

Homework #1(Chapter 8)**Question 1.** Exercise 8.8.

In the following proof, $\|\cdot\|_2$ is denoted by $\|\cdot\|$. According to **Definition 8.4** and **Lemma 8.5**, we have

$$(1) \quad \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} = \frac{\|A^{-1}\mathbf{r}\|}{\|A^{-1}\mathbf{f}\|}$$

Since

$$(2) \quad A^{-1}\mathbf{r} = A^{-1} \cdot \mathbf{r}, \quad r = A \cdot A^{-1}\mathbf{r},$$

we have

$$(3) \quad \frac{\|\mathbf{r}\|}{\|A\|} \leq \|A^{-1}\mathbf{r}\| \leq \|A^{-1}\mathbf{r}\|$$

Similarly,

$$(4) \quad \frac{\|\mathbf{f}\|}{\|A\|} \leq \|A^{-1}\mathbf{f}\| \leq \|A^{-1}\mathbf{f}\|$$

Substitute **3** and **4** into **1**, then we have

$$(5) \quad \frac{1}{\text{cond}(A)} \frac{\|\mathbf{r}\|_2}{\|\mathbf{f}\|_2} \leq \frac{\|\mathbf{e}\|_2}{\|\mathbf{u}\|_2} \leq \text{cond}(A) \frac{\|\mathbf{r}\|_2}{\|\mathbf{f}\|_2}$$

Question 2. Exercise 8.9.

According to **Example 8.2**, we have

$$(6) \quad \begin{aligned} \text{cond}(A) &= \frac{\max \lambda_k(A)}{\min \lambda_k(A)} = \frac{\sin^2 \frac{n-1}{2n} \pi}{\sin^2 \frac{1}{2n} \pi} \\ &= \frac{1}{\tan^2 \frac{1}{2n} \pi} \approx \frac{4n^2}{\pi^2} \end{aligned}$$

For $n = 8$, we have

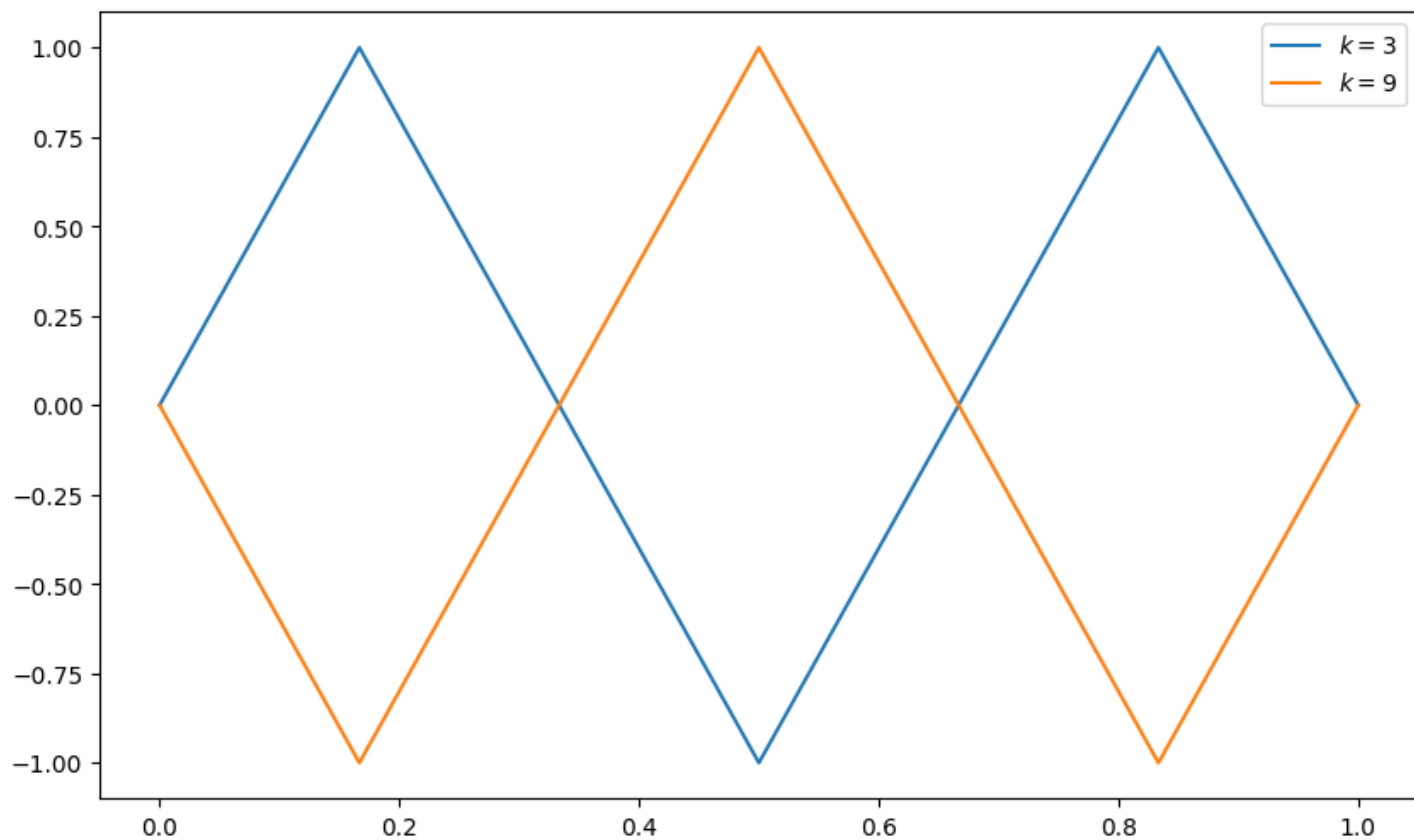
$$(7) \quad \text{cond}(A) \approx \frac{256}{\pi^2}$$

For $n = 1024$, we have

$$(8) \quad \text{cond}(A) \approx \frac{4194304}{\pi^2}$$

Question 3. Exercise 8.12.

Question 4. Exercise 8.15.



```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 def f1(x):
5     return np.sin(3*np.pi*x)
6 def f2(x):
7     return np.sin(9*np.pi*x)
8
9 x = np.arange(0, 1.001, 1/6)
10 value1 = []
11 value2 = []
12 for t in x:
13     value1.append(f1(t))
14     value2.append(f2(t))
15 plt.figure().set_size_inches(10,6)
16 plt.plot(x, value1, label='$k=3$')
17 plt.plot(x, value2, label='$k=9$')
18 plt.legend()
19 plt.savefig('../media/Ex8_15.png', bbox_inches='tight')

```

LISTING 1. code for plot

Question 5. Exercise 8.19.

According to **Definition 8.17**, we have

$$\begin{aligned}
 \mathbf{e}^{(l+1)} &= \mathbf{u}^{(l+1)} - \mathbf{u} = T\mathbf{u}^{(l)} + \mathbf{c} - \mathbf{u} \\
 &= T\mathbf{u}^{(l)} + \mathbf{c} - T\mathbf{u} - \mathbf{c} \\
 &= T(\mathbf{u}^{(l)} - \mathbf{u}) = T\mathbf{e}^{(l)}
 \end{aligned}
 \tag{9}$$

Thus $\mathbf{e}^{(l)} = T^l \mathbf{e}^{(0)}$.

To prove the proposition, it suffices to show that if $A \in \mathbb{C}^{n \times n}$, then

$$\lim_{k \rightarrow \infty} A^k = 0 \iff \rho(A) < 1
 \tag{10}$$

(1) \Rightarrow Assume $\lambda \in \lambda(A) : \rho(A) = |\lambda|$, then $\forall k$, we have $\lambda^k \in \rho(A^k)$. So

$$\rho(A)^k = |\lambda|^k \leq \rho(A^k) \leq \|A^k\|_2, \quad \forall k
 \tag{11}$$

Thus $\rho(A) < 1$.

(2) \Leftarrow If $\rho(A) < 1$, then for any matrix norm $\|\cdot\|$, we have $\|A\| < 1$. So

$$0 \leq \|A^k\| \leq \|A\|^k \rightarrow 0, \quad k \rightarrow \infty
 \tag{12}$$

Thus $\lim_{k \rightarrow \infty} A^k = 0$.

Question 6. Exercise 8.24.

According to **Definition 8.22**, we have

$$\begin{aligned}
 \mathbf{u}^{(l+1)} &= (1 - \omega)\mathbf{u}^{(l)} + \omega(T\mathbf{u}^{(l)} + \mathbf{c}) \\
 &= [(1 - \omega)I + \omega T]\mathbf{u}^{(l)} + \omega\mathbf{c}
 \end{aligned}
 \tag{13}$$

So

$$\begin{aligned}
 T_\omega &= (1 - \omega)I - \omega D^{-1}(L + U) \\
 &= I - \omega D^{-1}(L + U + D) = I - \omega D^{-1}A = I - \frac{\omega h^2}{2}A.
 \end{aligned}
 \tag{14}$$

Since

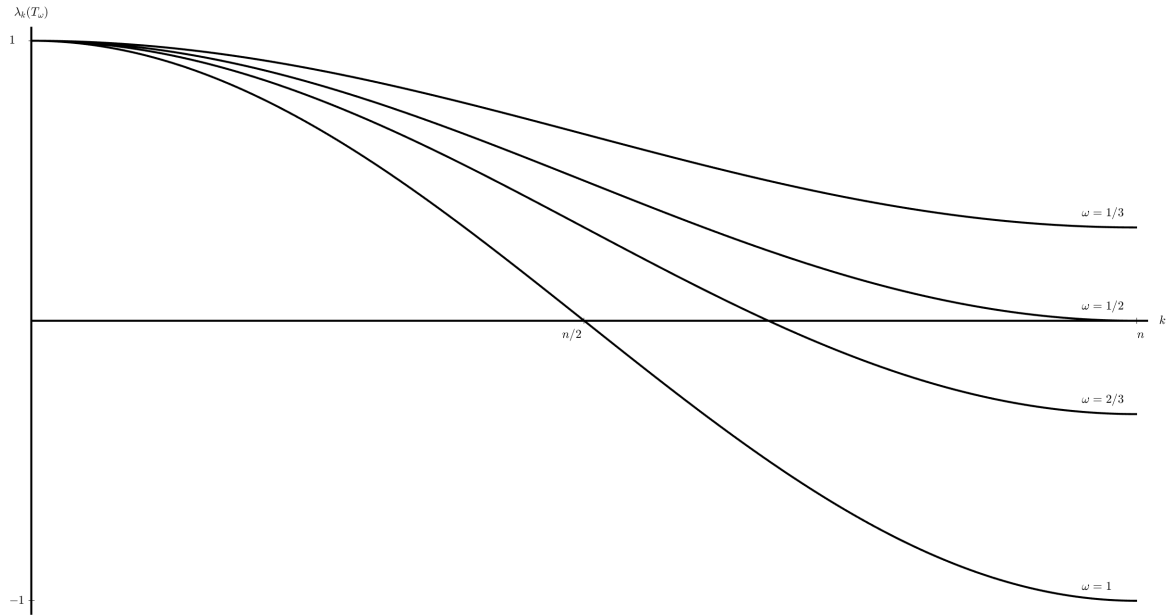
$$w_k T_\omega = w_k \left(I - \frac{\omega h^2}{2} A \right) = w_k - \frac{\omega h^2}{2} \lambda_k(A) w_k = \left(1 - \frac{\omega h^2}{2} \lambda_k(A) \right) w_k,
 \tag{15}$$

so

$$\lambda_k(T_\omega) = 1 - \frac{\omega h^2}{2} \lambda_k(A) = 1 - 2\omega \sin^2 \frac{k\pi}{2n},
 \tag{16}$$

and the eigenvector of $\lambda_k(T_\omega)$ is still w_k .

Question 7. Exercise 8.25.



```

1 syms k w
2 n = 100; t = 0:0.1:n;
3 f = 1 - 2*w*(sin(k*pi/(2*n)))^2;
4 f1 = subs(subs(f,w,1/3),k,t);
5 f2 = subs(subs(f,w,1/2),k,t);
6 f3 = subs(subs(f,w,2/3),k,t);
7 f4 = subs(subs(f,w,1),k,t);
8 fig = figure();
9 hold on
10 plot(t, f1, 'k','linewidth', 2); plot(t, f2, 'k','linewidth', 2)
11 plot(t, f3, 'k','linewidth', 2); plot(t, f4, 'k','linewidth', 2)
12 % set the axis
13 plot([0 0], [-1.05 1.05], 'k','linewidth', 2); plot([0, 101], [0, 0], 'k', 'linewidth', 2)
14 xlim([0 n+1]); ylim([-1.1 1.1]);
15 axis off
16 scatter(0,1,'k+'); scatter(0,-1,'k+'); scatter(50,0,'k+'); scatter(100,0,'k+');
17 text(-2,1,'$1$', 'interpreter', 'latex', 'fontsize', 12)
18 text(-2,-1,'$-1$', 'interpreter', 'latex', 'fontsize', 12)
19 text(48,-0.05,'$n/2$', 'interpreter', 'latex', 'fontsize', 12)
20 text(100,-0.05,'$n$', 'interpreter', 'latex', 'fontsize', 12)
21 text(95,1/3+0.05,'$\omega=1/3$', 'interpreter', 'latex', 'fontsize', 12)
22 text(95,0.05,'$\omega=1/2$', 'interpreter', 'latex', 'fontsize', 12)
23 text(95,-1/3+0.05,'$\omega=2/3$', 'interpreter', 'latex', 'fontsize', 12)
24 text(95,-1+0.05,'$\omega=1$', 'interpreter', 'latex', 'fontsize', 12)
25 text(102,0,'$k$', 'interpreter', 'latex', 'fontsize', 12)
26 text(0,1.1,'$\lambda_k(T_\omega)$', 'HorizontalAlignment', 'center', 'interpreter', 'latex', 'fontsize', 12)
27 % save figure
28 scrsz=get(0,'ScreenSize'); set(gcf,'Position',scrsz);
29 saveas(gcf, '../media/Ex8_25.png')

```

LISTING 2. code for the above figure

Since when $n = 64$

$$(17) \quad \min_{\omega \in [0,1]} \rho(T_\omega) = \min_{\omega \in [0,1]} \max_{k \in [1, n-1]} \lambda_k(T_\omega) = 1 - 2 \sin^2 \frac{\pi}{128} = 0.9988 > 0.9986,$$

then $\forall \omega \in [0, 1] \rho(T_\omega) \geq 0.9986$.

Question 8. Exercise 8.28.

To make the code more clear, we use Python instead.

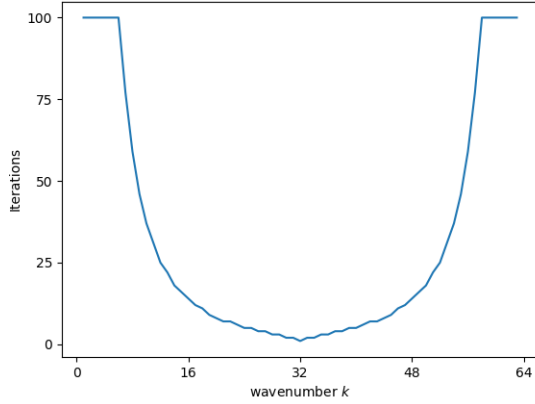


FIGURE 1. $\omega = 1$

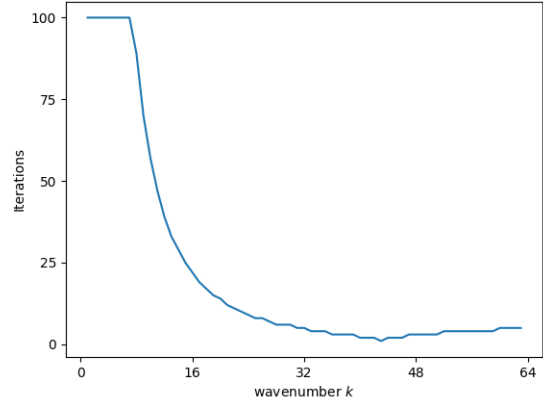


FIGURE 2. $\omega = 2/3$

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 def gen_A(n):
5     A = np.zeros([n-1, n-1])
6     for i in range(n-1):
7         for j in range(n-1):
8             if i == 0:
9                 A[i][0] = 2
10                A[i][1] = -1
11            elif i == n-2:
12                A[i][i] = 2
13                A[i][i-1] = -1
14            else:
15                A[i][i] = 2
16                A[i][i-1] = A[i][i+1] = -1
17    A = A * (n**2)
18    return A
19
20 def Jacobi(n, w0, omega):
21     A = np.eye(n-1) - gen_A(n) * (omega/(2*n**2))
22     w0 = -w0
23     err = np.linalg.norm(w0, ord=np.inf)
24     for i in range(100):
25         w0 = np.matmul(A, w0)
26         if(np.linalg.norm(w0, ord=np.inf) < err/100):
27             return i + 1
28     return 100
29
30 def Fourier(n, k):
31     w = []
32     for i in range(1, n):

```

```

33         w.append(np.sin(i/n*k*np.pi))
34     return np.array(w)
35
36 k = [i for i in np.arange(1, 64, 1)]
37 iter_1 = [] # omega = 1
38 iter_2 = [] # omega = 2/3
39 for i in k:
40     w0 = Fourier(64, i)
41     iter_1.append(Jacobi(64, w0, 1))
42     iter_2.append(Jacobi(64, w0, 2/3))
43 plt.figure()
44 plt.plot(k, iter_1)
45 plt.xlabel('wavenumber $k$')
46 plt.ylabel('Iterations')
47 plt.xticks(range(0,65,16))
48 plt.yticks(range(0,101,25))
49 plt.savefig(' ../media/Ex8_28_a')
50 plt.figure()
51 plt.plot(k, iter_2)
52 plt.xlabel('wavenumber $k$')
53 plt.ylabel('Iterations')
54 plt.xticks(range(0,65,16))
55 plt.yticks(range(0,101,25))
56 plt.savefig(' ../media/Ex8_28_b')

```

LISTING 3. code for the above figure