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Num PDE
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Homework #1(Chapter 8)

Question 1. Exercise 8.8.

In the following proof, $||\cdot||_2$ is denoted by $||\cdot||$. According to **Definition 8.4** and **Lemma 8.5**, we have

$$\frac{||\mathbf{e}||}{||\mathbf{u}||} = \frac{||A^{-1}\mathbf{r}||}{||A^{-1}\mathbf{f}||} \tag{1}$$

Since

$$A^{-1}\mathbf{r} = A^{-1}\cdot\mathbf{r}, \quad r = A\cdot A^{-1}\mathbf{r}, \tag{2}$$

we have

$$\frac{||\mathbf{r}||}{||A||} \le ||A^{-1}\mathbf{r}|| \le ||A^{-1}\mathbf{r}|| \tag{3}$$

Similarly,

$$\frac{||\mathbf{f}||}{||A||} \le ||A^{-1}\mathbf{f}|| \le ||A^{-1}\mathbf{f}|| \tag{4}$$

Substitute 3 and 4 into 1, then we have

$$\frac{1}{\mathrm{cond}(A)}\frac{\|\mathbf{r}\|_2}{\|\mathbf{f}\|_2} \le \frac{\|\mathbf{e}\|_2}{\|\mathbf{u}\|_2} \le \mathrm{cond}(A)\frac{\|\mathbf{r}\|_2}{\|\mathbf{f}\|_2} \tag{5}$$

Question 2. Exercise 8.9.

According to **Example 8.2**, we have

$$\begin{aligned} \mathrm{cond}(A) &= \frac{\max \lambda_k(A)}{\min \lambda_k(A)} = \frac{\sin^2 \frac{n-1}{2n}\pi}{\sin^2 \frac{1}{2n}\pi} \\ &= \frac{1}{\tan^2 \frac{1}{2n}\pi} \approx \frac{4n^2}{\pi^2} \end{aligned} \tag{6}$$

For $n = 8$, we have

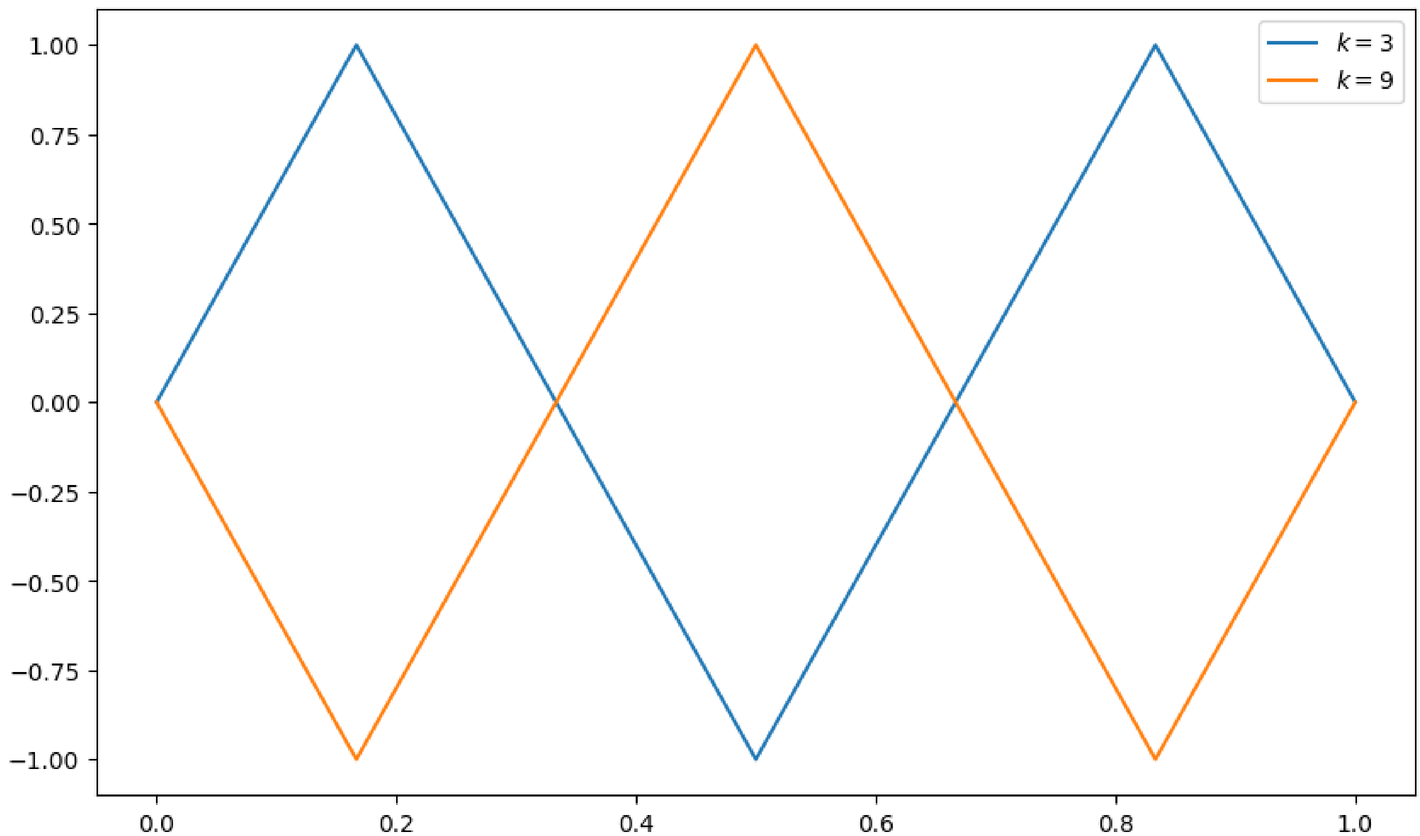
$$\mathrm{cond}(A) \approx \frac{256}{\pi^2} \tag{7}$$

For $n = 1024$, we have

$$\mathrm{cond}(A) \approx \frac{4194304}{\pi^2} \tag{8}$$

Question 3. Exercise 8.12.

Question 4. Exercise 8.15.




```python
import numpy as np
import matplotlib.pyplot as plt

def f1(x):
    return np.sin(3*np.pi*x)
def f2(x):
    return np.sin(9*np.pi*x)

x = np.arange(0, 1.001, 1/6)
value1 = []
value2 = []
for t in x:
    value1.append(f1(t))
    value2.append(f2(t))
plt.figure().set_size_inches(10,6)
plt.plot(x, value1, label='$k=3$')
plt.plot(x, value2, label='$k=9$')
plt.legend()
plt.savefig('../media/Ex8_15.png', bbox_inches='tight')
```

Listing 1. code for plot

Question 5. Exercise 8.19.

According to **Definition 8.17**, we have

$$
\begin{aligned}
\mathbf{e}^{(l+1)} &= \mathbf{u}^{(l+1)} - \mathbf{u} = T\mathbf{u}^{(l)} + \mathbf{c} - \mathbf{u} \\
&= T\mathbf{u}^{(l)} + \mathbf{c} - T\mathbf{u} - c \\
&= T(\mathbf{u}^{(l)} - u) = T\mathbf{e}^{(l)}
\end{aligned} \tag{9}
$$

Thus $\mathbf{e}^{(l)} = T^l\mathbf{e}^{(0)}$.

To prove the proposition, it suffices to show that if $A \in \mathbb{C}^{n\times n}$, then

$$
\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1 \tag{10}
$$

(1)"⇒" Assume $\lambda \in \lambda(A):\ \rho(A) = |\lambda|$, then $\forall k$, we have $\lambda^k \in \rho(A^k)$. So

$$
\rho(A)^k = |\lambda|^k \le \rho(A^k) \le ||A^k||_2, \quad \forall k \tag{11}
$$

Thus $\rho(A) < 1$.

(2)"⇐" If $\rho(A) < 1$, then for any matrix norm $||\cdot||$, we have $||A|| < 1$. So

$$
0 \le ||A^k|| \le ||A||^k \to 0,\ k \to \infty \tag{12}
$$

Thus $\lim_{k\to\infty} A^k = 0$.

Question 6. Exercise 8.24.

According to **Definition 8.22**, we have

$$
\begin{aligned}
\mathbf{u}^{(l+1)} &= (1-\omega)\mathbf{u}^{(l)} + \omega(T\mathbf{u}^{(l)} + \mathbf{c}) \\
&= [(1-\omega)I + \omega T]\mathbf{u}^{(l)} + \omega\mathbf{c}
\end{aligned} \tag{13}
$$

So

$$
\begin{aligned}
T_\omega &= (1-\omega)I - \omega D^{-1}(L+U) \\
&= I - \omega D^{-1}(L+U+D) = I - \omega D^{-1}A = I - \frac{\omega h^2}{2}A.
\end{aligned} \tag{14}
$$

Since

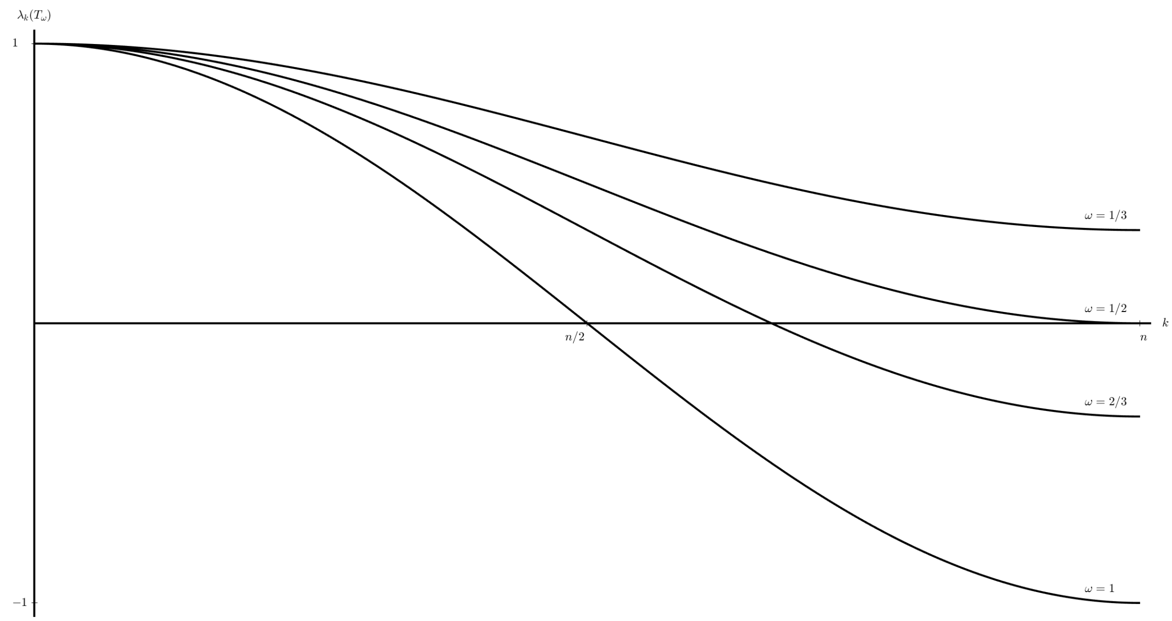
$$
w_k T_\omega = w_k(I - \frac{\omega h^2}{2}A) = w_k - \frac{\omega h^2}{2}\lambda_k(A)w_k = (1 - \frac{\omega h^2}{2}\lambda_k(A))w_k, \tag{15}
$$

so

$$
\lambda_k(T_\omega) = 1 - \frac{\omega h^2}{2}\lambda_k(A) = 1 - 2\omega\sin^2\frac{k\pi}{2n}, \tag{16}
$$

and the eigenvector of $\lambda_k(T_\omega)$ is still w_k.

Question 7. Exercise 8.25.




```matlab
syms k w
n = 100; t = 0:0.1:n;
f = 1 - 2*w*(sin(k*pi/(2*n)))^2;
f1 = subs(subs(f,w,1/3),k,t);
f2 = subs(subs(f,w,1/2),k,t);
f3 = subs(subs(f,w,2/3),k,t);
f4 = subs(subs(f,w,1),k,t);
fig = figure();
hold on
plot(t, f1, 'k','linewidth', 2); plot(t, f2, 'k','linewidth', 2)
plot(t, f3, 'k','linewidth', 2); plot(t, f4, 'k','linewidth', 2)
% set the axis
plot([0 0], [-1.05 1.05], 'k','linewidth', 2); plot([0, 101], [0, 0], 'k', 'linewidth', 2)
xlim([0 n+1]); ylim([-1.1 1.1]);
axis off
scatter(0,1,'k+'); scatter(0,-1,'k+'); scatter(50,0,'k+'); scatter(100,0,'k+');
text(-2,1,'$1$','interpreter', 'latex', 'fontsize', 12)
text(-2,-1,'$-1$','interpreter', 'latex', 'fontsize', 12)
text(48,-0.05,'$n/2$','interpreter', 'latex', 'fontsize', 12)
text(100,-0.05,'$n$','interpreter', 'latex', 'fontsize', 12)
text(95,1/3+0.05,'$\omega=1/3$','interpreter', 'latex', 'fontsize', 12)
text(95,0.05,'$\omega=1/2$','interpreter', 'latex', 'fontsize', 12)
text(95,-1/3+0.05,'$\omega=2/3$','interpreter', 'latex', 'fontsize', 12)
text(95,-1+0.05,'$\omega=1$','interpreter', 'latex', 'fontsize', 12)
text(102,0,'$k$','interpreter', 'latex', 'fontsize', 12)
text(0,1.1,'$\lambda_k(T_\omega)$','HorizontalAlignment', 'center', 'interpreter', 'latex', 'fontsize', 12)
% save figure
scrsz=get(0,'ScreenSize'); set(gcf,'Position',scrsz);
saveas(gcf, '../media/Ex8_25.png')
```

LISTING 2. code for the above figure

Since when $n = 64$

$$\min_{\omega\in[0,1]} \rho(T_\omega) = \min_{\omega\in[0,1]} \max_{k\in[1,n-1]} \lambda_k(T_\omega) = 1 - 2\sin^2\frac{\pi}{128} = 0.9988 > 0.9986, \tag{17}$$

then $\forall \omega \in [0,1]\ \rho(T_\omega) \geq 0.9986$.

Question 8. Exercise 8.28.

To make the code more clear, we use Python instead.

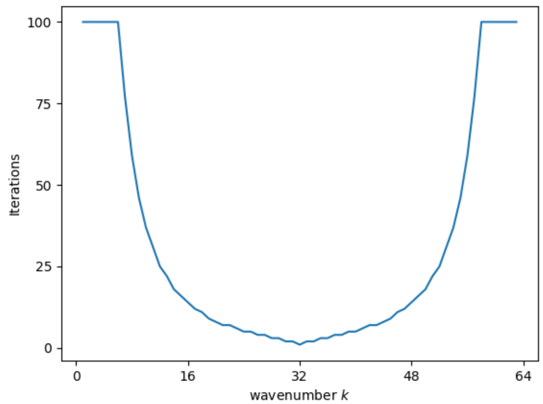


Figure 1. $\omega = 1$

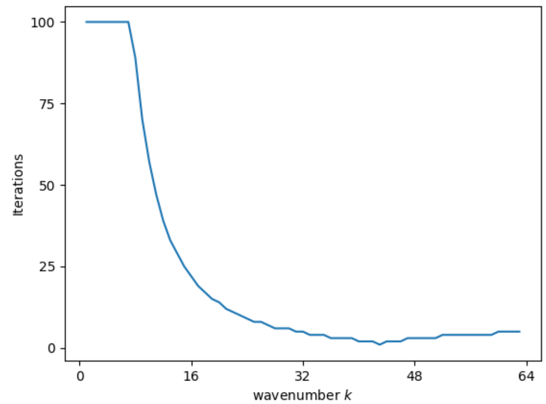


Figure 2. $\omega = 2/3$

```python
import numpy as np
import matplotlib.pyplot as plt

def gen_A(n):
    A = np.zeros([n-1, n-1])
    for i in range(n-1):
        for j in range(n-1):
            if i == 0:
                A[i][0] = 2
                A[i][1] = -1
            elif i == n-2:
                A[i][i] = 2
                A[i][i-1] = -1
            else:
                A[i][i] = 2
                A[i][i-1] = A[i][i+1] = -1
    A = A * (n**2)
    return A

def Jacobi(n, w0, omega):
    A = np.eye(n-1) - gen_A(n) * (omega/(2*n**2))
    w0 = -w0
    err = np.linalg.norm(w0, ord=np.inf)
    for i in range(100):
        w0 = np.matmul(A, w0)
        if(np.linalg.norm(w0, ord=np.inf) < err/100):
            return i + 1
    return 100

def Fourier(n, k):
    w = []
    for i in range(1, n):
```

```
        w.append(np.sin(i/n*k*np.pi))
    return np.array(w)

k = [i for i in np.arange(1, 64, 1)]
iter_1 = [] # omega = 1
iter_2 = [] # omega = 2/3
for i in k:
    w0 = Fourier(64, i)
    iter_1.append(Jacobi(64, w0, 1))
    iter_2.append(Jacobi(64, w0, 2/3))
plt.figure()
plt.plot(k, iter_1)
plt.xlabel('wavenumber $k$')
plt.ylabel('Iterations')
plt.xticks(range(0,65,16))
plt.yticks(range(0,101,25))
plt.savefig('../media/Ex8_28_a')
plt.figure()
plt.plot(k, iter_2)
plt.xlabel('wavenumber $k$')
plt.ylabel('Iterations')
plt.xticks(range(0,65,16))
plt.yticks(range(0,101,25))
plt.savefig('../media/Ex8_28_b')
```

LISTING 3. code for the above figure

Question 9. Exercise 8.41.

According to **Lemma 8.39**, we have

$$\begin{aligned}&\text{FMG computation cost}\\&=(\frac{2}{1-2^{-d}})(1+2^{-d}+2^{-2d}+\cdots+2^{-nd})\text{WU}<\frac{2}{(1-2^{-d})^2}\text{WU}\end{aligned}\tag{18}$$

Let $D=1,2,3$ respectively, we have

$$\begin{aligned}\text{computation cost}|_{D=1}&=8\\\text{computation cost}|_{D=2}&=\frac{32}{9}\\\text{computation cost}|_{D=3}&=\frac{128}{49}\end{aligned}\tag{19}$$

Question 10. Exercise 8.47.

For $k \ll \frac{n}{2}$, we know that $s_k = O(\frac{k^2}{n^2})$ and the exponent of λ_k and $\lambda_{k'}$ are both less than 1, thus c_1, c_2 are both small. Since $k' \in [\frac{n}{2}, n-1]$, then for large ν_1, $\lambda_{k'}^{\nu_1} \to 0$. So c_3, c_4 are both small.

Hence, all four c_i's are small, particularly for $k \ll \frac{n}{2}$ or as ν becomes large.

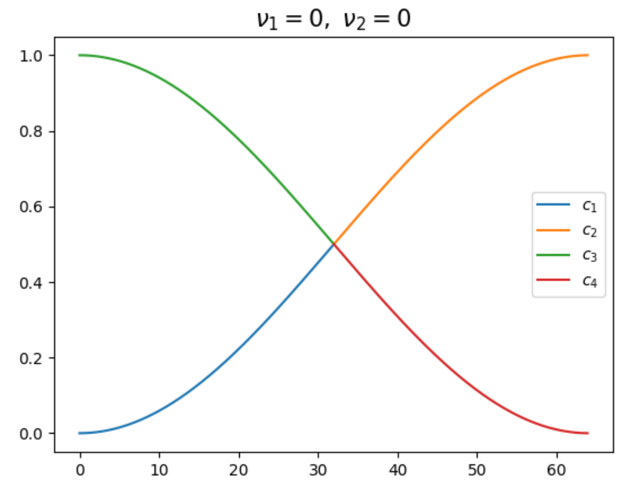


FIGURE 3. (a)

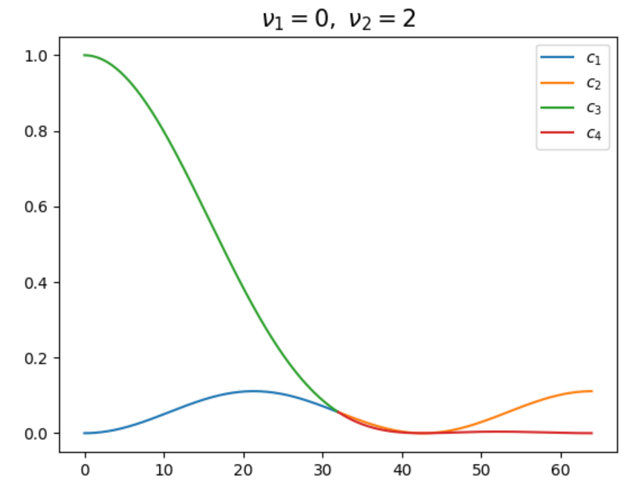


FIGURE 4. (b)

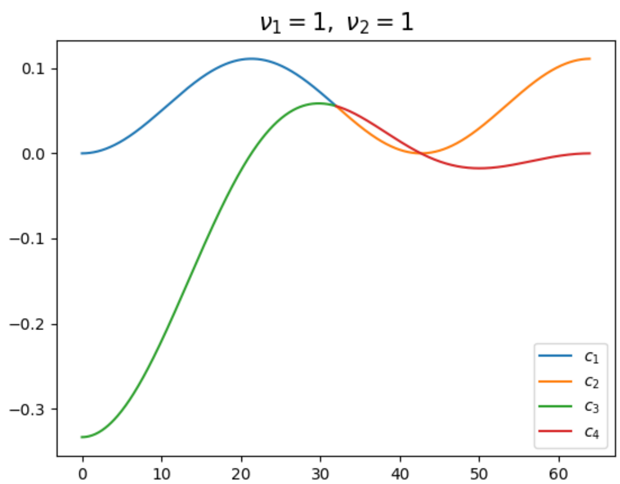


FIGURE 5. (c)

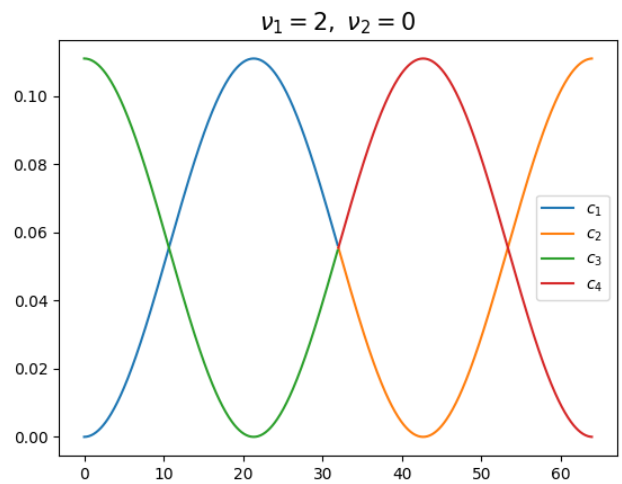


FIGURE 6. (d)

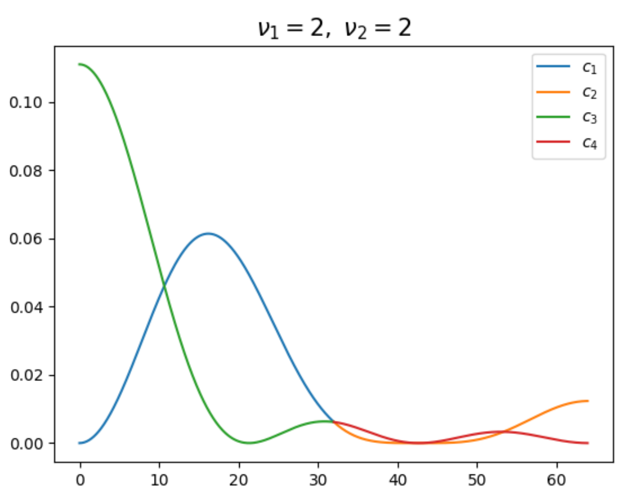


FIGURE 7. (e)

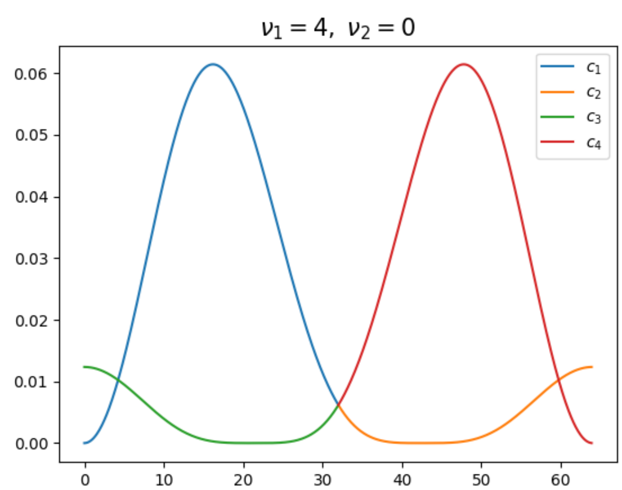


FIGURE 8. (f)

```python
import numpy as np
import matplotlib.pyplot as plt

class TG:
    def __init__(self, n, omega):
        self.n = n
        self.omega = omega

    def s(self, k):
        return (np.sin(k*np.pi/2/self.n))**2

    def c(self, k):
        return 1 - self.s(k)

    def lamb(self, k):
        return 1 - 2*self.omega*self.s(k)

    def plot(self, v1, v2):
        k1 = np.arange(0, self.n/2, 0.1)
        k2 = np.arange(self.n/2, self.n, 0.1)
        c1, c2, c3, c4 = [], [], [], []
        for k in k1:
            c1.append(np.power(self.lamb(k), v1+v2)*self.s(k))
            c3.append(np.power(self.lamb(self.n-k), v1)*np.power(self.lamb(k), v2)*self.c(k))
        for k in k2:
            c2.append(np.power(self.lamb(k), v1+v2)*self.s(k))
            c4.append(np.power(self.lamb(self.n-k), v1)*np.power(self.lamb(k), v2)*self.c(k))
        plt.figure()
        plt.title("$\\nu_1 = {v1},\ \\nu_2 = {v2}$".format(v1 = str(v1), v2 = str(v2)), fontsize = 15)
        plt.plot(k1, c1, k2, c2, k1, c3, k2, c4)
        plt.legend(['$c_1$', '$c_2$', '$c_3$', '$c_4$'], fontsize = 10)
        return plt.gcf()

def main():
    myTG = TG(64, 2/3)
    myTG.plot(0, 0)
    plt.savefig('../media/Ex8_47_00')
    myTG.plot(0, 2)
    plt.savefig('../media/Ex8_47_02')
    myTG.plot(1, 1)
    plt.savefig('../media/Ex8_47_11')
    myTG.plot(2, 0)
    plt.savefig('../media/Ex8_47_20')
    myTG.plot(2, 2)
    plt.savefig('../media/Ex8_47_22')
    myTG.plot(4, 0)
    plt.savefig('../media/Ex8_47_40')

if __name__ == '__main__':
    main()
```

Listing 4. code for the above figure

Question 11. Exercise 8.51.

According to **Lemmas 8.48** and **8.49**, we have

$$\mathcal{R}(I_h^{2h}) = \mathcal{R}(I_{2h}^h) = \frac{n}{2} - 1, \quad \mathcal{N}(I_h^{2h}) = n - 1 - \mathcal{R}(I_h^{2h}) = \frac{n}{2} \tag{20}$$