General single qubit state

- ⋆ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, or $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, $|\alpha|^2+|\beta|^2=1$
- ⋆ $|\psi\rangle$ is an eigenstate of $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ with eigenvalue 1

Evolution

- ⋆ $i\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle$ (Schrödinger)
- ⋆ if H time-independent then $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, $\quad U(t) = e^{-iHt}$

Pauli matrices

- ⋆ $\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
- ⋆ $\mathrm{Tr}[\sigma_0] = 2$, $\mathrm{Tr}[\sigma_i] = 0$ for $i = 1,2,3$ $\qquad S_i = \frac{\hbar}{2}\sigma_i$
- ⋆ $\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\epsilon_{ijk}\sigma_k \quad [\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$ $\qquad$ ⋆ $V = \boldsymbol{v}\cdot\boldsymbol{\sigma} \Rightarrow V^2 = \mathbb{I}$
- ⋆ $\{\sigma_i,\sigma_j\} = \sigma_i\sigma_J + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{I}$ $\qquad$ ⋆ Hermitian matrices
- ⋆ Paulis form an orthonormal basis with $\mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij}$
- ⋆ The Pauli α operator, $\alpha \in \{x,y,z\}$, rotates the state by π around the α-axis.
- ⋆ Paulis as generators: any single qubit hamiltonian can be written as $H = \omega\boldsymbol{n}\cdot\boldsymbol{\sigma}$
  $U = \exp(-i\omega\boldsymbol{\sigma}\cdot\boldsymbol{n}t) = \cos(\omega t)\mathbb{I} - i\sin(\omega t)\boldsymbol{n}\cdot\boldsymbol{\sigma}$
  Causes a qubit state to rotate around $\boldsymbol{n}$ at a rate $2\omega t$.
- ⋆ E.g. $R_z(\theta) = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)Z$
- ⋆ $\langle\psi|\sigma_x|\psi\rangle = \sin\theta\cos\phi \quad \langle\psi|\sigma_y|\psi\rangle = \sin\theta\sin\phi \quad \langle\psi|\sigma_z|\psi\rangle = \cos\theta$

Observables

- ⋆ $M = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ Hermitian
- ⋆ $\langle M\rangle = \langle\psi|M|\psi\rangle = \sum_k \lambda_k P_k$, $P_k = |\langle\lambda_k|\psi\rangle|^2$
- ⋆ $\Pi_k = |\lambda_k\rangle\langle\lambda_k|$ Projector $\Rightarrow P_k = \langle\psi|\Pi_k|\psi\rangle$, $\sum_k \Pi_k = 1$
- ⋆ Measurement $\Rightarrow$ State collapses to $\frac{\Pi_k|\psi_k\rangle}{\sqrt{P_k}}$

Composite systems $\quad \mathcal{H}_{ABC\ldots} = \mathcal{H}_a\otimes\mathcal{H}_b\otimes\mathcal{H}_c\ldots$

- ⋆ The resulting space has dimension $n_An_Bn_C\ldots$
- ⋆ Operators $T_{AB}|\lambda_ij\rangle = (T_a\otimes T_b)(|\mu_i\otimes\nu_j\rangle) = T_A|\mu_i\rangle\otimes|\nu_k\rangle$
- ⋆ $[T_A\otimes\mathbb{I}_B, \mathbb{I}_A\otimes T_B] = 0, \quad \{T_A\otimes\mathbb{I}_B, \mathbb{I}_A\otimes T_B\} = 2(T_A\otimes T_B)$
  $e^{A\otimes\mathbb{I}+\mathbb{I}\otimes B} = e^A\otimes e^B$
- ⋆ **Global** measurem. $T = T_A\otimes T_B \to T|T_i\rangle = t_i|T_i\rangle, \quad |\psi\rangle = \sum_i |T_i\rangle\langle T_i|\psi\rangle$
- ⋆ **Partial** measurement T_A: if $|\psi\rangle = \sum_{ij}|T_{A,i}\rangle\otimes|T_{B,j}\rangle = \sum_i |T_{A,i}\rangle\otimes|\phi_{B,i}\rangle$
  then $P_i = \sum_j |c_{ij}|^2$ and system collapses to $|\psi\rangle' \propto |T_{A,i}\rangle\otimes|\phi_{B,i}\rangle$
- ⋆ $P_{\hat{O}}(\lambda|\psi) = \langle\psi|\Pi^A_\lambda|\psi\rangle$ is the **probability** of a measurement of operator $\hat{O}$ yielding its eigenvalue λ, with A the subspace of the meas., $\Pi^A_\lambda = |\lambda\rangle\langle\lambda|_A\otimes\mathbb{I}_B$ and $|\lambda\rangle$ the eigenket corresponding to eigenvalue λ
  For measurements on both subspaces use $\Pi_\lambda = |\lambda\rangle\langle\lambda|_A\otimes|\kappa\rangle\langle\kappa|_B$
- ⋆ **Entangled** state: its coefficients cannot be written as the product of two independent coefficients. **Separable** state: the global wave function can be written as the product of two wavefunctions corresponding to subsystems A and B (measures performed on one part do not affect the other).
- ⋆ Condition of separability for 2 qubits: if $c_{ij} = c_i^{(A)}c_j^{(B)}$, with $i,j\in\{0,1\}$, then $\det\begin{pmatrix}c_{00} & c_{01}\\ c_{10} & c_{11}\end{pmatrix} = 0 \Leftrightarrow$ separable. E.g. in $(\alpha|0\rangle_A\otimes\beta|0\rangle_B)$, $c_{00} = \alpha\beta$.
- ⋆ $H_{AB} = H_A\otimes\mathbb{I}_B + \mathbb{I}_A\otimes H_B \Longrightarrow e^{-itH_{AB}} = e^{-itH_A}\otimes e^{-itH_B}$
  A separable unitary generates no entanglement when applied to a separable state. The reduced density matrices of each partition remain pure whenever the full state remains separable.

Quantum eraser

- ⋆ $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$, $|\swarrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$
- ⋆ $|H\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\swarrow\rangle)$, $|V\rangle = \frac{1}{\sqrt{2}}(|\nearrow\rangle - |\swarrow\rangle)$
- ⋆ $P(x) = \langle\psi(x,t)|(|x\rangle\langle x|\otimes\mathbb{I})|\psi(x,t)\rangle$ Probability density on screen

Bell states

- ⋆ $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A\otimes|0\rangle_B + |1\rangle_A\otimes|1\rangle_B\right)$ $\qquad$ Maximally entangled
  $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A\otimes|0\rangle_B - |1\rangle_A\otimes|1\rangle_B\right)$ $\qquad$ Non-separable
  $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A\otimes|1\rangle_B + |1\rangle_A\otimes|0\rangle_B\right)$
  $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A\otimes|1\rangle_B - |1\rangle_A\otimes|0\rangle_B\right)$
- ⋆ They form an orthonormal basis of the Hilbert space of the two spins $\mathcal{H} = \mathcal{H}_A\otimes\mathcal{H}_B$
- ⋆ Eigenstates of $\hat{H} = \mu_x\hat{S}_x^{(A)}\otimes\hat{S}_x^{(B)} + \mu_y\hat{S}_y^{(A)}\otimes\hat{S}_y^{(B)}$
- ⋆ $P_{\hat{S}_x^{(A)}}(\pm\frac{\hbar}{2}|\psi) = \frac{1}{2} \quad \forall|\psi\rangle$ a Bell state.

CHSH Inequality

- ⋆ Bipartite system with LHS measuring device, which can measure either A or A', and RHS device which can measure B or B'. The probability of a result combination is written as $P(l,r|L,R)$ with L,R the settings on LHS and RHS device and l and r the results of the measures (±1).
- ⋆ **Bell inequalities** define a correlation coefficient C and then place an upper bound on possible values this coefficient can take if you assume factorisability
- ⋆ **Factorisability**: $p(l,r|L,R) = \int P(l|L,\lambda)P(r|R,\lambda)P(\lambda)d\lambda$ where λ incorporates all effects from the system's shared history. Two **necessary conditions** for factorisability to hold: Setting Independence $P(l|L,B,\lambda) = P(l|L,B',\lambda)$ and Outcome Indipendence $P(l,A,R,r,\lambda) = P(l|A,R,r',\lambda)$.
- ⋆ $C := |\langle LR\rangle - \langle LR'\rangle| + |\langle LR\rangle + \langle L'R\rangle|$ with $\langle LR\rangle = \sum_{l,r=\pm1} lrP(l,r|L,R)$
- ⋆ $C\le 2$ in the classical case, violated in quantum case (Tsirelson's bound: $C\le 2\sqrt{2}$)
  Quantum Mechanics violates outcome indipendence.
- ⋆ A mixture of product states does not include non-classical correlations (through entanglement) that would allow to violate Bell inequality.

Reduced and mixed quantum states

- ⋆ **Density operator**: $\rho = |\psi\rangle\langle\psi| \Longrightarrow \langle O\rangle = \mathrm{Tr}(\rho O)$ for any observable O
- ⋆ **General single qubit**: $\rho = \begin{pmatrix}cos(\theta/2)^2 & \cos(\theta/2)\sin(\theta/2)e^{-i\phi}\\ \cos(\theta/2)\sin(\theta/2)e^{i\phi} & \sin(\theta/2)^2\end{pmatrix} =$
  $= \frac{1}{2}\sigma_0 + \frac{1}{2}\sum_{i=1}^3 v_i\sigma_i$ with $\boldsymbol{v} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$; $v_i = \mathrm{Tr}[\rho\sigma_i]$

  Proof: the density matrix of an arbitrary 2-level system can be written as $\rho = a\mathbb{I} + b\sigma_x + c\sigma_y + d\sigma_z = a\mathbb{I} + \boldsymbol{\sigma}\cdot\boldsymbol{v}'$ then use $\mathrm{Tr}\rho = 1$ to extract $a = \frac{1}{2}$ and define $\boldsymbol{v} = 2\boldsymbol{v}'$
- ⋆ The eigenvalues of ρ are $\frac{1}{2}(1\pm|\boldsymbol{v}|)$ Using $\langle\psi|\rho|\psi\rangle\ge 0$ this yields $|\boldsymbol{v}|\le 1$
- ⋆ $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ System prepared in state $|\psi_k\rangle$ with prob. p_k (**mixed** state)
- ⋆ Maximally mixed: $\frac{\mathbb{I}}{2} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$
- ⋆ If $\rho = p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|$ where ψ,ϕ have Bloch vectors $\boldsymbol{v},\boldsymbol{u}$ of pure states, **the mixed state has Bloch vector** $\boldsymbol{w} = p\boldsymbol{v} + (1-p)\boldsymbol{u}$; $|\boldsymbol{w}|^2\le 1$
- ⋆ More generally $\rho_{\text{mixed}} = \frac{1}{2}(1 + \boldsymbol{\sigma}\cdot\sum_i p_i\boldsymbol{r_i}) \Longrightarrow r_{\text{mixed}} = \sum_i p_i\boldsymbol{r_i}$
- ⋆ $\rho_A = \sum_{k=1}^{d_B}(\mathbb{I}_A\otimes\langle k|)\rho_{AB}(\mathbb{I}_A\otimes|k\rangle) = \mathrm{Tr}_B[\rho_{AB}]$ $\qquad$ **Reduced** state
- ⋆ $\mathrm{Tr}_B[|ij\rangle\langle kl|] = |i\rangle\langle k|\,\mathrm{Tr}[|j\rangle\langle l|]$ $\qquad$ **Properties of trace**
- ⋆ $\mathrm{Tr}\left(\hat{C}|\psi\rangle\langle\psi|\hat{D}\right) = \mathrm{Tr}\left(|\psi\rangle\langle\psi|\hat{D}\hat{C}\right) = \mathrm{Tr}\left(\langle\psi|\hat{D}\hat{C}|\psi\rangle\right) = \langle\psi|\hat{D}\hat{C}|\psi\rangle$
- ⋆ **Properties**: (i) $\rho_A^\dagger = \rho_A$ (self-adj.) $\quad$ (ii)$\mathrm{Tr}(\rho_A) = \sum_i\sum_\mu \alpha^*_{i,\mu}\alpha_{i,\mu} = |\psi|^2 = 1$
  (iii)$\langle\psi|\rho_A|\psi\rangle\ge 0$ for all $|\psi\rangle\in A$ i.e. positive or null eigenvalues
- ⋆ Conseq., $\rho_A = \sum_j p_j|k\rangle\langle j|$ where $p_j\ge 0$ and $\sum p_j = 1$.
  $\langle O\rangle = \mathrm{Tr}(\rho_A O) = \sum_j p_j\langle j|O|j\rangle = \sum_{p_j}\langle O\rangle_{|j\rangle}$
- ⋆ A **pure state** is a density matrix that has only one non zero eigenvalue. A **mixed state** can have more than one non-zero eigenvalue.
- ⋆ If a density op. describes a pure state, then it is a projector ($\rho^2 = \rho$) and $\mathrm{Tr}[\rho^2] = 1$ (indeed, given that $\sum p_n = 1$, $\mathrm{Tr}[\rho^2] = 1$ iff ρ is pure). If ρ is not pure then $\rho^2\ne\rho$ and $\mathrm{Tr}[\rho^2] < 1$ (*purity* of a state).
- ⋆ The Bloch vector for a pure state has norm 1. To prove this either expand ρ^2 and use $\mathrm{Tr}[\rho^2] = 1$ or use general single qubit and retrieve $\boldsymbol{r} = (\sin\theta\cos\phi,\ldots)$
- ⋆ A state of a system in $\mathcal{H}_1\otimes\mathcal{H}_2$ is separable if we can write its density matrix as $\rho_s = \sum_k p_k\rho_k^{(1)}\otimes\rho_k^{(2)}$
- ⋆ **Partial transpose**: $\rho^{T_B} = \sum_k \rho_k^{(1)}\otimes\left(\rho_k^{(2)}\right)^T$ i.e. $(\rho^{T_B})_{i\mu,j\nu} = \rho_{i\nu,j\mu}$
  If ρ is separable, then the partial transpose is a valid density matrix and in particular all its eigenvalues have to be non-negative. Therefore if at least one eigenvalue of ρ^{T_B} is negative, the state ρ must be entangled (**PPT criterion**).
- ⋆ When the measurement of an observable M on the system gives the result m then the density matrix reads $\rho' = \dfrac{P_m\rho P_m^\dagger}{\mathrm{Tr}(P_m^\dagger P_m\rho)}$ where P_m is the projector on subspace relative to m
- ⋆ **Evolution**: $\rho(t=0) = \sum_j \alpha_j|\psi_j(0)\rangle\langle\psi_j(0)|$ initial state
  $\Longrightarrow \rho(t) = \sum_j \alpha_j e^{-iHt}|\psi_j(0)\rangle\langle\psi_j(0)|e^{iHt} \Longrightarrow i\frac{\partial\rho}{\partial t} = [\hat{H},\rho]$
- ⋆ **Why is signaling impossible?** No matter what is performed upon the other partitions, the reduced density matrix is unchanged. Because the statistics of local measurements are informed entirely by expected values of operators upon the reduced density matrix, they are also independent of operations on other partitions.

Identical multi-particle systems

- ⋆ $\mathbb{P}_{1,2}\psi(\boldsymbol{r_1},\boldsymbol{r_2}) = \psi(\boldsymbol{r_2},\boldsymbol{r_1}) = \pm\psi(\boldsymbol{r_1},\boldsymbol{r_2}) \quad +1$: **bosons** $\quad -1$: **fermions**
- ⋆ $\mathbb{P}_{jk} = \mathbb{P}_{kj}$ $\qquad$ ⋆ $\mathbb{P}_{jk}^2 = \mathbb{I} \Leftrightarrow \mathbb{P}_{jk}^{-1} = \mathbb{P}_{jk}$ $\qquad$ ⋆ $\mathbb{P}_{jk} = \mathbb{P}_{jk}^\dagger$
- ⋆ $\langle\psi_{12}|O|\psi_{12}\rangle = \langle\psi_{12}|\mathbb{P}_{12}^\dagger O\mathbb{P}_{12}|\psi_{12}\rangle$ for all $\psi \Rightarrow [\mathbb{P}_{12},O] = 0, [\mathbb{P}_{12},H] = 0$
- ⋆ **Possible basis states** for a system of
  n bosons: $|\psi_{\boldsymbol{x}}\rangle = \mathcal{N}\sum_{\mathbb{P}\in S_n}\mathbb{P}|x_1,x_2,\ldots,x_n\rangle$ with $\mathcal{N} = \frac{1}{\sqrt{n!}\sqrt{\prod_k n_k!}}$
  n fermions $|\psi_{\boldsymbol{x}}\rangle = \frac{1}{\sqrt{n!}}\sum_{\mathbb{P}\in S_n}\mathrm{sign}(\mathbb{P})\mathbb{P}|x_1,x_2,\ldots,x_n\rangle$
- ⋆ How to use this formula: if $|\psi\rangle$ is a possible configuration of the system (e.g. $|001\rangle$ for three particles which can be in 0 or 1), apply formula to $|\psi\rangle$ and obtain state which respects (a)symmetry (e.g. $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ for bosons)

Second Quantization

- ⋆ Kets indicate the number of times a wave function is involved:
  for Bosons $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \to |11\rangle$; $|\uparrow\uparrow\rangle \to |20\rangle$; $|\downarrow\downarrow\rangle \to |02\rangle$
  for Fermions $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \to |11\rangle$ (only possible values 0,1)
- ⋆ **Creation and annihilation operators** to increase or decrease the number of particles: Bosons: $\begin{cases}\hat{c}_i^\dagger|n_1,\cdots,n_i,\cdots\rangle = \sqrt{n_i+1}\,|n_1,\cdots,n_i+1,\cdots\rangle\\ \hat{c}_i|n_1,\cdots,n_i,\cdots\rangle = \sqrt{n_i}\,|n_1,\cdots,n_i-1,\cdots\rangle\end{cases}$
  $[\hat{c}_i,\hat{c}_j] = [\hat{c}_i^\dagger,\hat{c}_j^\dagger] = 0; \qquad [\hat{c}_i,\hat{c}_j^\dagger] = \delta_{ij}$
  Fermions: $\begin{cases}\hat{c}_i^\dagger|n_1,\cdots,n_i,\cdots\rangle = (-1)^{n_1+\cdots+n_i-1}(1-n_i)\,|n_1,\cdots,n_i+1,\cdots\rangle\\ \hat{c}_i|n_1,\cdots,n_i,\cdots\rangle = (-1)^{n_1+\cdots+n_i-1}n_i\,|n_1,\cdots,n_i-1,\cdots\rangle\end{cases}$
  $\{\hat{c}_i,\hat{c}_j\} = \{\hat{c}_i^\dagger,\hat{c}_j^\dagger\} = 0; \qquad \{\hat{c}_i,\hat{c}_j^\dagger\} = \delta_{ij}$
- ⋆ **Mach-Zender interferometer**: we work in the Heisenberg picture, considering the action of a unitary process (the beamsplitter) on the creation and annihilation operators. First calculate the modified creation operator then apply it to $|0,0\rangle$ to obtain the state after passing through the beamsplitters

Perturbation Theory

- ⋆ $H = H_0 + \lambda V$, with $H_0 |\phi_n\rangle = \epsilon_n |\phi_n\rangle$ known, $\lambda \in \mathbb{R}^+$.
- ⋆ $H|\psi_n\rangle = E_n |\psi_n\rangle$ eigenspectrum unknown. The solution in the limit of small λ is $|\psi_n\rangle = |\phi_n\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots$
  $E_n = \epsilon_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$
- ⋆ S.E.: $(H_0 + \lambda V)(|\phi_n\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots) = (\epsilon_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots)(|\phi_n\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots)$ must be satisfied at each order in λ

Non-degenerate Time-Indipendent Perturbation Theory p.2

- ⋆ **Zero-th order** $H_0 |\phi_n\rangle = \epsilon_n |\phi_n\rangle$ unperturbed eigenvalue problem
- ⋆ **1st order** $E_n^{(1)} = \langle \phi_n | V | \phi_n \rangle$; $\quad |\psi_n^1\rangle = \sum_{m \neq n} \frac{\langle \phi_m | V | \phi_n \rangle}{\epsilon_n - \epsilon_m} |\phi_m\rangle$
- ⋆ **2nd order** $E_n^{(2)} = \sum_{m \neq n} \frac{|\langle \phi_m | V | \phi_n \rangle|^2}{\epsilon_n - \epsilon_m}$; for approx. to be val. $|E_n^{(2)}| \ll |E_n^{(1)}|$
- ⋆ satisfied as long as $\frac{1}{\Delta}(\langle \phi_n | V^2 | \phi_n \rangle - \langle \phi_n | V | \phi_n \rangle^2) \ll \langle \phi_n | V | \phi_n \rangle$,
  $\Delta = \min_m |\epsilon_n - \epsilon_m|$ or more restrictive: $\left| \frac{\langle \phi_m | V | \phi_n \rangle}{\epsilon_n - \epsilon_m} \right| \ll 1$

Degenerate Time-Indipendent Perturbation Theory

- ⋆ nth energy state has N-fold degeneracy $\Rightarrow H_0$ has energy ϵ_n with $\phi_{n_i}, i = 1 \dots N$
- ⋆ $\sum_j V_{ij} c_j = E_n^{(1)} c_i \iff \hat{V} \boldsymbol{c} = E_n^{(1)} \boldsymbol{c} \qquad V_{ij} = \langle \phi_{n_i} | V | \phi_{n_j} \rangle$
  eigenvalue problem, must diagonalise V matrix (in the degenerate subspace only!)
- ⋆ The eigenvectors are the corrected eigenstates (to 0th order), the eigenvalues are the 1st order correction to energy. If eigenvalues are the same, degeneracy is not lifted.
- ⋆ E.g. $V = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{diag}} \alpha, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}; -\alpha, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}; 0, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; 0, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
  Thus $|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$; $\quad |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$; $\quad |2\rangle, |3\rangle$ unchanged
  The corrected eigenstates have an energy correction $E^{(1)}$ of $\alpha, -\alpha$, 0,0.

Time-Dependent Hamiltonians

- ⋆ $U(t, t_0) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{t_0}^t \mathrm{d}t_1 \int_{t_0}^t \mathrm{d}t_2 \dots \int_{t_0}^t \mathrm{d}t_n T(H(t_1) \dots H(t_n))$
  where $T[H(t_1)H(t_2) \dots H(t_n)] = H(t_{i_1})H(t_{i_2}) \dots H(t_{i_n})$, $t_{i_1} \geq t_{i_2} \geq \dots$
- ⋆ Alternatively: $U(t, t_0) \approx \sum_j \exp(iH(t_j)\delta t)$ with an error of
  $|\int_{t_0}^t \mathrm{d}s H(s) - \delta t \sum_{r=1}^{n_t} H(t_0 + r\delta t)|^2$ (discretisation)

Interaction picture

- ⋆ Schrödinger picture: $i\frac{\partial}{\partial t} |\phi_S(t)\rangle = H(t) |\phi_S(t)\rangle, \quad O_S(t) = O_S$
- ⋆ Heisenberg picture: $O_H(t) = U_S(t, t_0)^\dagger O_S U_S(t, t_0), \quad |\phi_H(t)\rangle = |\phi_S(t_0)\rangle$
- ⋆ Interaction picture: $H(t) = H_H + V_S(t)$,
- ⋆ $O_I(t) = e^{iH_0(t-t_0)} O_S(t) e^{-iH_0(t-t_0)}$
- ⋆ $|\phi_I(t)\rangle = U_I(t, t_0) |\phi_I(t_0)\rangle, \quad U_I = e^{iH_0(t-t_0)} U_S(t, t_o)$
- ⋆ $\frac{\partial U_I}{\partial t} = -iV_I(t) U_I(t, t_0)$

Time-Dependent Perturbation Theory

- ⋆ $U_I(t, t_0) \approx \mathbb{I} - i\int_{t_0}^t \mathrm{d}t_1 V_I(t_1) + \dots$
- ⋆ The exact transition probability between two states is $|\langle \psi | \phi \rangle|^2$
- ⋆ $P_{i \to n}(t) = |\langle n | \phi_S(t) \rangle|^2 = |\langle n | U_I(t, t_0) | i \rangle|^2 =$
  $= \left| -i\int_{t_0}^t \mathrm{d}t_1 e^{i(E_n - E_i)(t - t_0)} \langle n | V(t_1, t_0) | i \rangle \right|^2$ general expression of **transition probabilities** between eigenstates $|i\rangle$ and $|n\rangle$ of H_0. Note: if $V = 0$ then a system in an eigenstate stays in an aigenstate.
- ⋆ For a constant potential $P_{i \to n}(t) \overset{t \to \infty}{=} 2\omega t |\langle n | V | i \rangle|^2 \delta(E_n - E_i)$
  $\frac{\partial}{\partial t} P_{i \to n}(t) = 2\pi |\langle n | V | i \rangle|^2 \delta(E_n - E_i)$

Variational method

- ⋆ $\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_0 \quad$ **Variational principle**
- ⋆ The idea is to come up with a parameterised guess for the state $|\psi\rangle$, and then we use the variational principle to find the parameter values that minimize ψ.
  It generalises to excited states orthogonal to the ground state ϕ_0(i.e. $\langle \phi_0 | \psi \rangle = 0$) by replacing E_0 with E_1. Limitation: the ground state should be known to ensure it is orthogonal to the excited state.
- ⋆ Steps: compute $\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = E(a)$ then minimise E with respect to the parameter.

Decoherence

If $|\psi\rangle = \sum_j c_j |E_j\rangle_A \otimes |\phi\rangle_B$ and $H_{AB} = \sum_j |E_j\rangle \langle E_j|_A \otimes H_B^{(J)}$ then
$\rho_{AB} = |\psi\rangle \langle \psi| = \sum_j \sum_i c_j c_i^* |E_j\rangle \langle E_i|_A \otimes |\phi\rangle \langle \phi|_B$

$\rho_a(t) = \sum_j \sum_i c_j c_i^* |E_j\rangle \langle E_i| \langle \phi | e^{itH_B^{(i)}} e^{-itH_B^{(j)}} |\phi\rangle$

$\rho_B(t) = \sum_j |c_j|^2 e^{-itH_B^{(j)}} |\phi\rangle \langle \phi| e^{itH_B^{(j)}}$

Particle in a box

- ⋆ $V(x) = \begin{cases} 0 & \text{if} |x| \leq L/2 \\ \infty & \text{else} \end{cases} \qquad \phi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos(\frac{n\pi}{L}x) & n \text{ odd} \\ \sqrt{\frac{2}{L}} \sin(\frac{n\pi}{L}x) & n \text{ even} \end{cases}$
- ⋆ $E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2}$

Harmonic oscillator

- ⋆ $H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2 \hat{x}^2 = \hbar\omega(\hat{N} + \frac{1}{2}), \quad \hat{N} = \hat{a}^\dagger \hat{a} \qquad$ ⋆ $\frac{1}{2m}\hat{p}^2 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- ⋆ $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right)$
- ⋆ $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a})$
- ⋆ $[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$
- ⋆ $\phi_0(x) = \left(\frac{m\omega}{\phi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2 \right)$
- ⋆ $\phi_n = \frac{1}{\sqrt{n!}} (a^\dagger)^n \phi_0 = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right)$
  with $H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} (e^{-z^2})$
- ⋆ $\hat{H}\phi_n = E_n \phi_n = \hbar\omega(n + \frac{1}{2})\phi_n$
- ⋆ $\hat{a}^\dagger \phi_n = \sqrt{n+1} \phi_{n+1} \quad \hat{a}\phi_n = \sqrt{n} \phi_{n-1}$

Hermitian and unitary operators

- ⋆ Hermitian operator: $M = M^\dagger \longrightarrow$ diagonalisable with real eigenvalues, linear
- ⋆ Unitary operators: $UU^\dagger = U^\dagger U = \mathbb{I} \longrightarrow \langle \psi | U^\dagger U | \psi \rangle = \mathbb{I}$ and linear

Tensor product

- ⋆ $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \end{pmatrix}^T$
- ⋆ $A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} \qquad$ ⋆ $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$
- ⋆ $R(g)^{\otimes k} = R(g) \otimes \dots \otimes R(g) \qquad$ ⋆ $\bigoplus_k R(g) = (R \bigoplus \dots \bigoplus R)(g)$
- ⋆ $f(\hat{A} \otimes \hat{B}) |a\rangle |\phi\rangle = |a\rangle \otimes f(a\hat{B}) |\phi\rangle$
- ⋆ $[A \otimes \mathbb{I}, \mathbb{I} \otimes B] = 0 \qquad$ ⋆ $\{A \otimes \mathbb{I}, \mathbb{I} \otimes B\} = 2A \otimes B$
- ⋆ $e^{A+B} = e^A e^B$ if $[A, B] = 0 \Longrightarrow e^{A \otimes \mathbb{I} + \mathbb{I} \otimes B} = e^A \otimes e^B$

Trigonometry

- ⋆ $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \qquad$ ⋆ $\sin(2a) = 2\sin a \cos a$
- ⋆ $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad$ ⋆ $\cos(2a) = \cos^2 a - \sin^2 a$
- ⋆ $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b) \qquad$ ⋆ $\sin(a/2) = \pm\sqrt{(1 - \cos(a))/2}$
- ⋆ $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b) \qquad$ ⋆ $\cos(a/2) = \pm\sqrt{(1 + \cos(a))/2}$
- ⋆ $2\cos(a)\sin(b) = \sin(a+b) - \sin(a-b) \qquad$ ⋆ $\sin^2 a = (1 - \cos 2a)/2$
- ⋆ $\cos(a) + \cos(b) = 2\cos((a+b)/2)\cos((a-b)/2)$ ⋆ $\cos^2 a = (1 + \cos 2a)/2$
- ⋆ $\sin(a) + \sin(b) = 2\sin((a+b)/2)\cos((a-b)/2)$
- ⋆ $\cos(a) - \cos(b) = -2\sin((a+b/2)\sin((a-b/2)$

Some useful integrals

- ⋆ $\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\alpha^2(x+\beta)^2 + Bx} = \frac{\sqrt{\pi}}{\alpha} e^{\frac{B^2}{4\alpha^2} - B\beta}$ with $\mathrm{Re}(\alpha^2) \geq 0,\ B, \beta \in \mathbb{C}$
- ⋆ $\int_{-\infty}^{+\infty} \mathrm{d}x\, x^n e^{-\frac{1}{2}Ax^2} = \begin{cases} (n-1)!!\sqrt{2\pi} A^{-\frac{n+1}{2}} & \text{If n is even} \\ 0 \quad \text{If n is odd} \end{cases}$
- ⋆ $\int_0^T \sin^2(\frac{x\pi}{T})\mathrm{d}x = \frac{T}{2}$ and same with $\cos^2$

Group Theory

- **Group**: Set G equipped with operation $*$ such that
  - G closed under $*$, i.e. if $a, b \in G$ then $a * b \in G$
  - Associative: $\forall a, b, c \in G$ one has $(a * b) * c = a * (b * c)$
  - Has identity, i.e an element e such that $e * a = a \forall a \in G$
  - Has inverse, i.e. $\forall a \in G$ it exists $b \in G$ such that $b * a = a * b = e$. $(b = a^{-1})$
- Any unitary that leaves a property invariant forms a grop with * matrix multiplic.
- **Finite group**: A group that contains a finite number of elements (the group *order*).
- Order 1 group (the only one, trivial group):

$*$	e
e	e

- Order 2: parity group

$*$	e	a
e	e	a
a	a	e

- The unique order 3 group is the cyclic $\mathbb{Z}_3$ group

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

- Order 4 has cyclic and symmetry of a rectangle
- Order 6 has cyclic group $\mathbb{Z}_6$ and the C3v group (on the right)

$*$	e	a	a^2	b	c	d
e	e	a	a^2	b	c	d
a	a	a^2	e	c	d	b
a^2	a^2	e	a	d	b	c
b	b	d	c	e	a^2	a
c	c	b	d	a	e	a^2
d	d	c	b	a^2	a	e

- The general cyclic group $\mathbb{Z}_n$ $(a^n = 1)$ is

$*$	e	a^1	a^2	$\dots$	a^{n-1}
$*$	e	a^1	a^2	$\dots$	a^{n-1}
a^1	a^1	a^2	a^3	$\dots$	e
a^2	a^2	a^3	a^4	$\dots$	a^1
$\vdots$					
a^{n-1}	a^{n-1}	e	a^1	$\dots$	a^{n-2}

- Symmetric permutation group S_n (all possible permutations of n objects) e.g. $S_3 = \{I, \text{SWAP}_{12}, \text{SWAP}_{13}, \text{SWAP}_{23}, \text{CYCLE}_{123}, \text{CYCLE}_{321}\}$ isomorphic to C3v
- **Lie group**: a continous group that depends analytically on some continous parameters λ. Examples: Real d-dimensional rotations $SO(d)$, the orthogonal group $O(d)$, the unitary group $U(d)$, the special unitary group $SU(d)$
- **Abelian group**: $a * b = b * a \; \forall a, b \in G$
- **Subgroup**: a subset H of the group G is a subgroup of G if and only if it is nonempty and itself forms a group. If it is neither the identity or G itself then it is a *proper* subgroup. **Lagrange Theorem:** the order of H divides the order og G. Thus if the order of a group is prime there is only one possible group
- **Group homomorphism**: an application from $(G, *)$ to $(G, **)$ such that $\forall x, y \in G \quad f(x * y) = f(x) ** f(y)$. An isomorphism sets a one-to-one correspondance.

Representations

- A **representation** R of a group G on a vector space V is a group homomorphism from G to a set of matrices that act on a vector space V. The dimension of a representation R is defined to be the dimension of the vector state V, i.e., $\dim(R) = \dim(V)$.
- All groups allow **trivial representation** $\forall g \in G,\ R(g) = \mathbb{I}$
- The **regular representation** is obtained by reordering the Caylei table so that only e fills the diagonal, then to every element assign the matrix obtained by replacing 1 in the positions where the element is in the table and 0 everywhere else.
- **Equivalent** reps are related by a similarity transformation $R'(g) = SR(g)S^{-1}$
- If R_1 and R_2 are two representations for G, then $R_1(g) \otimes R_2(g)$ is also a rep
- The direct sum $R_1 \oplus R_2$ is a rep of G acting on $V_1 \oplus V_2$
  $(R_1 \oplus R_2)(g) = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix}$
- Let U_g be a rep of G and H an Hermitian operator such that $[U_g, H] = 0 \; \forall g$. Then H and U_g are simultaneously block diagonalised (in the same basis).
  E.g. $U_g \otimes U_g$, the tensor product representation of $SU(2)$, commutes with SWAP and has therefore a symmetric $(d = 3)$ and an asymmetric $(d = 1)$ invariant subspace.
- **Reducible representation**: a rep $R(g)$ of G over a vector space V is reducible if there exists an invariant subspace, i.e. if there exists a non-trivial subspace W of V such that for all $|w\rangle \in W$ we have $R(g)|w\rangle \in W$ for any g.
- **Completely reducible rep**: if it splits into a direct sum of irreducible reps
- **Schur's 1st lemma**: Let $R_1(g)$ and $R_(g)$ be two non-equivalent irreducible reps of G, acting on vector spaces V_1, V_2. If there is a matrix A such that
  $AR_1(g) = R_2(g)A \; \forall g \quad$ then $A = 0$.
  Equivalently, if you can find A different than 0 that satisfies the eq., then the reps are reducible.
- **Schur's 2nd lemma**: Let R be an irreducible rep of G. If $AR(g) = R(g)A \; \forall g \in G$ then $A = \lambda\mathbb{I}$ for some $\lambda \in \mathbb{C}$.
  Equivalent to: if $[A, R(g)] = 0$ i.e. if they can be diagonalised in the same base.
- **Burnside's lemma**: for a finite group of order h there are only a finite number n of irreducible representations $a = 1 \dots n$ of dimension l_a and $\sum_{a=1}^{n} l_a^2 = h$
- In a group G, two elements g and g' are **equivalent** if there exists another element f such that $g' = f^{-1}gf$. This divides G in conjugacy classes.
- For a finite group, the number of (non-equiv.) irreps is equal to the number of conjugacy classes
- All irreducible reps of Abelian groups are scalar $(d = 1)$. An Abelian group of order n has n conjugacy classes and thus n irreducible reps.
- **Grand Orthogonality Theorem**: Let R_a and R_b be two non-equiv. unitary irreducible reps of a finite group G of order N. Let n_a and n_b be the dimensions of the vector space for R_a and R_b. Then
  $\sum_{g \in G} \dfrac{n_a}{N} \left[R_a(g)^\dagger\right]_{jk} [R_b(g)]_{lm} = \delta_{ab}\delta_{jm}\delta_{lk}$
  - if $a \neq b$ then $\sum_{g \in G} \left[R_a(g)^\dagger\right]_{jk} [R_b(g)]_{lm} = 0$ for all i, j, k, l
  - if $a = b$ then $\sum_{g \in G} \left[R_a(g)^\dagger\right]_{jk} [R_a(g)]_{lm} = 0$ if $j \neq m$ and/or $l \neq k$
  - if $a = b$ and $j = m$ and $l = k$ then $\sum_{g \in G} \left[R_a(g)^\dagger\right]_{jk} [R_a(g)]_{jk} = \dfrac{N}{n_a}$
- A finite group can only have a finite number of inequivalent irreducible representations. Specifically, the **maximum number of possible irreps** is given by the order of the group. Proof: irreps give us 'vectors of matrices' in a vector space of dimension $|G|$ and the theorem says they must be orthogonal. But there are at most $|G|$ orthogonal vectors in a vector space of dimension $|G|$
- **Group averaging**: if d is the dim. of vector space of rep and U_g is an irreducible representation then $\langle X \rangle_G = \frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{d} \mathrm{Tr}[X]\mathbb{I}$
- Proof: $\frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{N} \sum_g \left(\sum_{l,m} [U_g]_{lm} |l\rangle\langle m|\right) X \left(\sum_{k,j} [U_g^\dagger]_{kj} |k\rangle\langle j|\right)$
  $= \frac{1}{N} \sum_g \sum_{lmkj} [U_g]_{lm} X_{mk} [U^\dagger]_{kj} |l\rangle\langle j|$ then apply orthogonality
  $= \frac{1}{n_a} \sum_{lmkj} \delta_{lj}\delta_{mk} X_{mk} |l\rangle\langle j|$
- $\langle X \rangle_G = \int_G \mathrm{d}\mu(g) U_x(g) X U_x(g)^\dagger = \frac{1}{d} \mathrm{Tr}[X]\mathbb{I}$ for continous groups
- For reducible representations U_g we have
  $\langle X \rangle_G = \frac{1}{N} \sum_g U_g X U_g^\dagger = \sum_x \dfrac{\mathrm{Tr}[\Pi_x X]}{d_x} \Pi_x = \bigoplus_x \dfrac{\mathrm{Tr}[\Pi_x X]}{d_x} \mathbb{I}_x$
- Proof: any reducib. unitary can be writt. as $U(g) = \bigoplus_x U_x(g) = \sum_x U_x(g) \otimes I_{\bar{x}}$ where $\bar{x}$ is the subspace U_x does not act on.
  $\langle X \rangle_G = \frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{N} \sum_g \sum_{xx'} (U_x(g) \otimes I_{\bar{x}}) X (U_{x'}(g)^\dagger \otimes I_{\bar{x}}) = \frac{1}{N} \sum_g \sum_x (U_x(g) \otimes I_{\bar{x}}) X (U_x(g)^\dagger \otimes I_{\bar{x}}) = \frac{1}{d_x} \sum_x \mathrm{Tr}[X\Pi_x]\Pi_x \otimes I_{\bar{x}} = \frac{1}{d_x} \bigoplus_x \mathrm{Tr}[X\Pi_x]\Pi_x$
- In a representation R, all elements which are in the same conjugacy class have the same trace. Proof: $\mathrm{Tr}(R(u^{-1}yu)) = \mathrm{Tr}(R(u)R(u^{-1})R(y)) = \mathrm{Tr}R(y)$
- **Petit Orthogonality Theorem**: Let R_a and R_b be two non-equiv. unitary irreducible reps of a finite group of order N, then $\sum_{g \in G} \chi_a^*(g)\chi_b(g) = N\delta_{a,b}$
  where $\chi_R(g) = \mathrm{Tr}[R(g)]$, or equiv. $\sum_{\mu=1}^{N_c} \eta_\mu \chi_a^*(g)\chi_b(g) = N\delta_{a,b}$ with η_μ the nb of elements in class μ and N_c total nb of conjugacy classes
- The set of all traces $\{\chi_R(g)\}$ is the **character** of representation R. Two irreps are equivalent iff they have the same character. Proof by contradiction with Petit.
- Trace of all reps within a conj. class are the same $\rightarrow \chi_R(C_\mu) = \sum_a b_a \chi_a(C_\mu)$
- Assuming a decomposition in irreps $R(g) = \bigoplus_{a,x} R_{a,x}(g)$ for $x = 1 \dots b_a$, the number of irreps composing R_a is $b_a = \frac{1}{N} \sum_\mu \eta_\mu \chi_a^*(C_\mu)\chi_R(C_\mu)$
- A necessary and sufficient condition for a representation R to be an irrep is that $\sum_{\mu=1}^{N_c} \eta_\mu |\chi(C_\mu)|^2 = N$. Proof: decompose trace and use Petit.

Lie Algebras

- Rotation through infinitesimal angle $R(\theta) = \mathbb{I} + A$ and as $R^T R = \mathbb{I}$ we must have $A^T = -A \Longrightarrow$ e.g. in 2D $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \theta = J\theta$
- Then $R(\theta) = \lim_{N \to \infty} \left(R(\frac{\theta}{N})\right)^N = e^{\theta J}$
- What about 3D? 3 basis antisymmetric matrices:
  $J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} J_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
  and any 3x3 antisymmetric matrix can be written as $A = \theta_x J_x + \theta_y J_y + \theta_z J_z$ so that $R(\theta) = e^{\theta_x J_x + \theta_y J_y + \theta_z J_z}$
- Arbitrary dimension: $R(\theta) = e^{\sum_i \theta_i J_i}$ or $R(\theta) = e^{i \sum_i \theta_i \tilde{J}_i}$ with $\tilde{J} = -iJ$
- **Structure constants**: let $R = \mathbb{I} + A$, $R' = \mathbb{I} + B$. Then $RR'R^{-1} = \mathbb{I} + B + [A, B]$. $[A, B] = i^2 \sum_{i,j} \theta_i \theta'_j [\tilde{J}_i, \tilde{J}_j]$ measures how much they don't commute Generally we have $[T_a, T_b] = i f_{abc} T_c$ where f_{abc} are the structure constants of the algebra.
- **Lie algebra**: a linear space spanned by linear combinations $\sum_i \theta_i J_i$ of the generators of the associate Lie group G
- As Lie groups are differentiable, it is always possible to write an element g of a Lie group G as the exponential of an element J of the corresponding Lie Algebra $\mathfrak{g}$ i.e. $\mathfrak{g} = \{J | e^{iJ} \in G\}$
- The commutation relations of the generators J_j (i.e., a basis for $\mathfrak{g}$) are the structure constants of the group and can be used to identify the Lie Algebra $\mathfrak{g}$ (and thereby the corresponding Lie group G).
- What does it mean to represent an algebra? It means to find a set of matrices such that the defining commutation relations are satisfied.
- Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. If R is a representation of G on V, then there exists a unique representation r of $\mathfrak{g}$ on V given by
  $r(X) = \frac{\mathrm{d}}{\mathrm{d}\theta} (R(e^{\theta X}))\Big|_{\theta=0}$ for all $X \in \mathfrak{g}$. We call r the rep of $\mathfrak{g}$ induced by R.

Ladder operators

- $S_\pm = S_x \pm iS_y \qquad \star\ S_x = \frac{1}{2}(S_+ + S_-) \qquad \star\ S_y = \frac{1}{2i}(S_+ - S_-)$
- $S_+ |s, m\rangle = \sqrt{s(s+1) - m(m+1)}\, |s, m+1\rangle$
- $S_- |s, m\rangle = \sqrt{s(s+1) - m(m-1)}\, |s, m-1\rangle$
- For a composite system the ladder operators are $S_\pm = j_\pm \otimes \mathbb{I} + \mathbb{I} \otimes j_\pm$
- Obtaining **Clebsch-Gordan** coefficients: Start by the highest spin state (e.g. for two spin-1 states $|s = 1, m = 1\rangle \otimes |s = 1, m = 1\rangle = |S = 2, m = 2\rangle$), then apply lowering operator on both sides until you get 0. To find the other sectors, we must first find the set of states which are orthogonal to the previous ones, and find the highest weight among those.

Irreducible representations of C3v

(i) Trivial (ii) 2D irrep ↓

$$R(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; R(c_+) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \; R(c_-) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$R(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \; R(\sigma') = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \; R(\sigma'') = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(iii) $R(e) = 1, R(c_+) = 1, R(c_-) = 1, R(\sigma) = -1, R(\sigma') = -1, R(\sigma'') = -1$

⋆ Character table:

	e	$2C_3$	$3\sigma_v$
$R_{(i)}$	1	1	1
$R_{(ii)}$	2	−1	0
$R_{(iii)}$	1	1	−1

3D representation of C3v (reducible)

Triangle in a 3D space with vertex at positions $(1;0;0)$, $(0;1;0)$ and $(0;0;1)$

$$R(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; R(c_+) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; R(c_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; R(\sigma') = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; R(\sigma'') = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Other groups

- ⋆ $SU(2)$: single qubit rotations $\rightarrow e^{-i\sigma_x\theta_1}e^{-i\sigma_z\theta_2}e^{-i\sigma_x\theta_3}$
  2D irreducible representation: Pauli matrices
  3D irreducible representation: same as for $SO(3)$, obtain L_x, L_y, L_z from ladder operators acting on the basis $|l,m\rangle$
- ⋆ $U(1)$: symmetry group of rotations around one axis (e.g. x), such as phase of a laser. The associated representation is the set of rotations by an angle $\theta \in [0, 2\pi)$ about the axis $U(\theta) = e^{-i\theta\sigma_x/2}$. The representation is reducible in 1D reps $\{1\}$ and $\{e^{-i\theta}\}$. For example for σ_x $U_g = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = |+\rangle\langle+| + e^{-i\theta}|-\rangle\langle-|$
  Averaging a state ρ: $\langle\rho\rangle_G = \langle+|\rho|+\rangle|+\rangle\langle+| + \langle-|\rho|-\rangle|-\rangle\langle-|$ which is a projection onto the x-axis
- ⋆ $U(2)$: light polarisation $\rightarrow e^{-i\phi}e^{-i\sigma_x\theta_1}e^{-i\sigma_z\theta_2}e^{-i\sigma_x\theta_3}$
- ⋆ Possible dimensions of $SO(3)$ are odd
- ⋆ Any representation of $SO(3)$ is also a representation of $SU(2)$. The converse is not true. For each representation in $SO(3)$ there are two in $SU(2)$ ($SU(2)$ is a double cover of $SO(3)$)

Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Eigenspectrum of generic 2 x 2 hermitian matrix

- ⋆ Matrix of form $\begin{pmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{22} \end{pmatrix}$
- ⋆ Eigenvalues $E_\pm = \frac{1}{2}(V_{11}+V_{22}\pm\Delta E)$ where $\Delta E = \sqrt{(V_{11}-V_{22})^2 + 4|V_{12}|^2}$
- ⋆ The normalised eigenvectors are $\begin{pmatrix} \dfrac{V_{12}}{\sqrt{\Delta E\left(\dfrac{\Delta E}{2} \pm \dfrac{(V_{22}-V_{11})}{2}\right)}} e^{i\phi_\pm} \\ \pm\sqrt{\dfrac{\dfrac{\Delta E}{2} \pm \dfrac{(V_{22}-V_{11})}{2}}{\Delta E}}\, e^{i\phi_\pm} \end{pmatrix}$

  valid for any phase $\phi_\pm$

Eigenvectors of Pauli matrices

$$\psi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \psi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\psi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \psi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Miscellanea

- ⋆ To diagonalise off-diagonal matrix A take the square. A^2 will be diagonal and then from λ^2 you have candidates for λ

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