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Num PDE

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Homework #1(Chapter 8)

Question 1. Exercise 8.8.

In the following proof, $||\cdot||_2$ is denoted by $||\cdot||$. According to **Definition 8.4** and **Lemma 8.5**, we have

$$\frac{||\mathbf{e}||}{||\mathbf{u}||} = \frac{||A^{-1}\mathbf{r}||}{||A^{-1}\mathbf{f}||} \tag{1}$$

Since

$$A^{-1}\mathbf{r} = A^{-1}\cdot\mathbf{r}, \quad r = A\cdot A^{-1}\mathbf{r}, \tag{2}$$

we have

$$\frac{||\mathbf{r}||}{||A||} \leq ||A^{-1}\mathbf{r}|| \leq ||A^{-1}\mathbf{r}|| \tag{3}$$

Similarly,

$$\frac{||\mathbf{f}||}{||A||} \leq ||A^{-1}\mathbf{f}|| \leq ||A^{-1}\mathbf{f}|| \tag{4}$$

Substitute 3 and 4 into 1, then we have

$$\frac{1}{\mathrm{cond}(A)}\frac{\|\mathbf{r}\|_2}{\|\mathbf{f}\|_2} \leq \frac{\|\mathbf{e}\|_2}{\|\mathbf{u}\|_2} \leq \mathrm{cond}(A)\frac{\|\mathbf{r}\|_2}{\|\mathbf{f}\|_2} \tag{5}$$

Question 2. Exercise 8.9.

According to **Example 8.2**, we have

$$\begin{aligned}\mathrm{cond}(A) &= \frac{\max\lambda_k(A)}{\min\lambda_k(A)} = \frac{\sin^2\frac{n-1}{2n}\pi}{\sin^2\frac{1}{2n}\pi} \\ &= \frac{1}{\tan^2\frac{1}{2n}\pi} \approx \frac{4n^2}{\pi^2}\end{aligned} \tag{6}$$

For $n = 8$, we have

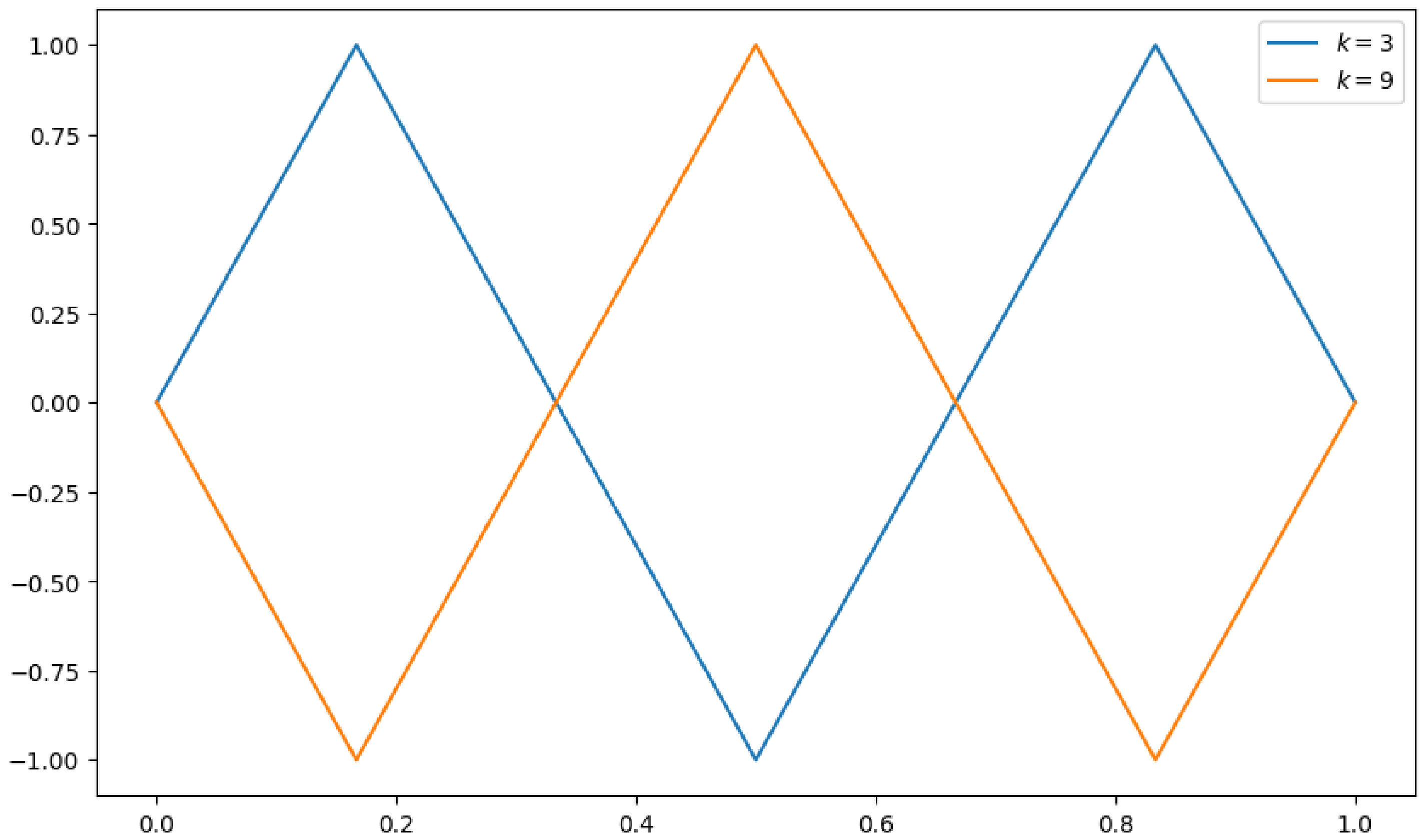
$$\mathrm{cond}(A) \approx \frac{256}{\pi^2} \tag{7}$$

For $n = 1024$, we have

$$\mathrm{cond}(A) \approx \frac{4194304}{\pi^2} \tag{8}$$

Question 3. Exercise 8.12.

Question 4. Exercise 8.15.




```
import numpy as np
import matplotlib.pyplot as plt

def f1(x):
    return np.sin(3*np.pi*x)
def f2(x):
    return np.sin(9*np.pi*x)

x = np.arange(0, 1.001, 1/6)
value1 = []
value2 = []
for t in x:
    value1.append(f1(t))
    value2.append(f2(t))
plt.figure().set_size_inches(10,6)
plt.plot(x, value1, label='$k=3$')
plt.plot(x, value2, label='$k=9$')
plt.legend()
plt.savefig('../media/Ex8_15.png', bbox_inches='tight')
```

Listing 1. code for plot

Question 5. Exercise 8.19.

According to **Definition 8.17**, we have

$$\begin{aligned} \mathbf{e}^{(l+1)} &= \mathbf{u}^{(l+1)} - \mathbf{u} = T\mathbf{u}^{(l)} + \mathbf{c} - \mathbf{u} \\ &= T\mathbf{u}^{(l)} + \mathbf{c} - T\mathbf{u} - c \\ &= T(\mathbf{u}^{(l)} - u) = T\mathbf{e}^{(l)} \end{aligned} \tag{9}$$

Thus $\mathbf{e}^{(l)} = T^l \mathbf{e}^{(0)}$.

To prove the proposition, it suffices to show that if $A \in \mathbb{C}^{n\times n}$, then

$$\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1 \tag{10}$$

(1)"⇒" Assume $\lambda \in \lambda(A): \ \rho(A) = |\lambda|$, then $\forall k$, we have $\lambda^k \in \rho(A^k)$. So

$$\rho(A)^k = |\lambda|^k \le \rho(A^k) \le ||A^k||_2, \quad \forall k \tag{11}$$

Thus $\rho(A) < 1$.

(2)"⇐" If $\rho(A) < 1$, then for any matrix norm $||\cdot||$, we have $||A|| < 1$. So

$$0 \le ||A^k|| \le ||A||^k \to 0, \ k \to \infty \tag{12}$$

Thus $\lim_{k\to\infty} A^k = 0$.

Question 6. Exercise 8.24.

According to **Definition 8.22**, we have

$$\begin{aligned} \mathbf{u}^{(l+1)} &= (1-\omega)\mathbf{u}^{(l)} + \omega(T\mathbf{u}^{(l)} + \mathbf{c}) \\ &= [(1-\omega)I + \omega T]\mathbf{u}^{(l)} + \omega\mathbf{c} \end{aligned} \tag{13}$$

So

$$\begin{aligned} T_\omega &= (1-\omega)I - \omega D^{-1}(L+U) \\ &= I - \omega D^{-1}(L+U+D) = I - \omega D^{-1}A = I - \frac{\omega h^2}{2}A. \end{aligned} \tag{14}$$

Since

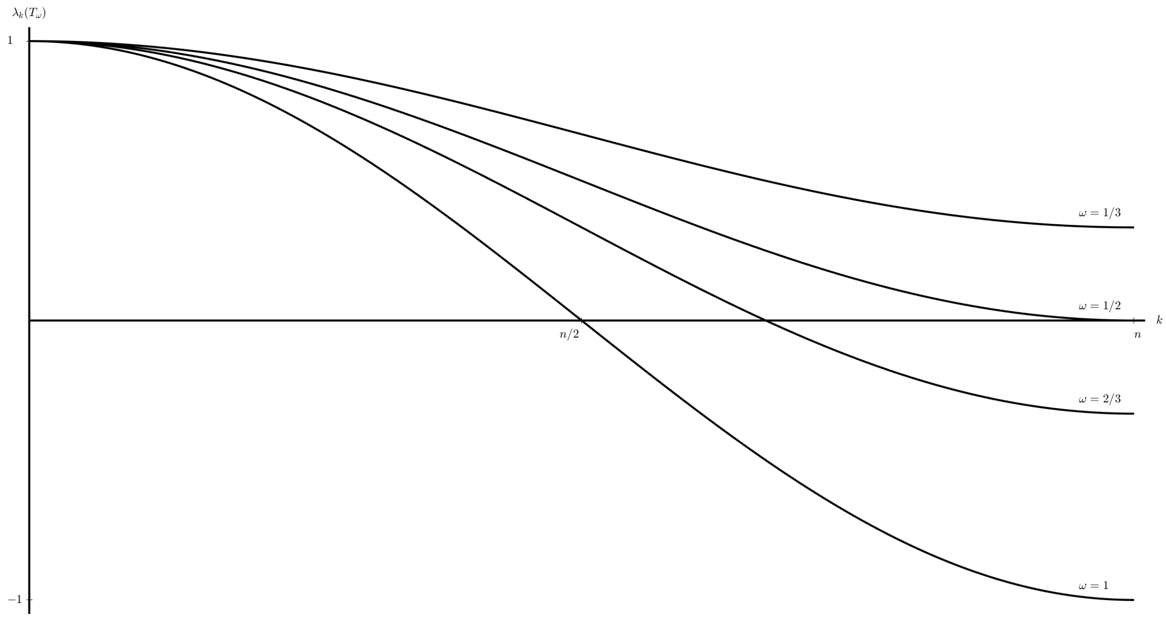
$$w_k T_\omega = w_k(I - \frac{\omega h^2}{2}A) = w_k - \frac{\omega h^2}{2}\lambda_k(A)w_k = (1 - \frac{\omega h^2}{2}\lambda_k(A))w_k, \tag{15}$$

so

$$\lambda_k(T_\omega) = 1 - \frac{\omega h^2}{2}\lambda_k(A) = 1 - 2\omega\sin^2\frac{k\pi}{2n}, \tag{16}$$

and the eigenvector of $\lambda_k(T_\omega)$ is still w_k.

Question 7. Exercise 8.25.




```
syms k w
n = 100; t = 0:0.1:n;
f = 1 - 2*w*(sin(k*pi/(2*n)))^2;
f1 = subs(subs(f,w,1/3),k,t);
f2 = subs(subs(f,w,1/2),k,t);
f3 = subs(subs(f,w,2/3),k,t);
f4 = subs(subs(f,w,1),k,t);
fig = figure();
hold on
plot(t, f1, 'k','linewidth', 2); plot(t, f2, 'k','linewidth', 2)
plot(t, f3, 'k','linewidth', 2); plot(t, f4, 'k','linewidth', 2)
% set the axis
plot([0 0], [-1.05 1.05], 'k','linewidth', 2); plot([0, 101], [0, 0], 'k', 'linewidth', 2)
xlim([0 n+1]); ylim([-1.1 1.1]);
axis off
str = {'$$\omega=1/3$$','$$\omega=1/2$$','$$\omega=2/3$$','$$\omega=1$$'};
scatter(0,1,'k+'); scatter(0,-1,'k+'); scatter(50,0,'k+'); scatter(100,0,'k+');
text(-2,1,'$1$','interpreter', 'latex', 'fontsize', 12)
text(-2,-1,'$-1$','interpreter', 'latex', 'fontsize', 12)
text(48,-0.05,'$n/2$','interpreter', 'latex', 'fontsize', 12)
text(100,-0.05,'$n$','interpreter', 'latex', 'fontsize', 12)
text(95,1/3+0.05,'$\omega=1/3$','interpreter', 'latex', 'fontsize', 12)
text(95,0.05,'$\omega=1/2$','interpreter', 'latex', 'fontsize', 12)
text(95,-1/3+0.05,'$\omega=2/3$','interpreter', 'latex', 'fontsize', 12)
text(95,-1+0.05,'$\omega=1$','interpreter', 'latex', 'fontsize', 12)
text(102,0,'$k$','interpreter', 'latex', 'fontsize', 12)
text(0,1.1,'$\lambda_k(T_\omega)$','HorizontalAlignment', 'center', 'interpreter', 'latex', 'fontsize', 12)
% save figure
scrsz=get(0,'ScreenSize'); set(gcf,'Position',scrsz);
saveas(gcf, '../media/Ex8_25.png')
```

Listing 2. code for the above figure

Since when $n = 64$

$$\min_{\omega\in[0,1]} \rho(T_\omega) = \min_{\omega\in[0,1]} \max_{k\in[1,n-1]} \lambda_k(T_\omega) = 1 - 2\sin^2 \frac{\pi}{128} = 0.9988 > 0.9986, \tag{17}$$

then $\forall\omega \in [0,1]\ \rho(T_\omega) \geq 0.9986$.