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Homework #2(Chapter 9)

Question 1. Exercise 9.9.

The NRM-1 holds obviously. For NRM-2, if $||\mathbf{v}|| = 0$, then $\forall \mathbf{x} \in \mathbb{F}^n, T\mathbf{x} = \mathbf{0}$, i.e. $T = \mathbf{0}$. NRM-3 holds, because for any$a \in \mathbb{F},\ \mathbf{x} \in \mathbb{F}^n$ we have $(aT)\mathbf{x} = T(a\mathbf{x})$. NRM-4 holds, because for any $T_1, T_2 \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ and $\mathbf{x} \in \mathbb{F}^n$, we have

$$||(T_1 + T_2)\mathbf{x}|| = ||T_1\mathbf{x} + T_2\mathbf{x}|| \leq ||T_1\mathbf{x}|| + ||T_2\mathbf{x}|| \leq (||T_1|| + ||T_2||)\mathbf{x} \tag{1}$$

Question 2. Exercise 9.14.

We verify each proposition in **Definition C.74** respectively.

(1) According to **Definition 9.8**, $d(T, S) > 0$.

(2) if $d(T, S) = ||T - S|| = \mathbf{0}$, then $T - S = \mathbf{0}$, i.e. $T = S$. Contrarily, if $T = S$, then $d(T, S) = 0$.

(3) $d(T, S) = ||T - S|| = ||S - T|| = d(S, T)$.

(4) For any $R, S, T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, then

$$d(R, S) + d(S, T) = ||R - S|| + ||S - T|| \geq ||(R - S) + (S - T)|| = ||R - T|| = d(R, T) \tag{2}$$

Question 3. Exercise 9.15. Suppose $A \in \mathbb{C}^{m\times n}$ is the matrix representation of T under standard basis vectors. Since $T\mathbf{e}_j \in \mathbb{R}^m$, then $A \in \mathbb{R}^{m\times n}$. Thus T carries $\mathbb{R}^n$ into $\mathbb{R}^m$. Obviously, we have

$$||T|| = \max_{\mathbf{z}\in\mathbb{C}^n:|\mathbf{z}|=1} |T\mathbf{z}| \tag{3}$$

Hence $||T|| = \sup_{\mathbf{z}\in\mathbb{C}^n:|\mathbf{z}|\leq 1} |T\mathbf{z}|$.

Suppose the real symmetric matrix A^TA has eigenvalues $\lambda_1, \cdots, \lambda_n$ and corresponding eigenvectors $v_1, \cdots, v_n$(WLOG, eigenvectors are orthonormal). Then

$$\forall \mathbf{z} \in \mathbb{C}^n : |\mathbf{z}| = 1\ \exists c_1, \cdots, c_n \in \mathbb{C},\ s.t. \begin{cases} |c_1|^2 + \cdots + |c_n|^2 = 1 \\ \mathbf{z} = c_1v_1 + \cdots + c_nv_n \end{cases} \tag{4}$$

Then

$$\begin{aligned} |A\mathbf{z}| &= \bar{\mathbf{z}}^TA^TAz = (\bar{c_1}v_1^T + \cdots + \bar{c_n}v_n^T)A^TA(c_1v_1 + \cdots + c_nv_n) \\ &= (\bar{c_1}v_1^T + \cdots + \bar{c_n}v_n^T)(\lambda_1c_1v_1 + \cdots + \lambda_nc_nv_n) \\ &= \lambda_1|c_1|^2 + \cdots + \lambda_n|c_n|^2 = |A(|c_1|v_1 + \cdots + |c_n|v_n)| =: |A\mathbf{x}|\ (\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1), \end{aligned} \tag{5}$$

which indicates that

$$\forall \mathbf{z} \in \mathbb{C}^n : |\mathbf{z}| = 1\ \exists \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1,\ s.t.\ |A\mathbf{z}| = |A\mathbf{x}|. \tag{6}$$

So

$$||T|| = \max_{\mathbf{z}\in\mathbb{C}^n:|\mathbf{z}|=1} |T\mathbf{z}| = \max_{\mathbf{x}\in\mathbb{R}^n:|\mathbf{x}|=1} |T\mathbf{x}| = \sup_{\mathbf{x}\in\mathbb{R}^n:|\mathbf{x}|\leq 1} |T\mathbf{x}| \tag{7}$$

Question 4. Exercise 9.17.

NRM-1,2,3 obviously hold. Suppose $T_1, T_2 \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, and A, B are the matrix of T_1, T_2 respectively. According to **Corollary 9.18** we know that

$$|T_1| = (\sum_{i,j} |a_{ij}|^2)^{\frac{1}{2}}, \quad |T_2| = (\sum_{i,j} |b_{ij}|^2)^{\frac{1}{2}}, \quad |T_1 + T_2| = (\sum_{i,j} |a_{ij} + b_{ij}|^2)^{\frac{1}{2}} \tag{8}$$

Then from 8, we see that $|T_1|, |T_2|$ can be regarded as the 2-norm of vectors

$$(a_{11}, \cdots, a_{1n}, \cdots, a_{m1}, \cdots, a_{mn}), \ (b_{11}, \cdots, b_{1n}, \cdots, b_{m1}, \cdots, b_{mn}) \tag{9}$$

respectively. Thus NRM-4 follows from the property of 2-norm.