cases} \frac{6c(2\pi-c)}{(2\pi)^3} & , \quad 0 < c < 2\pi \\ 0 & , \quad o/w \end{cases}$$

## Find: Distribution of the area

**Setup:** The area is  $a = \pi r^2$  so the cdf  $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi] = P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_R((a/\pi)^{1/2}) - F_R(-(a/\pi)^{1/2})$

**Steps: i)** Differentiate with respect to  $a$  to find an equation in terms of the pdf of  $x$ .

$$f_A(a) = \frac{d}{da} F_R((a/\pi)^{1/2}) - \frac{d}{da} F_R(-(a/\pi)^{1/2}) = f_R((a/\pi)^{1/2}) \frac{d}{da} (a/\pi)^{1/2} - f_R(-(a/\pi)^{1/2}) \frac{d}{da} (-(a/\pi)^{1/2})$$

**Result:** 
$$f_A(a) = \begin{cases} \frac{3(\sqrt{\pi}-\sqrt{a})}{\pi^{3/2}} & , \quad 0 < a < \pi \\ 0 & , \quad o/w \end{cases}$$

**06.04** Please double check the results of this solution

For  $X \sim WEI(\theta, \beta)$  we have the CDF as  $F_X = 1 - e^{-\frac{x}{\theta} \beta}$  and the pdf is  $f(x) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-\frac{x}{\theta} \beta}$

a) We make the transformation by the CDF method:

$$\begin{aligned} \Pr(Y \leq y) &= \Pr\left(\frac{X^\beta}{\theta} \leq y\right) \\ &= \Pr\left(X \leq \theta y^{\frac{1}{\beta}}\right) \\ &= F_X\left(\theta y^{\frac{1}{\beta}}\right) \\ &= 1 - e^{-\frac{\theta y^{\frac{1}{\beta}}}{\theta} \beta} \\ &= 1 - e^{-y}, \text{ where } 0 < y \end{aligned}$$

So we have our CDF. For the pdf we simply take the derivative of the above. So  $pdf = e^{-y}$  where  $0 < y$

b)  $W = \ln X$ . Again, the most simply method to get the CDF, and in turn the pdf is the CDF method.

$$\Pr(W \leq w) = \Pr(\ln X \leq w) \tag{1}$$

$$= \Pr(X \leq e^w) \tag{2}$$

$$= F(e^w) \tag{3}$$

$$= 1 - e^{-\frac{e^w}{\theta} \beta} \text{ where } 0 < w \tag{4}$$

$$\tag{5}$$

Again we simply differentiate to get the pdf. which turns out to be  $\beta e^{\beta w} \theta^{-\beta} e^{-\frac{e^w}{\theta} \beta}$ ,  $0 < w$   
c)

**06.10** Suppose  $X$  has pdf  $f_X(x) = \frac{1}{2}e^{-|x|}$  for all real  $x$ .

(a) Find the pdf of  $Y = |X|$ .

CDF Method

$$F_Y(y) = P[Y \leq y] = P[|x| \leq y] = P[-y \leq X \leq y] = F_X(y) - F_X(-y)$$

$$f_Y(y) = \frac{dF_X(y)}{dy} - \frac{dF_X(-y)}{dy}$$

$$f_Y(y) = f_X(y) \frac{dy}{dx} - f_X(-y) \left( \frac{-dy}{dy} \right)$$

$$f_Y = \frac{1}{2}e^{-y} + \frac{1}{2}e^{-y} = e^{-y} \quad y > 0$$

(b) Let  $W = 0$  if  $X \leq 0$  and  $W = 1$  if  $X > 0$ . Find the CDF of  $W$

$$F_W(w) = P[W = 0] = \frac{1}{2}$$

$$F_W(w) = P[W = 1] = \frac{1}{2}$$

$$F_W(w) =$$

$$\begin{cases} 0 & w \leq 0 \\ \frac{1}{2} & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

**06.13**  $X$  has pdf

$$f(x) = \begin{cases} \frac{x^2}{24} & -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

We want pdf of the CDF  $Y = X^2$  with regions:  $(-2, 0) \cup [0, 4)$

$$[F_X(\sqrt{y}) - F_X(-\sqrt{y})] = \left[ f_X(\sqrt{y}) \left( \frac{1}{2} \sqrt{y} \right) - f_X(-\sqrt{y}) \left( -\frac{1}{2} \sqrt{y} \right) \right]$$

$$f_Y(y) = \begin{cases} \frac{y}{48\sqrt{y}} + \frac{y}{48\sqrt{y}} & 0 < y < 4 \\ \frac{y}{48\sqrt{y}} & 4 \leq y \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\sqrt{y}}{24} & 0 < y < 4 \\ \frac{\sqrt{y}}{48} & 4 \leq y \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

**06.14**

**Given: Joint PDF**  $f(x, y) = \begin{cases} 4e^{-2(x+y)} & , \quad 0 < x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

**Find: a) CDF of W=X+Y**

**Setup:**  $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps:**

i) Express as a sum of probabilities, replace probabilities with binomials

ii) Simplify and Use Combinatorial Identity

**Result:**  $\binom{n+m}{k}$

**06.15** This is a simplified version of example 6.4.5.

$X_1, X_2 \sim POI(\lambda)$  so the MGF of both is  $e^{\lambda(e^t-1)}$ . Thus by theorem 6.4.4

$$M_Y(t) = e^{\lambda(e^t-1)} e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)} \sim POI(2\lambda)$$

The pdf then of Y is

$$f_Y(y) = \begin{cases} \frac{e^{-2\lambda}(2\lambda)^y}{y!} & y = 0, 1, 2, \dots \\ 0 & otherwise. \end{cases}$$

**06.16** Note: the pdf of  $f_{x_1, x_2} = \frac{1}{x_1^2} \frac{1}{x_2^2}$

a) We need to find  $f_{u,v} = f_{x_1, x_2}(x_1(u, v), x_2(u, v))|J|$  where J is our jacobian. First we let  $u = x_1 x_2$  and  $v = x_1$  thus  $x_1 = v$  and  $x_2 = \frac{u}{v}$ , now we can find J.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{1}{v}$$

Finally, our pdf is:

$$\begin{aligned} f_{U,V}(u, v) &= f_{x_1, x_2}\left(v, \frac{u}{v}\right) \left| \frac{1}{v} \right| \\ &= \frac{1}{v^2} \frac{1}{\left(\frac{u}{v}\right)^2} \left| \frac{1}{v} \right| \\ &= \frac{1}{u^2 v}, 1 < v < u < \infty \end{aligned}$$

b) We need to find  $f_u(u)$  given  $f_{U,V}(u, v) = \frac{1}{u^2 v}, 1 < v < u < \infty$

$$\begin{aligned}
f_u(u) &= \int_1^u \frac{1}{u^2 v} dv \\
&= \frac{1}{u^2} \ln(v) \Big|_1^u \\
&= \frac{1}{u^2} (\ln(u) - 0) \\
&= \frac{1}{u^2} \ln(u), 1 < u < \infty
\end{aligned}$$

**06.17 6.18** It is given that  $X$  and  $Y$  have a joint pdf given by

$$f(x, y) = e^{-y} \quad \text{if } 0 < x < y < \infty. \quad (6)$$

**(a): Find the joint pdf of  $S = X + Y$  and  $T = X$ .**

This can be done using the joint transformation method. By rearranging the above formulas we get  $X = T$  and  $Y = S - T$ . Then it is easy to get the jacobian

$$J = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad (7)$$

whose determinant is clearly one. Note that the order in which you take partial derivatives is unimportant provided you are consistent - you will get the same determinant either way. Then we substitute in  $X = T$  and  $Y = S - T$  into the pdf and multiply by the determinant of the jacobian:

$$f_{S,T}(s, t) = f_{X,Y}(x(s, t), y(s, t)) \times 1 = \begin{cases} e^{t-s} & \text{if } 0 < t < s/2 \\ 0 & \text{otherwise} \end{cases}. \quad (8)$$

The bounds of the function can be found in a few different ways. One way is to consider the bounds of the original function,  $0 < x < y < \infty$ . We can substitute in the new formulas for  $X$  and  $Y$  to get

$$0 < t < s - t < \infty. \quad (9)$$

Then it is apparent that

$$0 < 2t < s < \infty, \quad (10)$$

which then yields

$$0 < t < s/2, \quad (11)$$

the bounds of our new function.

**(b): Find the marginal pdf of  $T$ .**

The easiest way to do this is to "integrate out"  $S$  from the joint pdf we derived:

$$\begin{aligned}
f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s, t) ds = \int_{2t}^{\infty} e^{t-s} ds \\
&= e^t \int_{2t}^{\infty} e^{-s} ds = e^t (-e^{-s} \Big|_{2t}^{\infty}) \\
&= e^{-t} \quad \text{if } t > 0.
\end{aligned} \quad (12)$$

**(c): Find the marginal pdf of S.**

This is just like part (b), except this time "integrate out" T:

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)dt = \int_0^{s/2} e^{t-s} ds \\ &= e^{-s} \int_0^{s/2} e^t dt = e^{-s} (e^{t|_0^{s/2}}) \\ &= e^{-s} (e^{s/2} - 1) \quad \text{if } s > 0. \end{aligned} \tag{13}$$

**6.21** Let  $X$  and  $Y$  be continuous random variables with a joint density function given by

$$f_{X,Y}(x,y) = 2(x+y) \quad \text{if } 0 < x < y < 1 \quad \text{and } 0 \quad \text{otherwise.} \tag{14}$$

**(a) Find the joint density function of  $S = X$  and  $T = XY$ .**

We can solve for  $X$  and  $Y$  in terms of the new variables, to get  $X = S$  and  $Y = T/S$ . Then the jacobian is given by

$$J = \begin{pmatrix} 1 & 0 \\ -T/S^2 & 1/S \end{pmatrix}. \tag{15}$$

Then the new pdf is given by

$$f_{T,S}(s,t) = f_{X,Y}(x(s,t), y(s,t)) \times |1/s| = \begin{cases} 2(s + t/s) |1/s|, & 0 < s^2 < t < s < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The bounds of this equation can be interpreted in the following way: the old triangular region in the  $xy$  plane got transformed to the region in the  $st$  plane between the lines  $T = S$  and  $T = S^2$ .

**(b) Find the marginal pdf of T.**

To find the marginal of  $T$ ,  $s$  needs to be "integrated out."

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_{S,T}(s,t)ds \\ &= \int_t^{\sqrt{t}} 2(s + t/s) |1/s| ds = \int_t^{\sqrt{t}} 2(1 + t/s^2) ds \\ &= 2 \int_t^{\sqrt{t}} ds + 2t \int_t^{\sqrt{t}} 1/s^2 ds = 2(\sqrt{t} - t) + 2t(-1/s|_t^{\sqrt{t}}) \\ &= 2\sqrt{t} - 2t + 2 - 2\sqrt{t} = \begin{cases} 2 - 2t & t \in (0, 1) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{17}$$

**06.23** We will use the property that independent identically distributed random variables has the form of 6.4.4,  $M_Y(t) = [M_X(t)]^n$  where  $Y = X_1 + X_2 + \dots + X_n$ . then since  $X_i \sim \text{GEO}(p)$



$$\begin{aligned}
Mgf(Y) &= M_{X_1}(t)M_{X_2}(t)...M_{X_k}(t) \\
&= (M_X(t))^k \\
&= \left(\frac{pe^t}{1-qe^t}\right)^k \sim \text{NegativeBinomial}(k, p)
\end{aligned}$$

**06.25** First note,  $X_1, X_2, X_3, X_4$  are all independant, but they are not IID as only  $X_2, X_3, X_4 \sim POI(5)$  with  $X_1$  not being listed. So formula 6.4.5 does not hold. 6.4.4 does though.

A)

$$\begin{aligned}
Mgf(Y) &= M_{X_1}(t)M_{X_2+X_3+X_4}(t) \\
&= M_{X_1}(t)(M_{X_i}(t))^3
\end{aligned}$$

Since  $X_2, X_3, X_4$  are iid 6.4.5 holds for moving to this mgf

$$\begin{aligned}
&= M_{X_1}(t)(e^{\mu(e^t-1)})^3 \\
&= M_{X_1}(t)e^{3\mu(e^t-1)} \\
&= M_{X_1}(t)e^{15(e^t-1)} \\
e^{25(e^t-1)} &= M_{X_1}(t)e^{15(e^t-1)} \\
\frac{e^{25(e^t-1)}}{e^{15(e^t-1)}} &= M_{X_1}(t) \\
e^{10(e^t-1)} &= M_{X_1}(t) \sim POI(10)
\end{aligned}$$

B) For  $W = X_1 + X_2$  we have  $X_1 \sim POI(10)$  and  $X_2 \sim POI(5)$ . So  $POI(10 + 5) = POI(15)$   
**06.29**

**Given: PDF**  $f(x) = \begin{cases} \frac{1}{x^2} & , \quad 1 \leq x < \infty, 0 < y < \infty \\ 0 & , \quad o/w \end{cases}$

**Find: a) Joint PDF of the order statistics**

**Setup:**  $F_w(w) = P[W \leq w] = P[X + Y \leq w]$

**Steps: i)** Differentiate with respect to  $a$  to find an equation in terms of the pdf of  $x$ .  
 $f_A(a) = \frac{d}{da} F_R(a/\pi)^{1/2} - \frac{d}{da} F_R - (a/\pi)^{1/2} = f_R[(a/\pi)^{1/2}] \frac{d}{da} (a/\pi)^{1/2} - f_R - (a/\pi)^{1/2} \frac{d}{da} - (a/\pi)^{1/2}$

**ii)** Simplify and Use Combinatorial Identity

**Result:**  $\binom{n+m}{k}$

**Find: b) PDF of the smallest order statistic  $Y_1$**

**Setup:**

**Steps: i)**

**Result:**

**Find: c) PDF of the largest order statistic  $Y_n$**

**Setup:**

**Steps: i)**

**Result:**

**Find: d) PDF of the sample range  $R = Y_n - Y_1$ , for  $n = 2$**

**Setup:** The area is  $a = \pi r^2$  so the cdf  $F_A(a) = P[A \leq a] = P[\pi r^2 \leq a] = P[r^2 \leq a/\pi] = P[|r| \leq (a/\pi)^{1/2}] = P[-(a/\pi)^{1/2} \leq c \leq (a/\pi)^{1/2}] = F_r(a/\pi)^{1/2} - F_r(-(a/\pi)^{1/2})$

**Steps: i)**

**Result:**

**Find: e) PDF of the sample median  $R = Y_{(n+1)/2} - Y_1$ , for  $n$  odd so that  $r = (n+1)/2$**

**Setup:**

**Steps: i)**

**Result:** [11pt]article

a) The PDF of the smallest order statistic is provided by formula 6.5.4

$$g_1(y_1) = n[1 - F(y_1)]^{n-1}f(y_1)$$

In this case we have  $g_1(y_1) = n(1 - (1 - e^{-y})e^y)$  when  $y_1 > 0$

b) The PDF of the largest order statistic is provided by formula 6.5.6  $g_n(y_n) = n[F(y_n)]^{n-1}f(y_n)$

$g_n(y_n) = n[1 - e^{-y_n}]^{n-1}e^{-y_n}$  Simplifying provides

$ne^{-y_n}(1 - e^{-y_n})^{n-1}$  when  $y_n > 0$  c) Because the exponential distribution has the memoryless property, the difference between the first order statistic and the greatest order statistic won't be conditional on the value of the first order statistic (so we can treat it as zero). The probability that all the other observations  $(n-1)$  fall into the range is  $P(R < r) \in (0, r)$  So  $P(R < r) = [\int_0^r e^{-x} dx]^{n-1} = (1 - e^{-r})^{n-1}$  This is  $P(R \leq r)$  which is the CDF, differentiate to get the PDF:  $(n-1)(1 - e^{-r})^{n-2} e^{-r}$